5. COUNTING

§5.1. The Art of Counting

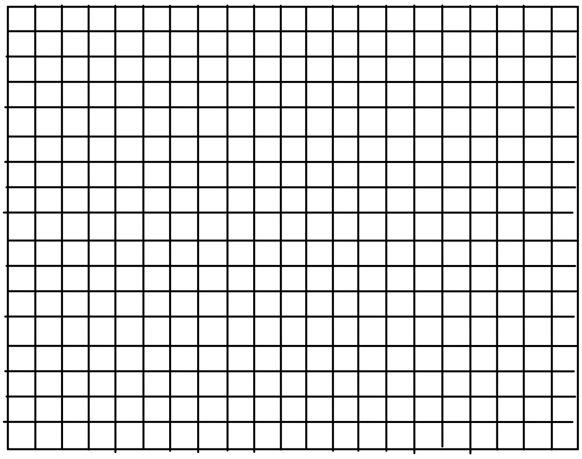
Counting was the first mathematical skill you ever mastered. Many mathematical theorems are based on counting. For example two sets can be proved to be different by simply showing that they have different sizes. And a subset can be shown to be the whole set by showing that they have the same size, though this only works if the sets are finite.

Counting is important in computing science. The complexity of an algorithm can be determined by counting the number of steps or amount of memory it takes. By examining how these numbers grow with the size of the task we can determine how useful our algorithm is. If we have an algorithm where the number of steps grows exponentially with the size of the problem it is of little use in practice.

It is quite elementary to count a finite set if the elements are listed. Counting an infinite set is quite another matter. You might think that it is sufficient to say that a set is infinite, or that its number of elements is ∞. But there are many infinite numbers – infinitely many in fact. Although the sets of integers, rational numbers and real numbers are all infinite, the number of elements in **Z** and **Q** is the same while the number of elements of R is bigger. Yes, a bigger infinite number!

But here we will be restricting our attention to finite sets. So what is there to say? A finite set can be very large and it would be too tedious to count each element individually. If the set has some structure to it we can usually count its elements using some mathematical calculation. A simple example is where we have a table with m rows and n columns. The number of cells is simply mn. There is no need to count them all.

Example 1: How many small squares are there in the following diagram?



There are 17 rows and 21 columns so there are $17 \times 21 = 357$ little squares.

We denote the number of elements of a set S by **#S**. If S is finite, #S is a non-negative integer. Note that $\#\varnothing = 0$, where $\varnothing$ denotes the empty set.

If S' is the complement of S in some larger set then $\#S' = N - \#S$, where N is the size of the larger set. If a subset contains most of the elements it is much easier to count the complement.

Example 2: How many of the numbers from 1 to 20 are not perfect squares.
Solution: The perfect squares are 1, 4, 9 and 16 so there are 4 perfect squares.
This leaves $20 - 4 = 16$ numbers that are not perfect squares.

Theorem 1: $\#(S \cup T) = \#S + \#T - \#(S \cap T)$.
Proof: If we add the sizes of the two sets we have double-counted those that are in both. So we need to subtract off $\#(S \cap T)$.
NOTE: This is a very simple case of the Inclusion-Exclusion Principle that we study later.

Theorem 2: $\#(S \times T) = \#S \times \#T$, where $S \times T$ is the set of ordered pairs (s, t), with $s \in S$ and $t \in T$.
Proof: This is equivalent to counting the individual cells in an array with m rows and n columns. The number is mn.

Theorem 3: $\#\wp(S) = 2^{\#S}$, where $\wp(S)$ is the set of subsets of S.
Proof: The number of subsets of a set of size n can be counted as follows.
For each of the n elements there is a choice of including it or of excluding it, that is 2 choices. Making n independent such choices there are 2^n combinations of choices. But each combination of choices corresponds to a subset and vice versa.

Example 3: How many subsets are there of the set {1, 2, 3}.
Solution: The answer is $2^3 = 8$. To help explain the above proof we shall list all 8 subsets together with the corresponding choice combinations (N = no, Y = yes).

subset	**choices**
∅	NNN
{1}	YNN
{2}	NYN
{3}	NNY
{1, 2}	YYN
{1, 3}	YNY
{2, 3}	NYY
{1, 2, 3}	YYY

§5.2. Binomial Coefficients

We define $\mathbf{n!} = n(n-1)(n-2) \dots 3.2.1$ This is called **factorial n**. This is the number of arrangements of n things.

