Case Study: Searching for Patterns

Problem: find all occurrences of pattern P of length m inside the text T of length n.

$\Rightarrow$ *Exact matching problem*

String Matching - Applications

- Text editing
- Term rewriting
- Lexical analysis
- Information retrieval
- And, bioinformatics

Exact matching problem

Given a string ***P*** (pattern) and a longer string ***T*** (text). Aim is to find all occurrences of pattern ***P*** in text ***T***.

The naive method:

T	a	g	c	a	g	a	a	g	a	g	t	a
P	a	a	a	a	a	a	a	g	a	g		

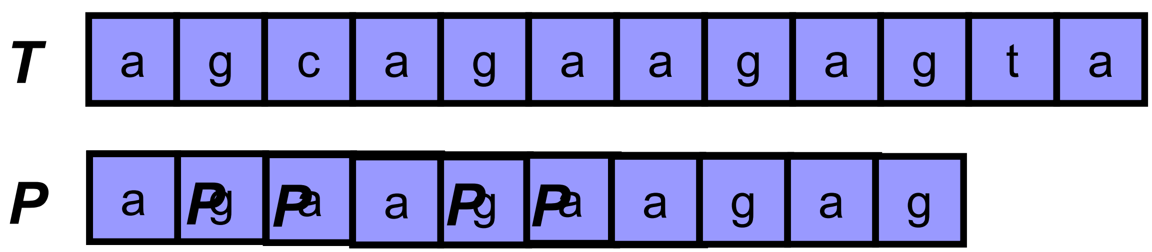
If m is the length of ***P***, and n is the length of ***T***, then

Time complexity = $O(m.n)$,

Space complexity = $O(m + n)$

Can we be more clever ?

- When a mismatch is detected, say at position *k* in the pattern string, we have already successfully matched *k*-1 characters.
- We try to take advantage of this to decide where to restart matching



The Knuth-Morris-Pratt Algorithm

Observation: when a mismatch occurs, we may not need to restart the comparison all way back (from the next input position).

What to do:

Constructing an array *h*, that determines how many characters to shift the pattern to the right in case of a mismatch during the pattern-matching process.

KMP (2)

The **key idea** is that if we have successfully matched the prefix P[1…i-1] of the pattern with the substring T[j-i+1,…,j-1] of the input string and P(i) ≠ T(j), then we do not need to reprocess any of the suffix T[j-i+1,…,j-1] since we know this portion of the text string is the prefix of the pattern that we have just matched.

The failure function h

For each position i in pattern P, define h_i to be the length of the longest proper suffix of $P[1,\ldots,i]$ that matches a prefix of P.

$i = 11$

	1	2	3	4	5	6	7	8	9	10	11	12	13
Pattern:	*a*	*b*	*a*	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*a*	*a*	*b*

Hence, $h(11) = 6$.

If there is no proper suffix of $P[1,\ldots,i]$ with the property mentioned above, then $h(i) = 0$

The KMP shift rule

The first mismatch in position *k=12* of *T* and in pos. *i+1=12* of *P*.

T:	a	b	a	a	b	a	b	a	a	b	a	c	a	b	a	a	b	a	b	a	a	b
P:	a	b	a	a	b	a	b	a	a	b	a	a	b									
P:						a	b	a	a	b	a	b	a	a	b	a	a	b				

