MA 214

Spring-2022

Tutorial 8

Problem 1: *Use Taylor polynomial P_4 and composite Simpson's rule with $n = 6$ to approximate the improper integral* $\int_0^1 \frac{e^{2x}}{\sqrt[5]{x^2}}\, dx$.

Solution: Let $f(x) = \frac{e^{2x}}{\sqrt[5]{x^2}}$ has a singularity at $x = 0$, it is an indefinite integral.

We use $P_4(x)$ to approximate e^{2x}

$$\int_0^1 f(x)\, dx = \int_0^1 \frac{e^{2x} - P_4(x)}{\sqrt[5]{x^2}}\, dx + \int_0^1 \frac{P_4(x)}{\sqrt[5]{x^2}}\, dx = I_1 + I_2$$

where $P_4(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!}$

Now computting $I_2 = \int_0^1 \frac{P_4(x)}{\sqrt[5]{x^2}}\, dx =$4.20119553

We compute I_1 using Simpson's one-third rule.

Let $G(x) = \frac{e^{2x} - P_4(x)}{\sqrt[5]{x^2}}$.

Now , $\lim_{x\to 0} G(x) = 0$

$$\int_0^1 G(x)\, dx \approx \frac{1}{18}\{G(0) + 2[G(x_2) + G_(x_4)] + 4[g(x_1) + G(x_3) + G(x_5)] + G(1)\} = 0.11782813$$

$$\therefore I = I_1 + I_2 = 4.31902366$$

Problem 2: *Use Taylor polynomial P_4 and composite Simpson's rule with $n = 6$ to approximate the improper integral* $\displaystyle\int_0^1 \frac{cos2x}{x^{\frac{1}{3}}}\,dx$.

Solution: Let $f(x) = \frac{cos2x}{x^{\frac{1}{3}}}$, it has a singularity at $x = 0$. Since we are using Simpson's rule, we use $P_4(x)$ to approximate $cos2x$.

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{cos2x - P_4(x)}{x^{\frac{1}{3}}}\,dx + \int_0^1 \frac{P_4(x)}{x^{\frac{1}{3}}}\,dx = I_1 + I_2$$

Here $P_4(x) = 1 - 2x^2 + (2/3)x^4$

$I_2 = \displaystyle\int_0^1 \frac{P_4(x)}{x^{\frac{1}{3}}}\,dx = 25/28 = 0.892857...$

Let $G(x) = cos(2x) - 1 + 2x^2 - (2/3)x^4$, and $I_1[0,1] = [0,1/3] \cup [1/3,2/3] \cup [2/3,1]$

So $I_1 = \sum_{i=1}^{3} I_i^{'}$ and

$I_1^{'} = (1/18)[G(0) + 4G(1/6) + G(1/3)]$

$I_2^{'} = (1/18)[G(1/3) + 4G(1/2) + G(2/3)]$

$I_3^{'} = (1/18)[G(2/3) + 4G(5/6) + G(1)]$

This gives

$I_1 = I_1^{'} + I_2^{'} + I_3^{'} = -0.00632808$

And $I = I_1 + I_2 = 0.88652905$

Problem 3: *Approximate the valie of the improper integral* $\int_1^\infty x^{\frac{-3}{2}} sin\frac{1}{x} dx.$

Solution: Given I $= \int_1^\infty x^{\frac{-3}{2}} sin\frac{1}{x} dx$
Put $t = 1/x \implies dt = \frac{-1}{x^2}dx \implies dx = -x^2 dt = \frac{-1}{t^2}dt$
Thus $x = 1 \to t = 1, x = \inf \to t = 0$

$$\begin{aligned} I &= \int_1^0 t\sqrt{t} sin(t)(\frac{-1}{t^2})dt \\ &= \int\limits_0^1 \frac{1}{\sqrt{t}} sin\, t\ dt \\ &= \int_0^1 f(t)\, dt \end{aligned}$$

where $f(t) = \frac{g(t)}{(t-a)^p}$; thus $g(t) = sin\, t, a = 0, p = 1/2$.

