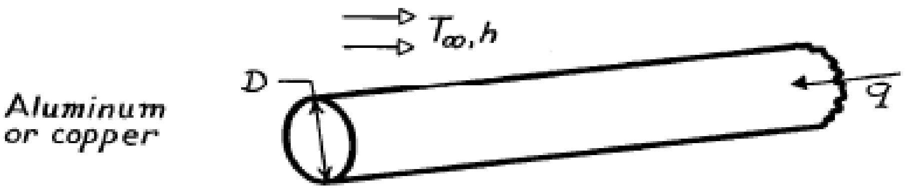
1. A long, circular aluminum rod is attached at one end to a heated wall and transfers heat by convection to a cold fluid.
(a) If the diameter of the rod is tripled, by how much would the rate of heat removal change?
(b) If a copper rod of the same diameter is used in place of the aluminum, by how much would the rate of heat removal change?

Solution:

KNOWN: Long, aluminum cylinder acts as an extended surface.

FIND: (a) Increase in heat transfer if diameter is tripled and (b) Increase in heat transfer if copper is used in place of aluminum.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Constant properties, (4) Uniform convection coefficient, (5) Rod is infinitely long.

PROPERTIES: *Table A-1,* Aluminum (pure): k = 240 W/m·K; *Table A-1,* Copper (pure): k = 400 W/m·K.

ANALYSIS: (a) For an infinitely long fin, the fin heat rate from Table 3.4 is

$$q_f = M = (hPkA_c)^{1/2}\theta_b$$

$$q_f = \left(h\,\pi\,D\,k\,\pi\,D^2/4\right)^{1/2}\theta_b = \frac{\pi}{2}(hk)^{1/2}D^{3/2}\,\theta_b.$$

where $P = \pi D$ and $A_c = \pi D^2/4$ for the circular cross-section. Note that $q_f \propto D^{3/2}$. Hence, if the diameter is tripled,

$$\frac{q_{f(3D)}}{q_{f(D)}} = 3^{3/2} = 5.2$$

and there is a 420% increase in heat transfer. <

(b) In changing from aluminum to copper, since $q_f \propto k^{1/2}$, it follows that

$$\frac{q_f\,(Cu)}{q_f\,(Al)} = \left[\frac{k_{Cu}}{k_{Al}}\right]^{1/2} = \left[\frac{400}{240}\right]^{1/2} = 1.29$$

and there is a 29% increase in the heat transfer rate. <

COMMENTS: (1) Because fin effectiveness is enhanced by maximizing $P/A_c = 4/D$, the use of a larger number of small diameter fins is preferred to a single large diameter fin.

(2) From the standpoint of cost and weight, aluminum is preferred over copper.

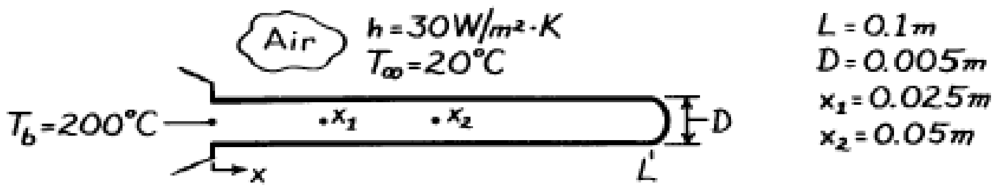
2. A brass rod 100 mm long and 5 mm in diameter extends horizontally from a casting at 200°C. The rod is in an air environment with T_∞ = 20°C and h = 30 W/m² K. What is the temperature of the rod 25, 50, and 100 mm from the casting?

Solution:

KNOWN: Length, diameter, base temperature and environmental conditions associated with a brass rod.

FIND: Temperature at specified distances along the rod.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Constant properties, (4) Negligible radiation, (5) Uniform convection coefficient h.

PROPERTIES: *Table A-1*, Brass $\left(\bar{T}=110^\circ C\right)$: $k=133\ W/m\cdot K$.

ANALYSIS: Evaluate first the fin parameter

$$m=\left[\frac{hP}{kA_c}\right]^{1/2}=\left[\frac{h\pi D}{k\pi D^2/4}\right]^{1/2}=\left[\frac{4h}{kD}\right]^{1/2}=\left[\frac{4\times 30\ W/m^2\cdot K}{133\ W/m\cdot K\times 0.005m}\right]^{1/2}$$

$$m=13.43\ m^{-1}.$$

Hence, $m\,L = (13.43)\times 0.1 = 1.34$ and from the results of Example 3.8, it is advisable not to make the infinite rod approximation. Thus from Table 3.4, the temperature distribution has the form

$$\theta=\frac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}\theta_b$$

Evaluating the hyperbolic functions, cosh mL = 2.04 and sinh mL = 1.78, and the parameter

$$\frac{h}{mk}=\frac{30\ W/m^2\cdot K}{13.43m^{-1}\ (133\ W/m\cdot K)}=0.0168,$$

with θ_b = 180°C the temperature distribution has the form

$$\theta=\frac{\cosh m(L-x)+0.0168\sinh m(L-x)}{2.07}\left(180^\circ C\right).$$

The temperatures at the prescribed location are tabulated below.

x(m)	cosh m(L-x)	sinh m(L-x)	θ	T(°C)	
x_1 = 0.025	1.55	1.19	136.5	156.5	<
x_2 = 0.05	1.24	0.725	108.9	128.9	<
L = 0.10	1.00	0.00	87.0	107.0	<

COMMENTS: If the rod were approximated as infinitely long: $T(x_1) = 148.7°C$, $T(x_2) = 112.0°C$, and $T(L) = 67.0°C$. The assumption would therefore result in significant underestimates of the rod temperature.

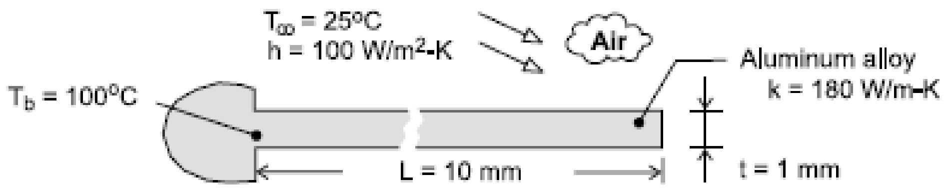
3. The extent to which the tip condition affects the thermal performance of a fin depends on the fin geometry and thermal conductivity, as well as the convection coefficient. Consider an alloyed aluminum (k = 180 W/m _ K) rectangular fin of length L = 10 mm, thickness t = 1 mm, and width $w >> t$. The base temperature of the fin is T_b = l00°C, and the fin is exposed to a fluid of temperature T_∞ = 25°C.
(a) Assuming a uniform convection coefficient of h = 100 W/m² K over the entire fin surface, determine the fin heat transfer rate per unit width q_f', efficiency η_f, effectiveness ε_f, thermal resistance per unit width $R_{t,f}'$, and the tip temperature $T(L)$. Contrast your results with those based on an *infinite fin* approximation.
(b) Explore the effect of variations in the convection coefficient on the heat rate for $10 < h < 1000$ W/m² K. Also consider the effect of such variations for a stainless steel fin (k = 15 W/m K).

Solution:

KNOWN: Thickness, length, thermal conductivity, and base temperature of a rectangular fin. Fluid temperature and convection coefficient.

