Laplace Transforms and use in Automatic Control

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Recap

- Fourier series
- Fourier transform: aperiodic
- Convolution integral
- Frequency response

Why Laplace transforms ??

- Integration and differentiation can be replaced by algebraic operations in complex plane.
- It converts complex functions such as sinusoidal, exponential into algebraic functions of complex variable 's'
- These algebraic equations can be easily solved in 's' for dependent variable whose time domain response can be found with inverse Laplace transforms

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- Q: cant we do these by using Fourier transform?

A: Not always. We will consider some cases later

Definition of L.T.

If $f(t)$ is a function of time , $f(t)=0$ for $t<0$
and 's' is a complex variable
and F(s) is the laplace transform of function $f(t)$

Then the laplace transform of $f(t)$ is given by,

$$F(s) = L\left[f(t)\right] = \int_0^\infty f(t).e^{-st}dt$$

What if s = jw ?????

Fourier transform formula

Inverse Laplace Transforms

If F(s) is the laplace transform of a function, then its time domain response is given by

$$L^{-1}\left[F(s)\right] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \quad \text{for} \quad t > 0$$

This is called inverse Laplace transform of F(s). c is a real number to avoid integration outside the region of convergence.

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Advantages of L.T over F.T

- We have already seen Fourier transform which is a function of the complex variable jw . Laplace transform is a function of the complex variable 's' denoting $'\sigma + jw'$ in which if σ = 0, then Laplace transforms equals Fourier transforms.
- What does this mean physically???
- Laplace transforms are introduced to fill the gaps which Fourier transform does not; We see these two cases with examples.

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Advantages of L.T over F.T

Case 1: Unstable systems

Stable system is one which gives bounded output for bounded input. Consider the unstable system $y(t) = t.u(t)$ whose output diverge for a unit step input.

Applying Fourier transforms on both sides, we get

$$Y(jw) = \left[\int_{-\infty}^{\infty} tu(t).e^{-jwt} dt \right]$$

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Advantages of L.T over F.T

But Dirichlets first condition states that [t.u(t)] should be absolutely integrable for the Fourier transform to exist. That is,

$$\int_{-\infty}^{\infty} | tu(t) | dt < \infty$$

which is violated in this case. Thus, it is clear that we can't find Fourier transform for unstable systems. Now, taking Laplace transforms,

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Advantages of L.T over F.T

$$Y(s) = \int_{-\infty}^{\infty} tu(t).e^{-st}\,dt$$

$$= \int_{0}^{\infty} t.e^{-st}\,dt \qquad = \left[-\frac{t.e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^{\infty}$$

$$= \frac{1}{s^2} \quad \text{for } \operatorname{Re}\{s\} > 0$$

So, it is clear that for unstable systems, even though Fourier transform does not exist, Laplace transform exits which helps to investigate the stability of a system

Advantages of L.T over F.T

Case 2:

Let us take the input signal as below instead of unit step

$$x(t) = e^{at}.u(t)$$

Since we need to find F.T of both the system and the input signal in order to obtain the output response, we need to find F.T of this input signal given above

$$X(jw) = \int_{-\infty}^{\infty} e^{at} u(t).e^{-jwt}\,dt$$

Advantages of L.T over F.T

$$\Rightarrow X(jw) = \int_0^{\infty} e^{at-jwt} dt$$

The right side part is tending to infinite, so we can't find F.T for this input signal. We can conclude this without any calculations directly from Dirichlet first condition as in case-1. Taking Laplace transforms,

$$X(s) = \int_{-\infty}^{\infty} e^{at} u(t).e^{-st} dt$$

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Advantages of L.T over F.T

$$\Rightarrow X(s) = \int_0^{\infty} e^{-t(s-a)} dt$$

$$= \frac{1}{s-a} \quad \text{for } \operatorname{Re}\{s\} > a$$

So, for some values of Re{s}, we are able to find L.T for the given signal. This is called 'Region of Convergence' (ROC). Here, it is the area in s-plane whose real part is greater than 'a'. In case-1, ROC in the right half of the S-plane.

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How to find laplace transform ??

Consider a sinusoidal function,

$$f(t) = 0, \quad \text{for } t < 0$$
$$= A\sin wt, \quad \text{for } t >= 0$$

But we know that,

$$\sin wt = \frac{1}{2j}(e^{jwt} - e^{-jwt})$$

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contd.......

Hence

$$L[A\sin wt] = \frac{A}{2j}\int_0^\infty (e^{jwt} - e^{-jwt})e^{-st}dt$$
$$= \frac{A}{2j}\cdot\frac{1}{s-jw} - \frac{A}{2j}\cdot\frac{1}{s+jw}$$
$$= \frac{Aw}{s^2+w^2}$$

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Laplace transforms for some commonly used functions

Function	Transform
Unit impulse	1
Unit step	$1/s$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin wt$	$\frac{w}{s^2+w^2}$

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contd.......

Function	Transform
$\cos wt$	$\frac{s}{s^2+w^2}$
$\sinh wt$	$\frac{w}{s^2-w^2}$
$\cosh wt$	$\frac{s}{s^2-w^2}$

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Some other functions

Observe how e^{-at} affects these functions,

Function	Transform
$e^{-at}t^n$	$\dfrac{n!}{(s+a)^{n+1}}$
$e^{-at}\sin wt$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos wt$	$\dfrac{s+a}{(s+a)^2+w^2}$

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Contd.......

