Set Theory

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Set Theory: Basics

Set is a correlation of mathematical objects taken from a suitable domain of disclosure.

Examples:

- $\mathbf{N} = \{0, 1, 2, 3, 4. . .\}$

  (set of **natural numbers)**

- $\mathbf{Z} = \{. . ., -3, -2, -1, 0, 1, 2, 3,. . .\}$

  (set of **integers)**

- $\mathrm{E} = \{0, 2, 4, 6. . .\}$

  (set of **even natural numbers)**

Set Operators

- *Union* of two sets A and B is the set of all elements in either set A or B.
  - Written $A \cup B$.
  - $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

- *Intersection* of two sets A and B is the set of all elements in both sets A or B.
  - Written $A \cap B$.
  - $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- *Difference* of two sets A and B is the set of all elements in set A which are not in set B.
  - Written $A - B$.
  - $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Set Operators

- ***Symmetric Difference*** of two sets A and B is the set of all elements which are either in "set A and not in B" or in "set B and not in A"
  - Written $A \Delta B$.
  - $A \Delta B = (A \setminus B) \cup (B \setminus A)$

- ***Complement*** of a set is the set of all elements not in the set.
  - Written A^c
  - Depends on the choice of superset/universal set S
  - $A^c = \{x \mid x \in \mathbf{S} / A\}$

Cartesain Product

- ***Cartesian Product***: Given two sets A and B, the set of
  - All *ordered pairs* of the form (a , b) where a is any element of A and b any element of B, is called the *Cartesian product* of A and B.
- Denoted as A x B
  - $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$
  - **Example**: Let A = {1,2,3}; B = {*x*,*y*}
    - A x B = {(1,*x*),(1,*y*),(2,*x*),(2,*y*),(3,*x*),(3,*y*)}
    - B x A = {(*x*,1),(*y*,1),(*x*,2),(*y*,2),(*x*,3),(*y*,3)}
    - B x B = B^2 = {(*x*,*x*),(*x*,*y*),(*y*,*x*),(*y*,*y*)}
- ***In general,***

$$A_1 \times A_2 \times \ldots\ldots \times A_n = \{(a_1, a_2 \ldots\ldots\ldots a_n) \mid a_1 \in A_1, \ldots\ldots\ldots a_n \in A_n\}$$

Example:

$$R^2 = R\ X\ R\ (2 - D\ Euclidean\ space)$$
$$= \{(\boldsymbol{x_1}, \boldsymbol{x_2}) : \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{R}\}$$
$$R^d = R\ X\ R \ldots\ldots\ldots X\ R(d - D\ Euclidean\ space)$$
$$= \{(\boldsymbol{x_1}, \boldsymbol{x_2} \ldots\ldots\ldots \boldsymbol{x_d}) : \boldsymbol{x_1}, \boldsymbol{x_2} \ldots\ldots\ldots \boldsymbol{x_d} \in \boldsymbol{R}\}$$

Topology in Euclidean Space

- Euclidean Metric – A measure of distance between two points
$$||x-y|| = \sqrt{(x_1-y_1)^2 + \dots\dots\dots(x_d-y_d)^2}$$

- Closed Ball
$$b(a,r) = \{x \in R^d : ||x-a|| \leq r\}$$

- Open Ball
$$b^{int}(a,r) = \{x \in R^d : ||x-a|| < r\}$$

- Bounded Set – Set A is bounded if there is a ball $b(a,r)$, such that $A \subset b(a,r)$

- A sequence $\{x_1, x_2 \dots\dots\dots\}$ is said to converge to x if $\lim_{n\to\infty} ||x_n - x|| = 0$

Topology in Euclidean Space

- **Open Set** - A set is said to be open if for each $x \in A$, a positive number ϵ can be found depending on x, such that $b(x, \epsilon) \subset A$
- *Example*: In case of d = 1, (u,v) is a open set
  - System of open sets of R^d is denoted by O.