FIND: (a) Heat rate per unit width, efficiency, effectiveness, thermal resistance, and tip temperature for different tip conditions, (b) Effect of convection coefficient and thermal conductivity on the heat rate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional conduction along fin, (3) Constant properties, (4) Negligible radiation, (5) Uniform convection coefficient, (6) Fin width is much longer than thickness (w >> t).

ANALYSIS: (a) The fin heat transfer rate for Cases A, B and D are given by Eqs. (3.72), (3.76) and (3.80), where $M \approx (2\,hw^2tk)^{1/2}\,(T_b - T_\infty) = (2 \times 100\ \mathrm{W/m^2\cdot K} \times 0.001\mathrm{m} \times 180\ \mathrm{W/m\cdot K})^{1/2}\,(75°C)\ w = 450\ w\ \mathrm{W}$, $m \approx (2h/kt)^{1/2} = (200\ \mathrm{W/m^2\cdot K}/180\ \mathrm{W/m\cdot K} \times 0.001\mathrm{m})^{1/2} = 33.3\mathrm{m^{-1}}$, $mL \approx 33.3\mathrm{m^{-1}} \times 0.010\mathrm{m} = 0.333$, and $(h/mk) \approx (100\ \mathrm{W/m^2\cdot K}/33.3\mathrm{m^{-1}} \times 180\ \mathrm{W/m\cdot K}) = 0.0167$. From Table B-1, it follows that $\sinh mL \approx 0.340$, $\cosh mL \approx 1.057$, and $\tanh mL \approx 0.321$. From knowledge of q_f, Eqs. (3.86), (3.81) and (3.83) yield

$$\eta_f \approx \frac{q_f'}{h(2L+t)\theta_b}, \quad \varepsilon_f = \frac{q_f'}{ht\,\theta_b}, \quad R_{t,f}' = \frac{\theta_b}{q_f'}$$

Case A: From Eq. (3.72), (3.86), (3.81), (3.83) and (3.70),

$$q_f' = \frac{M}{w}\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} = 450\ \mathrm{W/m}\,\frac{0.340 + 0.0167 \times 1.057}{1.057 + 0.0167 \times 0.340} = 151\ \mathrm{W/m}$$ <

$$\eta_f = \frac{151\,\text{W/m}}{100\,\text{W/m}^2\cdot\text{K}\left(0.021\text{m}\right)75^\circ\text{C}} = 0.96$$ <

$$\varepsilon_f = \frac{151\,\text{W/m}}{100\,\text{W/m}^2\cdot\text{K}\left(0.001\text{m}\right)75^\circ\text{C}} = 20.1,\ R'_{t,f} = \frac{75^\circ\text{C}}{151\,\text{W/m}} = 0.50\,\text{m}\cdot\text{K/W}$$ <

$$T(L) = T_\infty + \frac{\theta_b}{\cosh mL + (h/mk)\sinh mL} = 25^\circ\text{C} + \frac{75^\circ\text{C}}{1.057 + (0.0167)0.340} = 95.6^\circ\text{C}$$ <

Case B: From Eqs. (3.76), (3.86), (3.81), (3.83) and (3.75)

$$q'_f = \frac{M}{w}\tanh mL = 450\,\text{W/m}(0.321) = 144\,\text{W/m}$$ <

$$\eta_f = 0.92,\ \varepsilon_f = 19.2,\ R'_{t,f} = 0.52\ \text{m}\cdot\text{K/W}$$ <

$$T(L) = T_\infty + \frac{\theta_b}{\cosh mL} = 25^\circ\text{C} + \frac{75^\circ\text{C}}{1.057} = 96.0^\circ\text{C}$$ <

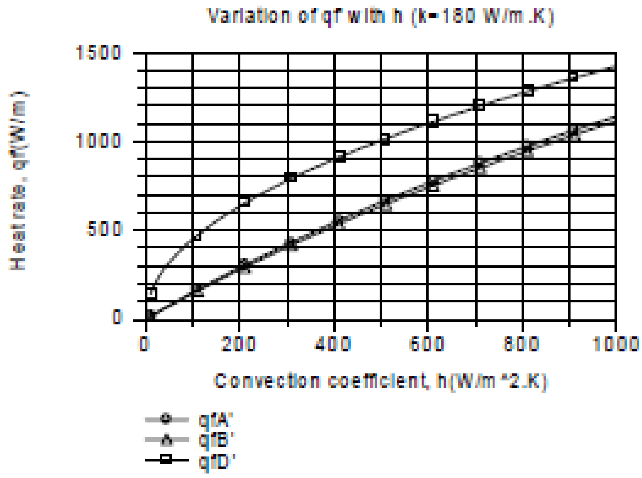
Continued .....

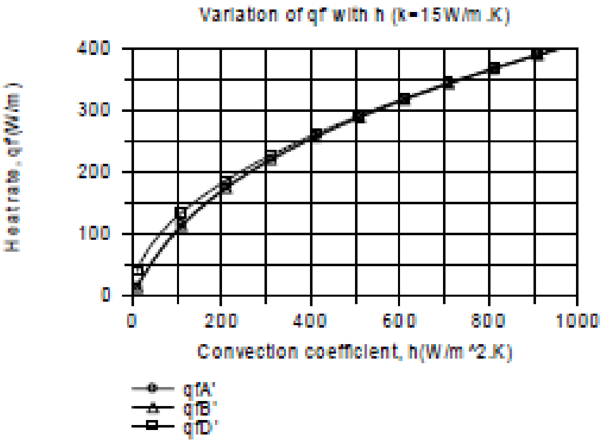
Case D ($L \to \infty$): From Eqs. (3.80), (3.86), (3.81), (3.83) and (3.79)

$$q'_f = \frac{M}{w} = 450\,\text{W/m}$$ <

$$\eta_f = 0,\ \varepsilon_f = 60.0,\ R'_{t,f} = 0.167\,\text{m}\cdot\text{K/W},\ T(L) = T_\infty = 25^\circ\text{C}$$ <

(b) The effect of h on the heat rate is shown below for the aluminum and stainless steel fins.





For both materials, there is little difference between the Case A and B results over the entire range of h. The difference (percentage) increases with decreasing h and increasing k, but even for the worst case condition (h = 10 W/m^2·K, k = 180 W/m·K), the heat rate for Case A (15.7 W/m) is only slightly larger than that for Case B (14.9 W/m). For aluminum, the heat rate is significantly over-predicted by the infinite fin approximation over the entire range of h. For stainless steel, it is over-predicted for small values of h, but results for all three cases are within 1% for h > 500 W/m^2·K.

COMMENTS: From the results of Part (a), we see there is a slight reduction in performance (smaller values of q_f', η_f and ε_f, as well as a larger value of $R'_{t,f}$) associated with insulating the tip.

Although $\eta_f = 0$ for the infinite fin, q_f' and ε_f are substantially larger than results for L = 10 mm, indicating that performance may be significantly improved by increasing L.

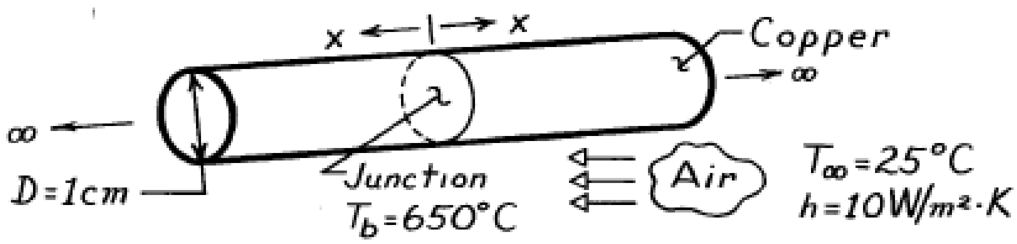
4. Two long copper rods of diameter *D* = 10 mm are soldered together end to end, with solder having a melting point of 650°C. The rods are in air at 25°C with a convection coefficient of 10 W/m^2 K. What is the minimum power input needed to effect the soldering?

