Tut-07

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1 Question 1

Use the composite trapezoidal rule with $n = 6$ to approximate the following integral:

$$1) \int_0^2 \frac{2}{x^2+4}\, dx \qquad\qquad 2) \int_0^\pi x^2 cos dx$$

Solution

Given $n = 6$, so we will have 6 partitions,
Now for every partition, $h = \frac{1}{3}$

$$\begin{aligned}\mathbf{I} = &\frac{1}{2} \times \frac{1}{3}\big[f(0) + f(\frac{1}{3})\big] + \frac{1}{2} \times \frac{1}{3}\big[f(\frac{1}{3}) + f(\frac{2}{3})\big] + \frac{1}{2} \times \frac{1}{3}\big[f(\frac{2}{3}) + f(1)\big] \\ &+ \frac{1}{2} \times \frac{1}{3}\big[f(1) + f(\frac{4}{3})\big] + \frac{1}{2} \times \frac{1}{3}\big[f(\frac{4}{3}) + f(\frac{5}{3})\big] + \frac{1}{2} \times \frac{1}{3}\big[f(\frac{5}{3}) + f(2)\big]\end{aligned}$$

$$1) \int_0^2 \frac{2}{x^2+4}\, dx$$

$$f(0) = \frac{2}{0^2+4} = \frac{1}{2} \qquad f(\frac{1}{3}) = \frac{2}{(\frac{1}{3})^2+4} = \frac{18}{37} \qquad f(\frac{2}{3}) = \frac{9}{20}$$

$$f(1) = \frac{2}{5} \qquad f(\frac{4}{3}) = \frac{9}{26} \qquad f(\frac{5}{3}) = \frac{18}{61} \qquad f(2) = \frac{1}{4}$$

On substituting values in the formula

$$\mathbf{I} \approx 0.7842$$

2) $\int_0^\pi x^2 \cos x dx$

As above $n = 6$ partitions i.e. $h = \frac{\pi}{6}$,

$$\mathbf{I} = \frac{1}{2} \times \frac{\pi}{6} \times \big[f(0) + 2f(\frac{\pi}{6}) + 2f(\frac{\pi}{3}) + 2f(\frac{\pi}{2}) + 2f(\frac{2\pi}{3}) + 2f(\frac{5\pi}{6}) + f(\pi)\big]$$

We have
$f(0) = 0,\ f(\frac{\pi}{6}) = 0.2374,\ f(\frac{\pi}{3}) = 0.5483,\ f(\frac{\pi}{2}) = 0,\ f(\frac{2\pi}{3}) = -2.1932,\ f(\frac{5\pi}{6}) = -5.9356,\ f(\pi) = -9.8696$
Hence

$$\mathbf{I} \approx -6.4287$$

2 Question 2

Use the composite simpson's rule to approximate the integrals:

$$1) f(x) = \frac{2}{x^2+4} \qquad\qquad 2) f(x) = x^2 \cos x$$

solution

1) $f(x) = \frac{2}{x^2+4}$

$$\begin{aligned}
\int_0^2 f(x)\,dx &\approx \sum_{j=0}^{5} \int_{\frac{j}{6}}^{\frac{j+1}{6}} f(x)\,dx \\
&= \frac{2}{6} \times \frac{1}{6}\Big[f(0) + 4f(\frac{1}{6}) + f(\frac{2}{6})\Big] + \frac{2}{6} \times \frac{1}{6}\big[f(\frac{2}{6}) + 4f(\frac{3}{6}) + f(\frac{4}{6})\big] + \cdots \\
&= \frac{2}{36}\big[f(0) + 4f(\frac{1}{6}) + 2f(\frac{2}{6}) + 4f(\frac{3}{6}) + 2f(\frac{4}{6}) + 4f(\frac{5}{6}) + 2f(1) + 4f(\frac{7}{6}) + \\
&\qquad 2f(\frac{8}{6}) + 4f(\frac{9}{6}) + 2f(\frac{10}{6}) + 4f(\frac{11}{6}) + f(2)\big] \\
&= 0.785398160076 \\
Error &= 0.785398163397 - 0.785398160076 = 0.000000007
\end{aligned}$$

2) $f(x) = x^2 \cos x$

$$\begin{aligned}
\int_0^{\pi} f(x)\,dx &\approx \frac{\pi}{6} \times \frac{1}{6}\big[f(0) + 4f(\frac{\pi}{12}) + 2f(\frac{2\pi}{12}) + \cdots + 2f(\frac{10\pi}{12}) + 4f(\frac{11\pi}{12}) + f(\pi)\big] \\
&= -5.400681848
\end{aligned}$$

