DISCRETE MATHEMATICS

W W L CHEN

Chapter 13

PRINCIPLE OF INCLUSION-EXCLUSION

13.1. Introduction

To introduce the ideas, we begin with a simple example.

Example 13.3.1. Consider the sets $S = \{1, 2, 3, 4\}$, $T = \{1, 3, 5, 6, 7\}$ and $W = \{1, 4, 6, 8, 9\}$. Suppose that we would like to count the number of elements of their union $S \cup T \cup W$. We might do this in the following way:

(1) We add up the numbers of elements of S, T and W. Then we have the count

$$|S| + |T| + |W| = 14.$$

Clearly we have over-counted. For example, the number 3 belongs to S as well as T, so we have counted it twice instead of once.

(2) We compensate by subtracting from $|S| + |T| + |W|$ the number of those elements which belong to more than one of the three sets S, T and W. Then we have the count

$$|S| + |T| + |W| - |S \cap T| - |S \cap W| - |T \cap W| = 8.$$

But now we have under-counted. For example, the number 1 belongs to all the three sets S, T and W, so we have counted it $3 - 3 = 0$ times instead of once.

(3) We therefore compensate again by adding to $|S| + |T| + |W| - |S \cap T| - |S \cap W| - |T \cap W|$ the number of those elements which belong to all the three sets S, T and W. Then we have the count

$$|S| + |T| + |W| - |S \cap T| - |S \cap W| - |T \cap W| + |S \cap T \cap W| = 9,$$

which is the correct count, since clearly $S \cup T \cup W = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

From the argument above, it appears that for three sets S, T and W, we have

$$|S \cup T \cup W| = \underbrace{(|S| + |T| + |W|)}_{\substack{\text{one at a time} \\ \text{3 terms}}} - \underbrace{(|S \cap T| + |S \cap W| + |T \cap W|)}_{\substack{\text{two at a time} \\ \text{3 terms}}} + \underbrace{(|S \cap T \cap W|)}_{\substack{\text{three at a time} \\ \text{1 term}}}.$$

13.2. The General Case

Suppose now that we have k finite sets $S_1, \ldots, S_k$. We may suspect that

$$|S_1 \cup \ldots \cup S_k| = \underbrace{(|S_1| + \ldots + |S_k|)}_{\substack{\text{one at a time} \\ \binom{k}{1} \text{ terms}}} - \underbrace{(|S_1 \cap S_2| + \ldots + |S_{k-1} \cap S_k|)}_{\substack{\text{two at a time} \\ \binom{k}{2} \text{ terms}}}$$
$$+ \underbrace{(|S_1 \cap S_2 \cap S_3| + \ldots + |S_{k-2} \cap S_{k-1} \cap S_k|)}_{\substack{\text{three at a time} \\ \binom{k}{3} \text{ terms}}} - \ldots + (-1)^{k+1} \underbrace{(|S_1 \cap \ldots \cap S_k|)}_{\substack{k \text{ at a time} \\ \binom{k}{k} \text{ terms}}}.$$

This is indeed true, and can be summarized as follows.

PRINCIPLE OF INCLUSION-EXCLUSION *Suppose that $S_1, \ldots, S_k$ are non-empty finite sets. Then*

$$\left| \bigcup_{j=1}^{k} S_j \right| = \sum_{j=1}^{k} (-1)^{j+1} \sum_{1 \le i_1 < \ldots < i_j \le k} |S_{i_1} \cap \ldots \cap S_{i_j}|, \tag{1}$$

where the inner summation

$$\sum_{1 \le i_1 < \ldots < i_j \le k}$$

is a sum over all the

$$\binom{k}{j}$$

distinct integer j-tuples $(i_1, \ldots, i_j)$ satisfying $1 \le i_1 < \ldots < i_j \le k$.