Solution:

KNOWN: Melting point of solder used to join two long copper rods.

FIND: Minimum power needed to solder the rods.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction along the rods, (3) Constant properties, (4) No internal heat generation, (5) Negligible radiation exchange with surroundings, (6) Uniform h, and (7) Infinitely long rods.

PROPERTIES: *Table A-1:* Copper $\bar{T} = (650+25)^{\circ}\,C = 600K$: $k = 379$ W/m·K.

ANALYSIS: The junction must be maintained at 650°C while energy is transferred by conduction from the junction (along both rods). The minimum power is twice the fin heat rat for an infinitely long fin,

$$q_{min} = 2q_f = 2\left(hPkA_c\right)^{1/2}\left(T_b - T_\infty\right).$$

Substituting numerical values,

$$q_{min} = 2\left[10\frac{W}{m^2\cdot K}(\pi\times 0.01m)\left[379\frac{W}{m\cdot K}\right]\frac{\pi}{4}(0.01m)^2\right]^{1/2}(650-25)^{\circ}\,C.$$

Therefore,

$$q_{min} = 120.9\ W.$$ <

COMMENTS: Radiation losses from the rods may be significant, particularly near the junction, thereby requiring a larger power input to maintain the junction at 650°C.

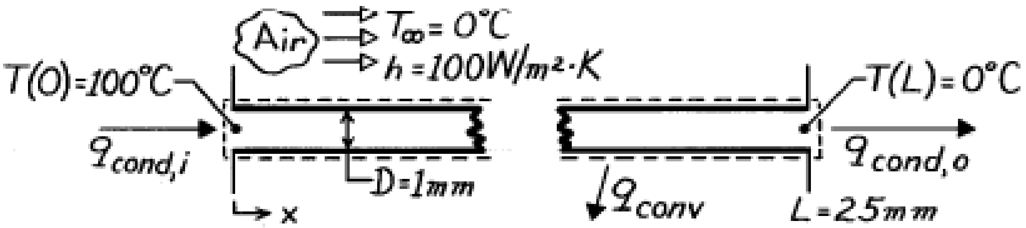
5. Circular copper rods of diameter D = 1 mm and length L = 25 mm are used to enhance heat transfer from a surface that is maintained at $T_{s,1}$ = 100°C. One end of the rod is attached to this surface (at x = 0), while the other end (x = 25 mm) is joined to a second surface, which is maintained at $T_{s,2}$ = 0°C. Air flowing between the surfaces (and over the rods) is also at a temperature of T_∞= 0°C, and a convection coefficient of h = 100 W/m² K is maintained.
(a) What is the rate of heat transfer by convection from a single copper rod to the air?
(b) What is the total rate of heat transfer from a 1-m by 1-m section of the surface at 100°C, if a bundle of the rods is installed on 4-mm centers?

Solution:

KNOWN: Dimensions and end temperatures of pin fins.

FIND: (a) Heat transfer by convection from a single fin and (b) Total heat transfer from a 1 m^2 surface with fins mounted on 4mm centers.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional conduction along rod, (3) Constant properties, (4) No internal heat generation, (5) Negligible radiation.

PROPERTIES: *Table A-1*, Copper, pure (323K): k = 400 W/m·K.

ANALYSIS: (a) By applying conservation of energy to the fin, it follows that

$$q_{conv} = q_{cond,i} - q_{cond,o}$$

where the conduction rates may be evaluated from knowledge of the temperature distribution. The general solution for the temperature distribution is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \qquad \theta \equiv T - T_\infty.$$

The boundary conditions are $\theta(0) \equiv \theta_o = 100°C$ and $\theta(L) = 0$. Hence

$$\theta_o = C_1 + C_2$$

$$0 = C_1 e^{mL} + C_2 e^{-mL}$$

Therefore, $$C_2 = C_1 e^{2mL}$$

$$C_1 = \frac{\theta_o}{1 - e^{2mL}}, \qquad C_2 = -\frac{\theta_o e^{2mL}}{1 - e^{2mL}}$$

and the temperature distribution has the form

$$\theta = \frac{\theta_o}{1 - e^{2mL}}\left[e^{mx} - e^{2mL-mx}\right].$$

The conduction heat rate can be evaluated by Fourier's law,

$$q_{cond} = -kA_c \frac{d\theta}{dx} = -\frac{kA_c\theta_o}{1 - e^{2mL}} m\left[e^{mx} + e^{2mL-mx}\right]$$

or, with $m = (hP/kA_c)^{1/2}$,

$$q_{cond} = -\frac{\theta_o (hPkA_c)^{1/2}}{1 - e^{2mL}}\left[e^{mx} + e^{2mL-mx}\right].$$

Hence at x = 0,

$$q_{cond,i} = -\frac{\theta_o (hPkA_c)^{1/2}}{1 - e^{2mL}}\left(1 + e^{2mL}\right)$$

at x = L

$$q_{cond,o} = -\frac{\theta_o (hPkA_c)^{1/2}}{1 - e^{2mL}}\left(2e^{mL}\right)$$

Evaluating the fin parameters:

$$m = \left[\frac{hP}{kA_c}\right]^{1/2} = \left[\frac{4h}{kD}\right]^{1/2} = \left[\frac{4 \times 100 \text{ W/m}^2 \cdot \text{K}}{400 \text{ W/m} \cdot \text{K} \times 0.001\text{m}}\right]^{1/2} = 31.62 \text{ m}^{-1}$$

$$(hPkA_c)^{1/2} = \left[\frac{\pi^2}{4} D^3 hk\right]^{1/2} = \left[\frac{\pi^2}{4} \times (0.001\text{m})^3 \times 100 \frac{\text{W}}{\text{m}^2 \cdot \text{K}} \times 400 \frac{\text{W}}{\text{m} \cdot \text{K}}\right]^{1/2} = 9.93 \times 10^{-3} \frac{\text{W}}{\text{K}}$$

$$mL = 31.62 \text{ m}^{-1} \times 0.025\text{m} = 0.791, \qquad e^{mL} = 2.204, \qquad e^{2mL} = 4.865$$

The conduction heat rates are

$$q_{cond,i} = \frac{-100\text{K}\left(9.93\times10^{-3}\text{ W/K}\right)}{-3.865}\times 5.865 = 1.507\text{ W}$$

$$q_{cond,o} = \frac{-100\text{K}\left(9.93\times10^{-3}\text{ W/K}\right)}{-3.865}\times 4.408 = 1.133\text{ W}$$

and from the conservation relation,

$$q_{conv} = 1.507\text{ W} - 1.133\text{ W} = 0.374\text{ W}. \qquad <$$

(b) The total heat transfer rate is the heat transfer from N = 250×250 = 62,500 rods and the heat transfer from the remaining (bare) surface ($A = 1\text{m}^2 - NA_c$). Hence,

$$q = N\, q_{cond,i} + hA\theta_o = 62{,}500\ (1.507\text{ W}) + 100\text{W/m}^2\cdot\text{K}\left(0.951\text{ m}^2\right)\ 100\text{K}$$

$$q = 9.42\times10^4\text{ W} + 0.95\times10^4\text{ W} = 1.037\times10^5\text{ W}.$$

COMMENTS: (1) The fins, which cover only 5% of the surface area, provide for more than 90% of the heat transfer from the surface.

