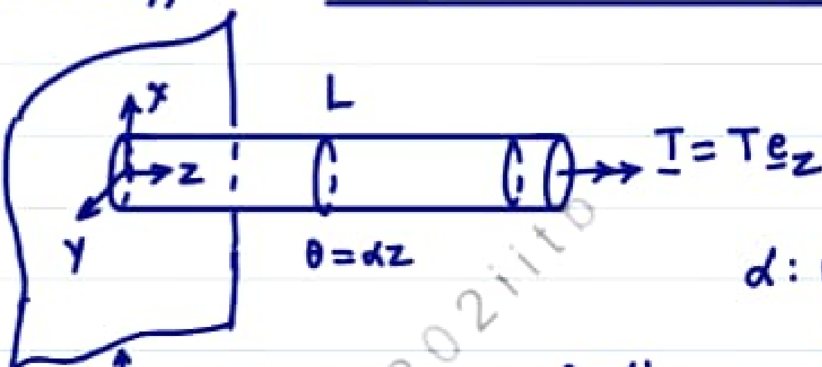
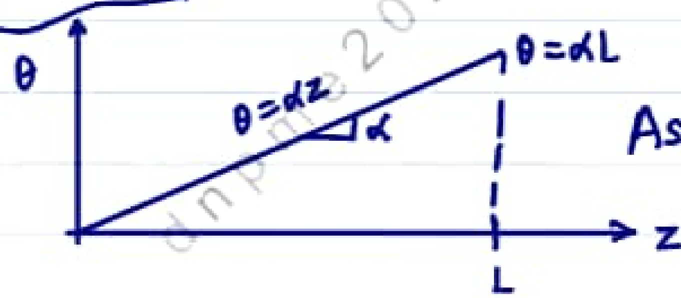
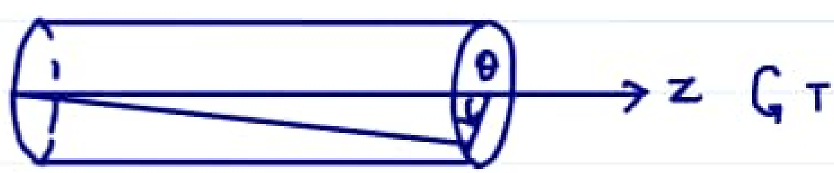
Previously, **TORSION OF CYL SHAFTS**



α: unit ∠ twist
rad/m



Assumption



Kinematic assumption
$$\begin{cases} u = -\alpha y z \\ v = +\alpha x z \\ w = 0 \end{cases}$$

Strains $\epsilon_{yz} = \alpha x/2$, $\epsilon_{xz} = -\alpha y/2$

Hooke's Law

Stresses $\sigma_{yz} = G\alpha x$, $\sigma_{xz} = -G\alpha y$

check if stresses are in equilibrium pointwise

3D

$$\Sigma F_x = 0 \Rightarrow \cancel{\frac{\partial \sigma_{xx}}{\partial x}} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}} + \cancel{b_x} = 0 \quad ✓$$

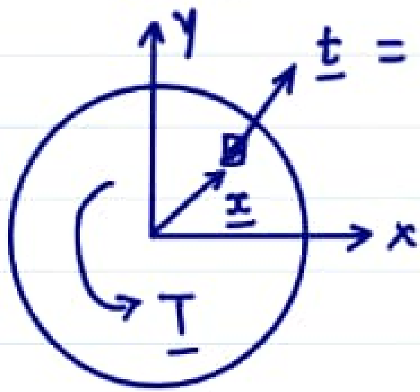
$$\Sigma F_y = 0 \Rightarrow \cancel{\frac{\partial \sigma_{yx}}{\partial x}} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}} + \cancel{b_y} = 0 \quad ✓$$

$$\Sigma F_z = 0 \Rightarrow \cancel{\frac{\partial \sigma_{xz}}{\partial x}} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \cancel{b_z} = 0 \quad ✓$$

Every c/s carries/transmits torque $\underline{T}$

$\underline{t} = \underset{\sim}{\sigma}\, \underline{n}$ $\quad N/m^2$

$$\int_{\Omega} \underline{x} \times \underline{t}\, da$$



$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

c/s at some z

$$\underline{T} = \int_{\Omega} \underline{x} \times \underline{t}\, da \qquad (\underline{T} = \underline{r} \times \underline{F})$$