Definition: $\binom{\mathbf{n}}{\mathbf{r}} = \dfrac{n!}{r!(n-r)!}$.
It is sometimes denoted by ${}^nC_r$ and is called a **binomial coefficient**. It is the number of subsets of size r of a set of size n. (This will be shown in the next section.)

Example 4: $5! = 5.4.3.2.1 = 120$.
$\binom{5}{3} = \dfrac{5.4.3.2.1}{(3.2.1).(2.1)} = \dfrac{5.4}{2.1} = 10$..

Although $\frac{n!}{r!(n-r)!}$ is a very neat formula for $\binom{n}{r}$ it involves a lot of unnecessary factor that get cancelled. An easier way to calculate $\binom{n}{r}$ is to start with n on the top of the fraction and r on the bottom and run each of them down by 1 until you reach 1 on the bottom. The numerator will have the same number of factors.

Example 4 (continued):
$\binom{5}{3} = \frac{5.4.3}{3.2.1}$. This involves much less unnecessary cancelling.

The calculation can be made even simpler by using the following fact.

Theorem 3: $\binom{n}{r} = \binom{n}{n-r+1}$.
Proof: An algebraic proof, using factorials, is not too difficult. But by far the easiest proof is to recognise that when we choose r things from n we are in effect choosing the n – r things to reject. The number of choices is the same.

Example 5: $\binom{100}{98} = \binom{100}{2} = \frac{100.99}{2.1} = 4950$. This is so much easier than $\frac{100.99.98.\ ...\ 3}{98.97.96.\ ...\ 1}$

Theorem 4: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.
Proof: The left hand side is the number of subsets of size r of the set {0, 1, 2, … , n}. We can separate these into those subsets that contain 0 and those that do not. Those that contain 0 will consist of 0 plus a further r – 1 elements of the set {1, 2, … , n} and clearly there will be $\binom{n}{r-1}$ of these. The subsets that do not contain 0 will be a subset of size r from the set {1, 2, … , n}. There will be $\binom{n}{r}$ of these. Hence $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.

Theorem 5: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

Proof: The number of subsets of {1, 2, 3, … , n} is 2^n and $\binom{n}{r}$ is the number of subsets of size r. The theorem simply counts the subsets by counting the numbers of each size.

§5.3. The Number of Choices

Theorem 6: The number of ways of choosing r things from n (where r ≤ n) depends on whether we allow repetitions and whether we want to distinguish between different orderings. The number of choices is given in the following table.

	ordered choices	**unordered choices**
repetitions allowed	n^r	$\binom{n+r-1}{r}$
no repetitions allowed	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Before giving a proof we will illustrate the four cases.

Example 6: How many ways are there choosing 3 things from {1, 2, 3, 4, 5}.
Repetitions allowed/ordered choices:
The choices are 111, 112, 113, 114, 115, 211, 212, …. , 555. There are 5 possibilities for the 1st choice. For each of these there are 5 possibilities for the 2nd choice, giving $5 \times 5 = 25$ possibilities for the 1st two choices. For each of these there are 5 possibilities for the 3rd choice, giving $5^3 = 125$ choices altogether.
No repetitions allowed/ordered choices:
The choices are 123, 124, 125, 132, 134, 135, 142, 143, 145, 151, 152, 153, 154, …. , 543. There are 5 possibilities for the 1st choice. For each of these there are only 4 possibilities for the 2nd choice, giving $5 \times 4 = 20$ possibilities for the 1st two choices. For each of these there are 3 possibilities left for the 3rd choice, giving $5 \times 4 \times 3 = 60$ choices altogether.

No repetitions allowed/unordered choices:
The choices are 123, 124, 125, 134, 135, 145, 234, 235, 245, 345.
The number of ordered choices is 60, but for each such choice there are 3! ways of arranging the 3 elements chosen. In the ordered case they must all be counted separately, but with the unordered case all 6 must be counted as one. So we must divide the 60 possibilities by 6 to get 10.

