MA 214: Introduction to numerical analysis
Lecture 15

Shripad M. Garge.
IIT Bombay

(shripad@math.iitb.ac.in)

2021-2022

Divided differences as a function

We are now considering the divided differences as a function of x by

$$f[x_0, \ldots, x_n, x].$$

Since the divided difference gives the difference between p_n and p_{n+1}, we have the following:

Let $f : [a, b] \to \mathbb{R}$, $x_0, x_1, \ldots, x_n \in [a, b]$ be given as usual and let P_n be the corresponding interpolating polynomial.

Then, for any point $x \in [a, b]$,

$$f(x) = p_n(x) + (x - x_0)(x - x_1) \cdots (x - x_n) f[x_0, x_1, \ldots, x_n, x].$$

Osculating polynomials

Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable for $r \gg 0$.

Taylor's theorem gives a polynomial $Q(x)$ at $x_0 \in [a, b]$ such that

$$\frac{d^k f(x_0)}{dx^k} = \frac{d^k Q(x_0)}{dx^k}$$

for $k = 0, \ldots, m$ for $m \leqslant r$.

Lagrange's theorem gives a polynomial P_k at distinct $x_0, \ldots, x_k$ in $[a, b]$ such that

$$f(x_i) = P_k(x_i)$$

for $i = 0, \ldots, k$.

Can we ask for both these properties in a single polynomial?

Osculating polynomials

Let f be as above, $x_0, \ldots, x_n$ be distinct nodes in $[a, b]$ and assume that for each node x_i an integer $m_i \geqslant 0$ is prescribed .

Is there a polynomial $P(x)$ such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$$

for $k = 0, \ldots, m_i$ and $i = 0, \ldots, n$?

Such a polynomial is called the osculating polynomial for f with the nodes x_i and the corresponding non-negative integers m_i.

If $n = 0$ then we have the Taylor's theorem while if $m_i = 0$ for all i then we have the Lagrange's theorem.

We are interested in the particular case of $m_i = 1$ for all i.

These are the Hermite polynomials.

Given a function f on nodes $x_0, x_1, \ldots, x_n$, the Hermite polynomial H_{2n+1} agrees with f on each x_i and, the derivative, H'_{2n+1} agrees with f' on each x_i.

Thus, at the points $(x_i, f(x_i))$, H_{2n+1} has the same shape as the function f because the tangents to f and to H_{2n+1} agree.

We will study the Hermite polynomials, see their construction and also study the corresponding errors.

Let f be given on nodes x_0 and x_1 and assume that we want to find a polynomial H such that

$$H(x_0) = f(x_0), H(x_1) = f(x_1), H'(x_0) = f'(x_0) \text{ and } H'(x_1) = f'(x_1).$$

Let L_0 and L_1 denote the Lagrange polynomials. We then define, for $i = 0, 1$:

$$H_i(x) = [1 - 2(x - x_i)L_i'(x_i)]L_i^2(x).$$

Then $H_i(x_j) = \delta_{i,j}$ and

$$H_i'(x_j) = -2L_i'(x_i)L_i^2(x_j) + [1 - 2(x_j - x_i)L_i'(x_i)]\big[2L_i(x_j)L_i'(x_j)\big] = 0.$$

We further define

$$\hat{H}_i(x) = (x - x_i)L_i^2(x).$$

Then $\hat{H}_i(x_j) = 0$ and

$$\hat{H}_i'(x_j) = L_i^2(x_j) + (x_j - x_i)2L_i(x_j)L_i'(x_j) = \delta_{i,j}.$$

Finally, we define

$$H = \left[f(x_0)H_0 + f(x_1)H_1\right] + \left[f'(x_0)\hat{H}_0 + f'(x_1)\hat{H}_1\right].$$

Note that the degree of H is $\leqslant 3$.

General case

Let f be defined on $x_0, \ldots, x_n$ and let L_i denote the Lagrange polynomials.

