Linear Algebra

Vector, matrix and tensor

Vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

n ↑ rows, 1 ↑ Column

Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ a_{31} & \cdots & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

$m \times n$

m ↑ rows, n ↑ Columns.

Vector, matrix and tensor

A_{ijk} a tensor of rank 3.

$x_i \rightarrow$ vector

a_{ij} is a matrix

Transpose, Addition, Subtraction and Scalar Multiplication

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

$$(A_{m\times n} + B_{m\times n})_{ij} = a_{ij} + b_{ij}$$

$$(c\,\underset{m\times n}{A})_{ij} = c\,a_{ij}$$

Matrix Multiplication

$$\underset{m\times p}{C} = \underset{m\times n}{A}\ \underset{n\times p}{B}$$

$$c_{ij} = \sum_k a_{ik} \times b_{kj} = a_{ik} b_{kj}$$

row $\times$ Column.

$$\boxed{AB \neq BA}$$

- $A(B+D) = AB + AD$
- $ABD = A(BD) = (AB)D$

vector Dot product.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} ; \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$$

Dot product

$$x \cdot y = \underbrace{x^T}_{1\times n} \underbrace{y}_{n\times 1} = \underbrace{\quad}_{1\times 1}$$

scalar

$$x^T y = y^T x$$

Einstein Summation Notation

$$\underset{m\times p}{C} = \underset{m\times n}{A}\ \underset{n\times p}{B}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$i \uparrow m \quad j \uparrow p \quad \uparrow m \quad \uparrow p$

- Summation is performed over repeated index
- No indices appear more than two times in the equation
- Indices which is summed over is called dummy indices appear only in one side of equation
- Indices which appear on both sides of the equation is free indices.

$$a_{ij} b_{jk} = \sum_{j} a_{ij} b_{jk}$$

$$a_{ii} = \sum_{i=1}^{d} a_{ii} = a_{11} + a_{22} + \cdots + a_{dd}$$

Matrix Multiplication

Dot Product:

$$x \cdot y = \underbrace{\begin{bmatrix} x_1, & x_2 & \cdots & x_n \end{bmatrix}}_{x^T} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{y} = \underbrace{x_i y_i}_{\text{scalar}}$$

Show that:

$$\underbrace{(x^T y)}_{x_i y_i} = \underbrace{(y^T x)}_{y_j x_j}$$

Show that:

$$\underbrace{(AB)^T} = \underbrace{B^T A^T}$$

Einstein

$$\left(a_{ij} \, b_{jk} \right)_{ik}$$

$$(A^T)_{ij} = A_{ji}$$

Square matrix, main diagonal, trace

$A_{m\times n}$ If $m = n$ $\Rightarrow$ sq. matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \rightarrow \text{Main diagonal}$$

Trace = $\sum$ of all elements of the main diagonal

$$= a_{ii} = \sum_{i=1}^{n} a_{ii}$$

Identity and Inverse Matrices

Identity matrix

$I_n \in \mathbb{R}^{n \times n}$ s.t.

$\forall x \in \mathbb{R}^n, \; I_n x = x$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Inverse Matrix of matrix A is A^{-1} if

$$A A^{-1} = I_n$$

$$Ax = b \Rightarrow \underbrace{A^{-1} A}_{I_n} x = A^{-1} b \Rightarrow \underbrace{I_n x}_{x} = A^{-1} b$$

$$\Rightarrow x = A^{-1} b$$

Linear Dependence and Span

$$\underset{\text{given}}{A}\ \overset{\text{Find}}{x} = \underset{\text{given}}{b}$$

For A^{-1} to exist, this should have exactly one sol.

If the above eqn. has ≥ 2 sol, then it should have ∞ sol.

If x, y are two sol; then

$z = \alpha x + (1-\alpha) y$ is also a sol. for any real α.

$$\underset{m \times n}{A}\ \underset{n \times 1}{x} = \underbrace{A_{:,j}\, x_j}_{\text{Linear Combination}}$$

$$(Ax)_i = A_{i,j}\, x_j = \sum_j A_{i,j}\, x_j$$

$$Ax = A_{:,j}\, x_j$$

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Linear Dependence and Span

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = A_{:j}\, x_j = \sum_{j=1}^{n} v^{(j)} x_j$$

} Linear Combination of columns.

$\{v^{(1)}, \ldots\ldots v^{(n)}\}$

$m \times 1$

Set of all possible points obtained by the linear Combination of the vectors is call span.