Proof. Consider an element x which belongs to precisely m of the k sets $S_1, \ldots, S_k$, where $m \le k$. Then this element x is counted exactly once on the left-hand side of (1). It therefore suffices to show that this element x is counted also exactly once on the right-hand side of (1). By relabelling the sets $S_1, \ldots, S_k$ if necessary, we may assume, without loss of generality, that $x \in S_i$ if $i = 1, \ldots, m$ and $x \notin S_i$ if $i = m+1, \ldots, k$. Then

$$x \in S_{i_1} \cap \ldots \cap S_{i_j} \qquad \text{if and only if} \qquad i_j \le m.$$

Note now that the number of distinct integer j-tuples $(i_1, \ldots, i_j)$ satisfying $1 \le i_1 < \ldots < i_j \le m$ is given by the binomial coefficient

$$\binom{m}{j}.$$

It follows that the number of times the element x is counted on the right-hand side of (1) is given by

$$\sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j} = 1 + \sum_{j=0}^{m} (-1)^{j+1} \binom{m}{j} = 1 - \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} = 1 - (1-1)^m = 1,$$

in view of the Binomial theorem. ○

13.3. Two Further Examples

The Principle of inclusion-exclusion will be used in Chapter 15 to study the problem of determining the number of solutions of certain linear equations. We therefore confine our illustrations here to two examples.

EXAMPLE 13.3.1. We wish to calculate the number of distinct natural numbers not exceeding 1000 which are multiples of 10, 15, 35 or 55. Let

$$S_1 = \{1 \le n \le 1000 : n \text{ is a multiple of } 10\}, \qquad S_2 = \{1 \le n \le 1000 : n \text{ is a multiple of } 15\},$$
$$S_3 = \{1 \le n \le 1000 : n \text{ is a multiple of } 35\}, \qquad S_4 = \{1 \le n \le 1000 : n \text{ is a multiple of } 55\},$$

so that

$$|S_1| = \left[\frac{1000}{10}\right] = 100, \qquad |S_2| = \left[\frac{1000}{15}\right] = 66, \qquad |S_3| = \left[\frac{1000}{35}\right] = 28, \qquad |S_4| = \left[\frac{1000}{55}\right] = 18.$$

Next,

$$\begin{aligned}
S_1 \cap S_2 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10 \text{ and } 15\} = \{1 \le n \le 1000 : n \text{ is a multiple of } 30\}, \\
S_1 \cap S_3 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10 \text{ and } 35\} = \{1 \le n \le 1000 : n \text{ is a multiple of } 70\}, \\
S_1 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10 \text{ and } 55\} = \{1 \le n \le 1000 : n \text{ is a multiple of } 110\}, \\
S_2 \cap S_3 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 15 \text{ and } 35\} = \{1 \le n \le 1000 : n \text{ is a multiple of } 105\}, \\
S_2 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 15 \text{ and } 55\} = \{1 \le n \le 1000 : n \text{ is a multiple of } 165\}, \\
S_3 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 35 \text{ and } 55\} = \{1 \le n \le 1000 : n \text{ is a multiple of } 385\},
\end{aligned}$$

so that

$$|S_1 \cap S_2| = \left[\frac{1000}{30}\right] = 33, \qquad |S_1 \cap S_3| = \left[\frac{1000}{70}\right] = 14, \qquad |S_1 \cap S_4| = \left[\frac{1000}{110}\right] = 9,$$
$$|S_2 \cap S_3| = \left[\frac{1000}{105}\right] = 9, \qquad |S_2 \cap S_4| = \left[\frac{1000}{165}\right] = 6, \qquad |S_3 \cap S_4| = \left[\frac{1000}{385}\right] = 2.$$

Next,

$$\begin{aligned}
S_1 \cap S_2 \cap S_3 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10,\ 15 \text{ and } 35\} \\
&= \{1 \le n \le 1000 : n \text{ is a multiple of } 210\}, \\
S_1 \cap S_2 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10,\ 15 \text{ and } 55\} \\
&= \{1 \le n \le 1000 : n \text{ is a multiple of } 330\}, \\
S_1 \cap S_3 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10,\ 35 \text{ and } 55\} \\
&= \{1 \le n \le 1000 : n \text{ is a multiple of } 770\}, \\
S_2 \cap S_3 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 15,\ 35 \text{ and } 55\} \\
&= \{1 \le n \le 1000 : n \text{ is a multiple of } 1155\},
\end{aligned}$$

so that

$$|S_1 \cap S_2 \cap S_3| = \left[\frac{1000}{210}\right] = 4, \qquad |S_1 \cap S_2 \cap S_4| = \left[\frac{1000}{330}\right] = 3,$$
$$|S_1 \cap S_3 \cap S_4| = \left[\frac{1000}{770}\right] = 1, \qquad |S_2 \cap S_3 \cap S_4| = \left[\frac{1000}{1155}\right] = 0.$$

