TRANSIENT HEAT CONDUCTION

STRUCTURE

- Introduction
- Lumped System Analysis
- Criteria Of The Lumped System Analysis
- Transient Heat Conduction In Large Plane Walls, Long Cylinders, And Spheres
- Transient Heat Conduction In Semi-infinite Solids
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TIME DEPENDENT CONDUCTION - **Temperature history inside a conducting body that is immersed suddenly in a bath of fluid at a different temperature.**

Ex: Quenching of special alloys

The temperature of such a body varies with time as well as position.

$$T(x, y, z, t)$$

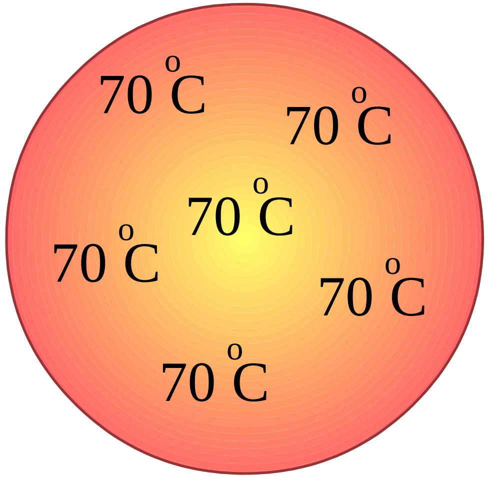
$(x, y, z,)$- **Variation in the *x,y* and *z* directions**

t **- Variation with time**

In this chapter, we consider the variation of temperature with *time* as well as *position* in one and multi-dimensional systems.

LUMPED SYSTEM ANALYSIS

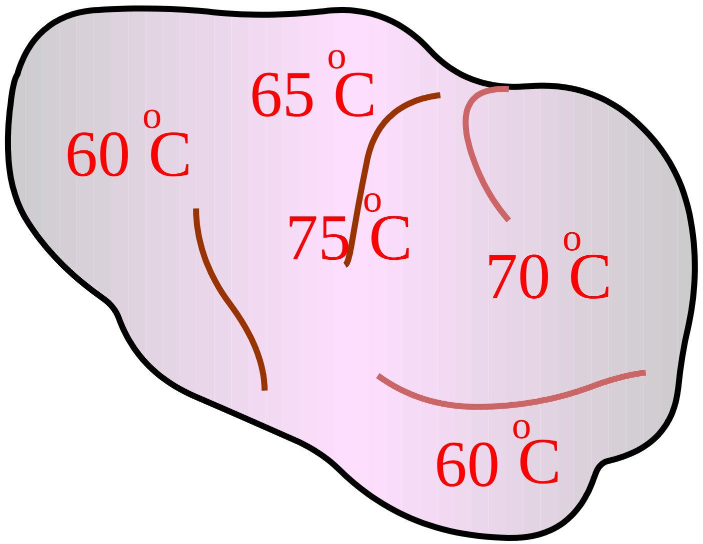
COPPER BALL WITH UNIFORM TEMPERATURE



Temperature of the copper ball changes with time, but it does not change with position at any given time.

Temperature of the ball remains uniform at all times

POTATA TAKEN FROM BOILING WATER

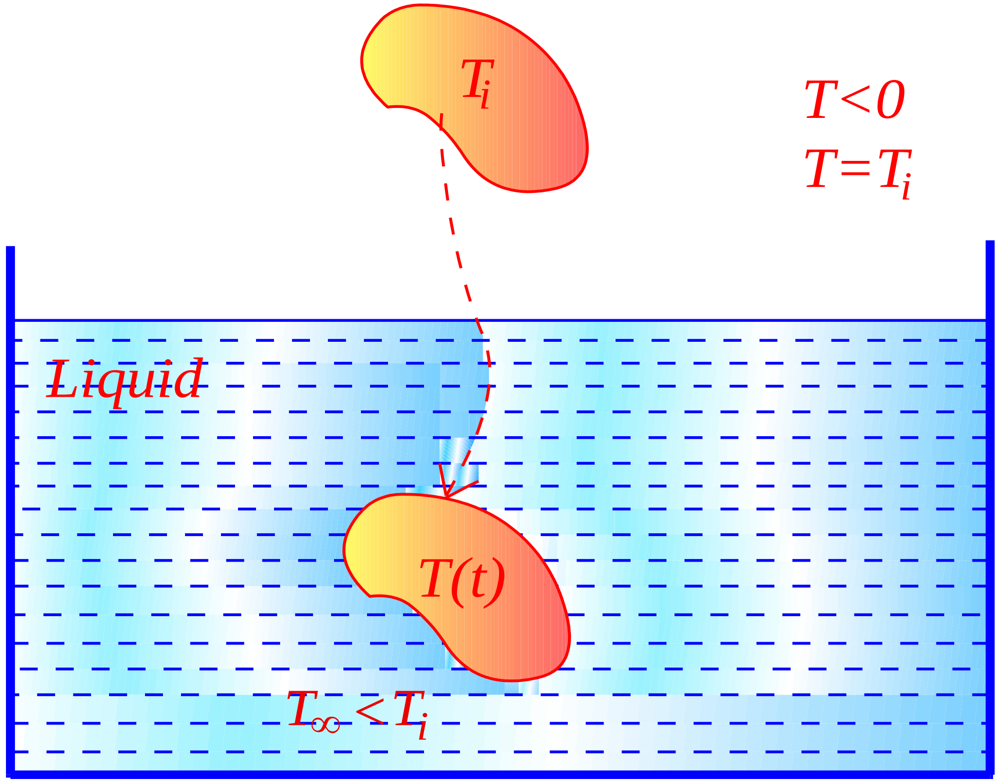


Large potato put in a vessel with boiling water.

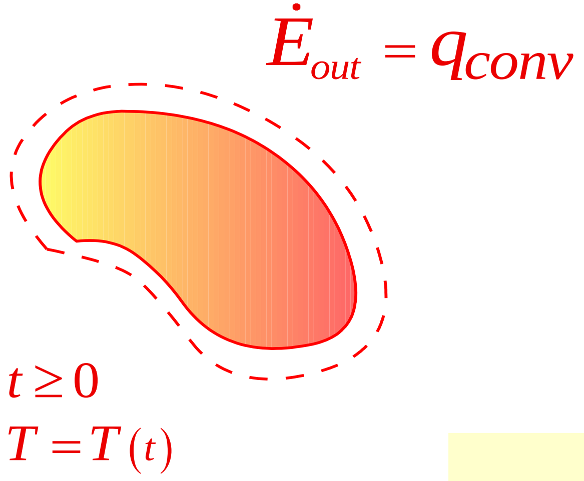
After few minutes, if you take out the potato, temperature distribution within the potato is not even close to being uniform.

Thus, lumped system analysis is not applicable in this case.

Hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$



During a differential time interval dt, the temperature of the body rises by a differential amount dT. An energy balance of the solid for the time interval dt can be expressed as

$$-\dot{E}_{out} = \dot{E}_{st}$$

$$-hA_s(T - T_\infty) = \rho V C_p \frac{dT}{dt}$$

$$\theta = T - T_\infty$$

$$-hA_s\theta = \rho V C_p \frac{d\theta}{dt}$$

V - Body volume

A_s - surface area

ρ - density of the body material

C_p - specific heat of the body material

$$-hA_s\theta = \rho VC_p \frac{d\theta}{dt}$$

$$\frac{\rho VC_p}{hA_s}\frac{d\theta}{\theta} = -dt$$

$$\frac{\rho VC_p}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

$$\frac{\rho VC_p}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

$$\theta_i = T_i - T_\infty$$

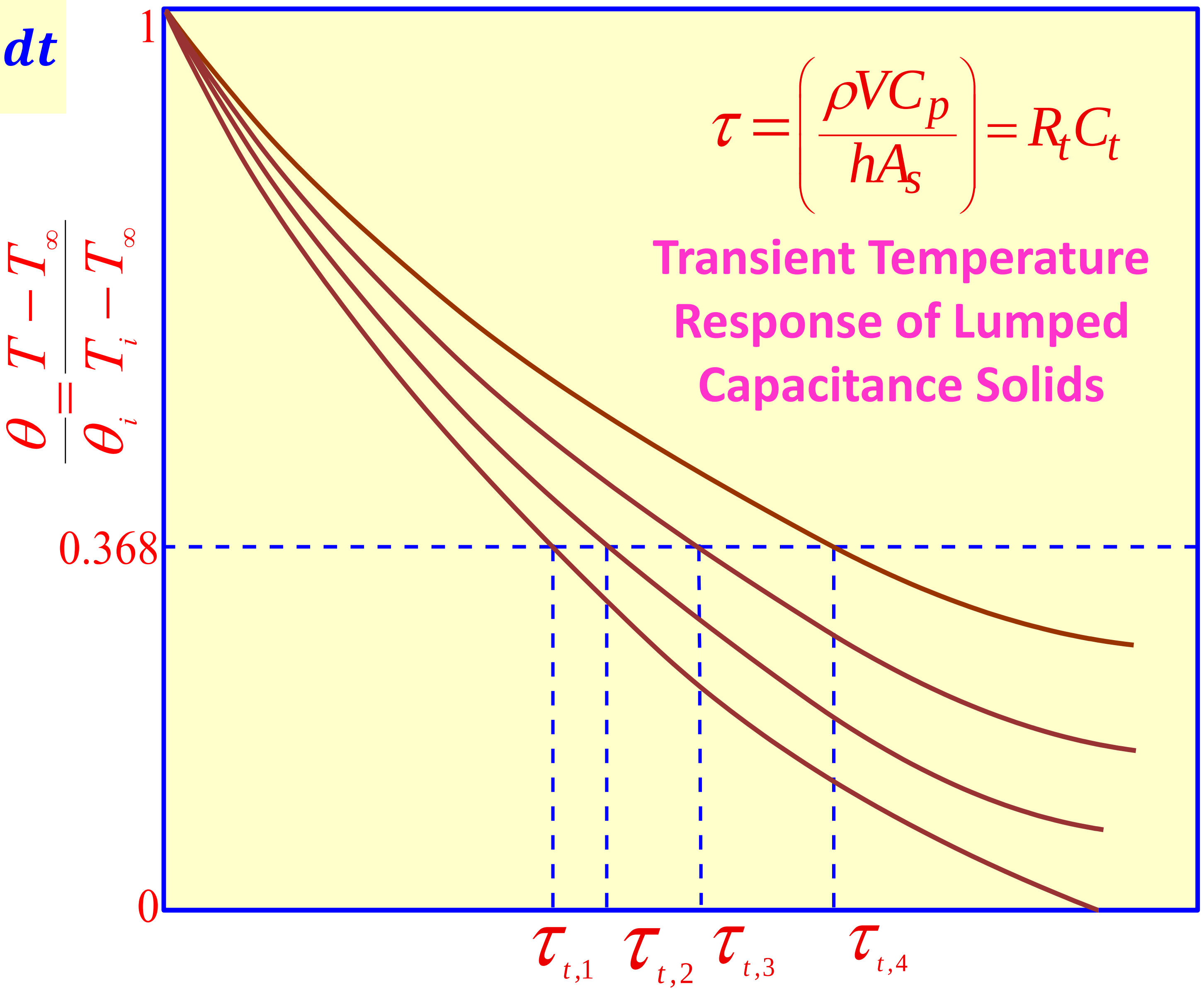
$$\theta = T - T_\infty$$

$$\tau \ln\frac{\theta}{\theta_i} = -t$$

$$\tau = \frac{\rho VC_p}{hA_s}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}}$$

$$\tau = \frac{\rho VC_p}{hA_s} = \left(\frac{1}{hA_s}\right)\rho VC_p = R_tC_t$$



R_t - Resistance to convection heat transfer

C_t - Lumped thermal capacitance of the solid

The rate of convection heat transfer between the body and its environment at any instant of time can be determined from Newton's law of cooling

$$q = hA_s[T(t) - T_\infty]$$

$$Q = \int_0^t q\,dt = \int_0^t hA_s[T(t) - T_\infty]\,dt$$

$$Q = \int_0^t hA_s\theta\,dt$$

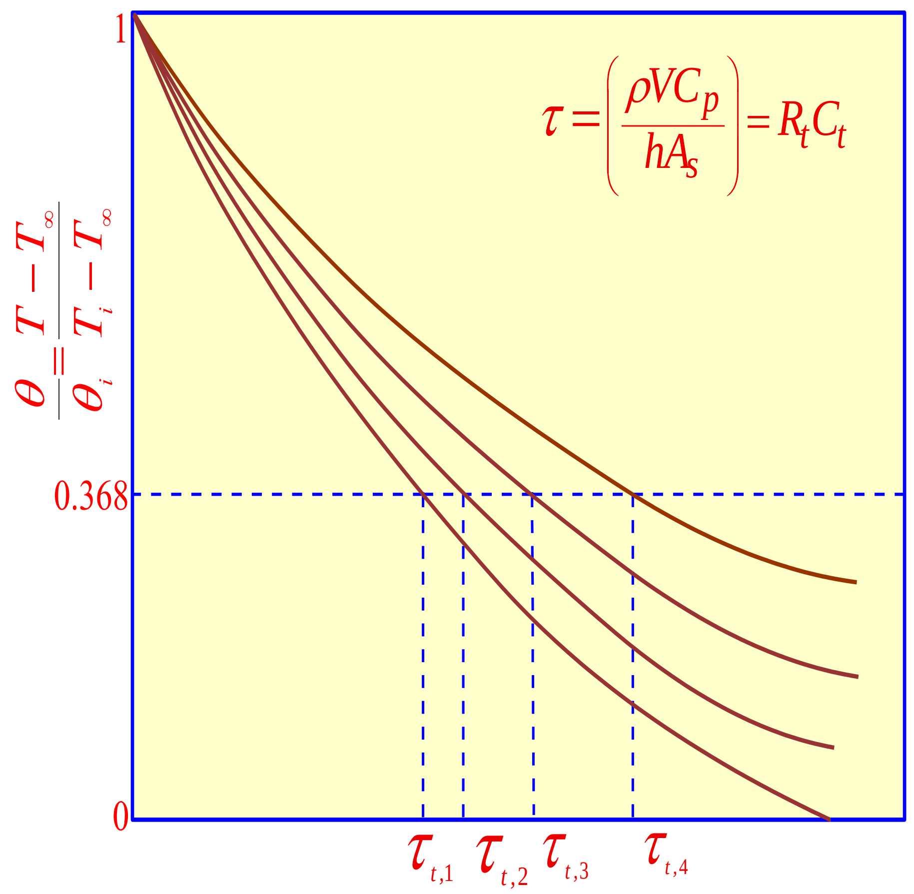
$$\theta = T - T_\infty$$

$$\theta = \theta_i e^{-\frac{t}{\tau}}$$

$$Q = \int_0^t hA_s\theta_i e^{-\frac{t}{\tau}}dt = hA_s\theta_i\int_0^t e^{-\frac{t}{\tau}}dt$$

$$\tau = \frac{\rho V C_p}{hA_s}$$

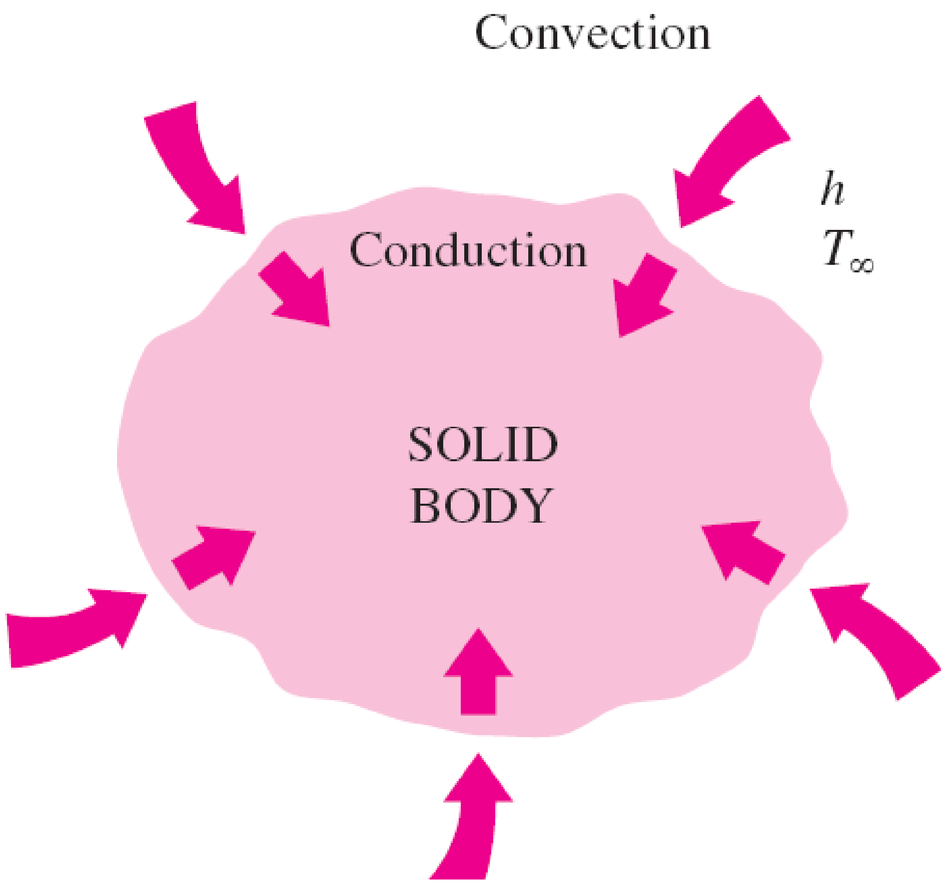
$$Q = hA_s\theta_i \left.\frac{e^{-\frac{t}{\tau}}}{-\frac{1}{\tau}}\right|_0^t = -hA_s\theta_i\frac{\rho V C_p}{hA_s}\left[e^{-\frac{t}{\tau}} - 1\right]$$



