1. LOGIC

There existed an egg who sat on a wall,
And the wall being short implies this story is tall.
Now if that fat egg had had a great fall
Or slipped off the top, but not jumped, then not all
The king's horses and all the king's men,
If they worked through the day and the evening, then
They could not succeed if and only if when
They attempted to put Humpty together again.

§1.1. The Role of Logic in Mathematics

Over many hundreds of years mathematics has played a vital role in the development of the sciences and so it is often regarded as a science itself. But mathematics is quite different to any of the other sciences in one important respect. It has no empirical facts. Physicists carry out experiments and formulate hypotheses. They use mathematics as a language to describe their hypotheses and as a tool to explore their logical consequences. But in the final analysis physical theories must be supported by experimental facts.

Where mathematics appears to come closest to being an experimental science is in geometry. We measure the angles of a triangle and they appear to total 180°. But when we do that we are acting as experimental physicists. We now know that there is no logical necessity for the sum of the angles of a triangle to be 180°. While this *is* a fact in Euclidean geometry, the geometry we all learnt at school, there are other logically consistent geometries where this is not the case. Could it not be that the angles of a triangle add up to slightly more than 180° or a little less. Perhaps the discrepancy is so minute that it has not yet been detected in our measurements. Who is to say whether Euclidean geometry is the right one to describe physical space? Only a physicist could do this with careful measurement – certainly not a mathematician.

In a very real sense, mathematics has no facts. It is concerned with the logical relationships *between* facts. Well then, is mathematics just the same as logic? No. Logic provides the tools for reasoning. Mathematics uses these logical tools to construct elaborate systems which can be used by science as models for different aspects of the real world.

If mathematics was to be viewed as a science, and there are those who argue that it is, then its laboratory is the human mind. Its instruments are the techniques of proof. This is why it is important for anyone studying mathematics to have an appreciation of logic.

§1.2. The Role of Logic in Computing Science

What was said about logic in mathematics is also true of computing science because in theoretical computer science there are theorems and proofs just as in mathematics. In fact there are those chauvinist mathematicians who dismiss theoretical computer science or theoretical physics as just particular branches of mathematics!

In one sense they are right. But computer science and theoretical physics are sufficiently integrated with experimental and practical aspects that for practical purposes they are quite properly studied in their own specialised departments. However you should never forget that the boundary between theoretical computing science and discrete mathematics is somewhat arbitrary. There is really a continuum that takes us all the way from mathematics to computer science and back again.

So, just as logic is important to mathematics, it is important to computer science. But for computer scientists, logic plays another very important role which affects some of the most applied areas of that subject.

It is no accident that at the heart of the central processing chip there is the "arithmetic logic unit". We talk about the logic of computer programs, for very good reasons. At a higher level still, to understand the theory of databases one needs to understand logic, and its partner, set theory. And another area, of interest to both mathematicians and computing scientists alike, is the area of automated theorem proving and proofs of program correctness.

§1.3. Propositions and Truth Functions

Definition: A **proposition** is a statement for which it is meaningful to say that it is true or false (but not both).

Definition The **truth value** of a proposition p is T or F according as the proposition is false or true.

T = TRUE F = FALSE

Now we do not need to ask the deep question "what is truth?" For our purposes truth values can be thought of as tags that are given to certain statements. Logic is concerned with *relative* truth, that is, the truth of a compound statement given the truth of its primitive constituents and the truth of those primitive statements is established by other means, or is simply assumed in axioms or definitions.

Example 1

"It rained today" is a proposition while "Will it rain today?" or "Come inside, it's raining." are not.

Some statements appear to be propositions, but on closer examination they are not because to attach a truth value to them leads to a logical paradox.

Example 2

THIS STATEMENT IS FALSE

is not a proposition because if we say that it is true then it is false and if we say that it is false then it is true.

It is the self-referential nature of this statement, the fact that it purports to be saying something about itself, which appears to cause the problem. However there are cases, involving no self-referentiality, that still lead to trouble.

