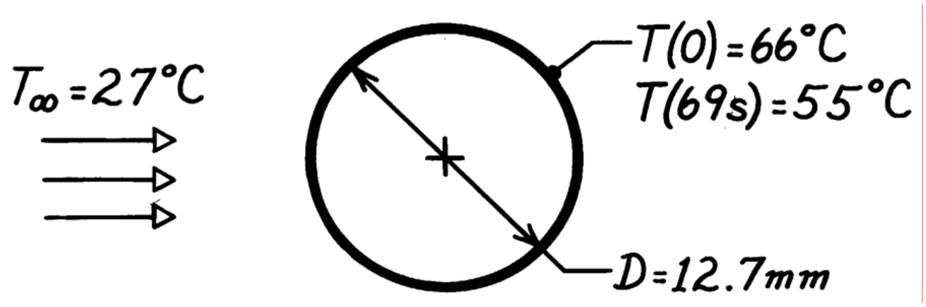
ME 346 S3 TUTORIAL 2

Q1. The heat transfer coefficient for air flowing over a sphere is to be determined by observing the temperature–time history of a sphere fabricated from pure copper. The sphere, which is 12.7 mm in diameter, is at 66 °C before it is inserted into an airstream having a temperature of 27 °C. A thermocouple on the outer surface of the sphere indicates 55 °C 69 s after the sphere is inserted into the airstream. Assume and then justify that the sphere behaves as a spacewise isothermal object and calculate the heat transfer coefficient.

Sol:

KNOWN: The temperature-time history of a pure copper sphere in an air stream.
FIND: The heat transfer coefficient between the sphere and the air stream.
SCHEMATIC:



ASSUMPTIONS: (1) Temperature of sphere is spatially uniform, (2) Negligible radiation exchange, (3) Constant properties.

PROPERTIES: *Table A-1*, Pure copper (333K): $\rho = 8933\ \text{kg/m}^3$, $c_p = 389$ J/kg·K, k = 398 W/m·K.

ANALYSIS: The time-temperature history is given by Eq. 5.6 with Eq. 5.7.

$$\frac{\theta(t)}{\theta_i} = \exp\left(-\frac{t}{R_t C_t}\right) \qquad \text{where} \qquad R_t = \frac{1}{hA_s} \qquad A_s = \pi D^2$$

$$C_t = \rho V c_p \qquad V = \frac{\pi D^3}{6}$$

$$\theta = T - T_\infty.$$

Recognize that when t = 69s,

$$\frac{\theta(t)}{\theta_i} = \frac{(55-27)^\circ \text{C}}{(66-27)^\circ \text{C}} = 0.718 = \exp\left(-\frac{t}{\tau_t}\right) = \exp\left(-\frac{69\text{s}}{\tau_t}\right)$$

and solving for τ_t find

$$\tau_t = 208\text{s}.$$

Hence,

$$h = \frac{\rho V c_p}{A_s \tau_t} = \frac{8933\ \text{kg/m}^3 \left(\pi 0.0127^3\ \text{m}^3/6\right) 389\text{J/kg}\cdot\text{K}}{\pi 0.0127^2 \text{m}^2 \times 208\text{s}}$$

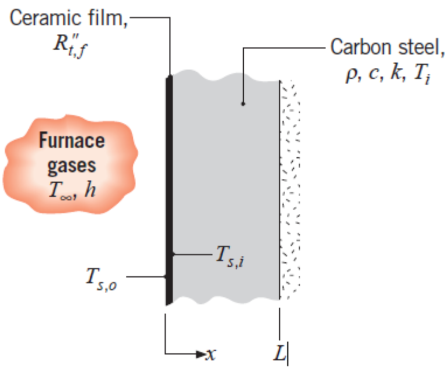
$$h = 35.3\ \text{W/m}^2 \cdot \text{K}.$$ <

COMMENTS: Note that with $L_c = D_o/6$,

$$\text{Bi} = \frac{hL_c}{k} = 35.3\ \text{W/m}^2 \cdot \text{K} \times \frac{0.0127}{6} \text{m}/398\ \text{W/m}\cdot\text{K} = 1.88 \times 10^{-4}.$$

Hence, Bi < 0.1 and the spatially isothermal assumption is reasonable.

Q2. A plane wall of a furnace is fabricated from plain carbon steel (k = 60 W/mK, ρ = 7850 kg/m3, c = 430 J/kg K) and is of thickness L = 10mm. To protect it from the corrosive effects of the furnace combustion gases, one surface of the wall is coated with a thin ceramic film that, for a unit surface area, has a thermal resistance of $R''_{t,f} = 0.01\ \text{m}^2$ K/W. The opposite surface is well insulated from the surroundings.



