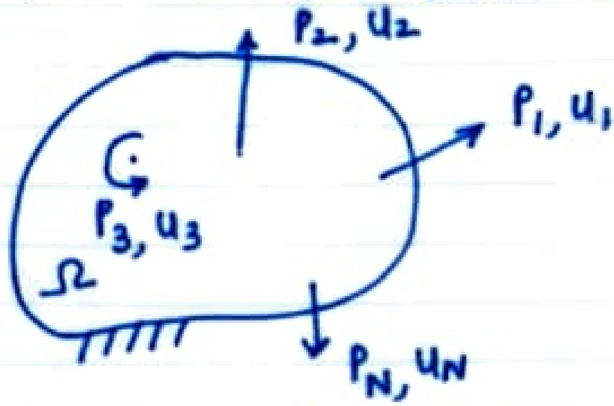
ME 202

LECTURE 37

13 APR 2022

Castigliano's Theorem II CT2 (w/o derivation)

Let Ω be a linear elastic solid **in static equilibrium**



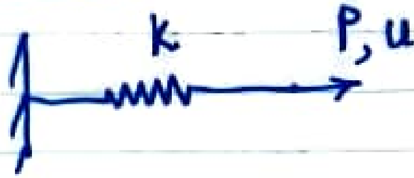
with applied forces/moments $P_1, P_2, \ldots, P_N$
and displacements $u_1, u_2, \ldots, u_N$
where in the direction of forces and at the point of application of the respective forces.

Let $U(P_1, P_2, \ldots, P_N)$ be the stored elastic energy / strain energy of the solid.

Then $$u_i = \frac{\partial U}{\partial P_i}$$

- P_i generalized force / moment
  u_i generalized displacement / angle i.e. angular displacement
- exact solution. not an approximation.
- u_i in the direction of P_i and at the point of application of P_i
- needs equilibrium / FBD analysis (see ahead)

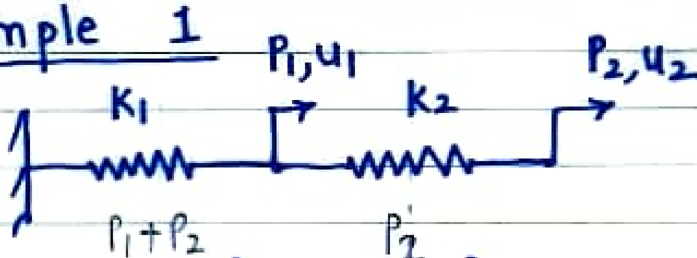
~~FEM~~ Example 0



$$U = \frac{1}{2}\frac{P^2}{k}$$

$$u = \frac{\partial U}{\partial P} = \frac{P}{k}$$

Example 1



$$U = \frac{1}{2}\frac{(P_1+P_2)^2}{k_1} + \frac{1}{2}\frac{P_2^2}{K_2}$$

$$u_1 = \frac{\partial U}{\partial P_1} = \frac{P_1+P_2}{k_1}, \quad u_2 = \frac{\partial U}{\partial P_2} = \frac{P_1+P_2}{k_1} + \frac{P_2}{k_2}$$

compare with PMPE method,

$$\Pi = \frac{1}{2}k_1(u_1)^2 + \frac{1}{2}k_2(u_2-u_1)^2 - P_1u_1 - P_2u_2$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial u_1} = 0 &\Rightarrow k_1u_1 - k_2(u_2-u_1) = P_1 \\ \frac{\partial \Pi}{\partial u_2} = 0 &\Rightarrow k_2(u_2-u_1) = P_2 \end{aligned}\right\} \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

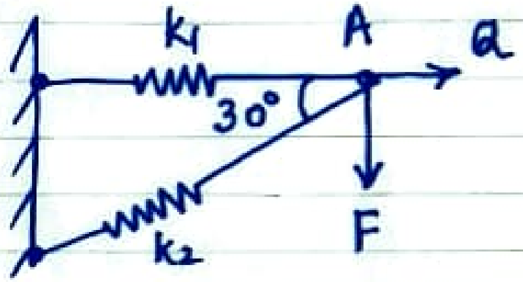
same ans as above

Note: we had to use $\underbrace{\text{equilibrium}}_{\text{FBDs}}$ to get forces in each component.

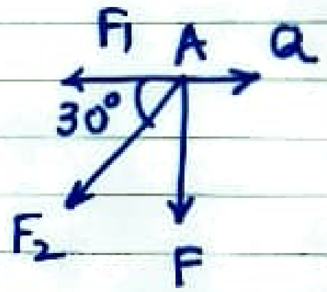
If asked to calculate disp at a point where no force is applied or in a direction where — Q

——— " ———, apply a dummy load at the appr. point in the appr. direction, use CT2 then set $Q=0$.