3 Question 3

Suppose that $f(0) = 1, f(0.5) = 2.5, f(1) = 2, f(0.25) = f(0.75) = \alpha$. Find α if the composite trapezoidal rule with $n = 4$ gives the value 1.75 for $\int_0^1 f(x)dx$.

solution

$f(0) = 1$; $f(0.5) = 2.5$; $f(1) = 2$; $f(0.25) = f(0.75) = \alpha$
We will calculate $\int_0^1 f(x)\,dx$ by trapezoidal rule with $n = 4$

$$\begin{aligned}
\mathbf{I} &= \frac{0.25}{2}\big[f(0) + f(0.25)\big] + \frac{0.25}{2}\big[f(0.25) + f(0.5)\big] \\
&\qquad\qquad + \frac{0.25}{2}\big[f(0.5) + f(0.75)\big] + \frac{0.25}{2}\big[f(0.75) + f(1)\big] \\
1.75 &= \frac{0.25}{2}\big[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\big] \\
1.75 &= \frac{0.25}{2}\big[1 + 2\alpha + 2 \times 2.5 + 2\alpha + 2\big] \\
14 &= 1 + 4\alpha + 5 + 2 \\
14 &= 8 + 4\alpha \\
6 &= 4\alpha \\
\alpha &= 1.5
\end{aligned}$$

4 Question 4

Use adaptive quadrature to compute the following integral with accuracy within 10^{-2}

$$f(x) = e^{2x} sin(3x)$$

solution

Given $f(x) = e^{2x} sin(3x)$

$$I = \int_1^3 f(x)dx$$

$$S(a,b) = \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$$

$$F(a,b) = \big|S(a,b) - S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b)\big|$$

Now if $F(1,3) = 67.0899164 > 15\epsilon$
so we divide $(1,3)$ into further as $(1,2)$ and $(2,3)$
$F(1,2) = 0.8008379 > \frac{15\epsilon}{2} = 0.075$
$F(1,1.5) = 0.0240668 < \frac{15\epsilon}{4} = 0.0375$
$F(1.5,2) = 0.011841 < \frac{15\epsilon}{4} = 0.0375$

For(2,3)
$F(2,3) = 6.5656500 > \frac{15\epsilon}{2} = 0.0375$ so we divide $(2,3)$ into $(2,2.5)$ and $(2.5,3)$
$F(2,2.5) = 0.1466767 > \frac{15\epsilon}{4} = 0.0375$
so we divide $(2,2.5)$ into $(2,2.25)$ and $(2.25,2.5)$
$F(2,2.25) = 0.0029153 < \frac{15\epsilon}{8} = 0.01875$

For $(2.5, 3)$
$F(2.5, 3) = 0.0.1439066 > \frac{15\epsilon}{4} = 0.0375$
so we divide $(2.5, 3)$ into $(2.5, 2.75)$ and $(2.75, 3)$
$F(2.5, 2.75) = 0.0069736 < \frac{15\epsilon}{8} = 0.01875$
$F(2.75, 3) = 0.0000401 < \frac{15\epsilon}{8} = 0.01875$
So the sub intervals that we should use to approximate are :
$(1, 1.5), (1.5, 2), (2, 2.25), (2.25, 2.5), (2.5, 2.75), (2.75, 3)$

$S(1, 1.5) + S(1.5, 2) + S(2, 2.25) + S(2.25, 2.5) + S(2.5, 2.75) + S(2.75, 3) = 108.5722885$
and

$$I = \int_1^3 f(x)dx = 108.5722885$$

so we get error as $0.006 < 0.01$.

5 Question 5

Use adaptive quadrature to compute the following integral with accuracy within 10^{-3}:

$$\int_0^5 2x \cos 2x - (x-2)^2 \, dx$$

solution

Approximate $\int_0^5 2x \cos 2x - (x-2)^2 \, dx$ using adaptive quadrature with $\epsilon = 10^{-3}$
First we check for $[0, 5]$ tolerance, we use simpson's rule

$$\begin{aligned}
\mathbf{I}_1 = S(a, b) = S(0, 5) &= \frac{5}{6}\big[f(0) + 4f(2.5) + f(5)\big] \\
&= -13.93123 \\
\mathbf{I}_2 = S(0, 2.5) + S(2.5, 5) &= \frac{5}{12}\big[f(0) + 4f(1.25) + 2f(2.5) + 4f(3.75) + f(5)\big] \\
&= -12.98603.
\end{aligned}$$