(2) The fin effectiveness, $\varepsilon = q_{cond,i} / hA_c\theta_o$, is ε = 192, and the fin efficiency, $\eta = (q_{conv} / h\pi\, DL\theta_o)$, is $\eta = 0.48$.

(3) The temperature distribution, $\theta(x)/\theta_o$, and the conduction term, $q_{cond,i}$, could have been obtained directly from Eqs. 3.77and 3.78, respectively.

(4) Heat transfer by convection from a single fin could also have been obtained from Eq. 3.73.

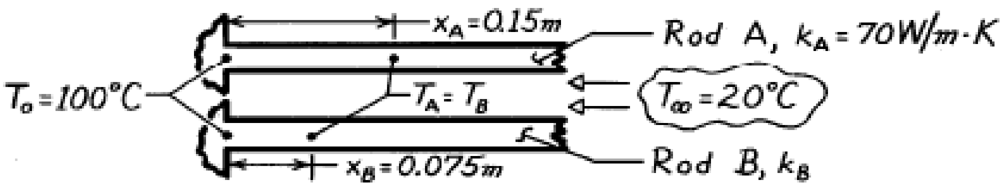
6. Consider two long, slender rods of the same diameter but different materials. One end of each rod is attached to a base surface maintained at 100°C, while the surfaces of the rods are exposed to ambient air at 20°C.
By traversing the length of each rod with a thermocouple, it was observed that the temperatures of the rods were equal at the positions $x_A = 0.15$ m and $x_B = 0.075$ m, where x is measured from the base surface. If the thermal conductivity of rod A is known to be $k_A = 70$ W/m K, determine the value of k_B for rod B.

Solution:

KNOWN: Positions of equal temperature on two long rods of the same diameter, but different thermal conductivity, which are exposed to the same base temperature and ambient air conditions.

FIND: Thermal conductivity of rod B, k_B.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Rods are infinitely long fins of uniform cross-sectional area, (3) Uniform heat transfer coefficient, (4) Constant properties.

ANALYSIS: The temperature distribution for the infinite fin has the form

$$\frac{\theta}{\theta_b} = \frac{T(x) - T_\infty}{T_o - T_\infty} = e^{-mx} \qquad m = \left[\frac{hP}{kA_c}\right]^{1/2}. \qquad (1,2)$$

For the two positions prescribed, x_A and x_B, it was observed that

$$T_A(x_A) = T_B(x_B) \qquad \text{or} \qquad \theta_A(x_A) = \theta_B(x_B). \qquad (3)$$

Since θ_b is identical for both rods, Eq. (1) with the equality of Eq. (3) requires that

$$m_A x_A = m_B x_B$$

Substituting for m from Eq. (2) gives

$$\left[\frac{hP}{k_A A_c}\right]^{1/2} x_A = \left[\frac{hP}{k_B A_c}\right]^{1/2} x_B.$$

Recognizing that h, P and A_c are identical for each rod and rearranging,

$$k_B = \left[\frac{x_B}{x_A}\right]^2 k_A$$

$$k_B = \left[\frac{0.075\text{m}}{0.15\text{m}}\right]^2 \times 70\ \text{W/m}\cdot\text{K} = 17.5\ \text{W/m}\cdot\text{K}. \qquad <$$

COMMENTS: This approach has been used as a method for determining the thermal conductivity. It has the attractive feature of not requiring power or temperature measurements, assuming of course, a reference material of known thermal conductivity is available.

7. A 40-mm-long, 2-mm-diameter pin fin is fabricated of an aluminum alloy (k = 140 W/m K).

(a) Determine the fin heat transfer rate for $T_b = 50^0$C, $T_\infty = 25^0$C, h = 1000 W/m^2 K, and an adiabatic tip condition.

(b) An engineer suggests that by holding the fin tip at a low temperature, the fin heat transfer rate can be increased. For $T(x = L) = 0^0$C, determine the new fin heat transfer rate. Other conditions are as in part (a).

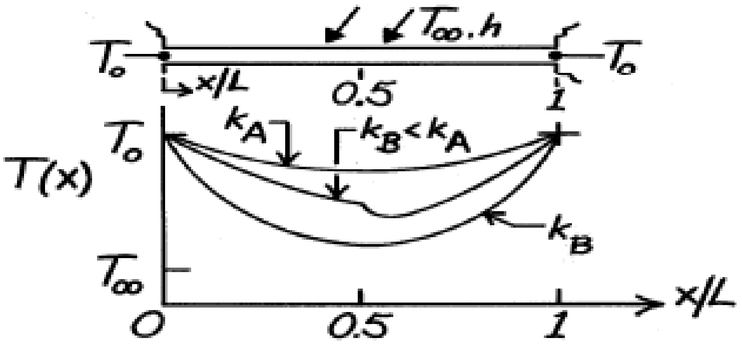
(c) Plot the temperature distribution, $T(x)$, over the range $0 \le x \le L$ for the adiabatic tip case and the prescribed tip temperature case. Also show the ambient temperature in your graph. Discuss relevant features of the temperature distribution.

Solution:

KNOWN: Slender rod of length L with ends maintained at T_o while exposed to convection cooling ($T_\infty < T_o$, h).

FIND: Temperature distribution for three cases, when rod has thermal conductivity (a) k_A, (b) $k_B < k_A$, and (c) k_A for $0 \le x \le L/2$ and k_B for $L/2 \le x \le L$.

SCHEMATIC:



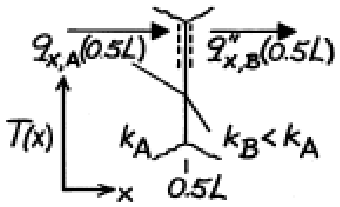
ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Constant properties, and (4) Negligible thermal resistance between the two materials (A, B) at the mid-span for case (c).

ANALYSIS: (a, b) The effect of thermal conductivity on the temperature distribution when all other conditions (T_o, h, L) remain the same is to reduce the minimum temperature with decreasing thermal conductivity. Hence, as shown in the sketch, the mid-span temperatures are $T_B(0.5L) < T_A(0.5L)$ for $k_B < k_A$. The temperature distribution is, of course, symmetrical about the mid-span.

(c) For the composite rod, the temperature distribution can be reasoned by considering the boundary condition at the mid-span.

$$q''_{x,A}(0.5L) = q''_{x,B}(0.5L)$$

$$-k_A \left.\frac{dT}{dx}\right)_{A,x=0.5L} = -k_B \left.\frac{dT}{dx}\right)_{B,x=0.5L}$$



Since $k_A > k_B$, it follows that

$$\left(\frac{dT}{dx}\right)_{A,x=0.5L} < \left(\frac{dT}{dx}\right)_{B,x=0.5L}.$$

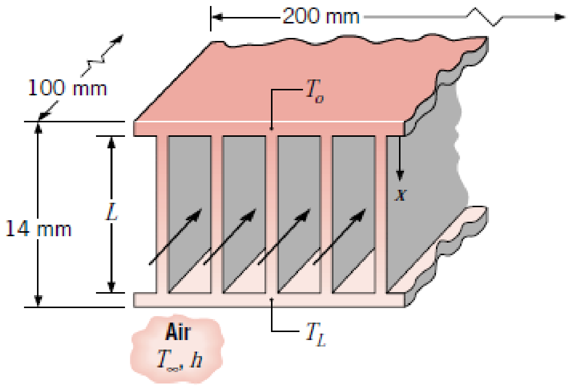
It follows that the minimum temperature in the rod must be in the k_B region, $x > 0.5L$, and the temperature distribution is not symmetrical about the mid-span.