$$Q = \rho V C_p\theta_i\left[1 - e^{-\frac{t}{\tau}}\right]$$

CRITERIA OF THE LUMPED SYSTEM ANALYSIS

Biot number Bi

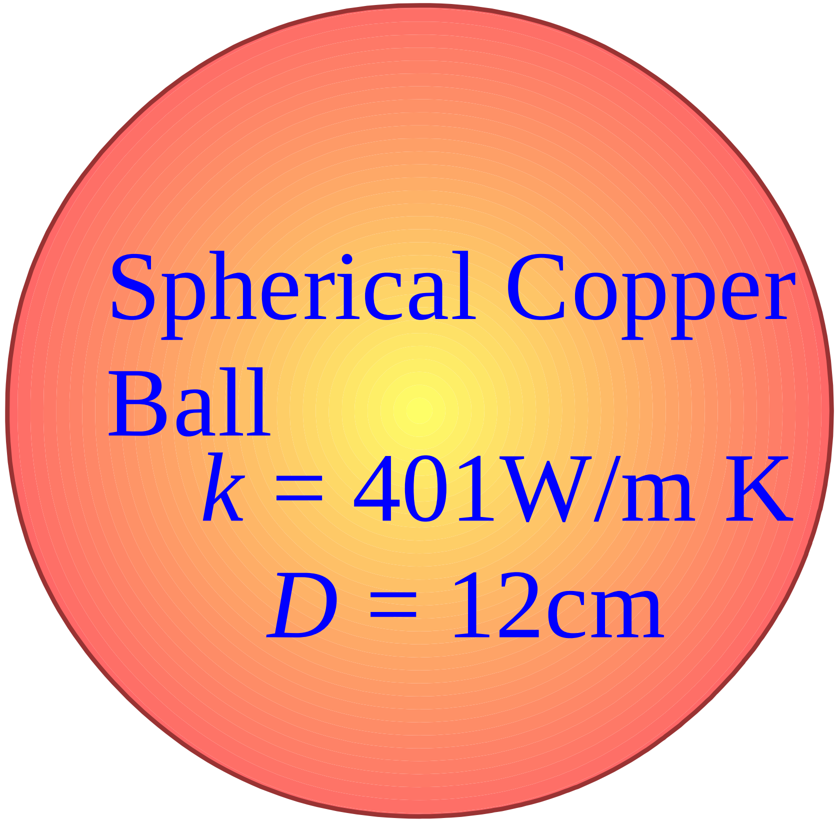


$$Bi = \frac{hL_c}{k}$$

$$Bi = \frac{h\Delta T}{\frac{k}{L_c}\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

$$Bi = \frac{\frac{L_c}{k}}{\frac{1}{h}} = \frac{\text{Conduction Resistance within the body}}{\text{Convective Reistance at the surface of the body}}$$

Lumped system analysis is exact when $Bi = 0$
Generally, accepted norm for assuming lumped system analysis, $Bi \leq 0.1$



$$L_c = \frac{V}{A_s} = \frac{\frac{\pi D^3}{6}}{\pi D^2} = \frac{D}{6} = \frac{12 \times 10^{-3}}{6} = 2 \times 10^{-3}$$

$$Bi = \frac{hL_c}{k} = \frac{15 \times 2 \times 10^{-3}}{401} = 75 \times 10^{-6} \ll 0.1$$

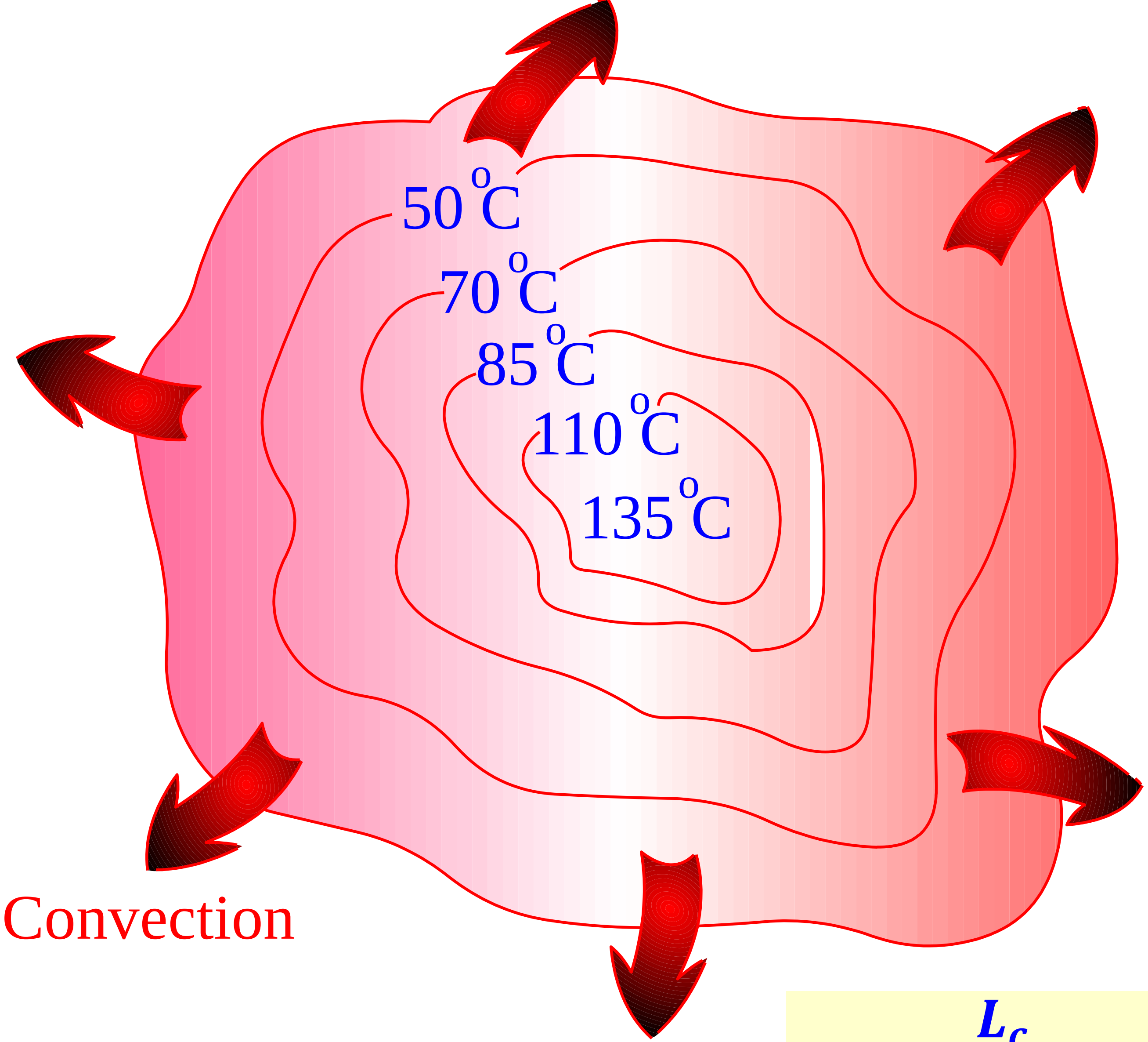
Hence, the spherical copper ball is lumped

Small bodies with higher thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis

Heat conduction in a specified direction n per unit surface area is expressed as

$$q'' = -k\frac{\partial T}{\partial n}$$

Larger the thermal conductivity implies the smaller the temperature gradient



$$h = 2000\,\mathrm{W/m°C}$$

$$Bi = \frac{hL_c}{k}$$

When the convection coefficient h is high and thermal conductivity k is low, large temperature differences occur between the inner and outer regions of a large solid

$$Bi = \frac{\frac{L_c}{k}}{\frac{1}{h}} = \frac{\text{Conduction Resistance within the body}}{\text{Convective Reistance at the surface of the body}}$$

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is known to be $h = 400\ W/m^2.K$ and the junction thermophysical properties are $k = 20\ W/m.K$, $C_p = 400\ J/kg.K$ and ρ = 8500 kg/m³. Determine the junction diameter needed for the thermocouple to have a time constant of one second. If the junction is at 25℃ and is placed in a gas stream that is at 200 ℃, how long will it take for the junction to reach 199 ℃?

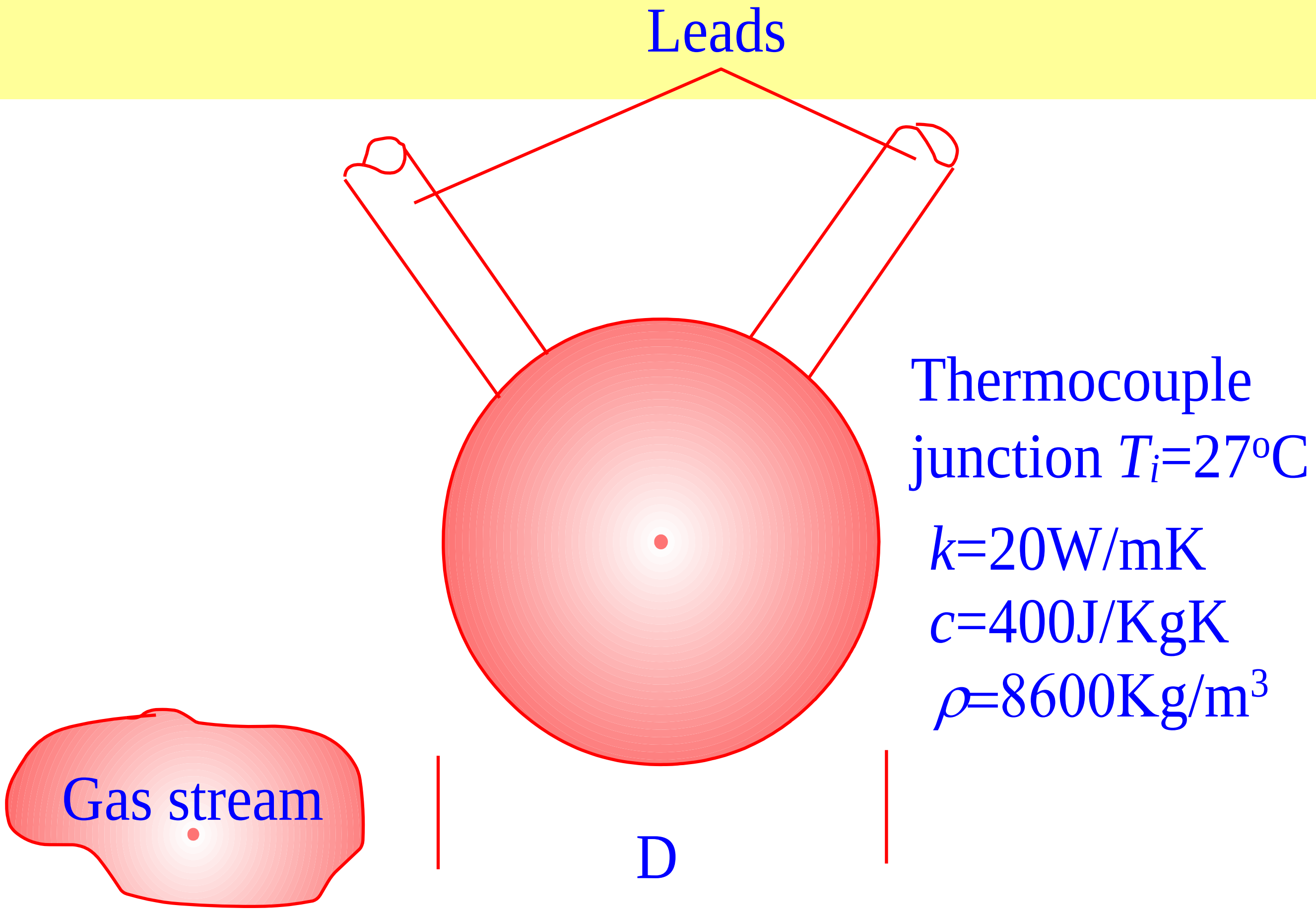
Known:

Thermophysical properties of thermocouple junction used to measure temperature of a gas stream

Find:

Junction diameter needed for a time constant of 1 second

Time required to reach 199 ℃ in gas stream at 200 ℃



Assumptions:

Temperature of the junction is uniform at any instant

Radiation exchange with the surroundings is negligible

Losses by conduction through the leads are negligible

Constant properties

Analysis:

Because the junction diameter is unknown, it is not possible to begin the solution by determining whether the criterion for using the lumped capacitance method, $Bi \le 0.1$.

However, a reasonable approach is to use the method to find the diameter and to then determine whether the criterion is satisfied.

$$\tau = \frac{\rho V C_p}{h A_s} = \frac{\rho C_p}{h}\frac{\frac{\pi D^3}{6}}{\pi D^2} = \frac{\rho C_p}{h}\frac{D}{6}$$

$$1 = \frac{8500(400)}{400}\frac{D}{6}$$

$$D = 7.06 \times 10^{-4}\, m$$

$$L_c = \frac{V}{A_s} = \frac{\frac{\pi D^3}{6}}{\pi D^2} = \frac{D}{6}$$

$$L_c = \frac{D}{6}$$

$$Bi = \frac{hL_c}{k} == \frac{400\left(\frac{7.06 \times 10^{-4}}{6}\right)}{20} = 2.35 \times 10^{-3}$$

Criterion for using the lumped capacitance method, $Bi \le 0.1$ is satisfied and the lumped capacitance method may be used to an excellent approximation.

The time required for the junction to reach $T = 199℃$

$$\frac{T - T_{\infty}}{T_i - T_{\infty}} = e^{-\frac{t}{\tau}}$$

$$\frac{199 - 200}{25 - 200} = e^{-\frac{t}{1}}$$

$$t = 5.17\ s$$

Comments:

Heat transfer due to radiation exchange between the junction and the surroundings and conduction through the leads would affect the time response of the junction and would, in fact, yield an equilibrium temperature that differs from T_{∞}.

SPATIAL EFFECTS

Variation of temperature with time and position in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.

No internal generation and constant properties

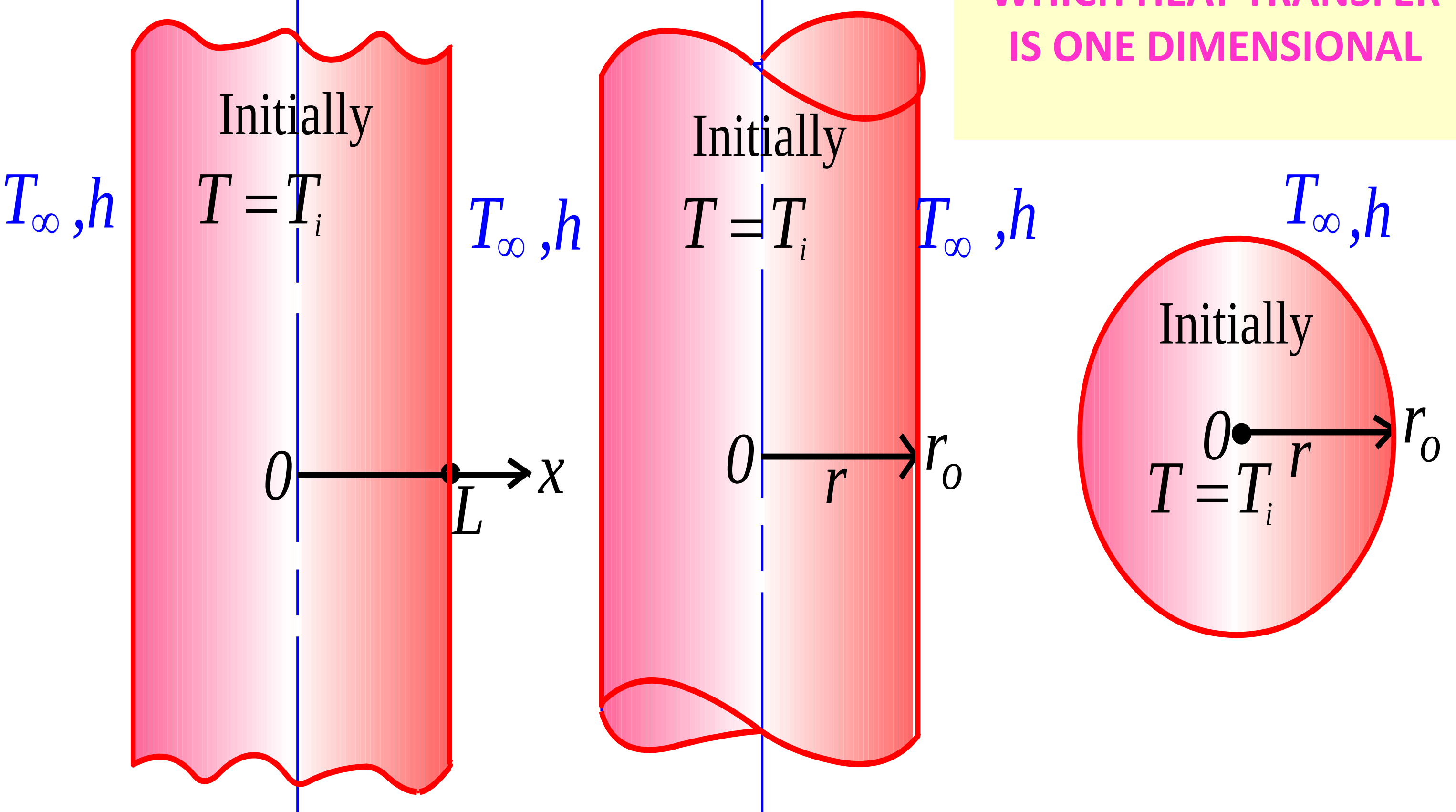
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$ **Initial Condition**

