MA 214: Introduction to numerical analysis
Lecture 24

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The Newton-Cotes formulae are generally unsuitable for use over large intervals.

High-degree formulae would be required, and the values of the coefficients in these formulas are difficult to obtain.

Also, the Newton-Cotes formulae are based on interpolatory polynomials that use equally-spaced nodes, a procedure that is inaccurate over large intervals because of the oscillatory nature of high-degree polynomials.

We now discuss a piecewise approach to numerical integration that uses the low-order Newton-Cotes formulae.

An example

We have $\int_0^4 e^x dx = e^4 - e^0 = 53.5981500331.$

Using Simpson's $\frac{1}{3}$-rule on $[0, 4]$:

$$\int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.7695829526.$$

The error is then -3.17143291943 which is rather unacceptable.

Using Simpson's $\frac{1}{3}$-rule to $[0, 2]$ and $[2, 4]$:

$$\begin{aligned} \int_0^4 e^x dx &= \int_0^2 e^x dx + \int_2^4 e^x dx \\ &= \frac{1}{3}(e^0 + 4e + e^2) + \frac{1}{3}(e^2 + 4e^3 + e^4) \\ &= 53.8638457459 \end{aligned}$$

The example, continued

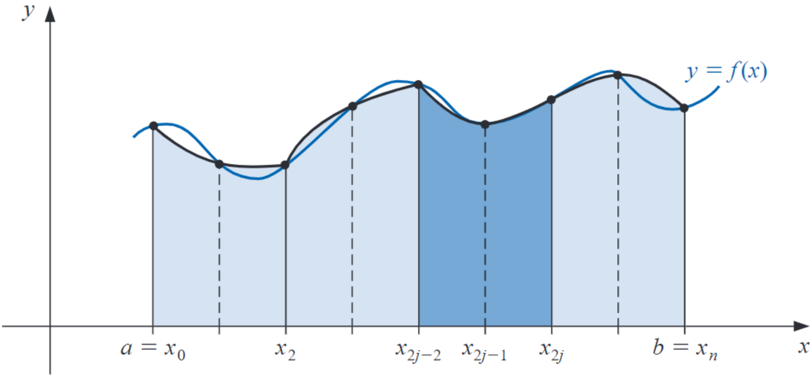
Now, we apply Simpson's $\frac{1}{3}$-rule to the subintervals of length 1:

$$\begin{aligned}\int_0^4 e^x dx &= \int_0^1 e^x dx + \int_1^2 e^x dx + \int_2^3 e^x dx + \int_3^4 e^x dx \\ &= \frac{1}{6}(e^0 + 4e^{1/2} + e) + \frac{1}{6}(e + 4e^{3/2} + e^2) \\ &\quad + \frac{1}{6}(e^2 + 4e^{5/2} + e^3) + \frac{1}{6}(e^3 + 4e^{7/2} + e^4) \\ &= 53.616220796\end{aligned}$$

The errors naturally decrease when we apply the Simpson's $\frac{1}{3}$-rule to smaller and smaller subintervals.

To generalize this procedure, we choose an even integer n, subdivide the interval $[a, b]$ into n subintervals, and apply Simpson's rule on each consecutive pair of subintervals.

General case



General case

In the general case, we have $h = (b-a)/n$ and $x_j = a + jh$ for each $j = 0, \ldots, n$. The formula then reads

$$\begin{aligned}\int_a^b f(x)dx &= \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx \\ &= \sum_{j=1}^{n/2} \left\{ \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\}\end{aligned}$$

for some ξ_j between x_{2j-2} and x_{2j}.

We simplify the above formula by observing the coefficients of terms which appear in more than one brackets and using some theorems to get a single μ in the error.

Let f be 4-times continuously differentiable on $[a,b]$, let n be an even integer, $h = (b-a)/n$ and $x_j = a + jh$ for $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j} + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

Notice that the above error term is of the order of h^4, whereas it was of the order of h^5 for the standard Simpson's rule.

However, for standard Simpson's rule we have h fixed at $h = (b-a)/2$ and for the composite rule we have a flexibility using n which permits us to reduce the value of h considerably.

