CONVECTION 2

DIFFERENTIAL ANALYSIS OF FLUID FLOW

Finite control volume approach is very practical and useful, since it does not generally require a detailed knowledge of the pressure and velocity variations within the control volume

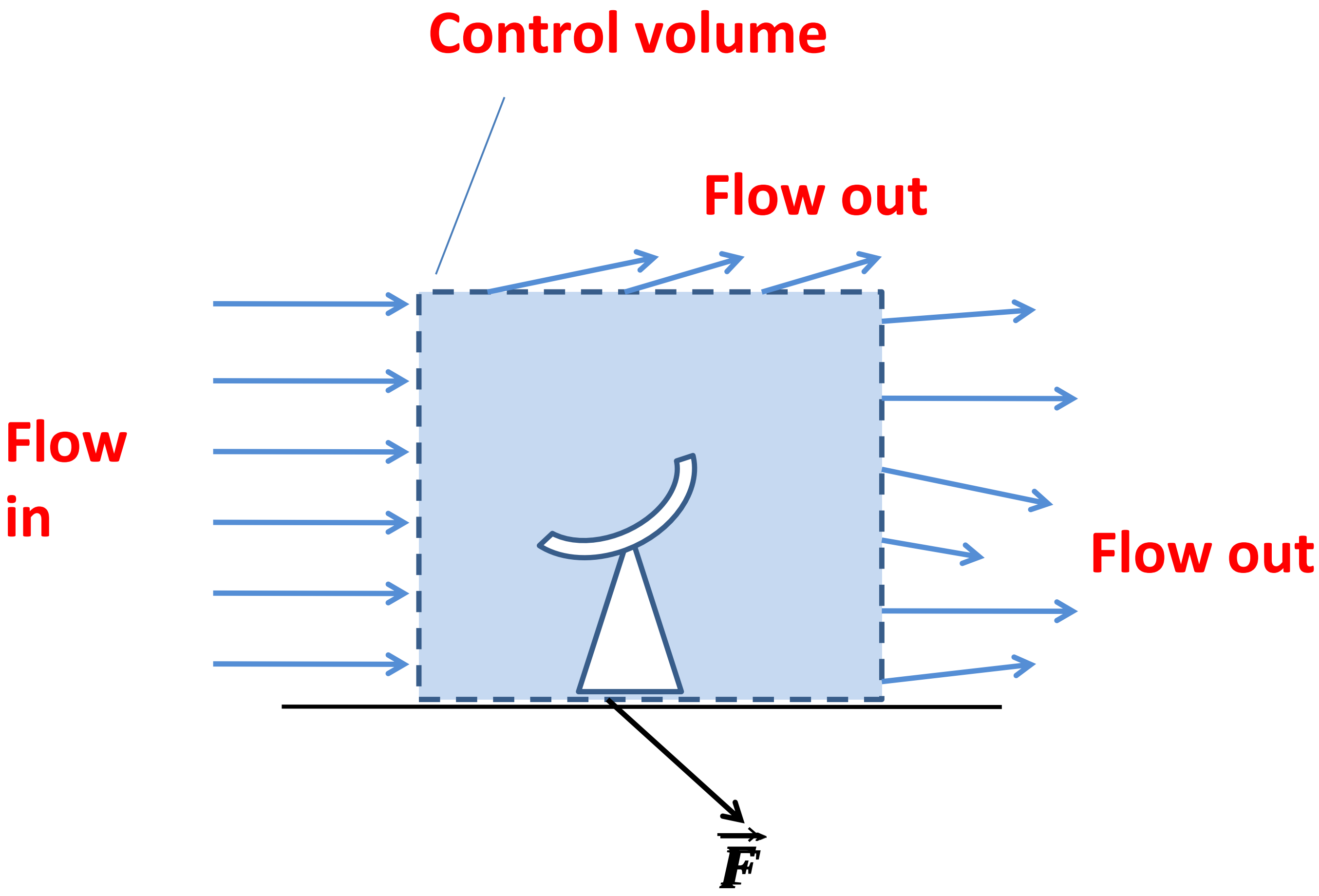
Problems could be solved without a detailed knowledge of the flow field