We define

$$H_i(x) = [1 - 2(x - x_i)L_i'(x_i)]L_i^2(x)$$

so that

$$H_i(x_j) = \delta_{i,j}$$

and

$$\begin{aligned} H_i'(x_j) &= -2L_i'(x_i)L_i^2(x_j) + [1 - 2(x_j - x_i)L_i'(x_i)]\big[2L_i(x_j)L_i'(x_j)\big] \\ &= 0. \end{aligned}$$

General case

Further

$$\hat{H}_i(x) = (x - x_i)L_i^2(x).$$

Then

$$\hat{H}_i(x_j) = 0$$

and

$$\begin{aligned} \hat{H}_i'(x_j) &= L_i^2(x_j) + (x_j - x_i)2L_i(x_j)L_i'(x_j) \\ &= \delta_{i,j}. \end{aligned}$$

Finally, we define

$$H_{2n+1}(x) = \sum_i f(x_i)H_i(x) + \sum_i f'(x_i)\hat{H}_i(x).$$

Note that the degree of H_{2n+1} is indeed $\leqslant 2n+1$.

Example

Find the Hermite polynomial that agrees with the following data:

x	$f(x)$	$f'(x)$
-1	2	-8
0	1	0
1	2	8

Here, $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Then

$$L_0(x) = \frac{1}{2}x(x-1), \quad L_1(x) = 1 - x^2, \quad L_2(x) = \frac{1}{2}x(x+1),$$

$$L_0'(x) = \frac{1}{2}(2x-1), \quad L_1'(x) = -2x, \quad L_2'(x) = \frac{1}{2}(2x+1).$$

Now, we compute the H and $\hat{H}$-polynomials.

We begin with

$$\begin{aligned} H_0(x) &= [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) \\ &= \left[1 - 2(x+1)\frac{-3}{2}\right]\frac{1}{4}x^2(x-1)^2 \\ &= \frac{(3x+4)x^2(x-1)^2}{4}. \end{aligned}$$

Similarly, $H_1(x) = (1 - x^2)^2$ and $H_2(x) = \dfrac{(-3x+4)x^2(x+1)^2}{4}$.

Further,

$$\hat{H}_0(x) = \frac{(x+1)x^2(x-1)^2}{4}, \ \hat{H}_1(x) = x(1-x^2)^2, \ \hat{H}_2(x) = \frac{(x-1)x^2(x+1)^2}{4}.$$

Example, continued

The final answer then is

$$
\begin{aligned}
H(x) &= \sum_i f(x_i)H_i(x) + \sum_i f'(x_i)\hat{H}_i(x) \\
&= 2\,\frac{(3x+4)x^2(x-1)^2}{4} + 1(1-x^2)^2 + 2\,\frac{(-3x+4)x^2(x+1)^2}{4} \\
&\quad +(-8)\,\frac{(x+1)x^2(x-1)^2}{4} + 8\,\frac{(x-1)x^2(x+1)^2}{4} \\
&= 3x^4 - 2x^2 + 1.
\end{aligned}
$$

One checks that

$$H(-1) = 2, \qquad H(0) = 1, \qquad H(1) = 2.$$

Further, $H'(x) = 12x^3 - 4x$ and hence

$$H'(-1) = -8, \qquad H'(0) = 0 \qquad H'(1) = 8.$$

MA 214: Introduction to numerical analysis
Lecture 16

Shripad M. Garge.
IIT Bombay

(shripad@math.iitb.ac.in)

2021-2022

Hermite polynomials

We are studying Hermite interpolating polynomials.

Given a function f on nodes $x_0, x_1, \ldots, x_n$, the Hermite polynomial H_{2n+1} agrees with f on each x_i and the derivative H'_{2n+1} agrees with f' on each x_i.

In fact,

$$H_{2n+1}(x) = \sum_i f(x_i) H_i(x) + \sum_i f'(x_i) \hat{H}_i(x)$$

where

$$H_i(x) = [1 - 2(x - x_i) L'_i(x_i)] L_i^2(x)$$

and

$$\hat{H}_i(x) = (x - x_i) L_i^2(x).$$

The L_i are, of course, the Lagrange polynomials for the data $(x_i, f(x_i))$.

