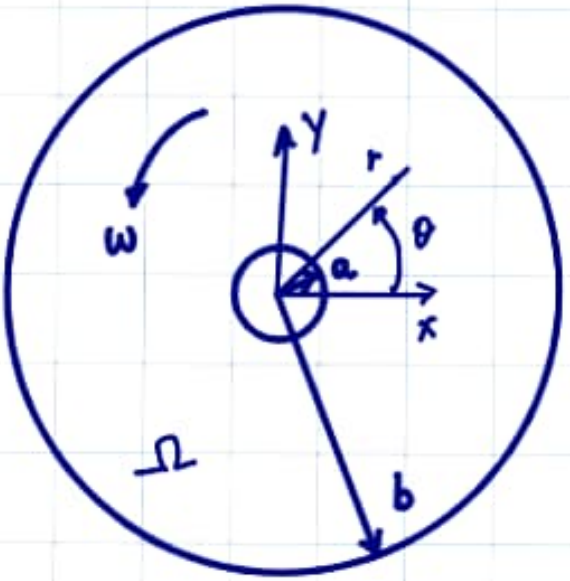
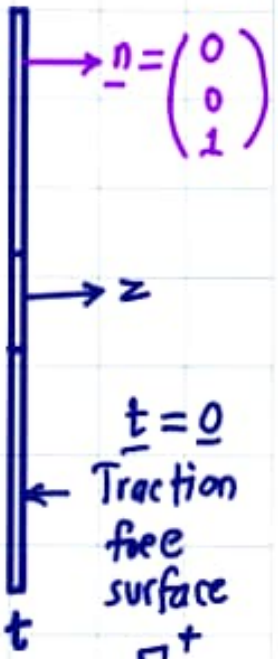
ME 202

LECTURE 27

MON 21 MAR 2022

Ref:
Srinath
Sadd





$t \ll a < b$

3D ⟶ 2D ⟶ 1D* Axisymmetric

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Cylindrical Coordinates r, θ, z

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{pmatrix}$$

$\underline{t} = \underset{\sim}{\sigma}\,\underline{n}$ on Γ

Static Equilibrium on a surface $\Sigma \underline{F} = \underline{0}$ with normal $\underline{n}$

$$\begin{pmatrix} t_r \\ t_\theta \\ t_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ & \sigma_{\theta\theta} & \sigma_{\theta z} \\ & & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

given

$0 = \sigma_{rz}\cdot 1,\ 0 = \sigma_{\theta z}\cdot 1,\ 0 = \sigma_{zz}\cdot 1$ on Γ^+

$0 = \sigma_{rz}(-1),\ 0 = \sigma_{\theta z}(-1),\ 0 = \sigma_{zz}(-1)$ on Γ^-

$\Rightarrow \sigma_{rz} = 0,\ \sigma_{\theta z} = 0,\ \sigma_{zz} = 0$ in Ω

Plane stress assumption

Hooke's Law in 3D

$$\epsilon_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu}{E}(\sigma_{\theta\theta} + \cancel{\sigma_{zz}}) = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}(\sigma_{rr} + \cancel{\sigma_{zz}}) = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$\epsilon_{zz} = \frac{\cancel{\sigma_{zz}}}{E} - \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{rz} = \frac{1+\nu}{E}\cancel{\sigma_{zr}} = 0$$

$$\epsilon_{\theta z} = \frac{1+\nu}{E}\cancel{\sigma_{\theta z}} = 0$$

$$\epsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}$$

$$= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\frac{\partial u_z}{\partial z} = \frac{-\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})$$

$$= \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$u_z \approx \frac{-\nu t}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & 0 \\ \epsilon_{r\theta} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}$$

