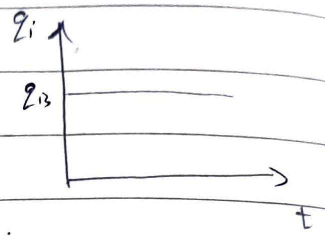
Homework: ME226 — 190100011

Q1. Step response:



Initial condition: $q_o(0) = 0$

$$\left.\frac{dq_o}{dt} = 0\right|_{t=0}$$

$$\frac{1}{\omega_n^2}\frac{d^2 q_o}{dt^2} + \frac{2\xi}{\omega_n}\frac{dq_o}{dt} + q_o = k q_{is}$$

Laplace transform

$$\frac{1}{\omega_n^2}\left[s^2 Q_o - s\cancel{q_o(0)}^{0} - \cancel{q_o'(0)}^{0}\right] + \frac{2\xi}{\omega_n}\left[sQ_o - \cancel{q_o(0)}^{0}\right] + Q_o = \frac{k q_{is}}{s}$$

$$Q_o = \frac{k q_{is}\,\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$= \frac{k q_{is}\,\omega_n}{\sqrt{1-\xi^2}} \times \underbrace{\frac{1}{s}}_{F(s)} \times \underbrace{\frac{\sqrt{1-\xi^2}\,\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}}_{G(s)}$$

$$\frac{b}{(s-a)^2 + b^2} \Rightarrow b = \omega_n\sqrt{1-\xi^2}$$

$$a = -\omega_n\xi$$

Inverse laplace using convolution theorem,

$$q_o = \frac{k q_{is}\,\omega_n}{\sqrt{1-\xi^2}} \times \int u(\tau)\, e^{-\xi\omega_n(t-\tau)} \sin\left[\sqrt{1-\xi^2}\,\omega_n(t-\tau)\right] d\tau$$

Look at the denominator : $s^2 + 2\xi \omega_n s + \omega_n^2$

<u>Case I</u> : $\xi > 1 \Rightarrow$ 2 real roots

$$s_1, s_2 = -\omega_n \xi \pm \omega_n \sqrt{\xi^2 - 1}$$

$$Q_o = kq_{i3} * \frac{s_1 s_2}{s(s-s_1)(s-s_2)} = kq_{i3} \underbrace{\frac{1}{s}}_{F(s)} \times \underbrace{\frac{s_1 s_2}{(s-s_1)(s-s_2)}}_{G(s)}$$

$$\cancel{q_o = kq_{i3} \left[1 - \left(\frac{s_2 e^{s_1 t} - s_1 e^{s_2 t}}{s_2 - s_1} \right) \right]}$$

$$q_o = kq_{i3} \int_0^t H(t-\tau) \left(\frac{e^{s_1 \tau} - e^{s_2 \tau}}{s_2 - s_1} \right) d\tau$$

$$= kq_{i3} \left[\frac{s_1 e^{s_1 t} - s_2 e^{s_2 t}}{s_2 - s_1} \right]_0^t$$

$$= kq_{i3} \left[\frac{s_1 e^{s_1 t} - s_2 e^{s_2 t}}{s_2 - s_1} + 1 \right]$$

$$= kq_{i3} \left[\frac{\omega_n \left(-\xi + \sqrt{\xi^2 - 1}\right) e^{\omega_n t \left(-\xi + \sqrt{\xi^2 - 1}\right)} - \omega_n \left(-\xi - \sqrt{\xi^2 - 1}\right) e^{\omega_n t \left(-\xi - \sqrt{\xi^2 - 1}\right)}}{-2\omega_n \sqrt{\xi^2 - 1}} + 1 \right]$$