Example 3

Consider the following infinite list of statements:

AT LEAST ONE OF THE FOLLOWING STATEMENTS IS FALSE
AT LEAST ONE OF THE FOLLOWING STATEMENTS IS FALSE
AT LEAST ONE OF THE FOLLOWING STATEMENTS IS FALSE
AT LEAST ONE OF THE FOLLOWING STATEMENTS IS FALSE

…………………………………………………………………….

There is no self-referentiality here. Each statement purports to be saying something about the remaining ones. So, although each reads identically to the next, they are all different statements and so they do not all have to have the same truth value.

But if any one of them is false then all the remaining ones are true and it is easy to see that this leads to a contradiction. But if they are all true we again get a contradiction.

Such problems can keep a logician awake at night, but we take the more practical view that we will ignore such potential problems and hope that we never meet them. That is a reasonable practical attitude because such logical paradoxes only exist in the laboratory. One has to go out of one's way to produce them.

§1.4. Compound Propositions

Compound propositions can be made up from simpler ones in such a way that their truth value can be determined from those of their constituents.

Not, And, Or

The simplest truth operator is "not" which only involves one constituent. Its truth table is:

p	not p
T	F
F	T

The truth tables for "and" and "or" are:

p	q	p and q
T	T	T
T	F	F
F	F	F
F	T	F

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

Notice that "p or q" is defined to be TRUE even when both are true. Sometimes in everyday English we use "or" in an exclusive sense, but in logic, and in mathematics, it always includes the possibility of both.

Richard Feynman, a famous physicist, was once asked by the Dean's wife at Princeton whether he wanted milk or lemon in his tea.. He replied "both". "Surely you're joking Mr Feynman." He was, but he was making a point about the mathematician's use of the word "or" – it *can* include both.

In order to work at the level of the underlying logical structure we denote primitive propositions (ones which are not built up from even simpler ones) by letters of the alphabet just as in algebra we represent numbers by letters.

The above three truth operators are denoted by special symbols $-$, $\wedge$, $\vee$ respectively.

$-$p denotes "not p";
p $\wedge$ q denotes "p and q";
p $\vee$ q denotes "p or q";

Several variations are in common use: "not p" is often denoted by p' , $\neg$p, or $\overline{p}$. Sometimes "p and q" is denoted by p.q and "p or q" by p + q.

Example 4

$(p \vee q) \wedge -(p \wedge q)$ denotes the compound statement "p or q but not both". This is the "exclusive or". Also note the use of the word "but" here. It is simply an alternative to "and". While there may be overtones of contrast or emphasis with "but" that we do not get with "and", at the fundamental level of logic their meaning is identical.

Implication

Our next truth operator is one that is very much misunderstood — implication. The definition of "p implies q" is given by its truth table:

p	**q**	**p implies q**
T	T	T
T	F	F
F	T	T
F	F	T

The problem that many people have with this definition is the third row which says that a false proposition implies a true one! In fact all it shows is that the technical definition of implication differs somewhat from the ordinary sense of the word. In normal usage implication involves a causal connection. It might be the case that I am wealthy and that I am honest. In the ordinary sense of the word we would not say however that "being wealthy implies that I am honest". Wealth does not cause honesty. However if both propositions are true for me then, in the sense of propositional logic, "I am wealthy" implies "I am honest".

Because propositional logic deals with isolated propositions it cannot express the notion of wealthy people *always* being honest (or its negation). That requires "quantifiers", something we will discuss later.

If you still feel uneasy about the above definition of implication you might like to ask yourself how else you might define it. For example if you decide that false statements cannot imply anything then you would want to change the last two rows of the truth table to become:

p	**q**	
T	T	T
T	F	F
F	T	F
F	F	F

But this is just the truth table for "and" and surely "implies" does not mean the same as "and". The final vindication for the accepted definition however only comes when we see implication in the context of quantifiers (see §1.8).

Notation: We denote "p implies q" by $p \rightarrow q$. Sometimes this is written as $p \Rightarrow q$ or $p \supset q$.

Equivalence

Definition: p is (logically) **equivalent to** q if they have the same truth value.

