SCALE ANALYSIS OF LAMINAR BOUNDARY LAYER

OBJECTIVES OF THIS LECTURE

- Write all the general governing equations in Cartesian coordinates
    - Conservation of mass
    - Conservation of momentum
    - Conservation of energy
- State the outcomes of these equations for an engineer
- Reduce the general equations for two dimensional, steady, incompressible flow
- Scale analysis of momentum equation to get
    - Hydrodynamic boundary layer thickness and skin friction coefficient
- Scale analysis of energy equation for Pr << 1 and Pr >> 1 to get
    - Thermal boundary layer thickness and Nusselt number
- Extend the discussion for Pr = 1
- Summarise the results and get the feel of all parameters estimated

GOVERNING EQUATIONS IN CARTESIAN COORDINATES

Conservation of mass

$$\frac{D\rho}{Dt} + \rho(\nabla.\vec{V}) = 0\ ;\ \left[\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right] + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Conservation of momentum

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + f_x$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{\partial}{\partial y}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + f_y$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{\partial}{\partial z}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + f_z$$

Conservation of energy

$$\rho C_p\frac{DT}{Dt} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{DP}{Dt} + \emptyset$$

GOVERNING EQUATIONS IN CARTESIAN COORDINATES

Conservation of momentum

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + f_x$$

Inertia forces	Pressure forces	Viscous forces	Body forces

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

GOVERNING EQUATIONS IN CARTESIAN COORDINATES

Conservation of energy

$$\rho C_p \frac{DT}{Dt} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{DP}{Dt} + \emptyset$$

Convection | Conduction | Pressure work | Viscous Dissipation

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + w\frac{\partial P}{\partial z}$$

$$\emptyset = 2\mu\left(\frac{\partial u}{\partial x}\right)^2 + 2\mu\left(\frac{\partial v}{\partial y}\right)^2 + 2\mu\left(\frac{\partial w}{\partial z}\right)^2 - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$
$$+ \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2$$

OUTCOMES OF THE GOVERNING EQUATIONS

Mass and momentum equations

Solve these equations to get

u, v, w and P

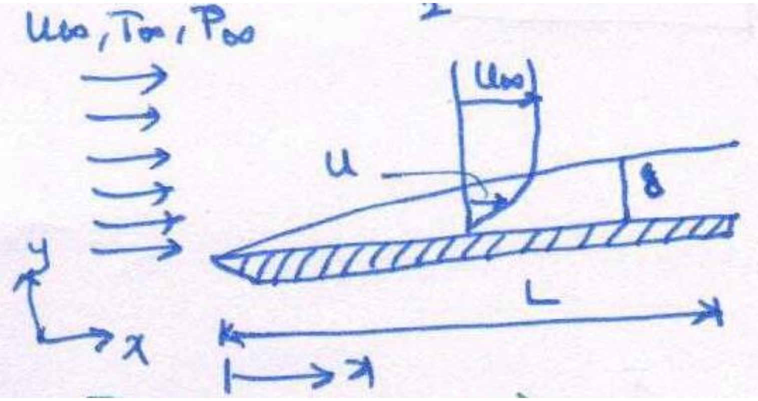
δ, C_f

δ – Hydrodynamic boundary layer thickness

C_f - Skin friction coefficient

(Engineering necessity)

$C_f = f(Re)$ - dimensional similarity



$$C_f = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$$

τ_w – Wall Shear Stress

ρ – Density of the fluid

u_∞ – Free Stream Velocity

Net force exerted by the stream on the plate

$$F = \int_0^L \tau_w W dx$$

W – Width of the flat Plate

OUTCOMES OF THE GOVERNING EQUATIONS

Energy equation

Solve this equation using u, v, w and P to get

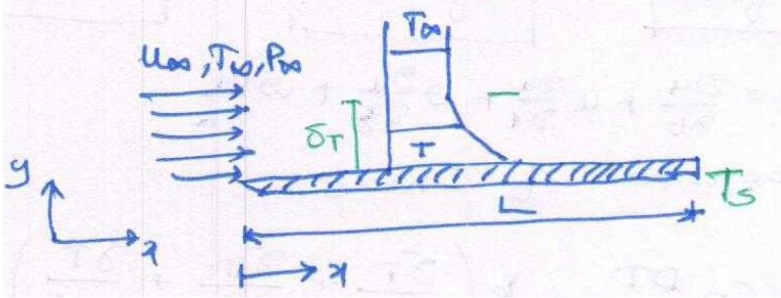
T

δ_T, Nu

δ_T – Thermal boundary layer thickness

Nu – Nusselt number (Engineering necessity)

$Nu = f(Re, Pr)$ - dimensional similarity



$$Nu = \frac{hL}{k_f}$$

$$h = \frac{-k_f \left.\frac{\partial T}{\partial y}\right|_{y=0}}{(T_s - T_\infty)}$$

h – heat transfer coefficient

L – Length of the Plate

k_f – Thermal conductivity of the fluid

T_s – Surface temperature of the Plate

T_∞ – Free stream temperature

Resistance to the transfer of heat from the plate to the stream

$$\dot{Q} = h(LW)(T_s - T_\infty)$$

W – Width of the flat Plate

LAMINAR BOUNDARY LAYER OVER A TWO DIMENSIONAL BODY

ASSUMPTIONS

- Properties are constant (k, ρ, C_p, μ)
- Flow is compressible (density does not vary with time and space)
- Flow is steady
- Laminar flow

Flow happens to be two dimensional when flow over a long flat plate is considered. Boundary layer thickness varies only with length but not with width

REDUCTION OF GOVERNING EQUATIONS

Conservation of mass

Incompressible flow - Density is constant and uniform

No, w

z does not exist

$$\frac{D\rho}{Dt} + \rho(\nabla.\vec{V}) = 0\ ;\ \cancel{\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right]} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}\right) = 0$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

REDUCTION OF GOVERNING EQUATIONS

Conservation of momentum – x direction

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + f_x$$

Zero, by continuity

$$\rho\left[\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right] = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}\right] + \cancel{f_x}$$

Steady flow

No, w

No, z

No body force

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

REDUCTION OF GOVERNING EQUATIONS

Conservation of momentum – y direction

$$\rho\frac{Dv}{Dt}=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]+\frac{\partial}{\partial y}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+f_y$$