Repetitions allowed/unordered choices:
The choices are 111, 112, 113, 114, 115, 122, 123, 124, 125, 133, 134, 135, 144, 145, 155, 222, 223, 224, 225, 233, 234, 235, 244, 245, 255, 333, 334, 335, 344, 345, 355, 444, 445, 455, 555. There are 35 such choices.

Proof of Theorem 6:
Repetitions allowed/ordered choices:
There are n possibilities for each of the r choices, giving n^r possibilities altogether.

No repetitions allowed/ordered choices:
There are N possibilities for the 1st choice and $n - 1$ possibilities for the 2nd choice giving $n(n - 1)$ possibilities for the 1st two choices. Continuing in the same way we get $n(n - 1)(n - 2) \dots (n - r + 1)$ possibilities altogether. Note that this product has r factors.
Finally, $n(n - 1)(n - 2) \dots (n - r + 1) = \dfrac{n(n-1)(n-2) \dots (n - r + 1)(n-r) \dots 3.2.1}{(n-r) \dots 3.2.1}$

$= \dfrac{n!}{(n - r)!}$.

No repetitions allowed/unordered choices:
As in example 4 we must divide the number in the previous case by the number of ways of arranging the r things chosen, that is, by r! This gives the number of choices as
$\dfrac{n!}{(n - r)!r!} = \dbinom{n}{r}$.

Repetitions allowed/unordered choices:
This case corresponds to partitions of n identical things into r compartments. If the k'th element of the set occurs a_k times then the k'th compartment contains a_k balls. Of course, if $a_k = 0$ the k'th compartment is empty.

For example the choice 133 corresponds to the partition X| | XX. Conversely the partition |XXX| has no elements in the 1st and 3rd compartments and 3 in the 2nd. This corresponds to the choice 222.

These partitions are written with r + (n − 1) symbols, r of them being X's and the n − 1 compartment dividers being |'s. Of the n + r − 1 places that contain one of these two symbols, we must choose r of them to be X's. The rest will be |'s and we will have a partition of n things into r compartments which, as we have seen, corresponds to a choice in this fourth case.

For example, if n = 10 and r = 5 we will have a sequence of 5 X's and 9 |'s, making a total of 14 symbols. From these 14 positions we must choose to contain the 5 X's. Suppose we choose 3, 4, 7, 12 and 13. This means that the 3rd, 4th, 7th, 12th and 13th positions will contain an X. The corresponding partition is | |XX| | X | | | |X X|. The corresponding choice is 33599.

Now, how many ways are there of choosing the r positions to contain an X out of the n + r − 1 total positions? Clearly the answer is $\binom{n+r-1}{r}$.

§5.4. The Binomial Theorem

We remember from school that $(a + b)^2 = a^2 + 2ab + b^2$. The Binomial Theorem generalizes this to $(a + b)^n$.

Theorem 7: $(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{r} a^{n-r}b^r + \ldots + \binom{n}{1} ab^{n-1} + b^n$.

Proof: When we expand (a + b)(a + b) … (a + b) we get a sum of terms where each one consists of a product of n factors. Each such factor is chosen from one of the n factors of $(a + b)^n$. A typical term will consist of r a's and n − r b's in some order. But since the order of the factors is irrelevant we write the term as a^rb^{n-r}. Typically we will have many terms that equate to a^rb^{n-r} corresponding to the different ways we can choose the r a's and the n − r b's. The number of terms that equate to a^rb^{n-r} will be the number of ways of choosing the r factors from $(a + b)^n$ from which we choose an "a" and this is $\binom{n}{r}$. This is then the coefficient of a^rb^{n-r} in the expansion of $(a + b)^n$.

Example 7:
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 4ab^4 + b^5$.

Notice the pattern that these expansions follow. We start with $a^n = a^nb^0$. Then in each successive term the power of "a" goes down by 1 and the power of "b" goes up by 1. The sum of these powers is n. The coefficients are the binomial coefficients.