Composite Trapezoidal rule

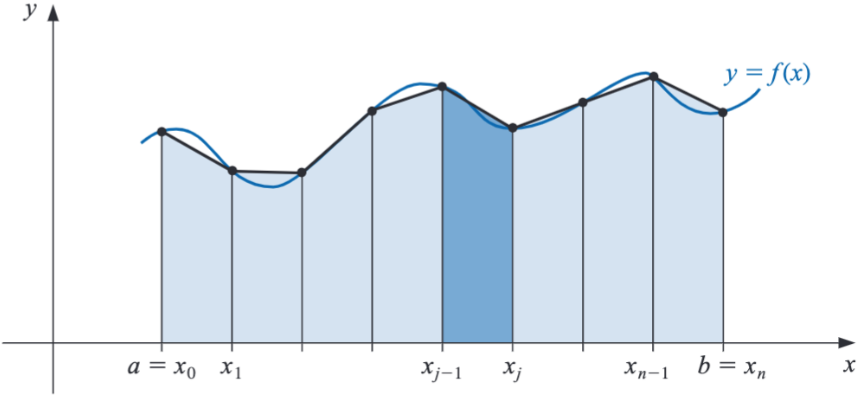
Let f be twice differentiable on $[a, b]$.

Let $h = (b - a)/n$ and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$.

There exists a $\mu \in (a, b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Composite Trapezoidal rule



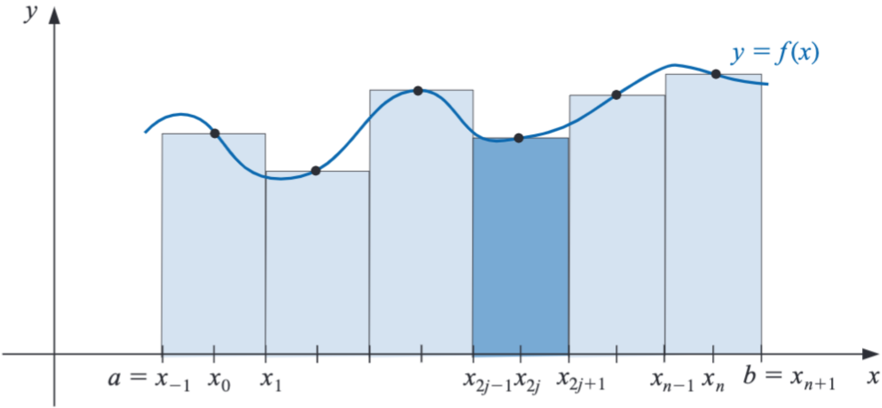
Let f be twice differentiable on $[a, b]$.

Let n be even, define $h = (b-a)/(n+2)$ and $x_j = a + (j+1)h$ for each $j = -1, 0, 1, \ldots, n+1$.

There exists a $\mu \in (a, b)$ for which the Composite Midpoint rule for $n+2$ subintervals can be written with its error term as

$$\int_a^b f(x)dx = 2h \sum_{j=0}^{n/2} f(x_{2j} + \frac{b-a}{6} h^2 f''(\mu).$$

Composite midpoint rule



Let us compute the integral $\int_0^{\pi} \sin x dx$ using the above two methods such that the error is less than 0.00002.

The error in the composite trapezoidal rule is given by

$$\left|\frac{\pi h^2}{12} f''(\mu)\right| = \frac{\pi h^2}{12} |\sin \mu|.$$

The condition on the error gives

$$\frac{\pi h^2}{12} |\sin \mu| \leqslant \frac{\pi h^2}{12} < 0.00002 \quad \text{so} \quad h < 0.00874038744.$$

Since $h = \pi/n$, this gives

$$n \geqslant 360.$$

The example, continued

We now apply the composite Simpson's rule. The error in this rule is given by

$$\left|\frac{\pi h^4}{180} f^{(4)}(\mu)\right| = \frac{\pi h^4}{180} |\sin \mu|$$

To ensure required accuracy we need

$$\frac{\pi h^4}{180} |\sin \mu| \leqslant \frac{\pi h^4}{180} < 0.00002 \quad \text{so} \quad h < 0.18398743167.$$

Since $h = \pi/n$, this gives

$$n \geqslant 18.$$

The composite Simpson's rule gives the required accuracy for a smaller number of subintervals.

