MA 214: Introduction to numerical analysis
Lecture 9

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A correction

In the third lecture, we did a computation using 2-digit arithmetic for $f(x) = x^3 + x^2 + x + 1$ at $x = 1.876$.

	x	x^2	x^3	$f(x)$
Exact	1.876	3.519376	6.602349376	12.997725376
3-d ch.	0.187×10^1	0.349×10^1	0.652×10^1	0.128×10^2
3-d ro.	0.188×10^1	0.353×10^1	0.664×10^1	0.13×10^2

This is actually 3-digit arithmetic. The absolute and the relative errors will also have to be calculated afresh.

I am sorry for this confusion and would like to thank Shreya for pointing it out to me.

Order of the fixed point iteration method

Consider the root-finding problem $g(x) = 0$ with $x = p$ as a solution.

Assume that it is converted to a fixed point problem with the function $f(x)$.

If $f'(p) \neq 0$ then the order of the fixed point iteration method is linear.

If $f'(p) = 0$ then the order of the fixed point iteration method is quadratic or higher.

If $x = p$ is a simple zero of g then Newton-Raphson works well to give quadratic convergence.

If $x = p$ is not a simple zero of g then Newton-Raphson may not give quadratic convergence.

Modified Newton-Raphson

Let $g : [a, b] \to \mathbb{R}$ be a function and let $x = p$ be a root of f.

Define $\mu(x) = \dfrac{g(x)}{g'(x)}$.

If the order of $x = p$ as a zero of g is m then $g(x) = (x - p)^m q(x)$, $g'(x) = m(x - p)^{m-1} q(x) + (x - p)^m q'(x)$ and hence

$$\mu(x) = (x - p) \ \frac{q(x)}{mq(x) + (x - p)q'(x)}.$$

Thus $x = p$ is a simple zero of $\mu(x)$.

Further, if $g(x)$ has no other zero in a neighbourhood of $x = p$ then $\mu(x)$ will also not have any other zero in that neighbourhood.

We now apply Newton-Raphson method to $\mu(x)$.

Newton-Raphson method for $\mu(x)$

We now define a fixed point iteration $p_{n+1} = f(p_n)$ where

$$\begin{aligned} f(x) &= x - \frac{\mu(x)}{\mu'(x)} \\ &= x - \frac{g(x)/g'(x)}{\left(g'(x)^2 - g(x)g''(x)\right)/g'(x)^2} \\ &= x - \frac{g(x)g'(x)}{\left(g'(x)^2 - g(x)g''(x)\right)} \end{aligned}$$

This iteration will converge to p with at least the quadratic order of convergence.

The only theoretical drawback with this method is that we now need to compute $g''(x)$ also at each step. Computationally, the denominator of the formula involves cancelling two nearly equal terms.

An earlier example

We had considered the function $g(x) = e^x - x - 1$ which has a zero at $x = 0$ of multiplicity 2.

The modified N-R method gives

$$\begin{aligned} p_1 &= p_0 - \frac{g(p_0)g'(p_0)}{g'(p_0)^2 - g(p_0)g''(p_0)} \\ &= 1 - \frac{(e-2)(e-1)}{(e-1)^2 - (e-2)e} \\ &= -2.3421061 \times 10^{-1} \end{aligned}$$

This is closer to 0 than the first term for Newton-Raphson, which was 0.58918.

The earlier example continued

n	p_n
0	1.0
1	0.58198
2	0.31906
3	0.16800
4	0.08635
5	0.04380
6	0.02206
7	0.01107
8	0.005545

n	p_n
1	$-2.3421061 \times 10^{-1}$
2	$-8.4582788 \times 10^{-3}$
3	$-1.1889524 \times 10^{-5}$
4	$-6.8638230 \times 10^{-6}$
5	$-2.8085217 \times 10^{-7}$

The table on the right gives values for the modified Newton-Raphson iteration method.

Another example

Of course, if $x = p$ is a simple zero for $g(x)$ then also the modified Newton-Raphson method will work well.

We used various fixed point iteration methods earlier to find a zero for $g(x) = x^3 + 4x^2 - 10$, namely $x = 1.36523001$.