These binomial coefficients follow a very nice pattern, Suppose we write them out without the a's and b's in a triangular shape.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
……………………

This triangle is called **Pascal's Triangle** after the mathematician Blaise Pascal. The $(n + 1)^{st}$ row gives the coefficients in the expansion of $(a + b)^n$. Notice the pattern. The first thing you will notice are the left and right edges which are all 1's. Then you will notice the left-right symmetry. In each row the numbers grow, reach a maximum in the middle, and then decrease again. A little less obvious is the fact that each number, apart from the 1's, is the sum of the two numbers immediately above it. For example, 10 = 4 + 6.

4 6
10

This fact is simply a consequence of Theorem 4. It enables the successive rows of the triangle to be generated very easily. For example we can continue the triangle as follows.

…………..
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
…………………………

§5.5. Counting Partitions and Equivalence Relations

A **partition** on a set S is a set of subsets $S_1, S_2, \ldots, S_n$ such that each pair of distinct subsets is disjoint and their union is S. Every element of S will therefore belong to exactly one of these subsets.

S_1	S_2	S_3	………………….	S_n

Transitive relations are rather difficult to count. But we can count equivalence relations using equivalence classes. Every equivalence relation on S corresponds to a partition of S into equivalence classes and every partition of S corresponds to an equivalence relation. So we simply need to count the partitions.

Example 8: The equivalence relation {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)} corresponds to the partition {{1, 4}, {2}, {3}}.

Example 9: How many equivalence relations are there on the set {1, 2, 3, 4}?
Solution: We begin by enumerating the types of partition. The above partition consists of a pair and two singles. We can represent this type by (××)(×)(×). With 4 elements the possible types of partition are: (××××), (×××)(×), (××)(××), (××)(×)(×) and (×)(×)(×)(×).

Now we have to count the numbers of partitions of each type. For (××××) there is only one partition – all in together. For (×××)(×) there are 4 choices for the singleton. Having chosen which one goes by itself there is no further choice – all the other three go in together. For the (××)(×)(×) type we have ${}^4C_2 = 6$ ways of choosing the two singletons and so 6 partitions of that type. And there is only one partition of the type (×)(×)(×)(×).

The partitions of the type (××)(××) are a little more complicated to count. To start with we have ${}^4C_2 = 6$ choices for the first pair, with no further choice for the second. But because the two pairs have the same size they're interchangeable and so each partition would have been counted twice. For example {1, 3}, {2, 4}} and{{2, 4}, {1, 3}} are the same partition. So we have just 3 partitions of this type.

We can summarise this as follows:

Type	Number
(××××)	1
(×××)(×)	4
(××)(××)	3
(×××)(×)(×)	6
(×)(×)(×)(×)	1
TOTAL	**15**

Thus there are exactly 15 equivalence relations on this set.

Theorem 8: The number of onto functions $f: S \to T$ is n! times the number of equivalence relations with n equivalence classes.

Proof: If we have an onto function $f: S \to T$ the equivalence relation $f(x) = f(y)$ has exactly n equivalence classes. Now for each partition of S into n equivalence classes there are n! different onto functions according to the n! ways in which we can assign the n elements of T to the n equivalence classes.

Example 10: Find the number of onto functions from {1, 2, 3, 4, 5, 6} to {a, b, c}.

Solution: The types of partitions with 3 classes, and the numbers of each type, are as follows:

Type	Number
(××××)(×)(×)	15
(×××)(××)(×)	60
(××)(××)(××)	15 $\quad$ (= ${}^6C_2.{}^4C_2/3!$)
TOTAL	**90**

There are 90 partitions of 6 elements into 3 classes and 3! onto functions for each of these partitions, giving 540 onto functions in all.

§5.6. The Inclusion-Exclusion Principle

Suppose we have a collection of subsets of some larger set. For example if we number the courses at some educational establishment 1, 2, … , n we could define S_r to be the set of students enrolled in course n. How many students are enrolled in at leat one course? The answer will be somewhat less than $\#S_1 + \#S_2 + \dots \#S_n$ because most students will be enrolled in several course and will be counted several times. So we must subtract the number who are enrolled in both S_r and S_t for various values of r and t. But then we may have over compensated. Students enrolled in three courses will be counted 3 times in the first sum but, their number will be subtracted off 3 times because they are enrolled in 3 pairs of courses. They must be added back.

