ME 202

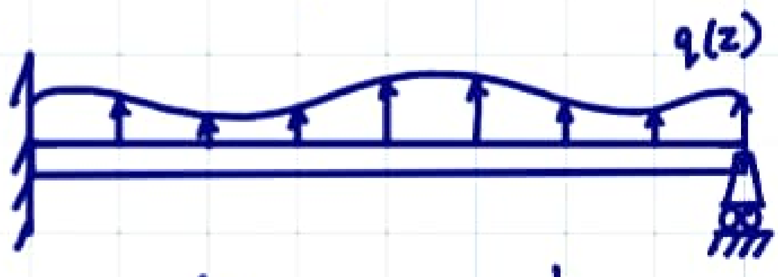
LECTURE 24

MON 14 MAR 2022

## Principle of Minimum Potential Energy PMPE

Goal:
- ODE $\longrightarrow$ Linear Algebraic Equations
- Approximate solutions

FEM
- Computationally efficient algos




0. $$\Pi[u(z)] = \int_0^L \frac{EI}{2}\left(\frac{d^2u}{dz^2}\right)^2 dz - \int_0^L q(z)u(z)\,dz$$

$$\Pi = \frac{1}{2}ku^2 + (-Pu)$$

BCs Boundary Conditions

Kinematic KBCs
slope
deflection
NON-NEGOTIABLE

Natural NBCs
shear force
bending moment

1. Approximate $u(z)$ by standard functions (polys, trigs, expos, etc.) with unknown coefficients
2. Ensure that approx obeys KBCs
3. This will give N unknown coefficients
   N = degrees of freedom $a_1, a_2, a_3, \ldots, a_N$
4. Plug this approx into $\Pi[u]$ and integrate
5. PMPE $\Rightarrow$ find $a_1, a_2, a_3$, etc to min $\Pi$
   i.e. $\frac{\partial \Pi}{\partial a_1} = 0, \frac{\partial \Pi}{\partial a_2} = 0, \ldots, \frac{\partial \Pi}{\partial a_N} = 0$

N equations for N unknowns

6. Get approx $u(z)$ in terms of N obtained coefficients

## Example 1


$u(0)=0$
$u'(0)=0$

$$\Pi = \int_0^L \frac{EI}{2} u''^2 \, dz - Q\,u(L) - M \underbrace{\theta(L)}_{u'(L)}$$

Approx

$$u = a_0 + a_1 z + a_2 z^2$$

$\text{KBCs} \Rightarrow a_0 = 0,\ a_1 = 0$

$u = a_2 z^2 \qquad N = 1 \qquad \text{SDOF}$

Use $\quad u = a_2 \left(\frac{z}{L}\right)^2, \quad u' = \frac{2a_2 z}{L^2}$

$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{2a_2}{L^2}\right)^2 dz - Q\,a_2 \;-\; M\frac{2a_2}{L}$$

$$\Pi = \frac{2EI}{L^3}\,a_2^2 - Qa_2 - M\frac{2a_2}{L}$$

$$\frac{\partial \Pi}{\partial a_2} = 0 \Rightarrow a_2 = \frac{L^2}{4EI}(2M + QL)$$

$$u(z) = \frac{L^2}{4EI}(2M + QL)\left(\frac{z}{L}\right)^2$$

$$u(L) = \frac{QL^3}{4EI} + \frac{ML^2}{2EI} \quad \text{approx.}$$

$$u(L) = \frac{QL^3}{3EI} + \frac{ML^2}{2EI} \quad \text{exact.}$$

Improve,

$$u = a_2\left(\frac{z}{L}\right)^2 + a_3\left(\frac{z}{L}\right)^3$$

$$\Pi = \frac{2EI}{L^3}\left(a_2^2 + 3a_2a_3 + 3a_3^2\right) - Q\,(a_2 + a_3) - M\left(\frac{2a_2}{L} + \frac{3a_3}{L}\right)$$

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$$\frac{\partial \Pi}{\partial a_2} = 0, \quad \frac{\partial \Pi}{\partial a_3} = 0$$

$$\Rightarrow \quad a_2 = (M + QL)\frac{L^2}{2EI}$$

$$a_3 = -\frac{QL^3}{6EI}$$

$$u = (M+QL)\frac{z^2}{2EI} - \frac{Qz^3}{6EI}$$

approx
||
exact

## Compare approx $\hat{u}(z)$ with exact $u(z)$

$$e(z) = u(z) - \hat{u}(z)$$

error


$L_\infty$ norm $\quad \max_L \{ e(z) \}$

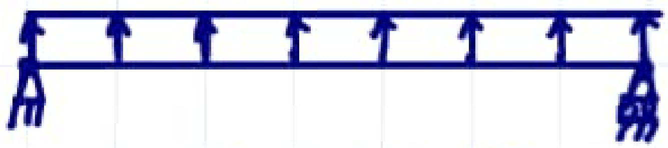
$L_1$ norm $\quad \frac{1}{L}\int_0^L |e(z)|\, dz$