The Newton-Raphson method is

$$p_n = p_{n-1} - \frac{p_{n-1}^3 + 4p_{n-1}^2 - 10}{3p_{n-1}^2 + 8p_{n-1}}$$

and the modified one is

$$p_n = p_{n-1} - \frac{\left(p_{n-1}^3 + 4p_{n-1}^2 - 10\right)\left(3p_{n-1}^2 + 8p_{n-1}\right)}{\left(3p_{n-1}^2 + 8p_{n-1}\right)^2 - \left(p_{n-1}^3 + 4p_{n-1}^2 - 10\right)\left(6p_{n-1} + 8\right)}.$$

The example continued

With the initial approximation $p_0 = 1$ we have

Newton-Raphson method:

$$p_1 = 1.37333333, \quad p_2 = 1.3652620, \quad p_3 = 1.36523001.$$

Modified Newton-Raphson method:

$$p_1 = 1.35689898, \quad p_2 = 1.3651958, \quad p_3 = 1.36523001.$$

Both the methods work fine but, of course, the second method requires substantially more calculations.

Any other methods?

There are many methods, some of them improve the order of convergence.

Suppose that $\{p_n\}$ converges to p linearly.

For large enough n, we have $(p_{n+1} - p)^2 \approx (p_n - p)(p_{n+2} - p)$ which further gives

$$p \approx p_n - \frac{\left(p_{n+1} - p_n\right)^2}{p_{n+2} - 2p_{n+1} + p_n} = \hat{p}_n.$$

This is called Aitken's Δ^2-method of accelerating convergence.

The sequence $\{p_n\}$, $p_n = \cos\left(\frac{1}{n}\right)$, converges linearly to 1. We apply Aitken's Δ^2-method and note the results in the following table.

Aitken's Δ^2-method

n	p_n	$\hat{p}_n$
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

We see that $\hat{p}_n \to 1$ more rapidly than p_n.

The second theme is complete

This brings us to the end of the second theme of our course:

Equations in one variable.

From our next lecture, we will begin the next theme:

Interpolation.

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Why interpolate?

This is the Census data of our nation:

Year	Population
1951	361,088,090
1961	438,936,918
1971	547,949,809
1981	685,184,692
1991	838,583,988
2001	1,028,737,436
2011	1,210,193,422

If we wanted to know an estimate of the population in, say 1983, then we find a function $f(x)$ that fits the given data and compute the value, $f(1983)$.

Polynomials

Polynomials are one of the most useful functions on earth, these are the functions of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

One reason for their importance is that they uniformly approximate continuous functions.

Given any continuous function, $f : [a, b] \to \mathbb{R}$, there exists a polynomial that is as "close" to the given function as desired.

This result is the Weierstrass approximation theorem.

Another reason to prefer polynomials is that P' is again a polynomial and, in general, computations with polynomials are easier.

Taylor polynomials

One natural choice for polynomials to approximate given functions would be to consider the polynomials given by Taylor's theorem, mentioned in Lecture 01.

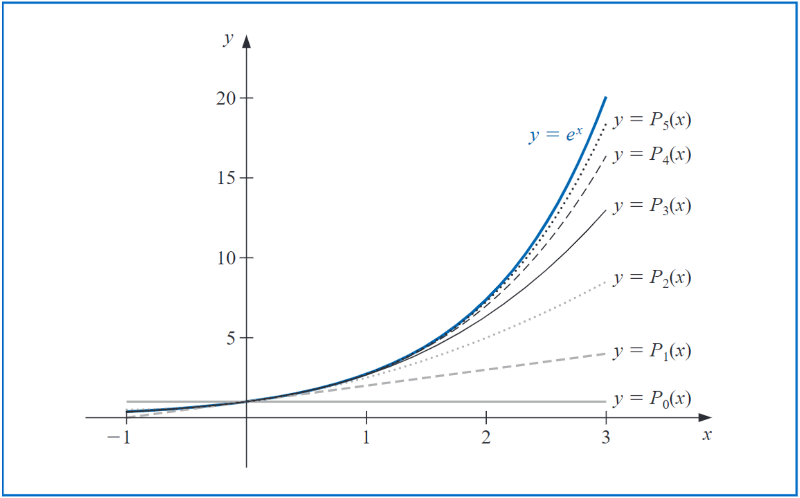
But these polynomials approximate the function only at a point. For example, consider $f(x) = e^x$ and the first 6 Taylor polynomials for f at $x = 0$:

$$P_0(x) = 1, \quad P_1(x) = 1 + x, \quad P_2(x) = 1 + x + \frac{x^2}{2},$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}, \qquad P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!},$$

$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

Taylor polynomials



Taylor polynomials

It may seem that the Taylor polynomials of higher degree approximate the function better, but this is not true for all functions.