Therefore

$$I = \int_0^1 f(t)\, ft = \int_0^1 \frac{(f(t) - P_4(t))}{t^{1/2}} dt + \int_0^1 \frac{P_4(t)}{t^{1/2}}$$

where

$$P_4(t) = \sum_{i=0}^4 \frac{g(0)}{i!}(t-0)^i = t - \frac{t^3}{6}$$

$$T_2 = \int_0^1 \frac{P_4(t)}{t^{1/2}}\, dt = \frac{13}{21}$$

Define $G(t) = \begin{cases} \frac{sin\, t - t + \frac{t^3}{6}}{t^{1/2}} & 0 < t \leq 1 \\ 0 & otherwise \end{cases}$ Using the composite Simpson's Rule,

$$T_1 = \int_0^1 G(t)dt \approx \frac{h}{3}(G(0) + 4G(1/4) + 2G(1/2) + 4G(1/3) + G(4))$$

Thus

$$\begin{aligned} T_1 = &\frac{1}{12}(0 + 4 * 0.0.00001625184 + 2 * 0.00036610020 + \\ &4 * 0.00225312100 + 0.00813765147) \\ = &\ 0.001496 \end{aligned}$$

Thus $I = T_1 + T_2 = 13/21 + 0.001496 = 0.620543$

Problem 4: *Use Euler's method with* $h = 0.25$ *to approximate the solution for the intial-value problem:* $y' = 1 + (t-y)^2,\ 2 \leq t \leq 3$ *and* $y(2) = 1$. *Compare the results with* $y(t) = t + \frac{1}{1-5}$.

Solution: Here $f(t,y) = 1 + (t-y)^2$ wherin $t_0 = 2$ and $w_0 = 1$

$t_1 = t_0 + h = 2.25$
$w_1 = w_0 + hf(t_0, w_0) = 1 + (0.25)(2) = 1.5$

$t_2 = t_0 + 2h = 2.5$
$w_2 = w_1 + hf(t_1, w_1) = 1.8906$

$t_3 = t_0 + 3h = 2.75$
$w_3 = w_2 + hf(t_2, w_2) = 2.2334$

$t_4 = t_0 + 4h = 3$
$w_4 = w_3 + hf(t_3, w_3) = 2.5501$

Thus we have the following approximations

$$\begin{aligned} y(2.25) &\approx w_1 = 1.15 \\ y(2.5) &\approx w_2 = 1.8906 \\ y(2.75) &\approx w_3 = 2.2334 \\ y(3) &\approx w_4 = 2.5501 \end{aligned}$$

We compare the above valies with the actual values given by $y(t) = t + 1/(1-t)$

t_i	w_i	y_i	$\lvert y_i - w_i \rvert$
2	1	1	0
2.25	1.5	1.45	0.05
2.5	1.8906	1.8333	0.0573
2.75	2.2334	2.1786	0.0548
3	2.5501	2.5	0.0501

Problem 5: *Use Euler's method with $h = 0.25$ to approximate the solution for the initial value problem: $y' = cos2t + sint2t, 0 \leq t \leq 1$ and $y(0) = 1$. Compare the results with $y(t) = \frac{1}{2}sin2t - \frac{1}{2}cos\,2t + \frac{3}{2}$.*

Solution:

$$y' = cos(2t) + sin(2t), 0 <= t <= 1$$

$$y(0) = 1$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

$$w_0 = y(0) = 1$$

$$f(t, y) = cos(2t) + sin(2t)$$

$t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, $t_4 = 1$

Using 5 digit chopping :