- **Closed Set** – A set is said to be closed if its complement A^c is open
- *Important:* For closed set we need a specific superset S to define A^c
- *Example:* Hypercubes

$$[\boldsymbol{u_1}, \boldsymbol{v_1}]\ \boldsymbol{X}\ [\boldsymbol{u_2}, \boldsymbol{v_2}]\ \boldsymbol{X}\ \dots\dots\dots [\boldsymbol{u_d}, \boldsymbol{v_d}]$$

Hyperplanes

$$\boldsymbol{x} = \{(\boldsymbol{x_1}, \dots\dots\dots \boldsymbol{x_d}) \in \boldsymbol{R^d} : \sum_{\boldsymbol{i-1}}^{\boldsymbol{d}} \boldsymbol{X_i a_i} = \boldsymbol{b}\}$$

where $b, a_1, \dots\dots\dots a_d$ are constants with a_i are not equal to 0

Topology in Euclidean Space

- **The Interior** A^{int} of a general set A is the **union** of all the open sets contained in A
  - $\mathbf{A^{int}}$ is the largest open set contained in A.

- **The Closure** A^{cl} of a general set A is the **intersection** of all the closed sets containing A
  - $\mathbf{A^{cl}}$ is the smallest closed set containing A

- Properties of $\mathbf{A^{cl}}$ and $\mathbf{A^{int}}$
  - $\mathbf{A^{int} \subset A \subset A^{cl}}$
  - Also, $A^{int} = ((A^c)^{cl})^c$
  - A set A is <u>open precisely</u> when $A^{int} = A$
  - A set A is <u>closed precisely</u> when $A^{cl} = A$
  - If $A = (A^{int})^{cl}$, then A is said to be **regular closed**
  - The boundary of a set A, $\partial A = A^{cl} \setminus A^{int}$
  - A set $K \epsilon R^d$ is said to be **Compact**, if it is both closed as well as bounded

Operations on Subsets of Euclidean Space

- **Addition:** $x + y = (x_1 + y_1, x_2 + y_2, \dots \dots \dots x_d + y_d)$

- **Translation:** $A_x = A + x = \{y + x : y \in A\}, for\ x\ and\ A \in R^d$

- **Scalar Multiplication** by $c \in R$,

  $c.x = cx = (cx_1, cx_2, \dots \dots \dots cx_d)$

- **Reflection**: Scalar Multiplication by c = -1,

  $\check{A} = -A = \{-x : x \in A\}\ for\ A \subset R^d$

Operations on Subsets of Euclidean Space

- **Minkowski Addition:** $A \oplus B = \{x + y : x \in A, y \in B\}\ for\ A, B$
  - It is both Associative and Commutative
  - $A_x = A \oplus \{x\}$
  - $A \oplus B = \bigcup_{y \in B} A_y = \bigcup_{x \in A} B_x$
  - $B \oplus A = \left\{x : B \cap \check{A}_x\ is\ not\ empty\right\}$
  - $A \oplus (B_1 \cup B_2) = A \oplus B_1 \cup A \oplus B_2$
  - $If\ A_1 \subset A_2, then\ A_1 \oplus B \subset A_2 \oplus B$