For instance, for $f(x) = \frac{1}{x}$, the Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

If we approximate $f(3) = 1/3$ by $P_n(3)$ then:

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

Taylor polynomials

One reason for this failure is that the only information that is used in computing the Taylor polynomials is the value of f and its various derivatives at the point x_0.

For ordinary computational purposes it is more efficient to use methods that include information at various points.

We will do that in the remaining part of this theme.

The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

Lagrange interpolating polynomials

Let f be a function with $f(x_0) = y_0$.

Is there a polynomial $P(x)$ with $P(x_0) = y_0$?

It would be simplest to take the constant polynomial: $P(x) = y_0$.

This is the function which takes the same value at every x.

If we have two points x_0 and x_1 with $f(x_0) = y_0$, $f(x_1) = y_1$ then clearly the constant function will not always work.

Let us search among the linear polynomials, the ones of the form

$$a_1 x + a_0.$$

Lagrange interpolating polynomials

We then have $a_1 x_0 + a_0 = y_0$ and $a_1 x_1 + a_0 = y_1$.

We solve these two linear equations in two unknowns to get

$$a_1 = \frac{y_0 - y_1}{x_0 - x_1}, \qquad a_0 = \frac{y_1 x_0 - y_0 x_1}{x_0 - x_1}$$

which further gives the interpolating polynomial

$$\frac{x - x_1}{x_0 - x_1}\, y_0 \; + \; \frac{x - x_0}{x_1 - x_0}\, y_1 .$$

The only thing we require here is that $x_0 \neq x_1$.

We generalise this problem and its solution in the next slides.

Lagrange interpolating polynomials

Let $x_0, x_1, \ldots, x_n$ be distinct $(n+1)$-points and let f be a function with $f(x_0) = y_0$, $f(x_1) = y_1$, $\ldots$, $f(x_n) = y_n$. We want to find a polynomial P with $P(x_i) = y_i$ for $i = 0, \ldots, n$.

We first solve $(n+1)$ special problems. We find polynomials $L_{n,i}$ with

$$L_{n,i}(x_j) = \delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

For a fixed i, $L_{n,i}(x_j) = 0$ for $j \neq i$. So $(x - x_j)$ divides $L_{n,i}(x)$ for each $j \neq i$. Since the points x_i are all distinct, we have that the product

$$(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)$$

divides $L_{n,i}(x)$.

Lagrange interpolating polynomials

If we define

$$L_{n,i}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

then $L_{n,i}(x_j) = \delta_{i,j}$.

Once these special problems are solved then the general polynomial is easily found by

$$P(x) = y_0 L_{n,0}(x) + y_1 L_{n,1}(x) + \cdots + y_n L_{n,n}(x).$$

We will check whether these are the same polynomials that we got in the early cases.

Case 0. $n = 0$, here we have $L_{0,0}(x) = 1$ and hence $P(x) = y_0$.

Case 1. $n = 1$, $L_{1,0}(x) = \dfrac{x - x_1}{x_0 - x_1}$ and $L_{1,1}(x) = \dfrac{x - x_0}{x_1 - x_0}$.

Then

$$P(x) = \frac{x - x_1}{x_0 - x_1}\, y_0 \;+\; \frac{x - x_0}{x_1 - x_0}\, y_1.$$

Case 2. $n = 2$,

$$P(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\, y_0 \;+\; \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\, y_1 \;+$$

$$\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\, y_2.$$

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Lagrange interpolation

We are studying interpolating functions by polynomials.

If $x_0, x_1, \ldots, x_n$ are distinct $(n+1)$-points and $f(x_i) = y_i$ for $i = 0, \ldots, n$ then Lagrange gives a polynomial P of degree at most n with the property that $f(x_i) = P(x_i)$ for all i as follows:

$$P(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

where

$$L_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

Let us do some examples.