Theorem 9 (Inclusion-Exclusion):

$\#(S_1 \cup S_2 \cup \dots S_n) = \#S_1 + \#S_2 + \dots + S_n$

$$
\begin{aligned}
&- \#(S_1 \cap S_2) - \ldots - \#(S_{n-1} \cap S_n) \\
&\qquad + \#(S_1 \cap S_2 \cap S_3) + \ldots \\
&\qquad\qquad - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad\qquad\qquad +(-1)^n \#(S_1 \cap S_2 \cap \ldots \cap S_n).
\end{aligned}
$$

Proof: The above discussion should be sufficient proof. However a more formal proof can be given by mathematical induction.

Example 11: Suppose the enrolments in 4 courses are as follows.

course 1	99
course 2	79
course 3	84
course 4	93
courses 1 and 2	44
courses 1 and 3	51
courses 1 and 4	60
courses 2 and 3	37
courses 2 and 4	47
courses 3 and 4	56
courses 1, 2, 3	13
courses 1, 2, 4	18
courses 1, 3, 4	33
courses 2, 3, 4	23
all 4 courses	3

How many students are enrolled in at least one course? The number is:
99 + 79 + 84 + 93 – 44 – 51 – 60 – 37 – 47 – 56 + 13 + 18 + 33 + 23 –3 = 144.

§5.7. Number of Non-Negative Solutions to $x_1 + x_2 + \ldots + x_n = r$.

Example 12: How many non-negative solutions are there to the equation x + y + z = 5?
Solution: The solutions are:
0 + 0 + 5 = 5
0 + 1 + 4 = 5
0 + 2 + 3 = 5
0 + 3 + 2 = 5
0 + 4 + 1 = 5
0 + 5 + 0 = 5
1 + 0 + 4 = 5
1 + 1 + 3 = 5
1 + 2 + 2 = 5
1 + 3 + 1 = 5
1 + 4 + 0 = 5
2 + 0 + 3 = 5
2 + 1 + 2 = 5
2 + 2 + 1 = 5
2 + 3 + 0 = 5
3 + 0 + 2 = 5
3 + 1 + 1 = 5
3 + 2 + 0 = 5

$4 + 0 + 1 = 5$
$4 + 1 + 0 = 5$
$5 + 0 + 0 = 5$.
Counting the solutions we see that there are 21 of them. It should be obvious that generating all the solutions is a very tedious of finding the number of solutions. Surely there is a more efficient technique.

Theorem 10: The number of non-negative integer solutions to the equation:

$$x_1 + x_2 + \ldots + x_n = r$$

$$\text{is } \binom{n + r - 1}{n - 1}.$$

Proof: Each such solution corresponds to a partition of r things into n compartments. The number of solutions is $\binom{n + r - 1}{r} = \binom{n + r - 1}{n - 1}$ by Theorem 6.

NOTE: The reason for using $n - 1$ instead of r in the binomial coefficient is simply because in most applications $n - 1$ is much smaller than r.

Example 12 (continued):
Using theorem 9 we can quickly find the answer to example 12. The number of solutions is $\binom{3 + 5 - 1}{2} = \binom{7}{2} = 21$.

Now suppose that, instead of all non-negative solutions to the equation

$$x + y + z = 5$$

we want the number of solutions where $x \geq 2$. Having taken the trouble listing all the solutions it is a simple exercise to count those where $x \geq 2$. The number is 10. However counting would not be feasible if the total was 55 instead of 5. There would be too many solutions.

If $x \geq 2$ then $u = x - 2 \geq 0$ and $u + y + z = 3$.

The non-negative solutions to $x + y + z = 5$ where $x \geq 2$ are in 1-1 correspondence with the non-negative solutions to $u + y + z = 3$. By theorem 9 this number is $\binom{3 + 3 - 1}{2} = \binom{5}{2} = 10$.