COMMENTS: (1) Recognize that the area under the curve on the T-x coordinates is proportional to the fin heat rate. What conclusions can you draw regarding the relative magnitudes of q_{fin} for cases (a), (b) and (c)?

(2) If L is increased substantially, how would the temperature distribution be affected?

8. Finned passages are frequently formed between parallel plates to enhance convection heat transfer in compact heat exchanger cores. An important application is in electronic equipment cooling, where one or more air-cooled stacks are placed between heat-dissipating electrical

components. Consider a single stack of rectangular fins of length L and thickness t, with convection conditions corresponding to h and T_∞.



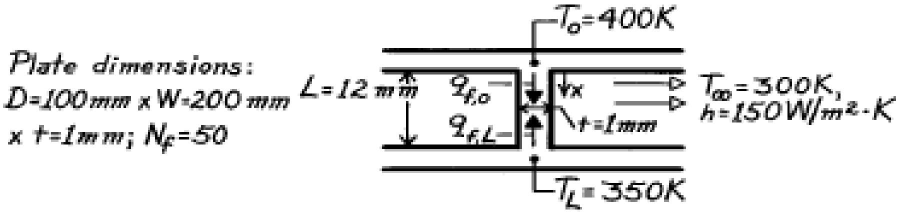
(a) Obtain expressions for the fin heat transfer rates, $q_{f,0}$ and $q_{f,L}$, in terms of the base temperatures, T_o and T_L.
(b) In a specific application, a stack that is 200 mm wide and 100 mm deep contains 50 fins, each of length L = 12 mm. The entire stack is made from aluminum, which is everywhere 1.0 mm thick. If temperature limitations associated with electrical components joined to opposite plates dictate maximum allowable plate temperatures of T_o = 400 K and T_L = 350 K, what are the corresponding maximum power dissipations if h = 150 W/m^2 K and T_∞ = 300 K?

Solution:

KNOWN: Arrangement of fins between parallel plates. Temperature and convection coefficient of air flow in finned passages. Maximum allowable plate temperatures.

FIND: (a) Expressions relating fin heat transfer rates to end temperatures, (b) Maximum power dissipation for each plate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in fins, (3) Constant properties, (4) Negligible radiation, (5) All of the heat is dissipated to the air, (6) Uniform h, (7) Negligible variation in T_∞, (8) Negligible contact resistance.

PROPERTIES: *Table A.1*, Aluminum (pure), 375 K: k = 240 W/m·K.

ANALYSIS: (a) The general solution for the temperature distribution in fin is

$$\theta(x) \equiv T(x) - T_\infty = C_1 e^{mx} + C_2 e^{-mx}$$

Boundary conditions: $\theta(0) = \theta_o = T_o - T_\infty$, $\theta(L) = \theta_L = T_L - T_\infty$.

Hence $\theta_o = C_1 + C_2$ $\theta_L = C_1 e^{mL} + C_2 e^{-mL}$

$$\theta_L = C_1 e^{mL} + (\theta_o - C_1) e^{-mL}$$

$$C_1 = \frac{\theta_L - \theta_o e^{-mL}}{e^{mL} - e^{-mL}} \qquad C_2 = \theta_o - \frac{\theta_L - \theta_o e^{-mL}}{e^{mL} - e^{-mL}} = \frac{\theta_o e^{mL} - \theta_L}{e^{mL} - e^{-mL}}.$$

Hence
$$\theta(x) = \frac{\theta_L e^{mx} - \theta_o e^{m(x-L)} + \theta_o e^{m(L-x)} - \theta_L e^{-mx}}{e^{mL} - e^{-mL}}$$

$$\theta(x) = \frac{\theta_o \left[e^{m(L-x)} - e^{-m(L-x)} \right] + \theta_L \left(e^{mx} - e^{-mx} \right)}{e^{mL} - e^{-mL}}$$

$$\theta(x) = \frac{\theta_o \sinh m(L-x) + \theta_L \sinh mx}{\sinh mL}.$$

The fin heat transfer rate is then

$$q_f = -kA_c \frac{dT}{dx} = -kDt \left[-\frac{\theta_o m}{\sinh mL} \cosh m(L-x) + \frac{\theta_L m}{\sinh mL} \cosh mx \right].$$

Hence
$$q_{f,o} = kDt \left(\frac{\theta_o m}{\tanh mL} - \frac{\theta_L m}{\sinh mL} \right) \qquad <$$

$$q_{f,L} = kDt \left(\frac{\theta_o m}{\sinh mL} - \frac{\theta_L m}{\tanh mL} \right). \qquad <$$

(b)
$$m = \left(\frac{hP}{kA_c} \right)^{1/2} = \left(\frac{50\ \text{W/m}^2 \cdot \text{K} \left(2 \times 0.1\ \text{m} + 2 \times 0.001\ \text{m} \right)}{240\ \text{W/m} \cdot \text{K} \times 0.1\ \text{m} \times 0.001\ \text{m}} \right)^{1/2} = 35.5\ \text{m}^{-1}$$

$$mL = 35.5\ \text{m}^{-1} \times 0.012\ \text{m} = 0.43$$

$$\sinh\ mL = 0.439 \qquad \tanh mL = 0.401 \qquad \theta_o = 100\ \text{K} \qquad \theta_L = 50\ \text{K}$$

$$q_{f,o} = 240\ \text{W/m} \cdot \text{K} \times 0.1\ \text{m} \times 0.001\ \text{m} \left(\frac{100\ \text{K} \times 35.5\ \text{m}^{-1}}{0.401} - \frac{50\ \text{K} \times 35.5\ \text{m}^{-1}}{0.439} \right)$$

$q_{f,o} = 115.4\ \text{W}$ (*from* the top plate)

$$q_{f,L} = 240\ \text{W/m} \cdot \text{K} \times 0.1\ \text{m} \times 0.001\ \text{m} \left(\frac{100\ \text{K} \times 35.5\ \text{m}^{-1}}{0.439} - \frac{50\ \text{K} \times 35.5\ \text{m}^{-1}}{0.401} \right)$$

$q_{f,L} = 87.8\ \text{W}.$ (*into* the bottom plate)

Maximum power dissipations are therefore

$$q_{o,max} = N_f q_{f,o} + \left(W - N_f t \right) D h \theta_o$$

$$q_{o,max} = 50 \times 115.4\ \text{W} + \left(0.200 - 50 \times 0.001 \right) \text{m} \times 0.1\ \text{m} \times 150\ \text{W/m}^2 \cdot \text{K} \times 100\ \text{K}$$

$$q_{o,max} = 5770\ \text{W} + 225\ \text{W} = 5995\ \text{W} \qquad <$$

$$q_{L,max} = -N_f q_{f,L} + \left(W - N_f t \right) D h \theta_o$$

$$q_{L,max} = -50 \times 87.8\,\text{W} + (0.200 - 50 \times 0.001)\,\text{m} \times 0.1\,\text{m} \times 150\,\text{W/m}^2 \cdot \text{K} \times 50\,\text{K}$$

$$q_{L,max} = -4390\,\text{W} + 112\,\text{W} = -4278\,\text{W}. \qquad <$$

COMMENTS: (1) It is of interest to determine the air velocity needed to prevent excessive heating of the air as it passes between the plates. If the air temperature change is restricted to $\Delta T_\infty = 5$ K, its flowrate must be

$$\dot{m}_{air} = \frac{q_{tot}}{c_p \Delta T_\infty} = \frac{1717\,\text{W}}{1007\,\text{J/kg} \cdot \text{K} \times 5\,\text{K}} = 0.34\,\text{kg/s}.$$