1.) Let $f(2) = 4$ and $f(5) = 1$.

Find a polynomial P interpolating f at these two points.

Here $x_0 = 2$, $y_0 = 4$, $x_1 = 5$ and $y_1 = 1$, so

$$L_0 = \frac{x-5}{2-5} = -\frac{1}{3}(x-5), \qquad L_1 = \frac{x-2}{5-2} = \frac{1}{3}(x-2).$$

This gives,

$$P = -\frac{4}{3}(x-5) + \frac{1}{3}(x-2) = -x + 6.$$

We were working with two points, x_0 and x_1, and our polynomial is a linear polynomial.

2.) Let $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$.

Find a polynomial interpolating $f(x) = 1/x$ at these three points.

We first determine L_0, L_1 and L_2:

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4),$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4),$$

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75),$$

then

$$P(x) = \frac{1}{2}L_0(x) + \frac{1}{2.75}L_1(x) + \frac{1}{4}L_2(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.$$

Examples

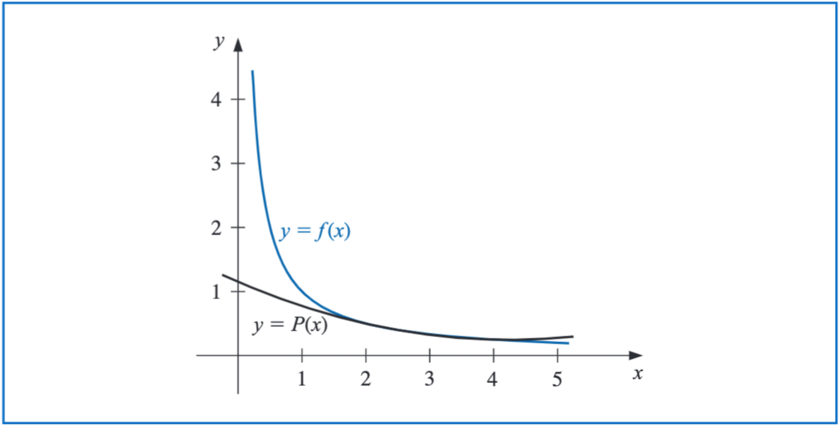
We had three points and the interpolating polynomial is of degree 2.

The value of the interpolating polynomial P at $x = 3$ is

$$P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = 0.329545454\cdots \approx 0.32955.$$

This is much better than the values of the Taylor polynomials.

We will see the graphs of $f(x)$ and $P(x)$ in the next slide to see that already for $n = 2$, $P(x)$ approximates $f(x)$ well-enough.


y
4
3
2
1
y = f(x)
y = P(x)
1
2
3
4
5
x

3.) Let us interpolate $f(1) = 1$ and $f(2) = 1$.

The Lagrange polynomials are

$$L_0(x) = \frac{x-2}{1-2} = -(x-2), \qquad L_1(x) = \frac{x-1}{2-1} = (x-1)$$

and the interpolating polynomial is

$$P(x) = -(x-2)\cdot 1 + (x-1)\cdot 1 = 1.$$

The interpolating polynomial is a constant polynomial!

Even though, we had two points, we do not get a linear polynomial but a constant one.

In general, for $(n+1)$-points, the interpolating polynomial will have degree at most n.

Uniqueness of the interpolating polynomial?

How many polynomials can we get to interpolate the given function at $(n+1)$-points?

Many, in fact, infinitely many!

But, if we put a restriction on the degree to be $\leqslant n$ then there will be a unique one.

Theorem

A polynomial of degree n has at most n distinct zeroes.

That is, a polynomial of degree $\leqslant n$ with $(n+1)$ zeroes is the zero polynomial.

This is why we got the constant polynomial in the last example above.

Error by the interpolating polynomial

Theorem

Let $f : [a, b] \to \mathbb{R}$ be $(n+1)$-times continuously differentiable.

Let $P(x)$ be the polynomial interpolating f at distinct $(n+1)$ points $x_0, x_1 \ldots, x_n \in [a, b]$.