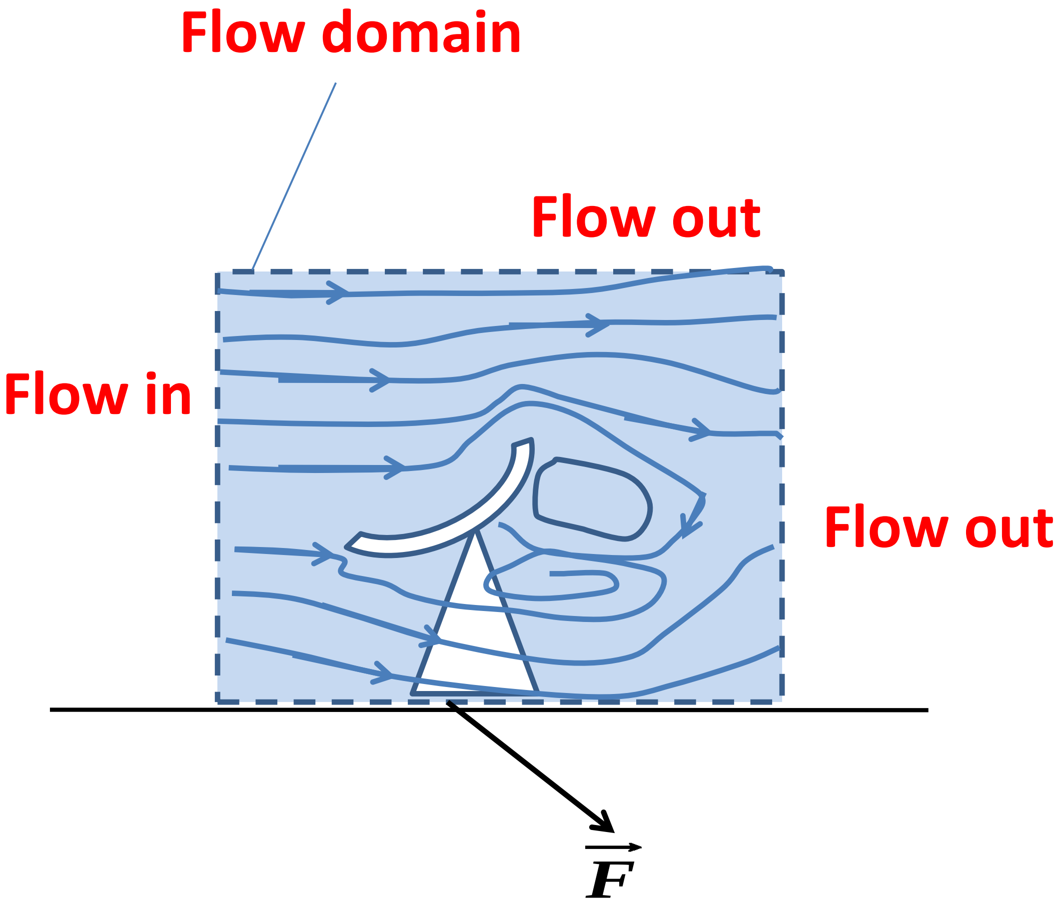
Unfortunately, there are many situations that arise in which details of the flow are important and the finite control volume approach will not yield the desired information

How the velocity varies over the cross section of a pipe, how the pressure and shear stress vary along the surface of an airplane wing

In these circumstances we need to develop relationships that apply at a point, or at least in a very small region infinitesimal volume within a given flow field. This approach - DIFFERENTIAL ANALYSIS