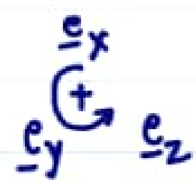
$$\begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \begin{pmatrix} \cancel{\sigma_{xx}} & \cancel{\sigma_{xy}} & \sigma_{xz} \\ \cancel{\sigma_{yx}} & \cancel{\sigma_{yy}} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \cancel{\sigma_{zz}} \end{pmatrix} \begin{pmatrix} n_x = 0 \\ n_y = 0 \\ n_z = 1 \end{pmatrix}$$

$$t_x = \sigma_{xz}, \quad t_y = \sigma_{yz}, \quad t_z = 0$$

$$\underline{x} = x\, \underline{e}_x + y\, \underline{e}_y, \quad \underline{t} = \sigma_{xz}\, \underline{e}_x + \sigma_{yz}\, \underline{e}_y$$

$$\underline{T} = T\underline{e}_z = \int_{\Omega} (x\, \underline{e}_x + y\, \underline{e}_y) \times (\sigma_{xz}\, \underline{e}_x + \sigma_{yz}\, \underline{e}_y)\, dx\, dy$$

$$\boxed{T = \int_{\Omega} (x\, \sigma_{yz} - y\, \sigma_{xz})\, dx\, dy}$$



Equate $\underline{e}_z$ cpts

$$T = \int_{\Omega} (x\, G\alpha x - y(-G\alpha y))\, dx\, dy$$

$$= G\alpha \int_{\Omega} (x^2 + y^2)\, dx\, dy$$

J = polar second moment of area = polar moment of inertia

$T = G\alpha J$ $\qquad$ $J \equiv$ geometry

$$\boxed{\alpha = \frac{T}{GJ}}$$

Recall, axial deformation

$$A = \int_\Omega dx\,dy$$

Torsion $J = \int_\Omega (x^2+y^2)\,dx\,dy$

Total angle of twist

$$\theta = \alpha L = \frac{TL}{GJ}$$

J for circular c/s. cylindrical

$$J = \int_\Omega (x^2+y^2)\,dx\,dy$$

$D = 2R$

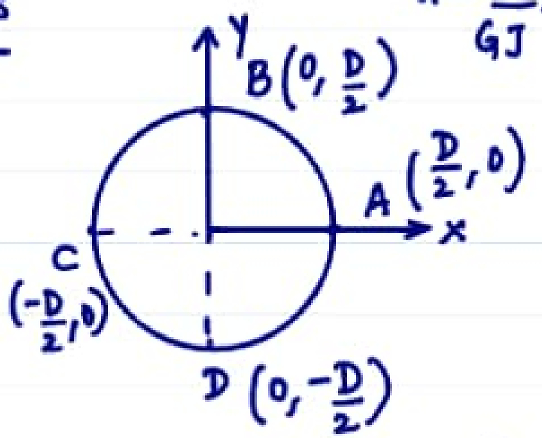
HW

$$J = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

<u>Max shear stress</u>

$$\alpha = \frac{T}{GJ} = \frac{32T}{G\pi D^4}$$

	A	B	C	D
$\tau_{zx} = -G\alpha y$	0	$-\tau$	0	$+\tau$
$\tau_{zy} = +G\alpha x$	$+\tau$	0	$-\tau$	0

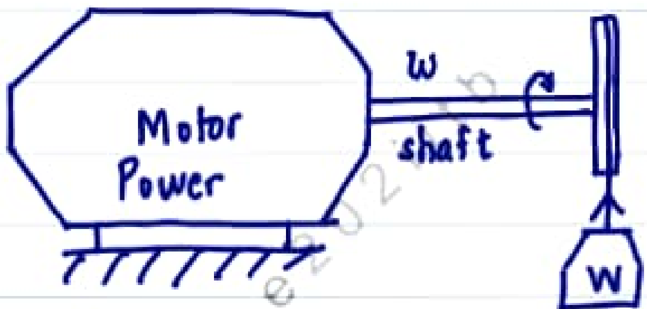


$$\tau = 16T/\pi D^3$$

Recall Axial deformation $\sigma = \frac{P}{A} = \frac{4P}{\pi D^2}$, $\sigma = \sigma_{zz}$