Zero, by continuity

$$\rho\left[\cancel{\frac{\partial v}{\partial t}}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\cancel{w\frac{\partial v}{\partial z}}\right]=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\cancel{\frac{\partial^2 v}{\partial z^2}}\right]+\frac{\partial}{\partial y}\left[\frac{\mu}{3}\cancel{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}\right]+\cancel{f_y}$$

Steady flow

No, w

No, z

No body force

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]$$

Conservation of momentum – z direction – does not exist because two dimensional flow

$$\rho\frac{Dw}{Dt}=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]+\frac{\partial}{\partial z}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+f_z$$

REDUCTION OF GOVERNING EQUATIONS

Conservation of energy

$$\rho C_p \frac{DT}{Dt} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{DP}{Dt} + \emptyset$$

Zero for subsonic flow

$$\rho C_p \left[\cancel{\frac{\partial T}{\partial t}} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \cancel{w\frac{\partial T}{\partial z}}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right] + \cancel{\frac{DP}{Dt}} + \cancel{\emptyset}$$

Steady flow

No *w* and z direction

No z direction

$$\boxed{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]}$$

GOVERNING EQUATIONS FOR FLOW OVER A TWO DIMENSIONAL BODY

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

- $\left.\begin{array}{c}\text{No slip } u = 0 \\ \text{Impermeability } v = 0 \\ \text{Wall temperature } T = T_s\end{array}\right\}$ at solid wall (y = 0)

At Infinitely far from solid in both x and y directions

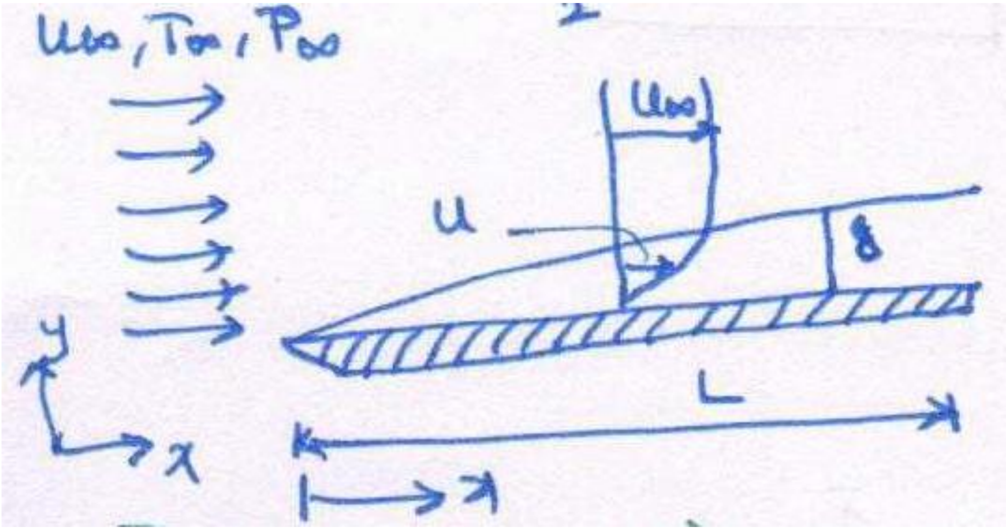
- $\left.\begin{array}{c}\text{Uniform flow } u = u_\infty \\ \text{Uniform flow } v = 0 \\ \text{Uniform temperature } T = T_\infty\end{array}\right\}$

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF MASS AND MOMENTUM EQUATIONS FOR FLOW OVER A FLAT PLATE

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$



Mass and momentum equations

Solve these equations to get

u, v, w and P

δ, C_f

δ – Hydrodynamic boundary layer thickness

C_f - Skin friction coefficient

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF MASS AND MOMENTUM EQUATIONS

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$u\sim u_\infty\ (m/s)$
$x\sim L\ (m)$
$y\sim\delta\ (mm)$

$$u_\infty\frac{u_\infty}{L},\frac{u_\infty\delta}{L}\frac{u_\infty}{\delta}\sim-\frac{1}{\rho}\frac{\partial P}{\partial x},\cancel{\nu\frac{u_\infty}{L^2}},\nu\frac{u_\infty}{\delta^2}$$

$$\nu\frac{u_\infty}{L^2}\ll\nu\frac{u_\infty}{\delta^2}$$

$$\frac{u_\infty^2}{L}\sim-\frac{1}{\rho}\frac{\partial P}{\partial x},\nu\frac{u_\infty}{\delta^2}-\ (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{u_\infty}{L}\sim\frac{v}{\delta}$$

$$v\sim\frac{u_\infty\delta}{L}-\ (1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]$$

$$u_\infty\frac{u_\infty\delta}{L}\frac{1}{L},\frac{u_\infty\delta}{L}\frac{u_\infty\delta}{L}\frac{1}{\delta}\sim-\frac{1}{\rho}\frac{\partial P}{\partial y},\nu\frac{v}{L^2},\nu\frac{v}{\delta^2}$$

$$\frac{u_\infty^2\delta}{L^2}\sim-\frac{1}{\rho}\frac{\partial P}{\partial y},\nu\frac{v}{\delta^2}$$

$$\frac{u_\infty^2\delta}{L^2}\sim-\frac{1}{\rho}\frac{\partial P}{\partial y},\nu\frac{1}{\delta^2}\frac{u_\infty\delta}{L}$$

$$\frac{u_\infty^2\delta}{L^2}\sim-\frac{1}{\rho}\frac{\partial P}{\partial y},\frac{\nu u_\infty}{\delta L}-\ (3)$$

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF MASS AND MOMENTUM EQUATIONS

$$dp = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\frac{dy}{dx}$$

$$1 = \frac{\frac{\partial P}{\partial x}}{\frac{dP}{dx}} + \frac{\frac{\partial P}{\partial y}\frac{dy}{dx}}{\frac{dP}{dx}} - (4) \Rightarrow$$

$$\frac{\frac{\partial P}{\partial y}\frac{dy}{dx}}{\frac{dP}{dx}} \sim \frac{\frac{\mu u_\infty}{\delta L}\frac{\delta}{L}}{\frac{\mu u_\infty}{\delta^2}} \sim \left(\frac{\delta}{L}\right)^2$$

