# Dynamic response of first-order instruments

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## Frequency response of first-order instruments

- Governing equation: $\tau \frac{dq_o}{dt} + q_o = Kq_i$
- Given: $q_i = A\cos(\omega t)$
- Given (for simplicity *only*): $q_o = 0$ at $t = 0$
- Easy to find $q_{oc} = Be^{-t/\tau}$
- Also find $q_{op} = C_1\cos(\omega t) + C_2\sin(\omega t)$
- No duplication. Putting in the governing equation to find $C_1$ and $C_2$

$$\tau[-C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t)] + [C_1\cos(\omega t) + C_2\sin(\omega t)] = KA\cos(\omega t)$$

## Frequency response

- Solving to get $C_1$ and $C_2$ as

$$C_1 = \frac{KA}{1+\tau^2\omega^2} \qquad C_2 = \frac{\tau\omega KA}{1+\tau^2\omega^2}$$

- Therefore, the complete solution is:

$$q_o = Be^{-t/\tau} + \frac{KA}{1+\tau^2\omega^2}[\cos(\omega t)+(\omega\tau)\sin(\omega t)]$$

- Use the given initial condition to find $B = -\frac{KA}{1+\tau^2\omega^2}$
- Can we solve this problem using Laplace transform?



## Step response of first-order instruments

- Governing equation: $\tau\frac{dq_o}{dt} + q_o = Kq_i$
- Given $q_i$ is a step input of magnitude $q_{is}$
- Given (for simplicity *only*): $q_o = 0$ at $t = 0$
- Taking Laplace transform: $L\left[\tau\frac{dq_o}{dt} + q_o\right] = L[Kq_i]$
- Get: $\tau[sQ - q_o(0)] + Q = \frac{Kq_{is}}{s}$ Recall: $L[H(t-a)] = \frac{e^{-as}}{s}$
- Or, $Q = \frac{Kq_{is}}{s(\tau s+1)} + \frac{\tau q_0(0)}{(\tau s+1)}$

$$q_0 = L^{-1}\left[\frac{Kq_{is}}{s(\tau s+1)}\right] + L^{-1}\left[\frac{\tau q_0(0)}{(\tau s+1)}\right]$$

# Step response

- Final solution:

$$q_o = Kq_{is}(1 - e^{-t/\tau}) + q_0(0)e^{-t/\tau}$$

- Speed of response depends on the value of τ
- Faster response for a smaller value of τ
- For $q_o(0) = 0$, get

$$\frac{q_o}{Kq_{is}} = (1 - e^{-t/\tau})$$