<u>Example 3</u>



Q not part of given problem



Get horiz. disp. of A.

F_1, F_2 tensile forces in springs k_1, k_2 resply.

$$F_1 + F_2 \cos 30^\circ = Q, \quad F_2 \sin 30^\circ + F = 0$$

$$\Rightarrow F_1 = Q + F\sqrt{3}, \quad F_2 = -2F$$

$$U(F,Q) = \frac{1}{2}\frac{F_1^2}{k_1} + \frac{1}{2}\frac{F_2^2}{k_2}$$

$$= \frac{1}{2}\frac{(Q+F\sqrt{3})^2}{k_1} + \frac{1}{2}\frac{(-2F)^2}{k_2}$$

$$\text{Horiz. disp @ } A = \frac{\partial U}{\partial Q} = \frac{Q+F\sqrt{3}}{k_1}$$

Now set $Q=0$,

$$\text{horiz. disp @ } A = \frac{\sqrt{3}\,F}{k_1}$$

$$\text{vert. disp @ } A = \frac{\partial U}{\partial F} = \left(\frac{Q+F\sqrt{3}}{k_1}\right)\sqrt{3} + \frac{4F}{k_2}$$

$$= \left(\frac{3}{k_1} + \frac{4}{k_2}\right)F \quad \text{after setting } Q=0.$$

Same problem using PMPE for comparison.

Example 4

$$M(z) = M_0 + P(L-z)$$

strain energy density $= \frac{1}{2}\sigma\epsilon$
(J/m^3)

$$U = \int_{\Omega} \frac{1}{2}\sigma\epsilon \, dV = \int_0^L dz \int_A \frac{1}{2}\sigma\epsilon \, da$$

$$\sigma = -\frac{Mx}{I}, \quad \epsilon = -\frac{Mx}{EI}$$

$$U = \int_0^L dz \frac{M^2}{2EI^2} \underbrace{\int_A x^2 \, da}_{I}$$

$$U = \int_0^L \frac{M^2 dz}{2EI} \quad \text{in general for bending}$$

in this case

$$U = \int_0^L \frac{1}{2EI}\left(M_0 + P(L-z)\right)^2 dz$$

$$= \left(P^2L^3 + 3L^2 M_0 P + 3M_0^2 L\right)/6EI$$

$$\frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{M_0 L^2}{2EI}$$ deflection at P, along P

$$\frac{\partial U}{\partial M_L} = \frac{PL^2}{2EI} + \frac{M_0 L}{EI}$$ deflection (angle) at M_0, along M_0

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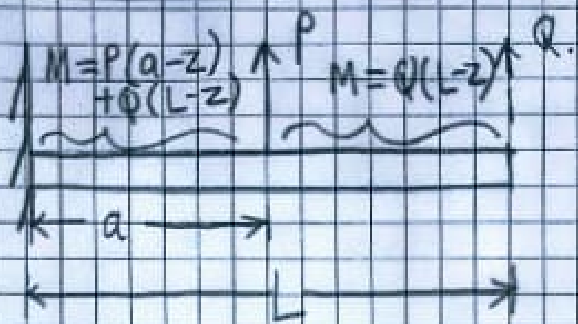
Castigliano Thm

Example 5.1



Cantilever. Load P applied at a. want deflection at L.

Apply dummy virtual load Q at L.

$$U = \int_0^L \frac{M^2}{2EI}\,dz$$

$$= \int_0^a \frac{(P(a-z)+Q(L-z))^2}{2EI}\,dz + \int_a^L \frac{(Q(L-z))^2}{2EI}\,dz$$

$$= \frac{(P+Q)^2 a^3}{6EI} + \frac{a(Pa+LQ)^2}{2EI} - \frac{a^2(P+Q)(Pa+LQ)}{2EI}$$

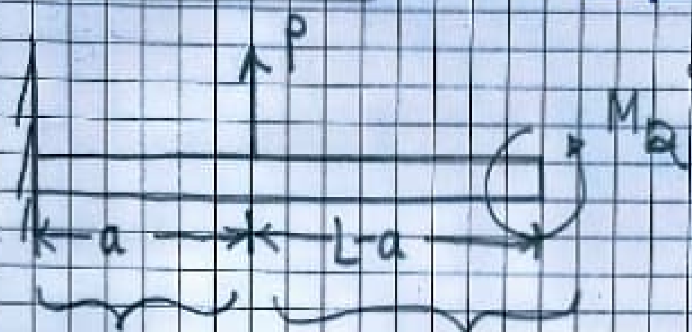
$$\frac{\partial U}{\partial Q} = \frac{a}{6EI}\left(6L^2Q - Pa^2 + 2Qa^2 + 3LPa - 6LQa\right)$$

$$\delta_L = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} = \frac{Pa^2}{6EI}(3L-a)$$

want slope at L

Example 5.2

virtual dummy moment



$$M_1 = P(a-z) + M_Q \qquad M_2 = M_Q.$$

$$U = \int_0^a \frac{M_1^2}{2EI}\,dz + \int_a^L \frac{M_2^2}{2EI}\,dz$$

$$= \frac{a\left(3M_Q^2 + 3M_Q Pa + P^2a^2\right)}{6EI} + \frac{M_Q^2\,(L-a)}{2EI}$$