Error for H_{2n+1}

Let $f : [a, b] \to \mathbb{R}$ be $(2n+2)$-times continuously differentiable.

Let $x_0, \ldots, x_n$ be distinct nodes in $[a, b]$ and let H_{2n+1} be the Hermite polynomial for the data $(x_i,\ f(x_i),\ f'(x_i))$.

Then for every $x \in [a, b]$ there exists $\xi(x) \in (a, b)$ such that

$$f(x) = H_{2n+1}(x) + \frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}\big(\xi(x)\big).$$

Let us check this for the example we considered in the last lecture.

Error for the example

Our data was:

x	$f(x)$	$f'(x)$
-1	2	-8
0	1	0
1	2	8

The error formula is

$$f(x) = H_5(x) + \frac{(x+1)^2x^2(x-1)^2}{6!}f^{(6)}(\xi(x)).$$

Here the function was $f(x) = x^8 + 1$, hence $f^{(6)}(x) = 20160x^2$ and we computed

$$H_5(x) = 3x^4 - 2x^2 + 1.$$

Error for the example

We compute

$$\begin{aligned} f(x) - H_5(x) &= (x^8+1) - (3x^4 - 2x^2 + 1) \\ &= x^2(x^6 - 3x^2 + 2) \\ &= x^2(x-1)^2(x^4 + 2x^3 + 3x^2 + 4x + 2) \\ &= x^2(x-1)^2(x+1)^2(x^2+2) \end{aligned}$$

Hence $\xi(x)^2 = \dfrac{x^2+2}{28}$ and

$$\xi(x) = \sqrt{\frac{x^2+2}{28}}.$$

For $x \in [-1, 1]$, $\xi(x) \in (-1, 1)$.

This is the error for our data with the function $f(x) = x^8 + 1$.

We can compute the Hermite polynomial H_{2n+1} using divided differences, which is where we see yet another application of the definition of the divided differences as a function.

Let $x_0, x_1, \ldots, x_n$ be distinct nodes and let the values of f and f' at these numbers be given.

Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by $z_{2i} = z_{2i+1} = x_i$ for all $i = 0, \ldots, n$. We then construct the table of divided differences for these nodes, z_i, with $f[z_{2i}, z_{2i+1}] = f[x_i, x_i] = f'(x_i)$.

We then compute the Hermite polynomial using the formula

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0) \cdots (x - z_{k-1}).$$

Divided differences and H_{2n+1}

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \dfrac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \dfrac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \dfrac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Our earlier example

Our data was $(-1, 2, -8)$, $(0, 1, 0)$ and $(1, 2, 8)$.

$$
\begin{array}{r|cccccc}
-1 & 2 & & & & & \\
 & & -8 & & & & \\
-1 & 2 & & 7 & & & \\
 & & -1 & & -6 & & \\
0 & 1 & & 1 & & 3 & \\
 & & 0 & & 0 & & 0 \\
0 & 1 & & 1 & & 3 & \\
 & & 1 & & 6 & & \\
1 & 2 & & 7 & & & \\
 & & 8 & & & & \\
1 & 2 & & & & &
\end{array}
$$

$$
\begin{aligned}
H_5(x) \;=\; & 2 + (-8)(x+1) \\
& + 7(x+1)^2 + (-6)(x+1)^2 x \\
& + 3(x+1)^2 x^2
\end{aligned}
$$

This gives $H_5(x) = 3x^4 - 2x^2 + 1$.

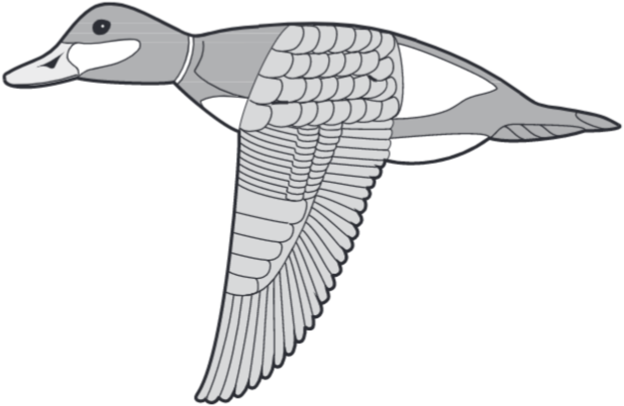
Error due to higher degree polynomials

Our study until now concerned the approximation of arbitrary functions on closed intervals using a single polynomial.