Function	Transform
$e^{-at}\sinh wt$	$\dfrac{w}{(s+a)^2-w^2}$
$e^{-at}\cosh wt$	$\dfrac{s+a}{(s+a)^2-w^2}$

i.e,

$$\boxed{L\left[e^{-at}.f(t)\right]=\int_0^{\infty} e^{-at}.f(t).e^{-st}dt=F(s+a)}$$

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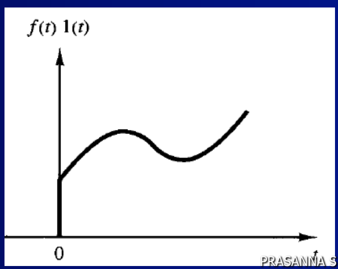
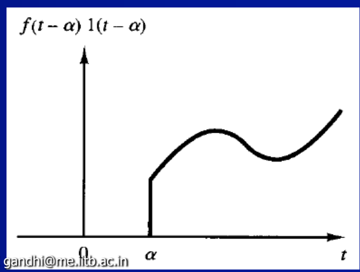
L.T of a translated function

let F(s) be the L.T of f(t) and l(t) is a function such that

$$f(t-\alpha).l(t-\alpha)=0 \quad \text{for} \quad t<\alpha$$





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then L.T of $f(t-\alpha).u(t-\alpha)$ is given by,

$$L[f(t-\alpha).l(t-\alpha)]=\int_0^\infty f(t-\alpha)l(t-\alpha).e^{-st}dt$$

$$=\int_{-\alpha}^\infty f(t)l(t).e^{-s(t+\alpha)}dt$$

$$=\int_0^\infty f(t).e^{-st}.e^{-\alpha t}dt$$

$$=e^{-\alpha s}.F(s)$$

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Change of time scale

If 't' is changed into t/α, where α is a positive constant. Then function $f(t)$ has changed into $f\left(\frac{t}{\alpha}\right)$, then L.T of $f\left(\frac{t}{\alpha}\right)$ is,

$$L\left[f\left(\frac{t}{\alpha}\right)\right]=\int_0^{\infty} f\left(\frac{t}{\alpha}\right).e^{-st}dt$$

$$=\alpha.\int_0^{\infty} f(t).e^{-\alpha st}dt \quad =\alpha F(\alpha s)$$

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Laplace transform theorems

The picture can't be displayed.

- Real differentiation theorem

$$L\left[\frac{d}{dt}f(t)\right]=sF(s)-f(0)$$

IMP for LTI Control syst

- Real integration theorem

$$L\left[\int f(t)dt\right]=\frac{F(s)}{s}+\frac{f^{-1}(0)}{s}$$

- Complex differentiation theorem

$$L\left[t.f(t)\right]=-\frac{d}{ds}F(s)$$

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Initial value theorem

- From this theorem, we can find the value of f(t) at t=0+ directly from the L.T of f(t).
- This theorem will not give the value of f(t) exactly at t=0 but slightly greater than zero

Def: If f(t) and $\frac{df(t)}{dt}$ are laplce transformable and if $\lim_{s\to\infty} sF(s)$ exists then,

$$f(0+) = \lim_{s\to\infty} sF(s)$$

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Final value theorem

- This theorem gives the value of f(t) when $t \to \infty$
- This theorem applies if and only if $\lim_{t\to\infty} f(t)$ exits
- All the poles of $sF(s)$ should lie in the left half of s-plane for $\lim_{t\to\infty} f(t)$ to exist.

Def: If f(t) and $\frac{df(t)}{dt}$ are laplce transformable and if $\lim_{t\to\infty} f(t)$ exists then,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

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Convolution Integral

- The integral $\int_0^t f_1(t-\tau)f_2(\tau)d\tau$ can also be written as $f_1(t)*f_2(t)$ which is known as convolution integral.
- If $F_1(s)$ & $F_2(s)$ are L.T's of $f_1(t)$ & $f_2(t)$ respectively, then L.T of convolution integral is given by,

$$L\left[\int_0^t f_1(t-\tau)f_1(t)d\tau\right] = F_1(s).F_2(s)$$

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Blocks can be multiplicative in Laplace domain

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Convolution Integral

- Linear, time-invariant (LTI) systems can be completely characterized by their impulse response
- Consider an LTI system whose impulse response is h(t), the response of the system to arbitrary input u(t) is given by:

y(t)=h(t)*u(t)

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau$$

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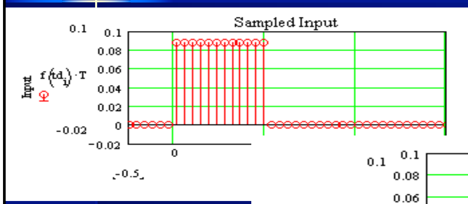
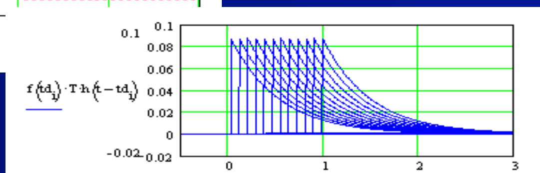
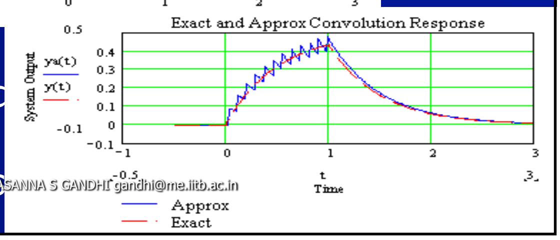
Convolution Integral Fundamentals



$$y(t) = \sum_{i=0}^{n} (f(i \cdot \Delta T) \cdot \Delta T) \cdot h(t - i \cdot \Delta T)$$



Convolution as sum of inverted and shifted impulse responses at various input points



L.T of product of two time functions

- If F(s) and G(s) are the laplace transforms of f(t) and g(t) respectively, then the laplace transform of $f(t) * g(t)$ is given by,

$$L[f(t) * g(t)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(p).G(s-p)dp$$