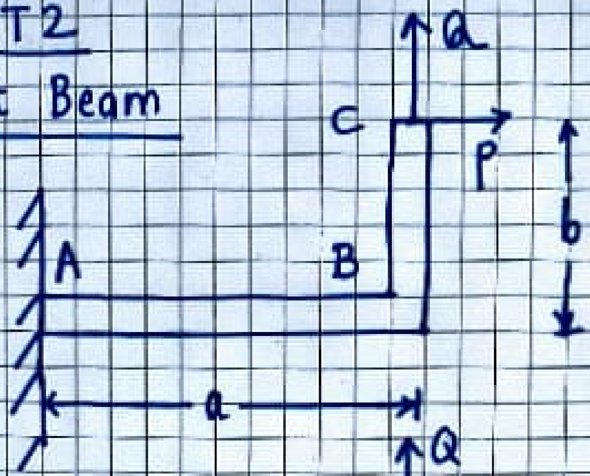
$$\frac{\partial U}{\partial M_Q} = \frac{Pa^2 + 2L\,M_Q}{2EI}$$

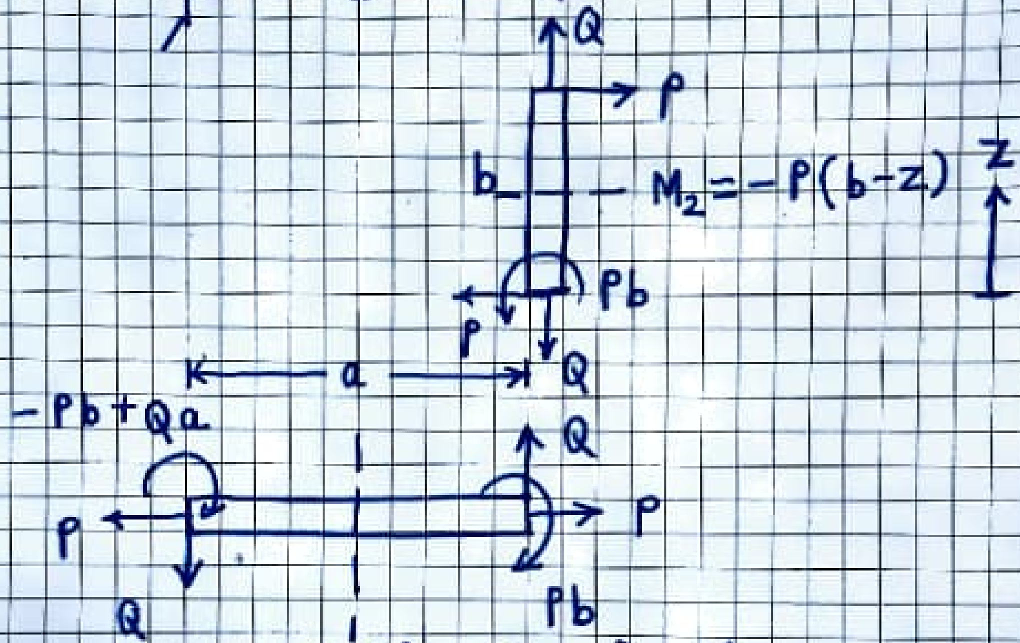
$$\theta_L = \left.\frac{\partial U}{\partial M_Q}\right|_{M_Q=0} = \frac{Pa^2}{2EI}$$

CT2

Bent Beam

Example 6





$$M_1(z) = Q(a-z) - Pb$$

$$U = \int_0^a \frac{M_1^2}{2EI}\,dz + \int_0^b \frac{M_2^2}{2EI}\,dz$$

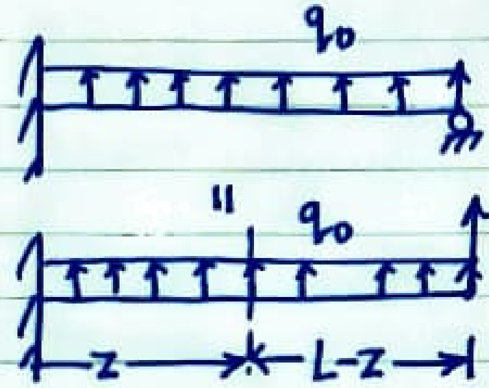
$$= \frac{P^2ab^2}{2EI} - \frac{PQa^2b}{2EI} + \frac{Q^2a^3}{6EI} + \frac{P^2b^3}{6EI}$$