However, high-degree polynomials can oscillate erratically, that is, a minor fluctuation over a small portion of the interval can induce large fluctuations over the entire range.

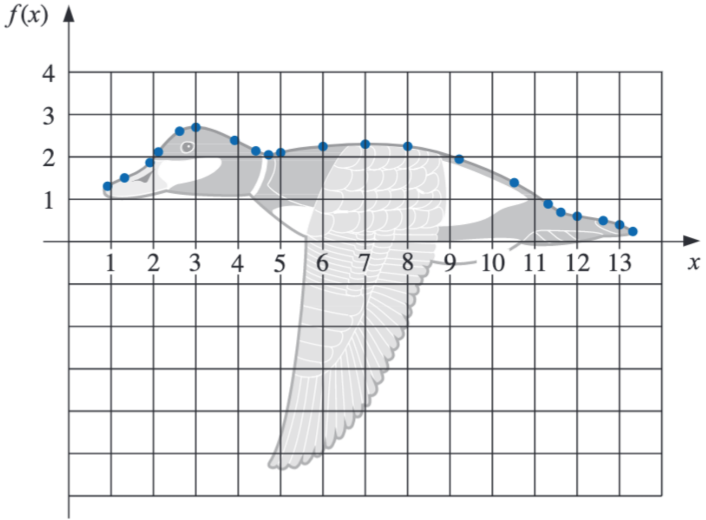
Let us see an example of this phenomenon.

A flying duck



To approximate the top profile of the duck, we choose points along the curve through which we want the approximating curve to pass.

A flying duck



More points are used when the curve is changing rapidly.

A flying duck

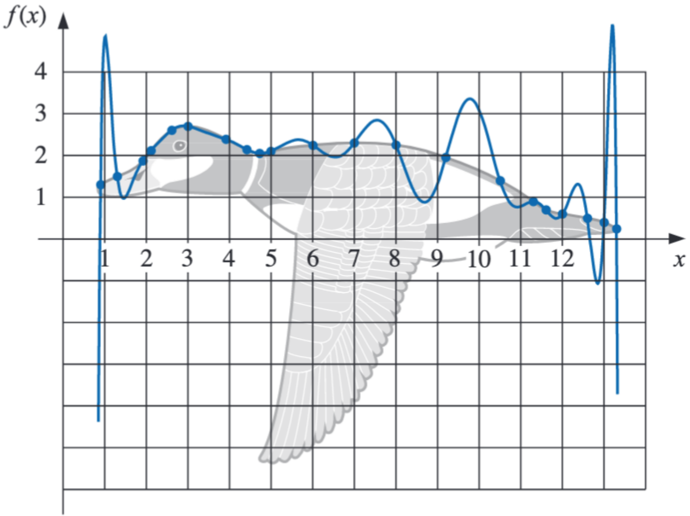
The nodes and the corresponding function values are

x	0.9	1.3	1.9	2.1	2.6	3.0	3.9
$f(x)$	1.3	1.5	1.85	2.1	2.6	2.7	2.4
x	4.4	4.7	5.0	6.0	7.0	8.0	9.2
$f(x)$	2.15	2.05	2.1	2.25	2.3	2.25	1.95
x	10.5	11.3	11.6	12.0	12.6	13.0	13.3
$f(x)$	1.4	0.9	0.7	0.6	0.5	0.4	0.25

This data gives a degree 20 interpolating polynomial.

Let us now superimpose the graph of this interpolating polynomial over the back of our flying duck.

A flying duck



It produces a very strange illustration of the back of our duck.