MA 214: Introduction to numerical analysis
Lecture 5

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Fixed points and roots

A fixed point of $f : [a, b] \to \mathbb{R}$ is $p \in [a, b]$ such that $f(p) = p$.

Note that $p \in [a, b]$ is a root of the equation $f(x) = 0$ if and only if p is a fixed point of $g(x) = f(x) - x$.

Fixed points of various functions are studied well in Mathematics. There are many nice results guaranteeing the existence of fixed points.

Fixed points

Fixed Point Theorem

- *If* $f : [a, b] \to [a, b]$ *is continuous then* f *has a fixed point.*
- *If, in addition,* $f'(x)$ *exists on* (a, b) *and* $|f'(x)| \leqslant k < 1$ *for all* $x \in (a, b)$ *then* f *has a unique fixed point in* $[a, b]$.

We will not rigorously prove this theorem.

The second part uses the Mean Value Theorem and the first part is easy to prove.

An example

Let us consider an example:

Consider the function $f(x) = (x^2 - 1)/3$ on $[-1, 1]$.

The extrema of f can be computed using calculus.

The maximum value of f is 0 attained at the points $x = 1$ and $x = -1$ while the minimum value of f is $-1/3$ attained at $x = 0$.

Thus, the image of f is contained in $[-1/3, 0] \subset [-1, 1]$.

Ergo, f has a fixed point in $[-1, 1]$.

Further, $|f'(x)| = \left|\frac{2x}{3}\right| \leqslant \frac{2}{3}$ on $[-1, 1]$.

Therefore, f has a unique fixed point in $[-1, 1]$.

The example, continued

Now, this fixed point of $f(x) = (x^2 - 1)/3$ can be computed by solving the quadratic equation

$$p^2 - 3p - 1 = 0$$

and one gets

$$p = \frac{1}{2}(3 \pm \sqrt{13}).$$

The number $p = (3 - \sqrt{13})/2$ lies in our interval $[-1, 1]$.

Note that the other root of the above quadratic $p = (3 + \sqrt{13})/2$ is in $[3, 4]$, however the function $f(x) = (x^2 - 1)/3$ does not satisfy the hypothesis of the above theorem.

One has $f(4) = 5$ and $f'(4) = \dfrac{8}{3} > 1$.

Fixed point iteration

We start with a continuous $f : [a, b] \to [a, b]$.

Take any initial approximation $p_0 \in [a, b]$ and generate a sequence $p_n = f(p_{n-1})$.

If the sequence $\{p_n\}$ converges to $p \in [a, b]$ then

$$f(p) = f(\lim_n p_n) = \lim_n f(p_n) = \lim_n p_{n+1} = p.$$

Thus, p is a fixed point for f.

This method is called the fixed point iteration method.

Another example

Consider the equation $x^3 + 4x^2 - 10 = 0$.

It has a unique root in $[1, 2]$ by the intermediate value theorem.

The equation can be rewritten in many ways and hence can be converted into fixed point problems in many ways:

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$,

(b) $x = g_2(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$,

(c) $x = g_3(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$.

We list the results of these fixed point iterations in the following table:

The example, continued

n	(a)	(b)	(c)
0	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768
2	6.732	2.9969	1.402540804
3	-469.7	$(-8.65)^{1/2}$	1.345458374
4	1.03×10^8		1.375170253
5			1.360094193
6			1.367846968
7			1.363887004
8			1.365916734
9			1.364878217
10			1.365410062
15			1.365223680
20			1.365230236
25			1.365230006
30			1.365230013

What went wrong?

The sequences given by iterations (a) and (b) are divergent.

The iteration given in (a) does not take the interval $[1, 2]$ to itself. Here, $g_1(1) = 6$ and $g_1(2) = -12$.

The same reason holds for $g_2(x)$ given in iteration (b) also.

In fact, the iteration (b) has taken the values outside the set of real numbers.

The iteration (c) takes the interval $[1, 2]$ to itself.

While the derivative $g'(x)$ fails to satisfy the condition in the fixed point theorem, a closer look tells us that it is enough to work on the interval $[1, 1.5]$ where the function g_3 is strictly decreasing.

A question

Question

How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

A possible answer to this is

Answer

Manipulate the root-finding problem into a fixed point problem that satisfies the conditions of fixed point theorem and has a derivative that is as small as possible near the fixed point.

Still better iterations

Check that the iterations

$$x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$$

and

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

converge more rapidly to the fixed point.