Applications Hollow Cylindrical Shafts

- minimize cost
- minimize self-weight



ω rad/sec

Power $= T\omega$

$$T = G\alpha J = G\alpha \int_{R_1}^{R_2} \int_{0}^{2\pi} \underbrace{(x^2+y^2)}_{r^2}\, r\,dr\,d\varphi$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$T = G\alpha \frac{\pi}{32}\left(D_2^4 - D_1^4\right)$$

$$= G\alpha \frac{\pi}{2}\left(R_2^4 - R_1^4\right)$$

Power Transmission by shafts

$$\text{Power} = T\omega = T\,\frac{2\pi N}{60}$$

T	Nm
ω	rad/s
Power	W
N	RPM

Tachometer

$$1\ \text{HP} \approx 746\ \text{W} = 550\ \text{ft lb/s}$$

$$\text{HP} = \frac{2\pi N T}{33{,}0000}$$

T	lb ft
N	RPM

1 ft = 12 in

$$\tau_{max} = \frac{16T}{\pi D^3}$$

psi

T	lb-in
D	in

Shear Traction/Shear Stress Due to Torque

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} = \begin{pmatrix} 0 & 0 & \sigma_{zx} \\ 0 & 0 & \sigma_{zy} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{t} = \sigma_{zx}\,\underline{e}_x + \sigma_{zy}\,\underline{e}_y$$

in-plane cpt only
No normal cpt.

normal cpt $\qquad\qquad$ shear cpt

$$\underline{t}_n = \underline{0}\ , \quad \underline{t}_s = \underline{t}$$

$\underline{t} = \underline{t}_n + \underline{t}_s$

$$\sigma_s = \|\underline{t}_s\|_2 = \sqrt{\underline{t}_s \cdot \underline{t}_s}$$

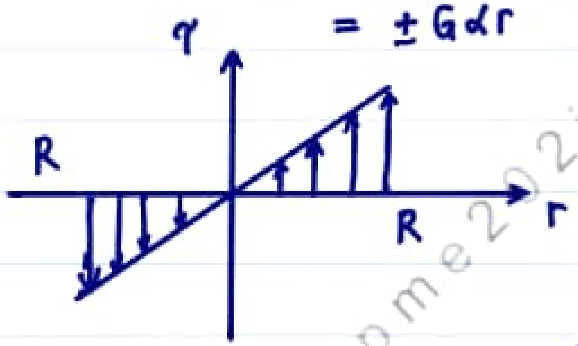
$$\sigma_s = \tau$$

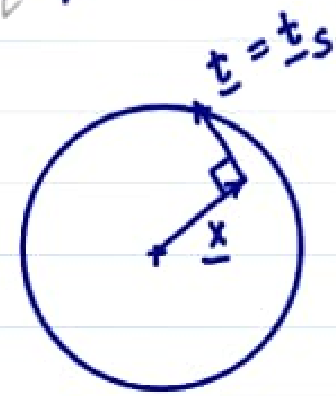
$$= \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}$$

$$= \sqrt{(-G\alpha y)^2 + (G\alpha x)^2}$$

$$= G\alpha\sqrt{x^2+y^2}$$

$$= \pm G\alpha r \qquad r = \text{radial coordinate}$$





$$\underline{t} = -G\alpha y\,\underline{e}_x + G\alpha x\,\underline{e}_y$$

$$\underline{x} = x\,\underline{e}_x + y\,\underline{e}_y$$

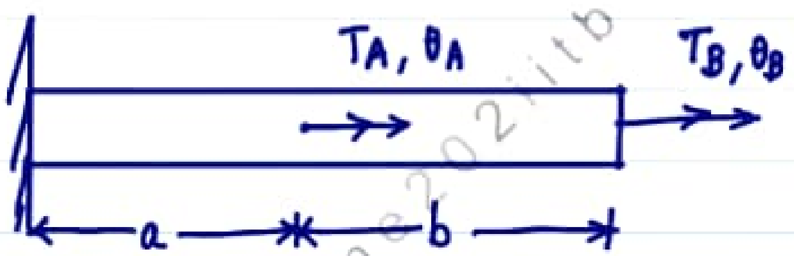
Note that $\underline{t}\cdot\underline{x} = 0$, $\underline{t} \perp \underline{x}$ in x-y plane

