ME 202
LECTURE 03
06 JAN 2022

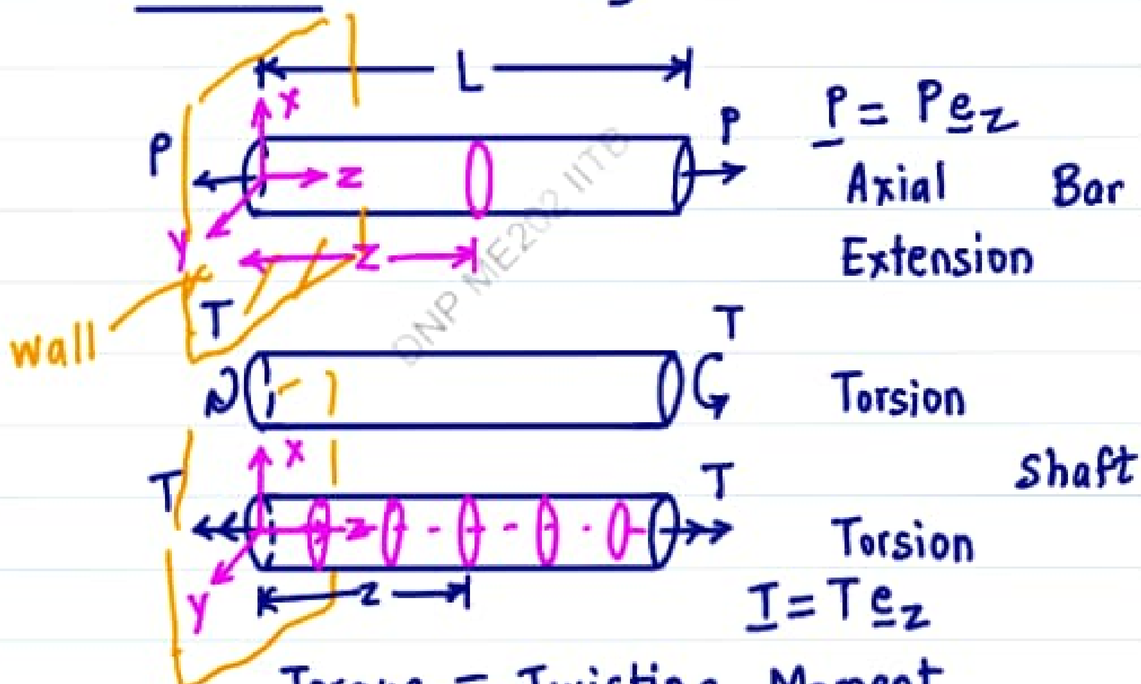
Previously,

- ☐ Equilibrium of stresses/tractions
- ☐ Kinematics / Strain-Displacement Relationship
- ☐ Stress-strain relationship/Hooke's Law
- ☐ Boundary conditions

Today's class,

- ☐ Axial deformation review
- ☐ Torsion

# TORSION Twisting




Torque $\equiv$ Twisting Moment

$$\underset{\hat{i}}{\underline{e}_x} = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad \underset{\hat{j}}{\underline{e}_y} = \begin{pmatrix}0\\1\\0\end{pmatrix},\quad \underset{\hat{k}}{\underline{e}_z} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

# Review of Axial Extension

$$\left.\begin{matrix} u \\ v \\ w \end{matrix}\right\} \text{disp. along} \quad \begin{matrix} \underline{e}_x, \ x \text{ axis} \\ \underline{e}_y, \ y \text{ axis} \\ \underline{e}_z, \ z \text{ axis} \end{matrix}$$

Assumption Kinematics $\quad u = 0$

$\quad v = 0$

$\quad w = c_1 z \qquad$ Ansatz

Strain $\quad \epsilon_{zz} = \dfrac{dw}{dz} = c_1$

Hooke's Law

Stress $\quad \sigma_{zz} = E c_1$



At $z = L$,


$$\underline{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{t} = \begin{pmatrix} 0 \\ 0 \\ P/A \end{pmatrix} \quad \text{applied traction}$$

From equilibrium at $z = L$,

resultant of tractions $\quad \displaystyle\int_{\Omega} \underline{t} \, da = \underline{P}$

Recall,

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{zz} \end{pmatrix} = \sigma_{zz}\,\underline{e}_z$$

$$\underset{\sim}{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} \leftarrow \text{uniaxial loading}$$

$$\int_{\Omega} \sigma_{zz}\, da\, \underline{e}_z = P\,\underline{e}_z$$

$$\int_{\Omega} \sigma_{zz}\, da = P$$

$$\int_{\Omega} E c_1\, da = P$$

$$E c_1 A = P \qquad A \text{ area c/s}$$

$$c_1 = \frac{P}{AE}$$

$$w(z) = \frac{Pz}{AE}, \qquad \delta = w(L) = \frac{PL}{AE}$$

$$\sigma_{zz} = \frac{P}{A}$$

Kinematic Assumptions

$\downarrow$

Strains $\xrightarrow{\text{Hooke's Law}}$ Stresses $\xrightarrow{\text{Equilibrium}}$ Ext. Force

Same logic for torsion.

