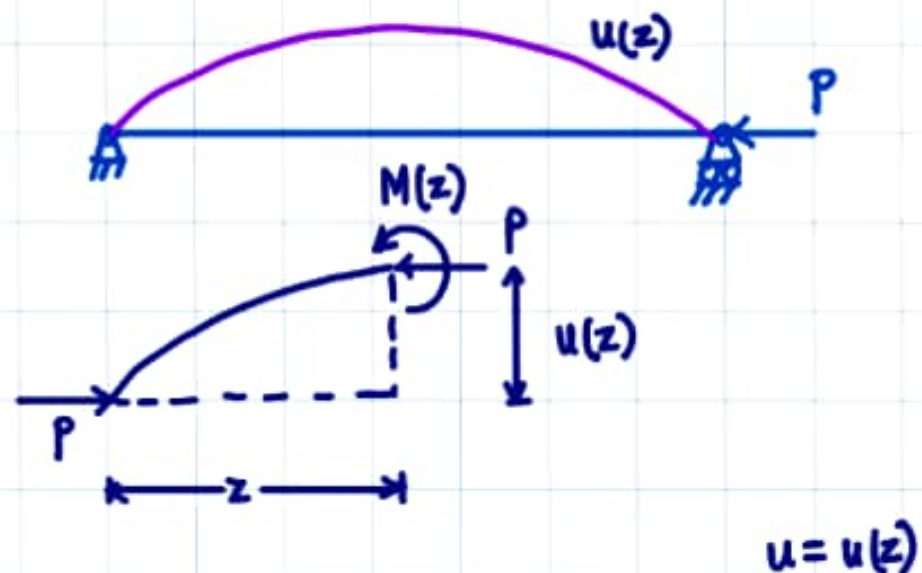


ME 202  
LECTURE 32  
THU 31 MAR 2022

BUCKLING OF ELASTIC COLUMNS/BEAMS



$$EI u'' = M(z) = -Pu,$$

$$u = u(z)$$

$$EI u'' + Pu = 0 \quad 2^{\text{nd}} \text{ order ODE}$$

$$-EI \frac{d^2}{dz^2} u = P u \quad \text{eigenvalue problem}$$

$P$  eigenvalues,  $u(z)$  eigenfunctions

$$u(0) = 0, \quad u(L) = 0$$

$$u'' + \lambda^2 u = 0 \quad \lambda^2 = \frac{P}{EI}$$

$$u = A \cos \lambda z + B \sin \lambda z$$

$$u(0) = 0 \Rightarrow A = 0$$

$$u(L) = 0 \Rightarrow B \sin \lambda L = 0$$

$$B = 0 \quad \text{or} \quad \sin \lambda L = 0$$

trivial soln

$$\lambda L = n\pi \quad n=1,2,3$$

$$P_1 = \frac{\pi^2 EI}{L^2}, \quad P_2 = \frac{4\pi^2 EI}{L^2}, \quad P_3 = \frac{9\pi^2 EI}{L^2}$$

$$u_1 = B \sin \frac{\pi z}{L}, \quad u_2 = B \sin \frac{2\pi z}{L}, \quad u_3 = B \sin \frac{3\pi z}{L}$$

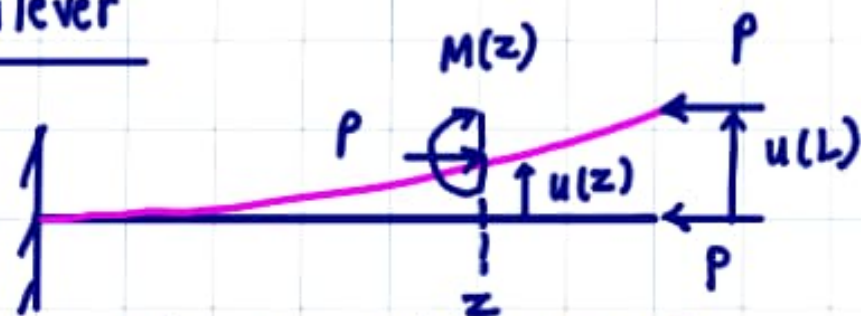
$B$  undetermined

Euler Load

$$\frac{\pi^2 EI}{L^2}$$



Cantilever



$$M(z) = P(u(L) - u(z)) = EI u''$$

$$EI u'' + Pu = Pu_L \quad u_L = u(L)$$

$$u = A \cos \lambda z + B \sin \lambda z + u_L$$

$$\lambda^2 = \frac{P}{EI}, \quad u' = -A\lambda \sin \lambda z + B\lambda \cos \lambda z$$