$Ax = b$ has a sol if 'b' is a part of the span of columns of A $\{v^{(1)}, v^{(2)}, \ldots\ldots v^{(n)}\}$

Linear Dependence and Span

For $\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$ to have a sol. 'b' should be a part of span of A.

$b \in R^m$

1. $\Rightarrow$ Column space of A to be $\mathbb{R}^m$

   $n \geq m$ } necessary but not sufficient.

2. Linear independence in the column vectors of A.

A has to be square matrix & have L.I. Column vectors.

$$A^{-1}(Ax) = A^{-1}b \Rightarrow \boxed{x = A^{-1}b}$$

Norms

$$\|x\|_p = \left[\sum_i |x_i|^p\right]^{1/p}$$

E.g. $\|x\|_2 = \sqrt{\sum x_i^2}$

Norm is a function f that satisfies

- $f(x) = 0 \Rightarrow x = 0$
- $f(x+y) \leq f(x) + f(y)$
- $\forall \alpha \in \mathbb{R}$; $f(\alpha x) = |\alpha| f(x)$

Symmetric matrix, unit vector and orthogonal

$A = A^T$

Unit Vector : $\|x\|_2 = 1$

Two vectors are orthogonal if $x^T y = 0$

Orthonormal

Orthogonal matrix : Columns are mutually orthonormal.

Row wise also orthonormal

$AA^T = I$

$A^T A = I = A A^T$ } $A^T = A^{-1}$

Kronecker delta

In 3 D space

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

$$\boldsymbol{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\delta_{ii} = 3$

$\delta_{ij}\delta_{jk} = \delta_{ik}$

$$\delta_{nm} = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \frac{k}{N}(n-m)}$$

Permutation tensor, also called the Levi-Civita tensor or isotropic tensor

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any two labels are the same} \\ 1, & \text{if } i,j,k \text{ is an even permutation of 1,2,3} \\ -1, & \text{if } i,j,k \text{ is an odd permutation of 1,2,3} \end{cases}$$

1, 2, 3 | 2, 3, 1 | 3, 1, 2 |

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Determinant

$$\det(\mathbf{A}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$ $$= \sum_i \sum_j \sum_k \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

Vector cross product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = \varepsilon_{ijk} \mathbf{e}_i a^j b^k$$

Define a third order tensor whose components are equal to zero unless all three indices are equal

$$\mathcal{H}_{ijk} = \begin{cases} 1 & \text{if } i=j=k \\ 0 & \text{otherwise} \end{cases}$$

Then you can use Einstein notation to write

$$\mathcal{D}_{ijk} = A_{ip}\mathcal{H}_{pjs}B_{sk}$$

This tensor is a useful addition to standard matrix algebra.

It can be used to generate a diagonal matrix A from a vector a (using a single-dot product)

$$A = \mathrm{Diag}(a) = \mathcal{H}\cdot a \quad\implies\quad A_{ij} = \mathcal{H}_{ijk}\,a_k$$

or to create a vector b from the main diagonal of a matrix B (using a double-dot product)

$$b = \mathrm{diag}(B) = \mathcal{H} : B \quad\implies\quad b_i = \mathcal{H}_{ijk}\,B_{jk}$$

or simply as a way to write $\mathcal{D}$ without resorting to index notation

$$\mathcal{D} = A\cdot\mathcal{H}\cdot B$$

Matrix decomposition

- Much as we can discover something about the true nature of an integer by decomposing it into prime factors, we can also decompose matrices in ways that show us information about their functional properties that is not obvious from the representation of the matrix as an array of elements

$$12 = 6 \times 2 = 3 \times 2 \times 2$$

$$A = C \underbrace{DE}$$

Eigen decomposition

- This decompose a matrix into a set of eigenvectors and eigenvalues.