Notation: We will denote "p is equivalent to q" by $p \leftrightarrow q$. Other notations in use are $p \equiv q$ and $p \Leftrightarrow q$.

This definition can be set out in a truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Summary of Truth Operators

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

§1.5. Tautologies

A **tautology** is a proposition built up from primitive propositions, which is always true irrespective of the truth values of the constituent propositions.

Tautologies are logical theorems. For example "(p and q) implies (q and p)" does not give us any information about the statements p and q. Rather it tells us about the symmetry of the "and" operator.

Example 5: The following three propositions are tautologies:
(1) $p \leftrightarrow p$
(2) $p \rightarrow (p \vee q)$
(3) $(p \wedge q) \leftrightarrow (q \wedge p)$
But $(p \vee q) \rightarrow (p \wedge q)$ is not a tautology.

§1.6. Translating From English

$p \rightarrow q$ might be expressed as:

if p then q
p implies q
q is implied by p
q, if p
p only if q
p is a sufficient condition for q
q is a necessary condition for p

$p \leftrightarrow q$ might be expressed as:

p is equivalent to q
p if and only if q (sometimes this is abbreviated to "p iff q")
p is a necessary and sufficient condition for q

$p \wedge q$ might be expressed as:

p and q
not only p, but q

$p \vee q$ might be expressed as:

p or q
unless p then q
p, unless q
at least one of p and q

Miscellaneous constructions:

p but not q	$p \wedge -q$
neither p nor q	$-(p \vee q)$
p or q but not both	$(p \vee q) \wedge -(p \wedge q)$

or more simply as $p \leftrightarrow -q$

[To see that these are equivalent use a truth table.]

§1.7. Laws of Logic

The following is a list of tautologies that are commonly used.
Examine them, think about them, but *do not attempt to learn the list*.

Commutative Laws:

(1) $(p \vee q) \leftrightarrow (q \vee p)$
(2) $(p \wedge q) \leftrightarrow (q \wedge p)$

Associative Laws:

(3) $((p \vee q) \vee r) \leftrightarrow (p \vee (q \vee r))$
(4) $((p \wedge q) \wedge r) \leftrightarrow (p \wedge (q \wedge r))$

Distributive Laws:

(5) $((p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$
(6) $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$

Idempotent Laws:

(7) $(p \vee p) \leftrightarrow p$
(8) $(p \wedge p) \leftrightarrow p$

De Morgan Laws:

(9) $-(p \vee q) \leftrightarrow (-p \wedge -q)$
(10) $-(p \wedge q) \leftrightarrow (-p \vee -q)$

Miscellaneous Laws:

(11) $(p \wedge q) \rightarrow p$
(12) $p \rightarrow (p \vee q)$
(13) $-(-p) \leftrightarrow p$
(14) $p \vee -p$
(15) $-(p \wedge -p)$

These last two assert that a proposition must either be true or false but not both.

Syllogism:

(16) $((p \to q) \wedge p) \to q$

A **syllogism** is a logical argument of the form:

$p \to q$
But p.
Therefore q.

Proof by Contradiction:

(17) $((p \to q) \wedge -q) \to -p$

Example 6:
"If you'd left the keys on the table they'd still be there. But the keys aren't there so you couldn't have left them there." which analyses formally to:

p = "You left your keys on the table."
q = "The keys are still there".

If p then q. But not q. Hence not p.
(Of course there are hidden assumptions such as "nobody else has come in" and "keys cannot spontaneously evaporate").

Transitive Property of Implication:

(18) $((p \to q) \wedge (q \to r)) \to (p \to r)$

§1.8. Quantifiers

A **predicate** is a statement that involves variables.

Predicates become propositions when particular objects (e.g. numbers) are substituted for the variables. The resulting propositions have truth values that depend on those elements.

An **n-ary predicate** is one which applies to a combination of n elements. Special terms are **unary** if $n = 1$, **binary** if $n = 2$ and **ternary** if $n = 3$.