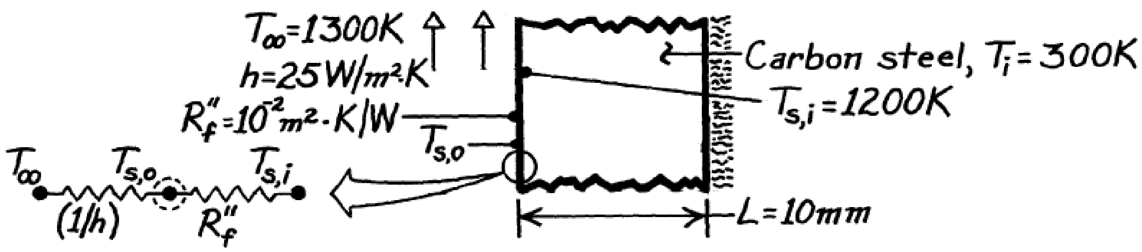
At furnace start-up the wall is at an initial temperature of T_i = 300 K, and combustion gases at T_∞ = 1300 K enter the furnace, providing a convection coefficient of h = 25 W/m^2 K at the ceramic film. Assuming the film to have negligible thermal capacitance, how long will it take for the inner surface of the steel to achieve a temperature of $T_{s,i}$ = 1200 K? What is the temperature $T_{s,o}$ of the exposed surface of the ceramic film at this time?

Sol:

KNOWN: Thickness and properties of furnace wall. Thermal resistance of film on surface of wall exposed to furnace gases. Initial wall temperature.

FIND: (a) Time required for surface of wall to reach a prescribed temperature, (b) Corresponding value of film surface temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible film thermal capacitance, (3) Negligible radiation.

PROPERTIES: Carbon steel (given): ρ = 7850 kg/m^3, c = 430 J/kg·K, k = 60 W/m·K.

ANALYSIS: The overall coefficient for heat transfer from the surface of the steel to the gas is

$$U = \left(R''_{tot}\right)^{-1} = \left(\frac{1}{h} + R''_f\right)^{-1} = \left(\frac{1}{25\ \text{W/m}^2\cdot\text{K}} + 10^{-2}\,\text{m}^2\cdot\text{K/W}\right)^{-1} = 20\ \text{W/m}^2\cdot\text{K}.$$

Hence,

$$Bi = \frac{UL}{k} = \frac{20\ \text{W/m}^2\cdot\text{K} \times 0.01\ \text{m}}{60\ \text{W/m}\cdot\text{K}} = 0.0033$$

and the lumped capacitance method can be used.

(a) It follows that

$$\frac{T-T_\infty}{T_i-T_\infty}=\exp\left(-t/\tau_t\right)=\exp\left(-t/RC\right)=\exp\left(-Ut/\rho Lc\right)$$

$$t=-\frac{\rho Lc}{U}\ln\frac{T-T_\infty}{T_i-T_\infty}=-\frac{7850\ \text{kg/m}^3\left(0.01\ \text{m}\right)430\ \text{J/kg}\cdot\text{K}}{20\ \text{W/m}^2\cdot\text{K}}\ln\frac{1200-1300}{300-1300}$$

$$t=3886\text{s}=1.08\text{h}.$$ <

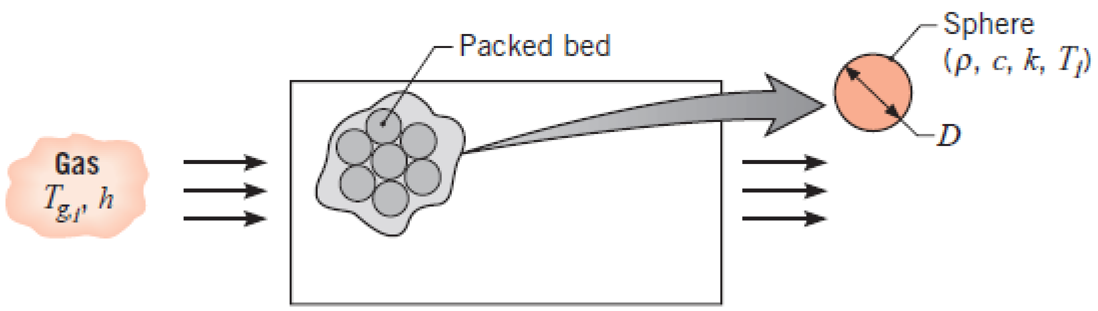
(b) Performing an energy balance at the outer surface (s,o),

$$h\left(T_\infty-T_{s,o}\right)=\left(T_{s,o}-T_{s,i}\right)/R''_f$$

$$T_{s,o}=\frac{hT_\infty+T_{s,i}/R''_f}{h+\left(1/R''_f\right)}=\frac{25\ \text{W/m}^2\cdot\text{K}\times1300\ \text{K}+1200\ \text{K}/10^{-2}\text{m}^2\cdot\text{K/W}}{\left(25+100\right)\text{W/m}^2\cdot\text{K}}$$

$$T_{s,o}=1220\ \text{K}.$$ <

Q3. Thermal energy storage systems commonly involve a packed bed of solid spheres, through which a hot gas flows if the system is being charged, or a cold gas if it is being discharged. In a charging process, heat transfer from the hot gas increases thermal energy stored within the colder spheres; during discharge, the stored energy decreases as heat is transferred from the warmer spheres to the cooler gas.