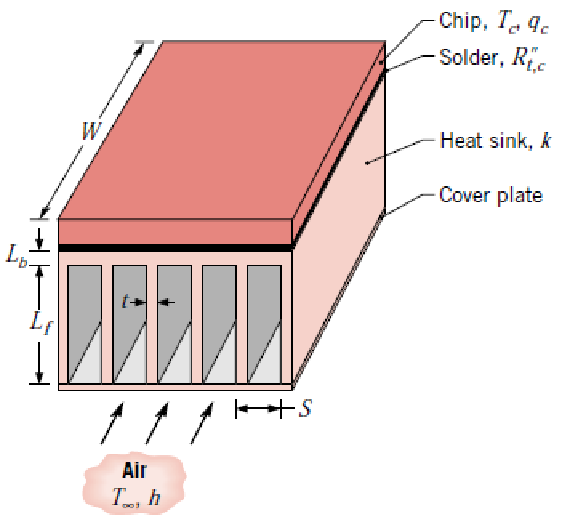
Its mean velocity is then

$$V_{air} = \frac{\dot{m}_{air}}{\rho_{air} A_c} = \frac{0.34\,\text{kg/s}}{1.16\,\text{kg/m}^3 \times 0.012\,\text{m}\,(0.2 - 50 \times 0.001)\,\text{m}} = 163\,\text{m/s}.$$

Such a velocity would be impossible to maintain. To reduce it to a reasonable value, e.g. 10 m/s, A_c would have to be increased substantially by increasing W (and hence the space between fins) and by increasing L. The present configuration is impractical from the standpoint that 1717 W could not be transferred to air in such a small volume.

(2) A negative value of $q_{L,max}$ implies that heat must be transferred from the bottom plate to the air to maintain the plate at 350 K.

9. An isothermal silicon chip of width W = 20 mm on a side is soldered to an aluminum heat sink (k = 180 W/m K) of equivalent width. The heat sink has a base thickness of L_b = 3 mm and an array of rectangular fins, each of length L_f = 15 mm. Air flow at T_∞ = 20^{0}C is maintained through channels formed by the fins and a cover plate, and for a convection coefficient of h = 100 W/m^2 K, a minimum fin spacing of 1.8 mm is dictated by limitations on the flow pressure drop. The solder joint has a thermal resistance of $R''_{t,c}$ = 2 X 10^{-6} m^2 K/W.



(a) Consider limitations for which the array has N = 11 fins, in which case values of the fin thickness t = 0.182 mm and pitch S = 1.982 mm are obtained from the requirements that W = (N - 1)S + t and S - t = 1.8 mm. If the maximum allowable chip temperature is T_c = 85^{0}C, what is the corresponding value of the chip power q_c? An adiabatic fin tip condition may be

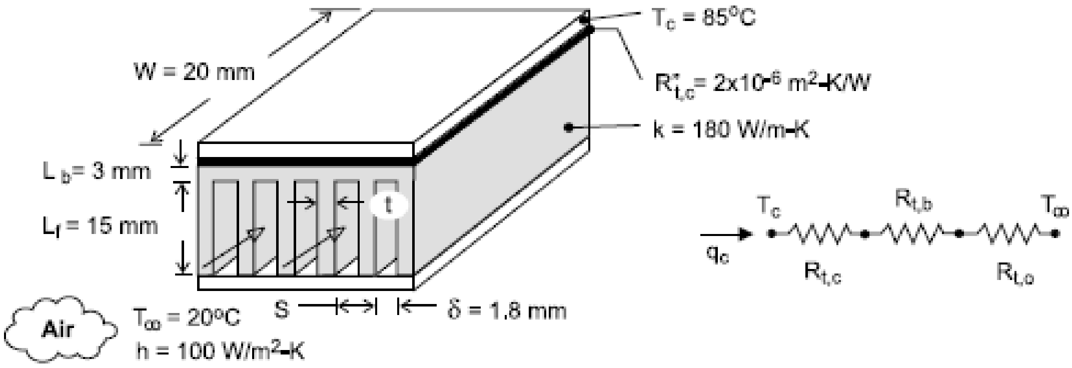
assumed, and air flow along the outer surfaces of the heat sink may be assumed to provide a convection coefficient equivalent to that associated with air flow through the channels.
(b) With (S - t) and h fixed at 1.8 mm and 100 W/m^2 K, respectively, explore the effect of increasing the fin thickness by reducing the number of fins. With N = 11 and S - t fixed at 1.8 mm, but relaxation of the constraint on the pressure drop, explore the effect of increasing the air flow, and hence the convection coefficient.

Solution:

KNOWN: Width and maximum allowable temperature of an electronic chip. Thermal contact resistance between chip and heat sink. Dimensions and thermal conductivity of heat sink. Temperature and convection coefficient associated with air flow through the heat sink.

FIND: (a) Maximum allowable chip power for heat sink with prescribed number of fins, fin thickness, and fin pitch, and (b) Effect of fin thickness/number and convection coefficient on performance.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional heat transfer, (3) Isothermal chip, (4) Negligible heat transfer from top surface of chip, (5) Negligible temperature rise for air flow, (6) Uniform convection coefficient associated with air flow through channels and over outer surfaces of heat sink, (7) Negligible radiation.

ANALYSIS: (a) From the thermal circuit,

$$q_c = \frac{T_c - T_\infty}{R_{tot}} = \frac{T_c - T_\infty}{R_{t,c} + R_{t,b} + R_{t,o}}$$

where $R_{t,c} = R''_{t,c} / W^2 = 2\times10^{-6} m^2 \cdot K/W / (0.02m)^2 = 0.005\ K/W$ and $R_{t,b} = L_b / k\left(W^2\right)$

$= 0.003m / 180\ W/m\cdot K\,(0.02m)^2 = 0.042\ K/W$. From Eqs. (3.103), (3.102), and (3.99)

$$R_{t,o} = \frac{1}{\eta_o h A_t}, \qquad \eta_o = 1 - \frac{N A_f}{A_t}(1-\eta_f), \qquad A_t = N A_f + A_b$$

where $A_f = 2WL_f = 2 \times 0.02m \times 0.015m = 6 \times 10^{-4}\ m^2$ and $A_b = W^2 - N(tW) = (0.02m)^2 - 11(0.182 \times 10^{-3}\ m \times 0.02m) = 3.6 \times 10^{-4}\ m^2$. With $mL_f = (2h/kt)^{1/2} L_f = (200\ W/m^2{\cdot}K/180\ W/m{\cdot}K \times 0.182 \times 10^{-3}m)^{1/2} (0.015m) = 1.17$, $\tanh mL_f = 0.824$ and Eq. (3.87) yields