A unary predicate is what we usually think of as a property, such as "x is even" or "x is female". We could write these symbolically as Ex for "x is even" and Fx for "x is female". There must be some underlying set over which the variables range. In the case of E it might be the set of integers and in the case of F it might be the set of all students in a particular class.

A binary predicate is what we usually think of as a relation, such as "$x < y$" or "x knows y". We could write these symbolically as xLy for "$x < y$" and xKy for "x knows y". The first of these example might have x and y ranging over the set of integers and the second example might have x and y ranging over the set of students in a particular class.

But x and y do not have to range over the same set. The relation "x has completed assignment y" could have x ranging over all the students in a particular class and y ranging over the set of assignment numbers.

Example 7:
"x is even" is a unary predicate that is TRUE for x = 2 and FALSE for x = 3.
"x loves y" is a binary predicate involving two elements x and y from a set of people.
"x is the distance between P and Q" is a ternary predicate involving a combination of a number and two points.

Notation: We commonly denote predicates by upper case letters and the variables by lower case letters. Often the variable names follow the predicate name such as Pn, Qxy and Rabc. But in the case of a binary predicate it is more usual to put one element on each side of the predicate name as in xRy. This what we usually do in mathematics with predicates such as "x = y" and "x < y".

If P is a unary predicate then **∀xPx** means "for all x, Px (is true)" and **∃xPx** denotes "for some x, Px", or "there exists x such that Px". "For all" is called the **universal** quantifier and "for some" is called the **existential** quantifier.

NOTES:
(1) "Some" means "at least one". Even if Px is true for just one x we are entitled to claim that ∃xPx.

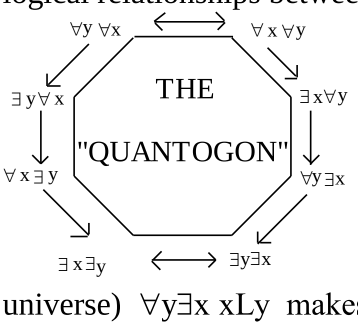
(2) There is some assumed non-empty universe over which we are quantifying.

(3) The variable "bound" by a quantifier is a "dummy" variable and can be replaced by another provided this is done consistently throughout the scope of the quantifier. This means that ∀x(Px → Qx) is equivalent to ∀t(Pt → Qt) or ∀w(Pw → Qw). This is analogous to the dummy variable of integration in a definite integral and to local variables in many computing languages.

(4) When working with quantifiers "Px implies Qx" usually has an implied universal quantifier and should be translated as ∀x(Px → Qx). It is for this reason that implication is defined to be true when the "antecedent" is false. The statement ∀x(x is even → x^2 is even) has to be true for *all* integers, including odd ones. The definition of implication makes it "vacuously true" in the case of odd integers.

§1.9. The Order of Quantifiers

With a predicate involving two variables there are eight ways they can be quantified: each of x, y can be quantified by ∀ or ∃ and they can be quantified in either order. The logical relationships between them are displayed by the following diagram:



Note carefully the difference between ∃x∀y and ∀y∃x. The first is the stronger of the two. It implies the second but not conversely. For example ∃x∀y[x > y] claims that there is an integer larger than every integer (FALSE) while ∀y∃x[x > y] makes the weaker (TRUE) claim that for every integer there is a larger one.

If xLy means "x loves y" (within an appropriate universe) ∀y∃x xLy makes the claim that "nobody is unloved" while ∃x∀y xLy comes close to making the religious claim "God loves everybody".

Example 8: $\forall m \exists n[n > m]$ is a true statement about numbers. For all numbers, there is a bigger one. If we reverse the order of the quantifiers we get $\exists n \forall m[n > m]$ which says that there is a number that is bigger than every number (even bigger than itself). This is clearly false.

