3. SETS, FUNCTIONS, RELATIONS AND STRINGS

If I see the moon, then the moon sees me
'Cos seeing's symmetric as you can see.
If I tell Aunt Maude and Maude tells the nation
Then I've told the nation 'cos the gossiping relation
Is transitive. And even if you're on the shelf
You're loved by someone, that's yourself,
'Cos love's reflexive. And if all three,
Reflex- and Trans- and Symmetry,
I'm sure you see, it makes good sense
To call it a relation of equivalence.

Ah, when I was young it came to pass
That I learnt all this in equivalence class.

§3.1. Sets in Mathematics and Computing Science

As we studied elementary mathematics we came to believe that the fundamental object in mathematics is the number. Then, as we began calculus, we learnt of a new type of mathematical object — the function. At first we thought of a function as a formula, or an algebraic expression, such as "the function x^2 ". But then they tried to tell us that functions are more than just formulae. They are rules which associate with every element in one set a unique element of another. Every function has to go from one set to another.

The deeper we get into mathematics the more we become involved with sets. We learn of vector spaces and groups — these are sets on which there is some algebraic structure. In geometry we no longer deal with shapes, but rather, sets of points. And in calculus we're increasingly concerned with domains of functions and regions of convergence and sets of solutions to differential equations. The set is paramount in mathematics.

In computing science, sets are one of many important data structures. In databases they are the fundamental data type.

§3.2. Defining a Set

A **set** is a collection of objects called **elements**.   We write $x \in S$ if x is an element of the set S and $x \notin S$ if it is not.

Generally elements are denoted by lower case letters and sets by capitals. However since there are sets whose elements are sets, and sets of sets of sets etc., it is not always possible to maintain this distinction.

There are three main ways of describing or defining a set:

(1) Use a standard name

N = the set of natural numbers 0, 1, 2, ... ;

NOTE: Some books exclude zero from this set and consider the natural numbers to be synonymous with positive integers.

Example 1:
$\mathbf{Z}$ = the set of integers 0, ±1, ±2, ... (from the German "Zahlen", meaning "numbers");
$\mathbf{Q}$ = the set of rational numbers ($\mathbf{Q}$ stands for "quotient");
$\mathbf{R}$ = the set of real numbers;
$\mathbf{C}$ = the set of complex numbers.

(2) List the elements
$\{x_1, x_2, \ldots, x_n\}$ denotes the set with the elements: $x_1, x_2, \ldots, x_n$.
Strictly speaking this notation can only be used for finite sets but if there is an obvious pattern we can indicate certain infinite sets this way.

Example 2:
{4, 5, 13} is the set whose elements are the integers 4, 5 and 13;
{0, 1, 4, 9, ... } indicates the set of square integers.

(3) Describe a property that characterises the elements.
$\{x \in S \mid Px\}$ denotes the set of all elements x, in S, for which the statement Px is true. If the set S is understood we often omit it and write $\{x \mid Px\}$.

Example 3:
$\{x \in \mathbf{Z} \mid x > 0\}$ is the set of positive integers, commonly denoted by $\mathbf{Z}^+$;
$\{x \in \mathbf{Z} \mid 3 < x < 7\} = \{4, 5, 6\}$;
$\{x \in \mathbf{R} \mid x^2 < 1\}$ denotes the open interval $(-1, 1)$ i.e. all real numbers *x* such that $-1 < x < 1$;
$\{n \mid \exists q[n = 7q]\}$ denotes the set of all multiples of 7.

NOTES:
(1) The symbol x in the notation $\{x \mid Px\}$ is a "dummy" variable. It can be replaced throughout by any other symbol not otherwise used. Thus $\{x \mid Px\} = \{r \mid Pr\}$.

(2) $y \in \{x \mid Px\} \leftrightarrow Py$, that is, y belongs to the set if and only if it has the defining property.

(3) When we describe a finite set by listing the elements the order does not matter. Also any repetitions are ignored. For example: {3, 1, 2, 4} = {1, 2, 3, 4} = {1, 2, 2, 3, 4, 4, 4}.

(4) It was once naively thought that for every property there must be a set, but this can lead to certain paradoxes. The most famous is the Russell Paradox named after the philosopher and mathematician Bertrand Russell. If $S = \{x \mid x \notin x\}$ then $S \in S$ if and only $S \notin S$. [If $S \in S$ it satisfies the defining property and if $S \notin S$ it also satisfies the defining property!] We clearly cannot allow a mathematical system in which such paradoxes exist. Those interested in the foundations of mathematics set up axioms for set theory in which there are precise restrictions on which properties are allowed to give rise to a set. Fortunately such deep problems are far removed from the coal-face of useful mathematics and we can safely ignore them.