Comparison of the composite rules

The composite Simpson's rule uses the same number of computations as the composite trapezoidal rule, yet the result is better.

The composite Simpson's rule gives the approximation:

$$\frac{\pi}{54}\left[2\sum_{j=1}^{8}\sin\left(\frac{j\pi}{9}\right)+4\sum_{j=1}^{9}\sin\left(\frac{(2j-1)\pi}{18}\right)\right]=2.0000104.$$

The composite trapezoidal rule with the same number of steps $n = 18$ gives

$$\frac{\pi}{36}\left[2\sum_{j=1}^{17}\sin\left(\frac{j\pi}{18}\right)+\sin 0+\sin\pi\right]=1.9949205.$$

The accuracy in the latter computation is about 5×10^{-3}.

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Round-off error

We saw an example in our last lecture where we computed the interval width, h, based on the required accuracy.

For the required accuracy, the composite Trapezoidal rule needed 360 intervals whereas the composite Simpson's rule needed only 18 intervals.

In addition to the fact that less computation is needed for the Simpson's technique, you might suspect that because of fewer computations this method would also involve less round-off error.

However, an important property shared by all the composite integration techniques is the stability with respect to the round-off error.

That is, the round-off error does not depend on the number of calculations performed.

Stability of composite integration rules

Let us assume that we apply the Composite Simpson's rule with n subintervals to a function f on $[a, b]$ and determine the maximum bound for the round-off error.

Assume that $f(x_i)$ is approximated by $\tilde{f}(x_i)$ and that

$$f(x_i) = \tilde{f}(x_i) + e_i \quad \text{for each } \; i = 0, 1, \ldots, n.$$

Then the accumulated error, $e(h)$, in the Composite Simpson's rule is

$$e(h) = \left| \frac{h}{3} \left[e_0 + 2 \sum_{j=1}^{(n/2)-1} e_{2j} + 4 \sum_{j=1}^{n/2} e_{2j-1} + e_n \right] \right|$$

$$\leqslant \frac{h}{3} \left[|e_0| + 2 \sum_{j=1}^{(n/2)-1} |e_{2j}| + 4 \sum_{j=1}^{n/2} |e_{2j-1}| + |e_n| \right].$$

Stability of composite integration rules

If the round-off errors are uniformly bounded by ϵ, then

$$e(h) \leqslant \frac{h}{3}\left[\epsilon + 2\left(\frac{n}{2} - 1\right)\epsilon + 4\left(\frac{n}{2}\right)\epsilon + \epsilon\right] = \frac{h}{3} 3n\epsilon = hn\epsilon$$

Hence

$$e(h) \leqslant (b-a)\epsilon$$

a bound independent of h (and n).

This means that, even though we may need to divide an interval into more parts to ensure accuracy, the increased computation does not increase the round-off error.

This result implies that the procedure is stable as $h \to 0$.

Recall that this was not true of the numerical differentiation procedures.

A problem with composite methods

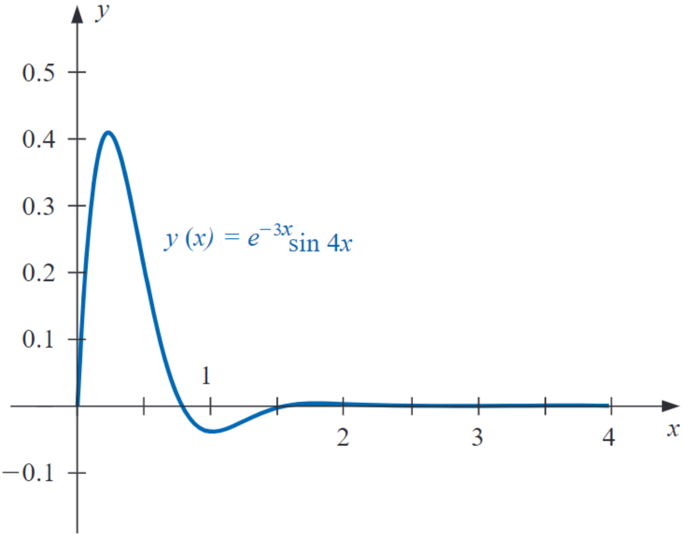
The composite formulae are very effective in most situations, but they suffer occasionally because they require the use of equally-spaced nodes.