DIFFERENTIAL ANALYSIS PROVIDES VERY DETAILED KNOWLEDGE OF A FLOW FIELD





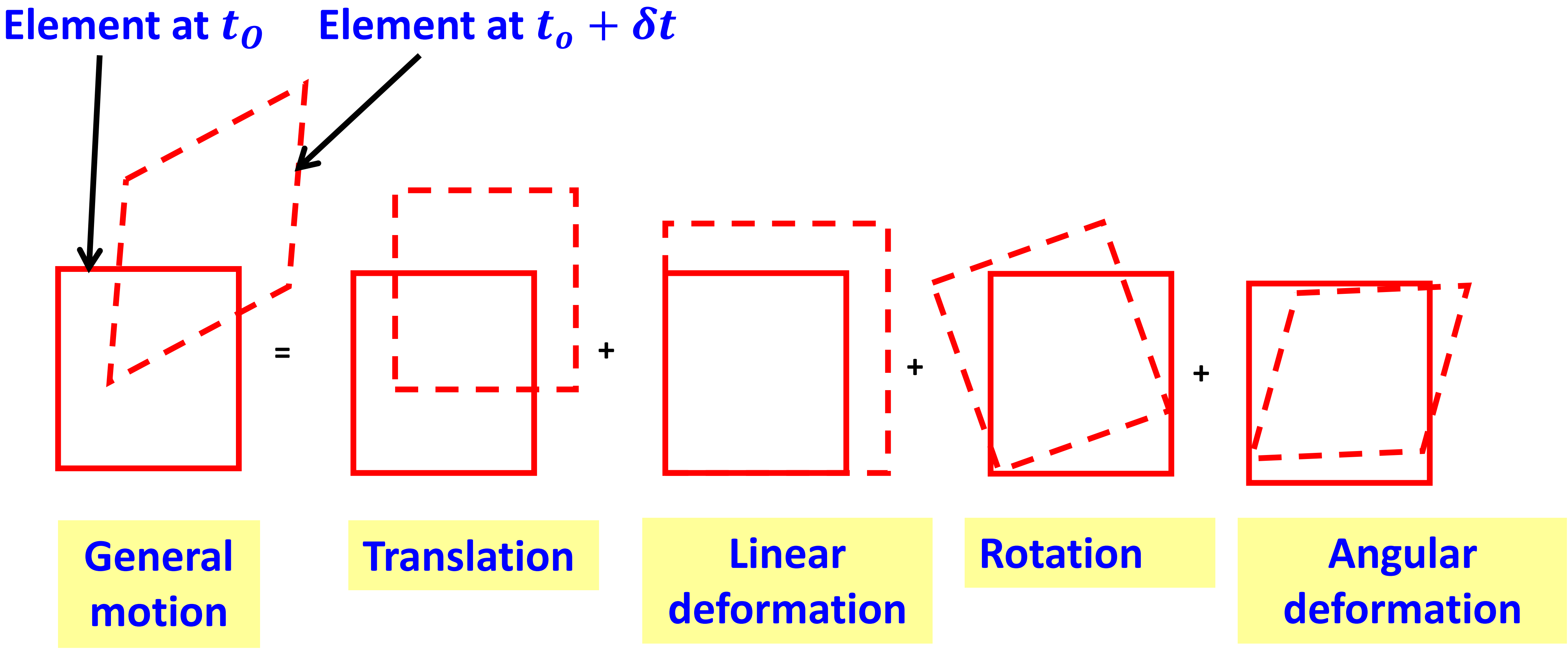
Control volume analysis

Interior of the CV is BLACK BOX

