# MA 214: Introduction to numerical analysis
# Lecture 13

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## Fixed points and roots

A fixed point of $f : [a, b] \to \mathbb{R}$ is $p \in [a, b]$ such that $f(p) = p$.

Note that $p \in [a, b]$ is a root of the equation $f(x) = 0$ if and only if $p$ is a fixed point of $g(x) = f(x) - x$. WRONG!

The sign of x should be positive, thank you Aryaman.

Fixed points of various functions are studied well in Mathematics. There are many nice results guaranteeing the existence of fixed points.

# Cumulative calculation of interpolating polynomials

The polynomials $P_0, Q_0$ and $P_1$ interpolating $f$ on $\{x_0\}$, $\{x_1\}$ and $\{x_0, x_1\}$, respectively, are related by

$$P_1(x) = \frac{(x - x_1)P_0(x) - (x - x_0)Q_0(x)}{(x_0 - x_1)}.$$

Further, $P_1, Q_1$ and $P_2$ interpolating $f$ on $\{x_0, x_1\}$, $\{x_1, x_2\}$ and $\{x_0, x_1, x_2\}$, respectively, are related by

$$P_2(x) = \frac{(x - x_2)P_1(x) - (x - x_0)Q_1(x)}{(x_0 - x_2)}.$$

If $P_2$ interpolates $f$ on $\{x_0, x_1, x_2\}$, $Q_2$ on $\{x_1, x_2, x_3\}$ and $P_3$ on $\{x_0, x_1, x_2, x_3\}$ then do we get

$$P_3(x) = \frac{(x - x_3)P_2(x) - (x - x_0)Q_2(x)}{(x_0 - x_3)}?$$

# Neville's formula

Let $f$ be defined on $\{x_0, x_1, \ldots, x_n\}$.

Choose two distinct nodes $x_i$ and $x_j$.

Let $Q_i$ be the polynomial interpolating $f$ on all nodes except $x_i$ and let $Q_j$ be the one interpolating $f$ on all nodes except $x_j$.

If $P$ denotes the polynomial interpolating $f$ on all nodes then

$$P(x) = \frac{(x - x_j)Q_j(x) - (x - x_i)Q_i(x)}{x_i - x_j}.$$

**Proof:** Just verify that $P(x_k) = f(x_k)$ for all $0 \leqslant k \leqslant n$.

# Neville's formula

In Neville's formula you can get the interpolating for higher degree from any two polynomials for two subsets of nodes which are obtained by removing a single node.

Let $P_{m_1,m_2,\ldots,m_k}$ denote the polynomial interpolating the given function on $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$ then

$$P_{0,1} = \frac{(x - x_0)P_1 - (x - x_1)P_0}{x_1 - x_0},$$

$$P_{1,2} = \frac{(x - x_1)P_2 - (x - x_2)P_1}{x_2 - x_1},$$

$$P_{0,1,2} = \frac{(x - x_0)P_{1,2} - (x - x_2)P_{0,1}}{x_2 - x_0} = \frac{(x - x_1)P_{0,2} - (x - x_0)P_{1,2}}{x_0 - x_1}$$

and so on.

# Neville's formula

We then have the following table:

| | | | | | |
|---|---|---|---|---|---|
| $x_0$ | $P_0$ | | | | |
| $x_1$ | $P_1$ | $P_{0,1}$ | | | |
| $x_2$ | $P_2$ | $P_{1,2}$ | $P_{0,1,2}$ | | |
| $x_3$ | $P_3$ | $P_{2,3}$ | $P_{1,2,3}$ | $P_{0,1,2,3}$ | |
| $x_4$ | $P_4$ | $P_{3,4}$ | $P_{2,3,4}$ | $P_{1,2,3,4}$ | $P_{0,1,2,3,4}$ |

Assume that we are given a function $f$ on $(n+1)$-nodes and that we want to compute $f(x)$ for some $x$.

We then go on computing various interpolating polynomials in the order $P_0$, $P_1$, $P_{0,1}$, $P_2$, $P_{1,2}$, $P_{0,1,2}$, $\ldots$ until a sufficient number of digits of the values of the interpolating polynomials of the two highest degrees at $x$ agree.

In this case, they are $P_{0,1,2,3}$, $P_{1,2,3,4}$ and $P_{0,1,2,3,4}$.