$|\mathbf{I}_2 - \mathbf{I}(f)| \approx \frac{1}{15}|\mathbf{I}_1 - \mathbf{I}_2| = 0.063 > 10^{-3}$
So we check for $[0.2, 5]$ and $[2.5, 5]$ independently with tolerance , $\frac{\epsilon}{2}$

check for $[0, 2.5]$

$$\mathbf{I}_1 = S(0, 2.5) = -5.45547$$
$$\mathbf{I}_2 = S(0, 1.25) + S(1.25, 2.5) = -5.48316$$

$|\mathbf{I}_2 - \mathbf{I}(f)| \approx \frac{1}{15}|\mathbf{I}_2 - \mathbf{I}_1| = 0.0018 > 0.0005$
So we apply Simpson's rule for interval $[0, 1.25]$ and $[1.25, 2.5]$. If we keep on going in this way we can confirm that for subinterval $[0, 0.625]$ with corresponding tolerance, $\frac{\epsilon}{8} = 0.000125$ satisfies for $[0, 0.625]$. We get

$$\mathbf{I}_1 = S(0, 0.625) = -1.54788$$

$$\mathbf{I}_2 = S(0, 0.3125) + S(0.3125, 0.625) = -1.54926$$

Clearly $\frac{|\mathbf{I}_1 - \mathbf{I}_2|}{15} = 1.213 \times 10^{-5} < 0.000125$.
Similarly we can also confirm that for sub-intervals
$[0.625, 1.25]$;$[1.25, 1.875]$; $[1.875, 2.1875]$; $[2.1875, 2.5]$; $[2..5, 3.125]$; $[3.125, 3.4375]$; $[3.4375, 3.75]$; $[3.75, 4.375]$; $[4.375, 4.6875]$; and $[4.6875, 5]$ satisfy for their corresponding tolerances.
So,

$$\int_0^5 f(x)\,dx = \int_0^5 \left[2x\cos 2x - (x-2)^2\right] dx \text{ is } \approx \text{ sum of } \mathbf{I}'s$$

of above subintervals which satisfy their corresponding tolerances such that occuracy of our approximation is within 10^{-3}.
So,

$$\begin{aligned}\int_0^5 f(x)\,dx &\approx -1.54926 + (-1.12907) + (-1.96954) + (-0.75058) + (-0.06526) \\ &\qquad + 2.21864 + 1.4210604 + 0.56188 + (-5.40321) \\ &= -15.306116\end{aligned}$$

Then the error is
$\int_0^5 f(x)\,dx = -15.3063 - 7 < 10^{-3}$.

6 Question 6

Let $T(a,b), T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)$ be the single and double applications of the trapezoidal rule to$\int_a^b f(x)\,dx$. Derive the relationship between

$$T(a,b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)$$

and

$$\int_a^b f(x)\,dx - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)$$

solution

From the composite trapezoidal rule, we have

$$\int_a^b f(x)\,dx = \frac{h}{2}\big[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\big] - \frac{b-a}{12}h^2 f''(\mu)$$

for some $\mu \in (a,b)$ and $h = \frac{b-a}{n}$
In this question we have $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b] \implies n = 2$.
Since

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{1}{2}(\frac{b-a}{2})\big[f(a) + 2f(\frac{a+b}{2}) + f(b)\big] - \frac{1}{2^2}\frac{(b-a)^3}{12}f''(\mu)\\ &= T(a, \frac{a+b}{2}) + T(\frac{a+b}{2}, b) - \frac{1}{4}\frac{(b-a)^3}{12}f''(\mu)\end{aligned}$$

$$|\int_a^b f(x)\,dx - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)\big| = \frac{1}{4}\frac{(b-a)^3}{12}|f''(\mu)| \tag{1}$$

Now let us go to adaptive quadrature method method we know that by putting $n = 1$ in composite Trapezoidal rule, we get

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{h}{2}\big[f(a) + f(b)\big] - \frac{(b-a)^3}{12}f''(\mu')\\ &= T(a,b) - \frac{(b-a)^3}{12}f''(\mu')\end{aligned}$$

Thus we have

$$T(a,b) - \frac{(b-a)^3}{12}f''(\mu') = T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b) - \frac{1}{4}\frac{(b-a)^3}{12}f''(\mu)$$

Take the opproximation $f''(\mu') \approx f''(\mu)$ the strength of our result depends on the validity of this approximation, we get

$$T(a,b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b) = \frac{3}{4}\frac{(b-a)^3}{12}f''(\mu) \tag{2}$$

Thus from equation (1) and (2) we get

$$\big|T(a,b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)\big| \approx 3\big|\int_a^b f(x)\,dx - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)\big|$$

7 Question 7

Approximate the integrals using Gaussian quadrature with $(n = 2)$ and compare from results to the exact values of the integrals.

