# MA 214: Introduction to numerical analysis (2021–2022)

## Tutorial 6

(March 09, 2022)

(1) Use the forward-difference and backward-difference formulae to determine each missing entry in the following table:

| $x$ | $0.5$ | $0.6$ | $0.7$ |
|---|---|---|---|
| $f(x)$ | $0.4794255386$ | $0.5646424734$ | $0.6442176872$ |
| $f'(x)$ | ? | ? | ? |

(2) Use the forward-difference and backward-difference formulae to determine each missing entry in the following table:

| $x$ | $0.0$ | $0.2$ | $0.4$ |
|---|---|---|---|
| $f(x)$ | $0.0$ | $0.7414027582$ | $1.3718246976$ |
| $f'(x)$ | ? | ? | ? |

(3) The data in the above problems were taken from the following functions. Compute the actual errors and find error bounds using the error formulae:

$$(1)\ f(x) = \sin x, \qquad (2)\ f(x) = e^x - 2x^2 + 3x - 1.$$

(4) If $f : [a, b] \to \mathbb{R}$ is continuously differentiable and $c_i \in \mathbb{R}$, $\theta_i \in (a, b)$ for $i = 0, \ldots, n$, then prove that there is a $\theta \in (a, b)$ such that

$$\sum_i c_i f'(\theta_i) = \left(\sum_i c_i\right) f'(\theta).$$

(5) Assume that for any sufficiently continuously differentiable function $f$, we have

$$f''(t) \approx Af(t + h) + Bf(t) + Cf(t - h)$$

where $A, B, C$ are constants, depending on $h$, to be determined. Replace $f(t \pm h)$ by the Taylor expansions. Ignoring the terms involving $h^3$ or higher powers of $h$, solve for $A, B, C$. Write the approximate formula for $f''(t)$ obtained thus.

(6) Derive Simpson's $\frac{1}{3}$-rd rule with error term by using

$$\int_{x_0}^{x_2} f(x)dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + kf^{(4)}(\xi).$$

Find $a_0, a_1$, and $a_2$ from the fact that the rule is exact for $f(x) = x^n$ when $n = 1, 2$, and $3$. Then find $k$ by applying the integration formula with $f(x) = x^4$.

(7) Approximate the following using the trapezoidal and Simpson's $\frac{1}{3}$-rd rule: Compute the actual error and compare it with the error given by the error formulae.

$$\int_{0.5}^{1} x^4 dx.$$

(8) Approximate the following using the trapezoidal and Simpson's $\frac{1}{3}$-rd rule: Compute the actual error and compare it with the error given by the error formulae.

$$\int_{0}^{0.5} \frac{2}{x-4} dx.$$

(9) The Trapezoidal rule applied to $\displaystyle\int_0^2 f(x)dx$ gives the value 4, and Simpson's $\frac{1}{3}$-rd rule gives the value 2. What is $f(1)$?

The Trapezoidal rule applied to $\displaystyle\int_0^2 f(x)dx$ gives the value 5, and the Mid-point rule gives the value 4. What value does Simpson's $\frac{1}{3}$-rd rule give?

Q1) Use the forward and backward difference formulas to determine each missing entry in the following table:

| $x$ | 0.5 | 0.6 | 0.7 |
| --- | --- | --- | --- |
| $f(x)$ | 0.4794255386 | 0.5646424734 | 0.6442176872 |
| $f'(x)$ | ? | ? | ? |

Ans) $$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$ with $h$ small

If $h>0$, it is forward difference
If $h<0$, it is backward difference.

For $x = 0.5$
We use forward difference

$$f'(0.5) \approx \frac{f(0.5+0.1) - f(0.5)}{0.1}$$

$$= \frac{0.5646424734 - 0.4794255386}{0.1}$$

$$= +0.85216935$$

For $x = 0.7$
We use backward difference

$$f'(0.7) \approx \frac{f(0.7-0.1) - f(0.7)}{-0.1} = \frac{0.5646424734 - 0.64421768}{0.1}$$

$$= +0.79575214$$

For $x = 0.6$

We can use either forward difference or backward difference.