$L_2$ norm
RMS

$$\sqrt{\frac{1}{L}\int_0^L e^2\,dz}$$

## Example 2

$q(z) = q_0$



$$\text{Exact } u = \frac{qz}{24EI}\left(z^3 - 2Lz^2 + L^3\right)$$

$$\text{Approx } u = a_1 \sin\frac{\pi z}{L} \qquad \text{KBCs } u(0) = 0, \quad u(L) = 0$$

$$\Pi = \int_0^L \frac{EI}{2} u''^2\,dz - \int_0^L q_0 u(z)\,dz$$

$$\frac{\partial \Pi}{\partial a_1} = 0 \Rightarrow a_1 = \frac{4L^4 q_0}{EI\pi^5}$$

Improv,

$$u = a_1 \sin\frac{\pi z}{L} + a_3 \sin\frac{3\pi z}{L} + a_5 \sin\frac{5\pi z}{L}$$

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$$u'' = -a_1\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi z}{L} - a_3\left(\frac{3\pi}{L}\right)^2 \sin\frac{3\pi z}{L}$$
$$- a_5\left(\frac{5\pi}{L}\right)^2 \sin\frac{5\pi z}{L}$$

<u>Orthogonality of sines</u>

$$\int_0^L \sin\frac{m\pi z}{L}\sin\frac{n\pi z}{L}\,dz = \begin{cases} \frac{L}{2} & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

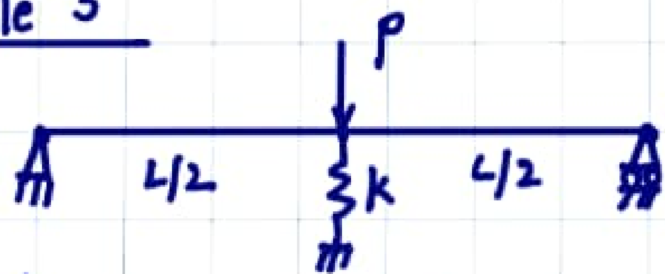
$N=3$

$$\Pi(a_1,a_3,a_5) = \frac{EI\pi^4}{4L^3}\left(a_1^2 + 81a_3^2 + 625a_5^2\right)$$
$$-\frac{2q_0L}{15\pi}\left(15a_1 + 5a_3 + 3a_5\right)$$

$$\frac{\partial \Pi}{\partial a_1} = 0 \Rightarrow a_1 = \frac{4q_0L^4}{EI\pi^5},\ a_3 = \frac{4q_0L^4}{EI(3\pi)^5}$$

$$a_5 = \frac{4q_0L^4}{EI(5\pi)^5}$$

## Example 3




$$\Pi = \int_0^L \frac{EI}{2} u''^2 dz + \frac{1}{2} k u^2\left(\frac{L}{2}\right) + \left(+ P u\left(\frac{L}{2}\right)\right)$$

Try $u = Az(z-L)$ obey KBCs
$u(0)=0$
$u(L)=0$

$$\Pi = \frac{EI}{2}\int_0^L (2A)^2 dz + \frac{1}{2} k \left(-\frac{AL^2}{4}\right)^2 - \frac{PAL^2}{4}$$

$$= EI\, 2A^2 L + \frac{K}{32} A^2 L^4 - \frac{PAL^2}{4}$$

$N=1$

$$\frac{d\Pi}{dA} = 0 \Rightarrow A = \frac{4PL}{64EI + KL^3}$$

HW Try same $u = A \sin \frac{\pi z}{L}$

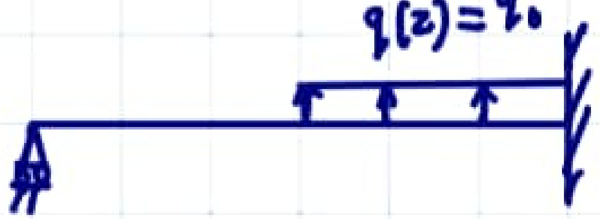
PMPE $\Pi$ min for exact solution.

If two approx $\hat{u}_1$, $\hat{u}_2$

$\Pi_{exact} \ldots < \Pi_1 < \Pi_2$

$\hat{u}_1$ is better than $\hat{u}_2$

<u>Example 3</u>




$$\Pi = \int_0^L \frac{EI}{2} u''^2 dz - \int_{L/2}^{L} q_0 u(z) dz$$

KBCs $u(0)=0$, $u(L)=0$, $u'(L)=0$

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$$u(z) = Az(L-z)^2 \qquad N=1$$

$$\text{Let } u(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

$$\frac{d\Pi}{dA} = 0 \Rightarrow \quad A = \frac{5 q_0 L}{768\, EI}$$

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