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Newton-Raphson method

This is a particular fixed point iteration method.

Assume that $f : [a, b] \to \mathbb{R}$ is twice differentiable.

Let $p \in [a, b]$ be a solution (to be found) of the equation $f(x) = 0$.

If p_0 is another point in $[a, b]$ then Taylor's theorem gives

$$0 = f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi)$$

for some ξ between p and p_0.

We assume that $|p - p_0|$ is very small, so that $(p - p_0)^2 \approx 0$.

Newton-Raphson method

This gives us

$$0 \approx f(p_0) + (p - p_0)f'(p_0) \quad \text{or} \quad p \approx p_0 - \frac{f(p_0)}{f'(p_0)}.$$

This sets the stage for Newton-Raphson method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

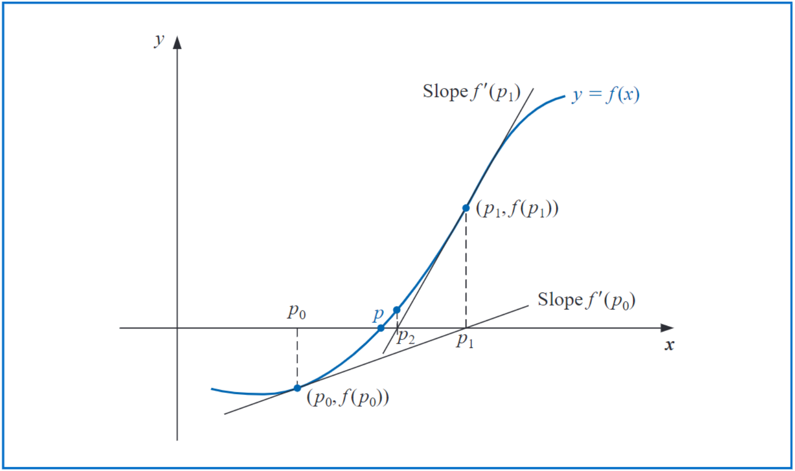
Note that the N-R iteration method stops if we get $f'(p_n) = 0$ for some n.

Geometric interpretation

The figure on the next slide illustrates how the approximations are obtained using successive tangents.

Starting with the initial approximation p_0, the approximation p_1 is the x-intercept of the tangent line to the graph of f at $(p_0, f(p_0))$.

The approximation p_2 is the x-intercept of the tangent line to the graph of f at $(p_1, f(p_1))$ and so on.


y
Slope $f'(p_1)$
$y = f(x)$
$(p_1, f(p_1))$
Slope $f'(p_0)$
p_0
p
p_2
p_1
x
$(p_0, f(p_0))$

An earlier example

We had, in our last lecture, considered the root-finding problem for

$$f(x) = x^3 + 4x^2 - 10$$

which we had converted into a fixed-point problem by

(a) $g_1(x) = x - x^3 - 4x^2 + 10,$ (b) $g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2},$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2},$ (d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2},$

(e) $x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

The last one is given by Newton and Raphson.

Table of approximations

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

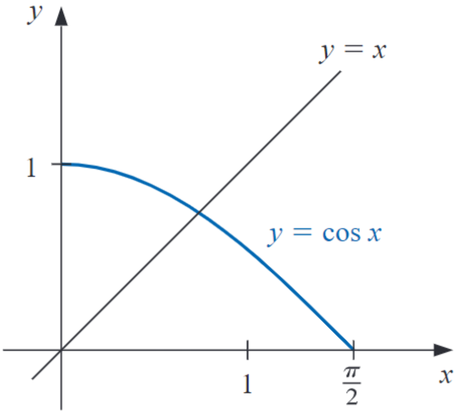
Consider the equation $f(x) = \cos x - x = 0$.

We convert the root-finding problem for this equation into a fixed point iteration problem in two ways:

1. $x = g_1(x) = \cos x$,
2. $x = g_2(x) = x - \dfrac{\cos x - x}{-\sin x - 1}$.

We need to decide on the initial approximation p_0.

Graphs of $y = \cos x$ and $y = x$



It seems reasonable to take the initial approximation $p_0 \approx \pi/4$.

Fixed point iterations for $\cos x - x$

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Fixed point iteration

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Newton-Raphson method

Newton-Raphson is better

The Newton-Raphson method is evidently better than the fixed point iteration method.

However, it is important to note that $|p - p_0|$ is needed to be small so that the term involving $(p - p_0)^2$ can be dropped from Taylor's polynomial.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be twice differentiable.