Negligibly small

$$1 = \frac{\frac{\partial P}{\partial x}}{\frac{dP}{dx}} \Rightarrow P = f(x) \ only \ but \ not \ y$$

$$v \sim \frac{u_\infty \delta}{L} - (1)$$

$$\frac{u_\infty^2}{L} \sim -\frac{1}{\rho}\frac{\partial P}{\partial x}, \nu\frac{u_\infty}{\delta^2} - (2)$$

$$\frac{u_\infty^2 \delta}{L^2} \sim -\frac{1}{\rho}\frac{\partial P}{\partial y}, \frac{\nu u_\infty}{\delta L} - (3)$$

$$\cancel{\frac{u_\infty^2 \delta}{L^2}} \sim -\cancel{\frac{1}{\rho}\frac{\partial P}{\partial y}}, \frac{\nu u_\infty}{\delta L} - (3)$$

P is not a function of y

Hence, y – momentum equation may be neglected

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF MASS AND MOMENTUM EQUATIONS FOR FLOW OVER A FLAT PLATE

To show that $\frac{1}{\rho}\frac{dP}{dx}$ = 0 for flow over a flat plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\left[\frac{\partial^2 u}{\partial y^2}\right]$$

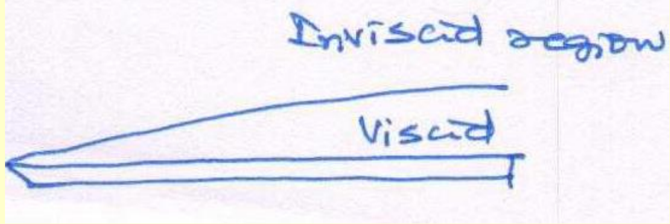
This $\frac{dP_\infty}{dx}$ is impressed upon the boundary layer, hence, $\frac{dP}{dx}$ = 0

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left[\frac{\partial^2 u}{\partial y^2}\right]$$

Flow can be divided into two domains

- Viscid region (boundary layer region)
- Inviscid region



In the inviscid region,

$u_\infty = constant$ does not vary with x

$v = 0$ and v = 0

Reducing the x-momentum equation for inviscid region

$$u_\infty\cancel{\frac{du_\infty}{dx}} + \cancel{v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\left[\cancel{\frac{\partial^2 u}{\partial y^2}}\right]$$

$$\frac{dP_\infty}{dx} = 0$$

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF MASS AND MOMENTUM EQUATIONS FOR FLOW OVER A FLAT PLATE

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v \sim \frac{u_\infty \delta}{L} \quad - \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left[\frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{u_\infty^2}{L} \sim -\frac{1}{\rho}\frac{\partial P}{\partial x}, \nu\frac{u_\infty}{\delta^2} \quad - \quad (2)$$

$$\frac{u_\infty^2}{L} \sim \nu\frac{u_\infty}{\delta^2}$$

$$\frac{\delta^2}{L} \sim \frac{\nu}{u_\infty}$$

$$\frac{\delta^2}{L^2} \sim \frac{\nu}{u_\infty L}$$

$$\frac{\delta}{L} \sim Re_L^{-\frac{1}{2}}$$

$$Re_L = \frac{u_\infty L}{\nu}$$

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$$

$$C_{fx} \sim \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2} \sim \frac{\mu\frac{\partial u}{\partial y}}{\rho u_\infty^2} \sim \frac{\nu\frac{u_\infty}{\delta}}{u_\infty^2} \sim \frac{\nu}{u_\infty \delta} \sim \frac{\nu}{u_\infty L}\frac{L}{\delta} \sim Re_L^{-1} Re_L^{\frac{1}{2}} \sim Re_L^{-\frac{1}{2}}$$

τ_w – Wall Shear Stress

ρ – Density of the fluid

u_∞ – Free Stream Velocity

$$C_{fx} \sim Re_L^{-\frac{1}{2}}$$

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF MASS AND MOMENTUM EQUATIONS FOR FLOW OVER A FLAT PLATE

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left[\frac{\partial^2 u}{\partial y^2}\right]$$

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$$

$$\frac{\delta}{L} \sim Re_L^{-\frac{1}{2}}$$

$$Re_L = \frac{u_\infty L}{\nu}$$

$$\frac{\delta}{L} = 4.92 Re_L^{-\frac{1}{2}}$$

$$C_{fx} \sim Re_L^{-\frac{1}{2}}$$

$$C_{fx} = 0.664 Re_L^{-\frac{1}{2}}$$

$\delta \sim L^{\frac{1}{2}}$ – Boundary layer thickness increases with the increase in the length

C_{fx} – decreases with the increase in the Reynolds number because inertia forces dominate over viscous forces

$$Re = \frac{Inertia\ Force}{Viscous\ Force} \sim \frac{u\frac{\partial u}{\partial x}}{\nu\left[\frac{\partial^2 u}{\partial y^2}\right]} \sim \frac{\frac{u_\infty^2}{L}}{\nu\left[\frac{u_\infty}{L^2}\right]} \sim \frac{u_\infty L}{\nu}$$

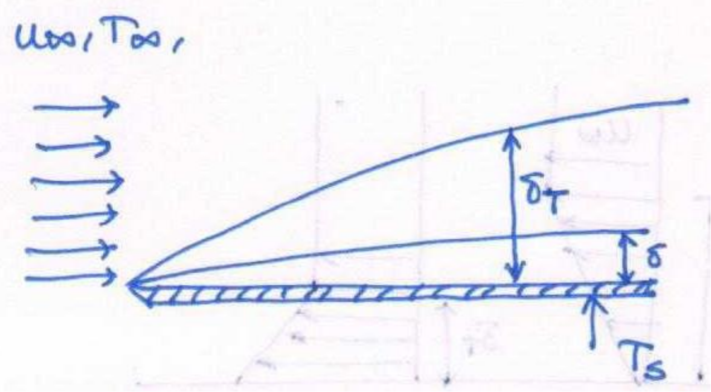
SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF ENERGY EQUATION FOR FLOW OVER A FLAT PLATE