This is inappropriate when integrating a function on an interval that contains regions with large functional variation and regions with small functional variation.

Such functions are commonplace, in mechanical engineering some of them describe certain features of spring and shock absorber systems, and in electrical engineering they are common solutions to elementary circuit problems, to mention a few examples.

One such function is $y(x) = e^{-3x} \sin 4x$ whose graph is shown in the next slide.

A problem with composite methods



A problem with composite methods

Let us assume that we need to compute $\int_0^4 y(x)dx$.

As per the graph, the integral on $[3, 4]$ seems to be very close to 0, and that on $[2, 3]$ is also not large.

However, on $[0, 2]$ there is significant variation of the function.

Beyond guessing that it would be positive, it is not at all clear what the integral is on this interval.

This is an example of a situation where composite integration would be inappropriate.

A very low order method could be used on $[2, 4]$, but a higher-order method would be necessary on $[0, 2]$.

Adaptive quadrature method

Question

How can we determine what technique should be applied on various portions of the interval of integration, and how accurate can we expect the final approximation to be?

If the approximation error for an integral on a given interval is to be evenly distributed, a smaller step size is needed for the large-variation regions than for those with less variation.

An efficient technique for this type of problem should predict the amount of functional variation and adapt the step size as necessary.

These methods are called adaptive quadrature methods.

Adaptive quadrature method

We now study one such method.

The method is based on the Composite Simpson's rule, but the technique is easily modified to use other composite procedures.

Suppose that we want to approximate $\int_a^b f(x)dx$ to within a specified tolerance $\epsilon > 0$.

The first step is to apply Simpson's rule with step size $h = (b-a)/2$. This gives, for some $\xi \in (a, b)$:

$$\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi)$$

where

$$S(a,b) = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right].$$

Adaptive quadrature method

Next, we apply the Composite Simpson's rule with $n = 4$ and step size $\dfrac{(b-a)}{4} = \dfrac{h}{2}$, giving

$$\int_a^b f(x)dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\tilde{\xi})$$

for some $\tilde{\xi} \in (a, b)$.

We now assume that $f^{(4)}(\xi) \approx f^{(4)}(\tilde{\xi})$, and the success of the technique depends on the accuracy of this assumption.

If it is accurate, then we equate the above two integrals with this approximation.

Adaptive quadrature method

$$S(a,b) - \frac{h^5}{90} f^{(4)}(\xi) \approx S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\xi)$$

or

$$\frac{h^5}{90} f^{(4)}(\xi) \approx \frac{16}{15}\left[S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right].$$

We now put this estimate in the formula given by Simpson's rule for $n = 4$

$$\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\xi)$$

$$\approx \frac{1}{15}\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right|.$$

This implies that $S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right)$ approximates $\displaystyle\int_a^b f(x)dx$ about 15-times better than it agrees with $S(a,b)$. Thus, if

$$\left| S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| < 15\epsilon$$

we expect

$$\left| \int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| < \epsilon$$

and $S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right)$ is assumed to be a sufficiently accurate approximation to $\displaystyle\int_a^b f(x)dx$.

Adaptive quadrature method

When $\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \not< 15\epsilon$, we apply the Simpson's rule individually to each subinterval, $[a, (a+b)/2]$ and $[(a+b)/2, b]$.

Then we use the error estimation procedure to determine if the approximation to the integral on each subinterval is within a tolerance of $\epsilon/2$. If so, we sum the approximations to produce an approximation to $\int_a^b f(x)dx$ within the tolerance ϵ.

If the approximation on one of the subintervals fails to be within the tolerance $\epsilon/2$, then that subinterval is subdivided, and the procedure is applied to the two subintervals to determine if the approximation on each subinterval is accurate to within $\epsilon/4$.

This halving procedure is continued until each portion is within the required tolerance.

Adaptive quadrature method

Problems can be constructed for which this tolerance will never be met.