Using forward difference:

$$f'(0.6) \approx \frac{f(0.7) - f(0.6)}{0.1}$$

$$= +0.79575214$$

Using backward difference:

$$f'(0.6) \approx \frac{f(0.6-0.1) - f(0.6)}{-0.1}$$

$$= \frac{f(0.6) - f(0.5)}{0.1}$$

$$= +0.85216935$$

We thus see that we could uniquely calculate the values of only the end points.
We can approximate

$$f'(0.6) \approx \frac{f'(0.6)_{\text{forward}} + f'(0.6)_{\text{backward}}}{2}$$

which is the same as saying

$$f'(0.6) \approx \frac{f(0.6+0.1) - f(0.6-0.1)}{2 \times 0.1}$$

In any case, we get the following table:

| $x$ | 0.5 | 0.6 | 0.7 |
|---|---|---|---|
| $f(x)$ | 0.479425538 6 | 0.564642473 9 | 0.644217687 2 |
| $f'(x)$ | 0.85216935 | 0.823960745 | 0.79575214 |

Q2

| $x$ | 0.0 | 0.2 | 0.4 |
|---|---|---|---|
| $f(x)$ | 0.0 | 0.7414027582 | 1.3718246976 |
| $f'(x)$ | ? | ? | ? |

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We have only three points to work with
$|h| = 0.2$

forward difference formula, $h > 0 \Rightarrow h = 0.2$
backward difference formula, $h < 0 \Rightarrow h = -0.2$

$\cancel{f'(x) = f}$ forword difference

$$f'(x=0) = \frac{f(0.2) - f(0)}{0.2} = 3.7070137915$$

$$f'(x=0.2) = \frac{f(0.4) - f(0.2)}{0.2} = 3.152109697$$

backward difference

$$f'(x=0.2) = \frac{f(0) - f(0.2)}{-0.2} = 3.7070137915$$

$$f'(0.4) = \frac{f(0.2) - f'(0.4)}{-0.2} = 3.152109697$$

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(Q.3) (1) $f(x) = \sin x$

$f'_{Am}$ = Approximation of $f'(x)$
$f'_{Ac}$ = Actual $f'(x)$

$f'(x) = \cos x$

we compute $f'_{Am}(x)$ using fwd-diff method with $h = 0.1$

| $x$ | 0.5 | 0.6 | 0.7 |
|---|---|---|---|
| $f(x)$ | 0.479425386 | 0.564624734 | 0.644217687 2 |
| $f'_{Am}(x)$ | 0.851691954 | 0.795929532 | 0.7313840369 95 |
| $f'_{Ac}(x)$ | $\cos(0.5) = 0.87758$ | $\cos(0.6) = 0.825336$ | $\cos(0.7) = 0.76484$ |
| Actual Error | 0.025888046 | 0.029406068 | 0.033455963 |

Error bound $\Rightarrow$ $|f'_{Am}(x) - f'_{Ac}(x)| \leq \left|\frac{Mh}{2}\right| \leq \left|\frac{f''(c)\,h}{2}\right|$

$f''(x) = -\sin x$

$\Rightarrow$ Error bound $= \frac{|h \sin x|}{2} = E$

$\boxed{h = 0.1}$

| $x$ | 0.5 | 0.6 | 0.7 |
|---|---|---|---|
| $E$ | $\left\lvert\frac{0.1 \cdot \sin(0.6)}{2}\right\rvert = 0.028232123669$ | $\left\lvert\frac{0.1 \cdot \sin(0.7)}{2}\right\rvert = 0.032210884361$ | $\left\lvert\frac{0.1 \cdot \sin(0.8)}{2}\right\rvert = 0.035867804544$ |