Two sets are defined to be **equal** if every element of one set is an element of the other. In symbols:

$$S = T \leftrightarrow \forall x[x \in S \leftrightarrow x \in T].$$

Example 4: $\{x \in \mathbf{Z} \mid x^2 < 2\} = \{-1, 0, 1\} = \{x \in \mathbf{R} \mid x^3 = x\}$

A set S is a **subset** of a set T if every element of S is an element of T. We write $S \subseteq T$ to denote the fact that S is a subset of T. In symbols this is very similar to the definition of equality of sets:

$$S \subseteq T \leftrightarrow \forall x\,[x \in S \rightarrow x \in T]$$

In particular every set is a subset of itself. Subsets S which are not the whole set T are called **proper** subsets. We denote this by $S \subset T$.

A very important set is the **empty** set. This is the set with no elements and can be described by an empty list {} or an impossible property $\{x \mid x \neq x\}$. Note that there is only one empty set. For example, the empty set of integers is the same as the empty set of triangles. The statement "mermaids don't exist" can be expressed by saying that "the set of mermaids is the empty set". The empty set is denoted by $\varnothing$. Like the number 0, it is an extremely useful concept. If we want to prove that a given property P can never hold we can consider the set $S = \{x \mid Px\}$ and then prove that $S = \varnothing$.

Another important set is the "universe" a set consisting of all the elements with which we are concerned in a given context. For example if we are considering sets of integers we would take our universe to be the set of *all* integers.

§3.3. Basic Set Functions

We now list a large number of definitions of functions which assign to each set S, or pair of sets S and T, a certain set.

Intersection:

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$

Union:

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

Difference:

$$S - T = \{x \mid x \in S \text{ and } x \notin T\}$$

Complement:

$$-S = \{x \mid x \notin S\} = U - S$$

where U is the universe in which we are working.

Example 5

Suppose $S = \{1, 2, 3\}$ and $T = \{1, 3, 5, 7\}$.

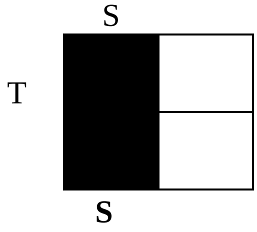
Then $S \cap T = \{1, 3\}$; $S \cup T = \{1, 2, 3, 5, 7\}$; $S - T = \{2\}$; $T - S = \{5, 7\}$

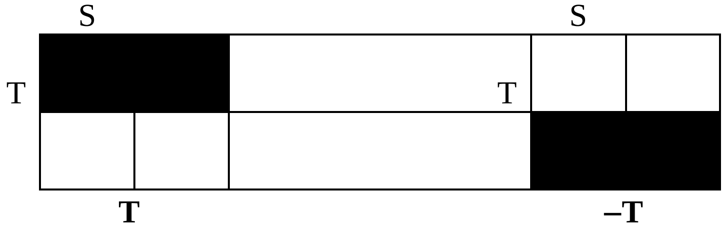
If our universe is the set of all integers and S is the set of even numbers, $-S$ is the set of all odd numbers.

§3.4. Venn Diagrams

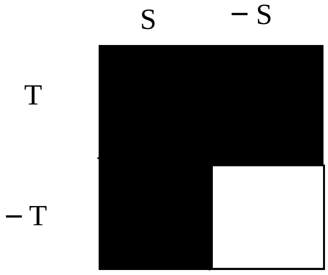
Some of these concepts can be illustrated by diagrams where sets are represented by regions drawn in the plane and elements by points inside them.

The simplest system is to divide a rectangle into two rows and two columns. The left side represents a set and the right side, its complement. Then the top half can represent a second set and the bottom, its complement.

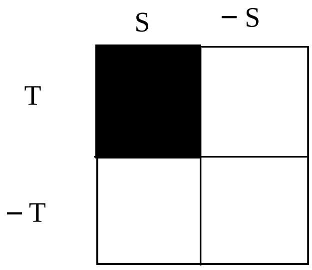




Example 6:

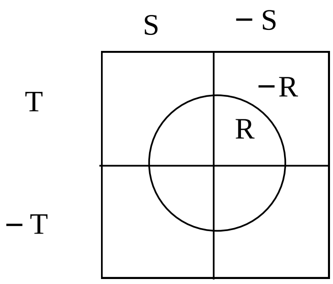


UNION



INTERSECTION

If there is a third set we can simply draw a circle in the middle, cutting across all four regions. The region inside the circle represents the third set and the outside represents its complement.



