ME 226

Homework

19010001l

Q1. Impulse function : $A\delta$

IC :
$q_o(0) = 0$
$\dot{q}_o(0) = 0$

$$\frac{1}{w_n^2}\frac{d^2q_o}{dt^2} + \frac{2\xi}{w_n}\frac{dq_o}{dt} + q_o = kA\delta$$

Taking laplace,

$$\frac{1}{w_n^2}\left[s^2Q_o - s\cancel{q_o(0)}^{0} - \cancel{q_o'(0)}^{0}\right] + \frac{2\xi}{w_n}\left[sQ_o - \cancel{q_o(0)}^{0}\right] + Q_o = kA$$

$$Q_o\left[\frac{s^2}{w_n^2} + \frac{2\xi s}{w_n} + 1\right] = kA$$

$$Q_o = \frac{kAw_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

$$= \frac{kAw_n^2}{(s+w_n\xi)^2 + [w_n^2 - \xi^2 w_n^2]}$$

Case 1 : $w_n^2 - \xi^2 w_n^2 > 0$

$$Q_o = \frac{kAw_n^2}{\sqrt{w_n^2 - \xi^2 w_n^2}} \frac{\sqrt{w_n^2 - \xi^2 w_n^2}}{(s+w_n\xi)^2 + (\sqrt{w_n^2 - \xi^2 w_n^2})^2}$$

Taking inverse laplace,

$$q_o(t) = \frac{kAw_n^2}{\sqrt{w_n^2 - \xi^2 w_n^2}} e^{-w_n\xi t} \sin(w_n t\sqrt{1-\xi^2})$$

Case 2 : $w_n^2 - \xi^2 w_n^2 = 0$

$$Q_o = \frac{kAw_n^2}{(s+w_n\xi)^2}$$

Taking inverse laplace,

$$q_o(t) = kAw_n^2 \times te^{-w_n t}$$

Case 3 : $w_n^2 - \xi w_n^2 < 0$

$$Q_o = \frac{kAw_n^2}{(s+w_n\xi)^2 - (\xi^2 w_n^2 - w_n^2)}$$

$$Q_o = \frac{kA\omega_n^2}{\sqrt{\xi^2\omega_n^2 - \omega_n^2}} \cdot \frac{\sqrt{\xi^2\omega_n^2 - \omega_n^2}}{(s+\xi\omega_n)^2 - \left(\sqrt{\xi^2\omega_n^2 - \omega_n^2}\right)^2}$$

Taking inverse laplace:

$$q_o(t) = \frac{kA\omega_n^2}{\sqrt{\xi^2\omega_n^2 - \omega_n^2}} \times e^{-\omega_n\xi t}\left[\frac{e^{\omega_n t\sqrt{\xi^2-1}} + e^{-\omega_n t\sqrt{\xi^2-1}}}{2}\right]$$

$$= \frac{kA\omega_n}{2\sqrt{\xi^2-1}} \; e^{-\omega_n\xi t}\left(e^{\omega_n t\sqrt{\xi^2-1}} + e^{-\omega_n t\sqrt{\xi^2-1}}\right)$$

$\boxed{Q2.}$ Ramp ~~impulse~~ response: $q_i t$

$$\frac{1}{\omega_n^2}\frac{d^2q_o}{dt^2} + \frac{2\xi}{\omega_n}\frac{dq_o}{dt} + q_o = kq_i t$$

Solving for homogeneous eq^n: $q_o = e^{\lambda t}$

$$\frac{1}{\omega_n^2}\lambda^2 e^{\lambda t} + \frac{2\xi}{\omega_n}\lambda e^{\lambda t} + e^{\lambda t} = 0$$

$$\frac{\lambda^2}{\omega_n^2} + \frac{2\xi\lambda}{\omega_n} + 1 = 0$$

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

$$\lambda = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

$$\therefore \quad q_{o,h} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Where, $\lambda_1 = \left(-\xi + \sqrt{\xi^2-1}\right)\omega_n$

