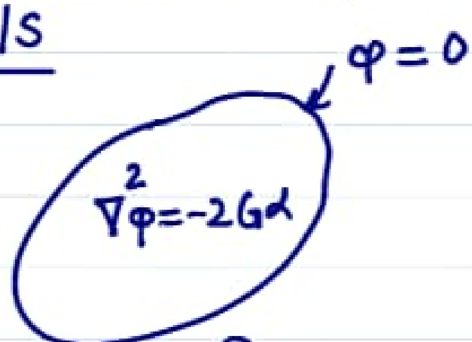
ME 202
LECTURE 11

TUE 25 JAN 2022

APPLICATION OF PMPE TO TORSION OF NON-CIRCULAR C/S



PMPE

- Lineor system

$$\underset{\sim}{A}\,\underline{x} = \underline{b} \qquad N \times N$$

$$T = 2\int_{\Omega} \varphi \, da$$

- Approximate method
- Computational method

$$\Pi_{total} = \Pi_{SE} + \underbrace{\Pi_{Ext\ \cancel{\text{Forces}}\ \text{Torques}}} \qquad L = 1$$

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$-T\alpha$

Recall,

strain/stored elastic energy density (J/m³) $= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\cdot\epsilon_{ij}$

$1 \; x$, $2 \; y$, $3 \; z$

$$= \cancel{\frac{1}{2}\sigma_{xx}\epsilon_{xx}} + \cancel{\frac{1}{2}\sigma_{xy}\epsilon_{xy}} + \cancel{\frac{1}{2}\sigma_{yx}\epsilon_{yx}}$$

$$+ \ldots + \frac{1}{2}\sigma_{xz}\epsilon_{xz} + \frac{1}{2}\sigma_{zx}\epsilon_{zx} + \frac{1}{2}\sigma_{yz}\epsilon_{yz}$$

$$+ \frac{1}{2}\sigma_{zy}\epsilon_{zy} + \cancel{\frac{1}{2}\sigma_{zz}\epsilon_{zz}}$$

Torsion $\underset{\sim}{\sigma} = \begin{pmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{pmatrix}$

Hooke's Law

$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}, \quad \epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

$$\sigma_{xz} = \frac{\partial\varphi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial\varphi}{\partial x}$$

$$SED = \frac{1}{2G}\Big[\underbrace{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}_{\underline{\nabla}\varphi\cdot\underline{\nabla}\varphi}\Big] \quad \frac{J}{m^3}$$

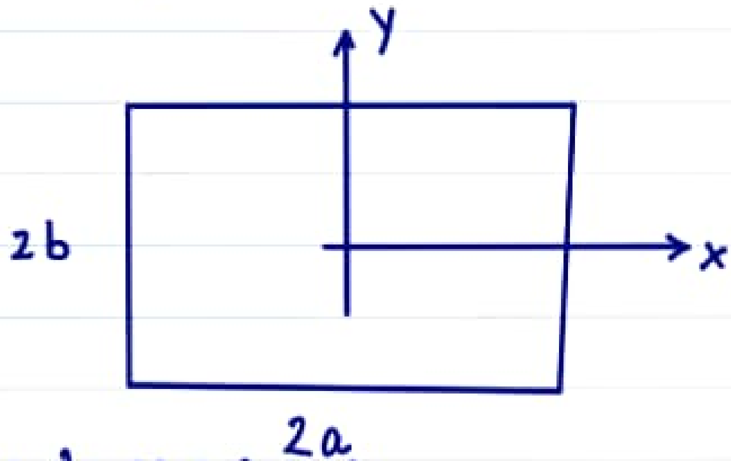
This is NOT $\nabla^2\varphi$

Total PE per unit length

$$\Pi = \int_{\Omega} \frac{1}{2G}\left[\underline{\nabla}\varphi \cdot \underline{\nabla}\varphi\right] da \;-\; 2\alpha \int_{\Omega} \varphi \, da$$

Find $\varphi(x,y)$ that approx. minimizes Π
and which obeys BC $\varphi = 0$.

Example



$$\varphi(x,y) = K\,(x^2 - a^2)(y^2 - b^2) \qquad \text{zero on } \partial\Omega$$

$$\frac{\partial\varphi}{\partial x} = -2Kx\,(b^2 - y^2), \quad \frac{\partial\varphi}{\partial y} = -2Ky\,(a^2 - x^2)$$

$$\Pi = \frac{1}{2G}\int_{x=-a}^{a} dx \int_{y=-b}^{b} dy \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]$$

$$-\,2\alpha \int_{x=-a}^{a} dx \int_{y=-b}^{b} dy \;\varphi$$

$$\Pi(K) \underset{\substack{\uparrow \\ \text{approx.}}}{=} \frac{64K^2a^3b^3(a^2+b^2)}{45G} - \frac{32Ka^3b^3}{9}\alpha$$