Theorem 11: The number of non-negative solutions of

$$x_1 + x_2 + \ldots + x_n = r$$

$$\text{with } x_i \geq k$$

is the number of non-negative solutions of

$$x_1 + x_2 + \ldots + x_n = r - k$$

which is $\binom{n + r - k - 1}{n - 1}$

Proof: The solutions to $x_1 + x_2 + \ldots + x_n = r$ with $x_i \geq k$ correspond to the solutions to $x_1 + x_2 + \ldots + x_n = r - k$ with only the non-negativity condition on all variables (by replacing x_i by $x_i - k$).
NOTE: To incorporate the condition $x_i \geq k$ simply reduce the top parameter in the binomial coefficient by k.

Example 13: How many non-negative integer solutions are there to the equation x + y + z = 5 for which $x \geq 2$ and $y \geq 1$?

Solution: Let u = x – 2 and v = y – 1. Then u + v + z = 2. The number of solutions for x, y and z, with the stated conditions, is the same as the number of solutions for u, v and z with no extra conditions. This is $\binom{4}{2} = 6$.

Example 14: How many non-negative integer solutions are there to the equation x + y + z = 5 for which $x \leq 2$?

Solution: We calculate the number of solutions where $x \geq 3$ and subtract from the total number of solutions. The total number of solutions is $\binom{7}{2} = 21$. The number of solutions where $x \geq 3$ is $\binom{4}{2} = 6$, so the number of solutions where $x \leq 2$ is 21 – 6 = 15.

Example 15: How many non-negative integer solutions are there to the equation x + y + z = 5 for which $x \leq 2$ and $y \leq 1$?

Solution: There are $\binom{7}{2}$ solutions altogether. We treat the two conditions separately. Let S_1 denote the set of solutions where $x \geq 3$ and let S_2 denote the set of solutions where $y \geq 2$. The solutions to be deleted are those in $S_1 \cup S_2$. So the number of solutions satisfying the given inequalities is $\binom{7}{2} - \#(S_1 \cup S_2)$.

Now $\#(S_1 \cup S_2) = \#S_1 + \#S_2 - \#(S_1 \cap S_2) = \binom{4}{2} + \binom{5}{2} - \binom{2}{2} = 6 + 10 - 1 = 15$.

Hence the required number of solutions is 21 – 15 = 6.

Example 16: How many non-negative integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 18$ for which each $x_i \leq 5$?

Solution: There are $\binom{21}{3}$ solutions altogether. Let S_i denote the set of solutions for which $x_i \geq 6$. Then for each i, $\#(S_i) = \binom{15}{3}$. For each distinct i, j $\#(S_i \cap S_j) = \binom{9}{3}$ and for each distinct i, j, k $\#(S_i \cap S_j \cap S_k) = \binom{3}{3}$. Finally $\#(S_1 \cap S_2 \cap S_3 \cap S_4) = 0$ because if each x_i is at least 6 their sum would be at least 24.

Now there are $\binom{4}{2}$ pairs of distinct i, j and $\binom{4}{3}$ distinct triples. Hence the required number of solutions is:

$$\binom{21}{3} - \binom{4}{1}\binom{15}{3} + \binom{4}{2}\binom{9}{3} - \binom{4}{3}\binom{3}{3} + 0$$

$= 1330 - 4.455 + 6.84 - 4.1 + 0$

$= 1330 - 1820 + 504 - 4$

$= 10.$

§5.9. Counting Arrangements of Symbols

Example 17: How many arrangements are there of the letters AAABCC?

Solution: There are 6 letters, and if all of them were different there would be 6! = 720 different arrangements. But the 3A's can be permuted in 3! ways and the 2C's can be permuted in 2! ways. Such permutations result in the same arrangement and must be counted just once. We must therefore divide the 6! by 3! and by 2!. The number of arrangements is therefore $\frac{6!}{3!2!}$ = 60.

Example 18: How many arrangements are there of the letters A, D, O, R, W, X, Z that include the string WORD.

Solution: Imagine the letters to be written on cards. Because the strings must contain WORD we write these four letters on a single card as WORD. The remaining 3 letters are each written on a separate card. So we have 4 cards that we can permute in any of the 4! arrangements, and all of them will contain the string WORD simply because they occur on the same card. So there are 4! = 24 such arrangements.