$$h_{11} = 6$$

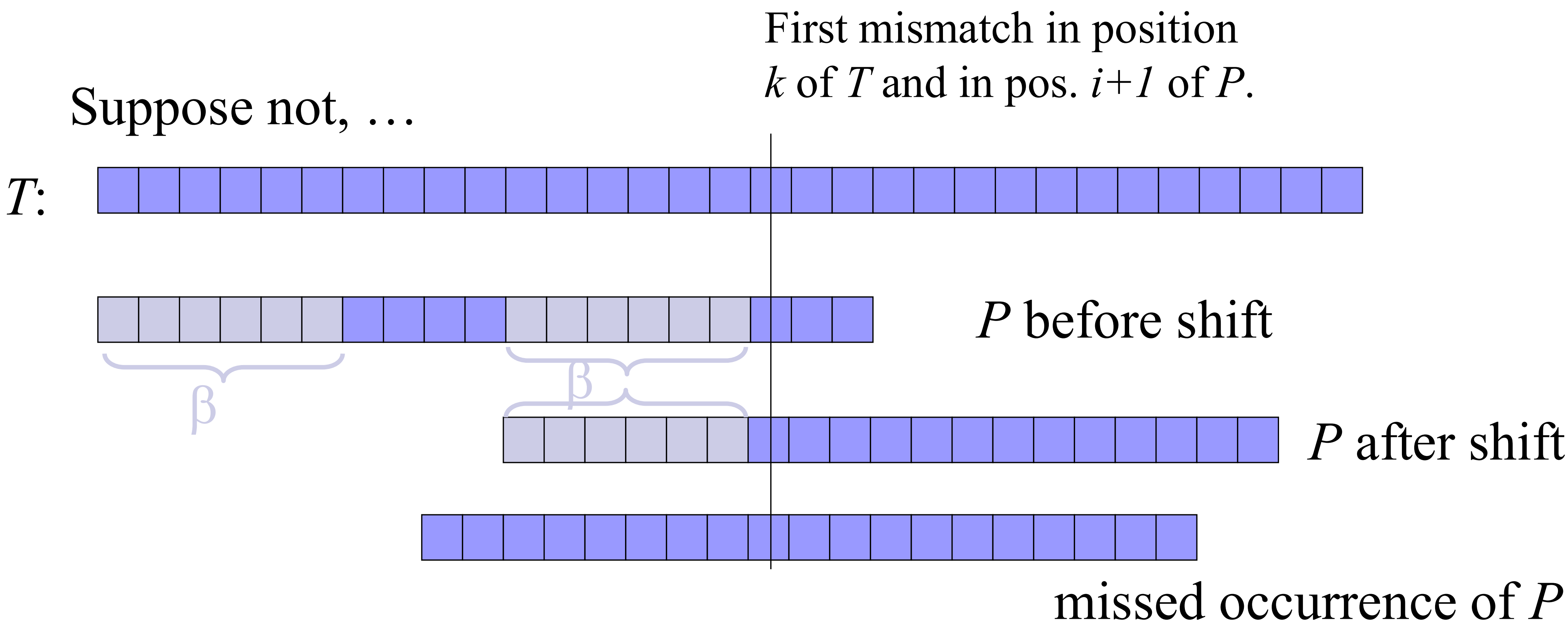
Shift P to the right so that P[1,…,h(i)] aligns with the suffix T[k- h(i),…,k-1].

They must match because of the definition of *h*.