Finally,

$$\begin{aligned}
S_1 \cap S_2 \cap S_3 \cap S_4 &= \{1 \le n \le 1000 : n \text{ is a multiple of } 10,\ 15,\ 35 \text{ and } 55\} \\
&= \{1 \le n \le 1000 : n \text{ is a multiple of } 2310\},
\end{aligned}$$

so that

$$|S_1 \cap S_2 \cap S_3 \cap S_4| = \left[\frac{1000}{2310}\right] = 0.$$

It follows that

$$\begin{aligned}|S_1 \cup S_2 \cup S_3 \cup S_4| = &(|S_1| + |S_2| + |S_3| + |S_4|)\\ &- (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|)\\ &+ (|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|)\\ &- (|S_1 \cap S_2 \cap S_3 \cap S_4|)\\ = &(100 + 66 + 28 + 18) - (33 + 14 + 9 + 9 + 6 + 2) + (4 + 3 + 1 + 0) - (0) = 147.\end{aligned}$$

EXAMPLE 13.3.2. Suppose that A and B are two non-empty finite sets with $|A| = m$ and $|B| = k$, where $m > k$. We wish to determine the number of functions of the form $f : A \to B$ which are not onto. Suppose that $B = \{b_1, \ldots, b_k\}$. For every $i = 1, \ldots, k$, let

$$S_i = \{f : b_i \not\in f(A)\};$$

in other words, S_i denotes the collection of functions $f : A \to B$ which leave out the value b_i. Then we are interested in calculating $|S_1 \cup \ldots \cup S_k|$. Observe that for every $j = 1, \ldots, k$, if $f \in S_{i_1} \cap \ldots \cap S_{i_j}$, then $f(x) \in B \setminus \{b_{i_1}, \ldots, b_{i_j}\}$. It follows that for every $x \in A$, there are only $(k - j)$ choices for the value $f(x)$. It follows from this observation that

$$|S_{i_1} \cap \ldots \cap S_{i_j}| = (k - j)^m.$$

Combining this with (1), we conclude that

$$|S_1 \cup \ldots \cup S_k| = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} (k - j)^m.$$

It also follows that the number of functions of the form $f : A \to B$ that are onto is given by

$$k^m - \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} (k - j)^m = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^m.$$

PROBLEMS FOR CHAPTER 13

1. Find the number of distinct positive integer multiples of $2, 3, 5, 7$ or 11 not exceeding 3000.

2. A natural number greater than 1 and not exceeding 100 must be prime or divisible by 2, 3, 5 or 7.
   a) Find the number primes not exceeding 100.
   b) Find the number of natural numbers not exceeding 100 and which are either prime or even.

3. Consider the collection of permutations of the set $\{1, 2, 3, \ldots, 8\}$; in other words, the collection of one-to-one and onto functions $f : \{1, 2, 3, \ldots, 8\} \to \{1, 2, 3, \ldots, 8\}$.
   a) How many of these functions satisfy $f(n) = n$ for every even n?
   b) How many of these functions satisfy $f(n) = n$ for every even n and $f(n) \neq n$ for every odd n?
   c) How many of these functions satisfy $f(n) = n$ for precisely 3 out of the 8 values of n?

4. For every $n \in \mathbb{N}$, let $\phi(n)$ denote the number of integers in the set $\{1, 2, 3, \ldots, n\}$ which are coprime to n. Use the Principle of inclusion-exclusion to prove that

$$\phi(n) = n \prod_{p} \left(1 - \frac{1}{p}\right),$$

where the product is over all prime divisors p of n.