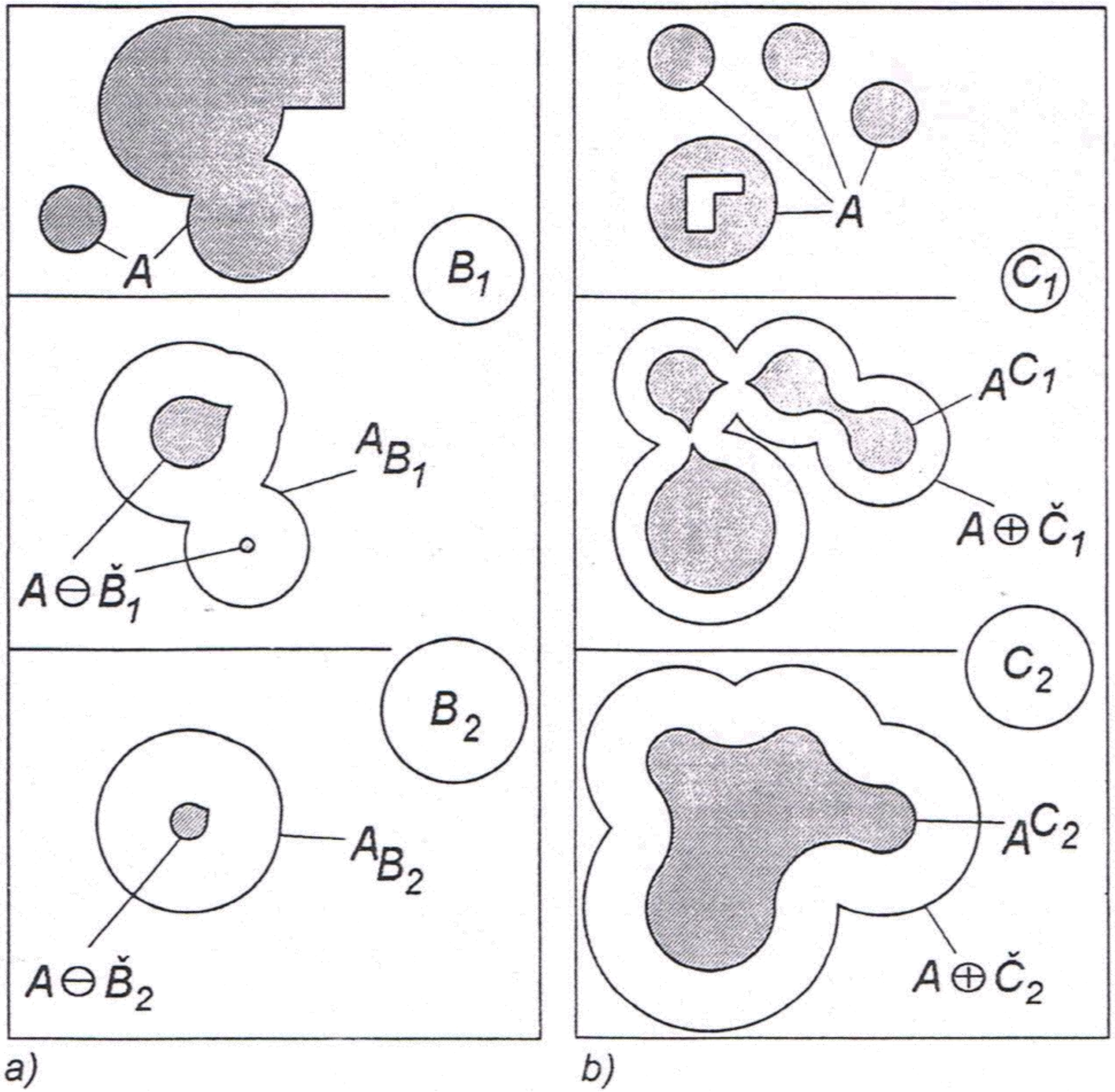
Operations on Subsets of Euclidean Space

- **Minkowski Subtraction:** $A \ominus B = \bigcap_{y \in B} A_y$
  - $or\ (A^c \oplus B)^c$
  - $(\mathrm{A} \ominus \check{\mathrm{C}}) \oplus C \subseteq A \subseteq (\mathrm{A} \oplus \check{\mathrm{C}}) \ominus C$

- **Dilation:** $\boldsymbol{A} \longmapsto \boldsymbol{A} \oplus \check{\mathrm{C}}$

- **Erosion:** $\boldsymbol{A} \longmapsto \boldsymbol{A} \ominus \check{\mathrm{C}}$
  - Opening of A by C (Erosion followed by Dilation)
  - Closing of A by C (Dilation followed by Erosion)

Operations on Subsets of Euclidean Space



(a) The operations of erosion and opening applied to a set. Components that overlap are separated while small components and roughnesses vanish or are reduced.