P.S.

check static equilibrium equation in 1D

$$\frac{d}{dz}\sigma_{zz} + b = 0$$ along z direction

$$\underline{\nabla} \cdot \underset{\sim}{\sigma} + \underline{b} = \underline{0}$$

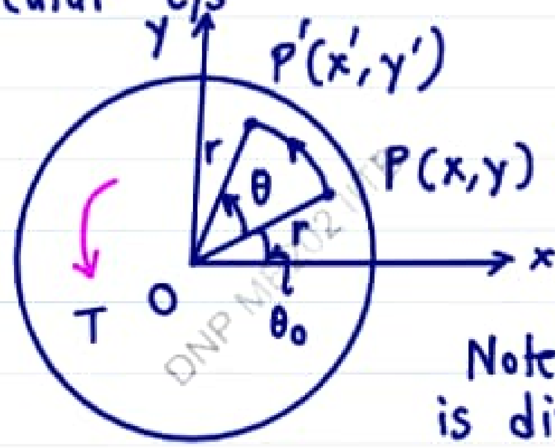
$$\frac{d}{dz}(Ec_1) + 0 = 0 \checkmark$$

## Questions

1. How much does the shaft deflect?
   Angular deflection
2. What is the max torque
   that can applied?

## Kinematic Assumptions

1. Circular c/s




Rigid Body Rotation for each c/s

Note: $\theta = \alpha z$
is diff for every c/s

c/s of shaft at some z coordinate

2. Let $\alpha$ unit angle of twist for the entire shaft $\quad \alpha \equiv$ rad/m

$\theta(z) =$ angle of twist for c/s at z
$= \alpha z$

Total angle of twist $= \theta(L) = \alpha L$

3. Each c/s twists as if it undergoes a rigid body rotation about z-axis through angle $\theta(z)$.

$$x = r\cos\theta_0, \quad y = r\sin\theta_0$$
$$x' = r\cos(\theta+\theta_0), \quad y' = r\sin(\theta+\theta_0)$$

$$u = x' - x = \cancel{r\cos\theta\cos\theta_0} - r\sin\theta\sin\theta_0 - \cancel{r\cos\theta_0}$$
$$v = y' - y = r\sin\theta\cos\theta_0 + \cancel{r\cos\theta\sin\theta_0} - \cancel{r\sin\theta_0}$$
$w = 0$ assume, no warping



3. small angular deflections

$\theta$ small, $\cos\theta \simeq 1$, $\sin\theta \simeq \theta$

$$u = -r\sin\theta_0 . \theta = -y\alpha z = -\alpha yz$$
$$v = r\cos\theta_0 . \theta = +x.\alpha z = +\alpha xz$$
$$w = 0$$

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## Strains from displacements

2D → 3D extend

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 0, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0 \quad , \quad \epsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(-\alpha y + 0\right) = -\frac{\alpha y}{2}$$

$$\epsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\alpha x + 0\right) = \frac{\alpha x}{2}$$

strain at any point in shaft

$$\underset{\sim}{\epsilon} = \begin{pmatrix} 0 & 0 & -\alpha y/2 \\ 0 & 0 & \alpha x/2 \\ -\frac{\alpha y}{2} & \frac{\alpha x}{2} & 0 \end{pmatrix}$$



Get stresses from Hooke's Law

Isotropic. $\sigma_{xx} = 0, \ \sigma_{yy} = 0, \ \sigma_{zz} = 0$

$$\sigma_{xy} = 2G\epsilon_{xy} = 0 \qquad \mu = G = \frac{E}{2(1+\nu)}$$

$$\sigma_{xz} = 2G\epsilon_{xz} = -\cancel{2}G\frac{\alpha y}{\cancel{2}} = -G\alpha y$$

$$\sigma_{yz} = 2G\epsilon_{yz} = 2G\frac{\alpha x}{2} = G\alpha x$$

Stress / Stress tensor at any point in the shaft

$$\underset{\sim}{\sigma} = \begin{pmatrix} 0 & 0 & -G\alpha y \\ 0 & 0 & +G\alpha x \\ -G\alpha y & +G\alpha x & 0 \end{pmatrix}$$

Note: $\alpha$ unknown as of now.

Note: Hooke's Law for isotropic body in 3D

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{zz}$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\epsilon_{yz} = \frac{\sigma_{yz}}{2G} = \frac{1+\nu}{E}\sigma_{yz}$$

$$\epsilon_{zx} = \frac{\sigma_{zx}}{2G} = \frac{1+\nu}{E}\sigma_{zx}$$

Note: We use tensor shear strains in this course and not engineering shear strains

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\gamma_{xy}}{2}$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{\gamma_{yz}}{2}$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{\gamma_{zx}}{2}$$