$f(x) = e^x - 2x^2 + 3x - 1$

$f'(x) = e^x - 4x + 3$

$f''(x) = e^x - 4$

$f'(x) = \frac{f(x+0.2) - f(x)}{0.2}$

| $x$ | 0.0 | 0.2 | 0.4 |
|---|---|---|---|
| $f(x)$ | 0.0 | 0.7414027582 | 1.3718246976 |
| $f'_{Am}(x)$ | 3.707013791 | 3.152109697 | 2.6514705 1395 |
| $f'_{Ac}(x)$ | $e^0 - 4 \times 0 + 3 = 4$ | $e^{0.2} - 0.8 + 3 = 1.221402 + 2.2 = 3.421402758$ | $e^{0.4} + 1.4 = 1.491824697 + 1.4 = 2.891824697$ |
| Actual Error | 0.292986209 | 0.2692930611 6 | 0.2403541 8305 |

Error Bound $= \left| \frac{f''(\varepsilon)\, h}{2} \right| = E$

$\boxed{h = 0.2}$

| $x$ | 0 | 0.2 | 0.4 |
|---|---|---|---|
| $f''(\varepsilon)$ | $\lvert 1-4 \rvert = \lvert -3 \rvert = 3$ | $\lvert 1.221402 - 4 \rvert = 2.778598$ | $\lvert 1.491824697 - 4 \rvert = 2.508175303$ |
| $E$ | $3 \times \frac{0.2}{2} = 0.3$ | $2.778598 \times \frac{0.2}{2} = 0.2778598$ | $2.508175303 \times \frac{0.2}{2} = 0.2508175303$ |

Note:- $f(0.6) = e^{0.6} - 2 \times 0.36 + 1.8 - 1$

$= 1.8221188 0039 - 0.72 + 0.8$

$= 1.90211880039$

4. Given $f : [a,b] \longrightarrow \mathbb{R}$ is continuously differentiable

- Also $c_i \in \mathbb{R}$, $c_i \geq 0$ & $\theta_i \in (a,b)$ for $i = 0, \ldots n$

To show, that there is a $\theta \in (a,b)$ such that

$$\sum_i c_i f'(\theta_i) = \left(\sum_i c_i\right) f'(\theta)$$

We'll use induction over the value of $n$ to show that above statement is true.

- Base case ($n = 0$)

LHS = $c_0 f'(\theta_0)$

RHS = $c_0 f'(\theta)$

LHS = RHS for $\theta = \theta_0$

$\therefore$ Base case holds.

- Induction step

Suppose the given statement holds for $n = k$

ie.

$\exists\, \theta = \theta'$ in $(a,b)$ such that

$$\sum_{i=1}^{k} c_i f'(\theta_i) = \left(\sum_{i=1}^{k} c_i\right) f'(\theta') \quad -(1)$$

Given that the above statement holds, we now need to prove that:-

$\exists\, \theta = \theta''$ in $(a,b)$ such that

$$\sum_{i=1}^{k+1} c_i f'(\theta_i) = \left(\sum_{i=1}^{k+1} c_i\right) f'(\theta'')$$

$$LHS = \sum_{i=1}^{K+1} c_i f'(\theta_i)$$

$$= \underbrace{\left(\sum_{i=1}^{K} c_i f'(\theta_i)\right)} + c_{K+1} f'(\theta_{K+1})$$

Substituting from ① we get

$$LHS = \left(\sum_{i=1}^{K} c_i\right) f'(\theta') + c_{K+1} f'(\theta_{K+1})$$

Let $\sum_{i=1}^{K} c_i = A$ $\{A \geq 0$ as $c_i \geq 0$ for $i = 0, \ldots K\}$

- Case 1: $A + c_{K+1} = 0$

$\Rightarrow A = 0$ & $c_{K+1} = 0$

$\Rightarrow c_i = 0$ for $i = 0, \ldots K+1$

$LHS = 0$ & $RHS = 0$

$\therefore LHS = RHS$ holds.