§3.5. Cartesian Product

The **Cartesian Product** of two sets S, T is:

$$S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}.$$

It is the set of all **ordered pairs** whose components come from S and T respectively

$$S \times T \times R = \{(x, y, z) \mid x \in S, y \in T, z \in R\} \text{ etc.}$$

The **Cartesian Power**, S^n, is $S \times S \times ... \times S$ (n factors)

Example 7:
Suppose S = {1, 2, 3} and T = {1, 3, 5, 7}. Then
$S \times T = \{(1, 1), (1, 3), (1, 5), (1, 7), (2, 1), (2, 3), (2, 5), (2, 7), (3, 1), (3, 3), (3, 5), (3, 7)\}$
$(S \cap T)^3 = \{(1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 3), (3, 1, 1), (3, 1, 3), (3, 3, 1), (3, 3, 3)\}$

NOTES: (1) While the order of the elements of a set does not matter there is often a particular order that is more natural than the others. The advantage of using a systematic order is that it ensures that every combination has been accounted for. Look carefully at the order in which we have listed the elements in the above example.

(2) Venn Diagrams cannot be used to depict the Cartesian Product.

The **cardinal number** of a finite set is its number of elements. We denote the cardinal number of S by $\#S$. For example $\#\varnothing = 0$.

Example 8: If $S = \{1, 2, 3\}$ and $T = \{3, 5\}$ then:
$\#S = 3$, $\#T = 2$, $\#(S \cap T) = 1$, $\#(S \cup T) = 4$, $\#(S - T) = 2$, $\#(S \times T) = 6$, $\#\wp(S) = 8$.

The following properties describe how the various set functions affect cardinal numbers.

(1) $\#(S \cup T) = \#S + \#T - \#(S \cap T)$;
(2) $\#(S \times T) = (\#S).(\#T)$;
(3) $\#(S - T) = \#S - \#(S \cap T)$;
(4) $\#\varnothing = 0$.

§3.6. Relations

"Jack is my mother's uncle." That is what we usually think of when we talk about relations — people with whom we are related. But it is the relationship itself that mathematicians would call a *relation*.

A **relation** is a statement involving two variables, or two gaps, where the names of people or things can be inserted.

So "x is the mother of y", or just "mother of" is one such relation. In mathematics there are numerous relations such as "is less than", "is parallel to", "is a power of ".

We can denote the fact that x has the relation with y by xRy. This reflects the word order in natural language, and indeed where we have invented symbols for particular relations in mathematics we usually use this format, such as "$x < y$".

But a relation, as well as being a connection between two sets, can also be considered as a set in itself. Suppose R is a relation such that xRy makes sense for $x \in S$ and $y \in T$. We can represent R by the set of ordered pairs (x, y) for which the relation holds, that is $\{(x, y) \mid xRy\}$.

Now $S \times T$ denotes the set of all ordered pairs (x, y) with $x \in S, y \in T$. So the set of ordered pairs for which xRy (is true) is just a subset of $S \times T$. This gives us an alternative definition of a relation.

A **relation from a set S to a set T** is any subset of $S \times T$ and a **relation on a set S** is a relation from S to S i.e. a subset of $S \times S$.

Example 9: If $S = \{1, 2, 3, 4\}$ then the relation normally written $x < y$ would be:

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

and the relation "$x = y^n$ for some integer $n > 1$" becomes:

$$\{(2, 2), (3, 3), (4, 2), (4, 4)\}.$$

When a computer makes reference to a relation such as “<” it certainly doesn't consult an array of ordered pairs. But when dealing with the relation “student x is enrolled in course y” we might store this information in a database as an array of ordered pairs.

Example 10:

STUDENT	COURSE
AARON Adam	MATHS 1
AARON Adam	PHYSICS 1
ANTON Tom	MATHS 1

In mathematical notation we would write this relation as
{(AARON Adam, MATHS 1), (AARON Adam, PHYSICS 1), (ANTON Tom, MATHS 1)}

§3.7. Equivalence Relations

Like all mathematical objects, relations can be classified according to whether they satisfy certain properties. The three most important properties for a relation R on a set S are the **reflexive**, **symmetric** and **transitive** properties.

R is **reflexive** if xRx for all x.
R is **symmetric** if $xRy \rightarrow yRx$ for all x, y.
R is **transitive** if xRy and $yRz \rightarrow xRz$ for all x, y, z.

Example 11: $x < y$ on the set of integers.
Not Reflexive: For example $1 < 1$ is FALSE.
Not Symmetric: For example $1 < 2$ is TRUE but $2 < 1$ is FALSE.
Transitive: If $x < y$ and $y < z$ it follows that $x < z$.