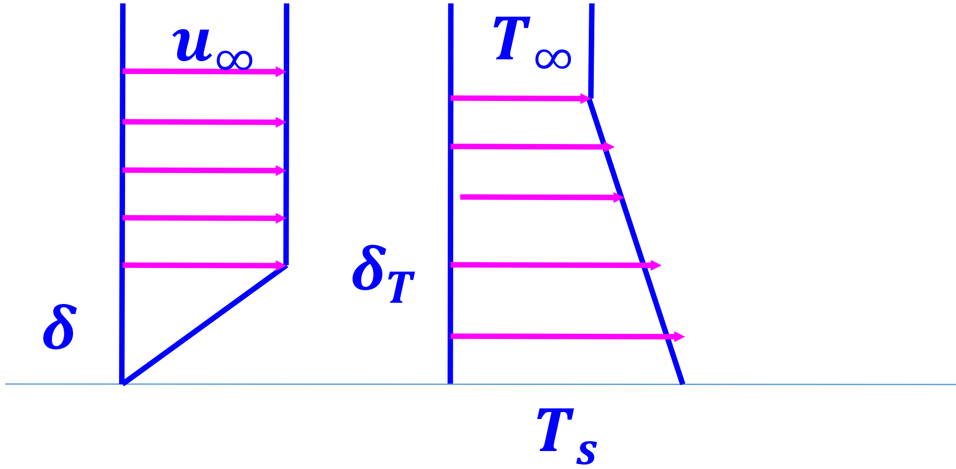
Pr << 1 (Liquid Metals – Na, Hg) THICK THERMAL BOUNDARY LAYER

$$Pr \ll 1 \Rightarrow \nu \ll \alpha \Rightarrow \delta << \delta_T$$

$$Pr = \frac{\nu}{\alpha}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$





$$u_\infty\frac{\Delta T}{L},\ \cancel{u_\infty\frac{\delta}{L}\frac{\Delta T}{\delta_T}} \sim \cancel{\alpha\frac{\Delta T}{L^2}},\ \alpha\frac{\Delta T}{\delta_T^2}$$

$$\frac{\delta}{\delta_T} \ll 1$$

Negligbly small

As $\delta << \delta_T \Rightarrow$ u is u_∞ within the thermal Boundary layer thickness

$$u_\infty\frac{\Delta T}{L} \sim \alpha\frac{\Delta T}{\delta_T^2}$$

$$\frac{\delta_T^2}{L} \sim \frac{\alpha}{u_\infty}$$

$$\frac{\delta_T^2}{L^2} \sim \frac{\nu}{u_\infty L}\frac{\alpha}{\nu}$$

$$\left(\frac{\delta_T}{L}\right)^2 \sim \frac{1}{Re_L}\frac{1}{Pr}$$

$$\frac{\delta_T}{L} \sim Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{2}}$$

TYPICAL RANGES OF PRANDTL NUMBERS FOR COMMON FLUIDS

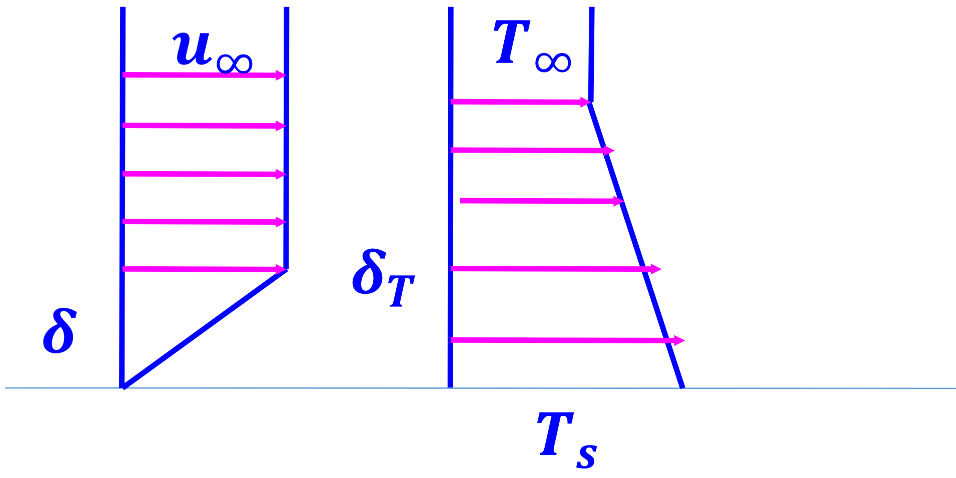
Fluid	Pr
Liquid Metals	0.004 – 0.03
Gases	0.7 – 1.0
Water	1.7 – 13.7
Light organic fluids	5 – 50
Oils	50 – 100,000
Glycerin	2000 – 1,00,000

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF ENERGY EQUATION FOR FLOW OVER A FLAT PLATE

Pr << 1 (Liquid Metals – Na, Hg) THICK THERMAL BOUNDARY LAYER

$$Pr = \frac{\nu}{\alpha}$$

$$Pr \ll 1 \Rightarrow \nu \ll \alpha \Rightarrow \delta << \delta_T$$



$$\frac{\delta_T}{L} \sim Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{2}}$$

$$Nu = \frac{hL}{k_f} = \frac{-\frac{k_f}{T_s - T_\infty}\frac{\partial T}{\partial y}L}{k_f} \sim \frac{1}{\Delta T}\frac{\Delta T}{\delta_T}L \sim \frac{L}{\delta_T} \sim \left(\frac{\delta_T}{L}\right)^{-1}$$

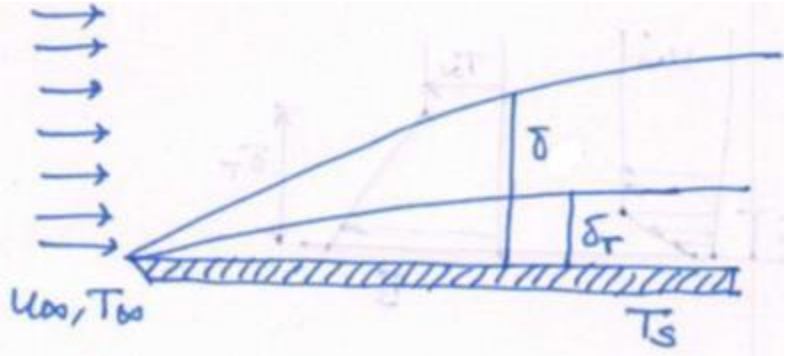
$\delta_T \sim L^{\frac{1}{2}}$ – Thermal Boundary layer thickness increases with the increase of *L*