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty]$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$ **Boundary conditions**

SIMPLE GEOMETRIES IN WHICH HEAT TRANSFER IS ONE DIMENSIONAL



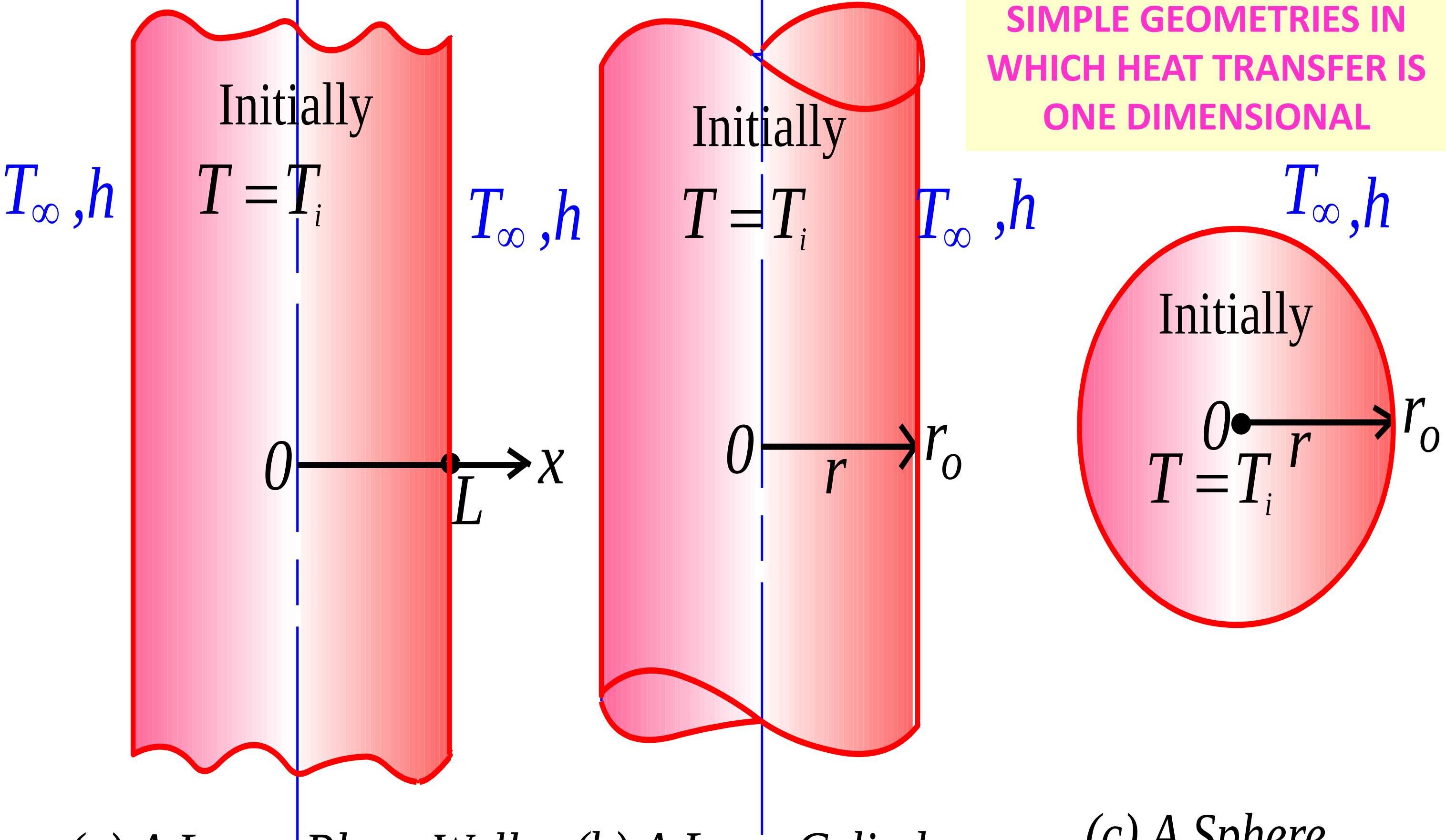
(a) A Large Plane Wall (b) A Long Cylinder (c) A Sphere

TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES

Variation of temperature with time and position in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.

No internal generation and constant properties

SIMPLE GEOMETRIES IN WHICH HEAT TRANSFER IS ONE DIMENSIONAL

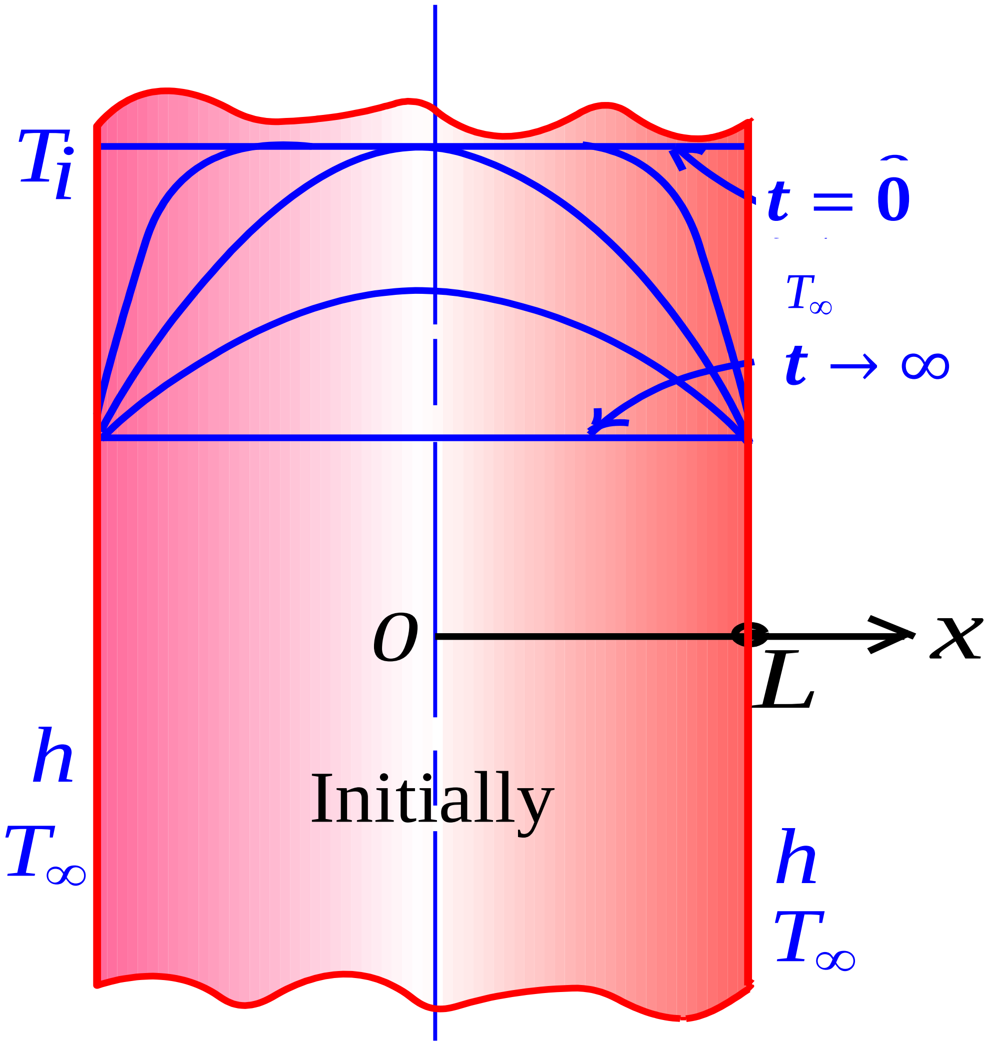


(a) A Large Plane Wall (b) A Long Cylinder (c) A Sphere

All three cases possess geometric and thermal symmetry: The plane wall is symmetric about its *center* plane $(x = 0)$, the cylinder is symmetric about its *center line* $(r = 0)$, and the sphere is symmetric about its *center point* $(x = 0)$.

Neglect *radiation* heat transfer between these bodies and their surrounding surfaces, or incorporate the radiation effect into the convection heat transfer coefficient h.

Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$



(a) A Large Plane Wall

- When the wall is first exposed to the surrounding medium at $T_i > T_\infty$ at $t = 0$, the entire wall is at its initial temperature T_i.
- But the wall temperature at and near the surfaces starts to drop as a result of heat transfer from the wall to the surrounding medium.
- This creates a temperature gradient in the wall and initiates heat conduction from the inner parts of the wall toward its outer surfaces.
- Note that the temperature at the center of the wall remains at T_i until $t = t_2$, and that the temperature profile within the wall remains symmetric at all times about the center plane.
- The temperature profile gets flatter and flatter as time passes as a results of heat transfer, and eventually becomes uniform at $T = T_\infty$. The wall reaches thermal equilibrium with its surroundings. At that point, the heat transfer stops since there is no longer a temperature difference. Similar discussions can be given for the long cylinder or sphere.

Non-dimensionalisation of governing equation and boundary conditions

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \qquad X = \frac{x}{L}$$

$$\frac{\partial^2 \theta}{\partial X^2}\frac{(T_i - T_\infty)}{L^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}(T_i - T_\infty)$$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial \theta}{\partial t}$$

$$Fo = \frac{\alpha t}{L^2} \qquad Fo \sim \frac{\alpha t}{L^2} \sim \frac{\frac{m^2}{s}.s}{m^2}$$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial Fo}$$

$$T(x,0) = T_i$$

$$\theta(x,0) = \frac{T(x,0) - T_\infty}{T_i - T_\infty} = \frac{T_i - T_\infty}{T_i - T_\infty} \qquad \theta(X,0) = 1$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{(T_i - T_\infty)}{L}\frac{\partial \theta}{\partial X}\right|_{X=0} = 0 \qquad \left.\frac{\partial \theta}{\partial X}\right|_{X=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty] \qquad \theta(L,t) = \frac{T(L,t) - T_\infty}{T_i - T_\infty}$$

$$-k\left.\frac{(T_i - T_\infty)}{L}\frac{\partial \theta}{\partial X}\right|_{X=1} = h\theta(L,t)(T_i - T_\infty)$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{X=1} = -\frac{hL}{k}\theta(L,t) \qquad \left.\frac{\partial \theta}{\partial X}\right|_{X=1} = -Bi\,\theta(1,Fo)$$

Non-dimensionalisation of governing equation and boundary conditions

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty]$$

$$T = \phi(x, t, k, h, L, \alpha, T_i, T_\infty)$$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial Fo}$$

$$\theta(X,0) = 1$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{X=0} = 0$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{X=1} = -Bi\,\theta(1, Fo)$$

$$\theta = f(X, Bi, Fo)$$

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

$$X = \frac{x}{L}$$

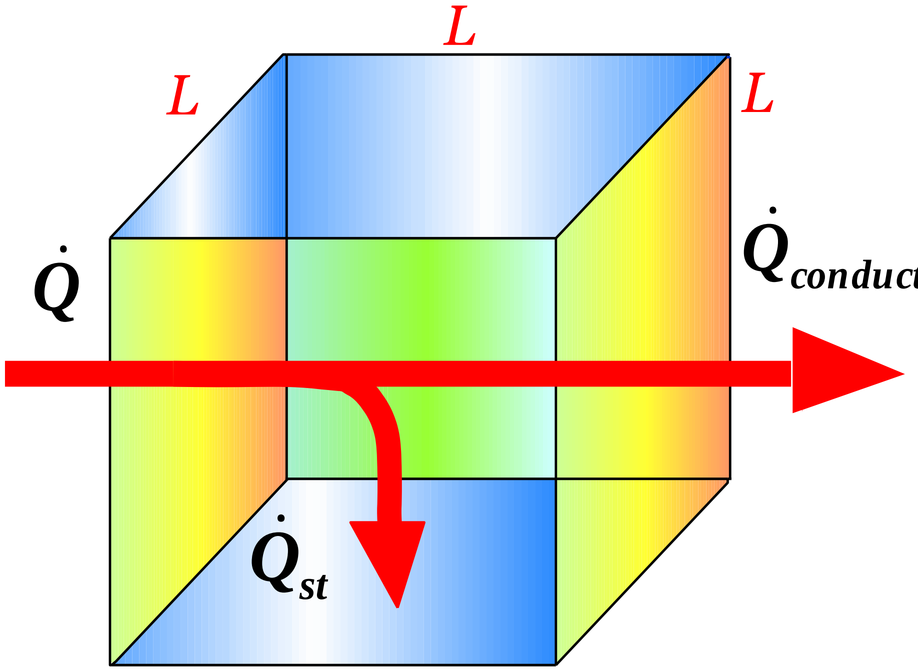
$$Fo = \frac{\alpha t}{L^2}$$

$$Bi = \frac{hL}{k}$$

Dimensional similarity approach

Physical significance of the Fourier number

$$Fo = \frac{\alpha t}{L^2} = \frac{kL^2 \frac{\Delta T}{L} t}{\rho C_p L^3 \frac{\Delta T}{t}} = \frac{\text{Rate at which heat is conducted across L of a body of volume } L^3}{\text{Rate at which heat is stored in a body of volume } L^3}$$



$$Fo = \frac{\alpha t}{L^2} = \frac{\dot{Q}_{conducted}}{\dot{Q}_{stored}}$$

What constitutes an infinitely large plate or an infinitely long cylinder ?

A plate whose thickness is small relative to the other dimensions can be modeled as an infinitely large plate, except very near the outer edges.

But the edge effects on large bodies are usually negligible, and thus a large plane wall such as the wall of a house can be modeled as an infinitely large wall for heat transfer purposes. Similarly, a long cylinder whose diameter is small relative to its length can be analyzed as an infinitely long cylinder.

Exact solution for a Plane Wall

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 Fo} cos\left(\frac{\lambda_n x}{L}\right)$$

$$A_n = \frac{4sin\lambda_n}{2\lambda_n + sin(2\lambda_n)}$$

$$Fo = \frac{\alpha t}{L^2}$$

The discrete values of λ_n are the positive roots of the transcendental equation

$$Bi = \lambda_n Tan\lambda_n$$

$$Bi = \frac{hL}{k}$$

The discrete values of λ_n (first four roots are shown in the table) are positive roots of the transcendental equation

The first four roots of the Transcendental equation

$$Bi = \lambda_n Tan \lambda_n$$

For transient conduction in a plane wall

Bi	λ_1	λ_2	λ_3	λ_4
0	0	3.1416	6.2832	9.4248
0.001	0.0316	3.1419	6.2833	9.4249
0.002	0.0447	3.1422	6.2835	9.425
0.004	0.0632	3.1429	6.2838	9.4252
0.006	0.0774	3.1435	6.2841	9.4254
0.008	0.0893	3.1441	6.2845	9.4256
0.01	0.0998	3.1448	6.2848	9.4258
0.02	0.141	3.1479	6.2864	9.4269
0.04	0.1987	3.1543	6.2895	9.429
0.06	0.2425	3.1606	6.2927	9.4311
0.08	0.2791	3.1668	6.2959	9.4333
0.1	0.3111	3.1731	6.2991	9.4354
0.2	0.4328	3.2039	6.3148	9.4459
0.3	0.5218	3.2341	6.3305	9.4565
0.4	0.5932	3.2636	6.3461	9.467

Bi	λ_1	λ_2	λ_3	λ_4
0.5	0.6533	3.2923	6.3616	9.4775
0.6	0.7051	3.3204	6.377	9.4879
0.7	0.7506	3.3477	6.3923	9.4983
0.8	0.791	3.3744	6.4074	9.5087
0.9	0.8274	3.4003	6.4224	9.519
1	0.8603	3.4256	6.4373	9.5293
1.5	0.9882	3.5422	6.5097	9.5801
2	1.0769	3.6436	6.5783	9.6296
3	1.1925	3.8088	6.704	9.724
4	1.2646	3.9352	6.814	9.8119
5	1.3138	4.0336	6.9096	9.8928
6	1.3496	4.1116	6.9924	9.9667
7	1.3766	4.1746	7.064	10.0339
8	1.3978	4.2264	7.1263	10.0949
9	1.4149	4.2694	7.1806	10.1502
10	1.4289	4.3058	7.2281	10.2003

Bi	λ_1	λ_2	λ_3	λ_4
15	1.4729	4.4255	7.3959	10.3898
20	1.4961	4.4915	7.4954	10.5117
30	1.5202	4.5615	7.6057	10.6543
40	1.5325	4.5979	7.6647	10.7334
50	1.54	4.6202	7.7012	10.7832
60	1.5451	4.6353	7.7259	10.8172
80	1.5514	4.6543	7.7573	10.8606
100	1.5552	4.6658	7.7764	10.8871
inf	1.5708	4.1724	7.854	10.9956
15	1.4729	4.4255	7.3959	10.3898
20	1.4961	4.4915	7.4954	10.5117
30	1.5202	4.5615	7.6057	10.6543
40	1.5325	4.5979	7.6647	10.7334
50	1.54	4.6202	7.7012	10.7832
60	1.5451	4.6353	7.7259	10.8172

It can be shown that for values of $Fo > 0.2$, the infinite series solution can be approximated by the first term of the series

Plane wall

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \cos\left(\frac{\lambda_1 x}{L}\right)$$

Cylinder

$$\theta(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} J_o\left(\frac{\lambda_1 r}{r_o}\right)$$

Sphere

$$\theta(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \frac{\sin\left(\frac{\lambda_1 r}{r_o}\right)}{\left(\frac{\lambda_1 r}{r_o}\right)}$$

The constants A_1 and λ_1 are functions of the Bi number only, and their values are listed in Table . 1 against the Bi number for all three geometries.