Example 12: $xRy \leftrightarrow x = y^n$ for some $n \in \mathbf{N}$ on the set $\mathbf{N}$.
Reflexive: Since $x = x^1$ for all x.
Not symmetric: Now 4R2 (as $4 = 2^2$) but it is not true that 2R4 ($2 \neq 4^n$ for any positive n).
Transitive: Suppose xRy and yRz.
Then $x = y^n$ for some $n \in \mathbf{N}$ and $y = z^m$ for some $m \in \mathbf{N}$.
(We mustn't fall into the trap of assuming that the power is the same in each case.)
Hence $x = (z^m)^n = z^{mn}$. Since $mn \in \mathbf{N}$ it follows that xRz.

Example 13: $xRy \leftrightarrow |x - y| < 3$
Reflexive: Since for all x, $|x - x| = 0$ which is less than 3.
Symmetric: Since $|y - x| = |x - y|$.
Not Transitive: For example 1R3 and 3R5 but it is not true that 1R5.

Example 14: $xRy \leftrightarrow x - y$ is even, defined on the set $\mathbf{Z}$.
Reflexive : For all x, $x - x = 0$ which is even.
Symmetric: Suppose xRy.
This means that $x - y$ is even, say $x - y = 2h$ for some $h \in \mathbf{Z}$.
Then $y - x = -2h = 2(-h)$ which is even, so yRx.
Transitive: Suppose xRy and yRz.
This means that $x - y = 2h$ for some $h \in \mathbf{Z}$ and $y - z = 2k$ for some $k \in \mathbf{Z}$.
Now $x - z = (x - y) + (y - z) = 2h + 2k = 2(h+k)$ which is even. Hence xRz.

An **equivalence relation** is a relation on a set that is reflexive, symmetric and transitive. Example 14 is an example of an equivalence relation.

§3.8. Equivalence Classes

If R is an equivalence relation and $x \in S$, we define

$$\mathbf{[x]_R} = \{ y \in S \mid xRy \},$$

that is, the set of all elements of S that are equivalent to x under the relation R.

We call $[x]_R$ the **equivalence class** containing x. Often we omit the subscript R.
NOTE: The fact that [x] does indeed contain x follows from the reflexive property.

To find the equivalence classes for a given equivalence relation use the following simple algorithm. Choose an element x which has not yet been included in an equivalence class (any element will do to begin with). List all the elements that are related to x. These form an equivalence class. Now choose another element not yet listed. Continue until all elements have been included.

Equivalence classes are somewhat like families. Those in the same class are related – those in different classes are not. Human families do not provide a perfect analogy with equivalence classes since being related is not an equivalence relation. (The relation of being related is not transitive since my cousin has cousins who I would not consider as being related to me.)

Example 15: Consider the set of railway stations of the world with xRy meaning that you can travel by rail from station x to station y (possibly having to change trains several times). The equivalence classes are the connected railway networks. One would be the government railways of mainland Australia. Another would be the Tasmanian Railways (unfortunately this has become the empty set). Some privately owned railways would form little equivalence classes. Until recently the British Rail network in England, Scotland and Wales formed an entire equivalence class, but with the opening of the Channel Tunnel they now form only part of a larger network that extends across Europe.

Example 16: Let $S = \{1,2,3\}$ and let $R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$.
It is easy to show that R is an equivalence relation. The equivalence classes are:
$[1] = \{1,3\} = [3]$ and $[2] = \{2\}$.

Theorem 1: $x \in [y] \leftrightarrow [x] = [y]$
Proof: Suppose that $[x] = [y]$. Since $x \in [x]$, $x \in [y]$.
Conversely, suppose that $x \in [y]$. Then yRx (and by the symmetric property, xRy). We must show that $[x] = [y]$. We first show that $[x] \subseteq [y]$.
Suppose that $z \in [x]$. Then xRz.
Since yRx, it follows from the transitive property that yRz
and hence zRy by the symmetric property. Thus $z \in [y]$, and so $[x] \subseteq [y]$.
Similarly, $[y] \subseteq [x]$, and so $[x] = [y]$.

Theorem 2: For all $x, y \in S$, either $[x] \cap [y] = \varnothing$ or $[x] = [y]$.
Proof: Suppose that [x], [y] are not disjoint. Let $z \in [x] \cap [y]$.
Then $z \in [x]$ and $z \in [y]$. By the previous theorem, $[x] = [z]$ and $[z] = [y]$ and so $[x] = [y]$.

We have proved that any two distinct equivalence classes are disjoint. So:

IF *R* IS AN EQUIVALENCE RELATION ON A SET S THEN *S* IS THE DISJOINT UNION OF ITS EQUIVALENCE CLASSES

This means that S can be chopped up into non-overlapping equivalence classes so that elements are equivalent if and only if they belong to the same class.