Then, for each $x \in [a, b]$, there exists $\xi(x) \in (a, b)$ with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n).$$

We will use Lagrange polynomials in numerical differentiation and numerical integration, so this error formula will be very useful then.

An earlier example

We earlier computed the interpolating $P(x)$ for $f(x) = 1/x$ at $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$.

The error for this polynomial is

$$\frac{f'''(\xi(x))}{3!}(x-2)(x-2.75)(x-4).$$

For $f(x) = x^{-1}$, we have $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$ and $f'''(x) = -6x^{-4}$.

The error form then becomes $-\big(\xi(x)\big)^{-4}(x-2)(x-2.75)(x-4)$.

Let us compute the maximum possible error on $[2, 4]$, by that we mean the maximum absolute value of the error.

The example continued

The maximum value for $\big(\xi(x)\big)^{-4}$ on $[2,4]$ is $2^{-4} = 1/16$.

We now compute the maximum value of the polynomial

$$Q(x) = (x-2)(x-2.75)(x-4).$$

We have $Q'(x) = 3x^2 - 17.5x + 24.5$, the critical points then are

$$x = \frac{7}{3},\ x = \frac{7}{2} \quad \text{with} \quad Q\left(\frac{7}{3}\right) = \frac{25}{108},\ Q\left(\frac{7}{2}\right) = -\frac{9}{16}.$$

The maximum absolute value of the error on $[2,4]$ is less than or equal to

$$\frac{1}{16}\cdot\frac{9}{16} = \frac{9}{256} = 0.03515625.$$

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Error form for Lagrange polynomials

If $f(x) : [a, b] \to \mathbb{R}$ is $(n+1)$-times differentiable then for a polynomial P interpolating f at $x_0, x_1, \ldots, x_n$ the error term is given by

$$f(x) - P(x) = \frac{f^{(n+1)}\big(\xi(x)\big)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $\xi(x)$ depends on x.

We can use this formula to prepare a data for a function within the prescribed error.

Example

Question

Assume that we want to prepare a table for values of

$$f(x) = e^x : [0,1] \to \mathbb{R}.$$

Let the nodes be $x_0 = 0, x_1 = h, \ldots, x_i = ih, \ldots.$

We interpolate the function between each pair of consecutive points, x_i *and* x_{i+1}*, by a linear polynomial.*

What step size h *will ensure that this interpolation gives an absolute error of at most* 10^{-4} *for all* $x \in [0,1]$*?*

Choose j such that $x_j \leqslant x \leqslant x_{j+1}$ and let P be the linear polynomial interpolating f on $[x_j, x_{j+1}]$. Then

$$\begin{aligned}
|f(x) - P(x)| &= \left| \frac{f^{(2)}(\xi)}{2}(x - x_j)(x - x_{j+1}) \right| \\
&= \frac{|f^{(2)}(\xi)|}{2} |(x - jh)(x - (j+1)h)| \\
&\leqslant \frac{\max_{\xi \in [0,1]} e^{\xi}}{2} \max_{x_j \leqslant x \leqslant x_{j+1}} |(x - jh)(x - (j+1)h)| \\
&= \frac{e}{2} \max_{jh \leqslant x \leqslant (j+1)h} |(x - jh)(x - (j+1)h)|
\end{aligned}$$

We now find the $\max_{jh \leqslant x \leqslant (j+1)h} |(x - jh)(x - (j+1)h)|$.

Example

If $g(x) = (x - jh)(x - (j+1)h)$ then

$$g'(x) = (x - (j+1)h) + (x - jh) = 2\left(x - jh - \frac{h}{2}\right).$$

The only critical point of g is $x = jh + h/2$ with the value

$$g(jh + h/2) = -(h/2)^2 = -h^2/4.$$

This is the maximum value of $|g|$ on $[jh, (j+1)h]$ as g is zero on the end-points of the interval. Then

$$|f(x) - P(x)| \leqslant \frac{e}{2}\frac{h^2}{4} = \frac{eh^2}{8}.$$

For the error to be $\leqslant 10^{-4}$, we must take $h \leqslant 0.0171$, so $h = 1/59$ will work.

A practical difficulty

In the above example, we did not need the explicit linear polynomial interpolating the function, however, we made full use of the information about f.