Torsional Stiffness

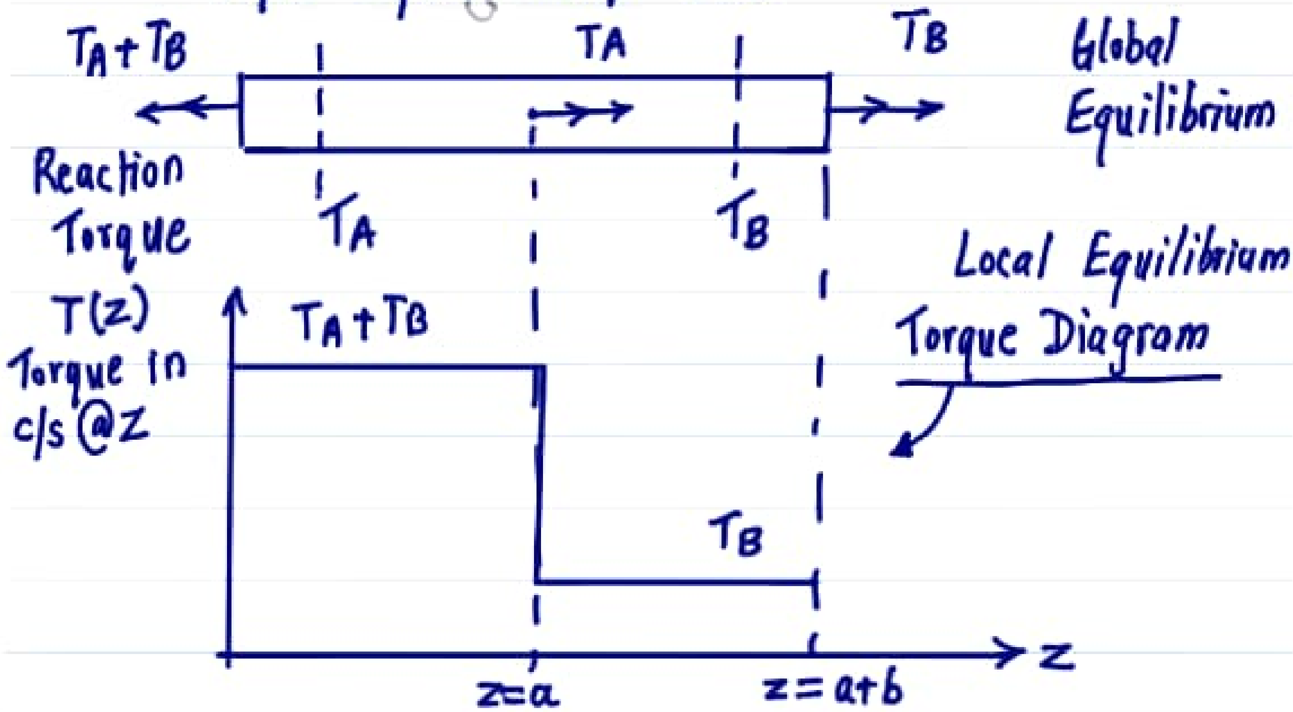
$$\theta = \frac{TL}{GJ}, \qquad k_T = \frac{T}{\theta} = \frac{GJ}{L}$$

Multiple Torques

Linear $\equiv$ whole = sum of its parts



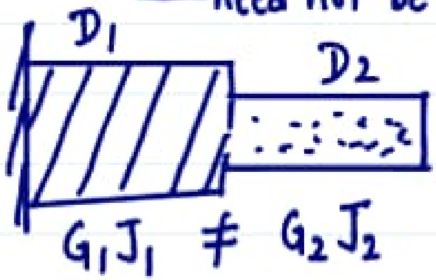
Torque Equilibrium/ FBD



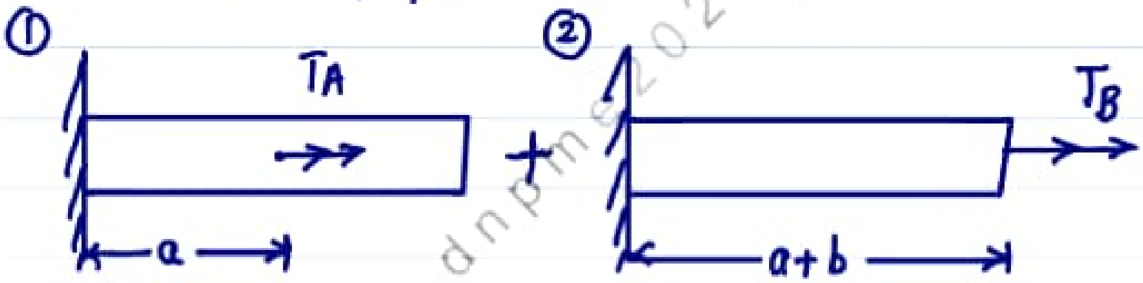
Total angle of twist at $z = a+b$ $\qquad \frac{TL}{GJ}$

$$= \frac{(T_A+T_B)a}{\underbrace{GJ}} + \frac{T_B\, b}{\underbrace{GJ}}$$

need not be equal



<u>Linear Superposition</u> Given problem =



① $\rightarrow$ $\frac{T_A\, a}{GJ} + \frac{T_B(a+b)}{GJ}$ $\leftarrow$ ②

Total angle of twist at free end