The function J_o is the zeroth order Bessel function of the first kind, whose value can be determined from Table . 2.

Noting that $\cos(0) = J_o(0) = 1$ and the limit of $\frac{\sin x}{x}$ is also 1, the above relations simplify to the following at the center of a plane wall, cylinder, or sphere.

Table 1 Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders and spheres ($Bi = hL/k$ for plane wall of thickness $2L$, and $Bi = hr_o/k$ for a cylinder or sphere of radius r_o)

	Plane wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.173	1.003
0.02	0.141	1.0033	0.1995	1.005	0.2445	1.006
0.04	0.1987	1.0066	0.2814	1.0099	0.345	1.012
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.013	0.396	1.0197	0.486	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.617	1.0483	0.7593	1.0592
0.3	0.5218	1.045	0.7465	1.0712	0.9208	1.088
0.4	0.5932	1.058	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.791	1.1016	1.149	1.1724	1.432	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5	1.3138	1.2403	1.9898	1.5029	2.5704	1.787
6	1.3496	1.2479	2.049	1.5253	2.6537	1.8338
7	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8	1.3978	1.257	2.1286	1.5526	2.7654	1.892
9	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10	1.4289	1.262	2.1795	1.5677	2.8363	1.9249
20	1.4961	1.2699	2.288	1.5919	2.9857	1.9781
30	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50	1.54	1.2727	2.3572	1.6002	3.0788	1.9962
100	1.552	1.2731	2.3809	1.6015	3.1102	1.999
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2

$$Bi = \frac{hL}{k}$$ plane wall of thickness $2L$

$$Bi = \frac{hr_o}{k}$$ cylinder or sphere of radius r_o

Table 2 The zeroth and first order Bessel functions of the first kind

λ_1	$J_0(\lambda_1)$	$J_1(\lambda_1)$	λ_1	$J_0(\lambda_1)$	$J_1(\lambda_1)$
0.0	1.0000	0.0000	1.5	0.5118	0.5579
0.1	0.9975	0.0499	1.6	0.4554	0.5699
0.2	0.9900	0.0995	1.7	0.3980	0.5778
0.3	0.9776	0.1483	1.8	0.3400	0.5815
0.4	0.9604	0.1960	1.9	0.2818	0.5812
0.5	0.9385	0.2423	2.0	0.2239	0.5767
0.6	0.9120	0.2867	2.1	0.1666	0.5683
0.7	0.8812	0.3290	2.2	0.1104	0.5560
0.8	0.8463	0.3688	2.3	0.0555	0.5399
0.9	0.8075	0.4059	2.4	0.0025	0.5202
1.0	0.7652	0.4400	2.6	-0.0968	-0.4708
1.1	0.7196	0.4709	2.8	-0.1850	-0.4097
1.2	0.6711	0.4983	3.0	-0.2601	-0.3391
1.3	0.6201	0.5220	3.2	-0.3202	-0.2613
1.4	0.5669	0.5419	1.5	0.5118	0.5579

Center of Plane wall $(x = 0)$: $Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

Center of Cylinder $(r = 0)$: $Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

Center of sphere $(r = 0)$: $Fo > 0.2$

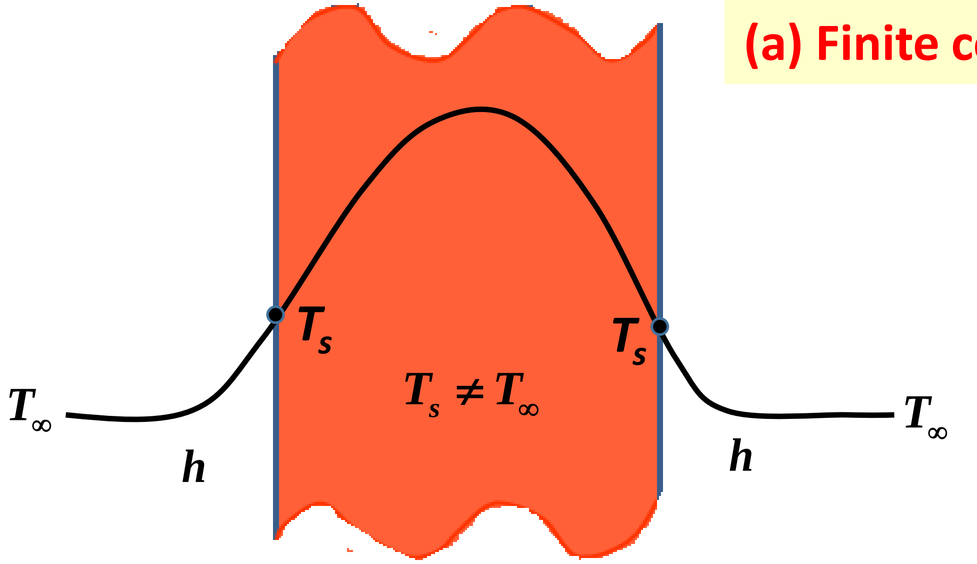
$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

Specified surface temperature corresponds to the case of convection to an environment at T_∞ with a convection coefficient h that is infinite

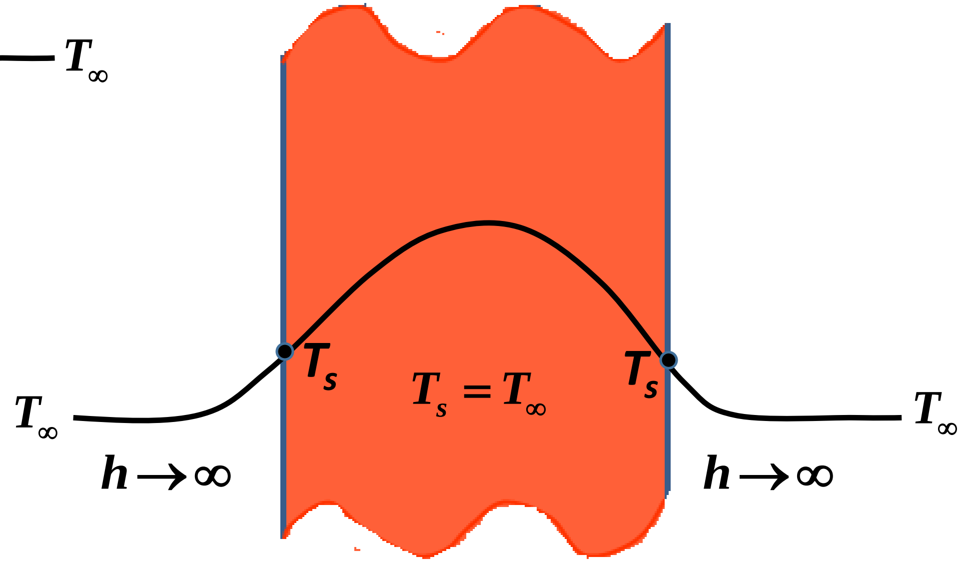
Physical Significance of $h \to \infty$

$$\frac{1}{Bi} = \frac{k}{hL} = 0$$

Real life situation corresponding to $h \to \infty$ is boiling and condensation



(a) Finite convection coefficient



(b) Infinite convection coefficient

Surfaces of the body are suddenly brought to the temperature T_∞ at $t = 0$ and kept at T_∞ at all times can be handled by setting h to infinity.

TOTAL ENERGY TRANSFERRED FROM THE WALL

$$E_{in} - E_{out} = \Delta E_{st}$$

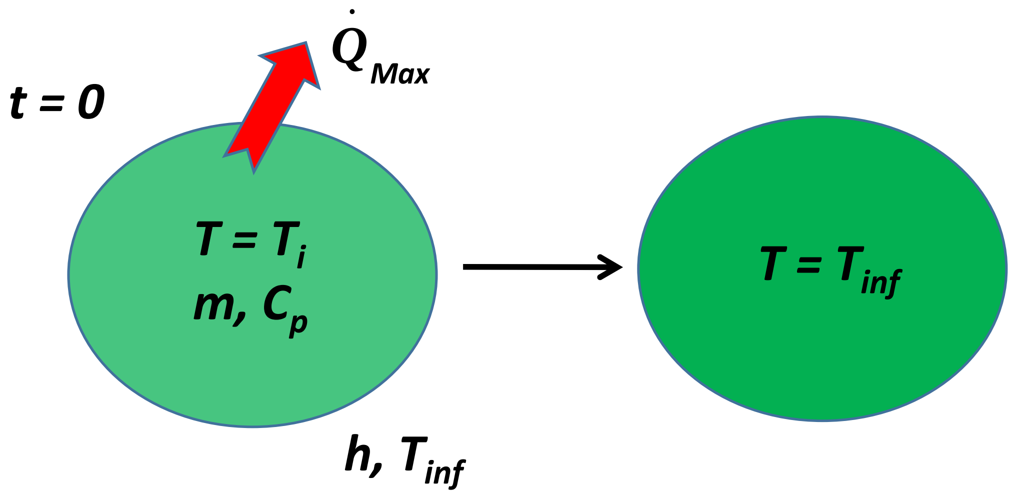
$$E_{in} = 0$$

$$\Delta E_{st} = E(t) - E(t = 0)$$

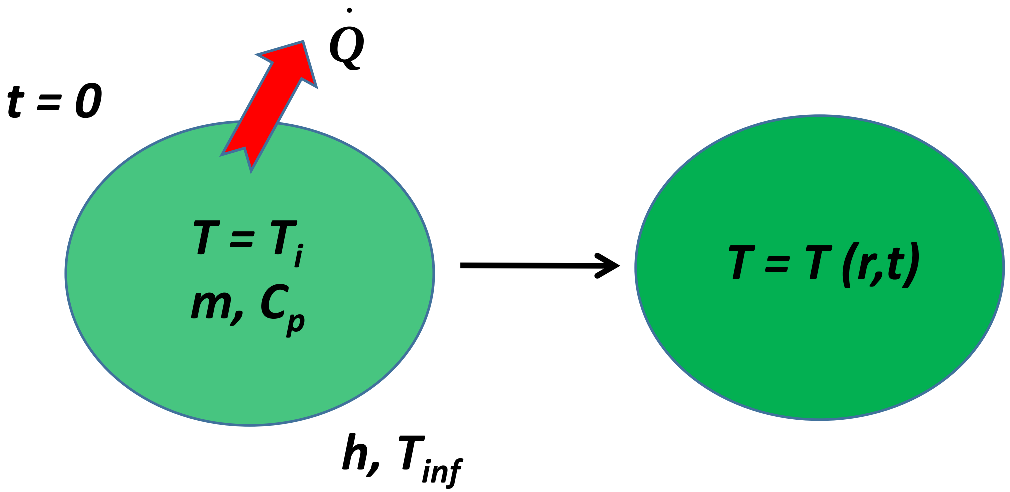
$$Q = \int \rho V C_p [T(x,t) - T_i]\, dV$$

Integration is performed over the volume of the wall

Negative sign indicates that the heat is leaving the body



Maximum heat transfer ($t \rightarrow \infty$)



b) Actual heat transfer for time *t*

The temperature of the body changes from the initial temperature T_i to the temperature of the surroundings T_∞ at the end of the transient heat conduction process.
Maximum amount of heat that a body gain (or lose if $T_i > T_\infty$) is simply the change in the energy content of the body.

$$Q_{max} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i)$$

where m is the mass, V is the volume, ρ is the density, and C_p is the specific heat of the body. Q_{max} represents the amount of heat transfer for $t \to \infty$.

The amount of heat transfer Q at a finite time t will obviously be less than this maximum.

Ratio Q/Q_{max} is obtained by one term approximations given later for the large plane wall, long cylinder and sphere, respectively.

Note that once the *fraction* of heat transfer Q/Q_{max} has been determined from these relations for the given t, the actual amount of heat transfer by that time can be evaluated by multiplying this fraction by Q_{max}. A *negative* sign for Q_{max} indicates that heat is *leaving* the body.

The fraction of heat transfer can also be determined from the following relations, which are based on the one-term approximations which are valid for $Fo > 0.2$

Plane wall

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{o,wall}\left(\frac{sin\lambda_1}{\lambda_1}\right)$$

It can be shown that for values of $Fo > 0.2$, the infinite series solution can be approximated by the first term of the series

Cylinder

$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{o,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere

$$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - 3\theta_{o,sph}\frac{sin\lambda_1 - \lambda_1 cos\lambda_1}{\lambda_1^3}$$

The use of the one-term solutions is limited to the following conditions

- Body is initially at a uniform temperature,
- Temperature of the medium surrounding the body and the convection heat transfer coefficient are constant and uniform
- There is no energy generation in the body

An ordinary egg can be approximated as a 5 cm diameter sphere. The egg is initially at a uniform temperature of 5^0 C and is dropped into boiling water at 95^0 C. Taking the convection heat transfer coefficient to be h = 1200 $W/m^2.^0$ C, determine how long it will take for the center of the egg to reach 70^0 C.

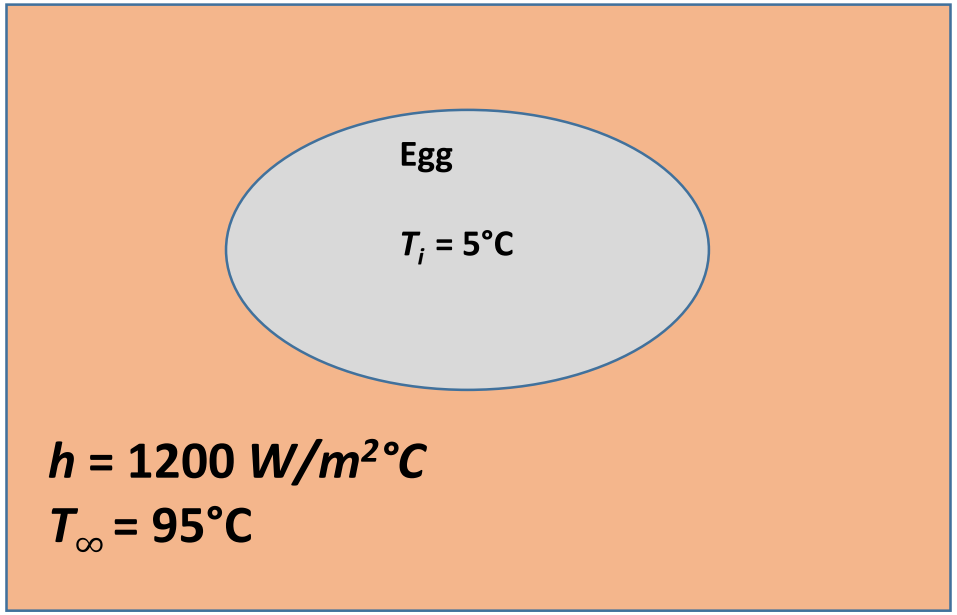
Known:

Temperature drop in the egg, convection heat transfer coefficient

Find:

The time taken for the center of the egg to reach 70^0 C

Schematic:



Assumptions

The egg is spherical in shape with a radius of $r_o = 5\ cm$.

Heat conduction in the egg is one dimensional because of thermal symmetry about the mid point

The thermal properties of the egg and the heat transfer coefficient are constant

The Fourier number is $Fo > 0.2$ so that one term approximate solutions are applicable

Properties:

The water content of the eggs is about 74%, and thus the thermal conductivity and diffusivity of the eggs can be approximated by those of water at the average temperature of (5+70)/2 = 37.5^0 C; k = 0.627 W/m.^{0} C and $\alpha = k/\rho C_p = 0.151 \times 10^{-6}$ m^2/s.

Analysis:

The temperature within the egg varies with radial distance as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts or the one-term solutions. Here we will use the latter to demonstrate their use. The Biot number for this problem is

$$Bi = \frac{hL_c}{k} = \frac{1200 \times \frac{5 \times 10^{-2}}{6}}{0.627} = 15.95$$

which is much greater than 0.1, and thus the lumped system analysis is not applicable.

For the analysis in which the constants λ_1 and A_1 are to be found out, the Biot number is defined on the basis of the radius of the sphere not the characteristic length as L_c

$$L_c = \frac{V}{A_s} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{d}{6} = \frac{5 \times 10^{-2}}{6}$$

$$Bi = \frac{hr_o}{k} = \frac{1200 \times 2.5 \times 10^{-2}}{0.627} = 47.8$$

Let us assume that the Fourier number is greater than 0.2 and solve it using the one term solution. After we find the time, we shall check whether the Fourier number is greater than 0.2 or not

The coefficients λ_1 and A_1 for a sphere corresponding to this $Bi = 47.8$ are from Table $\lambda_1 = 3.0753$ and $A_1 = 1.9958$

Then the cooking time is determined from the definition of the Fourier number to be

Center of sphere $(r = 0)$: $Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \qquad \frac{70 - 95}{5 - 95} = 1.9958 e^{-(3.0753)^2 Fo}$$

$$Fo = 0.209$$

which is greater than 0.2, and thus the one term solution is applicable with an error of less than 2 %.

Bi	λ_1	A_1
1	1.5708	1.2732
2	2.0288	1.4793
3	2.2889	1.6227
4	2.4556	1.7202
5	2.5704	1.787
6	2.6537	1.8338
7	2.7165	1.8673
8	2.7654	1.892
9	2.8044	1.9106
10	2.8363	1.9249
20	2.9857	1.9781
30	3.0372	1.9898
40	3.0632	1.9942
50	3.0788	1.9962
100	3.1102	1.999
∞	3.1416	2

Then the cooking time is determined from the definition of the Fourier number to be

$$Fo = \frac{\alpha t}{L^2} = \frac{0.151 \times 10^{-6} t}{(2.5 \times 10^{-2})^2} = 0.209$$

$$t = 865\ s = 14.4\ mins$$

Comments:

The time taken for the center of the egg to be heated from 5^0 C to 70^0 C would be around 15 min.