Typically, we will have only have some data and we will need to work with that without any information of the function f:

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Let us say that we want to estimate $f(1.5)$ by the various interpolating polynomials.

A practical difficulty

Here $1.0 \leqslant 1.3 \leqslant \mathbf{1.5} \leqslant 1.6 \leqslant 1.9 \leqslant 2.2$.

We have a linear polynomial P_1 interpolating f on $[1.3, 1.6]$ with $P_1(1.5) = 0.5102968$.

There are two quadratic polynomials P_2 and Q_2 interpolating f on $[1, 1.6]$ and on $[1.3, 1.9]$ with

$$P_2(1.5) = 0.5124715 \quad \text{and} \quad Q_2(1.5) = 0.5112857.$$

Similarly, there are two cubic polynomials P_3 and Q_3 interpolating f on $[1, 1.9]$ and on $[1.3, 2.2]$ with

$$P_3(1.5) = 0.5118127 \quad \text{and} \quad Q_3(1.5) = 0.5118302.$$

Finally, there is the quartic polynomial P_4 using the whole data with $P_4(1.5) = 0.5118200$.

A practical difficulty

Since the last three values agree to 4 decimal places, we can take them to be reasonable approximations to $f(1.5)$.

The value $P_4(1.5) = 0.5118200$ can be taken to be the most accurate one, since it is almost the average of the other two "correct" values and also because P_4 uses the whole data.

One practical difficulty with these computations is that the computations of the smaller degree interpolating polynomials did not quite help in computing those of higher degree.

We would like to find a method that helps in computing the interpolating polynomials cumulatively.

Cumulative calculation of interpolating polynomials

Let us assume that f is given on distinct nodes $x_0, x_1, \ldots, x_n$.

The constant polynomial for the node x_0 will be $P_0(x) = f(x_0)$ and that for the node x_1 will be $Q_0(x) = f(x_1)$.

The linear polynomial for the nodes x_0 and x_1 is

$$\begin{aligned} P_1(x) &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \\ &= \frac{(x - x_1)P_0(x) - (x - x_0)Q_0(x)}{(x_0 - x_1)} \end{aligned}$$

It is difficult to build the general case from this calculation. We will do the quadratic case also.

Cumulative calculation of interpolating polynomials

The linear polynomial for nodes x_0 and x_1 is

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

while the one for nodes x_1 and x_2 is

$$Q_1(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2).$$

The quadratic polynomial for nodes x_0, x_1, x_2 is

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Cumulative calculation of interpolating polynomials

Note that

$$
\begin{aligned}
P_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) \\
&\quad + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \\
&= \frac{(x-x_2)}{(x_0-x_2)} \left[\frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1) \right] \\
&\quad - \frac{(x-x_0)}{(x_0-x_2)} \left[\frac{(x-x_2)}{(x_1-x_2)} f(x_1) + \frac{(x-x_1)}{(x_2-x_1)} f(x_2) \right] \\
&= \frac{(x-x_2)P_1(x) - (x-x_0)Q_1(x)}{(x_0-x_2)}
\end{aligned}
$$

Cumulative calculation of interpolating polynomials

The polynomial P_0 interpolated f on $\{x_0\}$ and Q_0 interpolated f on $\{x_1\}$, then P_1 interpolating f on $\{x_0, x_1\}$ is given by

$$P_1(x) = \frac{(x - x_1)P_0(x) - (x - x_0)Q_0(x)}{(x_0 - x_1)}.$$

Further, P_1 interpolated f on $\{x_0, x_1\}$ and Q_1 interpolated f on $\{x_1, x_2\}$, then P_2 interpolating f on $\{x_0, x_1, x_2\}$ is given by

$$P_2(x) = \frac{(x - x_2)P_1(x) - (x - x_0)Q_1(x)}{(x_0 - x_2)}.$$

If P_2 interpolates f on $\{x_0, x_1, x_2\}$, Q_2 on $\{x_1, x_2, x_3\}$ and P_3 on $\{x_0, x_1, x_2, x_3\}$ then do we get

$$P_3(x) = \frac{(x - x_3)P_2(x) - (x - x_0)Q_2(x)}{(x_0 - x_3)}?$$