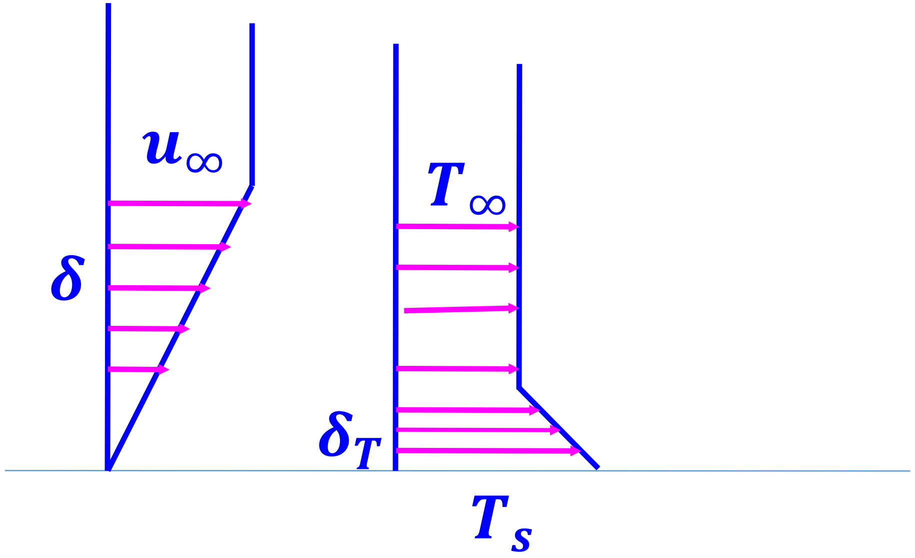
Nusselt number increases with the increase in the Reynolds number

$$Nu \sim Re_L^{\frac{1}{2}} Pr^{\frac{1}{2}}$$

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF ENERGY EQUATION FOR FLOW OVER A FLAT PLATE

Pr >> 1 (Oils) THIN THERMAL BOUNDARY LAYER





$$Pr = \frac{\nu}{\alpha}$$

$$Pr \gg 1 \Rightarrow \nu \gg \alpha \Rightarrow \delta >> \delta_T$$

As $\delta >> \delta_T \Rightarrow$ u is not u_∞

Assume velocity distribution within the hydrodynamic Boundary layer to be linear

$$u = my + C$$

$$u = 0 \; at \; y = 0 \Rrightarrow C = 0$$

$$u = u_\infty \; at \; y = \delta \Rrightarrow u_\infty = m\delta \Rrightarrow m = \frac{u_\infty}{\delta}$$

Velocity at $y = \delta_T$ $\quad u = \frac{u_\infty}{\delta}\delta_T$

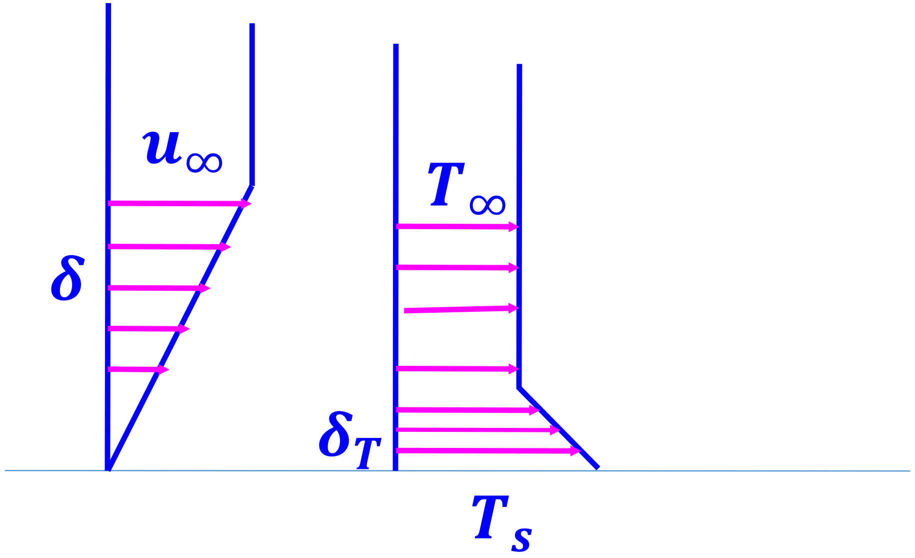
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{\delta}\delta_T \frac{1}{L} \sim \frac{v}{\delta_T}$$

$$v \sim \frac{u_\infty}{\delta}\delta_T \frac{\delta_T}{L}$$

SCALE ANALYSIS (ORDER OF MAGNITUDE ANALYSIS) OF ENERGY EQUATION FOR FLOW OVER A FLAT PLATE

Pr >> 1 (Oils) THIN THERMAL BOUNDARY LAYER



$$Pr = \frac{\nu}{\alpha}$$

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$$u = \frac{u_\infty}{\delta}\delta_T$$

$$v \sim \frac{u_\infty}{\delta}\delta_T\frac{\delta_T}{L}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

$$\frac{u_\infty}{\delta}\delta_T\frac{\Delta T}{L}, \frac{u_\infty}{\delta}\delta_T\frac{\delta_T}{L}\frac{\Delta T}{\delta_T} \sim \cancel{\alpha\frac{\Delta T}{L^2}}, \alpha\frac{\Delta T}{\delta_T^2}$$

$$\frac{u_\infty}{\delta}\delta_T\frac{\Delta T}{L} \sim \alpha\frac{\Delta T}{\delta_T^2}$$

$$\frac{\delta_T^3}{L} \sim \alpha\frac{\delta}{u_\infty}$$

$$\frac{\delta_T^3}{L^3} \sim \frac{\nu}{u_\infty L}\frac{\alpha}{\nu}\frac{\delta}{L}$$

$$\left(\frac{\delta_T}{L}\right)^3 \sim \frac{1}{Re_L}\frac{1}{Pr}Re_L^{-\frac{1}{2}}$$

$$\left(\frac{\delta_T}{L}\right)^3 \sim Re_L^{-\frac{3}{2}}Pr^{-1}$$

$$\frac{\delta_T}{L} \sim Re_L^{-\frac{1}{2}}Pr^{-\frac{1}{3}}$$

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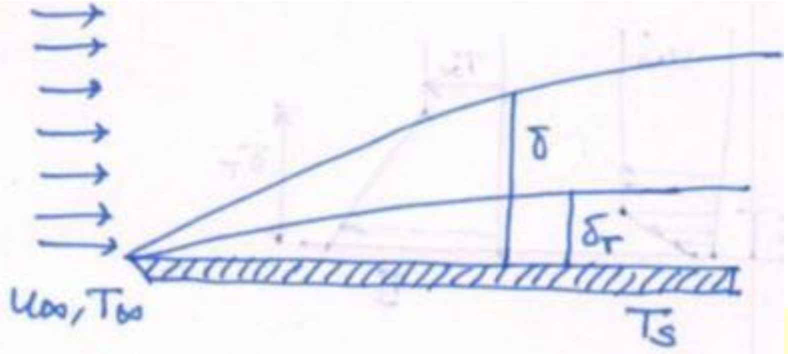