Disp at Q in dir of Q $= \dfrac{\partial U}{\partial Q} = \dfrac{Qa^3}{3EI} - \dfrac{Pba^2}{2EI}$

Disp at P in dir of P $= \dfrac{\partial U}{\partial P} = \dfrac{Pb^3}{3EI} + \dfrac{Pab^2}{2EI} - \dfrac{Qa^2b}{2EI}$

Example 47

Obtain reaction of a statically indeterminate beam.



R = reaction at roller

$$M(z) = R(L-z) + q_0(L-z)\frac{(L-z)}{2}$$

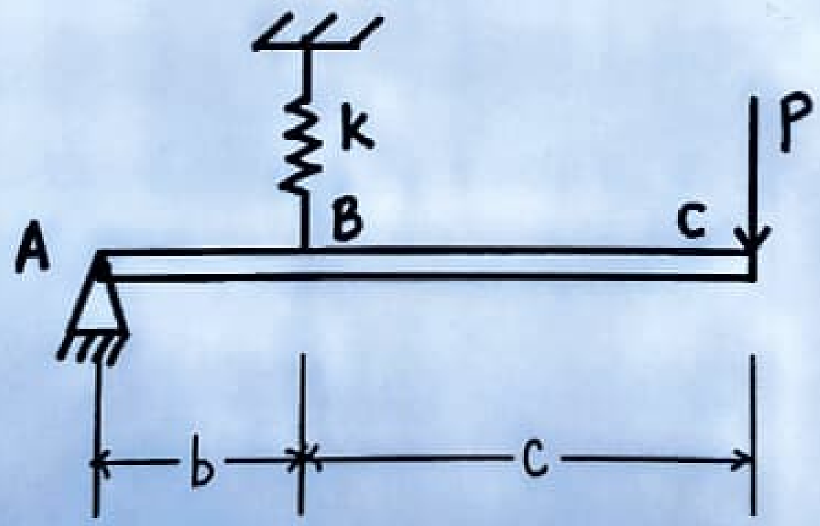
$$U = \int_0^L \frac{1}{2EI} M^2(z)\,dz$$

$$= \frac{L^3}{120EI}\left(3L^2 q_0^2 + 15LRq_0 + 20R^2\right)$$

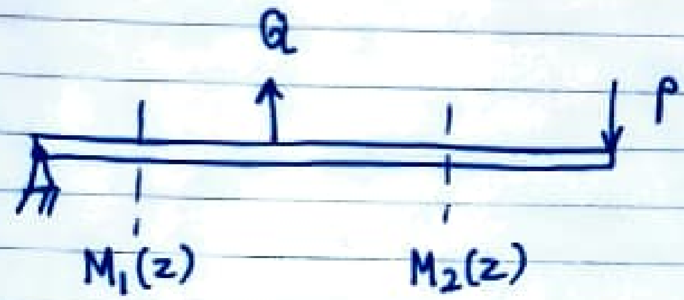
$$\frac{\partial U}{\partial R} = q_0\frac{L^4}{8EI} + \frac{RL^3}{3EI} = 0 \quad \text{as roller @ } z = L$$

$$\Rightarrow R = \frac{-3Lq_0}{8}$$

3. (8 points) A beam ABC with flexural rigidity EI is simply supported at A and held by a linear spring with stiffness k at B. A load P acts at the free end C. Find the deflection at C due to the applied load.



Solution by Castigliano Thm — Example 8



$L = b + c$

$$M_1(z) = -P(L-z) + Q(L-z-c)$$

$$M_2(z) = -P(L-z)$$

$$Qb = P(b+c)$$

$$U(P) = \int_0^b \frac{M_1^2}{2EI}\,dz + \int_b^{b+c} \frac{M_2^2}{2EI}\,dz + \frac{1}{2}\frac{Q^2}{K}$$

$$= \frac{P^2c^3}{6EI} + \frac{P^2c^2b}{6EI} + \frac{P^2(b+c)^2}{2b^2K}$$

$$\delta_c = \frac{\partial U}{\partial P}$$

$$= \frac{Pc^3}{3EI} + \frac{Pbc^2}{3EI} + \frac{P(b+c)^2}{b^2K}$$

$$= P(b+c)\left[\frac{c^2}{3EI} + \frac{b+c}{b^2K}\right]$$ ✓