Find K that minimizes Π

$$\frac{\partial \Pi}{\partial K} = 0 \Rightarrow \frac{128Ka^3b^3(a^2+b^2)}{45G} - \frac{32\alpha a^3 b^3}{9} = 0$$

$$K = \frac{5G\alpha}{4(a^2+b^2)}$$

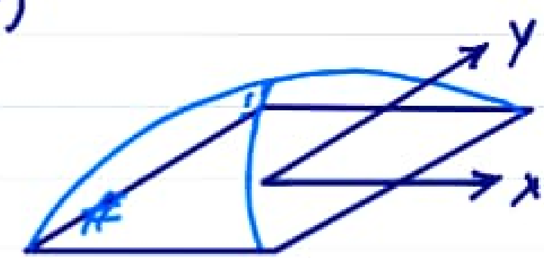
$$\varphi = \frac{5G\alpha}{4(a^2+b^2)}(x^2-a^2)(y^2-b^2)$$

$$T = 2\int_{x=-a}^{a} dx \int_{y=-b}^{b} dx \ \varphi = \frac{40G\alpha a^3 b^3}{9(a^2+b^2)}$$

$$T = \frac{40Ga^3b^3}{9(a^2+b^2)} \cdot \alpha$$

Recall,

soap film analogy



☐ zero on $\partial\Omega$

☐ agrees with analogy ☐ easy to integrate

Pick $\varphi = K \cos\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{2b}\right)$

$$\frac{\partial \varphi}{\partial x} = -\frac{K\pi}{2a} \sin\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{2b}\right)$$

$$\frac{\partial \varphi}{\partial y} = -\frac{K\pi}{2b} \sin\left(\frac{\pi y}{2b}\right) \cos\left(\frac{\pi x}{2a}\right)$$

$$\Pi(K) = \frac{K^2 \pi^2 (a^2+b^2)}{8Gab} - \frac{32\alpha a b K}{\pi^2}$$

PMPE $\Rightarrow \Pi'(K) = 0 \Rightarrow K = \frac{128 G a^2 b^2 \alpha}{\pi^4 (a^2+b^2)}$

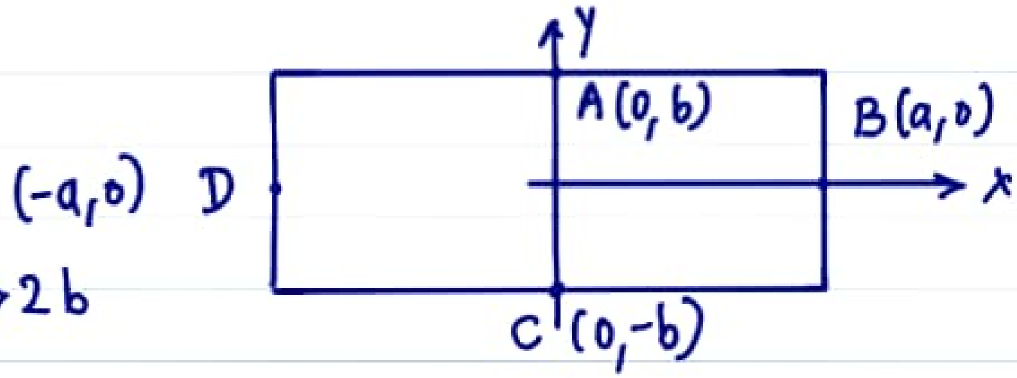
$$T = 2\int \varphi \, dx\, dy$$

$$T = \frac{4096\, G a^3 b^3}{\pi^6 (a^2+b^2)} \alpha$$

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2}$$

Use orthogonality of cosines to simplify integrals

PMPE $\dfrac{\partial \Pi}{\partial K_{11}} = 0, \ \dfrac{\partial \Pi}{\partial K_{12}} = 0, \ \ldots\ldots, \text{etc.}$



$2a > 2b$

$$\tau = \frac{64\, G\alpha}{\pi^3 (a^2+b^2)} \sqrt{a^2 b^4 \cos^2\left(\frac{\pi y}{2b}\right) \sin^2\left(\frac{\pi x}{2a}\right) + b^2 a^4 \cos^2\left(\frac{\pi x}{2a}\right) \sin^2\left(\frac{\pi y}{2b}\right)}$$

From soap film analogy, expect τ to hit max at A, C.

$$\tau_A = \tau_C = \frac{64 a^2 b\, G\alpha}{\pi^3 (a^2+b^2)}$$ max shear stress

$$\tau_B = \tau_D = \frac{64 a b^2\, G\alpha}{\pi^3 (a^2+b^2)}$$

Exercise 1 check τ at corners of rectangle

Exercise 2 $\varphi(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K_{mn} \cos\left(\frac{m\pi x}{2a}\right) \cos\left(\frac{n\pi y}{2b}\right)$