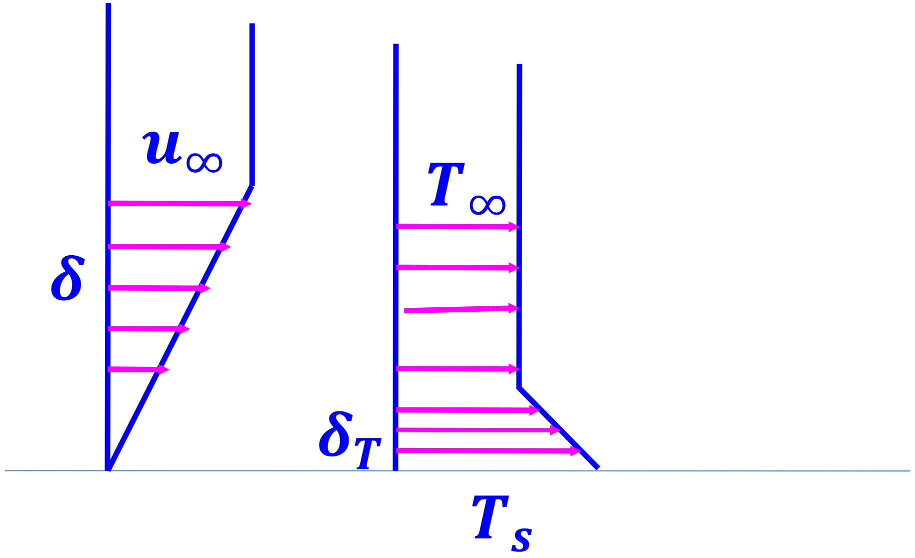

$$\frac{\delta_T}{L} \sim Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{3}}$$

$$Nu = \frac{hL}{k_f} = \frac{-\frac{k_f}{T_s - T_\infty}\frac{\partial T}{\partial y} L}{k_f} \sim \frac{1}{\Delta T}\frac{\Delta T}{\delta_T} L \sim \frac{L}{\delta_T} \sim \left(\frac{\delta_T}{L}\right)^{-1}$$

$\delta_T \sim L^{\frac{1}{2}}$ – Thermal Boundary layer thickness increases with the increase of L

Nusselt number increases with the increase in the Reynolds number

$$Nu \sim Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

SUMMARY OF SCALE ANALYSIS

$$\frac{\delta}{L} \sim Re_L^{-\frac{1}{2}}$$

$$C_{fx} \sim Re_L^{-\frac{1}{2}}$$

$$Pr = \frac{\nu}{\alpha}$$

$$Re_L = \frac{u_\infty L}{\nu}$$

$$Pr \ll 1 \Rightarrow \ \ll \alpha \Rightarrow \ \delta \ << \delta_T$$

$$\frac{\delta_T}{L} \sim Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{2}}$$

$$Nu \sim Re_L^{\frac{1}{2}} Pr^{\frac{1}{2}}$$

$$Pr \gg 1 \Rightarrow \nu \gg \alpha \Rightarrow \ \delta \ >> \delta_T$$

$$\frac{\delta_T}{L} \sim Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{3}}$$

$$Nu \sim Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

$$\frac{\delta}{L} = 4.92 Re_L^{-\frac{1}{2}}$$

$$C_{fx} = 0.664 Re_L^{-\frac{1}{2}}$$

$$Pr \ll 1 \Rightarrow \nu \ll \alpha \Rightarrow \ \delta \ << \delta_T$$

$$\frac{\delta_T}{L} = 1.77 Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{2}}$$

$$Nu = 0.565 \ Re_L^{\frac{1}{2}} Pr^{\frac{1}{2}}$$

$$Pr \gg 1 \Rightarrow \nu \gg \alpha \Rightarrow \ \delta \ >> \delta_T$$

$$\frac{\delta_T}{L} = 3.01 \ Re_L^{-\frac{1}{2}} Pr^{-\frac{1}{3}}$$

$$Nu = 0.332 \ Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

SUMMARY OF SCALE ANALYSIS

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$$Re = (100)^2 = 10000$$
$$L = 1\text{ m}$$

$$\frac{\delta}{L} = \frac{4.92}{\sqrt{Re_L}} \Rightarrow \frac{\delta}{1000} = \frac{4.92}{100} \Rightarrow \delta = 4.92\text{ mm}$$

$$Pr \ll 1 \Rightarrow Pr = (0.1)^2 = 0.01$$

$$\frac{\delta_T}{L} = \frac{1.77}{\sqrt{Re_L}\sqrt{Pr}} \Rightarrow \frac{\delta_T}{1000} = \frac{1.77}{100 \times 0.1} \Rightarrow \delta_T = 177\; mm$$

$$Pr \gg 1 \Rightarrow Pr = (100)^3 = 1000000$$

$$\frac{\delta_T}{L} = \frac{1.77}{\sqrt{Re_L}\;\sqrt[3]{Pr}} \Rightarrow \frac{\delta_T}{1000} = \frac{3.01}{100 \times 100}$$

$$\delta_T = 0.3\; mm$$

NONDIMENSIONALIZED CONVECTION AND SIMILARITY

When viscous dissipation is negligible, the continuity, momentum, and energy equations for steady incompressible, laminar flow of a fluid with constant properties

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial P}{\partial x}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

With the boundary conditions

$x=0 \qquad u(0,y)=u_\infty \qquad T(0,y)=T_\infty$

$y=0 \qquad u(x,0)=0 \qquad T(x,0)=T_s$

$y\rightarrow\infty \qquad u(x,\infty)=u_\infty \qquad T(x,u_\infty)=T_\infty$

$$x^* = \frac{x}{L},\ y^* = \frac{y}{L}, u^* = \frac{u}{u_\infty}, v^* = \frac{v}{u_\infty}, P^* = \frac{P}{\rho u_\infty^2}, T^* = \frac{T - T_s}{T_\infty - T_s}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{u_\infty}{L}\frac{\partial u^*}{\partial x^*} + \frac{u_\infty}{L}\frac{\partial v^*}{\partial y^*} = 0 \qquad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}$$

$$\frac{\rho u_\infty^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = \mu\frac{u_\infty}{L^2}\frac{\partial^2 u^*}{\partial {y^*}^2} - \frac{\rho u_\infty^2}{L}\frac{\partial P^*}{\partial x^*}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{L}{\rho u_\infty^2}\mu\frac{u_\infty}{L^2}\frac{\partial^2 u^*}{\partial {y^*}^2} - \frac{L}{\rho u_\infty^2}\frac{\rho u_\infty^2}{L}\frac{\partial P^*}{\partial x^*}$$