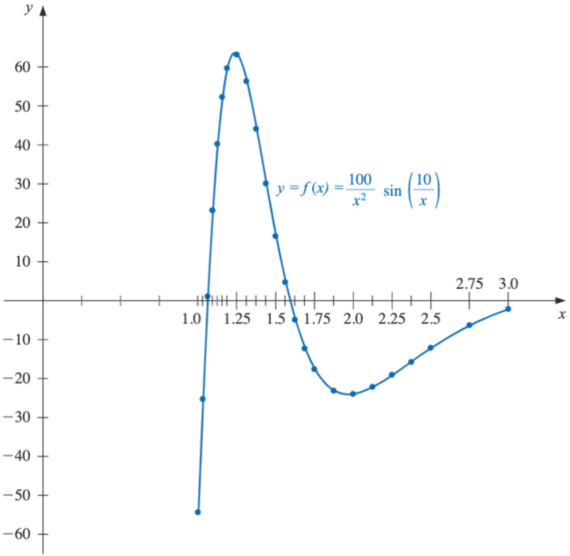
But the technique is usually successful because each subdivision typically increases the accuracy of the approximation by a factor of 16 while requiring an increased accuracy factor of only 2.

Let us consider the function $f(x) = (100/x^2)\sin(10/x)$, $x \in [1, 3]$.

Using the Adaptive quadrature method with tolerance 10^{-4} to approximate $\int_1^3 f(x)dx$ produces -1.426014, a result that is accurate to within 1.1×10^{-5}.

The approximation required that Simpson's rule with $n = 4$ be performed on the 23 subintervals whose endpoints are shown in the next slide.

Adaptive quadrature method



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Lecture 26

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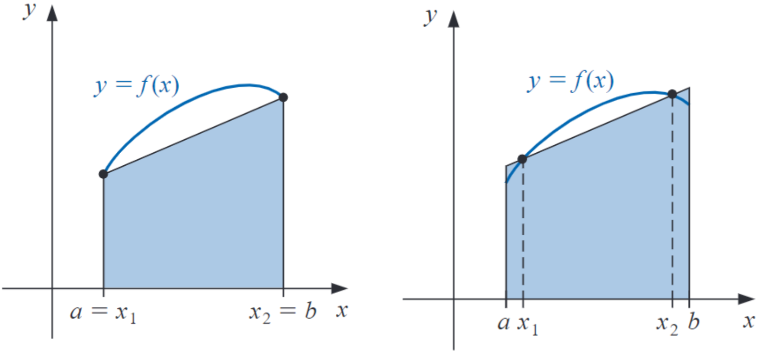
All the Newton-Cotes formulae use values of the function at equally-spaced points.

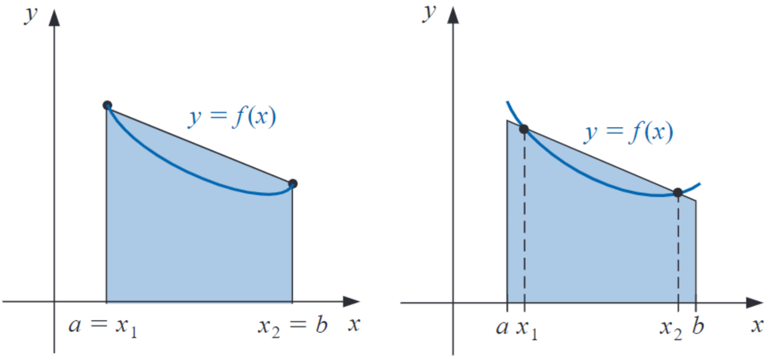
This restriction is convenient when the formulae are combined to form the composite rules, but it can significantly decrease the accuracy of the approximation.

The Trapezoidal rule approximates the integral of the function by integrating the linear function that joins the endpoints of the graph of the function. But this is often not the best approximation.

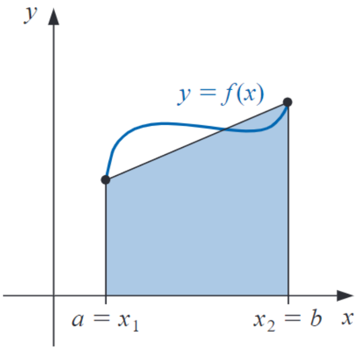
Consider, for example, the Trapezoidal rule applied to determine the integrals of the functions whose graphs are shown in the next slides. We also show some lines in each case which are likely to give much better approximations.