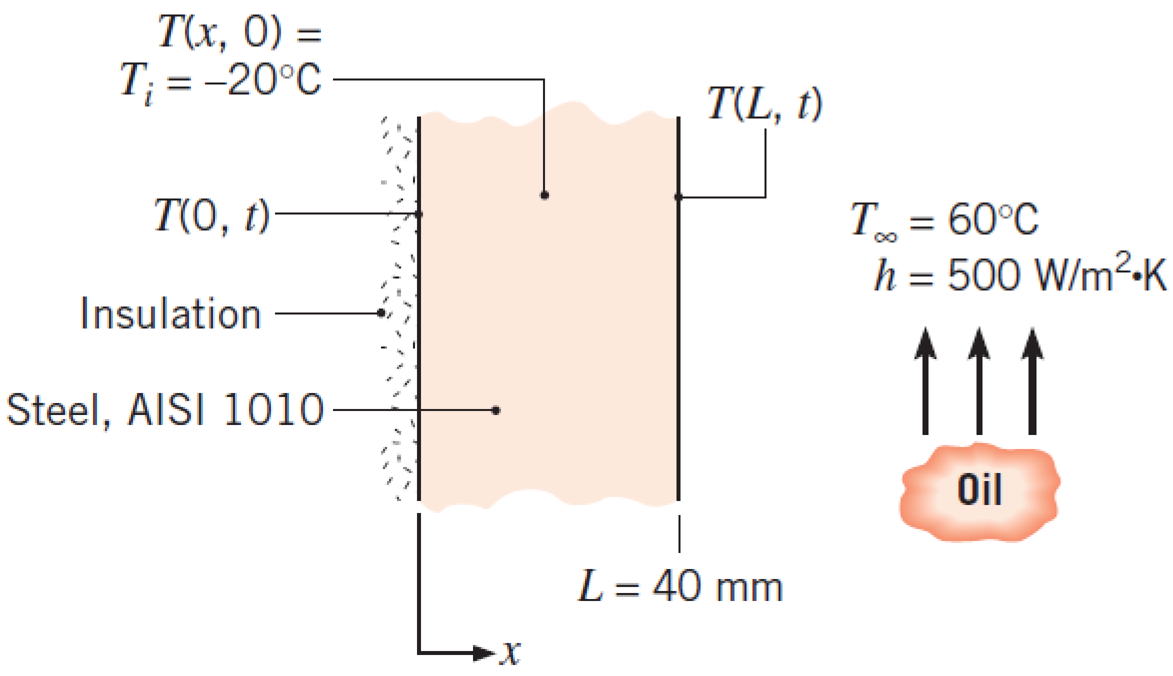
Consider a steel pipeline (AISI 1010) that is 1 m in diameter and has a wall thickness of 40 mm. The pipe is heavily insulated on the outside, and, before the initiation of flow, the walls of the pipe are at a uniform temperature of 20°C. With the initiation of flow, hot oil at 60° C is pumped through the pipe, creating a convective condition corresponding to h = 500 $W/m^2 \cdot K$ at the inner surface of the pipe.

1. What are the appropriate Biot and Fourier numbers 8 min after the initiation of flow?
2. At t = 8 min, what is the temperature of the exterior pipe surface covered by the insulation?
3. What is the heat flux q(W/m^2) to the pipe from the oil at t = 8min?
4. How much energy per meter of pipe length has been transferred from the oil to the pipe at t 8min?

Known: Wall subjected to sudden change in convective surface condition.

Find:

1. Biot and Fourier numbers after 8 min.
2. Temperature of exterior pipe surface after 8 min.
3. Heat flux to the wall at 8 min.
4. Energy transferred to pipe per unit length after 8 min.



Assumptions:

1. **Pipe wall can be approximated as plane wall, since thickness is much less than diameter.**
2. **Constant properties.**
3. **Outer surface of pipe is adiabatic**

Properties: **Table A.1, steel type AISI 1010** $[T = -20 + 60/2 = 20 = 293\ K]$:

$\rho = 7832\ kg/m^3 \quad C_p = 434\ J/kg.K \quad \alpha = 18.8 \times 10^{-6}\ m^2/s$

Analysis:

1. At t = 8 min, the Biot and Fourier numbers are computed with $L_c = L$

Properties: Table A.1, steel type AISI 1010 [$T = -20 + 60/2 = 20 = 293\ K$]:

$\rho = 7832\ kg/m^3 \quad C_p = 434\ J/kg.K \quad \alpha = 18.8 \times 10^{-6}\ m^2/s$

$$Bi = \frac{hL}{k} = \frac{500 \times 0.04}{63.9} = 0.313$$

$$Fo = \frac{\alpha t}{L^2} = \frac{18.8 \times 10^{-6} \times 8 \times 60}{(0.04)^2} = 5.64$$

1. What are the appropriate Biot and Fourier numbers 8 min after the initiation of flow?
2. At *t* = 8 min, what is the temperature of the exterior pipe surface covered by the insulation?

With $Bi = 0.313$, use of the lumped capacitance method is inappropriate. However, since $Fo > 0.2$ and transient conditions in the insulated pipe wall of thickness L correspond to those in a plane wall of thickness $2L$ experiencing the same surface condition, the desired results may be obtained from the one-term approximation for a plane wall.

The midplane temperature can be determined from Equation

Center of Plane wall $(x = 0)$: $\quad Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

1. What are the appropriate Biot and Fourier numbers 8 min after the initiation of flow?
2. At t = 8 min, what is the temperature of the exterior pipe surface covered by the insulation?

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

$Bi = 0.313$ $Fo = 5.64$

$A_1 = 1.047$ $\lambda_1 = 0.531$

$Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - 60}{-20 - 60} = 1.047 e^{-(0.531)^2(5.64)}$$

$\theta(0,t) = 0.214$ $T(0,t) = 42.9℃$

Hence after 8 min, the temperature of the exterior pipe surface, which corresponds to the midplane temperature of a plane wall, is 42.9℃

	Plane wall	
Bi	λ_1	A_1
0.01	0.0998	1.0017
0.02	0.141	1.0033
0.04	0.1987	1.0066
0.06	0.2425	1.0098
0.08	0.2791	1.013
0.1	0.3111	1.0161
0.2	0.4328	1.0311
0.3	0.5218	1.045
0.4	0.5932	1.058
0.5	0.6533	1.0701
0.6	0.7051	1.0814
0.7	0.7506	1.0918
0.8	0.791	1.1016
0.9	0.8274	1.1107

3. What is the heat flux q(W/m^2) to the pipe from the oil at t = 8min?

Heat transfer to the inner surface at $x = L$ is by convection, and at any time t the heat flux may be obtained from Newton's law of cooling. Hence at $t = 480\ s$,

$$q''_x(L, 480s) = q''_L = h[T(L, 480s) - T_\infty]$$

Using the one-term approximation for the surface temperature

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} cos\left(\frac{\lambda_1 x}{L}\right)$$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = \theta(0,t) cos\left(\lambda_1 \frac{L}{L}\right)$$

$$\theta(0,t) = 0.214$$

$$\lambda_1 = 0.531$$

$$\theta(x,t) = \frac{T(L, 8\ min) - 60}{-20 - 60} = 0.214 cos\left(0.531 \times \frac{180}{\pi}\right)$$

$$T(L, 8\ min) = 45.2°\mathrm{C}$$

$$q''_x(L, 480s) = q''_L = h[T(L, 480s) - T_\infty] = 500(45.2 - 60) = -7400\ W/m^2$$

$$q''_x(L, 480s) = q''_L = -7400\ W/m^2$$

4. How much energy per meter of pipe length has been transferred from the oil to the pipe at $t = 8\ min$?

The energy transfer to the pipe wall over the 8-min interval may be obtained from

Plane wall

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{o,wall}\left(\frac{sin\lambda_1}{\lambda_1}\right)$$

$$\theta(0,t) = 0.214$$

$$\lambda_1 = 0.531$$

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - 0.214\left(\frac{sin\left(\frac{0.531 \times 180}{\pi}\right)}{0.531}\right)$$

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 0.8$$

$$Q_{max} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i)$$

$$Q_{max} = \rho(\pi DL)C_p(T_\infty - T_i) = 7832(\pi \times 1 \times 0.04) \times 434 \times (-20 - 60)$$

$$Q_{max} = -3.4125 \times 10^7\ \frac{J}{m}$$

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 0.8$$

$$\left(\frac{Q}{-3.4125 \times 10^7}\right)_{wall} = 0.8$$

$$Q = -2.73\ \times 10^7\ \frac{J}{m}$$

$$V = \frac{\pi}{4}\left(d_o^2 - d_i^2\right)L = \frac{\pi}{4}\left(d_o - d_i\right)\left(d_o + d_i\right)L$$

$$V = \frac{\pi}{4}\left(d_o^2 - d_i^2\right)Length = \pi\,\frac{\left(d_o - d_i\right)}{2}\,\frac{\left(d_o + d_i\right)}{2}\,Length$$

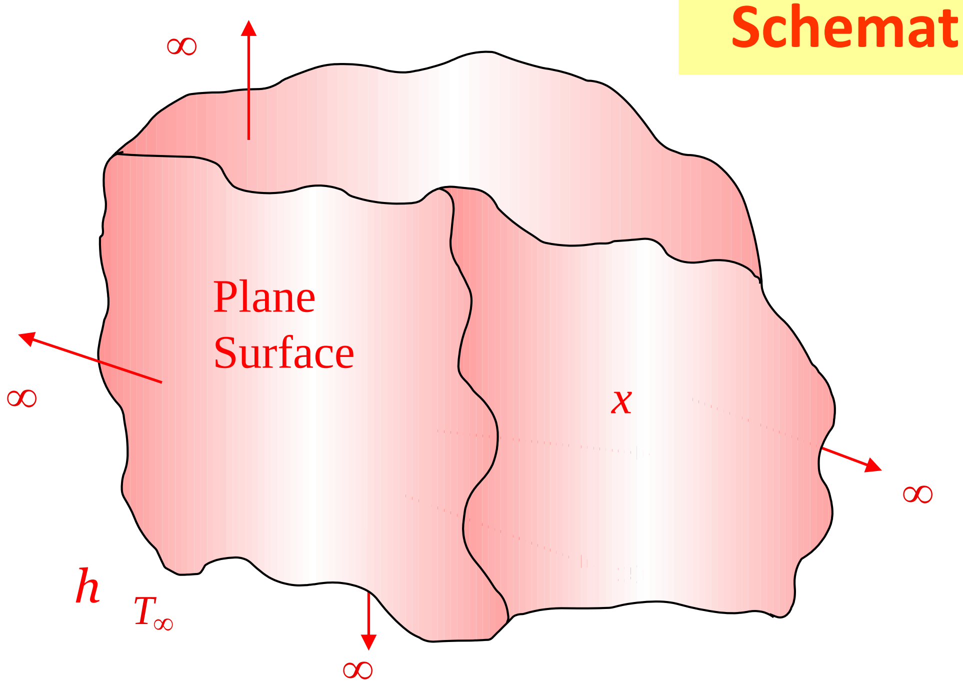
$$V = \frac{\pi}{4}\left(d_o^2 - d_i^2\right)Length = \pi\, dL Length$$

TRANSIENT HEAT CONDUCTION IN SEMI-INFINITE SOLIDS

- **A semi-infinite solid is an idealised body that has a single plane surface and extends to infinity in all directions.**
- **This idealised body is used to indicate that the temperature change in the part of the body in which we are interested (the region close to the surface) is due to the thermal conditions on a single surface.**
- **Ex: Earth – temperature variation near its surface**

  Thick wall – temperature variation near one of its surfaces
- **For short periods of time, most bodies can be modeled as semi-infinite solids since heat does not have sufficient time to penetrate deep into the body and the thickness of the body does not enter into the heat transfer analysis.**

Schematic of the semi-infinite medium



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$ **Initial Condition**

$$T(0,t) = T_s$$

$$T(x \to \infty, t) = T_i$$

Boundary conditions

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Convert partial differential equation into ordinary differential equation by combining the two independent variables x and t into a single variable η

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = \frac{dT}{d\eta}\frac{x}{\sqrt{4\alpha}}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) = -\frac{dT}{d\eta}\frac{x}{2t\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial t} = -\frac{dT}{d\eta}\frac{x}{2t\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dT}{d\eta}\frac{1}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{1}{\sqrt{4\alpha t}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{d}{d\eta}\left(\frac{dT}{d\eta}\frac{1}{\sqrt{4\alpha t}}\right)\frac{1}{\sqrt{4\alpha t}} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2} = \frac{1}{\alpha}\left(-\frac{dT}{d\eta}\frac{x}{2t\sqrt{4\alpha t}}\right)$$

$$\frac{d^2 T}{d\eta^2} = -\frac{2x}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{d^2 T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$$

$$T(\eta \to \infty) = T_i$$

$$T(\eta = 0) = T_s$$

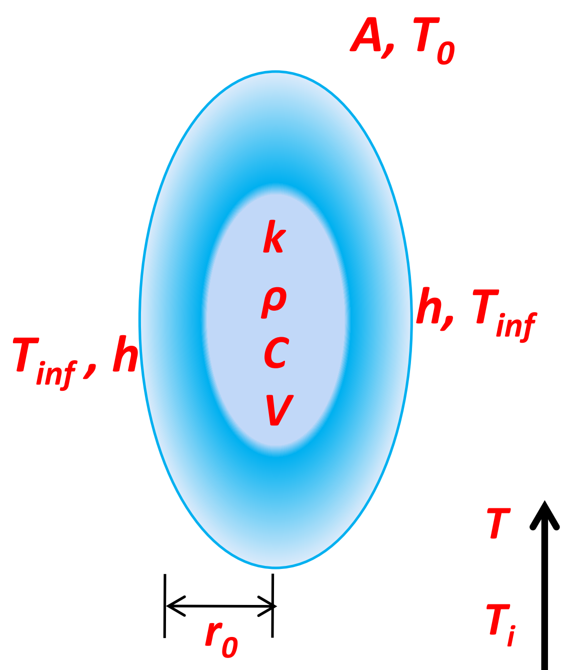
$T(x,0) = T_i$ $\quad$ $t = 0 \Rightarrow \eta \to \infty$ $\quad$ $T(\eta \to \infty) = T_i$

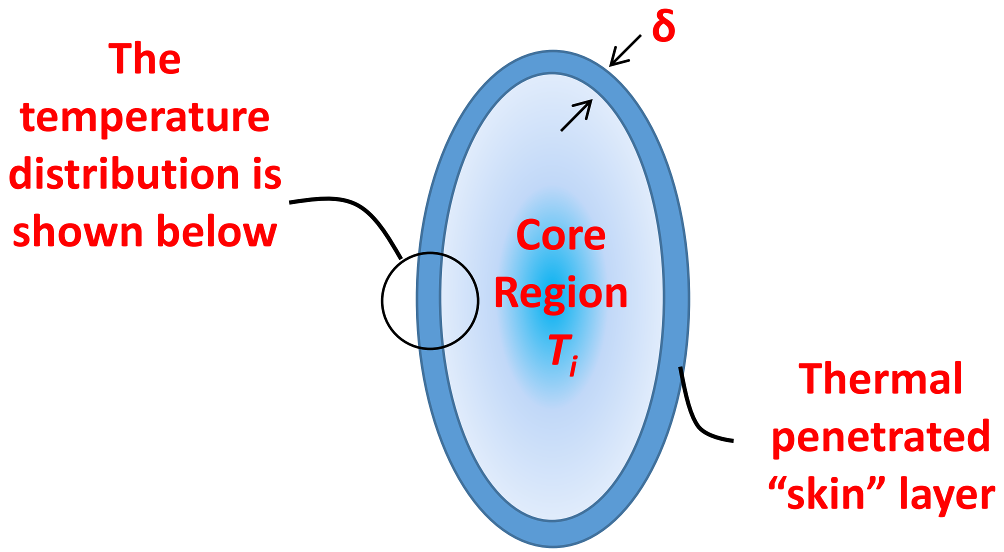
$T(0,t) = T_s$ $\quad$ $x = 0 \Rightarrow \eta = 0$ $\quad$ $T(\eta = 0) = T_s$

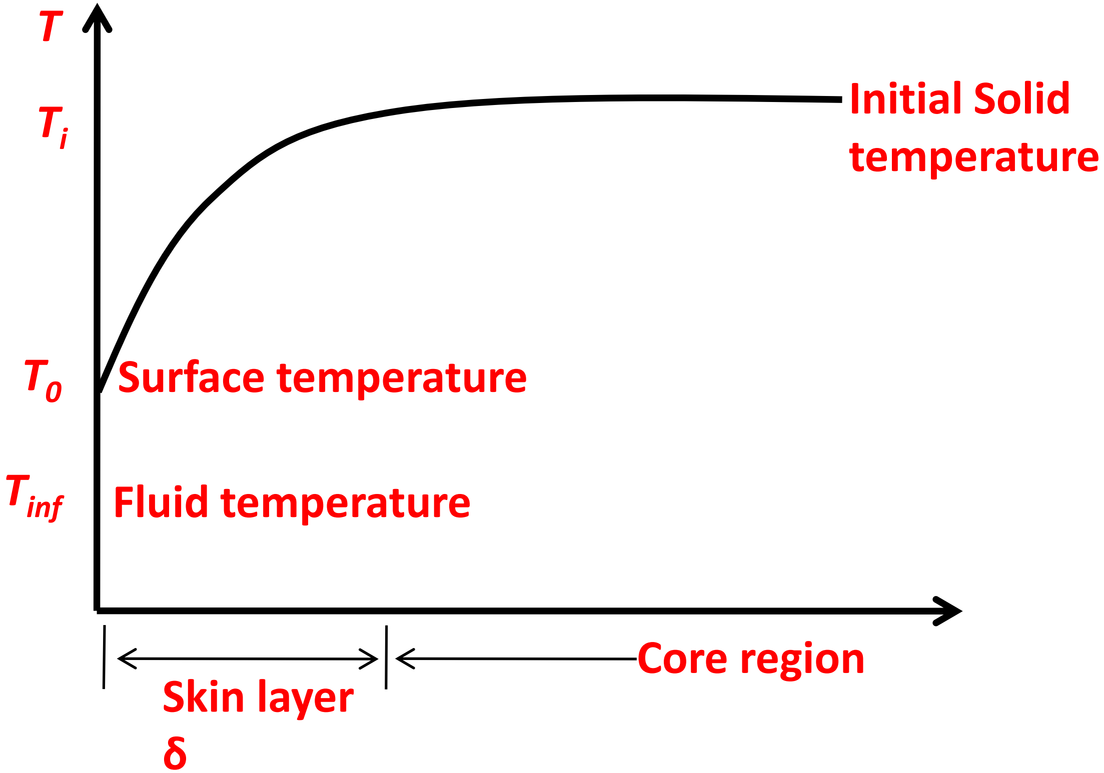
$T(x \to \infty, t) = T_i$ $\quad$ $x \to \infty \Rightarrow \eta \to \infty$ $\quad$ $T(\eta \to \infty) = T_i$