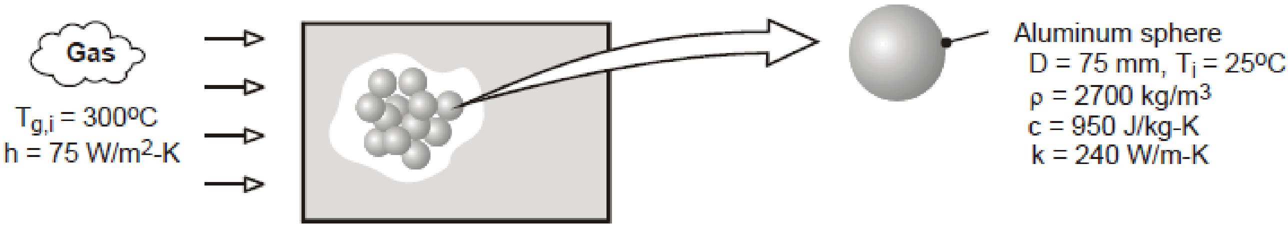
Consider a packed bed of 75-mm-diameter aluminum spheres (ρ = 2700 kg/m3, c = 950 J/kg K, k = 240 W/m K) and a charging process for which gas enters the storage unit at a temperature of Tg,i = 300°C. If the initial temperature of the spheres is Ti = 25°C and the convection coefficient is h =75 W/m2 K, how long does it take a sphere near the inlet of the system to accumulate 90% of the maximum possible thermal energy? What is the corresponding temperature at the center of the sphere? Is there any advantage to using copper instead of aluminum?

Sol:

KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere to acquire 90% of maximum possible thermal energy and the corresponding center temperature. Potential advantage of using copper in lieu of aluminum.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute Bi = $h(r_o/3)/k = 75\ W/m^2{\cdot}K\ (0.025m)/150\ W/m{\cdot}K = 0.013 < 0.1$. Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time. From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$\frac{Q}{\rho c V \theta_i} = 0.90 = 1 - \exp\left(-t/\tau_t\right)$$

where $\tau_t = \rho Vc/hA_s = \rho Dc/6h = 2700\,kg/m^3 \times 0.075m \times 950\,J/kg\cdot K/6 \times 75\,W/m^2\cdot K = 427s$. Hence,

$$t = -\tau_t \ln\left(0.1\right) = 427s \times 2.30 = 984s \qquad <$$

From Eq. (5.6), the corresponding temperature at any location in the sphere is

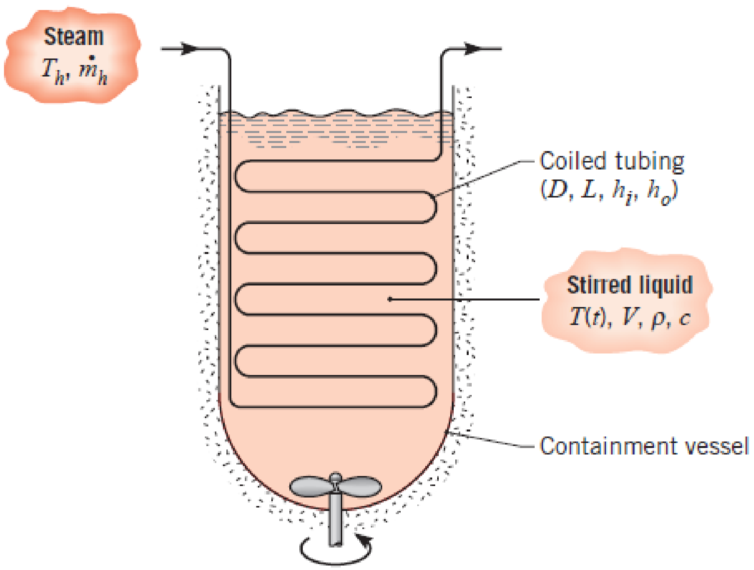
$$T\left(984s\right) = T_{g,i} + \left(T_i - T_{g,i}\right)\exp\left(-6ht/\rho Dc\right)$$

$$T\left(984s\right) = 300°C - 275°C\ \exp\left(-6 \times 75\,W/m^2\cdot K \times 984s/2700\,kg/m^3 \times 0.075m \times 950\,J/kg\cdot K\right)$$

$$T\left(984\right)s = 272.5°C \qquad <$$

Obtaining the density and specific heat of copper from Table A-1, we see that $(\rho c)_{Cu} \approx 8900\ kg/m^3 \times 400\ J/kg{\cdot}K = 3.56 \times 10^6\ J/m^3{\cdot}K > (\rho c)_{Al} = 2.57 \times 10^6\ J/m^3{\cdot}K$. Hence, for an equivalent sphere diameter, the copper can store approximately 38% more thermal energy than the aluminum.