§3.9. Functions

A **function** f from a set S to a set T is the pair of sets (S, T) together with a rule that associates with each element $x \in S$ a unique element of T, written f(x). This element is called the **image** of f.

We indicate that f is a function from S to T by writing $f:S \rightarrow T$. The set S is called the **domain** of f and T is called the **codomain**. Other words that are used instead of "function" are "map", "operator" and "transformation".

Example 17: $f: S \rightarrow S$ where S is the set of people who are now living or who have died, with f(x) defined to be "the father of x". But note that "the son of x" does not define a function on S because, although everyone has a unique (genetic) father, not everyone has a unique son and many people in S have no sons at all.

The simplest way to describe a function from one finite set to another is by means of a table of values.

Example 18: f::{1, 2, 3, 4}→{1, 2, 3, 4} where f is defined by means of the following table of values:

x	f(x)
1	2
2	1
3	3
4	2

You do not have to discover any pattern in this table before you can call f a function. The rule here is simply to look up the table. The second column could have been filled up in any way, as long as no symbols other than 1, 2, 3, 4 is used.

Example 19: There are 8 functions from {1, 2, 3} to {A, B} viz.

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
1	A	A	A	A	B	B	B	B
2	A	A	B	B	A	A	B	B
3	A	B	A	B	A	B	A	B

Notice the systematic way in which these functions have been listed.

A function $f:S \rightarrow T$ is **1-1** (pronounced "**one to one**") if

$$f(x) = f(y) \rightarrow x = y.$$

The **image** of a function $f:S \rightarrow T$ is $\{f(x) \mid x \in S\}$. It is denoted by **im f**. A function $f:S \rightarrow T$ is **onto** if $t \in T \rightarrow \exists s[f(s) = t$ and $s \in S]$, that is, if every element of T gets mapped to. A **permutation** on a set S is a 1-1 and onto function f from a set to itself.

Example 20: Consider the following functions from $\mathbf{R} \rightarrow \mathbf{R}$:

$$f(x) = x + 1;$$
$$g(x) = x^2 + 1;$$
$$h(x) = x^3 + x^2;$$
$$k(x) = e^x.$$

Then f is 1-1 and onto, g is neither h is onto but not 1-1 and k is 1-1 but not onto.

If $f:S \rightarrow U$ and $g:T \rightarrow S$ are functions then we can compose them together to get a function from S to T. The **composition** of f, g is $\mathbf{f \circ g}$: $T \rightarrow U$ where $(f \circ g)(x)$ is defined to be $f(g(x))$. Note that, although we write it as $f \circ g$, it is in fact g that is applied first.

Sometimes we talk about the **product** of the two functions. This is where we write the functions as **gf**, in the order in which the functions are to be carried out. So $(gf)(x) = f(g(x))$.

§3.10. Strings and Languages

If A is any set of symbols (it is usually finite and is called an **alphabet**) then a **string** on A is a finite sequence of symbols juxtaposed without any separators: $a_1a_2 \ldots a_n$.

Example 21: The following are some strings on the alphabet {1, 2, +, =}:

$$\alpha = \text{“1+1=2”},$$
$$\beta = \text{“2+2+2+2+2+2=12”},$$
$$\gamma = \text{“22+12=11”},$$
$$\delta = \text{“++=”}$$

Strings α and β represent true statements in ordinary arithmetic. String γ makes sense but is false. String δ is pure nonsense. At least it does not make sense in ordinary arithmetic, though it could be given a meaning. But *all* four are valid strings on the alphabet.

The **length** of the string $\alpha = a_1a_2 \ldots a_n$ is defined to be n and is denoted by $|\alpha|$.

Example 22: For the strings in example 1, $|\alpha| = 5$, $|\beta| = 14$, $|\gamma| = 8$, $|\delta| = 3$.

The **null** string is the unique string of length 0 and is denoted by the symbol $\boldsymbol{\lambda}$. It might seem a bit ridiculous to invent a word, and a symbol, for a string of no symbols. But people once thought that about the number zero. “Nothing”, whether it is the number zero, the empty set, or the null string, is a very useful concept. Life would be so much more complicated without it.

The **product** of two strings

$\alpha = a_1a_2 \ldots a_m$,

$\beta = b_1b_2 \ldots b_n$.,

is defined to be the string

$\alpha\beta = a_1a_2 \ldots a_mb_1b_2 \ldots b_n$.

This operation is called **concatenation**.

Example 23: Suppose α, β, γ and δ are the strings defined in example 1, and λ is the null string. Then $\alpha\delta$ = "1+1=2++=" while $\delta\alpha$ = "++=1+1=2". Neither makes much sense but both are nevertheless valid strings.