# Example

Compute $f(2.1)$ using Neville's method on the following data:

| $x$ | $f(x)$ |
|---|---|
| 2.0 | 0.6931 |
| 2.2 | 0.7885 |
| 2.3 | 0.8329 |

The computations give

| $x_i$ | $P_i(x)$ | $P_{i,i+1}(x)$ | $P_{i,i+1,i+2}(x)$ |
|---|---|---|---|
| 2.0 | 0.6931 | | |
| 2.2 | 0.7885 | 0.7410 | |
| 2.3 | 0.8329 | 0.7441 | 0.7420 |

Neville's method gives the values of interpolating polynomials at a specific point, without having to compute the polynomials themselves.

# Divided differences

We will see another method to construct the interpolating polynomials.

Given the function $f$ on distinct $(n+1)$-nodes, $x_0, \ldots, x_n$, there is a unique polynomial $P_n$ interpolating $f$ on these nodes.

We define $f[x_0, \ldots, x_n]$ to be the coefficient of $x^n$ in $P_n$.

Now, it follows readily that the value of $f[x_0, \ldots, x_n]$ does not depend on the ordering of the nodes $x_i$.

We will get a recurrence formula for the coefficients $f[x_0, \ldots, x_n]$.

# Divided differences

Let $P_{n-1}$ and $Q_{n-1}$ be the polynomials interpolating $f$ on the nodes $x_0, \ldots, x_{n-1}$ and $x_1, \ldots, x_n$, respectively:

$$f(x_0) = P_{n-1}(x_0), f(x_1) = P_{n-1}(x_1), \ldots, f(x_{n-1}) = P_{n-1}(x_{n-1}),$$

and

$$f(x_1) = Q_{n-1}(x_1), f(x_2) = Q_{n-1}(x_2), \ldots, f(x_n) = Q_{n-1}(x_n).$$

By Neville's method,

$$P_n(x) = \frac{(x - x_0)Q_{n-1}(x) - (x - x_n)P_{n-1}(x)}{x_n - x_0}.$$

# Divided differences

The coefficient of $x^n$ in $P_n$ is then

$$\frac{(\text{the coefficient of } x^{n-1} \; in \; Q_{n-1}) - (\text{the coefficient of } x^{n-1} \; in \; P_{n-1})}{x_n - x_0}$$

$$= \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Let us now see whether we can get the recurrence relation for the polynomials $P_n$ in terms of the divided differences.

# Recurrence relation for $P_n$

We note that for $i < n$, $P_n(x_i) = P_{n-1}(x_i)$.

In other words, $P_n - P_{n-1}$ has a zero at each of the points $x_0, x_1, \ldots, x_{n-1}$. Hence

$$P_n - P_{n-1} = \alpha(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where $\alpha$ is a real number.

Now, $\alpha$ has to be the coefficient of the monomial $x^n$ in $P_n$, as the degree of $P_{n-1}$ is $\leqslant n-1$.

Hence $f[x_0, x_1, \ldots, x_n] = \alpha$ and we have

$$P_n = P_{n-1} + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, x_1, \ldots, x_n].$$

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# Divided differences

We are studying divided differences, introduced by Newton, to construct the interpolating polynomials recursively.

For the polynomial $P_n$ interpolating a given function $f$ on nodes $x_0, \ldots, x_n$ we define

$$f[x_0, \ldots, x_n]$$

to be the coefficient of $x^n$ in $P_n$.

If $P_{n-1}$ interpolates $f$ on the nodes $x_0, \ldots, x_{n-1}$ then

$$P_n - P_{n-1} = f[x_0, \ldots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

We also have

$$f[x_0, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

# Properties of the divided differences

The divided differences can be computed in the following way

$$
\begin{array}{cccccc}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & \ddots & \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \dots, x_n] \\
 & & \vdots & & \iddots & \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}
$$

Since everything is independent of the order of the points, we can construct the polynomial $p_n$ in the forward way as well as in the backward way.