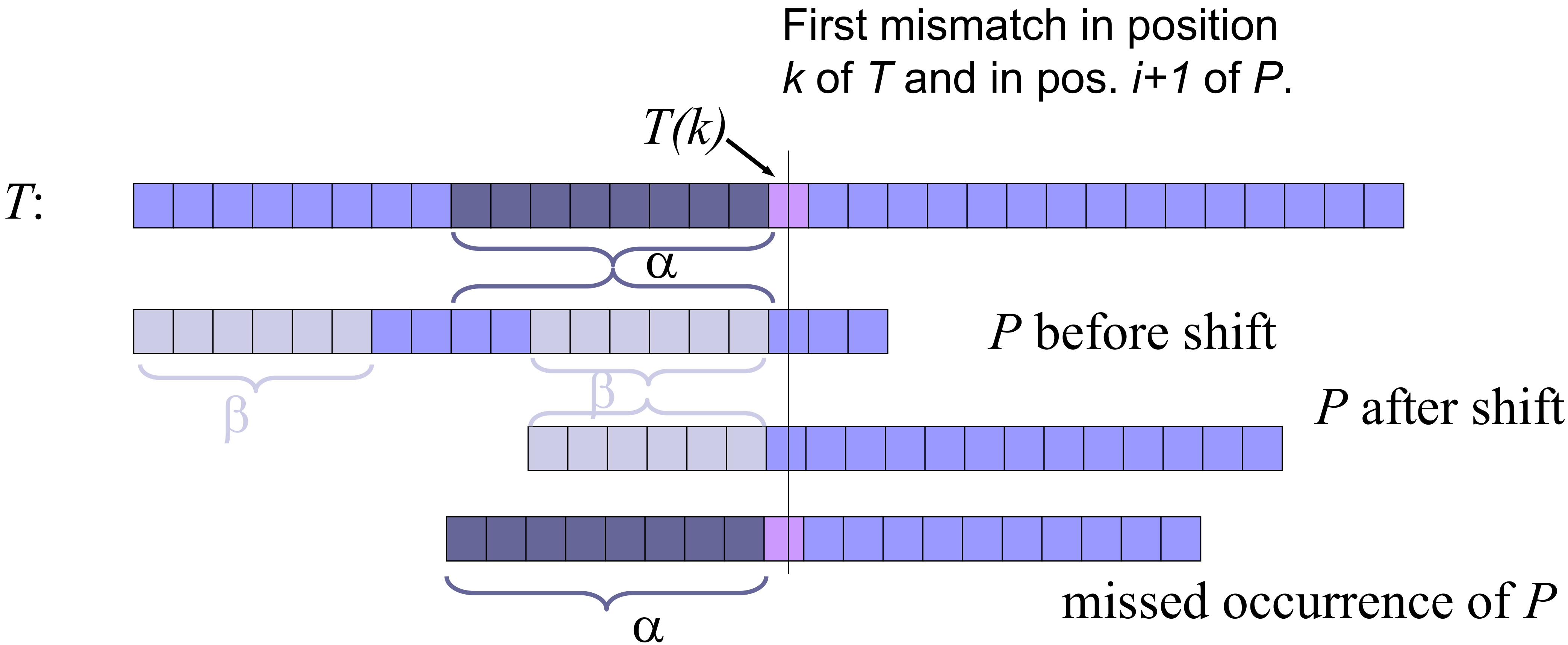
In other words, shift *P* exactly *i – h(i)* places to the right.

If there is no mismatch, then shift *P* by *m – h(m)* places to the right.

The KMP algorithm finds all occurrences of *P* in *T*.



Correctness of KMP.



$|\alpha| > |\beta| = h(i)$

It is a contradiction.

An Example

Given:

	1	2	3	4	5	6	7	8	9	10	11	12	13
Pattern:	*a*	*b*	*a*	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*a*	*a*	*b*
Array *h*:	0	0	1	1	2	3	2	3	4	5	6	4	5

Input string: abaababaabacabaababaabaab.

Scenario 1:

$i+1 = 12$

a b a a b a b a a b a a b

a b a a b a b a a b a c a b a a b a b a a b a a b

$k = 12$

What is $h(i) = h(11) = ?$ $h(11) = 6 \Rightarrow i - h(i) = 11 - 6 = 5$

Scenario 2:

$i = 6,\ h(6) = 3$ $i+1$

a b a a b a b a a b a a b

a b a a b a b a a b a c a b a a b a b a a b a a b

k

An Example

Scenario 3: *i = 3, h(3) = 1*

```
                      i+1
                       ↓
                a b a a b a b a a b a a b
a b a a b a b a a b a c a b a a b a b a a b a a b
                       ↑
                       k
```