Observe that $\alpha\delta \neq \delta\alpha$. In general, strings do not commute. But note that $\alpha\lambda = \lambda\alpha$ since both give just "1+1=2". In fact concatenating the null string on either end of *any* string leaves that string unchanged.

Theorem 3: For all strings on a given alphabet:

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$
$$\alpha\lambda = \lambda\alpha$$
$$|\alpha\beta| = |\alpha| + |\beta|$$
$$|\lambda| = 0$$

The set of all strings on a given alphabet is **closed** under the operation of concatenation, the **associative** law holds and there is an **identity**, namely the null string. However strings do not have **inverses** (apart from the null string which is its own inverse) because concatenation cannot make a string shorter.

An abstract structure in which there is a binary operation that satisfies both the closure law and the associative law, and which has an identity element, is known as a **semigroup**. A **group** is a semigroup in which every element has an inverse. The structure formed by the set of all strings on a given alphabet by the concatenation operation, is therefore a semigroup, but not a group because of the lack of inverses.

Normally we think of languages as special systems where certain strings are given meaning. There are the natural languages with which we communicate with one other in our daily lives such as English, French or Swahili. Then there are the specialised programming languages with which we communicate with computers such as BASIC, PASCAL and C++.

But there are many other specialised systems that we do not usually think of as languages. Morse Code is a language that is built on top of a natural language. Chess enthusiasts and knitters have developed specialised languages in which they describe their activities.

There are two aspects to any language — *syntax* and *semantics*. Syntax is concerned with determining whether a given string has a valid form while semantics is concerned with the meanings that are given to acceptable strings. Semantics can only be discussed in the context of a particular language but syntax is more formal and can be discussed abstractly.

The sentence "The silver prejudice accelerated within the complicated sausage." is a syntactically correct English sentence. The verbs, nouns and adjectives are all in correct positions relative to one another. But semantically it is nonsense. It lacks meaning. On the other hand the syntax of the sentence "Is near post-office where?" is incorrectly structured for English, but if a foreign tourist stopped you in the street with this question you might be able to guess that he wanted to be directed to the nearest post-office. So correct syntax is not always necessary for communication. But it certainly makes communication much easier!

Decimal notation for real numbers is a mini-language that uses the ten digits, the minus sign and the decimal point. The semantics of this language involves concepts such as powers of 10 and limits of sequences, but the syntax can be expressed by a few simple rules:

(1) If a minus sign is included it must be at the front.

(2) At most one decimal point may be included and if included it must have a digit on either side.

(3) The first digit cannot be 0 unless it is the only symbol or a decimal point follows.

The above examples are languages that are used in practice, but when one studies languages mathematically the word **language** refers to any (finite or infinite) set of strings. One of the fundamental problems is to describe an infinite language by giving a finite description. This is discussed in some depth in *Languages and Machines* in this series of notes.

EXERCISES FOR CHAPTER 3

Exercise 1: Let $A = \{1, 3, 5, 9, 11\}$, $B = \{n \in \mathbf{N} | n < 10\}$, $C = \{n \in \mathbf{N} | n \text{ is prime}\}$,
$D = A \cap C$ and $E = A - C$.
List the elements of:

(a) B;
(b) D;
(c) E;
(d) $B - A$;
(e) $A \cup (B \cap C)$;
(f) $D \times E$.

Exercise 2: Simplify each of the following (each answer is either S or $\varnothing$):

(a) $S \cap S$;
(b) $S - S$;
(c) $S \cap \varnothing$;
(d) $S \times \varnothing$;
(e) $S \cup \varnothing$

Exercise 3: Which of the following statements are true for all sets R, S and T?

(a) $S \cap T = T \cap S$;
(b) $S - T = T - S$;
(c) $S \times T = T \times S$;
(d) $R \cap (S \cap T) = (R \cap S) \cap T$;
(e) $-(S \cap T) = -S \cap -T$;
(f) $-(S \cup T) = -S \cap -T$;
(g) $R \cap (S - T) = (R \cap S) - (R \cap T)$;
(h) $R \cup (S - T) = (R \cup S) - (R \cup T)$

Exercise 4: If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$ write down:

(a) $A \cap B$;
(b) $A \cup B$;
(c) $A \times B$.

Exercise 5: If $A = \{1, 2, 3\}$, $B = \{0, 1\}$ and $C = \{0, 2, 4\}$ write down:

(a) $A \cap (B \cup C)$;
(b) $B \times C$.