$$w_1 = 1 + 0.25f(0, 1) = 1 + 0.25 * 1 = 1.25$$

$$w_2 = 1.25 + 0.25f(0.25, 1.25) = 1.25 + 0.25 * 1.35700 = 1.58925$$

$$w_3 = 1.58925 + 0.25f(0.5, 1.58925) = 1.58925 + 0.25 * 1.38177 = 1.93469$$

$$w_4 = 1.93469 + 0.25f(0.75, 1.93469) = 1.93469 + 0.25 * 1.06823 = 2.20174$$

Comparing with

$$y(t) = \frac{sin(2t)}{2} - \frac{cos(2t)}{2} + \frac{3}{2}$$

$$y(0) = 1.0 \ (w_0 = 1)$$

$$y(0.25) = 1.30092 \ (w_1 = 1.25)$$

$$y(0.5) = 1.65058 \ (w_2 = 1.58925)$$

$$y(0.75) = 1.96337 \ (w_3 = 1.93469)$$

$$y(1) = 2.16272 \ (w_4 = 2.20174)$$

Relative error in approximating $y(1) = \frac{|2.20174-2.16272|}{2.16272} = 1.8\%$

Problem 6: *Consider the initial-value problem:* $y' = -10y$, $0 \le t \le 2$, $y(0) = 1$ *with solution* $y(t) = e^{-10t}$. *What happens when Euler's method is applied to this problem with* $h = 0.1$*? Does this behavior violate the erroe bound?*

Solution: For the given initial value problem, by Euler's Method
$w_{i+1} = w_i + hf(t_i, w_i)$ which implies $w_{i+1} = w_i(1 - 10h)$
Thus $w_n = w_0(1-10h)^n$. But since $h = 0.1$, all the $w_n = 0$ for $n = 1, 2, \ldots, 20$
The given $f(t, y)$ satisfies the Lipschitz condition as $|f(t, y_2) - f(t, y_1)| = |10(y_1 - y_2)| \le 10|y_1 - y_2| \implies L = 10$ which is our Lipschitz constant.
The error bound between $y(t_i)$ and w_i is given by

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right)$$

where $h = 0.1, L = 10, M = \max_{t \in [0,2]} |y''(t)| = 100$

For $i = 0$ this relation holds as $|w_0 - w_0| = 0$
For $1 \le i \le 20$, let us look at the error inequality
$t_i = 0 + ih = 0.1i$
$y(t_i) = e^{-10t_i} = e^{-1} > 0$ for $i \in \{1, 2, \ldots, 20\}$, $\;w_i = 0$; $\;e^{L(t_i - a)} = e^i$
For $i > 1$, $e^i - 1 > 0 \implies 0.5e^i - 0.5 - e^{-i} \ge 0$ has to hold for error bound to be accurate.
Let $g(i) = \frac{e^i}{2} - e^{-i} - 0.5$
$g(1) = 0.49$ and $g'(i) > 0$ for $i = 1, 2, \ldots, 20$. Therefore $g(i) > 0$ for all $1 \le i \le 20$
Thus the error bound holds for all $i = 0, 1, 2, \ldots 20$.

Problem 7: *Use Taylor's Method of order 2 and 4 with* $h = 0.25$ *to approximate the solution for the intial value problem:* $y' = 1 + \frac{y}{t},\ 1 \leq t \leq 2, y(1) = 2$

Solution: Given $h = 0.25$, and the initial value problem $y' = 1 + \frac{y}{t}$, we are required to approximate the solution for $1 \leq t \leq 2$. So $f(t, y(t)) = 1 + \frac{y}{t}$ and $w_{i+1} = w_i + hT(t_i, w_i)$

Part 1: $\mathbf{n = 2}$

$$T(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$
$$f'(t, y(t)) = \frac{ty' - y}{t^2} = \frac{t\left(1 + \frac{y}{t}\right) - y}{t^2} = \frac{1}{t}$$

So, we get the following expression for $T(t_i, w_i)$,

$$T(t_i, w_i) = 1 + \frac{w_i}{t_i} + \frac{h}{2}\frac{1}{t_i}$$

So we have $w_{i+1} = 0.25 + w_i\left(1 + \frac{0.25}{t_i}\right) + \frac{0.0625}{2t_i}$ Thus, we get the following values (with $w_0 = y(1) = 2$),

t_i	w_i
1	2
1.25	2.78125
1.5	3.6125
1.75	4.4854
2	5.394

Part 2: $n = 4$

$$T(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i)$$

From previous part, $f'(t, y(t)) = \frac{1}{t}$, so we have, $f''(t, y(t)) = -\frac{1}{t^2}$ and $f'''(t, y(t)) = \frac{2}{t^3}$. Using these expressions in the above equation, we get,

$$T\left(t_i, w_i\right) = 1 + \frac{w_i}{t_i} + \frac{h}{2}\frac{1}{t_i} - \frac{h^2}{6}\frac{1}{t_i^2} + \frac{h^3}{12}\frac{1}{t_i^3}$$

