Midterm Cheatsheet

Bayes' Rule for Events

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$

where, by the Law of Total Probability,

$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

Posterior Model

$f(\pi|y) = \frac{f(\pi)L(\pi|y)}{f(y)} \propto f(\pi)L(\pi|y)$

Beta Model

$\pi \sim \text{Beta}(\alpha, \beta)$

The Beta model is specified by continuous pdf

$f(\pi) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\pi^{\alpha-1}(1-\pi)^{\beta-1} \quad \text{for } \pi \in [0,1], \alpha > 0, \text{and } \beta > 0$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ and $\Gamma(z+1) = z\Gamma(z)$. Fun fact: when z is a positive integer, then $\Gamma(z)$ simplifies to $\Gamma(z) = (z-1)!$.

Beta Descriptives

$E(\pi) = \frac{\alpha}{\alpha+\beta}$

$\text{Mode}(\pi) = \frac{\alpha-1}{\alpha+\beta-2}$

$\text{Var}(\pi) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The Beta-Binomial Model

Let $\pi \sim \text{Beta}(\alpha, \beta)$ and $X|n \sim \text{Bin}(n, \pi)$ then

$\pi|(Y = y) \sim \text{Beta}(\alpha + y, \beta + n - y)$

Gamma Prior

$\lambda \sim \text{Gamma}(s, r)$ where $s > 0$ and $r > 0$:

The Gamma distribution is specified by continuous pdf $f(\lambda) = \frac{r^s}{\Gamma(s)}\lambda^{s-1}e^{-r\lambda} \quad \text{for } \lambda \in [0, \infty)$

Gamma Descriptives

$E(\lambda) = \frac{s}{r}$

$\text{Mode}(\lambda) = \frac{s-1}{r}$ where $s \geq 1$

$\text{Var}(\lambda) = \frac{s}{r^2}$

Poisson Likelihood

$f(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!} \quad \text{for } y \in \{0, 1, 2, \ldots, n\}$

The Gamma-Poisson Model

If $f(\lambda) \sim \text{Gamma}(s, r)$

and if $y_i \sim iid$ Poissson$(\lambda)$for $i \in 1, \ldots, n$

then $f(\lambda|\vec{y}) \sim \text{Gamma}(s + \sum y_i, r + n)$.

Normal Prior

If $\mu \sim N(\theta, \tau^2)$ then

$$f(\mu) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left[-\frac{(\mu-\theta)^2}{2\tau^2}\right] \quad \text{for } \mu \in (-\infty, \infty)\ . \tag{1}$$

Normal Likelihood

If $Y \sim N(\mu, \sigma^2)$ then

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] \quad \text{for } y \in (-\infty, \infty) \tag{2}$$

$$L(\mu|\vec{y}) \propto \prod_{i=1}^{n} \exp\left[-\frac{(y_i-\mu)^2}{2\sigma^2}\right] = \exp\left[-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}\right].$$

The Normal Posterior

If

$$\begin{aligned} Y_i|\mu &\overset{ind}{\sim} N(\mu, \sigma^2) \\ \mu &\sim N(\theta, \tau^2) \end{aligned}$$

then

$$\mu|\vec{y} \ \sim \ N\left(\frac{\theta\sigma^2/n + \bar{y}\tau^2}{\tau^2 + \sigma^2/n},\ \frac{\tau^2\sigma^2/n}{\tau^2 + \sigma^2/n}\right). \tag{3}$$