- Case 2: $A + c_{K+1} \neq 0$ $\{A + c_{K+1} > 0\}$

$$LHS = A f'(\theta') + c_{K+1} f'(\theta_{K+1})$$

{Multiplying & dividing by $(A + c_{K+1})$}

$$LHS = (A + c_{K+1}) \left( \frac{A f'(\theta') + c_{K+1} f'(\theta_{K+1})}{A + c_{K+1}} \right)$$

1. Let $B = \dfrac{A f'(\theta') + c_{K+1} f'(\theta_{K+1})}{A + c_{K+1}}$

WLOG, let $f'(\theta') \leq f'(\theta_{K+1})$, then we know $B$ is the weighted average of $f'(\theta')$ & $f'(\theta_{K+1})$

$\Rightarrow B \in [f'(\theta'), f'(\theta_{K+1})]$

2. Give $f$ is continuously differentiable over $[a,b]$

$\Rightarrow f'$ is continuous over the interval $(a,b)$

As $\theta', \theta_{k+1} \in (a,b)$

By the intermediate value theorem, we know $f'$ will take every value between $f'(\theta')$ & $f'(\theta_{k+1})$ at some point.

$\therefore$ Combining the above two points, we get $\exists\ \theta''' \in (a,b)$ such that

$$f'(\theta''') = B$$

Thus the LHS becomes $(A + C_{k+1})\, f'(\theta''')$

$$\text{LHS} = (A + C_{k+1})\, f'(\theta''')$$

$$= \left(\sum_{i=1}^{k+1} C_i\right) f'(\theta''')$$

LHS = RHS for $\theta'' = \theta'''$.

$\therefore$ The given statement is TRUE.

Q5

$$f''(t) \approx Af(t+h) + Bf(t) + Cf(t-h)$$
$$f(t+h) \approx f(t) + hf'(t) + \frac{h^2}{2}f''(t)$$
$$f(t-h) \approx f(t) - hf'(t) + \frac{h^2}{2}f''(t)$$

Comparing coefficients of $f(t), f'(t)$ *and* $f''(t)$ respectively :

$$A + B + C = 0$$
$$hA - hC = 0 \leftrightarrow A = C$$
$$\frac{h^2}{2}A + \frac{h^2}{2}C = 1$$

This gives us :

$$A = C = \frac{1}{h^2}$$

and

$$B = -\frac{2}{h^2}$$

Thus :

$$f''(t) \approx \frac{1}{h^2}f(t+h) - \frac{2}{h^2}f(t) + \frac{1}{h^2}f(t-h)$$

Q6) Simpson's $\frac{1}{3}$rd rule.

$$\int_{x_0}^{x_2} f(x)\,dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^4(\eta)$$

$$x_0 < \eta < x_2$$

Given formula is exact for $f(x) = x, x^2, x^3$

C-I: $f(x) = x$

$$\int_{x_0}^{x_2} x\,dx = a_0 x_0 + a_1 x_1 + a_2 x_2$$

$$= \frac{x_2^2}{2} - \frac{x_0^2}{2}$$

C-II: $f(x) = x^2$

$$\int_{x_0}^{x_2} x^2\,dx = a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2$$

$$= \frac{x_2^3}{3} - \frac{x_0^3}{3}$$

C-III: $f(x) = x^3$

$$\int_{x_0}^{x_2} x^3\,dx = a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3$$

$$= \frac{x_2^4}{4} - \frac{x_0^4}{4}$$

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \\ x_0^3 & x_1^3 & x_2^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2^2}{2} - \frac{x_0^2}{2} \\ \frac{x_2^3}{3} - \frac{x_0^3}{3} \\ \frac{x_2^4}{4} - \frac{x_0^4}{4} \end{bmatrix}$$