$$Re_L = \frac{\rho u_\infty L}{\mu}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\mu}{\rho u_\infty L}\frac{\partial^2 u^*}{\partial {y^*}^2} - \frac{\partial P^*}{\partial x^*} \qquad u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial {y^*}^2} - \frac{\partial P^*}{\partial x^*}$$

$$x^* = \frac{x}{L}, \; y^* = \frac{y}{L}, u^* = \frac{u}{u_\infty}, v^* = \frac{v}{u_\infty}, P^* = \frac{P}{\rho u_\infty^2}, T^* = \frac{T - T_s}{T_\infty - T_s}$$

$$\rho C_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$\rho C_p \frac{u_\infty (T_\infty - T_s)}{L} \left(u^* \frac{\partial T^*}{\partial x^*} + v\frac{\partial T^*}{\partial y^*} \right) = k\frac{(T_\infty - T_s)}{L^2} \left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} \right)$$

$$u^* \frac{\partial T^*}{\partial x^*} + v\frac{\partial T^*}{\partial y^*} = k\frac{(T_\infty - T_s)}{L^2} \frac{L}{\rho C_p u_\infty (T_\infty - T_s)} \left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}} \right)$$

$$u^* \frac{\partial T^*}{\partial x^*} + v\frac{\partial T^*}{\partial y^*} = \frac{k}{\rho C_p u_\infty L} \left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}} \right)$$

$$\frac{1}{Re_L Pr} = \frac{\mu}{\rho u_\infty L} \frac{k}{\mu C_p} = \frac{k}{\rho C_p u_\infty L}$$

$$u^* \frac{\partial T^*}{\partial x^*} + v\frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr} \left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}} \right)$$

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, u^* = \frac{u}{u_\infty}, v^* = \frac{v}{u_\infty}, P^* = \frac{P}{\rho u_\infty^2}, T^* = \frac{T - T_s}{T_\infty - T_s}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\partial P^*}{\partial x^*}$$

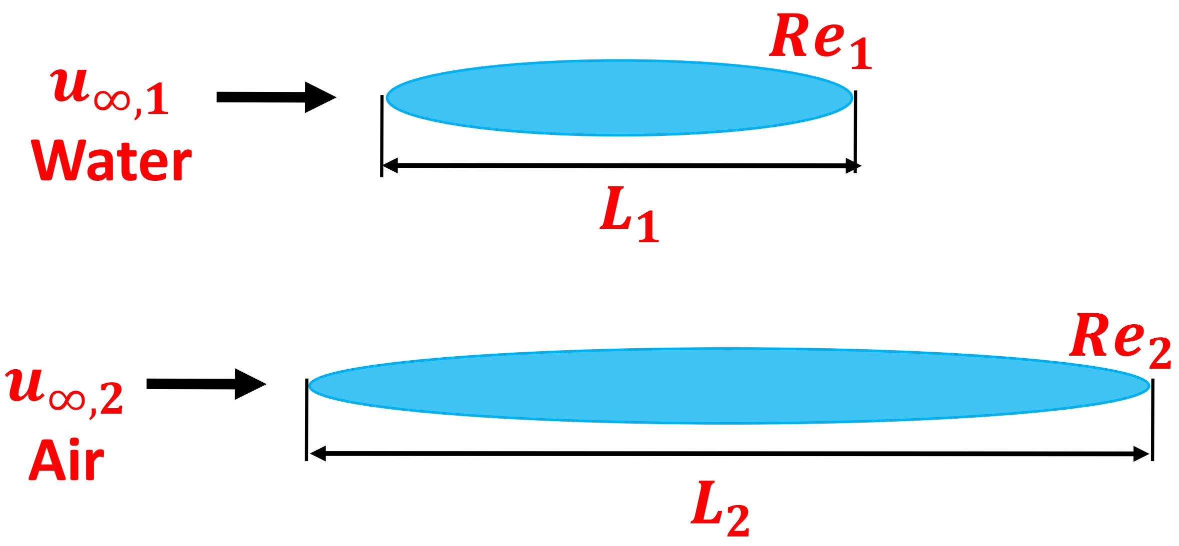
$$Re_L = \frac{\rho u_\infty L}{\mu}$$

$$u^* \frac{\partial T^*}{\partial x^*} + v \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr} \left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}} \right)$$

$$Pr = \frac{\mu}{C_p k}$$

$x = 0$	$u(0, y) = u_\infty$	$T(0, y) = T_\infty$
$y = 0$	$u(x, 0) = 0$	$T(x, 0) = T_s$
$y \to \infty$	$u(x, \infty) = u_\infty$	$T(x, \infty) = T_\infty$

$x^* = 0$	$u^*(0,\ y^*) = 1$	$T^*(0,\ y^*) = 1$
$y^* = 0$	$u^*(x^*, 0) = 0$	$T^*(x^*, 0) = 0$
$y^* \to \infty$	$u^*(x^*, \infty) = 1$	$T(x^*, \infty) = 1$



$$Re_1 = Re_2 \qquad Pr_1 = Pr_2$$

$$C_{f,1} = C_{f,2} \qquad Nu_1 = Nu_2$$

Parameters before nondimensionalizing

$$L, u_\infty, T_\infty, \mu, \rho, k, C_p$$

Parameters after nondimensionalizing

$$Re, Pr$$

The number of parameters is reduced greatly by non-dimensionalising the convection equations

For a given geometry, the solution for u^* can be expressed as

$$x^* = \frac{x}{L},\ y^* = \frac{y}{L}, u^* = \frac{u}{u_\infty}$$

$$u^* = f_1(x^*, y^*, Re_L)$$

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu u_\infty}{L}\frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0} = \frac{\mu u_\infty}{L} f_2(x^*, Re_L)$$

$$C_{f,x} = \frac{\tau_s}{\frac{\rho u_\infty^2}{2}} = \frac{\frac{\mu u_\infty}{L}}{\frac{\rho u_\infty^2}{2}} f_2(x^*, Re_L) = \frac{2}{Re_L} f_3(x^*, Re_L)$$

$$C_{f,x} = \phi(x^*, Re_L)$$

Friction coefficient for a given geometry can be expressed in terms of the Reynolds number Re_L and the dimensionless space variable x^* alone (instead of being expressed in terms of x, L, u_∞, ρ *and* μ).

This is a very significant finding, and shows the value of nondimensionalized equations.

