Dynamic response of instruments

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General relation

- In general, we assume that the relation between any particular input and output is of the form:

$$a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-1} q_0}{dt^{n-1}} + .. + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + .. + b_1 \frac{dq_i}{dt} + b_0 q_i$$

where q_0: output quantity

q_i: input quantity

t: time

a^s & b^s: system physical parameters

- Writing D = d/dt, the above equation becomes:

$$\left(a_n D^n + a_{n-1} D^{n-1} + .. + a_1 D + a_0\right) q_0 = \left(b_m D^m + b_{m-1} D^{m-1} + .. + b_1 D + b_0\right) q_i$$

Solution of q_0

- The complete solution for q_o is given as:

  $q_o = q_{op} + q_{oc}$

  where q_{op}: particular part of the solution,

  q_{oc}: complementary/homogenous part of the solution, given by setting RHS to zero

$$\left(a_n D^n + a_{n-1} D^{n-1} + .. + a_1 D + a_0\right) q_{0C} = 0$$

Solution of q_{0c}

- Since a^s are constant, we can assume solution to be of the form

  exp(λt)
- Putting it in the governing equation, we obtain

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + .. + a_1 \lambda + a_0 = 0$$

  as the characteristic equation
- Let $\lambda_1, \lambda_2 .. \lambda_n$ be the roots of this equation
- Then $q_{0c} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + .. + c_n e^{\lambda_n t}$
- There should be n independent coefficients (c^s) for it to be a general solution

Solution of q_{0c} (contd.)

- So, what if the roots are (real and) repeated?
- Case 2: If a given root is repeated p times, then the solution for *that* root is
  $$\left(C_1 + C_2 t + .. + C_p t^{p-1}\right) e^{\lambda_p t}$$
- Then $q_{0c} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + .. + \left(C_1 + C_2 t + .. + C_p t^{p-1}\right) e^{\lambda_p t} + .. + c_n e^{\lambda_n t}$
- Case 3: Complex roots (unrepeated): $\lambda_1 = a + bj$
- Recall: complex roots come in pair $\lambda_2 = a - bj$
- Solution for each pair of complex roots can be written as $Ce^{at}\sin(bt+\phi)$

  (with C and phi as the unknowns)
- Therefore, $q_{0c} = Ce^{at}\sin(bt+\phi) + c_3 e^{\lambda_3 t} + .. + c_n e^{\lambda_n t}$

Solution of q_{0c} (contd.)

- Case 4: Complex roots, repeated

$$q_{0c} = C_0 e^{at}\sin(bt+\phi_0) + C_1 e^{at}\sin(bt+\phi_1) + c_5 e^{\lambda_5 t} + .. + c_n e^{\lambda_n t}$$

- Unknowns: C_0, C_1, phi_0, phi_1, c_5,..c_n (total n)
- Similarly, you can handle if a complex root is repeated thrice (or more) times
- Also, if more than a single pair of repeated complex roots

Finding of particular solution – Method of Undetermined Coefficients

- Conditions for the method to apply:
  - Linear, constant coefficient type of equation
  - Repeated differentiation of each function on RHS yields only a finite number of linearly independent terms
  - For example,

$x^2 -> \{x^2, 2x, 2, 0, 0..\}$ Finite

$\frac{1}{x} -> \{\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}..\}$ Not finite

$\sin(2x) -> \{\sin(2x), 2\cos(2x), -4\sin(2x), -8\cos(2x)..\}$ Finite or not?