If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that for any $p_0 \in [p - \delta, p + \delta]$, the Newton-Raphson method generates a sequence $\{p_n\}$ converging to p.

Newton-Raphson is better

This theorem states that, under reasonable assumptions, Newton-Raphson method converges provided that a sufficiently accurate initial approximation is chosen.

This result is important for the theory of the method, but it is seldom applied in practice because it does not tell us how to determine the constant δ.

In a practical application, an initial approximation is selected and successive approximations are generated by the method.

These will generally either converge quickly to the root, or it will be clear that convergence is unlikely.

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Problems with Newton-Raphson method

If $f : [a, b] \to \mathbb{R}$ is a twice differentiable function then the Newton-Raphson fixed point iteration to finding a root of f is given by an initial approximation p_0 and

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

One major problem with this otherwise powerful method is that we need to compute the value of f' at each step.

Typically, f' is far more difficult to compute and needs more arithmetic operations to calculate than f.

We introduce a slight variation to circumvent this problem.

By definition,

$$f'(a) = \lim_{x \to 0} \frac{f(a) - f(x)}{a - x}.$$

If we assume that p_{n-2} is reasonably close to p_{n-1} then

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation to $f'(p_{n-1})$ the Newton-Raphson method gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

This method is called the Secant method

Geometric interpretation

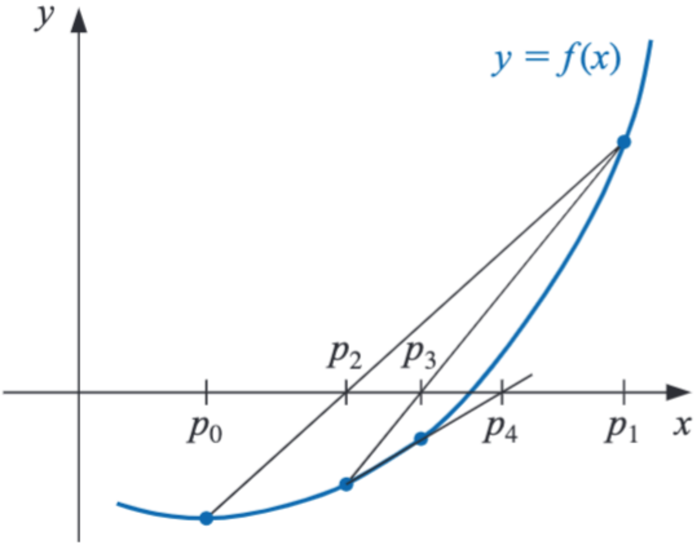
Starting with the two initial approximations p_0 and p_1, the approximation p_2 is the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

The approximation p_3 is the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$, and so on.

Note that only one function evaluation is needed per step for the Secant method after p_2 has been determined.

In contrast, each step of Newton's method requires an evaluation of both the function and its derivative.

Geometric interpretation



An earlier example

We use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those found using the Newton-Raphson method.

Since we need two initial approximations for the Secant method, let us take $p_0 = 0.5$ and $p_1 = \pi/4$.

The further approximations are obtained by the formula

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}.$$

We list the results in the following table.

The example, continued

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

Newton-Raphson method

Secant method

The method of false position

The Newton-Raphson or the Secant method may give successive approximations which are on one side of the root.

That is $f(p_{n-1}) \cdot f(p_n)$ need not be negative.

Then the root may not lie between p_{n-1} and p_n.

We can modify this by taking the pair of approximations which are on both sides of the root.

This gives the regula falsi method or the method of false position.

The method of false position

We choose initial approximations p_0 and p_1 with $f(p_0) \cdot f(p_1) < 0$.

The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

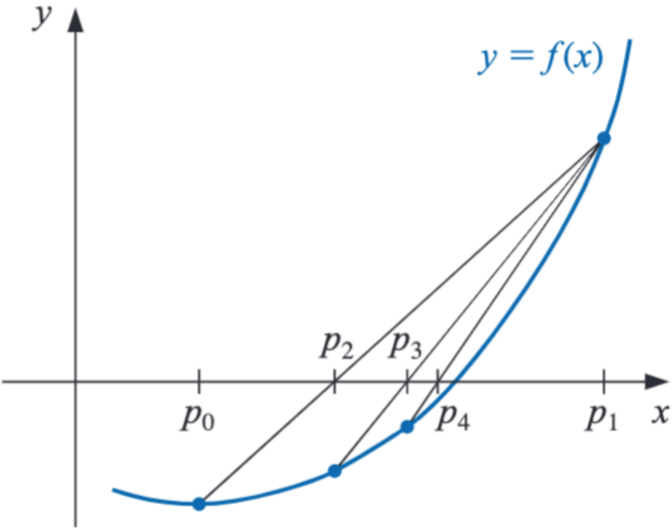
We then compute $f(p_2) \cdot f(p_1)$.