<u>Case II</u> : $\xi = 1 \Rightarrow s_1 = s_2 = -\omega_n \xi = -\omega_n$

$$Q_o = kq_{i3} \, \omega_n^2 \times \frac{1}{\underbrace{s}_{F(s)} \underbrace{(s+\omega_n)^2}_{G(s)}}$$

classmate

Date

Page

$$q_0 = k q_{i3} w_n^2 \int_0^t H(t-\tau) \times \tau e^{-w_n \tau} d\tau$$

$$= k q_{i3} w_n^2 \left[\tau (-1/w_n) e^{-w_n \tau} - 1/w_n^2 \times e^{-w_n \tau} \right]_0^t$$

$$= k q_{i3} \left[-t w_n e^{-w_n t} - e^{-w_n t} - 1 \right]$$

Case III : $\xi < 1$

$$Q_0 = \frac{k q_{i3} w_n}{\sqrt{1-\xi^2}} \times \underbrace{\frac{1}{s}}_{F(s)} \times \underbrace{\frac{\sqrt{1-\xi^2}\, w_n}{s^2 + 2\xi w_n + w_n^2}}_{G(s)}$$

$$q_0 = \frac{k q_{i3} w_n}{\sqrt{1-\xi^2}} \int_0^t H(\cancel{t-\tau}) e^{-\xi w_n (\cancel{t-}\tau)} \sin\left(\sqrt{1-\xi^2}\, w_n (\cancel{t-}\tau)\right) d\tau$$

$$= \frac{k q_{i3} w_n}{\sqrt{1-\xi^2}} \int_0^t e^{-\xi w_n \tau} \sin\left(\sqrt{1-\xi^2}\, w_n \tau\right) d\tau$$

$$= \frac{k q_{i3} w_n}{\sqrt{1-\xi^2}} \int_0^t e^{a\tau} \sin(b\tau)\, d\tau$$

$$a = -\xi w_n \quad , \quad b = \sqrt{1-\xi^2}\, w_n$$

$$q_0 = \frac{k q_{i3} w_n}{\sqrt{1-\xi^2}} \left[\frac{e^{a\tau}}{a^2+b^2} (a \sin b\tau - b \cos b\tau) \right]_0^t$$

$$\frac{q_0}{k q_{i3}} = 1 - \frac{e^{\xi w_n t}}{\sqrt{1-\xi^2}} \left(\xi \sin(\sqrt{1-\xi^2}\, w_n t) - \sqrt{1-\xi^2} \cos\left(w_n t \sqrt{1-\xi^2}\right) \right)$$

classmate

Date

Page

Homework : ME 226 190100011

Q2.

Partial fractions :

$$Q_o = \frac{k q_{i3} w_n^2}{s(\underbrace{s^2 + 2\xi w_n s + w_n^2}))}$$

Roots for this quadratic are :

$$s_1, s_2 = \frac{-2\xi w_n \pm \sqrt{4\xi^2 w_n^2 - 4w_n^2}}{2}$$

$$= -\xi w_n \pm w_n\sqrt{\xi^2 - 1}$$

$$= w_n \{ \xi \pm \sqrt{\xi^2 - 1} \}$$

So,

$$s_1 = w_n(\xi + \sqrt{\xi^2 - 1})$$

$$s_2 = w_n(\xi - \sqrt{\xi^2 - 1})$$

Case I : $\xi > 1$

$$\frac{1}{s(s^2 + 2\xi w_n s + w_n^2)} = \frac{A}{s} + \frac{B}{s - s_1} + \frac{C}{s - s_2}$$

$$\Rightarrow A(s-s_1)(s-s_2) + Bs(s-s_2) + Cs(s-s_1) = 0$$

$$A + B + C = 0 \longrightarrow (1)$$

$$A(s_1 + s_2) + Bs_2 + Cs_1 = 0 \longrightarrow (2)$$

$$As_1 s_2 = 1 \longrightarrow (3)$$

$$A = \frac{1}{s_1 s_2} = \frac{1}{\omega_n^2}$$

$$\text{Eq}\left[(2) - s_1 \times (1)\right] \Rightarrow As_2 + B(s_2 - s_1) = 0$$

$$B = \frac{-As_2}{s_2 - s_1}$$

$$= \frac{1}{s_1(s_1 - s_2)}$$

$$B = \frac{1}{s_1(s_1 - s_2)} = \frac{1}{\omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)} \times \frac{1}{2\omega_n \sqrt{\xi^2 - 1}}$$

$$= \frac{1}{2\omega_n^2\left(\xi + \sqrt{\xi^2 - 1}\right)\sqrt{\xi^2 - 1}} = \frac{-1}{2\omega_n^2} \frac{\left(\xi + \sqrt{\xi^2 \ldots}\right.}{\sqrt{\xi^2 - 1}}$$

$$C = -A - B$$

$$= \frac{1}{2\omega_n^2} \frac{\left(\xi - \sqrt{\xi^2 - 1}\right)}{\sqrt{\xi^2 - 1}}$$