Method of Undetermined Coefficients

- Consider:

$$y'''' - y'' = 3x^2 - \sin(2x)$$

- First obtain the homogeneous solution
- Characteristic equation is:

$$\lambda^4 - \lambda^2 = 0$$

- with roots: $\lambda = \pm 0, \pm 1$
- Therefore, $y_h = (c_1 + c_2 x) + c_3 e^x + c_4 e^{-x}$
- For function, f_2 = -sin(2x) consider y_{p2} = Dsin(2x) + Ecos(2x) (where D, E will be determined)

Method of Undetermined Coefficients (contd)

- Putting the proposed solution in the governing equation
- Get: (16D sin(2x) + 16E cos(2x)) – (-4D sin(2x) – 4E cos(2x)) = -sin(2x)
- Solve to get: D = -1/20, E = 0
- Therefore, y_{p2} = -sin(2x) /20
- For function, $f_1 = 3x^2$ consider $y_{p1} = Ax^2+Bx+C$ (where A, B, C will be determined)
- Note, duplication between y_{p1} and y_h (x is a common term)
- Multiply y_{p1} by x to remove duplication: $y_{p1} = Ax^3+Bx^2+Cx$
- However, duplication between y_{p1} and y_h is still there
- Multiply y_{p1} by x again: $y_{p1} = Ax^4+Bx^3+Cx^2$
- No duplication – hence ready to proceed

Method of Undetermined Coefficients (contd)

- Put the proposed solution in the governing equation
- Obtain: 24 A – (12A x^2 + 6Bx + 2C) = $3x^2$
- Solve to get: A = -1/4; B = 0; C = (-)3
- Therefore, $y_{p1} = -\frac{1}{4}x^4 - 3x^2$
- Therefore, the general solution of the equation is: $y = (c_1 + c_2 x) + c_3 e^{x} + c_4 e^{-x} - \frac{x^4}{4} - 3x^2 - \frac{\sin(2x)}{20}$
- Note, c_1..c_4 still need to be evaluated from given initial/boundary conditions
- Find the solution, if instead of sin(2x), f_2 = 2sinh(x)?
- What if the input function is not continuous?

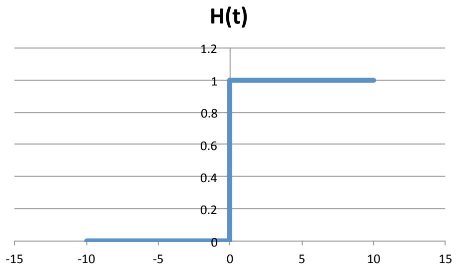
24A - 2C = 0
A=-1/4
-6-2C=0 => C = -3

Step function

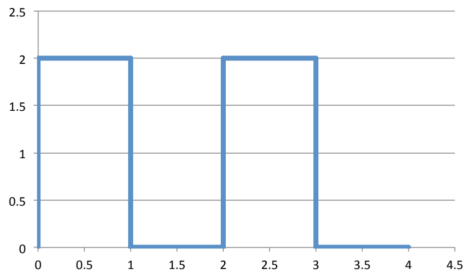
- H(t) = 1 for t > 0

  = 0 otherwise
- H(t-a) = 1 for t > a

  = 0 otherwise



- Consider square wave of magnitude b
- f(t) = (H(t) – H(t-1))*2 + (H(t-2) – H(t-3))*2 + ..



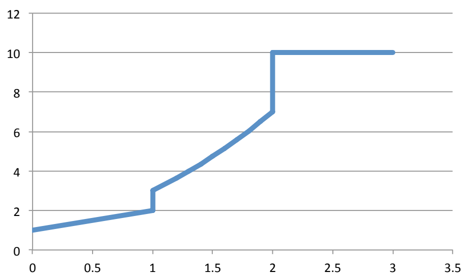
Expressing discontinuous functions in terms of step functions

- f(t) = t + 1 for 0 < t < 1

  = t^2 + t + 1 for 1 < t < 2

  = 10 for t > 2
- f(t) = (t+1)H(t) + (t^2 + t + 1)H(t-1) + 10H(t-2)

  - (t+1)H(t-1) - (t^2 + t + 1)H(t-2)



Laplace Transform

- Discontinuous input could be in the form of impulse, step, ramp, square wave
- We shall employ Laplace transform for discontinuous inputs $F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$
- Existence of Laplace transform of function f(t) is guaranteed (theorem exists) under the following conditions:
  - f(t) is piecewise continuous on $0 \le t \le A$ for every A > 0
  - f(t) is of exponential order as t -> inf. That is, there exists K,c,T such that $|f(t)| \le Ke^{cT}$ for every t >= T