Both the differential equation and the boundary conditions depend only on η and are independent on x and t

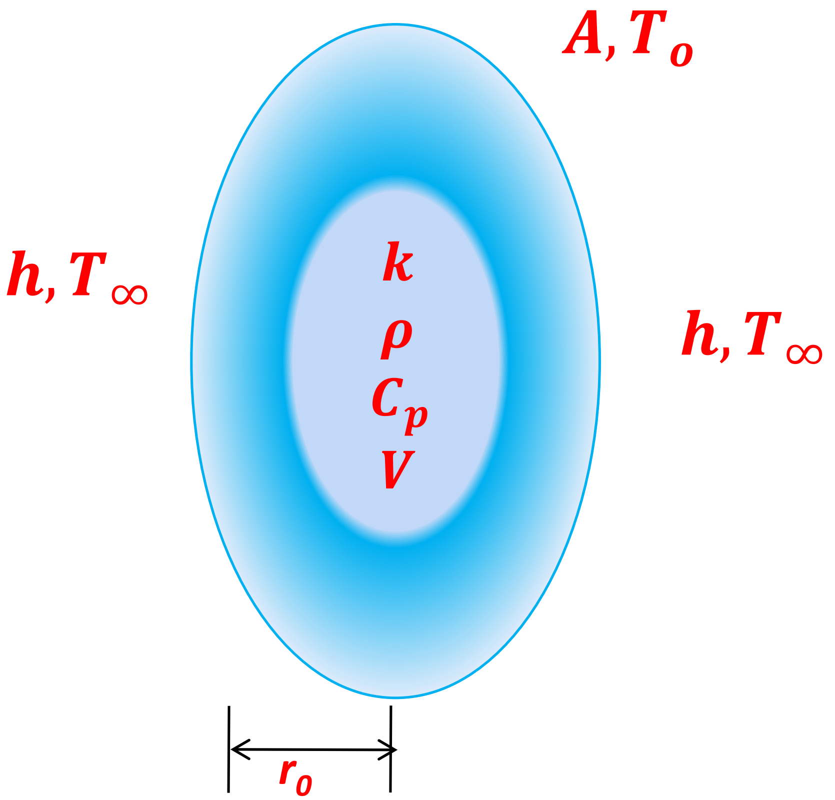
Formation of a thermally penetrated skin layer under the surface immersed in a fluid

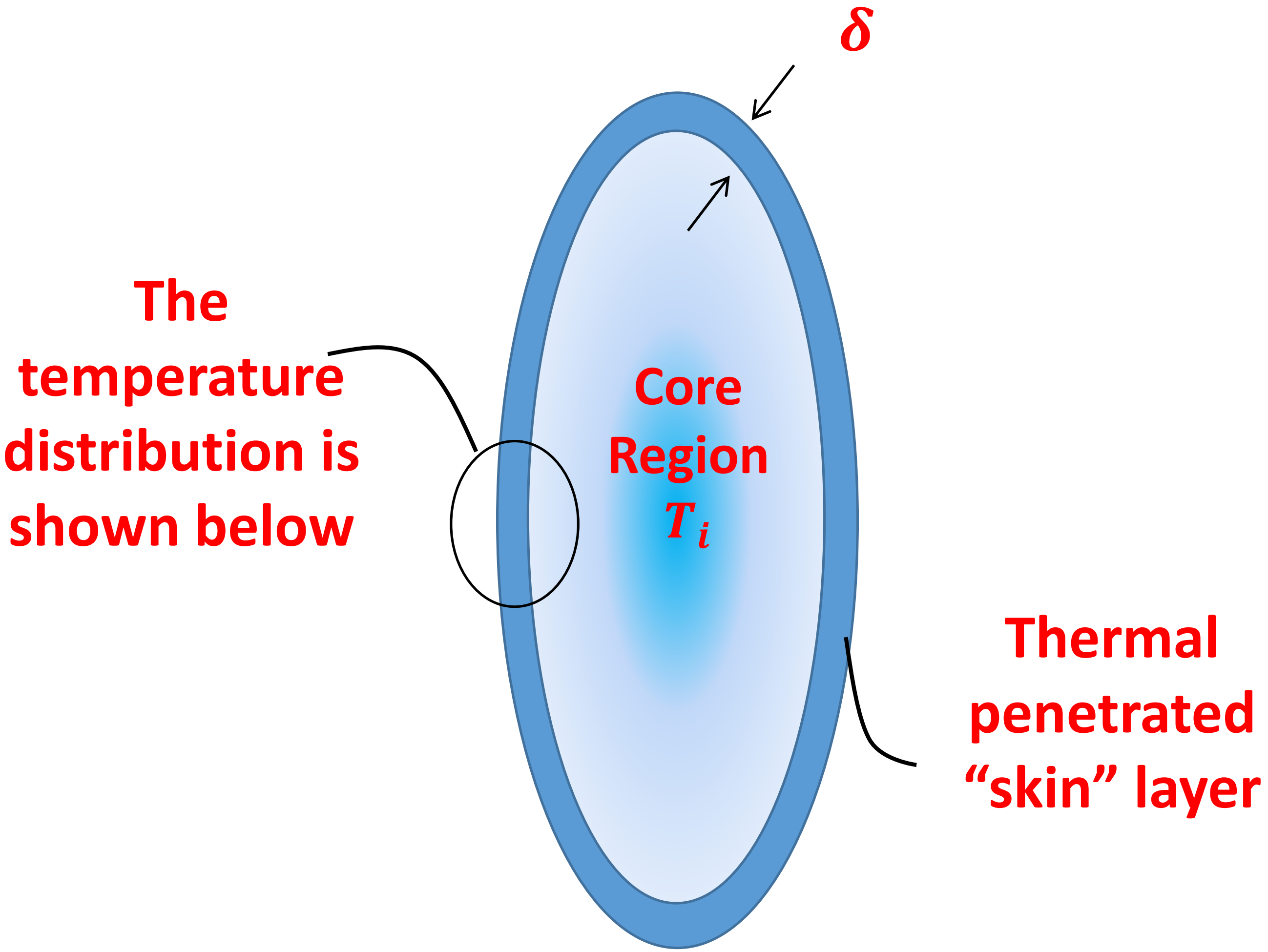




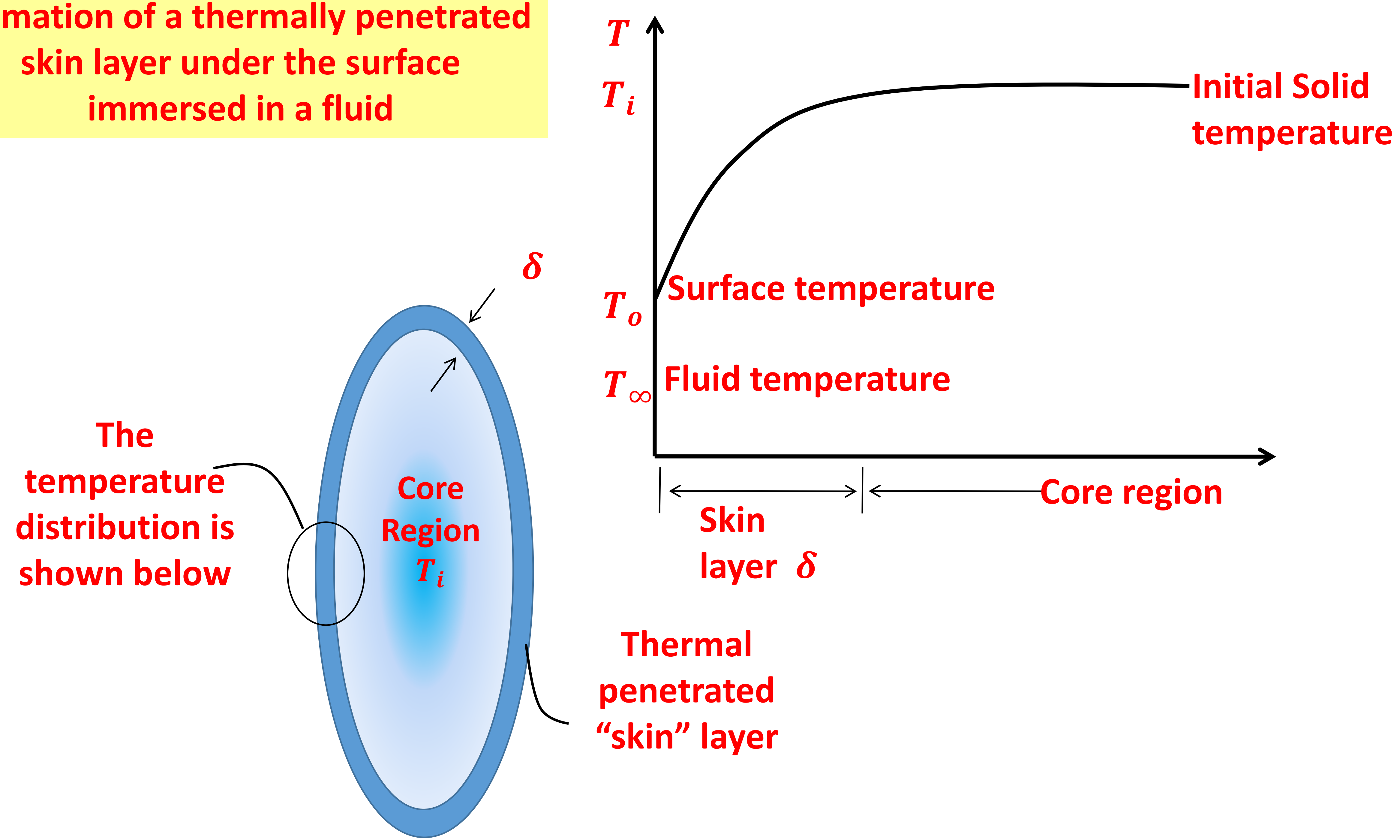


Formation of a thermally penetrated skin layer under the surface immersed in a fluid





Formation of a thermally penetrated skin layer under the surface immersed in a fluid



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\left(\frac{\partial T}{\partial x}\right)_{x\sim\delta} - \left(\frac{\partial T}{\partial x}\right)_{x\sim 0}}{\delta - 0}$$

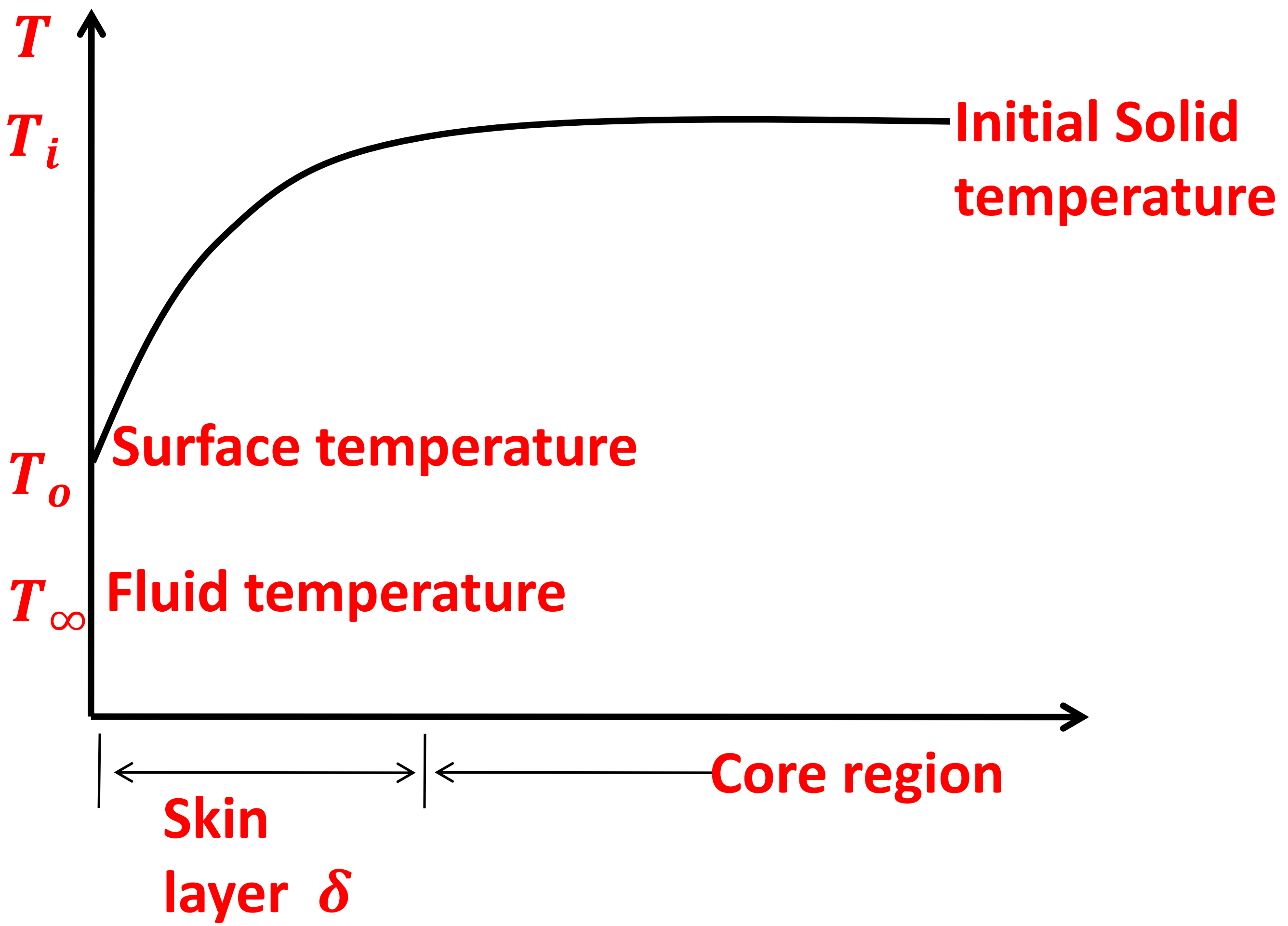
$$\left(\frac{\partial T}{\partial x}\right)_{x\sim\delta} \sim 0 \qquad \left(\frac{\partial T}{\partial x}\right)_{x\sim 0} \sim \frac{T_i - T_o}{\delta - 0}$$

$$\frac{\partial^2 T}{\partial x^2} \sim -\frac{T_i - T_o}{\delta^2}$$

$$\frac{\partial T}{\partial t} \sim -\frac{T_i - T_o}{0 - t} \quad \frac{\partial T}{\partial t} \sim \frac{T_o - T_i}{t}$$