- An **eigenvector** of a square matrix *A* is a non-zero vector *v* such that multiplication by *A* alters only the scale of *v:*

$$Av = \lambda v$$

$$\boxed{\underset{n\times n}{A}\ \underset{n\times 1}{v} = \underset{n\times 1}{\lambda v}}$$

Eigenvector

$$\frac{dy}{dt} = \underset{n\times n}{A}\,\underset{n\times 1}{Y}$$

n linear DE.

$$\underset{n\times 1}{y(t)} = e^{\lambda t}\,\underset{n\times 1}{X}$$

$$\lambda e^{\lambda t} X = A e^{\lambda t} X \Rightarrow \lambda X = A X$$

$$Ax = \lambda X \quad \text{or} \quad (A - I\lambda)x = 0$$

Eigenvector

Eg:

$$\left.\begin{aligned} y_1' &= 5y_1 + y_2 \\ y_2' &= 3y_1 + 3y_2 \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

This gives two eigenvalue and eigenvectors

$$\lambda_1 = 6,\quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\qquad \lambda_2 = 2,\quad X_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2$$

If B.C. are that at $t=0$, $y(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Eigenvector

If we know the e.v.s for

$A x = \lambda_1 x$ then we also know the e.v. of

$A^2 x = \lambda_2 x$

$A^2 x = A(Ax) = A(\lambda_1 x) = \lambda_1 (Ax)$

$= \lambda_1 (\lambda_1 x)$

$= \lambda_1^2 x$

$\Rightarrow \lambda_1^2 x = \lambda_2 x \Rightarrow \lambda_2 = (\lambda_1)^2$

Eigenvector

Similarly $(A + cI)X = \lambda_3 X$

$$\lambda_3 = (\lambda_1 + c)$$

$$A^n X = \lambda_n X$$

$$\Rightarrow \lambda_n = (\lambda_1)^n$$

Eigen decomposition

The E.D of A is given by

$$\underset{n\times n}{A} = \underset{n\times n}{V}\ \underset{n\times n}{\text{diag}(\lambda)}\ \underset{n\times n}{V^{-1}}$$

$$V = [\underset{n\times 1}{v^{(1)}},\ v^{(2)} \ldots\ v^{(n)}]$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$\text{diag}(\lambda) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 \cdots \\ \cdots & & & \lambda_n \end{bmatrix}_{n\times n}$$

Eigen decomposition

$$\boxed{A = V \, \mathrm{diag}(\lambda) \, V^{-1}}$$

$$V = \left[v^{(1)}, v^{(2)} \dots v^{(n)} \right]_{n \times n}$$

$$\mathrm{diag}(\lambda) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 \cdots \\ \cdots & & & \lambda_n \end{bmatrix}_{n \times n}$$

$$V \, \mathrm{diag}(\lambda) = \left[\underbrace{v^{(1)} \lambda_1}, v^{(2)} \lambda_2 \dots v^{(n)} \lambda_n \right]_{n \times n}$$

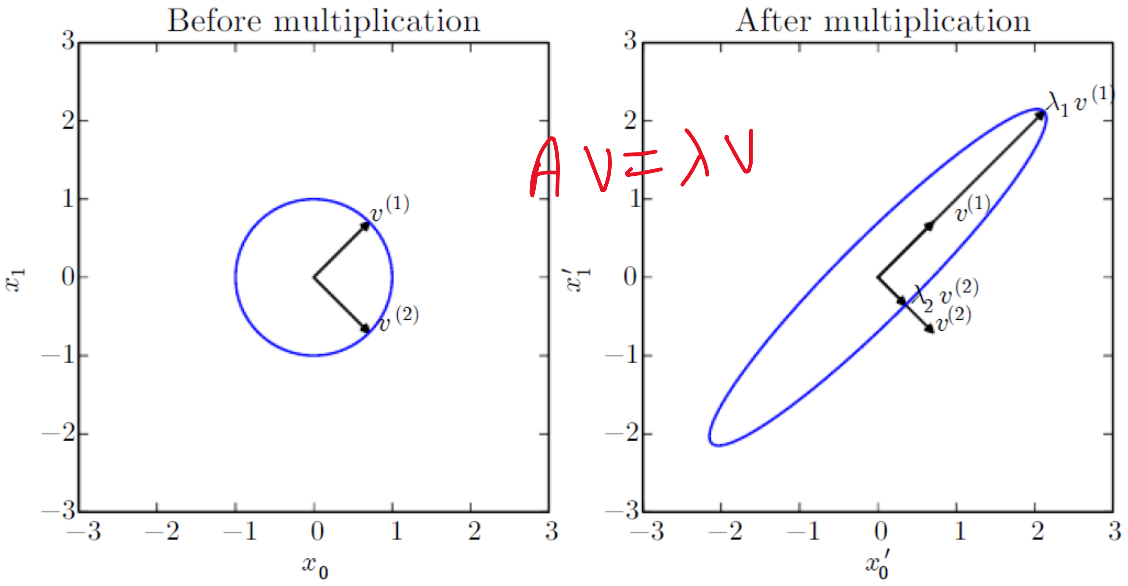
$$\left[A v^{(1)} \quad A v^{(2)} \dots A v^{(n)} \right] = A \left[v^{(1)}, v^{(2)} \dots v^{(n)} \right]$$

$$\Rightarrow V \, \mathrm{diag}(\lambda) = A V \Rightarrow V \, \mathrm{diag}(\lambda) \, V^{-1} = A V V^{-1} = A$$

Eigenvector and eigenvalues

$Av = \lambda v$

That is, the eigenvectors are the vectors that the linear transformation **A** merely elongates or shrinks, and the amount that they elongate/shrink by is the eigenvalue.