(b) The operations of dilation and closing applied to a set. Gaps are closed up, concavities vanish or are reduced, and clusters of small particles are merged

Monty Hall Problem Puzzle

- The host of a game show, offers the guest a choice of three doors. Behind one is a expensive car, but behind the other two are goats.
  After you have chosen one door, he reveals one of the other two doors behind which is a goat (he wouldn't reveal a car).

  Now he gives you the chance to switch to the other unrevealed door or stay at your initial choice. You will then get what is behind that door.

  You cannot hear the goats from behind the doors, or in any way know which door has the prize.

  Should you stay, or switch, or doesn't it matter?
-

  Your first choice has a 1/3 chance of having the car, and that does not change.
  The other two doors HAD a combined chance of 2/3, but now a Goat has been revealed behind one, all the 2/3 chance is with the other door.

Probability Of Second Girl Child

- A family has two kids, one of them is a girl. Assume safely that the probability of each gender is 1/2. What is the probability that the other kid is also a girl?

- ANS:

1/3

This is a famous question in understanding conditional probability, which simply means that given some information you might be able to get a better estimate.

The following are possible combinations of two children that form a sample space in any earthly family:
Girl – Girl
Girl – Boy
Boy – Girl
Boy - Boy

Since we know one of the children is a girl, we will drop the Boy-Boy possibility from the sample space.
This leaves only three possibilities, one of which is two girls. Hence the probability is 1/3

Hard Logic Probabilty Puzzle

- Bruna was first to arrive at a 100 seat theater.
  She forgot her seat number and picks a random seat for herself.

  After this, every single person who get to the theater sits on his/her seat if its available else chooses any available seat at random.
  Neymar is last to enter the theater and 99 seats were occupied.

  Whats the probability what Neymar gets to sit in his own seat ?

- ANS: 1/2

  one of two is the possibility
  1. If any of the first 99 people sit in neymar seat, neymar will not get to sit in his own seat.
  2. If any of the first 99 people sit in Bruna's seat, neymar will get to sit in his seat.

Probability Space

- Probability space, which has three components (Ω; F; P), respectively the sample space, event space, and probability function

- **Ω : sample space**. Set of outcomes of an experiment

- In probability theory, the event space **F** is modelled as a σ-algebra (or σ – field) of **Ω**, which is a collection of subsets of with the following properties:

  (1) **Ω** ∈ F

  (2) If an event A ∈ F, then B ∈ F, then A – B ∈ F and

  (3) If $A_i \in F$, then $\cup A_i \in F$ (closed under countable union). A countable sequence can be indexed using the natural integers.

- Probability function **P** assigns a number ("probability") to each event in F. It is a function mapping F ⟶ [0,1] satisfying:

  1. P(A) ≥ 0, for all A ∈ B.
  2. P(**Ω**) = 1
  3. Countable additivity: If $A_i \in B$ are pairwise disjoint (i.e., Ai ∩ Aj = ϕ, for all i ≠ j), then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Probability Space (Sample Space and Event)

Probability space, which has three components (Ω; F; P), respectively the sample space, event space, and probability function

Ω : sample space. Set of all possible outcomes of an experiment in the sample space. An element of **Ω** is denoted by ω.

Example: Driving to work, a commuter passes through a sequence of three intersections with traffic lights. At each light, she either stops, s, or continues, c. The sample space is the set of all possible outcomes:

Ω = {ccc, ccs, css, csc, sss, ssc, scc, scs}

where csc, for example, denotes the outcome that the commuter continues through the first light, stops at the second light, and continues through the third light.