Example 19: How many arrangements are there of the letters A, D, O, R, W, X, Z that do not include the string WORD.

Solution: There are 7 letters altogether and so 7! = 5040 arrangements. Of these 24 will contain the string WORD. Hence there are 5040 − 24 = 5016 arrangements that do not include the string WORD.

Example 20: The 26 letters A-Z are each written on a card. In how many ways can these cards be arranged so that the words ONE, TWO, … TEN do not occur.

Solution: The strings to be excluded are: ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE and TEN. Of these THREE, SEVEN and NINE will be excluded automatically because they involve repeated letters.

The sizes of $S_1, \dots, S_{10}$ are given as follows. [The number is $(26 - n + 1)!$ except in those cases where the string is impossible.]

	#
S_1	24!
S_2	24!
S_3	0
S_4	23!
S_5	23!
S_6	24!
S_7	0
S_8	22!
S_9	0
S_{10}	24!

For r = 1, 2, … , 10 let S_r be the set of all arrangements that include the string that represents the number r. As observed above $S_3 = S_7 = S_9 = \varnothing$.

If a string contains both ONE and TWO they must occur together as TWONE so the arrangements in $S_1 \cap S_2$ are those that contain TWONE. $S_1 \cap S_4 = \varnothing$ because the "O" in FOUR cannot be part of ONE.

In the following table we consider $S_i \cap S_i$ for $i < j$. If the entry is $\varnothing$ this indicates at $S_i \cap S_j$ is empty as both words cannot occur. Otherwise there is listed one or two strings which must appear. Where only one is listed it is because the two words must run together sharing a letter.

	S_2	**S_4**	**S_5**	**S_6**	**S_8**	**S_{10}**
S_1	TWONE	$\varnothing$	$\varnothing$	ONE, SIX	ONEIGHT	$\varnothing$
S_2		$\varnothing$	TWO, FIVE	TWO, SIX	EIGHTWO	$\varnothing$
S_4			$\varnothing$	FOUR, SIX	FOUR, EIGHT	FOUR, TEN
S_5				$\varnothing$	$\varnothing$	$\varnothing$
S_6					$\varnothing$	SIX, TEN
S_8						$\varnothing$

The number of arrangements in each $S_i \cap S_j$ is given as follows.

	S_2	**S_4**	**S_5**	**S_6**	**S_8**	**S_{10}**
S_1	22!	0	0	22!	20!	0
S_2		0	21!	22!	20!	0
S_4			0	21!	19!	21!
S_5				0	0	0
S_6					0	22!
S_8						0

Where there is just one composite string to be included, of length n, the number of "cards" being permuted is $(26 - n + 1)! = (25 - n)!$. Where there are two separate strings to be included, with n letters between them, the number of "cards" being permuted is $(26 - n + 2)! = (24 - n)!$

The only cases where $S_i \cap S_j \cap S_k$ is non-empty, for $i < j < k$, are shown below. The required string(s) and the number of elements are given.

$S_1 \cap S_2 \cap S_6$	TWONE, SIX	20!
$S_1 \cap S_2 \cap S_8$	EIGHTWONE	16!
$S_4 \cap S_6 \cap S_{10}$	FOUR, SIX, TEN	19!

The only intersection of 4 or more subsets that is non-empty is $S_1 \cap S_2 \cap S_6 \cap S_8$.

$S_1 \cap S_2 \cap S_6 \cap S_8$	EIGHTWONE, SIX	16!

The number of arrangements that do not contain any of the strings ONE, … , TEN is:
$26! - 4.24! - 2.23! - 22! + 4.22! + 3.21! + 2.20! + 19! - 20! - 19! - 16! + 16!$
$= 26! - 4.24! - 2.23! + 3.22! + 3.21! + 20!$

This is as far as you would be expected to carry the calculations. But, just this once, armed with a high-precision calculator, we will continue as follows:
$= 20!(165765600 - 1020096 - 21252 + 1386 + 63 + 1)$
$= 164725702 \times 20!$
$= 400761491194106764001280000.$