If $f(p_2) \cdot f(p_1) < 0$, then apply the Secant method to the pair (p_1, p_2).

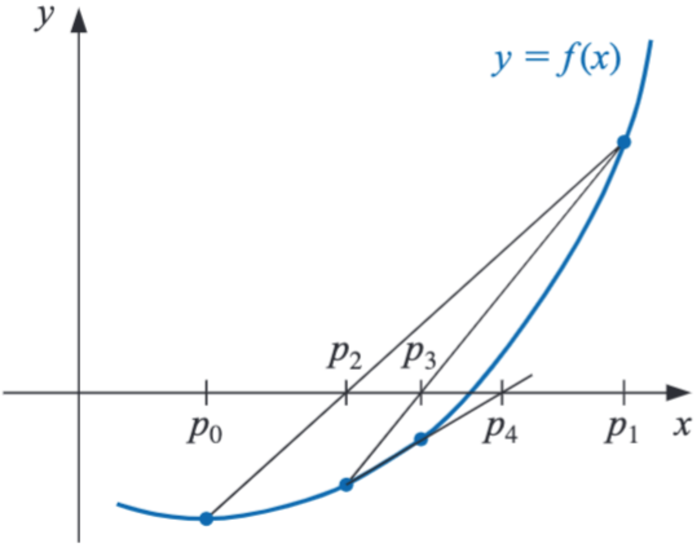
If not, then $f(p_2) \cdot f(p_0)$ must be negative and hence we apply the Secant method to the pair (p_0, p_2).

This relabelling ensures that the root is always bracketed by the successive approximations.

Regula falsi method



Secant method



The earlier example

We use the earlier example, $f(x) = \cos x - x$ to illustrate the regula falsi method.

For that, we use the same initial approximations that were used in the Secant method, $p_0 = 0.5$ and $p_1 = \pi/4$.

The added requirement of the regula falsi method results in more calculations than the Secant method, just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations.

Regula $\leftrightarrow$ Secant $\leftrightarrow$ Newton-Raphson

n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

Regula falsi　　Secant　　Newton-Raphson

Comparison of all root finding methods

The bisection method guarantees a sequence converging to the root but it is a slow method.

The other methods are sure to work, once the sequence is convergent.

This typically depends on the initial approximations being very close to the root.

It is therefore, a normal practise to use the bisection method first, to get within a reasonable neighbourhood of the root, and then use the Newton-Raphson or the Secant method.

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Order of convergence

We want to compare the convergence rates of various fixed point iterations.

Let $\{p_n\}$ be a sequence that converges to p with $p_n \neq p$ for any n. If there are positive constants λ and α such that

$$\lim_n \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

then the order of convergence of $\{p_n\}$ to p is α with asymptotic error λ.

An iterative technique of the form $p_n = g(p_{n-1})$ is said to be of order α if the sequence $\{p_n\}$ converges to the solution $p = g(p)$ with order α.

Order of convergence

In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.

The asymptotic constant affects the speed of convergence but not to the extent of the order.

Two cases of order are given special attention:

1. If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.
2. If $\alpha = 2$, the sequence is quadratically convergent.

We will now see examples to compare these two orders of convergence.

Suppose that $p_n \to 0$ with linear order of convergence with asymptotic error 0.5 and $q_n \to 0$ with quadratic order of convergence and the same asymptotic error 0.5:

$$\lim_n \frac{|p_{n+1} - 0|}{|p_n - 0|^1} = 0.5, \qquad \lim_n \frac{|q_{n+1} - 0|}{|q_n - 0|^2} = 0.5$$

This gives

$$|p_n| \approx (0.5)|p_{n-1}| \approx (0.5)^2|p_{n-2}| \approx \cdots \approx (0.5)^n|p_o|$$

and

$$|q_n| \approx (0.5)|q_{n-1}|^2 \approx (0.5)\left((0.5)|q_{n-2}|^2\right)^2 = (0.5)^3|q_{n-2}|^4$$

$$\approx (0.5)^7|q_{n-3}|^8 \approx \cdots \approx (0.5)^{2^n-1}|q_o|^{2^n}.$$

We have $|p_n| \approx (0.5)^n |p_o|$ and $|q_n| \approx (0.5)^{2^n-1}|q_o|^{2^n}$. With $p_0 = q_0 = 1$, here are the relative speeds of convergence:

n	$(0.5)^n$	$(0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

The q_n is within 10^{-38} of 0 by the seventh term. At least 126 terms of p_n are needed to ensure this accuracy.