# The forward formula

$$
\begin{array}{llllll}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & \ddots & \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \dots, x_n] \\
 & & \vdots & & \iddots & \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}
$$

$$
P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +
$$

$$
\cdots + f[x_0, x_1, \dots, x_n](x - x_0) \cdots (x - x_{n-1}).
$$

# The backward formula

$$\begin{array}{cccccc}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & \ddots & \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \dots, x_n] \\
 & & \vdots & & \iddots & \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}$$

$$P_n(x) = f(x_n) + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1})$$

$$+ \cdots + f[x_0, x_1, \dots, x_n](x - x_n) \cdots (x - x_1).$$

# Example

Find the polynomial interpolating $f$ on $\{0, 1, 2\}$ with $f(0) = 1$, $f(1) = 4$ and $f(2) = 15$. The forward table is

$$\begin{array}{llll} 0 & 1 & & \\ & & 3 & \\ 1 & 4 & & 4 \\ & & 11 & \\ 2 & 15 & & \end{array}$$

hence
$$\begin{aligned} p_2(x) &= 1 + 3x + 4x(x-1) \\ &= 4x^2 - x + 1. \end{aligned}$$

The backward table is

$$\begin{array}{llll} 0 & 1 & & \\ & & 3 & \\ 1 & 4 & & 4 \\ & & 11 & \\ 2 & 15 & & \end{array}$$

then
$$\begin{aligned} p_2(x) &= 15 + 11(x-2) + 4(x-2)(x-1) \\ &= 4x^2 - x + 1. \end{aligned}$$

# Nested form of the interpolating polynomial

The forward formula is

$$P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)\cdots(x - x_{n-1}).$$

This polynomial can be expressed in a nested form as follows:

$$P_n(x) = f(x_0) + (x - x_0)\Big[f[x_0, x_1] + (x - x_1)\big[f[x_0, x_1, x_2] + \cdots + (x - x_{n-1})f[x_0, x_1, \ldots, x_n]\cdots\big]\Big].$$

# Nested form of the interpolating polynomial

In particular, we have

$$P_2(x) = f(x_0) + (x - x_0)\Big[f[x_0, x_1] + (x - x_1) f[x_0, x_1, x_2]\Big]$$

$$\begin{aligned} P_3(x) = f(x_0) \quad &+ \quad (x - x_0)\Bigg[f[x_0, x_1] \\ &+ \quad (x - x_1)\Big[f[x_0, x_1, x_2] \\ &+ \quad (x - x_2) f[x_0, x_1, x_2, x_3]\Big]\Bigg]. \end{aligned}$$

This nested form of the interpolating polynomial is useful for computing the polynomials $P_n$ effectively.

# Divided differences as a function

In the definition of $f[x_0, \ldots, x_n]$, we need that the nodes $x_i$ be all distinct.

We now give the definition of the divided differences when some of the nodes may be equal to each other.

By the Mean Value Theorem, $f[x_0, x_1] = f'(\xi)$ for some $\xi$ between $x_0$ and $x_1$. In fact, we also have the following theorem:

**Theorem**

*If $f$ is n-times continuously differentiable on $[a, b]$ then*

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

*for some $\xi \in [a, b]$.*

# Divided differences as a function

Since $f[x_0, x_1] = f'(\xi)$ for some $\xi$ between $x_0$ and $x_1$, we define

$$f[x_0, x_0] = f'(x_0).$$

This gives

$$f[x_0, x_0] = \lim_{x_1 \to x_0} f[x_0, x_1].$$

We define $f[x_0, \ldots, x_n]$ in a similar way when the nodes are not necessarily distinct, by taking limits. For instance,

$$f[x_0, x_1, x_0] = f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

$$= \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}.$$

And $f[x_0, x_0, x_0] = \dfrac{f^{(2)}(x_0)}{2}$.

# Divided differences as a function

We have thus defined $f[x_0, \ldots, x_n]$ in general.

Now, by letting the last $x_n$ as a variable $x$, we get a function of $x$:

$$f[x_0, x_1, \ldots, x_{n-1}, x].$$

This function is continuous. Indeed,

$$f[x_0, x] = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & x \neq x_0 \\ f'(x_0) & x = x_0 \end{cases}$$

which implies continuity of $f[x_0, x]$.

# Divided differences as a function

In general,

$$f[x_0, x_1, \ldots, x_{n-2}, x_{n-1}, x] = f[x_0, x_1, \ldots, x_{n-2}, x, x_{n-1}]$$

$$= \frac{f[x_1, \ldots, x_{n-2}, x, x_{n-1}] - f[x_0, \ldots, x_{n-2}, x]}{x_{n-1} - x_0}$$

which gives continuity by induction.

We need to take care when there are equalities among the nodes.