$$\eta_f = \frac{\tanh mL_f}{mL_f} = \frac{0.824}{1.17} = 0.704$$

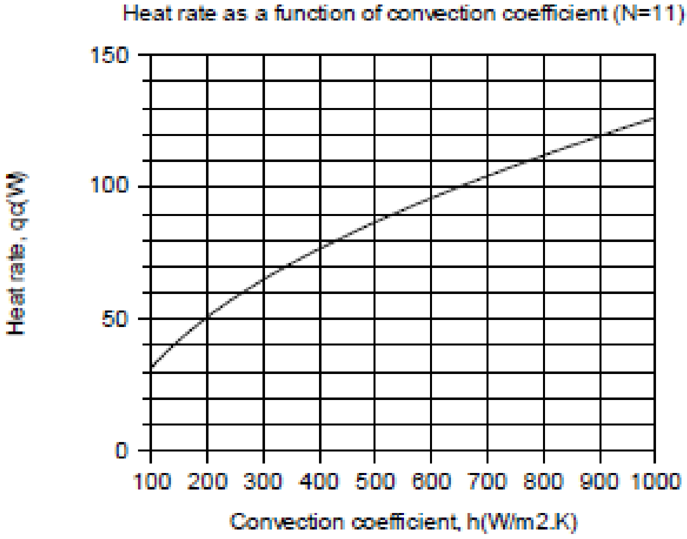
It follows that $A_t = 6.96 \times 10^{-3}\ m^2$, $\eta_o = 0.719$, $R_{t,o} = 2.00$ K/W, and

$$q_c = \frac{(85-20)°C}{(0.005+0.042+2.00)K/W} = 31.8\ W$$ <

(b) The following results are obtained from parametric calculations performed to explore the effect of decreasing the number of fins and increasing the fin thickness.

N	t(mm)	η_f	$R_{t,o}$ (K/W)	q_c (W)	A_t (m^2)
6	1.833	0.957	2.76	23.2	0.00378
7	1.314	0.941	2.40	26.6	0.00442
8	0.925	0.919	2.15	29.7	0.00505
9	0.622	0.885	1.97	32.2	0.00569
10	0.380	0.826	1.89	33.5	0.00632
11	0.182	0.704	2.00	31.8	0.00696

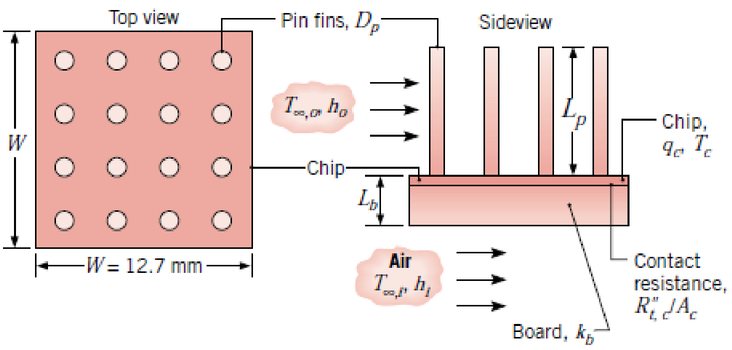
Although η_f (and η_o) increases with decreasing N (increasing t), there is a reduction in A_t which yields a minimum in $R_{t,o}$, and hence a maximum value of q_c, for N = 10. For N = 11, the effect of h on the performance of the heat sink is shown below.



With increasing h from 100 to 1000 $W/m^2 \cdot K$, $R_{t,o}$ decreases from 2.00 to 0.47 K/W, despite a decrease in η_f (and η_o) from 0.704 (0.719) to 0.269 (0.309). The corresponding increase in q_c is significant.

COMMENTS: The heat sink significantly increases the allowable heat dissipation. If it were not used and heat was simply transferred by convection from the surface of the chip with h = 100 $W/m^2 \cdot K$, R_{tot} = 2.05 K/W from Part (a) would be replaced by $R_{cnv} = 1/hW^2$ = 25 K/W, yielding q_c = 2.60 W.

10. As more and more components are placed on a single integrated circuit (chip), the amount of heat that is dissipated continues to increase. However, this increase is limited by the maximum allowable chip operating temperature, which is approximately 75°C. To maximize heat dissipation, it is proposed that a 4 X 4 array of copper pin fins be metallurgically joined to the outer surface of a square chip that is 12.7 mm on a side.



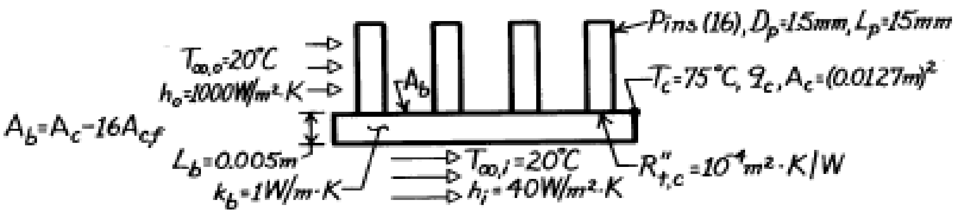
(a) Sketch the equivalent thermal circuit for the pin– chip–board assembly, assuming one-dimensional, steady-state conditions and negligible contact resistance between the pins and the chip. In variable form, label appropriate resistances, temperatures, and heat rates.
(b) For the conditions prescribed in Problem 6, what is the maximum rate at which heat can be dissipated in the chip when the pins are in place? That is, what is the value of q_c for T_c = 75°C? The pin diameter and length are D_p = 1.5 mm and L_p _ 15 mm.

Solution:

KNOWN: Geometry and cooling arrangement for a chip-circuit board arrangement. Maximum chip temperature.

FIND: (a) Equivalent thermal circuit, (b) Maximum chip heat rate.

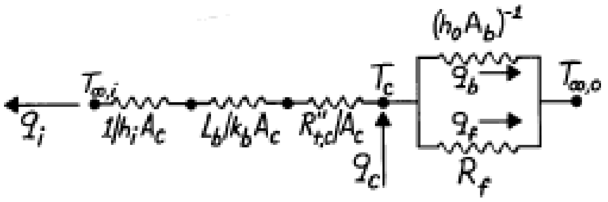
SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer in chi board assembly, (3) Negligible pin-chip contact resistance, (4) Constant properties, (5) Negligible chip thermal resistance, (6) Uniform chip temperature.

PROPERTIES: *Table A.1*, Copper (300 K): k = 400 W/m·K.

ANALYSIS: (a) The thermal circuit is



$$R_f = \frac{\theta_b}{16 q_f} = \frac{\cosh mL + (h_o/mk)\sinh mL}{16\left(h_o PkA_{c,f}\right)^{1/2}\left[\sinh mL + (h_o/mk)\cosh mL\right]}$$

(b) The maximum chip heat rate is

$$q_c = 16q_f + q_b + q_i.$$

Evaluate these parameters

$$m = \left(\frac{h_o P}{kA_{c,f}}\right)^{1/2} = \left(\frac{4h_o}{kD_p}\right)^{1/2} = \left(\frac{4\times 1000\ \text{W/m}^2\cdot\text{K}}{400\ \text{W/m}\cdot\text{K}\times 0.0015\ \text{m}}\right)^{1/2} = 81.7\ \text{m}^{-1}$$

$$mL = \left(81.7\ \text{m}^{-1}\times 0.015\ \text{m}\right) = 1.23, \quad \sinh mL = 1.57, \quad \cosh mL = 1.86$$

$$(h/mk) = \frac{1000\ \text{W/m}^2\cdot\text{K}}{81.7\ \text{m}^{-1}\times 400\ \text{W/m}\cdot\text{K}} = 0.0306$$

$$M = \left(h_o\pi\ D_p k\pi\ D_p^2/4\right)^{1/2}\theta_b$$

$$M = \left[1000\ \text{W/m}^2\cdot\text{K}\left(\pi^2/4\right)(0.0015\ \text{m})^3\ 400\ \text{W/m}\cdot\text{K}\right]^{1/2}\left(55^\circ\text{C}\right) = 3.17\ \text{W}.$$