So we have, $w_{i+1} = 0.25 + w_i\left(1 + \frac{0.25}{t_i}\right) + \frac{0.25^2}{2t_i} - \frac{0.25^3}{6t_i^2} + \frac{0.25^4}{12t_i^3}$ Thus we get the following values with $w_0 = y(1) = 2$.

t_i	w_i
1	2
1.25	2.77897
1.5	3.60826
1.75	4.47941
2	5.38639

Problem 8: *Use Taylor's Method of order 2 and 4 with* $h = 0.25$ *to approximate the solution for the intial value problem:* $y' = cos2t + sin3t, \ 0 \leq t \leq 1, y(0) = 1$

Solution: Taylor's Method of Order 2
$y(x_i + h) = y(x_0) + hy'(x_i) + \frac{h^2}{2}y''(x_i)$ where $x_0 = 0, h = 0.25,$
$x_{i+1} = x_i + h, y'' = -2sin2t + 3cos3t$

Thus $y(0.25) = y(0) + hy'(0) + \frac{h^2}{2}y''(0) = 1.34375$

Now for $x_0 = 0.25$
$y(0.5) = y(0.25) + hy'(0.25) + \frac{h^2}{2}y''(0.25) = 1.7722$

Now for $x_0 = 0.5$
$y(0.75) = y(0.5) + hy'(0.5) + \frac{h^2}{2}y''(0.5) = 2.1107$

Now for $x_0 = 0.75$
$y(1) = y(0.75) + hy'(0.75) + \frac{h^2}{2}y''(0.75) = 2.2017$

Taylor's Method of Order 2
$y(x_i + h) = \sum_{i=0}^{4} \frac{h^i}{i!}y^{(i)}(x_i)$ where $h = 0.25$, $x_{i+1} = x_i + h$ and $x_0 = 0$.
Thus we have

$$\begin{aligned} y(0.25) &= 1.3289 \\ y(0.5) &= 1.7296 \\ y(0.75) &= 2.0372 \\ y(1) &= 2.1133 \end{aligned}$$

Problem 9: *Use Taylor's method of order 2 and 4 with $h = 0.5$ to approximate the solution for the initial-value problem: $y' = te^{3t} - 2y$, $0 \leq t \leq 1$, $y(0) = 0$.*

Solution: Given $h = 0.5$, $y' = te^{3t} - 2y$, $0 \leq t \leq 1$, $y(0) = 0$
Let $w_{i+1} = w_i + hT(t_i, w_i)$
For $n = 2$,
$T(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) = t_i e^{3t_i} - 2w_i + \frac{h}{2}(e^{3t_i} + t_i e^{3t_i} + 4y)$
This gives $w_{i+1} = \frac{1}{2} t_i e^{3t_i} + \frac{1}{8} e^{3t_i} + \frac{1}{8} t_i e^{3t_i} + \frac{w_i}{2}$

t_i	w_i
0	0.125
0.5	2.02324
1	16.07572

For $n = 4$,

$$\begin{aligned} T(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i) \\ &= t_i e^{3t_i} - 2w_i + \frac{0.5}{2}(e^{3t_i} + t_i e^{3t_i} + t_i w_i) + \\ &\frac{(0.5)^2}{6}(4e^{3t_i} + 7t_i e^{3t_i} - 8w_i) + \frac{(0.5^3)}{24}(19e^{3t_i} + 13t_i e^{3t_i} 16w_i) \end{aligned}$$

t_i	w_i
0	0.765625
0.5	9.535
1	35.7840877