§1.10. Quantifiers in Mathematics

Suppose that the universe of quantification is **N** = {0, 1, 2,...}, the set of natural numbers and that the primitive predicate is $x = y$, supplemented by the functions of addition and multiplication. Other predicates can be built up from these as follows:

$x \neq y$	$-(x = y)$
x is even	$\exists y\ (x = 2y)$
$x \leq y$	$\exists z\ (y = x + z)$
$x < y$	$(x \leq y) \wedge (x \neq y)$ i.e. $\exists z\ (y = x + z) \wedge -(x = y)$
x divides y	$\exists z\ (y = xz)$
x is prime	$(x \neq 1) \wedge [\forall y\ [y \text{ divides } x \rightarrow ((y = 1) \vee (y = x))]$ i.e. $-(x = 1) \wedge \forall y\ (\exists q[x = yq] \rightarrow ((y = 1) \vee (y = x)))$
x is a prime power	$\exists p[p \text{ is prime} \wedge \forall q[(q \text{ is prime}) \wedge (q \text{ divides } x) \rightarrow (q = p)]]$

These last two statements can be expanded until they are expressed entirely in terms of equality, addition and multiplication.

The famous *Goldbach Conjecture* (whose truth is strongly suspected but has never been proved) asserts that "every even number bigger than 2 is the sum of two primes". This can be written as:

$$\forall x\ (((x > 2) \wedge (x \text{ is even})) \rightarrow \exists y\ \exists z((y \text{ is prime}) \wedge (z \text{ is prime}) \wedge (x = y + z)))$$

§1.11. Negation Rules

To prove a theorem by contradiction requires the negation of the theorem to be assumed in order to reach a contradiction. If the statement has a complicated logical structure it may be necessary to rewrite this negation more simply. One can do this by using the following rules describing the way negation interacts with the other truth operators and the two quantifiers.

PROPOSITION	**NEGATION**
$-p$	p
$p \wedge q$	$-p \vee -q$
$p \vee q$	$-p \wedge -q$
$p \rightarrow q$	$p \wedge -q$
$p \leftrightarrow q$	$p \leftrightarrow -q$
$\forall x P x$	$\exists x\ -Px$
$\exists x P x$	$\forall x\ -Px$

Example 9:
The negation of $\exists x[Px \rightarrow (Qx \wedge -Rx)]$ is

$$\begin{aligned}
&-\exists x[Px \rightarrow (Qx \wedge -Rx)]\\
&\leftrightarrow \forall x\, -[Px \rightarrow (Qx \wedge -Rx)]\\
&\leftrightarrow \forall x\, [Px \wedge -(Qx \wedge -Rx)]\\
&\leftrightarrow \forall x[Px \wedge (-Qx \vee - -Rx)]\\
&\leftrightarrow \forall x[Px \wedge (-Qx \vee Rx)].
\end{aligned}$$

EXERCISES FOR CHAPTER 1

Exercise 1: *If you're not staying on Great Keppel Island you're not having a perfect holiday.* Express this in symbols . What does it say about the island?

Exercise 2: Write out in symbols the sentence: "exactly one of p, q and r is true".

Exercise 3: (a) Let Pxy denote the statement $xyx = x$. If the universe of quantification is the set of all real numbers, which of the following are TRUE?

(i) $\forall x \exists y\, Pxy$;
(ii) $\exists x \forall y\, Pxy$;
(iii) $\exists y \forall x\, Pxy$;
(iv) $\forall x \forall y[Pxy \rightarrow Pyx]$;
(v) $\exists x\, Pxx$.

(b) Repeat where the universe of quantification is the set of positive real numbers.
(c) Repeat when the universe of quantification is the set of positive integers.

Exercise 4: Explain in words why the statement $x > y$ (for real numbers) can be written as

$$\forall z[-(y = x + zz)].$$

Exercise 5: Express the following statement about real numbers symbolically, using only logical symbols and equations (no words or inequalities):

for all positive real numbers there is a smaller one

[**NOTE:** The only symbols you are allowed to use are the logical symbols, variables, brackets, "+" and "=". You may *not* use any other symbols such as $\mathbf{R}^+$ or inequalities. The Universe of Quantification is $\mathbf{R}$, which you don't have to mention explicitly.]