The fin heat rate is

$$q_f = M\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} = 3.17\ \text{W}\frac{1.57 + 0.0306\times 1.86}{1.86 + 0.0306\times 1.57}$$

$$q_f = 2.703\ \text{W}.$$

The heat rate from the board by convection is

$$q_b = h_o A_b\theta_b = 1000\ \text{W/m}^2\cdot\text{K}\left[(0.0127\ \text{m})^2 - (16\pi/4)(0.0015\ \text{m})^2\right]55^\circ\text{C}$$

$$q_b = 7.32\ \text{W}.$$

The convection heat rate is

$$q_i = \frac{T_c - T_{\infty,i}}{\left(1/h_i + R''_{t,c} + L_b/k_b\right)(1/A_c)} = \frac{(0.0127\ \text{m})^2\left(55^\circ\text{C}\right)}{\left(1/40 + 10^{-4} + 0.005/1\right)\text{m}^2\cdot\text{K/W}}$$

$$q_i = 0.29\ \text{W}.$$

Hence, the maximum chip heat rate is

$$q_c = \left[16(2.703) + 7.32 + 0.29\right]\text{W} = \left[43.25 + 7.32 + 0.29\right]\text{W}$$

$$q_c = 50.9\ \text{W}. \qquad <$$

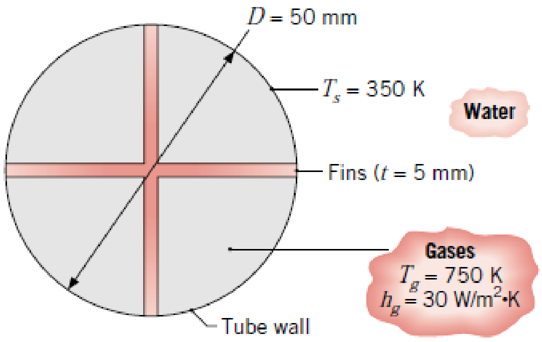
COMMENTS: (1) The fins are extremely effective in enhancing heat transfer from the chip (assuming negligible contact resistance). Their effectiveness is $\varepsilon = q_f/\left(\pi\ D_p^2/4\right)h_o\theta_b = 2.703$ W/0.097 W = 27.8

(2) Without the fins, q_c = 1000 W/m^2·K(0.0127 m)^{2}55°C + 0.29 W = 9.16 W. Hence the fins provide for a (50.9 W/9.16 W) × 100% = 555% enhancement of heat transfer.

(3) With the fins, the chip heat flux is 50.9 W/(0.0127 m)^2 or $q_c'' = 3.16\times10^5$ W/m^2 = 31.6 W/cm^2.

(4) If the infinite fin approximation is made, $q_f = M = 3.17$ W, and the actual fin heat transfer is overestimated by 17%.

12. Water is heated by submerging 50-mm diameter, thin-walled copper tubes in a tank and passing hot combustion gases (T_g = 750 K) through the tubes. To enhance heat transfer to the water, four straight fins of uniform cross section, which form a cross, are inserted in each tube. The fins are 5 mm thick and are also made of copper (k = 400 W/m K).



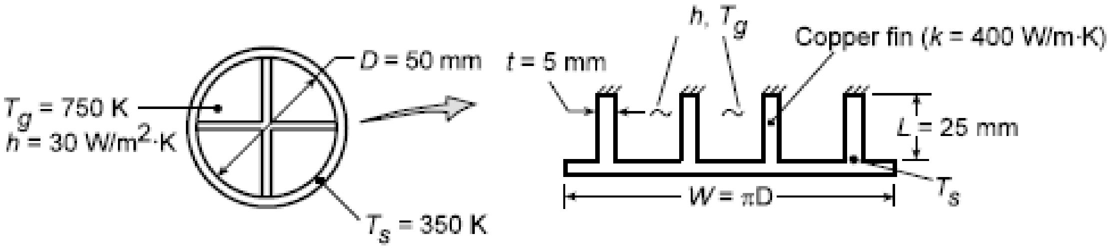
If the tube surface temperature is T_s = 350 K and the gas-side convection coefficient is h_g = 30 W/m^2 K, what is the rate of heat transfer to the water per meter of pipe length?

Solution:

KNOWN: Diameter and internal fin configuration of copper tubes submerged in water. Tube wall temperature and temperature and convection coefficient of gas flow through the tube.

FIND: Rate of heat transfer per tube length.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional fin conduction, (3) Constant properties, (4) Negligible radiation, (5) Uniform convection coefficient, (6) Tube wall may be unfolded and represented as a plane wall with four straight, rectangular fins, each with an adiabatic tip.

ANALYSIS: The rate of heat transfer per unit tube length is:

$$q_t' = \eta_o h A_t' \left(T_g - T_s\right)$$

$$\eta_o = 1 - \frac{NA_f'}{A_t'}\left(1 - \eta_f\right)$$

$$NA_f' = 4\times 2L = 8(0.025\text{m}) = 0.20\text{m}$$

$$A_t' = NA_f' + A_b' = 0.20\text{m} + (\pi D - 4t) = 0.20\text{m} + (\pi\times 0.05\text{m} - 4\times 0.005\text{m}) = 0.337\text{m}$$

For an adiabatic fin tip,

$$\eta_f = \frac{q_f}{q_{max}} = \frac{M\tanh mL}{h(2L\cdot 1)(T_g - T_s)}$$

$$M = [h2(1\text{m}+t)k(1\text{m}\times t)]^{1/2}(T_g - T_s) \approx \left[30\,\text{W/m}^2\cdot\text{K}(2\text{m})400\,\text{W/m}\cdot\text{K}\left(0.005\text{m}^2\right)\right]^{1/2}(400\text{K}) = 4382\text{W}$$

$$mL = \{[h2(1\text{m}+t)]/[k(1\text{m}\times t)]\}^{1/2}L \approx \left[\frac{30\,\text{W/m}^2\cdot\text{K}(2\text{m})}{400\,\text{W/m}\cdot\text{K}\left(0.005\text{m}^2\right)}\right]^{1/2} 0.025\text{m} = 0.137$$

Hence, tanh mL = 0.136, and

$$\eta_f = \frac{4382\text{W}(0.136)}{30\,\text{W/m}^2\cdot\text{K}\left(0.05\text{m}^2\right)(400\text{K})} = \frac{595\text{W}}{600\text{W}} = 0.992$$

$$\eta_o = 1 - \frac{0.20}{0.337}(1-0.992) = 0.995$$

$$q_t' = 0.995\left(30\,\text{W/m}^2\cdot\text{K}\right)0.337\text{m}(400\text{K}) = 4025\,\text{W/m}$$

COMMENTS: Alternatively, $q_t' = 4q_f' + h(A_t' - A_f')(T_g - T_s)$. Hence, q' = 4(595 W/m) + 30 W/m²·K (0.137 m)(400 K) = (2380 + 1644) W/m = 4024 W/m.