$\lambda_2 = -\left(\xi + \sqrt{\xi^2-1}\right)\omega_n$

Solving for particular sol^n: $q_{o,p} = At + B$

$$0 + \frac{2\xi A}{\omega_n} + At + B = kq_i t$$

$A = kq_i$

$B = -\frac{2\xi A}{\omega_n} = -\frac{2\xi}{\omega_n}kq_i$

$$q_o(t) = q_{o,h} + q_{o,p}$$

$$= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + kq_i t - \frac{2\xi_s}{\omega_n} kq_i$$

Imposing initial conditions:

$$q_o(0) = c_1 + c_2 - \frac{2\xi_s}{\omega_n} kq_i = 0$$

$$q_o'(0) = \lambda_1 c_1 + \lambda_2 c_2 + kq_i = 0$$

We get,

$$c_1 = \frac{\lambda_2\left(\frac{-2\xi_s}{\omega_n} kq_i\right) - kq_i}{\lambda - \lambda_2} = \frac{kq_i}{2\omega_n}\left[2\xi_s + \frac{2\xi_s^2 - 1}{\sqrt{\xi_s^2 - 1}}\right]$$

$$c_2 = \frac{\lambda_1\left(-\frac{2\xi_s}{\omega_n} kq_i\right) - kq_i}{\lambda_2 - \lambda_1} = \frac{kq_i}{2\omega_n}\left[2\xi_s - \frac{2\xi_s^2 - 1}{\sqrt{\xi_s^2 - 1}}\right]$$

So finally, we get

$$q_o(t) = \frac{kq_i}{2\omega_n}\left[2\xi_s + \frac{2\xi_s^2 - 1}{\sqrt{\xi_s^2 - 1}}\right] e^{\omega_n t(-\xi_s + \sqrt{\xi_s^2 - 1})} + \frac{kq_i}{2\omega_n}\left[2\xi_s - \frac{2\xi_s^2 - 1}{\sqrt{\xi_s^2 - 1}}\right] e^{-\omega_n t(\xi_s + \sqrt{\xi_s^2 - 1})}$$

$$+ kq_i\left(t - \frac{2\xi_s}{\omega_n}\right)$$

Case 2: $\xi_s = 1$

$$\lambda_1 = \lambda_2 = -\xi \omega_n$$

$$q_{o,h} = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

$$q_o(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} + kq_i\left(t - \frac{2\xi_s}{\omega_n}\right)$$

Imposing IC:

$$q_o(0) = c_1 - \frac{2kq_i \xi_s}{\omega_n} = 0 \quad \rightarrow \quad c_1 = kq_i \times \frac{2\xi_s}{\omega_n} = \frac{2kq_i}{\omega_n}$$

$$q_o'(0) = \lambda c_1 + c_2 + kq_i = 0$$

$$-\omega_n \xi_s \times \frac{2kq_i}{\omega_n} + c_2 \quad kq_i = 0$$

$$c_2 = kq_i$$

$$q_o(t) = \left(\frac{2kq_i}{w_n} + t\,kq_i\right)e^{-w_n t} + kq_i\left(t - \frac{2}{w_n}\right)$$

Case 3 : $\xi < 1$

$$\lambda = -\xi w \pm i w\sqrt{1-\xi^2}$$

$$q_o(t) = e^{-\xi w_n t}\left[c_1 \cos\left(w_n t\sqrt{1-\xi^2}\right) + c_2 \sin\left(w_n t\sqrt{1-\xi^2}\right)\right] + kq_i\left(t - \frac{2\xi_s}{w_n}\right)$$

Imposing IC :

$$q_o(0) = c_1 - kq_i \times \frac{2\xi_s}{w_n} = 0 \Rightarrow c_1 = kq_i\,\frac{2\xi_s}{w_n}$$

$$\dot{q}_o(0) = -\xi w_n c_1 + c_2 w_n\sqrt{1-\xi^2} + kq_i = 0 \Rightarrow c_2 = \frac{(2\xi_s^2 - 1)\,kq_i}{w_n\sqrt{1-\xi^2}}$$

So finally,

$$\frac{q_o(t)}{kq_i} = e^{-\xi w_n t}\left[\frac{2\xi_s}{w_n}\cos\left(w_n t\sqrt{1-\xi^2}\right) + \frac{(2\xi^2-1)}{w_n\sqrt{1-\xi^2}}\sin\left(w_n t\sqrt{1-\xi^2}\right)\right] + t - \frac{2\xi_s}{w_n}$$