Dimensionless temperature T^* for a given geometry

$$x^* = \frac{x}{L}, \; y^* = \frac{y}{L}, u^* = \frac{u}{u_\infty}, T^* = \frac{T - T_s}{T_\infty - T_s}$$

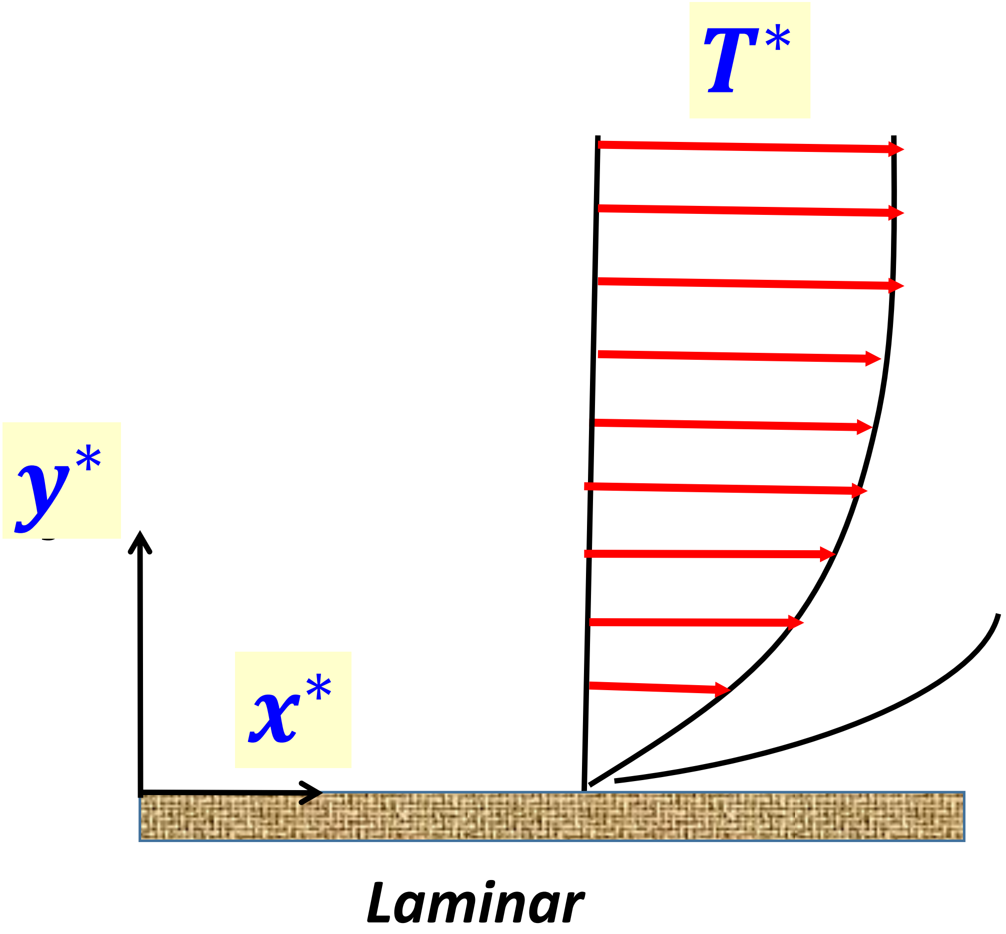
$$T^* = g(x^*, y^*, Re_L, Pr)$$

$$h = \frac{-k \left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_s - T_\infty} = \frac{-k(T_\infty - T_s)}{(T_s - T_\infty)L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = \frac{k}{L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$Nu = \frac{hL}{k} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = g_2(x^*, Re_L, Pr)$$

Note that the Nusselt number is equivalent to the dimensionless temperature gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient

Nusselt number is equivalent to the dimensionless temperature gradient at the surface



$$\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = Nu$$

Local Nusselt number $Nu_x = g_2(x^*, Re_L, Pr)$

Average Nusselt number $Nu_L = g_3(Re_L, Pr)$

A common form of Nusselt number:

$$Nu = CRe_L^m Pr^n$$

ANALOGIES BETWEEN MOMENTUM AND HEAT TRANSFER

REYNOLDS ANALOGY $Pr = 1$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

No pressure gradient

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\partial P^*}{\partial x^*}$$

$$u^*\frac{\partial T^*}{\partial x^*} + v\frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L}\left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right)$$

$Pr = 1$

$$u^*\frac{\partial T^*}{\partial x^*} + v\frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr}\left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right)$$

$$C_{f,x} = \frac{2}{Re_L} f_3(x^*, Re_L)$$

$$Nu_x = g_2(x^*, Re_L, Pr)$$

$Pr = 1$

$$f_3(x^*, Re_L) = g_2(x^*, Re_L, Pr)$$

$$Nu_x = C_{f,x}\frac{Re_L}{2}$$

REYNOLDS ANALOGY

$$St = \frac{h}{\rho C_p u_\infty} = \frac{Nu_x}{Re_L\, Pr}$$

$$Nu_x = C_{f,x}\frac{Re_L}{2}$$

$$\frac{Nu_x}{Re_L Pr} = \frac{C_{f,x}}{2}$$

$Pr = 1$

$$St_x = \frac{C_{f,x}}{2}$$

CHILTON-COLBOURN ANALOGY

Laminar flow over a flat plate

$$C_{f,x} = 0.664 Re_x^{-\frac{1}{2}}$$

$$Nu_x = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

$$\frac{C_{f,x}}{Nu_x} = \frac{0.664 Re_x^{-\frac{1}{2}}}{0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}}$$

$$\frac{C_{f,x}}{2} = \frac{Nu_x}{Re_x Pr^{\frac{1}{3}}}$$

$$\frac{C_{f,x}}{2} = \frac{Nu_x}{Re_x Pr} \frac{Pr}{Pr^{\frac{1}{3}}}$$

$$\frac{C_{f,x}}{2} = St_x Pr^{\frac{2}{3}} = j_H$$

For $0.6 < Pr < 60$, j_H is called the Colburn j-factor.

Although this relation is developed using relations for laminar flow over a flat plate (for which $\frac{\partial P^*}{\partial x^*} = 0$), experimental studies show that it is also applicable approximately for turbulent flow over a surface, even in the presence of pressure gradients.

For laminar flow, however, the analogy is not applicable unless $\frac{\partial P^*}{\partial x^*} = 0$. Therefore, it does not apply to laminar flow in a pipe