So,

$$Q_o = kq_i \omega_n^2 \left[\frac{A}{s} + \frac{B}{s - s_1} + \frac{C}{s - s_2}\right]$$

Inverse laplace

$$q_o = kq_i \omega_n^2 \left(A + Be^{s_1 t} + Ce^{s_2 t}\right)$$

$$q_0 = k q_{i3} \left[1 - \frac{(\xi + \sqrt{\xi^2-1})}{2\sqrt{\xi^2-1}} e^{\omega_n t(-\xi + \sqrt{\xi^2-1})} + \frac{\xi - \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} e^{\omega_n t(-\xi - \sqrt{\xi^2-1})} \right]$$

Case II : $\xi = 1$

Equal roots

~~Product of roots~~

$$s_1 = s_2 = -\omega_n$$

$$Q_0 = \frac{k q_{i3} \omega_n^2}{s(s+\omega_n)^2}$$

$$\frac{1}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs = 1$$

$$A + B = 0$$

$$A = 1/\omega_n^2$$

$$B = -1/\omega_n^2$$

$$2A\omega_n s + B\omega_n s + Cs = 0$$

$$C = -1/\omega_n$$

$$Q_o = k q_{i3} \omega_n^2 \left[\frac{1}{\omega_n^2 s} - \frac{1}{\omega_n^2 (s+\omega_n)} - \frac{1}{\omega_n (s+\omega_n)^2} \right]$$

$$= k q_{i3} \left[1 - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \right]$$

Inverse laplace :

$$q_o = k q_{i3} \left[1 - e^{-\omega_n t} - \omega_n t \, e^{-\omega_n t} \right]$$

Case III : $\xi < 1$

Imaginary roots :

$$Q_o = \frac{k q_{i3} \omega_n^2}{s(s^2 + 2\xi \omega_n s + \omega_n^2)}$$

$$\frac{1}{s(s^2 + 2\omega_n \xi s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\omega_n \xi s + \omega_n^2)}$$

$$A(s^2 + 2\omega_n \xi s + \omega_n^2) + (Bs + C)s = 1$$

$$A = 1/\omega_n^2$$

$$B = -A = -1/\omega_n^2$$

$$C = -2A\xi\omega_n = -\frac{2\xi}{\omega_n}$$

$$Q_o = kq_{i3}\left[\frac{1}{s} - \frac{(s+2\xi w_n)}{s^2+2\xi w_n s+w_n^2}\right]$$

$$= kq_{i3}\left[\frac{1}{s} - \frac{(s+2\xi w_n)}{(s+w_n\xi)^2+w_n^2(1-\xi^2)}\right]$$

$$= kq_{i3}\left[\frac{1}{s} - \frac{(s+\xi w_n)}{(s+w_n\xi)^2+w_n^2(1+\xi^2)} - \frac{\xi w_n}{(s+w_n\xi)^2+w_n^2(1+\xi^2)}\right]$$

Let $\Rightarrow$ $w_d = w_n\sqrt{1-\xi^2}$

$$Q_o = kq_{i3}\left[\frac{1}{s} - \underbrace{\frac{(s+\xi w_n)}{(s+w_n\xi)^2+w_d^2}}_{\frac{s-a}{(s-a)^2+b^2}} - \underbrace{\frac{\xi w_d}{(s+w_n\xi)^2+w_d^2}\times\left(\frac{1}{\sqrt{1-\xi^2}}\right)}_{\frac{a}{(s-a)^2+b^2}}\right]$$

Inverse laplace

$$q_o = kq_{i3}\left[1 - e^{-\xi w_n t}\cos(w_d t) - \frac{\xi e^{-\xi w_n t}}{\sqrt{1-\xi^2}}\sin(w_d t)\right]$$

Let, $\phi = \sin^{-1}\left(\sqrt{1-\xi^2}\right)$

$\sin\phi = 1-\xi^2$

$$\sqrt{1-\xi^2}\cos(w_d t) + \xi\sin(w_d t) = \sin(w_d t+\phi)$$

$$q_o = kq_{i3}\left[1 - \frac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}}\times\sin\left(\sqrt{1-\xi^2}\,w_n t+\phi\right)\right]$$