Exercise 6: Express the statement "m and n have no common factor" in symbolic form using only logical symbols and primitive arithmetic statements of the form $x = yz$. (The universe of quantification is the set of positive integers.)

Exercise 7: Which of the following are true? Give brief reasons.

(i) $\forall x\, \exists y\, [y < x]$ where the universe of quantification is $\mathbf{R}^+$ the set of all positive real numbers;
(ii) $\exists y\, \forall x\, [y < x]$ where the universe of quantification is $\mathbf{R}^+$, the set of all positive real numbers.

Exercise 8: Which of the following are true? Give brief reasons.

(i) $\forall x\ \exists y\ [x + y = 100]$ where the universe of quantification is **Z** the set of all (positive, negative and zero) integers;

(ii) $\exists y\ \forall x\ [x + y = 100]$ where the universe of quantification is **Z** the set of all (positive, negative and zero) integers;

(iii) $\forall x\ \exists y\ [x\,y + x^2 = 0]$ where the universe of quantification is **Z** the set of all (positive, negative and zero) integers;

(iv) $\exists y\ \forall x\ [x^2\,y < x + 1]$ where the universe of quantification is **Z** the set of all (positive, negative and zero) integers;

Exercise 9: Negate the following statement, expressing the negation in its simplest form:

$$\forall x\ [Px \to \exists y[xQy \vee -yQx]]$$

Exercise 10: Negate the following statement, expressing the negation in its simplest form:

$$\exists x\ [xAx \to \forall y[-xAy \vee yAx]]$$

Exercise 11: Negate the following $\forall[Px \to \exists y(Qxy \wedge -Py)]$.

Exercise 12: Express the negation of $\exists y[Py \to \forall z[-yQz \wedge Pz]]$ in such a way that the negation operator is only attached to primitive statements such as Px or xQy. Now write this in an equivalent, but simpler, form which does not use the negation operator at all.

SOLUTIONS FOR CHAPTER 1

Exercise 1: Let g = "staying on Great Keppel Island" and let p = "having a perfect holiday". The statement says that $-g \to -p$ which is equivalent to $p \to g$. It says you won't have a perfect holiday anywhere else, but makes no claim about Great Keppel itself. Perhaps there's no such thing as a perfect holiday anywhere!

Exercise 2: $(p \vee q \vee r) \wedge -(p \wedge q) \wedge -(p \wedge r) \wedge -(q \wedge r)$

Exercise 3: (a) (i) is TRUE. If $x \neq 0$, let $y = 1/x$. If $x = 0$, let $y =$ anything. Then $xyx = x$.
(ii) is TRUE. Take $x = 0$.
(iii) is FALSE. Putting $x = 1$ gives $y = 1$. Putting $x = 2$ gives $y = ¼$.
So there is no such y.
(iv) is FALSE. Let $x = 0$ and $y = 1$. Then $xyx = x = 0$, but $yxy = 0 \neq y$.
(v) is TRUE. Let $x = 1$.

(b) (i) is TRUE. Let $y = 1/x$.
(ii) is FALSE, for taking $x = 1$ gives $y = 1$. But this would mean that $x^2 = x$ for all $x > 0$, which is not so.
(iii) is FALSE, as above.
(iv) is TRUE. If $xyx = x$ then $y = 1/x$ and $yxy = y$.
(v) is TRUE as above.

(c) (i) is FALSE. Take $x = 2$. The corresponding y would have to satisfy $4y = 2$, which has no solution in the positive integers.
(ii) is FALSE, as above.
(iii) is FALSE since (i) is FALSE.
(iv) is TRUE. If $xyx = x$ then $xy = 1$ and so $x = y = 1$ (only solution for positive integers). Thus $yxy = y$.
(v) is TRUE as above.