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Example: Driving to work, a commuter passes through a sequence of three intersections with traffic lights. At each light, she either stops, s, or continues, c. The sample space is the set of all possible outcomes:
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We are often interested in particular subsets of **Ω**, which in probability language are called **events**. In the above example, the event that the commuter stops at the first light is the subset of denoted by
A = {sss, ssc, scc, scs}

The algebra of set theory carries over directly into probability theory. The union of two events, A and B, is the event C that either A occurs or B occurs or both occur:
C = A ∪ B. For example, if A is the event that the commuter stops at the first light (listed before), and if B is the event that she stops at the third light,
B = {sss, scs, ccs, css}
then C is the event that she stops at the first light or stops at the third light and consists of the outcomes that are in A or in B or in both:
C = {sss, ssc, scc, scs, ccs, css}

Probability Space (Sample Space and Event)

Similar to the union of two events, intersection of two events and complement of an event is also defined.
The laws of set theory work for the events:

Commutative Laws:

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Associative Laws:

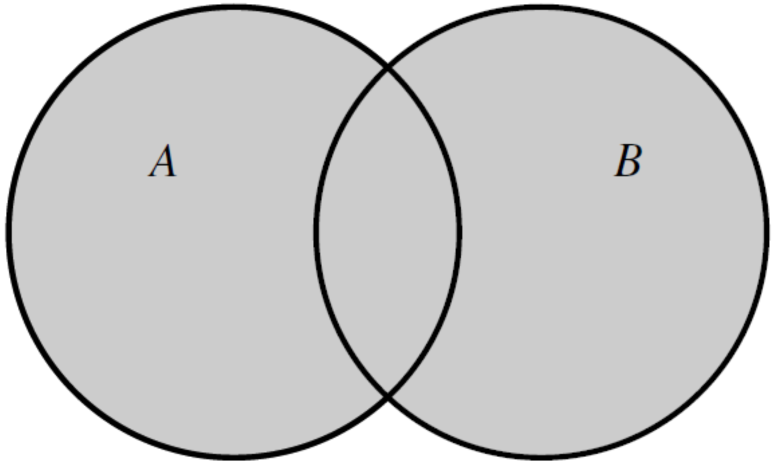
$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

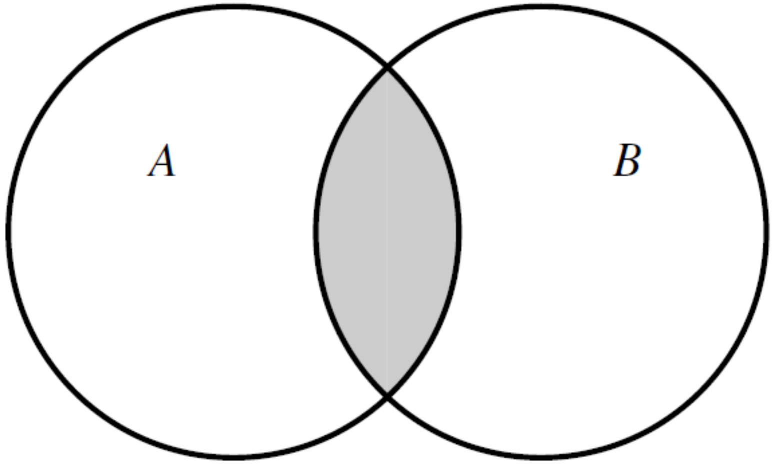
Distributive Laws:

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$



$A \cup B$



$A \cap B$

Probability Space (Event space F)

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Random Variable

- A random variable is a function from the sample space to the real numbers, $X(r)$, $r \in \Omega$, and measurable with respect to P, i.e for every real number x, the set $\{r: X(r) < x\}$ is an event in F

- CDF and PDF
  - For a random variable X on (R, B(R), P_x), we define its cumulative distribution function (CDF) $F_X(x) \equiv P_x(X \leq x)$, for all x (note that all the sets $X \leq x$ are in B(R)).

- Important: F(x) is a CDF if
  1. $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$
  2. F(x) is non-decreasing
  3. F(x) is right-continuous: for every x0, $\lim_{x \to x_o+} F(x) = F(x_0)$