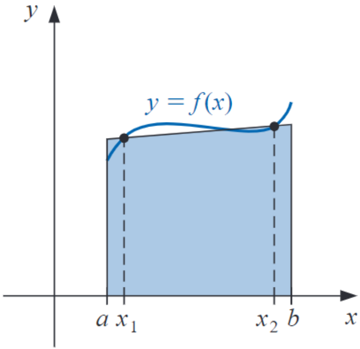
Gaußian quadrature





y
y = f(x)
a = x1
x2 = b
x
y
y = f(x)
a x1
x2 b
x


y
y = f(x)
a = x1
x2 = b
x


y
y = f(x)
a
x1
x2
b
x

Gaußian quadrature

Gaußian quadrature chooses the points for evaluation in an optimal, rather than equally-spaced, way.

The nodes $x_1, x_2, \ldots, x_n$ in the interval $[a, b]$ and coefficients $c_1, c_2, \ldots, c_n$ are chosen to minimize the expected error obtained in the approximation

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i).$$

To measure this accuracy, we assume that the best choice of these values produces the exact result for the largest class of polynomials, that is, the choice that gives the greatest degree of precision.

The coefficients $c_1, c_2, \ldots, c_n$ in the approximation formula are arbitrary, and the nodes $x_1, x_2, \ldots, x_n$ are restricted only by the fact that they must lie in $[a, b]$.

This gives us $2n$ parameters to choose.

If the coefficients of a polynomial are considered parameters, the class of polynomials of degree at most $2n - 1$ also contains $2n$ parameters.

This, then, is the largest class of polynomials for which it is reasonable to expect a formula to be exact.

With the proper choice of the values and constants, exactness on this set can be obtained.

We illustrate this method by means of an example where $n = 2$ and the interval is $[-1, 1]$.

We will then discuss the more general situation.

We want to determine c_1, c_2, x_1, and x_2 so that the integration formula

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

gives the exact result whenever $f(x)$ is a polynomial of degree $2(2) - 1 = 3$ or less, that is, when

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

for some a_0, a_1, a_2, a_3.

Clearly, it is enough to work with special cases of polynomials, namely, $f(x) = 1,\ x,\ x^2$ or x^3. The first equation is

$$c_1 + c_2 = \int_{-1}^{1} 1dx = 2.$$

Similarly, we have

$$c_1x_1 + c_2x_2 = \int_{-1}^{a} x dx = 0, \qquad c_1x_1^2 + c_2x_2^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

and $c_1x_1^3 + c_2x_2^3 = \displaystyle\int_{-1}^{1} x^3 dx = 0$. We can solve these equations to obtain $c_1 = 1 = c_2$, $x_1 = \dfrac{-\sqrt{3}}{3}$ and $x_2 = \dfrac{\sqrt{3}}{3}$.

The approximation formula then reads

$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

which is correct for polynomials of degree $\leqslant 3$.

There is another method to compute the nodes and coefficients.

There exist polynomials $\{P_n(x)\}$ for $n = 0, 1, \ldots$ satisfying:

(1) $P_n(x)$ is a monic polynomial of degree n,

(2) $\int_{-1}^{1} P(x)P_n(X) = 0$ whenever the degree of $P(x)$ is less than n.

These are called Legendre polynomials.

The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3},$$

$$P_3(x) = x^3 - \frac{3}{5}x, \quad \text{and} \quad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

The roots of these polynomials are distinct, lie in the interval $(-1,1)$, have a symmetry with respect to the origin, and, most importantly, are the correct choice for determining x_i and c_i.

Theorem

Let $x_1, x_2, \ldots, x_n$ be the roots of $P_n(x)$ and define for each $i = 1, 2, \ldots, n$,

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx.$$

If $P(x)$ is a polynomial of degree $< 2n$ then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i).$$

The nodes x_i and the coefficients c_i can be computed from the definition of Legendre polynomials.

But they have also been tabulated extensively. We can just use them off the shelf.

For instance, for P_3 the nodes and the coefficients are as follows:

x_i	c_i
0.7745966692	0.5555555556
0.0000000000	0.8888888889
-0.7745966692	0.5555555556

Let us use these values to compute $\int_{-1}^{1} e^x \cos x dx$.