Fixed point iteration method

Let $f : [a,b] \to [a,b]$ be differentiable. Assume further that $|f'(x)| \leqslant k < 1$ on (a,b) and that f' is continuous on (a,b).

Consider the fixed point iteration: $p_{n+1} = f(p_n)$. The above assumptions ensure that $p_n \to p$, a fixed point of f.

The Mean Value Theorem gives

$$p_{n+1} - p = f(p_n) - f(p) = f'(\xi_n)(p_n - p)$$

where ξ_n lies between p_n and p, hence $\lim_n \xi_n = p$. Hence

$$\lim_n \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_n |f'(\xi_n)| = |f'(p)|.$$

The convergence of a fixed point iteration method is thus linear if $f'(p) \neq 0$.

Higher order of convergence

It is thus clear that for obtaining a higher order of convergence, we need to have $f'(p) = 0$.

Theorem

Let p be a solution of the equation $x = f(x)$.

Let $f'(p) = 0$ and f'' be continuous with $|f''(x)| < M$ nearby p.

Then there exists a $\delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$, the sequence defined $p_n = f(p_n - 1)$ converges at least quadratically to p.

Moreover, for sufficiently large values of n

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

For quadratically convergent fixed-point methods, we should search for functions whose derivatives are zero at the fixed point.

We should have $f(p) = p$ and $f'(p) = 0$.

If we have the root-finding problem for $g(x) = 0$ then the easiest way to construct a fixed-point problem would be

$$f(x) = x - \phi(x)g(x)$$

where ϕ is a differentiable function, to be chosen later. Now

$$f'(x) = 1 - \phi'(x)g(x) - \phi(x)g'(x)$$

$$0 = f'(p) = 1 - \phi(p)g'(p) \quad \text{or} \quad \phi(p) = \frac{1}{g'(p)}.$$

Newton-Raphson method

It is therefore instructive to define $\phi(x) = \dfrac{1}{g'(x)}$ which gives us

$$p_{n+1} = f(p_n) = p_n - \frac{g(p_n)}{g'(p_n)}.$$

This is, of course, our Newton-Raphson method to find a root of the equation $g(x) = 0$.

We have assumed that $g'(p) \neq 0$ in the above analysis.

The Newton-Raphson and the Secant method will generally not work if $g'(p) = 0$.

Multiplicity of a zero

Let $g : [a, b] \to \mathbb{R}$ be a function and let $p \in [a, b]$ be a zero of g.

We say that p is a zero of multiplicity m of g if for $x \neq p$, we can write $g(x) = (x - p)^m q(x)$ with $\lim_{x \to p} q(x) \neq 0$.

Whenever g has a simple zero at p, that is, if $m = 1$ in the above analysis, then the Newton-Raphson method works well for g.

We will see an example to see that the Newton-Raphson method does not give a quadratic convergence if the order of the zero is more than 1.

An example

Let $g(x) = e^x - x - 1$.

We then have $g'(x) = e^x - 1$ and $g''(x) = e^x$. Therefore, $g(0) = g'(0) = 0$ but $g''(0) = 1 \neq 0$.

Thus, $x = 0$ is a zero of g with multiplicity 2.

We take $p_0 = 1$ and apply the Newton-Raphson method. Then

$$p_1 = p_0 - \frac{g(p_0)}{g'(p_0)} = 1 - \frac{e-2}{e-1} \approx 0.58198$$

and

$$p_2 = p_1 - \frac{g(p_1)}{g'(p_1)} \approx 0.31906.$$

The example, continued

n	p_n
0	1.0
1	0.58198
2	0.31906
3	0.16800
4	0.08635
5	0.04380
6	0.02206
7	0.01107
8	0.005545

n	p_n
9	2.7750×10^{-3}
10	1.3881×10^{-3}
11	6.9411×10^{-4}
12	3.4703×10^{-4}
13	1.7416×10^{-4}
14	8.8041×10^{-5}
15	4.2610×10^{-5}
16	1.9142×10^{-6}

This sequence is definitely converging to 0 but not quadratically.