Q4. Batch processes are often used in chemical and pharmaceutical operations to achieve a desired chemical composition for the final product and typically involve a transient heating operation to take the product from room temperature to the desired process temperature. Consider a situation for which a chemical of density ρ = l200 kg/m3 and specific heat c =2200 J/kg K occupies a volume of V = 2.25m^3 in an insulated vessel. The chemical is to be heated from room temperature, Ti = 300 K, to a process temperature of T = 450 K by passing saturated steam at Th = 500 K through a coiled, thin-walled, 20-mm-diameter tube in the vessel. Steam condensation within the tube maintains an interior convection coefficient of hi = 10,000 W/m^2 K, while the highly agitated liquid in the stirred vessel maintains an outside convection coefficient of ho = 2000 W/m^2 K.



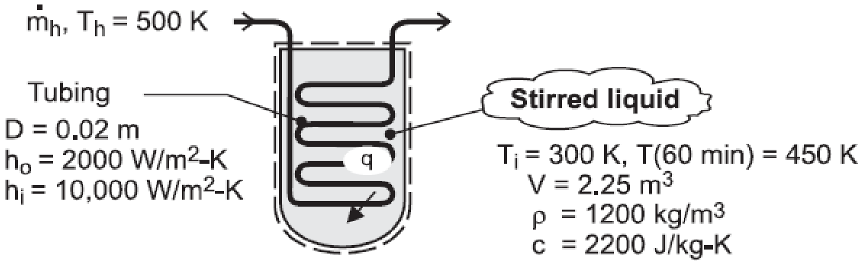
If the chemical is to be heated from 300 to 450 K in 60 min, what is the required length L of the submerged tubing?

Sol:

KNOWN: Volume, density and specific heat of chemical in a stirred reactor. Temperature and convection coefficient associated with saturated steam flowing through submerged coil. Tube diameter and outer convection coefficient of coil. Initial and final temperatures of chemical and time span of heating process.

FIND: Required length of submerged tubing. Minimum allowable steam flowrate.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible heat loss from vessel to surroundings, (3) Chemical is isothermal, (4) Negligible work due to stirring, (5) Negligible thermal energy generation (or absorption) due to chemical reactions associated with the batch process, (6) Negligible tube wall conduction resistance, (7) Negligible kinetic energy, potential energy, and flow work changes for steam.

ANALYSIS: Heating of the chemical can be treated as a transient, lumped capacitance problem, wherein heat transfer from the coil is balanced by the increase in thermal energy of the chemical. Hence, conservation of energy yields

$$\frac{dU}{dt} = \rho V c \frac{dT}{dt} = UA_s\left(T_h - T\right)$$

Integrating, $$\int_{T_i}^{T} \frac{dT}{T_h - T} = \frac{UA_s}{\rho V c}\int_o^t dt$$

$$-\ln\frac{T_h - T}{T_h - T_i} = \frac{UA_s t}{\rho V c}$$

$$A_s = -\frac{\rho V c}{U t}\ln\frac{T_h - T}{T_h - T_i} \qquad (1)$$

$$U = \left(h_i^{-1} + h_o^{-1}\right)^{-1} = \left[(1/10,000) + (1/2000)\right]^{-1} \mathrm{W/m^2 \cdot K}$$

$$U = 1670\,\mathrm{W/m^2 \cdot K}$$

$$A_s = -\frac{\left(1200\,\mathrm{kg/m^3}\right)\left(2.25\mathrm{m^3}\right)\left(2200\,\mathrm{J/kg \cdot K}\right)}{\left(1670\,\mathrm{W/m^2 \cdot K}\right)(3600\mathrm{s})}\ln\frac{500-450}{500-300} = 1.37\mathrm{m^2}$$

$$L = \frac{A_s}{\pi D} = \frac{1.37\mathrm{m^2}}{\pi(0.02\mathrm{m})} = 21.8\mathrm{m}$$ <

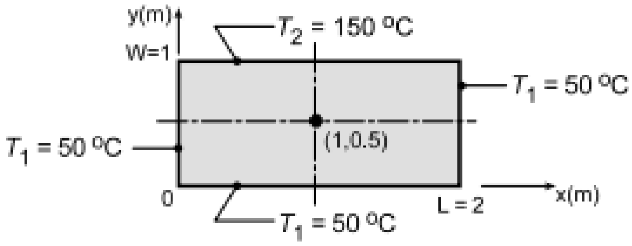
COMMENTS: Eq. (1) could also have been obtained by adapting Eq. (5.5) to the conditions of this problem, with T_∞ and h replaced by T_h and U, respectively.