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \\ x_0^3 & x_1^3 & x_2^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2^2}{2} - \frac{x_0^2}{2} \\ \frac{x_2^3}{3} - \frac{x_0^3}{3} \\ \frac{x_2^4}{4} - \frac{x_0^4}{4} \end{bmatrix} \rightsquigarrow A \cdot x = B$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{(h+x_0)(2h+x_0)}{2h^2 x} & \left(\frac{-3}{2hx_0} - \frac{1}{h^2}\right) & \frac{1}{2h^2 x_0} \\ \frac{-x(2h+x_0)}{h^2(h+x_0)} & \frac{2}{h^2} & -\frac{1}{h^2(h+x_0)} \\ \frac{x(h+x_0)}{2h^2(2h+x_0)} & \frac{-(\frac{h}{2}+x_0)}{h^2(2h+x_0)} & \frac{1}{2h^2(2h+x_0)} \end{bmatrix}}_{A^{-1}} \cdot B$$

To make calculations easier, consider

$x_0 = x_0$ for $a_0$

$x_0 = x_1 - h$, $x_2 = x_1 + h$ for $a_1$

& $x_0 = x_2 - 2h$, $x_1 = x_2 - h$ for $a_2$

$$\boxed{a_0 = \frac{h}{3}, \quad a_1 = \frac{4h}{3}, \quad a_2 = \frac{h}{3}}$$

For $k$, consider $f(x) = x^4$, $f^{IV}(\eta) = 4! = 24$

$$\int_{x_0}^{x_2} x^4 \, dx = \frac{h}{3} x_0^4 + \frac{4h}{3} x_1^4 + \frac{h}{3} x_2^4 + 24k$$

$$= \frac{x_2^5}{5} - \frac{x_0^5}{5}$$

$$24k = 0.2\left[(x_1+h)^5 - (x_1-h)^5\right] - \frac{h}{3}(x_1-h)^4 - \frac{4h}{3}x_1^4 - \frac{h}{3}(x_1+h)^4$$

$$= \cancel{-0.26667h^5} = -\frac{8}{30}h^5$$

$$\boxed{k = -\frac{h^5}{90}}$$

# MA214 Tutorial 6

March 5, 2022

## 1 Q7

Given, $f(x) = x^4$, we are required to compute $\int_{0.5}^{1} f(x)dx$ using the trapezoidal and Simpson's rule
To calculate the true value,

$$\int_a^b f(x)dx = \frac{x^5}{5}|_{0.5}^{1} = \frac{1 - 0.03125}{5} = 0.19375$$

**Trapezoidal Rule**

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12}f''(\xi)$$

where $x_0 = a$, $x_1 = b$. For our case, $f(x_0) = 0.0625$, $f(x_1) = 1$ and $h = 0.5$. Substituting values, we get,

$$\int_a^b f(x)dx \approx \frac{0.5}{2}(0.0625 + 1) = 0.265625$$

True error $= 0.265625 - 0.19375 = 0.071875$
Error from formula $= \frac{0.5^3}{12}f''(\xi)$. $f''(x) = 12x^2$. In the given range the max value of $f''(x) = 12$.
So the max error possible from the formula $= 0.125$

**Simpson's rule**

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) - \frac{h^5}{90}f^{(4)}(\xi)$$

where $x_1 = \frac{x_0+x_1}{2}$, For our case, $x_0 = 0.5$, $x_1 = 0.75$, $x_2 = 1$, $f(x_0) = 0.0625$, $f(x_1) = 0.3164$, $f(x_2) = 1$ and h=0.25. Substituting these values, we get.

$$\int_a^b f(x)dx \approx \frac{0.25}{3}(0.0625 + 1.2656 + 1) = 0.19401$$

True error $= 0.19401 - 0.19375 = 0.00026$
Error from formula $= \frac{h^5}{90}f^{(4)}(\xi)$. $f^{(4)} = 24$. So, the max error possible from the formula is $= 24\frac{0.25^5}{90} = 0.00026$