An example

We have

$$\begin{aligned}\int_{-1}^{1} e^x \cos x dx \quad &\approx \quad 0.5555555556 \ e^{0.7745966692} \cos(0.7745966692) \\ &+ 0.8888888889 \\ &+ 0.5555555556 \ e^{-0.7745966692} \cos(-0.7745966692) \\ &= \quad 1.9333904.\end{aligned}$$

The actual answer is

$$\int_{-1}^{1} e^x \cos x dx = 1.9334214$$

so the absolute error is less than 3.2×10^{-5}.

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What next in numerical integration?

We have learnt some numerical methods to compute definite integrals.

We then learnt the amazing method of adaptive qudratures.

The icing on the cake was the Gaußian quadrature method.

We now want to venture outside the 1-dimensional realm.

Yup! That's right. We are going to do multiple integrals using our numerical techniques.

Multiple integrals

We start with

$$\iint\limits_R f(x,y)dA$$

where

$$R = \{(x,y) : a \leqslant x \leqslant b, c \leqslant y \leqslant d\},$$

is a rectangular region in the plane.

We have

$$\iint\limits_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy \right) dx.$$

We start with the composite trapezoidal rule.

Multiple trapezoidal rule

If we let $k = (d-c)/2$ and $h = (b-a)/2$ then the composite trapezoidal rule gives

$$\int_c^d f(x,y)dy \approx \frac{k}{2}\left[f(x,c) + f(x,d) + 2f\left(x, \frac{c+d}{2}\right)\right].$$

We further get

$$\int_a^b \int_c^d f(x,y)dydx \;\approx\; \int_a^b \left(\frac{d-c}{4}\right)\left[f(x,c) + f(x,d) + 2f\left(x, \frac{c+d}{2}\right)\right]$$

$$= \frac{(b-a)(d-c)}{16}\left[f(a,c) + f(a,d) + f(b,c) + f(b,d) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right.$$

$$\left. +2\left\{f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right)\right\}\right].$$

Multiple trapezoidal rule

The procedure is quite straightforward.

But the number of function evaluations grows with the square of the number required for a single integral.

In a practical situation, we would not expect to use a method as elementary as the composite trapezoidal rule. Instead, we will employ the composite Simpson's $\left(\frac{1}{3}\text{-rd}\right)$ rule to illustrate the general approximation technique, although any other composite formula could be used in its place.

To apply the composite Simpson's rule, we divide the region R by partitioning both $[a, b]$ and $[c, d]$ into an even number of subintervals.

To simplify the notation, we choose even integers n and m, and partition $[a, b]$ and $[c, d]$ with the evenly spaced mesh points $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_m$, respectively.

These subdivisions determine step sizes $h = (b-a)/n$ and $k = (d-c)/m$.

Writing the double integral as the iterated integral

$$\iint\limits_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

we first approximate $\displaystyle\int_c^d f(x,y)dy$, treating x as constant, using the composite Simpson's rule.

Multiple Simpson's rule

Let $y_j = c + jk$, for each $j = 0, 1, \ldots, m$. Then

$$\int_c^d f(x,y)dy = \frac{k}{3}\Bigg[f(x,y_0) + 2\sum_{j=1}^{(m/2)-1} f(x,y_{2j}) + 4\sum_{j=1}^{m/2} f(x,y_{2j-1}) + f(x,y_m)\Bigg] - \frac{(d-c)k^4}{180}\frac{\partial^4 f}{\partial y^4}(x,\mu)$$

for some $\mu \in (c,d)$.

The final double integral is a sum of $\displaystyle\int_a^b f(x,y_j)$.

With $x_i = a + ih$, for each $i = 0, 1, \ldots, n$ we apply the composite Simpson's rule to each $\displaystyle\int_a^b f(x,y_j)$ for $j = 0, 1, \ldots, m$.