Exercise 6: Let A, B be the following binary languages (sets of binary strings):
A = {1, 10, 101}, B = {1, 11, 011}. Find each of the following sets:

(a) $A \cap B$;
(b) $A \cup B$;
(c) $A - B$;
(d) $A \times B$;
(e) AB.

Exercise 7: Let R be the relation on the set of real numbers defined by xRy if $y = xq$ for some positive rational number q. Prove that R is an equivalence relation. Find an equivalence class with a finite number of elements.

Exercise 8: Let $\approx$ be the relation on the set **R** where $x \approx y$ means "$x^m = y^n$ for some positive integers m, n". Prove that $\approx$ is an equivalence relation. Which equivalence classes are finite?

Exercise 9: The relation R on the set {2, 3, 4, 8, 15} defined by $xRy \leftrightarrow$ "x + y is not prime" is an equivalence relation. Find the equivalence classes.

Exercise 10: For each of the following relations determine which of the reflexive, symmetric and transitive properties hold. Which, if any, are equivalence relations?

(a) R on the set **N** where $xRy \leftrightarrow x \geq y$;
(b) R on the set **N** where $xRy \leftrightarrow x^2 = y^2$;
(c) R on the set $\mathbf{R}^+$ where $xRy \leftrightarrow \log(y/x)$ is an integer.

Exercise 11: Let R be the relation on the set of integers defined by xRy if x + y is even. Prove that R is an equivalence relation.

Exercise 12: For each of the following relations determine which of the reflexive, symmetric and transitive properties hold. Which, if any, are equivalence relations?

(a) R on the set S of people where $xRy \leftrightarrow$ x is the brother of y;
(b) R on the set **Q** where $xRy \leftrightarrow y = x.2^n$ for some $n \in \mathbf{Z}$.

Exercise 13: Define the relation R on the set S = {1, 2, 3, ..., 100} by:

$$aRb \leftrightarrow a|2^r b \text{ for some } r \in \mathbf{N} \text{ and } b|2^s a \text{ for some } s \in \mathbf{N}.$$

(a) Prove that R is an equivalence relation.
(b) Find the number of equivalence classes.

Exercise 14: Suppose that the relation $\approx$ is defined on the set of all binary strings by:
$A \approx B$ if there exist strings X, Y such that AX = XB and YA = BY.
(a) Prove that $\approx$ is an equivalence relation.
(b) Find the equivalence class containing 11001.

Exercise 15: Let R be the relation on N defined by xRy if 3x + 4y is a multiple of 7. Prove that R is an equivalence relation.

Exercise 16: Let S = {1, 2} and T = {1, 2, 3}. Are there more functions from S to T or functions from T to S? What about 1-1 functions? What about onto functions?

Exercise 17: Let L be the set of lotteries conducted in N.S.W. during a given year and let N be the set of all the numbers of the tickets. Define $w:L\rightarrow N$ by $w(x)$ = number of the ticket winning 1st prize in lottery x. Which do you think is more likely, for w to be 1-1 or for it to be onto? Discuss.

Exercise 18: Let α = “string” and β = “length”.
Find (a) $\alpha\beta$; (b) $\beta\alpha$; (c) $\beta\lambda$; (d) strings γ, δ such that $|\gamma| = 2$ and $\gamma\delta = \alpha$.

SOLUTIONS FOR CHAPTER 3

Exercise 1: (a) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; (b) 3, 5, 11; (c) 1, 9; (d) 0, 2, 4, 6, 7, 8;
(e) 1, 2, 3, 5, 7, 9, 11; (f) (3,1), (3,9), (5,1), (5,9), (11,1), (11,9).

Exercise 2: (a)(e) S; (b)(c)(d) $\varnothing$.

Exercise 3: (a), (d), (f), (g).

Exercise 4: (a) $A \cap B = \{2\}$; (b) $A \cup B = \{1, 2, 3, 4\}$;
(c) $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$.

Exercise 5: (a) $A \cap (B \cup C) = \{1, 2, 3\} \cap \{0, 1, 2, 4\} = \{1, 2\}$;
(b) $B \times C = \{(0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4)\}$.

Exercise 6: (a) {1}; (b) {1, 10, 11, 011, 101}; (c) {10, 101};
(d) {(1, 1), (1, 11), (1, 011), (10, 1), (10, 11), (10, 011), (101, 1), (101, 11), (101, 011);
(e) {11, 111, 1011, 101, 10011, 10111, 101011}.

Exercise 7: *Reflexive:* x = x1 and 1 is rational.
Symmetric: Suppose xRy. Then $y = xq$ for some $q \in \mathbf{Q}^+$ and so $x = yq^{-1}$ and since $q^{-1} \in \mathbf{Q}^+$, yRx.
Transitive: Suppose xRy and yRz. Then $y = xq$ and $z = yr$ for some $q, r \in \mathbf{Q}^+$.
Then $z = (xq)r = x(qr)$. Since $qr \in \mathbf{Q}^+$, xRz.
{0} is the only finite equivalence class.