$$-\frac{T_i - T_o}{\delta^2} = \frac{1}{\alpha}\frac{T_o - T_i}{t}$$

$$\delta^2 \sim \alpha t \qquad \delta \sim \sqrt{\alpha t}$$



$$\eta \sim \frac{x}{\delta} \qquad \eta = \frac{x}{\sqrt{4\alpha t}}$$

The constant four helps in further mathematical simplifications

$$\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$$

$$T(\eta \to \infty) = T_i$$

$$T(\eta = 0) = T_s$$

$$\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$$

$$\frac{d}{d\eta}\left(\frac{dT}{d\eta}\right) = -2\eta\frac{dT}{d\eta}$$

$$p = \frac{dT}{d\eta}$$

$$\frac{dp}{d\eta} = -2\eta p$$

$$\frac{dp}{p} = -2\eta d\eta$$

Integrating

$$\frac{dp}{p} = -2\eta d\eta$$

$$lnp = -2\frac{\eta^2}{2} + C$$

$$ln\frac{dT}{d\eta} = -\eta^2 + C$$

$$\frac{dT}{d\eta} = C_1 e^{-\eta^2}$$

Integrating

$$\frac{dT}{d\eta} = C_1 e^{-\eta^2}$$

$$T = C_1 \int_0^\eta e^{-\eta^2} d\eta + C_2$$

$$T(\eta = 0) = T_s \Rightarrow C_2 = T_s$$

$$erf(0) = 0$$

$$C_2 = T_s$$

$$T = C_1 \int_0^\eta e^{-\eta^2} d\eta + T_s$$

$$T(\eta \to \infty) = T_i$$

$$T_i = C_1 \int_0^\infty e^{-\eta^2} d\eta + T_s$$

$$C_1 = \frac{T_i - T_s}{\int_0^\infty e^{-\eta^2} d\eta}$$

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2} dt$$

$$erf(\infty) = 1$$

$$1 = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2} dt$$

$$C_1 = \frac{T_i - T_s}{\frac{\sqrt{\pi}}{2}}$$

$$C_1 = \frac{2(T_i - T_s)}{\sqrt{\pi}}$$

ERROR FUNCTION $erf(z)$

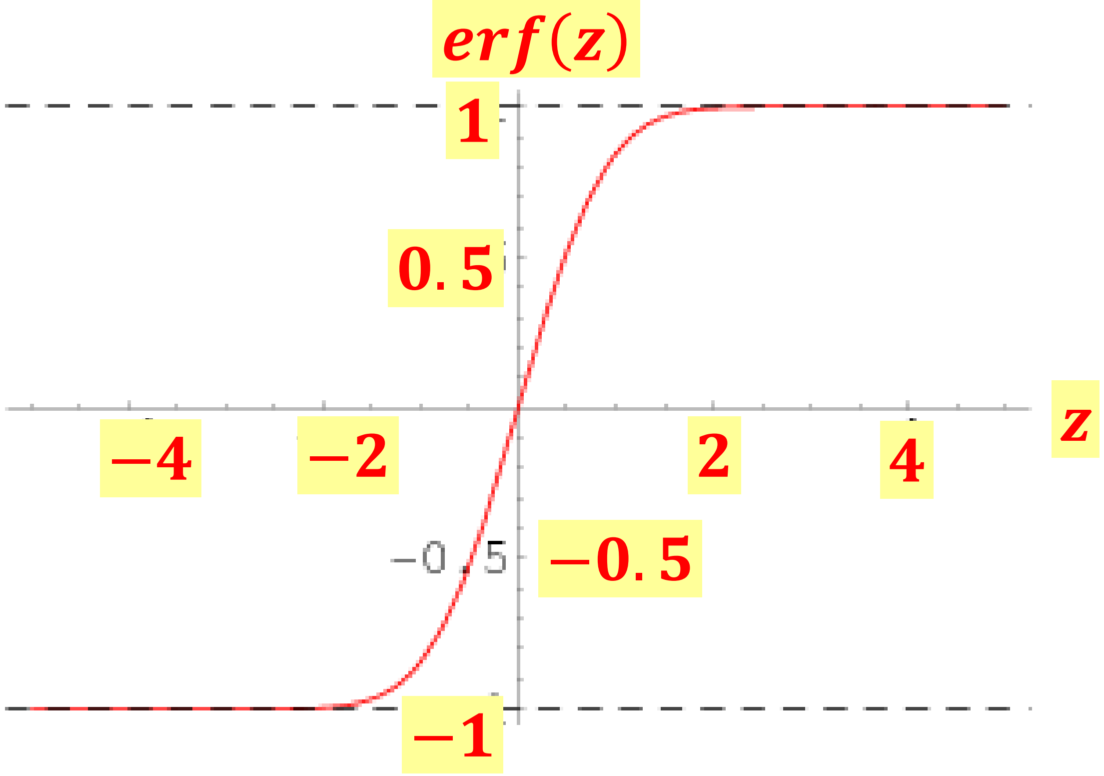
$erf(z)$ is the error function encountered in integrating the normal distribution (which is normalised form of the Gaussian distribution). It is defined by

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)}$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} + \cdots \right)$$

$erf(0) = 0$ $erf(\infty) = 1$



$erf(z)$ is an odd function

$erf(-z) = -erf(z)$

$erf(z) + erfc(z) = 1$

$erfc(z)$ is called the complementary error function

$$T = C_1 \int_0^{\eta} e^{-\eta^2} d\eta + C_2$$

$$C_2 = T_s$$

$$C_1 = \frac{2(T_i - T_s)}{\sqrt{\pi}}$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-t^2} dt$$

$$T = \frac{2(T_i - T_s)}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta + T_s$$

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta = erf(\eta)$$

$$\frac{T - T_s}{T_i - T_s} = erf(\eta)$$

The surface heat flux may be obtained by Fourier's law at $x = 0$

$$q'' = -k \frac{\partial T}{\partial x}\bigg|_{x=0} = -k(T_i - T_s) \frac{d(erf\eta)}{d\eta} \frac{\partial \eta}{\partial x}\bigg|_{\eta = 0} = k(T_s - T_i) \frac{2}{\sqrt{\pi}} e^{-\eta^2} \frac{\partial \eta}{\partial x}\bigg|_{\eta = 0}$$

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

$$q'' = k(T_s - T_i) \frac{2}{\sqrt{\pi}} e^{-\eta^2} \frac{1}{\sqrt{4\alpha t}}\bigg|_{\eta = 0}$$

$$q'' = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 1 Constant surface temperature $T(0,t) = T_s$

$$\frac{T(x,t) - T_s}{T_i - T_s} = erf\left(\frac{x}{\sqrt{4\alpha t}}\right) \qquad q''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2 Constant Surface Heat Flux $q'' = q''_o$

$$T(x,t) - T_i = \frac{2q''_o \left(\frac{\alpha t}{\pi}\right)^{\frac{1}{2}}}{k} exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k} erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case 3 Surface convection

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = erfc\left(\frac{x}{\sqrt{4\alpha t}}\right) - \left[exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\right]\left[erfc\left(\frac{x}{\sqrt{4\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

$$erf(z) = \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} + \cdots\right) \qquad erf(z) + erfc(z) = 1$$

w	erf w	w	erf w	w	erf w
0.00	0	0.36	0.38933	1.04	0.85865
0.02	0.02256	0.38	0.40901	1.08	0.87333
0.04	0.04511	0.40	0.42839	1.12	0.88679
0.06	0.06762	0.44	0.46622	1.16	0.8991
0.08	0.09008	0.48	0.50275	1.20	0.91031
0.10	0.11246	0.52	0.5379	1.30	0.93401
0.12	0.13476	0.56	0.57162	1.40	0.95228
0.14	0.15695	0.60	0.60386	1.50	0.96611
0.16	0.17901	0.64	0.63459	1.60	0.97635
0.18	0.20094	0.68	0.66378	1.70	0.98379
0.20	0.2227	0.72	0.69143	1.80	0.98909
0.22	0.2443	0.76	0.71754	1.90	0.99279
0.24	0.267	0.80	0.7421	2.00	0.99532
0.26	0.2869	0.84	0.76514	2.20	0.99814
0.28	0.30788	0.88	0.78669	2.40	0.99931
0.30	0.32863	0.92	0.80677	2.60	0.99976
0.32	0.34913	0.96	0.82542	2.80	0.99992
0.34	0.36936	1.00	0.8427	3.00	0.99998

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)}$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} + \cdots \right)$$

$erf(0) = 0$ $erf(\infty) = 1$

$erf(z)$ is an odd function

$erf(-z) = -erf(z)$

$erf(z) + erfc(z) = 1$

$erfc(z)$ is called the complementary error function

In areas where the air temperature remains below $0°C$ for prolonged periods of time, the freezing of water in underground pipes is a major concern. Fortunately, the soil remains relatively warm during those periods, and it takes weeks for the subfreezing temperatures to reach the water mains in the ground. Thus, the soil effectively serves as an insulation to protect the water from subfreezing temperatures in winter.

The ground at a particular location is covered with snow pack at $-10°C$ for a continuous period of three months. The average soil properties at that location are $k = 0.4\ W/m.°C$ and $\alpha = 0.15 \times 10^{-6}\, m^2/s$. Assuming an initial uniform temperature of $15°C$ for the ground, determine the minimum burial depth to prevent the waterpipes from freezing.

Known:

$T(x, 0) = 0°C$

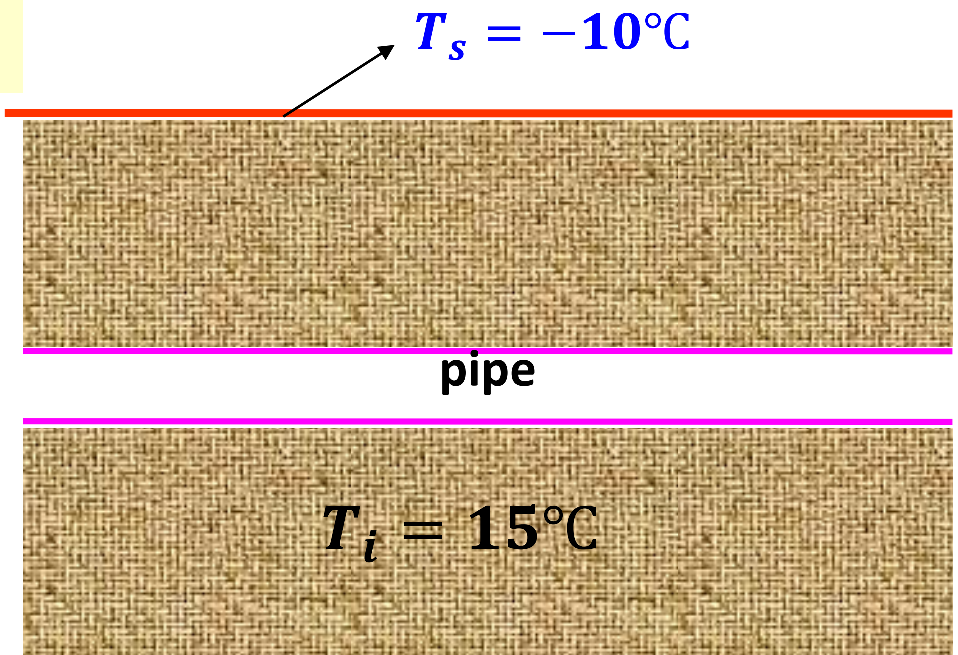
$T_s = -10°C$

$T_i = 15°C$

$k = 0.4\ W/m.°C$

$\alpha = 0.15 \times 10^{-6}\, m^2/s$

SCHEMATIC



Assumptions:

1. The temperature in the soil is affected by the thermal conditions at one surface only and thus the soil can be considered as semi-infinite medium
2. The thermal properties of the soil are constant

Known:

$T(x,t) = 0°C$

$T_s = -10°C$

$T_i = 15°C$

$k = 0.4\ W/m.°C$

$\alpha = 0.15 \times 10^{-6}\ m^2/s$

ANALYSIS

$$\frac{T(x,t) - T_s}{T_i - T_s} = erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

$$\frac{0-(-10)}{15-(-10)} = erf\left(\frac{x}{\sqrt{4 \times 0.15 \times 10^{-6}(3 \times 30 \times 24 \times 60 \times 60)}}\right)$$

$$0.4 = erf\left(\frac{x}{\sqrt{4 \times 0.15 \times 10^{-6}(3 \times 30 \times 24 \times 60 \times 60)}}\right)$$

$$0.3708 = \frac{x}{\sqrt{4 \times 0.15 \times 10^{-6}(3 \times 30 \times 24 \times 60 \times 60)}}$$

$$x = 0.8\,m$$

Water pipes must be buried to a depth of atleast 80 cm to avoid freezing under the specified conditions

TRANSIENT CONDUCTION IN MULTI-DIMENSIONAL SYSTEMS

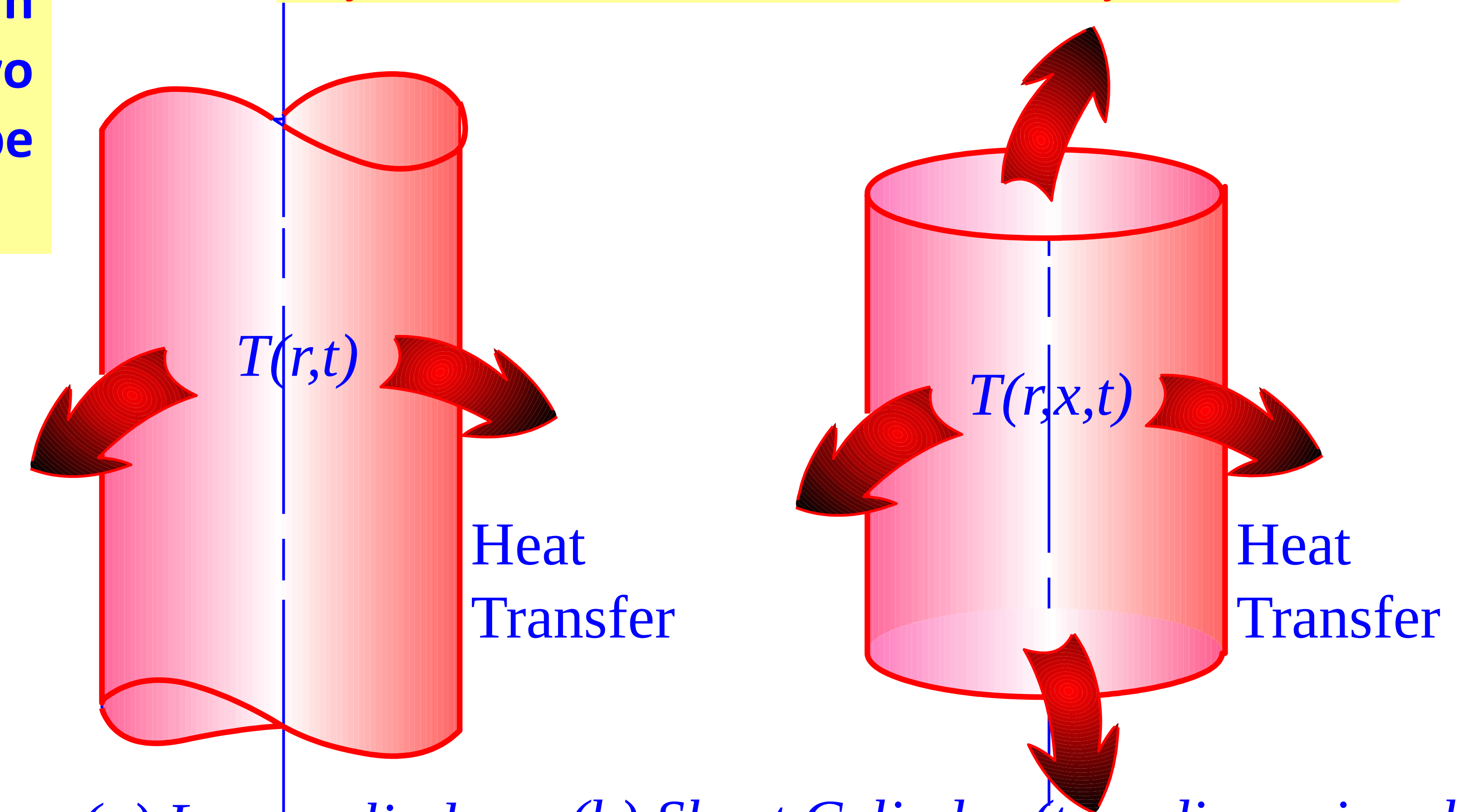
The transient temperature distribution presented earlier can be used to determine the temperature distribution and heat transfer in one-dimensional heat conduction problems associated with a large plane wall, a long cylinder, a sphere, and a semi-infinite medium.

Using a superposition principle called the product solution, these distributions can also be used to construct solutions for the two dimensional transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, or a semi-infinite cylinder or plate, and even three dimensional problems associated with geometries such as a rectangular prism or a semi-infinite rectangular bar, provided that all surfaces of the solid are subjected to convection to the same fluid at temperature, with the same heat transfer coefficient h, and the body involves no heat generation.