(Q8)

$$f(x) = \frac{2}{x-4} \quad , \quad a = 0 \quad , b = 0.5$$

trapezoidal rule $\Rightarrow$ $\int_a^b f(x)dx \approx \frac{(b-a)}{2}(f(a)+f(b))$

$f(0) = -0.5$

$f(0.5) = \frac{2}{-3.5} = -0.5714$

$$\Rightarrow \int_a^b f(x)dx \approx -\frac{0.5}{2}(0.5+0.5714)$$

$$= -0.26785$$

Simpson's $\frac{1}{3}$ Rule $\Rightarrow$ $\int_a^b f(x)dx = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$$= \frac{0.5}{6}\left[f(0) + 4f(0.25) + f(0.5)\right]$$

$$= -\frac{0.5}{6}\left[0.5 + 4\times 0.5333 + 0.5714\right]$$

$$= -0.26706$$

Actual Integral Value

$$\int_0^{0.5} \frac{2}{x-4}\,dx = 2\ln(|x-4|)\Big]_0^{0.5}$$

$$= -0.26706$$

Error

Trapezoidal - Actual error $= -0.26706 - (-0.26785)$

$= 0.00079$

Formula $\Rightarrow$ $E = \dfrac{(b-a)^3}{12} \|f''\|$

$$= \frac{(0.5)^3}{12} \times \max_{x \in [0,0.5]} \left( \left| \frac{4}{(x-4)^3} \right| \right)$$

$$= \frac{(0.5)^3}{12} (0.09329)$$

$$= 0.00097$$

Simpson's Rule - Actual Error $= -0.26706 - (-0.26706)$

(upto 5 decimal places) $= 0$

Formula = $$E = \frac{(b-a)^5}{90} \|f^{(4)}\|$$

$$= \frac{(0.5)^5}{90} \max_{x \in [0, 0.5]} \left( \left| \frac{48}{(x-4)^5} \right| \right)$$

$$= \frac{(0.5)^5}{90} \times (0.09139)$$

$$= \underline{0.0000317}$$

we are approximating the $f''(\xi)$ and $f^{(4)}(\xi)$ in the error formula with corresponding max norms since that gives the maximum/upper bound on the error.

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Tut6

Q9 From trapezoidal Rule we have

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(x_0)+f(x_1)\right)$$

where $h = b-a$

$$\Rightarrow \int_0^2 f(x)\,dx = 4 \approx \frac{(2-0)}{2}\left(f(0)+f(2)\right)$$

$$\Rightarrow f(0)+f(2) = 4 \quad \text{—— (1)}$$

From Simpson's $1/3^{rd}$ Rule we have

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

where $h = (b-a)/2$

$$\Rightarrow \int_0^2 f(x)\,dx = 2 \approx \frac{((2-0)/2)}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

$$6 = f(x_0)+4f(x_1)+f(x_2)$$

$$\Rightarrow \boxed{f(x_1) = 1/2} \quad \text{(using (1))}$$

From Midpt-Rule we have

$$\int_a^b f(x)\,dx \approx 2h\,f\left(\frac{a+b}{2}\right) \quad \text{where } h = (b-a)/2$$

$$\int_0^2 f(x)\,dx = 4 \approx 2 * \frac{(2-0)}{2} * f(1)$$

$$\Rightarrow f(1) = 2$$

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using trapezoidal Rule

$$\int_0^2 f(x)dx = 5 \approx \frac{(2-0)}{2}\left(f(0)+f(2)\right)$$

$$\rightarrow f(0)+f(2) = 5$$

Putting these values in Simpson's Rule

$$\int_0^2 f(x)dx = \frac{(b-a)}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{(2-0)}{6}\left[f(0)+f(2)+4f(x_1)\right] = \frac{1}{3}\left[5+8\right]$$

$$= 13/3$$