$$\text{BCs} \quad u(0) = 0, \quad u'(0) = 0, \quad u(L) = u_L$$

$$u_L + A = 0$$

$$B\lambda = 0 \Rightarrow B = 0$$

$$\cancel{u(L)} + A \cos \lambda L + \cancel{B \sin \lambda L} = \cancel{u(L)}$$

$$A \cos \lambda L = 0 \Rightarrow \lambda L = \frac{n\pi}{2}$$

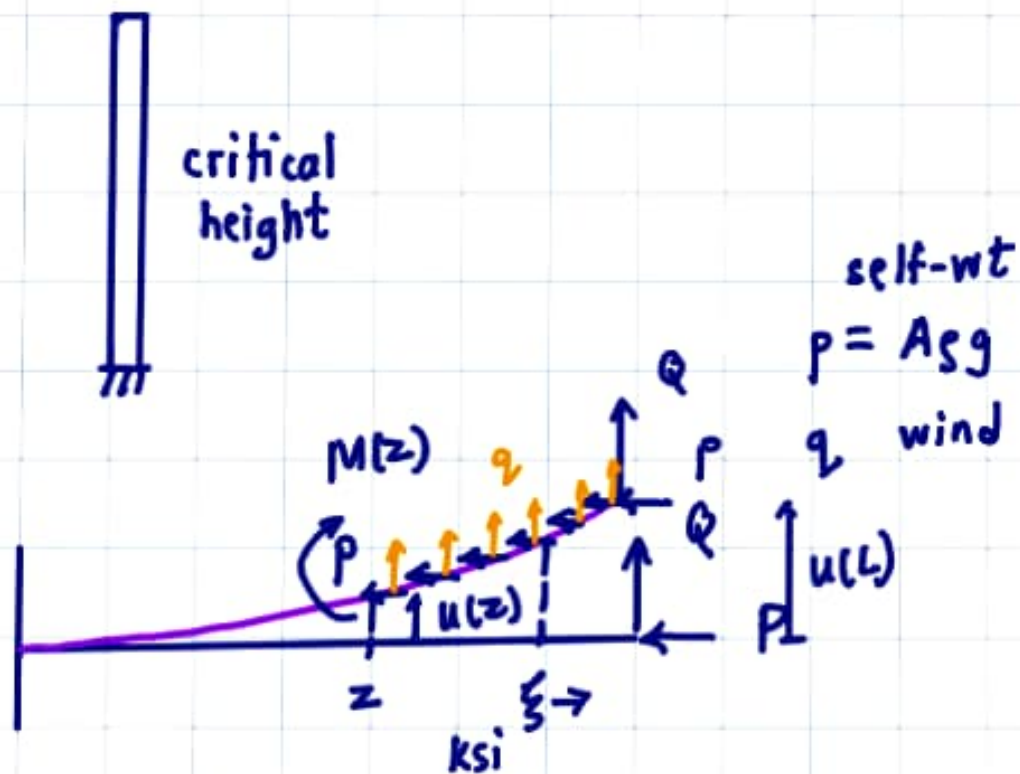
$n$  odd

$$u(z) = -A + A \cos \frac{n\pi z}{2L}$$

$$u(z) = A \left( 1 - \cos \frac{n\pi z}{2L} \right)$$

$$P_n = \frac{n^2 \pi^2 EI}{4L^2}$$

### Self-buckling (under own weight)



$$M(z) = Q(L-z) + P(u(L) - u(z)) + \int_z^L q(\xi)(\xi-z)d\xi + \int_z^L P(\xi)(u(\xi) - u(z))d\xi = EI u''$$

Take derivs wrt  $z$

DUIS / Leibniz / Reynolds Transport Thm  
Newton



$$\frac{d}{dz} \int_{\varphi_1(z)}^{\varphi_2(z)} F(\xi, z) d\xi = \int_{\varphi_1(z)}^{\varphi_2(z)} \frac{\partial F}{\partial z} d\xi + F(\varphi_2, z) \frac{d\varphi_2}{dz} - F(\varphi_1, z) \frac{d\varphi_1}{dz}$$

$$\frac{d}{dz} \int_z^L F d\xi = \int_z^L -\frac{du}{dz} d\xi + F(L, z) \frac{dL}{dz} - F(z, z) \frac{dz}{dz}$$