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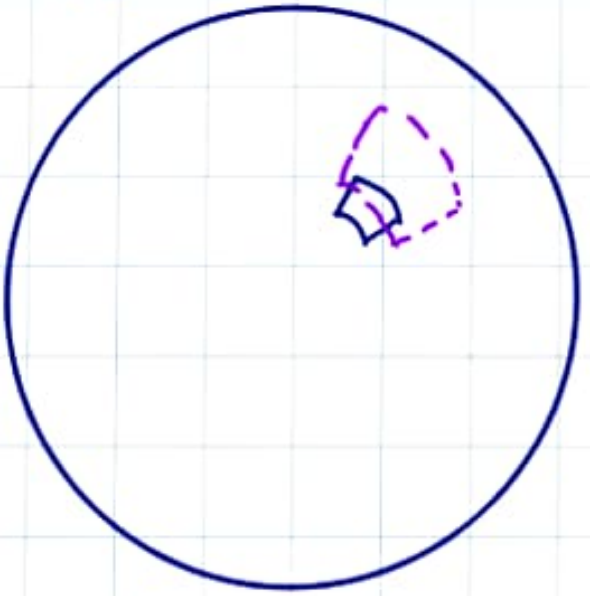
$u_\theta = 0 \qquad \frac{\partial \sigma}{\partial \theta} = 0$

$$\rho \frac{\partial^2 u_r}{\partial t^2} = -\rho r \omega^2$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + b_\theta = 0 \quad \checkmark$$

$$\frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{zr} + b_z = 0 \quad \checkmark$$



$$\frac{d}{dr}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho r \omega^2 = 0$$

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Stress Formulation

ODE in $\sigma_{rr}(r)$

Depends on BCs.

Displacement Formulation

ODE in $u_r(r)$

① Spinning freely at constant ω

Ensemble

BCs

Inner radius traction free

Outer ——————"——————

@ $r = a$.

$$\underline{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow 0 = \sigma_{rr}(-1) \Rightarrow \sigma_{rr}(a) = 0$$

$$\sigma_{rr}(b) = 0$$

Aside $\omega = 0$, internal pressure $= P_a$
external pressure $= P_b$

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} \Rightarrow \begin{pmatrix} +P_a \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{rr} & 0 \\ 0 & \sigma_{\theta\theta} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\sigma_{rr}(a) = -P_a$$

$$\sigma_{rr}(b) = -P_b$$

$$u(r) = u_r(r)$$

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right) = \frac{du}{dr}$$

$$\epsilon_{\theta\theta} = \frac{u}{r} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)$$

equate

$$u = \frac{r}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) \Rightarrow \frac{du}{dr} =$$

$$\Rightarrow \quad \sigma_{\theta\theta} = \sigma_{rr} + \nu\sigma_{rr}' \qquad ' \equiv \frac{d}{dr}$$

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Plug into eqm

$$r\frac{d^2\sigma_{rr}}{dr^2} + 3\frac{d\sigma_{rr}}{dr} = 0$$

$$\sigma_{rr} = C_1 + \frac{C_2}{r^2}, \quad \sigma_{\theta\theta} = C_1 - \frac{C_2}{r^2}$$

C_1, C_2 from TBCs on σ_{rr}

$$\sigma_{rr} = \underbrace{\frac{P_a a^2 - P_b b^2}{b^2 - a^2}} - \underbrace{\frac{a^2 b^2}{r^2}\left(\frac{P_a - P_b}{b^2 - a^2}\right)}$$

$$\sigma_{\theta\theta} = \qquad\qquad + \qquad\qquad$$

$$u = \frac{1-\nu}{E}\frac{P_a a^2 - P_b b^2}{b^2 - a^2} r + \frac{1+\nu}{E}\frac{a^2 b^2}{r^2}\frac{P_a - P_b}{b^2 - a^2}$$

Displacement Approach

Invert Hooke's Law

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$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\underbrace{\epsilon_{rr}}_{\frac{du}{dr}} + \nu\,\underbrace{\epsilon_{\theta\theta}}_{\frac{u}{r}}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\epsilon_{\theta\theta} + \nu\,\epsilon_{rr}\right)$$

→ Into eqm

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = 0$$

$$u = C_1 r + \frac{C_2}{r}$$

Useful for DBCs.