ME 202

LECTURE 33

TUE 05 APR 2022

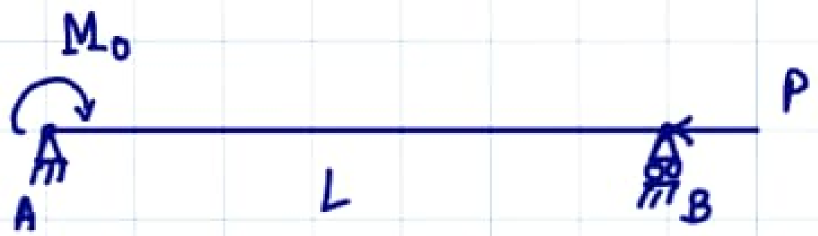
Approaches to buckling,

- [x] Non-trivial deformations of a perfect (perfectly symmetrical) system
  Eigenvalue problem. Equilibrium.

- [x] Uncontrollably large (theoretically ∞) deformations of an imperfect (asymmetric) system. Equilibrium.
- [ ] Stability switching in potential energy

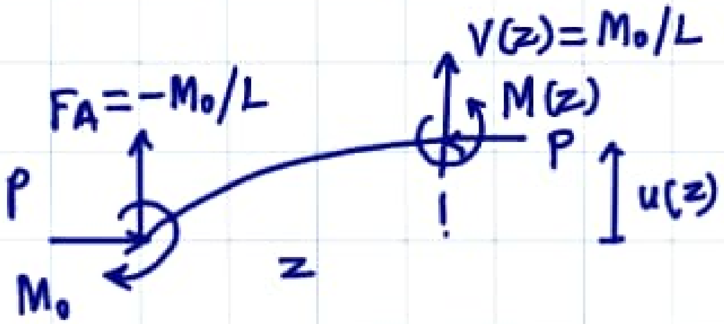
Energy approach useful in obtaining approximate solutions to buckling problems.

Imperfect Column



$$F_A + F_B = 0, \quad M_0 = F_B L, \quad F_B = \frac{M_0}{L}$$

$$F_A = -\frac{M_0}{L}$$



$$\Sigma M_A = 0$$

$$M(z) + \frac{M_0}{L} z + P u(z) - M_0 = 0$$

$$M(z) = EIu''(z)$$

$$EIu''(z) + Pu(z) = M_0\left(1 - \frac{z}{L}\right)$$

$$u'' + \underbrace{\frac{P}{EI}}_{\lambda^2} u = \frac{M_0}{EI}\left(1 - \frac{z}{L}\right)$$

$$u = A\cos\lambda z + B\sin\lambda z + \frac{M_0}{P}\left(1 - \frac{z}{L}\right)$$

BC $\quad u(0) = 0, \quad \frac{M_0}{P} + A = 0, \; A = -\frac{M_0}{P}$

$$u(L) = 0, \; -\frac{M_0}{P}\cos\lambda L + B\sin\lambda L = 0$$

$$B = \frac{M_0}{P}\frac{\cos\lambda L}{\sin\lambda L}$$

$$u(z) = \frac{M_0}{P}\left[1 - \frac{z}{L} - \cos\lambda z + \frac{\cos\lambda L}{\sin\lambda L}\sin\lambda z\right]$$