5. A two-dimensional rectangular plate is subjected to prescribed boundary conditions. Solve for the temperature distribution and calculate the temperature at the midpoint (1, 0.5) by considering the first five nonzero terms of the infinite series that must be evaluated. Assess the error resulting from using only the first three terms of the infinite series. Plot the temperature distributions $T(x, 0.5)$ and $T(1.0, y)$.

KNOWN: Two-dimensional rectangular plate subjected to prescribed uniform temperature boundary conditions.

FIND: Temperature at the mid-point using the exact solution considering the first five non-zero terms; assess error resulting from using only first three terms. Plot the temperature distributions T(x,0.5) and T(1,y).

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, the temperature distribution is

$$\theta(x,y) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi x}{L}\right) \cdot \frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (1,4.19)$$

Considering now the point (x,y) = (1.0,0.5) and recognizing x/L = 1/2, y/L = 1/4 and W/L = 1/2,

$$\theta(1,0.5) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{\sinh(n\pi/4)}{\sinh(n\pi/2)}.$$

When n is even (2, 4, 6 ...), the corresponding term is zero; hence we need only consider n = 1, 3, 5, 7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi}\left\{2\sin\left(\frac{\pi}{2}\right)\frac{\sinh(\pi/4)}{\sinh(\pi/2)} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh(3\pi/4)}{\sinh(3\pi/2)} + \right.$$

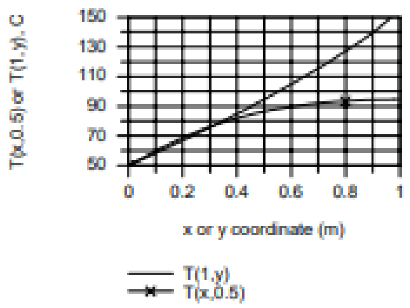
$$\left. \frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh(5\pi/4)}{\sinh(5\pi/2)} + \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh(7\pi/4)}{\sinh(7\pi/2)} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh(9\pi/4)}{\sinh(9\pi/2)}\right\}$$

$$\theta(1,0.5) = \frac{2}{\pi}[0.755 - 0.063 + 0.008 - 0.001 + 0.000] = 0.445 \qquad (2)$$

$$T(1,0.5) = \theta(1,0.5)(T_2 - T_1) + T_1 = 0.445(150-50)+50 = 94.5^\circ C. \qquad <$$

If only the first three terms of the series, Eq. (2), are considered, the result will be θ(1,0.5) = 0.46; that is, there is less than a 0.2% effect.

Using Eq. (1), and writing out the first five terms of the series, expressions for θ(x,0.5) or T(x,0.5) and θ(1,y) or T(1,y) were keyboarded into the IHT workspace and evaluated for sweeps over the x or y variable. Note that for T(1,y), that as y → 1, the upper boundary, T(1,1) is greater than 150°C. Upon examination of the magnitudes of terms, it becomes evident that more than 5 terms are required to provide an accurate solution.



6. For two dimension heat conduction in a plate, as shown in figure 3, find the temperature distribution T (x,y) by solving the boundary value problem. Use the steady state heat conduction equation.

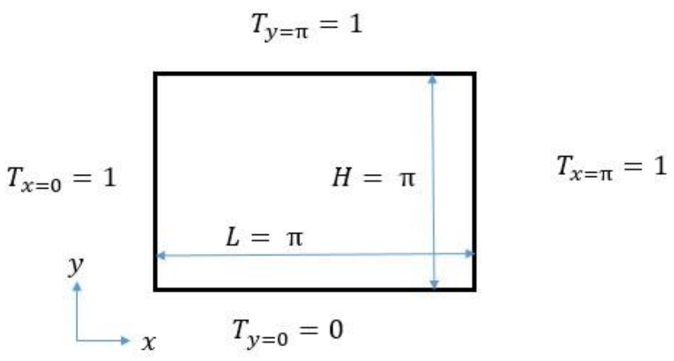


Figure 3

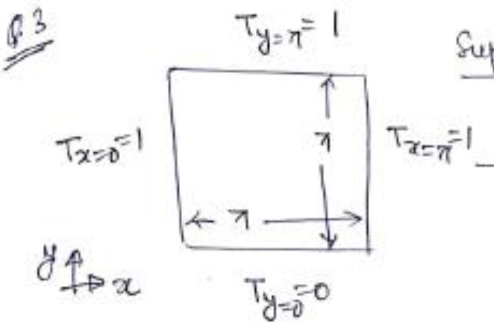
Known: Square plate prescribed with known boundary conditions on sides.