Exercise 4: $(x > y) \leftrightarrow (y - x < 0)$. Negative real numbers are precisely those that have no square roots, so we can write this as:

$$x > y \leftrightarrow -\exists z\, [\, y - x = zz] \leftrightarrow \forall z[-(y - x = zz)] \leftrightarrow \forall z[-(y = x + zz)].$$

Exercise 5: The statement can be expressed as $\forall x\, [x > 0 \rightarrow \exists y[(y > 0) \wedge (y < x)]]$.
Using Ex 1B3, we can write it in more primitive terms as:

$$\forall x\, [\forall z[-(x + zz = 0)] \rightarrow \exists y[\forall z[-(y + zz = 0)] \wedge \forall z[-(y = x + zz)]].$$

Exercise 6:
The statement can be expressed as $\forall d[((d \text{ divides } m) \wedge (d \text{ divides } n)) \rightarrow (d = 1)]$.
This can be expressed in more primitive terms as $\forall d[\, (\exists q[m = dq] \wedge \exists q[n = dq]) \rightarrow (d = 1)]$
NOTE: Since q is only a "local variable" in each place we are able to use the same symbol each time, even though the q that works for m will generally be different to that which works for n.

Exercise 7: (i) is TRUE. For example take $y = x/2$. (ii) is FALSE. If there was such a y, then taking $x = y$ we would get $y < y$ which is impossible.

Exercise 8: (i) is TRUE. Take $y = 100 - x$.; (ii) is FALSE. If it was TRUE then all integers would be equal $(= 100 - y)$; (iii) is TRUE. Take $y = -x$.; (iv) is TRUE. Take $y = -1$. Then $x^2y - x - 1 = -(x^2 + x + 1)$ which is always negative.

Exercise 9: The negation is:

$$\begin{aligned}-\forall x[Px \rightarrow \exists y[xQy \vee -yQx]] &\leftrightarrow \exists x[-[Px \rightarrow \exists y[xQy \vee -yQx]]]\\ &\leftrightarrow \exists x[Px \wedge -\exists y[xQy \vee -yQx]]\\ &\leftrightarrow \exists x[Px \wedge \forall y[-[xQy \vee -yQx]]]\\ &\leftrightarrow \exists x[Px \wedge \forall y\, [-xQy \wedge yQx]]\end{aligned}$$

Exercise 10: The negation is:

$$\begin{aligned}-\exists x\, [xAx \rightarrow \forall y[-xAy \vee yAx]] &\leftrightarrow \forall x\, -[xAx \rightarrow \forall y[-xAy \vee yAx]]\\ &\leftrightarrow \forall x\, [xAx \wedge -\forall y[-xAy \vee yAx]]\\ &\leftrightarrow \forall x\, [xAx \wedge \exists y\, -[-xAy \vee yAx]]\\ &\leftrightarrow \forall x\, [xAx \wedge \exists y\, [xAy \wedge -yAx]]\end{aligned}$$

Exercise 11: The negation is:

$$\begin{aligned}-\forall x[Px \rightarrow \exists y(Qxy \wedge -Py)] &\leftrightarrow \exists x\, -[Px \rightarrow \exists y(Qxy \wedge -Py)]\\ &\leftrightarrow \exists x\, [Px \wedge -\exists y(Qxy \wedge -Py)]\\ &\leftrightarrow \exists x\, [Px \wedge \forall y\, -(Qxy \wedge -Py)]\\ &\leftrightarrow \exists x[Px \wedge \forall y(-Qxy \vee Py)].\end{aligned}$$

Exercise 12: $-\exists y[Py \rightarrow \forall z[-yQz \wedge Pz]] \leftrightarrow \forall y\, -[Py \rightarrow \forall z[-yQz \wedge Pz]]$

$$\leftrightarrow \forall y\, [Py \wedge\!\rightarrow -\forall z[-yQz \wedge Pz]]$$
$$\leftrightarrow \forall y\, [Py \wedge\!\rightarrow \exists z\, -[-yQz \wedge Pz]]$$
$$\leftrightarrow \forall y\, [Py \wedge\!\rightarrow \exists z\, [yQz \vee -Pz]]$$
$$\leftrightarrow \forall y\, [Py \wedge\!\rightarrow \exists z\, [yQz]].$$

This last step follows because if Py is true for all y then in particular so is Pz and so the $-Pz$ option is redundant.