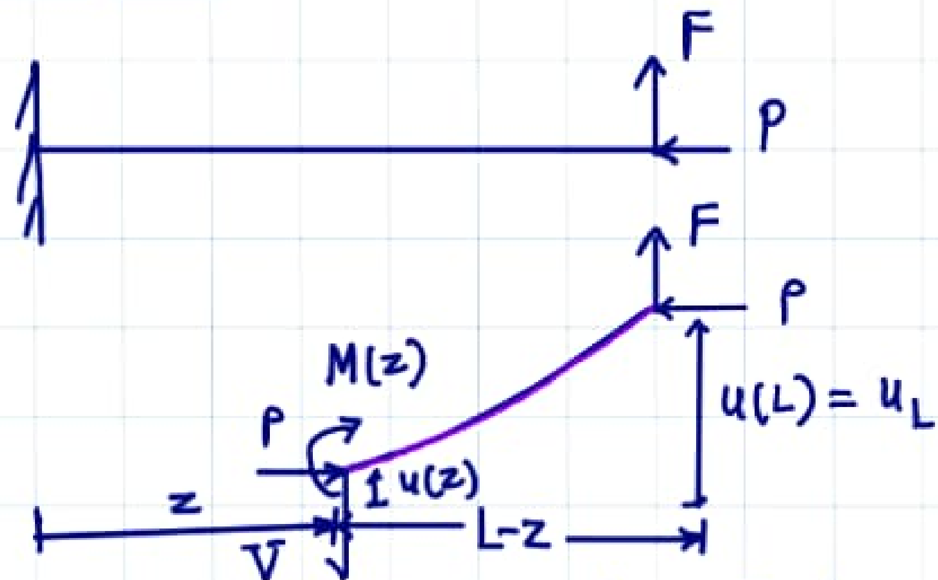
As $\lambda L \to \pi$, $u \to \infty$

As

$$\sqrt{\frac{P}{EI}}\, L \to \pi, \quad P \to \frac{\pi^2 EI}{L^2}$$

$$\lambda L \to n\pi, \quad P \to \frac{n^2\pi^2 EI}{L^2}$$

Example 2



$$M(z) = F(L-z) + P(u_L - u(z)) = EI\,u''$$

$$u'' + \frac{P}{EI} u = Pu_L + FL - Fz$$

$$u = A\cos\lambda z + B\sin\lambda z + u_L + \frac{FL}{P} - \frac{Fz}{P}$$

$$\lambda^2 = \frac{P}{EI}$$

Cantilever BCs $u(0) = 0$, $u'(0) = 0$

$u(L) = u_L$

$$u_L + \frac{FL}{P} + A = 0$$

$$\frac{-F}{P} + B\lambda = 0$$

$$u_L + A\cos\lambda L + B\sin\lambda L = u_L$$

$$A = -\frac{F}{P\lambda}\tan\lambda L, \quad B = \frac{F}{P\lambda}$$

$$u_L = \frac{F}{P\lambda}\tan\lambda L - \frac{FL}{P}$$

DNYANESH PAWASKAR

As $P \to 0$, $u_L \to \dfrac{FL^3}{3EI}$?

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \ldots \qquad |\theta| < \pi/2$$

$$u_L = \frac{F}{P\lambda}\underbrace{\left(\cancel{\lambda L} + \frac{(\lambda L)^3}{3}\right)}_{} - \frac{F\cancel{L}}{P}$$

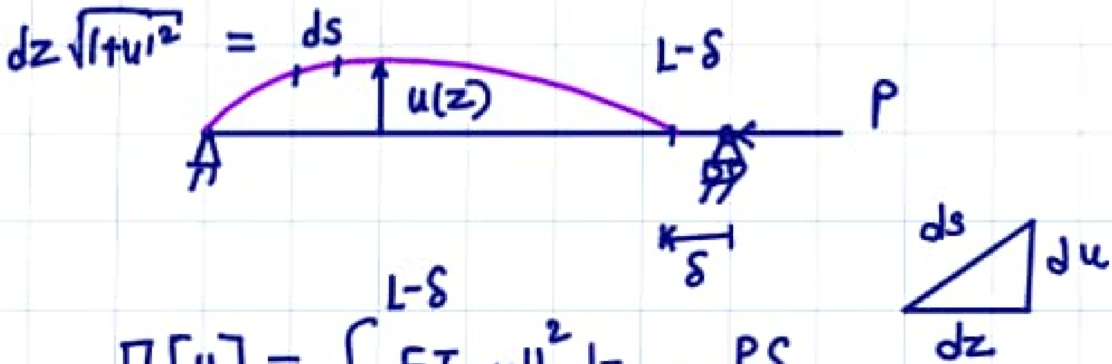
$$= \frac{FL^3}{3EI} \qquad \begin{matrix} \lambda L \to 0 \\ P \to 0 \end{matrix}$$

$$u(z) = \frac{F}{P}\tan\lambda L\,(1 - \cos\lambda z) - \frac{F}{P\lambda}(\lambda z - \sin\lambda z)$$