Find: Temperature distribution in the plate

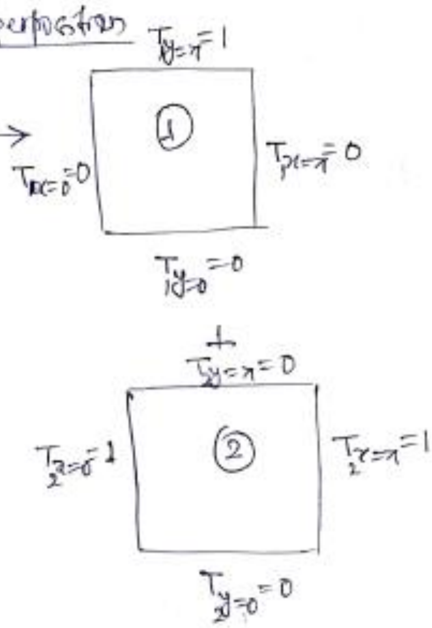
Assumptions: 1. 2 D heat conduction, 2. Steady-state, 3. Constant properties

Analysis: Use superposition to split the problem into simpler sub-problems

Q 3



Superposition



Such that

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

can be split into,

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0$$

$T_1(0,y) = T_1(\pi,y) = 0$ | $T_2(0,y) = T_2(\pi,y) = 1$

$T_1(x,0) = 0,\ T_1(x,\pi) = 1$ | $T_2(x,0) = 0,\ T_2(x,\pi) = 0$

For problem 1, $T_1(x,y) = X(x)\,Y(y)$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2}\cdot\frac{1}{X} + \frac{\partial^2 Y}{\partial y^2}\cdot\frac{1}{Y} = -\lambda^2$$

$$\Rightarrow X'' + \lambda^2 X = 0 \quad \text{and} \quad Y'' - \lambda^2 Y = 0$$

It gives,

$$X(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x) \qquad \text{— (1)}$$

$$Y(y) = C_3\cosh(\lambda y) + C_4\sinh(\lambda y) \qquad \text{— (2)}$$

Putting $T_1(0,y) = T_1(\pi,y) = 0 \Rightarrow X(0) = X(\pi) = 0$ in (1),

$$C_1 = 0 \text{ and } \lambda = \frac{n\pi}{\pi} = n\ ;\ n = 0, 1, 2 \dots$$

So, $X(x) = \sum_{n=0}^{\infty} C_{2,n} \sin nx$

and, putting $T_1(x,0) = 0 \Rightarrow Y(0) = 0$ in (2),

$C_3 = 0$

Now, $Y(y) = \sum_{n=0}^{\infty} C_{4,n} \sinh(ny)$

So, $T_1(x,y) = \sum_{n=0}^{\infty} C_n \sinh(ny) \cdot \sin(nx)$

Putting $T_1(x,\pi) = 1$

$$\Rightarrow 1 = \sum_{n=0}^{\infty} C_n \sinh(n\pi) \cdot \sin nx$$

$$\Rightarrow C_n = \frac{\frac{2}{\pi} \cdot \int_0^{\pi} \sin(nx)\, dx}{\sinh(n\pi)}$$

Finally for problem (1),

$$T_1(x,y) = \frac{2}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{\sinh(n\pi)} \int_0^{\pi} \sin(nx)\,dx \right) \sinh(ny) \sin nx \quad \text{— (A)}$$

Now, for problem (2),

~~$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$~~

homogeneous direction is y-direction.

So, $X'' - \lambda^2 X = 0$ and, $Y'' + \lambda^2 Y = 0$

and, $Y(y) = C_1 \cos\lambda y + C_2 \sin\lambda y$ — (3)

$X(x) = C_3 \cosh(\lambda x) + C_4 \sinh(\lambda x)$ — (4)

putting,

$T_2(x,0) = T_2(x,\pi) = 0 \Rightarrow Y(0) = Y(\pi) = 0$

it gives, $C_1 = 0$, and $\lambda = n$; $n = 0,1,2,\dots$

so, $Y(y) = \sum_{n=0}^{\infty} C_{2,n} \sin ny$

and, $T_2(x,y) = \sum_{n=0}^{\infty} \left(A_n \cosh(nx) + B_n \sinh(nx)\right) \sinh(ny)$ — (5)

at $x = 0$, $T_2(0,y) = 1$

$$\Rightarrow \quad 1 = \sum_{n=0}^{\infty} A_n \cdot \sinh(ny)$$

$$\Rightarrow A_n = \frac{2}{\pi} \int_0^{\pi} \sinh(ny)\, dy \qquad \text{—— (6)}$$

at $x = \pi$, $T_2(\pi,y) = 1$

$$\Rightarrow \quad 1 = \sum_{n=0}^{\infty} \left(A_n \cosh(n\pi) + B_n \sinh(n\pi)\right) \sinh(ny)$$

$$\Rightarrow A_n \cosh(n\pi) + B_n \sinh(n\pi) = \frac{2}{\pi} \int_0^{\pi} \sinh(ny)\, dy$$

$$\Rightarrow B_n = \frac{1}{\sinh(n\pi)} \cdot \left[\frac{2}{\pi} \int_0^{\pi} \sinh(ny)\, dy - A_n \cdot \cosh(n\pi)\right] \text{ — (7)}$$

Eqn (A) ~~and~~, (5) (6) (7) constitutes soln to problem (2).