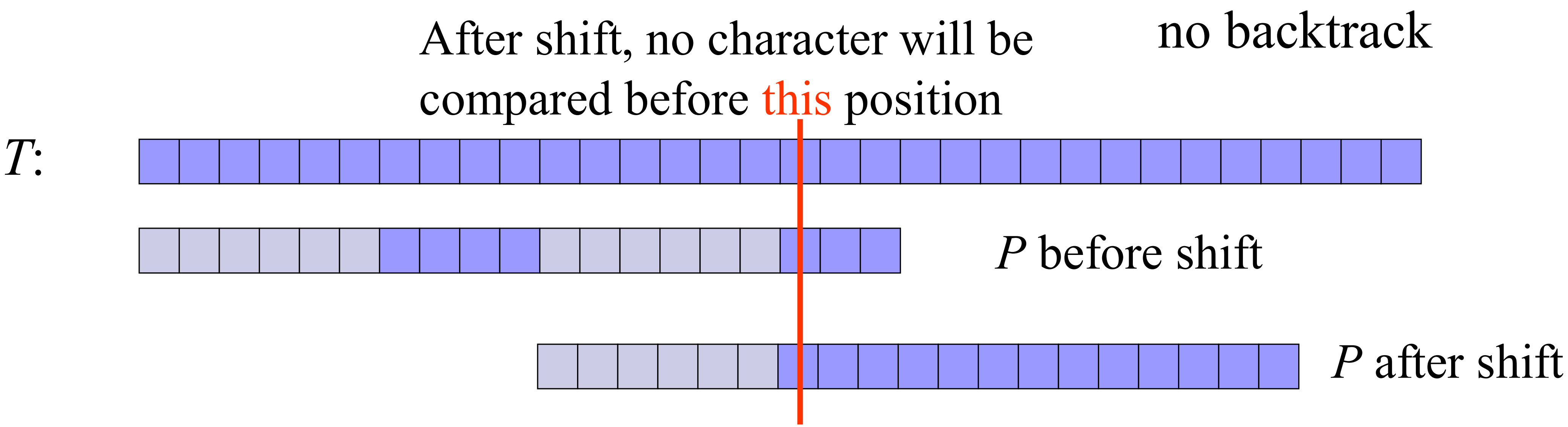
Subsequently *i = 2, 1, 0*

Finally, a match is found:

```
                        i+1
                         ↓
                         a b a a b a b a a b a a b
a b a a b a b a a b a c a b a a b a b a a b a a b
                         ↑
                         k
```

Complexity of KMP

In the KMP algorithm, the number of character comparisons is at most $2n$.



In any shift at most one comparison involves a character of T that was previously compared.

Hence #comparisons $\leq$ #shifts + $|T| \leq 2|T| = 2n$.

Computing the failure function

- We can compute h(i+1) if we know h(1)..h(i)
- To do this we run the KMP algorithm where the text is the pattern with the first character replaced with a $.
- Suppose we have successfully matched a prefix of the pattern with a suffix of T[1..i]; the length of this match is h(i).
- If the next character of the pattern and text match then h(i+1)=h(i)+1.
- If not, then in the KMP algorithm we would shift the pattern; the length of the new match is h(h(i)).
- If the next character of the pattern and text match then h(i+1)=h(h(i))+1, else we continue to shift the pattern.
- Since the no. of comparisons required by KMP is length of pattern+text, time taken for computing the failure function is O(n).

Computing h: an example

Given:	1	2	3	4	5	6	7	8	9	10	11	12	13
Failure function *h*:	0	0	1	1	2	3	2	3	4	5	6	4	5

	1	2	3	4	5	6	7	8	9	10	11	12	13	
Text:	*$*	*b*	*a*	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*a*	*a*	*b*	
Pattern:						*a*	*b*	*a*	*a*	*b*	*a*	*b*		h(11)=6
Pattern									*a*	*b*	*a*	*a*	*b*	h(6)=3

- h(12)= 4 = h(6)+1 = h(h(11))+1
- h(13)= 5 = h(12)+1

KMP - Analysis

- The KMP algorithm never needs to backtrack on the text string.

preprocessing ***searching***

Time complexity = $O(\underbrace{m}_{\textit{preprocessing}} + \underbrace{n}_{\textit{searching}})$

Space complexity = $O(m + n)$,

where $m = |P|$ and $n = |T|$.