Exercise 8: *Reflexive:* $x^1 = x^1$ so $x \approx x$ for all x.
Symmetric: Suppose $x \approx y$. Thus, $x^m = y^n$ for some positive integers m, n.
Then since $y^n = x^m$, $y \approx x$.
NOTE: The fact that the definition says “ *for some* positive integers m, n” means that they don’t have to be the same ones each time. We do not have to show that $x^m = y^n \rightarrow y^m = x^n$.
Transitive: Suppose $x \approx y$ and $y \approx z$. Thus $x^m = y^n$ and $y^s = z^t$ for some positive integers m, n, s, t.
Then $x^{x^{ms}} = (x^m)^s = (y^n)^s = (y^s)^n = z^{nt}$. Hence $x \approx z$.
The only finite equivalence classes are {0} and {1, –1}. All others are infinite.
eg $[2] = \{\pm 2^n \mid n \in \mathbf{Z}\}$.

Exercise 9: {2, 4, 8}, {3, 15}.

Exercise 10: (a) RT; (b) RST; (c) none; (d) RST. (b)(d) are equivalence relations.

Exercise 11: *Reflexive:* $x + x = 2x$ is even for all integers x;
Symmetric: if $x + y$ is even then $y + x$ is even;
Transitive: Suppose $x + y$ and $y + z$ are even. Then $(x + y) + (y + z) = x + z + 2y$ is even. Hence $x + z$ is even.

Exercise 12: (a) S only [Not R since I am not my own brother. Not T: e.g. take x = me, y = my brother and z = me.]
(b) RST (i.e. an equivalence relation).

Exercise 13: (a) *Reflexive:* Let $a \in S$. The since $a|2a$, aRa. Hence R is reflexive.
Symmetric: R is clearly symmetric as defined.
Transitive: Suppose aRb and bRc. Then $a|2^r b$, $b|2^s a$, $b|2^u c$, $c|2^v b$ for some $r, s, u, v \in \mathbf{N}$. Hence $a|2^{r+u}c$ and $c|2^{v+s}a$. Since $r + u$ and $v + s$ are both natural numbers it follows that aRc. Hence R is transitive.
Thus we have shown that R is an equivalence relation.
(b) Two numbers are equivalent under R if the are equal, give or take some factors of 2. Thus each equivalence class contains exactly one odd number and so the number of equivalence classes is exactly 50.

Exercise 14: (a) *Reflexive:* Let A be a binary string. Since $A\lambda = \lambda A$ we have $A \approx A$.
Symmetric: Suppose $A \approx B$. Then for some strings X, Y, $AX = XB$ and $YA = BY$.
Hence $BY = YA$ and $XB = AX$ whence $B \approx A$.
Transitive: Suppose $A \approx B$ and $B \approx C$. Then for some strings X, Y, S, T:
$AX = XB$ and $YA = BY$ and $BS = SC$ and $TB = CT$.
Hence $A(XY) = XBY = (XY)A$ and $(TY)A = TBY = C(TY)$. Hence $A \approx C$.
(b) The equivalence class containing 1011 is {1011, 0111, 1110, 1101}. Note that these are just the cyclic rearrangements of 1011.

Exercise 15: *Reflexive:* Let $x \in \mathbf{N}$. Then $3x + 4x = 7x$ is a multiple of 7 and so xRx.
Symmetric: Suppose xRy. Then $7 | 3x + 4y$. Thus $7 | 7(x + y) - (3x + 4y)$, that is, $7 | 4x + 3y$. Hence yRx.
Transitive: Suppose xRy and yRz. Then $7 | 3x + 4y$ and $7 | 3y + 4z$.
Hence $7 | 3x + 7y + 4z$ and so $7 | 3x + 4z$. So xRz.

Exercise 16: There are more functions from S to T than from T to S, more 1-1 functions from S to T than from T to S but more onto functions from T to S than from S to T. The actual numbers are:

	S→T	**T→S**
all functions	9	8
1-1	6	0
onto	0	6

Exercise 17: The statement "w is 1-1" means that nobody wins twice. While this is not certain it's a highly likely event. The statement "w is onto" means that every ticket number will win 1st prize at least once during the year — an impossible event since quite clearly there are more tickets in each lottery than there are lotteries during the year.

Exercise 18: (a) "stringlength"; (b) "lengthstring"; (c) "length"; (d) γ = "st", β = "ring".