Final soln is,

$$\boxed{T(x,y) = T_1(x,y) + T_2(x,y)}$$ Ans

7. Obtain the temperature distribution T (x,y) for the square plate subjected to two-dimensional steady-state heat conduction. Referring to figure 2, the left side wall is subjected to $T = T_0$, and both the right and bottom sides are insulated such that $\frac{\partial T}{\partial x}\ (x = L) = \frac{\partial T}{\partial y}\ (y = 0) = 0$. The top side is cooled by convection, which can be expressed as $-k\frac{\partial T}{\partial y}\ (y = th/2) = h(T_{y=th/2} - T_\infty)$.

Use the steady state heat conduction equation as: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.

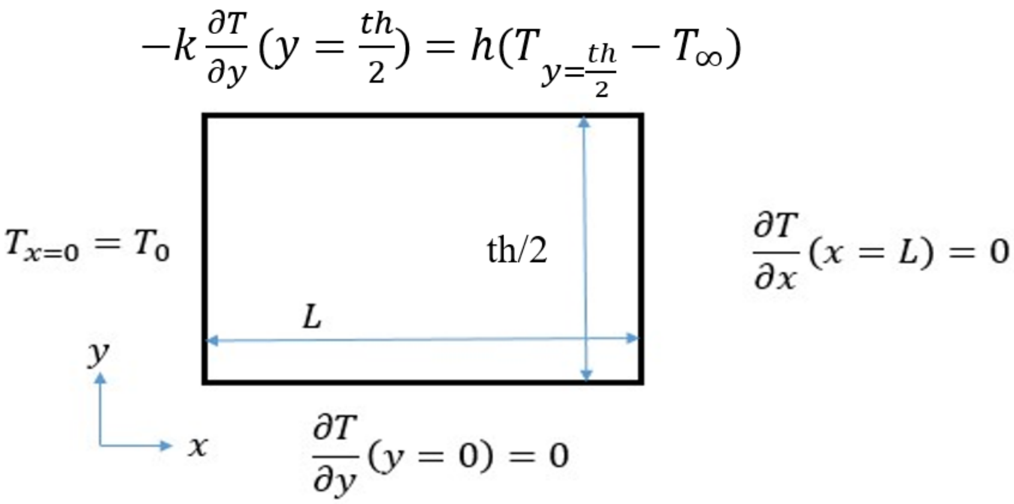


Figure 2

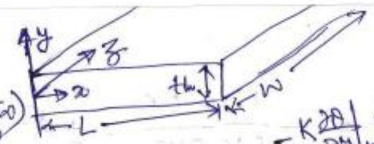
Known: Square plate prescribed with known boundary conditions on sides.

Find: Temperature distribution in the plate

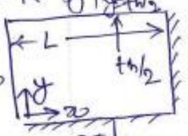
Assumptions: 1. 2 D heat conduction, 2. Steady-state, 3. Constant properties

Analysis:

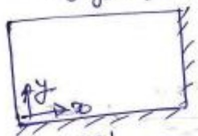
Q.② Solution



$$-K\left.\frac{\partial T}{\partial y}\right|_{y=th/2} = h\left(T_{y=th/2} - T_\infty\right) \qquad -K\left.\frac{\partial \theta}{\partial y}\right|_{y=th/2} = h\left(\theta_{y=th/2} - \theta_\infty\right)$$



$$T|_{x=0} = T_b \qquad \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \qquad \left.\frac{\partial T}{\partial y}\right|_{y=0} = 0$$



$$\theta_{x=0} = 0 \qquad \left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0 \qquad \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = 0$$

$$\theta = T - T_b$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial T}{\partial x} \;\&\; \frac{\partial \theta}{\partial y} = \frac{\partial T}{\partial y}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \theta_\infty = T_\infty - T_b \qquad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

So, x - dirn is homogeneous direction.

Governing eqn, $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$ —— ①

Boundary conditions,

$$\theta_{x=0} = 0 \;;\; \left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0 \;\&\; \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = 0 \;;\; -K\left.\frac{\partial \theta}{\partial y}\right|_{y=th/2} = h\left(\theta_{y=th/2} - \theta_\infty\right)$$

taking, $\theta(x,y) = \theta X(x) \cdot \theta Y(y)$

Putting in eqn ①,

$$\theta Y(y)\frac{\partial^2 \theta X(x)}{\partial x^2} + \theta X(x)\frac{\partial^2 \theta Y(y)}{\partial y^2} = 0$$

$$\Rightarrow \boxed{\frac{1}{\theta X}\frac{\partial^2 \theta X}{\partial x^2} + \frac{1}{\theta Y}\frac{\partial^2 \theta Y}{\partial y^2} = 0}$$

$$\Rightarrow \frac{1}{\theta X}\frac{\partial^2 \theta X}{\partial x^2} = -\frac{1}{\theta Y}\frac{\partial^2 \theta Y}{\partial y^2} = \pm\lambda^2$$

$$\Rightarrow \frac{\partial^2 \theta X}{\partial x^2} \pm \lambda^2 \theta X = 0 \quad \&$$

$$\frac{\partial^2 \theta Y}{\partial y^2} \mp \lambda^2 \theta Y = 0$$

$\because$ x-dirⁿ is homogeneous dirⁿ.