The solution in such multi-dimensional geometries can be expressed as the product of the solutions for the one-dimensional geometries whose intersection is the multidimensional geometry

When the properties are assumed to be constant, it can be shown that the solution of this two dimensional problem can be expressed as

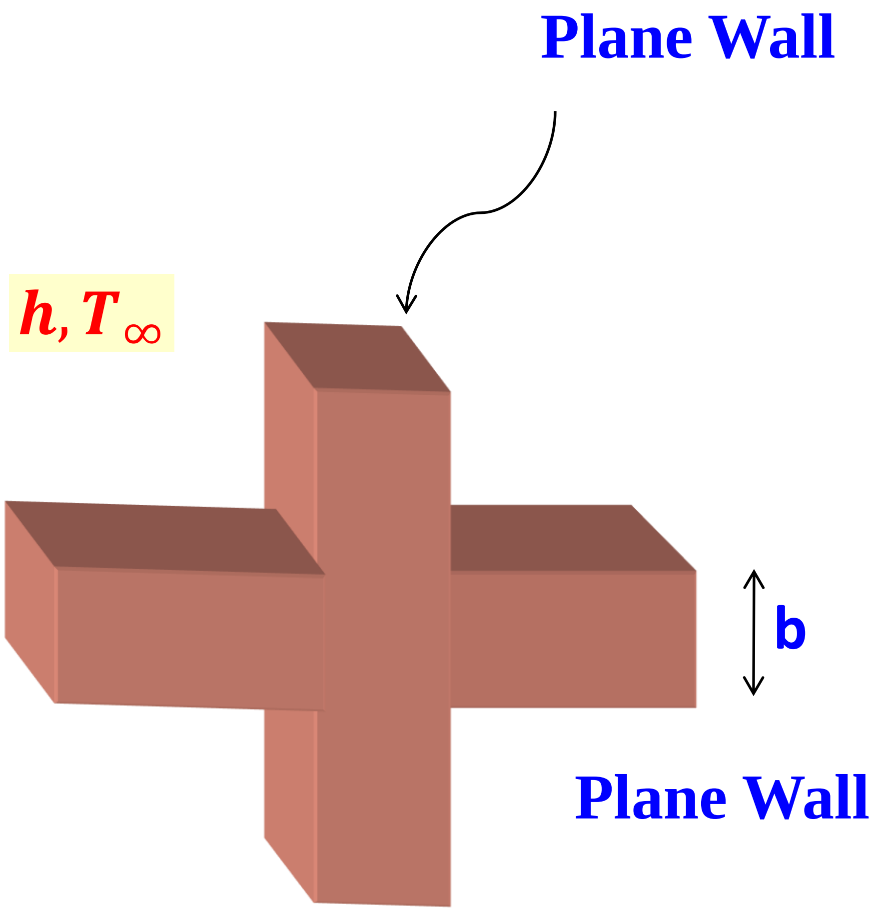
$$\theta_{\substack{short\\cylinder}}(r,x,t)=\theta_{\substack{Plane\\wall}}(x,t)\,\theta_{\substack{short\\cylinder}}(r,t)$$



(a) Long cylinder (b) Short Cylinder (two-dimensional)

$$\left(\frac{T(r,x,t)-T_i}{T_i-T_\infty}\right)_{\substack{Short\\Cylinder}}=\left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{Plane\\Wall}}+\left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{Infinite\\Cylinder}}$$

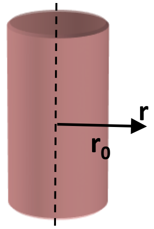
The solution for a long solid bar whose cross section is an $a \times b$ rectangle is the intersection of the two infinite plane walls of thicknesses a and b



Transient temperature distribution for this rectangular bar

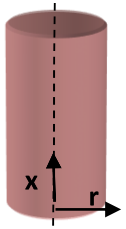
$$\left(\frac{T(x,y,t) - T_\infty}{T_i - T_\infty}\right)_{\substack{Rectangular \\ Bar}} = \theta_{wall}(x,t)\,\theta_{wall}(y,t)$$

$$\left(\frac{T(x,y,t) - T_\infty}{T_i - T_\infty}\right)_{\substack{Rectangular \\ Bar}} = \left(\frac{T(x,t) - T_\infty}{T_i - T_\infty}\right)_{\substack{Plane \\ Wall}} + \left(\frac{T(y,t) - T_\infty}{T_i - T_\infty}\right)_{\substack{Plane \\ Wall}}$$



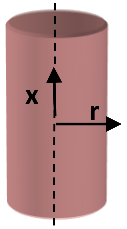
$$\theta(r,t)=\theta_{cyl}(r,t)$$

Infinite cylinder



$$\theta(x,r,t)=\theta_{cyl}(r,t)\theta_{semi-inf}(x,t)$$

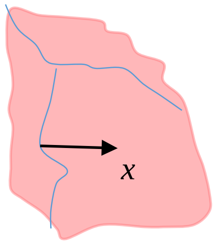
Semi-infinite cylinder



$$\theta(x,r,t)=\theta_{cyl}(r,t)\theta_{wall}(x,t)$$

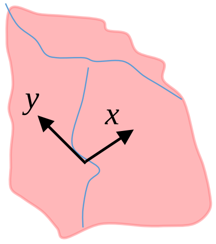
Short cylinder

***x*-coordinate is measured from the surface in a semi-infinite solid, and from the midplane in a plane wall. The radial distance *r* is always measured from the centerline.**



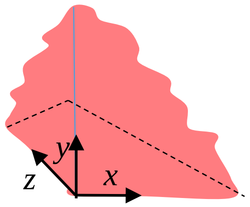
$$\theta(x,t)=\theta_{semi-inf}(x,t)$$

Semi-infinite medium



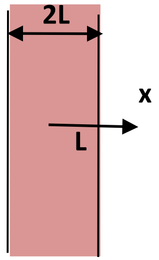
$$\theta(x,y,t)=\theta_{semi-inf}(x,t)\theta_{semi-inf}(y,t)$$

Quarter-infinite medium



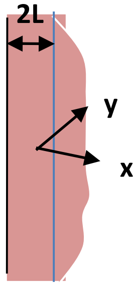
$$\theta(x,y,z,t)=\theta_{semi-inf}(x,t)\theta_{semi-inf}(y,t)\theta_{semi-inf}(z,t)$$

Corner region of a large medium



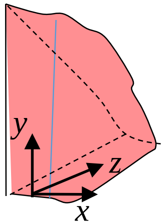
$$\theta(x,t)=\theta_{wall}(x,t)$$

Infinite Plate (or plane wall)



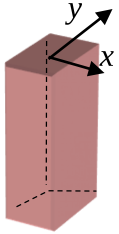
$$\theta(x,y,t)=\theta_{wall}(x,t)\theta_{semi-inf}(y,t)$$

Semi-infinite plate



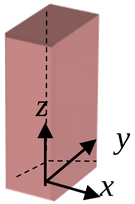
$$\theta(x,y,z,t)=\theta_{wall}(x,t)\theta_{semi-inf}(y,t)\theta_{semi-inf}(z,t)$$

Quarter-infinite plate



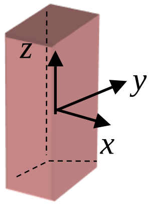
$$\theta(x,y,t)=\theta_{wall}(x,t)\theta_{wall}(y,t)$$

Infinite rectangular bar



$$\theta(x,y,z,t)=\theta_{wall}(x,t)\theta_{wall}(y,t)\theta_{semi-inf}(z,t)$$

Semi-infinite rectangular bar



$$\theta(x,y,z,t)=\theta_{wall}(x,t)\theta_{wall}(y,t)\theta_{wall}(z,t)$$

Rectangular parallelepiped

A modified form of the product solution can also be used to determine the total transient heat transfer to or from a multidimensional geometry by using the one dimensional values, as shown by L.S.Langston in 1982.

The transient heat transfer for a two dimensional geometry formed by the intersection of two one dimensional geometries 1 and 2 is

$$\left(\frac{Q}{Q_{max}}\right)_{total,2D} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]$$

Transient heat transfer for a three dimensional body formed by the intersection of three one dimensional bodies 1,2 and 3 is given by

$$\left(\frac{Q}{Q_{max}}\right)_{total,3D} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right] + \left(\frac{Q}{Q_{max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{max}}\right)_2\right]$$

Problem: A short brass cylinder to diameter D = 10 cm and height H = 12 cm is initially at a uniform temperature $T_i = 120^0$ C. The cylinder is now placed in atmospheric air at 25^0 C, where heat transfer takes place by convection, with a heat transfer coefficient of $h = 60$ W/m^2.^{0}C. Calculate the temperature at (a) the center of the cylinder (b) the center of the top surface of the cylinder 15 min after the start of the cooling (c) Determine the total heat transfer from the short brass cylinder ($\rho = 8530$ kg/m^3, $C_p = 0.38$ kJ/kg.^{0} C.

Known: Initial temperature of the short cylinder, dimensions of the cylinder, convective boundary conditions

Find: Temperature at the center of the cylinder & the center of the top surface of the cylinder 15 min after the start of the cooling and the total heat transfer from the short brass cylinder

Assumptions:

Heat conduction in the short cylinder is two dimensional, and thus the temperature varies in both the axial *x*- and the radial *r*- directions.

The thermal properties of the cylinder and the heat transfer coefficient are constant.

The Fourier number is $\tau > 0.2$ so that the one term approximation solutions are applicable

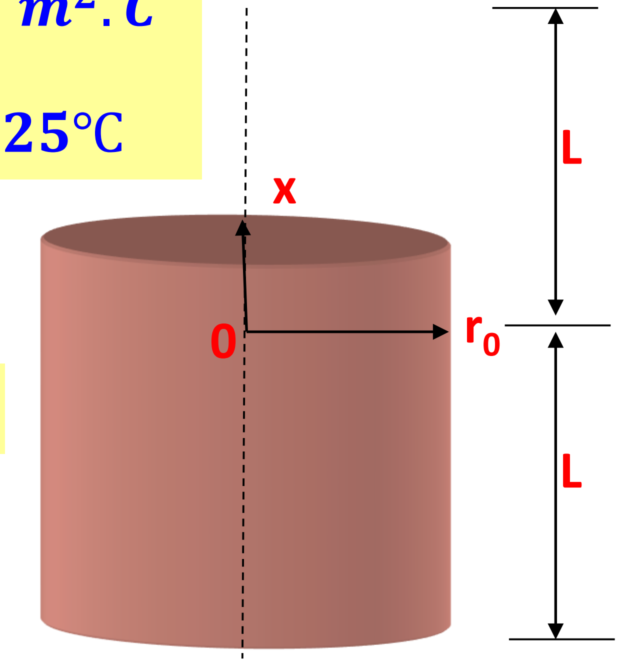
Properties:

The properties of the brass at room temperature are k = 110 W/m.^{0} C, ρ = 8530 kg/m^3, C_p = 380 J/kg.^{0} C, and α = 33.9 × 10^{-6} m^2/s. More accurate results are obtained by using properties at average temperature.

$$h = 60\ \frac{W}{m^2.C}$$

$$T_\infty = 25°C$$

$$T_i = 120°C$$



This short cylinder can physically be formed by the intersection of a long cylinder of radius r_0 = 5 cm and a plane wall of thickness $2L$ = 12 cm

$$Bi = \frac{hL}{k} = \frac{60 \times 6 \times\ 10^{-2}}{110} = 0.03268$$

$$Fo = \frac{\alpha t}{L^2} = \frac{3.39\ \times\ 10^{-6} \times 15 \times 60}{(6 \times\ 10^{-2})^2} = 8.48$$

This short cylinder can physically be formed by the intersection of a long cylinder of radius r_0 = 5 cm and a plane wall of thickness $2L$ = 12 cm

$Bi = 0.03268$ $Fo = 8.48$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

$Bi = 0.03268$ $A_1 = 1.0055$ $\lambda_1 = 0.1785$

$Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 1.0055 e^{-(0.1785)^2(8.48)}$$

$$\theta_{wall}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 0.8052$$

	Plane wall	
Bi	λ_1	A_1
0.01	0.0998	1.0017
0.02	0.141	1.0033
0.04	0.1987	1.0066
0.06	0.2425	1.0098
0.08	0.2791	1.013
0.1	0.3111	1.0161
0.2	0.4328	1.0311
0.3	0.5218	1.045
0.4	0.5932	1.058
0.5	0.6533	1.0701
0.6	0.7051	1.0814
0.7	0.7506	1.0918
0.8	0.791	1.1016
0.9	0.8274	1.1107

This short cylinder can physically be formed by the intersection of a long cylinder of radius r_0 = 5 cm and a plane wall of thickness $2L$ = 12 cm

$$Bi = \frac{hr_o}{k} = \frac{60 \times 5 \times 10^{-2}}{110} = 0.02727$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{3.39 \times 10^{-6} \times 15 \times 60}{(5 \times 10^{-2})^2} = 12.2$$

$Fo = 12.2$

$Bi = 0.02727$ $A_1 = 1.0068$ $\lambda_1 = 0.2293$

Center of Cylinder $(r = 0)$: $Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 1.0068 e^{-(0.2293)^2(12.2)}$$

$$\theta_{cyl}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 0.53$$

Bi	λ_1	A_1
0.01	0.1412	1.0025
0.02	0.1995	1.005
0.04	0.2814	1.0099
0.06	0.3438	1.0148
0.08	0.396	1.0197
0.1	0.4417	1.0246
0.2	0.617	1.0483
0.3	0.7465	1.0712
0.4	0.8516	1.0931
0.5	0.9408	1.1143
0.6	1.0184	1.1345
0.7	1.0873	1.1539
0.8	1.149	1.1724
0.9	1.2048	1.1902

$$\theta_{wall}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 0.8052$$

$$\theta_{cyl}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 0.53$$

$$\theta_{\substack{\text{short}\\ \text{cylinder}}}(0,t) = \frac{T(0,0,t) - T_\infty}{T_i - T_\infty} = \theta_{wall}(0,t)\theta_{cyl}(0,t)$$

$$\frac{T(0,0,t) - 25}{120 - 25} = (0.8052)(0.53) = 0.42684$$

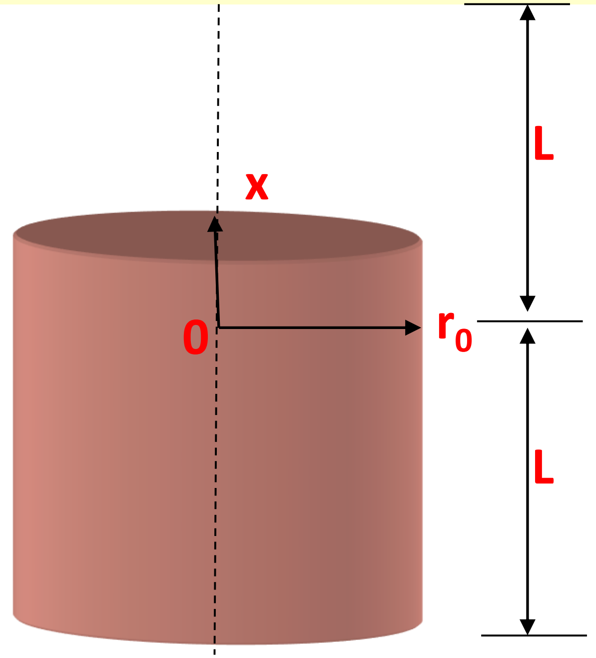
$$T(0,0,t) = 65.55℃$$

This is the temperature at the center of the short cylinder, which is also the center of both the long cylinder and the plate.

The center of the top surface of the cylinder is still at the center of the long cylinder (r = 0), but at the outer surface of the plane wall (*x = L*). Therefore, we first need to find the surface temperature of the wall. Noting that *x = L = 0.06 m*,

$Bi = 0.03268$ $Fo = 8.48$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$



$Bi = 0.03268$ $A_1 = 1.0055$ $\lambda_1 = 0.1785$

$Fo > 0.2$

$$\theta(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 1.0055 e^{-(0.1785)^2(8.48)}$$

$$\theta_{wall}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 0.8052$$

Plane wall

$$\theta_{wall}(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} cos\left(\frac{\lambda_1 x}{L}\right) = \theta(0,t) cos\left(\frac{\lambda_1 x}{L}\right)$$

$Bi = 0.03268$ $A_1 = 1.0055$ $\lambda_1 = 0.1785$ $\theta_{wall}(0,t) = 0.8052$

$$\theta_{wall}(L,t) = \frac{T(L,t) - T_\infty}{T_i - T_\infty} = \theta(0,t) cos\left(\frac{\lambda_1 L}{L}\right)$$

$$\theta_{wall}(L,t) = \frac{T(L,t) - T_\infty}{T_i - T_\infty} = 0.8052 cos\left(0.1785 \times \frac{180}{\pi}\right) = 0.8052 \times 0.9841$$

$$\theta_{wall}(L,t) = \frac{T(L,t) - T_\infty}{T_i - T_\infty} = 0.8052 cos\left(0.1785 \times \frac{180}{\pi}\right) = 0.7924$$

$$\theta_{wall}(L,t) = 0.7924$$

$$\theta_{\substack{short\\cylinder}}(0,t) = \frac{T(L,0,t) - T_\infty}{T_i - T_\infty} = \theta_{wall}(L,t)\theta_{cyl}(0,t)$$

$$\theta_{wall}(L,t) = 0.7924 \qquad \theta_{cyl}(0,t) = 0.53$$

$$\theta_{\substack{short\\cylinder}}(0,t) = \frac{T(L,0,t) - T_\infty}{T_i - T_\infty} = 0.7924 \times 0.53$$

$$\theta_{\substack{short\\cylinder}}(0,t) = \frac{T(L,0,t) - 25}{120 - 25} = 0.7924 \times 0.53$$

$$T(L,0,t) = 64.9℃$$

This is the temperature at the center of the top surface of the cylinder

We must first determine the maximum heat that can be transferred from the cylinder, which is the sensible energy content of the cylinder relative to its environment:

$$m = \rho V = \rho \pi r_o^2 L = 8350 \times \pi\left(5 \times 10^{-2}\right)^2\left(6 \times 10^{-2}\right) = 4.02\ kg$$

$$Q_{max} = mC_p(T_\infty - T_i) = 4.02 \times 0.38(120 - 25) = 145.1\ kJ$$

Then we determine the dimensionless heat transfer ratios for both geometries. For the plane wall,

$Bi = 0.03268$ $A_1 = 1.0055$ $\lambda_1 = 0.1785$ $\theta_{wall}(0,t) = 0.8052$

Plane wall

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{o,wall}\left(\frac{sin\lambda_1}{\lambda_1}\right)$$

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - 0.8052\left(\frac{sin\left(\frac{0.1785 \times 180}{\pi}\right)}{0.1785}\right)$$

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 0.199$$

Cylinder

$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - \theta_{o,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

$$\theta_{cyl}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = 0.53$$

$Bi = 0.02727$ $A_1 = 1.0068$ $\lambda_1 = 0.2293$

$$J_1(\lambda_1) = J_1(0.2293) = 0.1138$$

$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - \theta_{o,cyl}\frac{J_1(\lambda_1)}{\lambda_1} = 1 - (0.53)\frac{0.1138}{0.2293}$$

$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 0.47394 \quad \left(\frac{Q}{Q_{max}}\right)_1 = 0.199 \quad \left(\frac{Q}{Q_{max}}\right)_2 = 0.47394$$

λ_1	$J_0(\lambda_1)$	$J_1(\lambda_1)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419

Heat transfer ratio for the short cylinder

$$\left(\frac{Q}{Q_{max}}\right)_{total,2D} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2\left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]$$

$$\left(\frac{Q}{Q_{max}}\right)_{total,2D} = 0.199 + 0.47394[1 - 0.199] = 0.5786$$

Therefore, the total heat transfer from the cylinder during the first 15 min of cooling is

$$\left(\frac{Q}{Q_{max}}\right)_{total,2D} = 0.5786$$

$$Q_{max} = 145.1\ kJ$$

$$\left(\frac{Q}{145.1}\right)_{total,2D} = 0.5786$$

$$Q = 83.96\ kJ$$