$$F = u(\xi) - u(z)$$

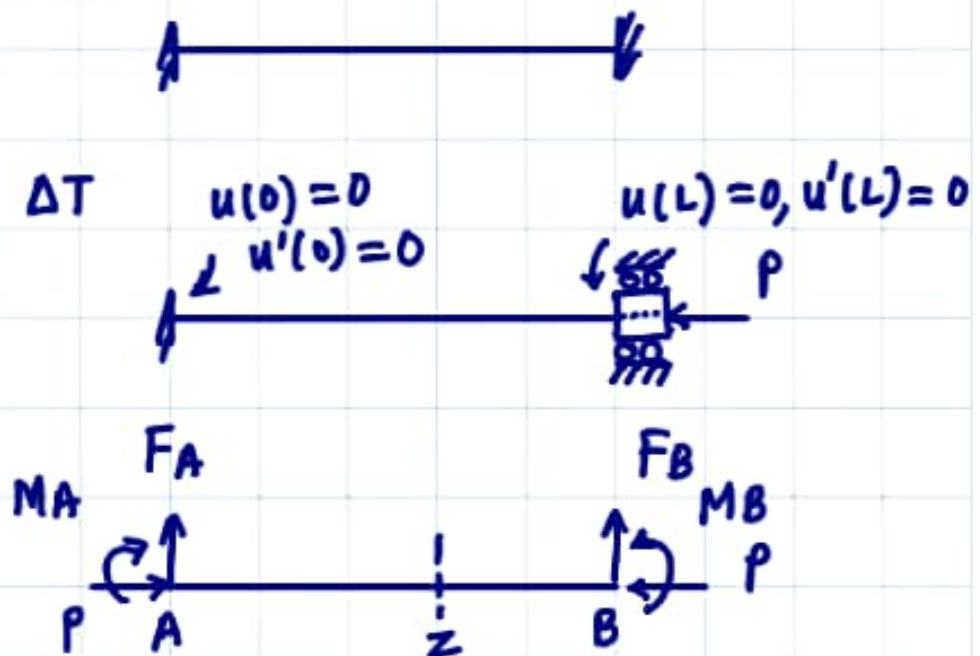
$$= (L-z) \left( -\frac{du}{dz} \right)$$

$$EI \frac{d^3 u}{dz^3} + Q + P \frac{du}{dz} + q(L-z)$$

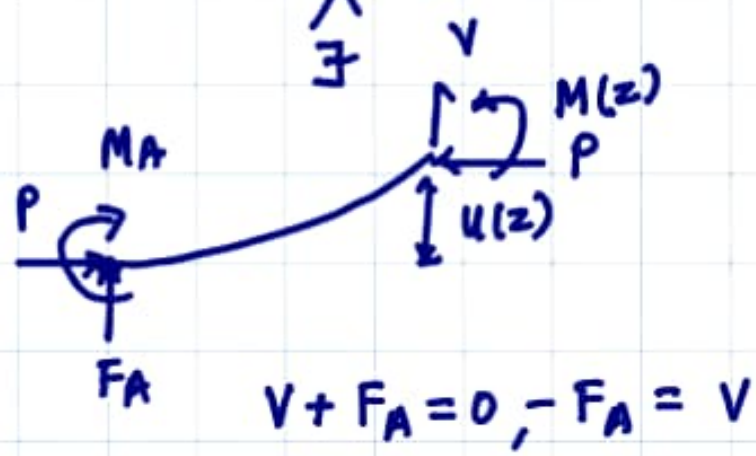
$$+ \underbrace{P(L-z)}_{\text{variable coefficient}} \frac{du}{dz} = 0$$

variable coefficient

## Example



Find  $P$  for which  $\exists$  non-trivial  $u(z)$



$$-M_A - F_A z + M + P u(z) = 0$$

$$M = -P u(z) + M_A + F_A z = EI u''$$

$$EI u'' + P u = M_A + F_A z$$

$$\lambda^2 = \frac{P}{EI}$$

$$u(z) = A \cos \lambda z + B \sin \lambda z + \frac{M_A}{P} + \frac{F_A}{P} z$$

$$A, B, M_A, F_A$$

$$4 \text{ BCs } \quad u(0)=0, \quad u'(0)=0, \quad u(L)=0, \\ u'(L)=0$$

$$\frac{M_A}{P} + A = 0, \quad M_A = -AP$$

$$\frac{F_A}{P} + B\lambda = 0, \quad F_A = -B\lambda$$



$$-A - B\lambda L + A \cos \lambda L + B \sin \lambda L = 0$$

$$-B\lambda - A\lambda \sin \lambda L + B\lambda \cos \lambda L = 0$$

$$\underbrace{\begin{pmatrix} \cos \lambda L - 1 & \sin \lambda L - \lambda L \\ -\lambda \sin \lambda L & \lambda \cos \lambda L - \lambda \end{pmatrix}}_{\det = 0} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$