$$\boxed{\frac{\partial^2 \theta X}{\partial x^2} + \lambda^2 \theta X = 0} \;\text{②} \quad \& \quad \boxed{\frac{\partial^2 \theta Y}{\partial y^2} - \lambda^2 \theta Y = 0} \;\text{③}$$

for, eqⁿ ②,

$$\theta X = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$$

& for eqⁿ ③,

$$\theta Y = C_3 \sinh(\lambda y) + C_4 \cosh(\lambda y)$$

Solving for x-dirⁿ,

$$\theta X = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$$

BCs, $\theta_{x=0} = 0 \Rightarrow \theta X\big|_{x=0} = 0$

& $\frac{\partial \theta}{\partial x}\Big|_{x=L} = 0 \Rightarrow \frac{\partial \theta X}{\partial x}\Big|_{x=L} = 0$

$\Rightarrow$ for, $\theta X|_{x=0} = 0$

$\Rightarrow C_1 \cancel{\sin(0)}^{0} + C_2 \cos(0) = 0 \Rightarrow \boxed{C_2 = 0}$

for, $\left.\frac{\partial \theta X}{\partial x}\right|_{x=L} = 0 \Rightarrow$

$\Rightarrow C_1 \lambda \cos(\lambda L) = 0 \Rightarrow \cos(\lambda L) = 0 \quad (\because C_1 \neq 0)$

$\Rightarrow \underbrace{\lambda_i L = \frac{(1+2i)}{2}\pi}_{\text{eigenvalues.}}$ where, $i = 0, 1, 2, \ldots$

Eigen functions,

$$\theta X_i = C_{1,i} \sin(\lambda_i x) \; ; \; i = 0, 1, 2, \ldots$$

Now, in <u>y-dirn</u>

$$\theta Y_i = C_{3,i} \sinh(\lambda_i y) + C_{4,i} \cosh(\lambda_i y)$$

$\because \theta_i = \theta X_i \cdot \theta Y_i$

$$= C_{1,i} \sin(\lambda_i x) \left[C_{3,i} \sinh(\lambda_i y) + C_{4,i} \cosh(\lambda_i y) \right]$$

$$\Rightarrow \theta = \sum_{i=0}^{\infty} \theta_i = \sum_{i=0}^{\infty} C_{1,i} \sin(\lambda_i x) \left[C_{3,i} \sinh(\lambda_i y) + C_{4,i} \cosh(\lambda_i y) \right]$$

$$\boxed{\theta = \sum_{i=0}^{\infty} \sin(\lambda_i x) \left[C_{5,i} \sinh(\lambda_i y) + C_{6,i} \cosh(\lambda_i y) \right]}$$

Now, using 3rd BC, in y-dirn

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=0} = 0$$

$$\Rightarrow \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = \sum_{i=0}^{\infty} \sin(\lambda_i x)\left[C_{5,i}\lambda_i \cancel{\cosh(0)}^{1} + C_{6,i}\lambda_i \cancel{\sinh(0)}^{0}\right] = 0$$

$$\Rightarrow \sum_{i=0}^{\infty} \sin(\lambda_i x)\cdot C_{5,i} = 0 \Rightarrow \boxed{C_{5,i} = 0}$$

hence, we have now,

$$\boxed{\theta = \sum_{i=0}^{\infty} C_i \sin(\lambda_i x)\cosh(\lambda_i y)}$$

Putting last BC,

$$-K\left.\frac{\partial \theta}{\partial y}\right|_{y=th/2} = h\left(\theta_{y=th/2} - \theta_\infty\right)$$

$$\Rightarrow -K\left[\sum_{i=0}^{\infty} \sin(\lambda_i x)\cdot C_i \lambda_i \sinh(\lambda_i th/2)\right] = h\left[\sum_{i=0}^{\infty} C_i \sin(\lambda_i x)\cosh(\lambda_i th/2) - \theta_\infty\right]$$

$$\Rightarrow +\sum_{i=0}^{\infty} C_i \sin(\lambda_i x)\left\{\frac{K\lambda_i}{h}\sinh(\lambda_i th/2) + \cosh(\lambda_i th/2)\right\} = \theta_\infty$$

Now, using orthogonality,

$$\Rightarrow \boxed{C_i = \frac{\theta_\infty \int_0^L \sin(\lambda_i x)\,dx}{\left[\frac{K\lambda_i}{h}\sinh\left(\lambda_i t_{h/2}\right) + \cosh\left(\lambda_i t_{h/2}\right)\right] \int_0^L \sin^2(\lambda_i x)\,dx}}$$

Ans