$$1)\int_1^{1.5} x^2 \log x\,dx \qquad\qquad 2)\int_0^1 x^2 e^{-x}\,dx$$

solution

Gaussian quadrature rule for $n = 2$,

$$\int_{-1}^{1} f(x)\,dx \approx f(\frac{-\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) \tag{1}$$

First we need to transform our integral $\int_a^b f(x)\,dx$ into an integral defined over $[-1, 1]$.Let

$$x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$$
$$dx = \frac{(b-a)}{2}dt$$

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{-1}^{1} f\big(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\big)\big(\frac{b-a}{2}\big)\,dt \\
\int_1^{1.5} f(x)\,dx &= \int_{-1}^{1} f\big(\frac{1}{2}(1.5+1) + \frac{1}{2}(1.5-1)t\big)\big(\frac{1.5-1}{2}\big)\,dt \\
&= \frac{1}{4}\int_{-1}^{1} f(\frac{t+5}{4})\,dt \\
&= \frac{1}{4}\int_{-1}^{1} (\frac{t+5}{4})^2 \log\big(\frac{t+5}{4}\big)\,dt \\
Now using (1) \\
&= \frac{1}{4}\big[(\frac{\frac{-\sqrt{3}}{3}+5}{4})^2 \log\big(\frac{\frac{-\sqrt{3}}{3}+5}{4}\big)\big] + \big[(\frac{\frac{\sqrt{3}}{3}+5}{4})^2 \log\big(\frac{\frac{\sqrt{3}}{3}+5}{4}\big)\big] \\
&= 0.1922687
\end{aligned}$$

Exact value of the integral

$$\int_1^{1.5} x^2 \log x\,dx = 0.1922594$$

$$|\text{estimate value - exact value}| = 9.3 \times 10^{-6}$$

2) $\int_0^1 x^2 e^{-x}\,dx$, using the similar process as in (I) we get

$$\text{estimate value} = 0.1593326$$

and

$$\text{Exact value } = 0.0012702$$
$$|\text{estimate value - exact value}| = 0.0012702.$$

8 Question 8

Approximate the integrals using Gaussian quadrature with $n = 4$ and compare from results to the exact values of the integrals.
I) $\int_1^{1.5} x^2 \log x \, dx$

solution

1) For interval $[-1, 1]$, $n = 4$, by Gaussian quadrature

$$\begin{array}{ccccc} x_i, & \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} & -\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} & \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} & -\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} \\ C_i, & \frac{18+\sqrt{30}}{36} & \frac{18+\sqrt{30}}{36} & \frac{18-\sqrt{30}}{36} & \frac{18-\sqrt{30}}{36} \end{array}$$

1)

$$\int_1^{1.5} x^2 \log x \, dx = \int_{-1}^{1} (\frac{t+5}{4})^2 \log (\frac{t+5}{4}) \, dx$$

By sustituting $x = \frac{t+5}{4}$.
Actual value of the integral calculated using $\int_1^{1-5} f(x)g(x) \, dx$ is $\frac{x^3}{9}(3\log x - 1) + c$. Actual ans from 1 to 1.5 = 0.1922593577
Using Gaussian quadrature in interval $[-1, 1]$ for transformed integral

$$\text{approx value} = \sum_{i=1}^{4} \frac{c_i}{4} \left(\frac{x_i+5}{4}\right)^2 \log\left(\frac{x_i+5}{4}\right) = 0.1922593578$$

2)

$$\int_0^1 x^2 e^{-x} \, dx = \int (\frac{t+1}{2})^2 e^{-\frac{t+1}{2}} \frac{dt}{2}$$

by substituting $x = \frac{t+1}{2}$,

Actual value of integral calculated using $\int f(x)g(x) \, dx$ is $-(x^2+2x+2)e^{-x}+c$
Actual answer from 0 to 1 = 0.160602794
Using Gaussian quadrature in interval $[-1, 1]$

$$\sum_{i=1}^{4} \frac{c_i}{2} (\frac{x_i+1}{2})^2 e^{-\left(\frac{x_i+1}{2}\right)} = 0.160602777$$

Note:- we could have also solved for $\{x_i\}_{i=1}^4$ $\{c_i\}_{i=1}^4$ for both functions.

9 Question 9

Determine constants a, b, c, d that produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3.

solution

For $f(x) = 1$ we have

$$2 = a + b$$

For $f(x) = x$ we have

$$0 = -a + b + c + d$$

For $f(x) = x^2$

$$2/3 = a + b - 2c + 2d$$

For $f(x) = x^3$

$$0 = -a + b + 3c + 3d$$

We solve this system of equations to get:

$$a = 1$$

$$b = 1$$

$$c = 1/3$$

$$d = 1/3$$