As $\lambda L \to \dfrac{\pi}{2}$, as $L\sqrt{\dfrac{P}{EI}} \to \dfrac{\pi}{2}$

$$P \to \underbrace{\frac{\pi^2 EI}{4L^2}}_{P^*,\ P_{cr}} \quad \text{for cantilever}$$

Potential Energy Method



$$\Pi[u] = \int_0^{L-\delta} \frac{EI}{2} u''^2 \, dz - P\delta$$

Assume incompressibility N.A.

$$L = \int_0^{L-\delta} \underbrace{\sqrt{1+u'^2}}_{\approx 1+\frac{1}{2}u'^2} \, dz, \quad |u'| \ll 1$$

$$L = \int_0^{L-\delta} \left(1+\frac{1}{2}u'^2\right) dz = L-\delta + \int_0^{L-\delta} \frac{1}{2}u'^2 \, dz$$

$$\delta = \int_0^{L-\cancel{\delta}} \frac{1}{2}u'^2 \, dz, \quad \delta \ll L$$

$$\delta = \int_0^L \frac{1}{2} u'^2 \, dz$$

$$\Pi = \int_0^L \left(\frac{EI}{2} u''^2 - \frac{P}{2} u'^2 \right) dz$$

Approx solutions.

Pinned-pinned $\quad u(0)=0, \; u(L)=0$

$u = Az(L-z) \quad$ approx.

$u' = AL - 2Az$

$u'' = -2A$

$$\Pi = 2EIA^2L - \frac{P}{6}A^2L^3$$

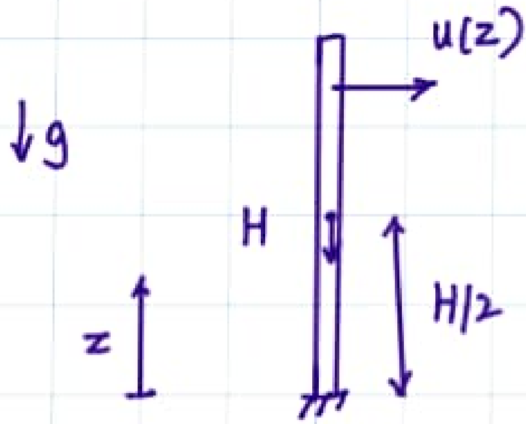
Equilibrium $\dfrac{\partial \Pi}{\partial A} = 0$, $\quad \dfrac{2AL^3}{6}\left(\dfrac{12EI}{L^2} - P \right) = 0$

for nontrivial A, $\quad P = \dfrac{12EI}{L^2}$ approx. buckling load

DNYANESH PAWASKAR

$$P_{exact} = \frac{\pi^2 EI}{L^2}, \quad \pi^2 \approx 9.8696....$$

Max Height of a Tall Column/Tower



$W = Mg$

KBCs $u(0) = 0$

$u'(0) = 0$

Quick $\Pi = \int_0^H \frac{EI}{2} u''^2 dz - \int_0^{H/2} \frac{Mg}{2} u'^2 dz$

Assume $u = \cancel{A_0} + \cancel{A_1} z + A_2 z^2$

Correct Way

$$\int_0^H \frac{EI}{2} u''^2 \, dz - \int_0^H \frac{P}{2} u'^2 \, dz$$

$$P = mg(H-z)$$

$m =$ Mass / height

$= \rho A$