$AV = \lambda V$

Eigenvector and eigenvalues

- $Av = \lambda v$

$$(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{v} = 0$$

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

- We call $p(\lambda)$ the characteristic polynomial, and the equation, called the characteristic equation, is an Nth order polynomial equation in the unknown λ. This equation will have N_λ distinct solutions, where $1 \leq N_\lambda \leq N$.

Eigen decomposition

- The **eigendecomposition** of a square matrix A is then given by

$$A = V \operatorname{diag}(\lambda) V^{-1}$$

- Where matrix V *is made* with one eigenvector per column, and λ *is* concatenation of the eigenvalues to form a vector

Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD)

Singular value decomposition is a generalization of the eigen- decomposition of a square Matrix to a non-square matrix.

$$M = U\Sigma V \hat{} *$$

Where,

M is a real of complex m x n matrix

U is an m x m, real or complex unitary matrix (conjugate transpose, U^* is also its inverse)

Σ is an m x n rectangular diagonal matrix with non-negative real numbers

V^* is an n x n, real or complex unitary matrix

If $M$is real then U and $V$are real orthogonal matrices

The diagonal values of Σ are know as the singular values. By convention they are written in descending order. In this case Σ (but not always $U$and V^*) is uniquely determined by M.

Singular Value Decomposition (SVD)

$$M = U\Sigma V^*$$

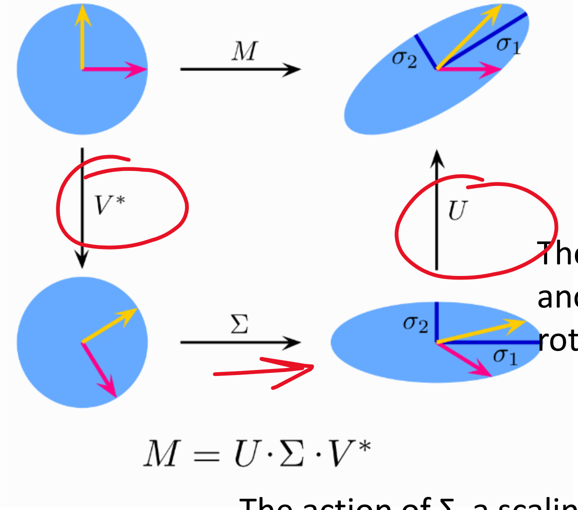
Illustration of the singular value decomposition UΣV* of a real 2×2 matrix M.

$$Me_1 = (U \Sigma V^*)\, e_1 = U \Sigma\, (V^* e_1)$$

$$(\Sigma \;\sim\;)$$

$$U(\;\sim\;)$$

The action of M, indicated by its effect on the unit disc D and the two canonical unit vectors e1 and e2.



The action of V*, a rotation, on D, e1, and e2.

The action of U, another rotation.

The action of Σ, a scaling by the singular values σ1 horizontally and σ2 vertically.

Compact Singular Value Decomposition

Compact singular value decomposition is similar to SVD with a square diagonal matrix Σ_{d} .

$$M = U_c \Sigma_{\mathrm{d}} V_c^*$$

Where,

M is a real of complex m x n matrix

U_c is an m x r, semi-unitary matrix ($\mathrm{U}_{\mathrm{c}}^* U_c = I_{rxr}$)

Σ_{d} is an r x r square diagonal matrix with positive real numbers

V_c is an n x r , semi-unitary matrix ($\mathrm{V}_{\mathrm{c}}^* \mathrm{V}_c = I_{rxr}$)

$r \leq \min\{n, m\}$ is the rank of Mmatrix, and Σ_{d} has only the non-zero singular values of M.

Thus, $\Sigma = \begin{bmatrix} \Sigma_d & 0 \\ 0 & 0 \end{bmatrix}$