The final answer then is

$$\iint_R f(x,y)dA \approx \frac{hk}{9}\Bigg\{\left[f(x_0,y_0)+2\sum_{i=1}^{(n/2)-1} f(x_{2i},y_0)+4\sum_{i=1}^{n/2} f(x_{2i-1},y_0)+f(x_n,y_0)\right]$$

$$+2\left[\sum_{j=1}^{(m/2)-1} f(x_0,y_{2j})+2\sum_{j=1}^{(m/2)-1}\sum_{i=1}^{(n/2)-1} f(x_{2i},y_{2j})+4\sum_{j=1}^{(m/2)-1}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j})\right.$$

$$\left.+\sum_{j=1}^{(m/2)-1} f(x_n,y_{2j})\right]+4\left[\sum_{j=1}^{m/2} f(x_0,y_{2j-1})+2\sum_{j=1}^{m/2}\sum_{i=1}^{(n/2)-1} f(x_{2i},y_{2j-1})\right.$$

$$\left.+4\sum_{j=1}^{m/2}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j-1})+\sum_{j=1}^{m/2} f(x_n,y_{2j-1})\right]$$

$$+\left[f(x_0,y_m)+2\sum_{i=1}^{(n/2)-1} f(x_{2i},y_m)+4\sum_{i=1}^{n/2} f(x_{2i-1},y_m)+f(x_n,y_m)\right]\Bigg\}$$

Let us do an example.

An example

Use composite Simpson's rule with $n = 4$ and $m = 2$ to approximate

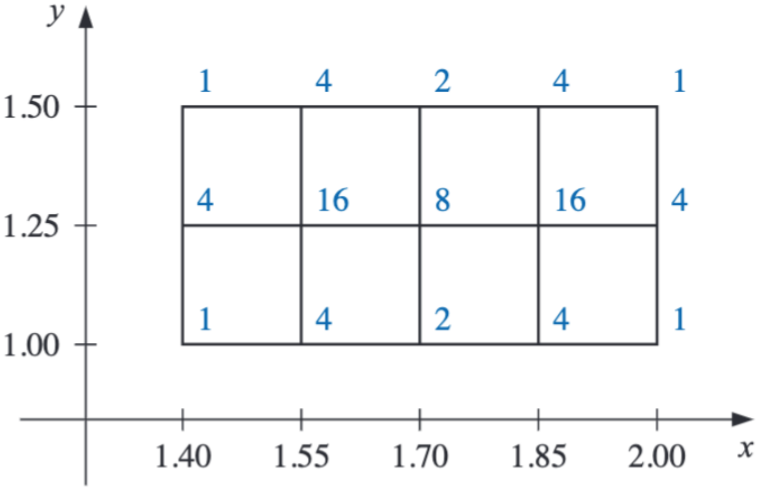
$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)dydx$$

The step sizes for this application are $h = (2.0 - 1.4)/4 = 0.15$ and $k = (1.5 - 1.0)/2 = 0.25$.

The region of integration R is shown in the next slide, together with the nodes (x_i, y_j), where $i = 0, 1, 2, 3, 4$ and $j = 0, 1, 2$.

It also shows the coefficients $w_{i,j}$ of $f(x_i, y_j) = \ln(x_i + 2y_j)$ in the sum that gives the approximation to the integral.

The example, continued



The example, continued

The final approximation then is

$$\frac{(0.15)(0.25)}{9}\sum_{i=0}^{4}\sum_{j=0}^{2} w_{i,j}\ln(x_i+2y_j) = 0.4295524387.$$

The actual integral is

$$\int_{1.4}^{2.0}\int_{1.0}^{1.5}\ln(x+2y)dydx = 0.4295545265$$

so the error is 2.1×10^{-6}.

Error formula

The error in the formula for $\displaystyle\iint_R f(x,y)\,dA$ is

$$E = -\frac{(d-c)(b-a)}{180}\left[h^4\,\frac{\partial^4 f}{\partial x^4}(\bar\eta,\bar\mu) \;+\; k^4\,\frac{\partial^4 f}{\partial y^4}(\hat\eta,\hat\mu)\right]$$

for some $(\bar\eta,\bar\mu),(\hat\eta,\hat\mu)\in R$.

The error predicted by this formula in the above example is

$$\begin{aligned}
|E| &\leqslant \frac{(0.5)(0.6)}{180}\left[(0.15)^4\max_R\frac{6}{(x+2y)^4} + (0.25)^4\max_R\frac{96}{(x+2y)^4}\right]\\
&\leqslant 4.72\times 10^{-6}
\end{aligned}$$