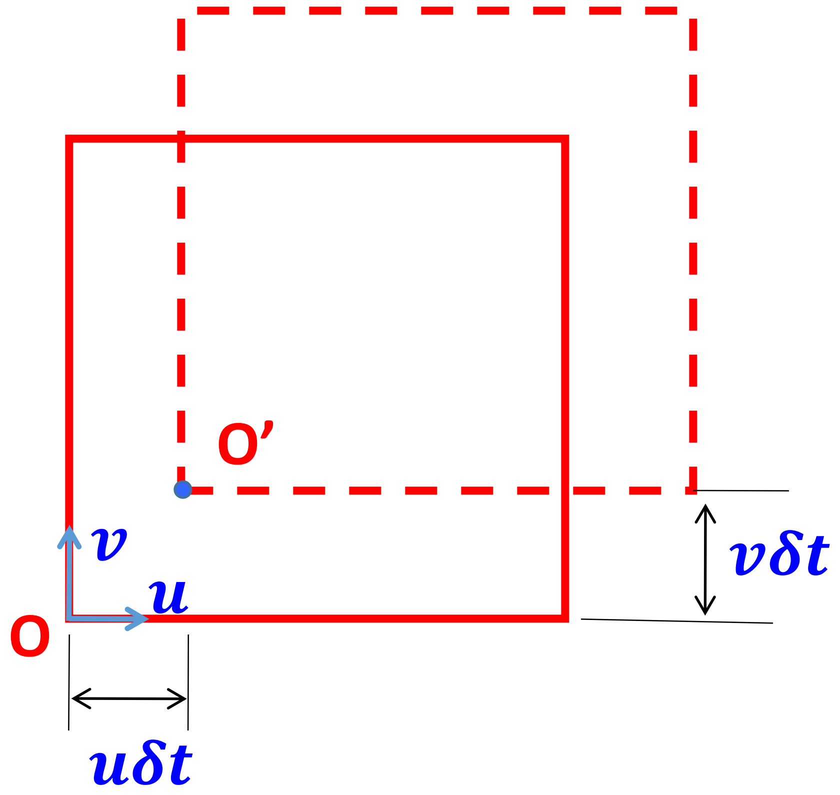
Differential analysis

All the details of the flow are solved at every point within the flow domain

LINEAR MOTION AND DEFORMATION

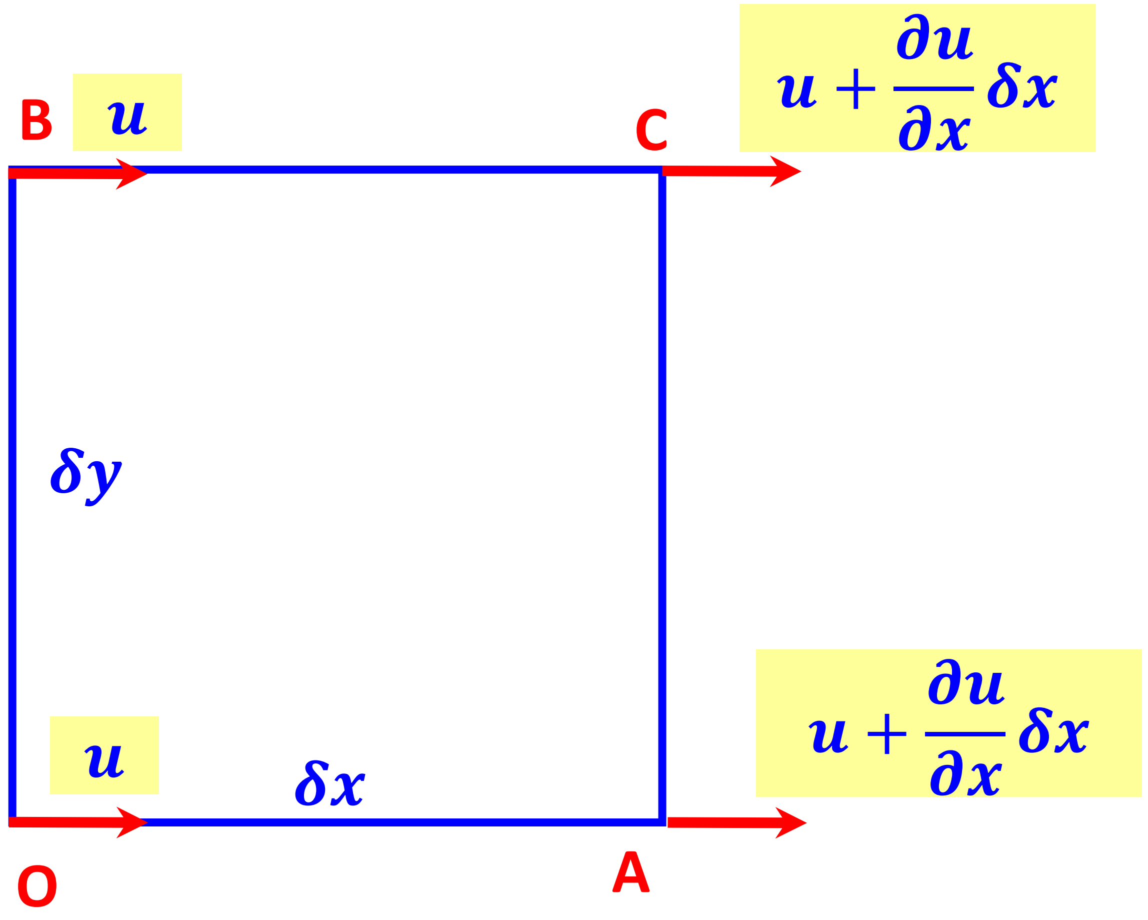


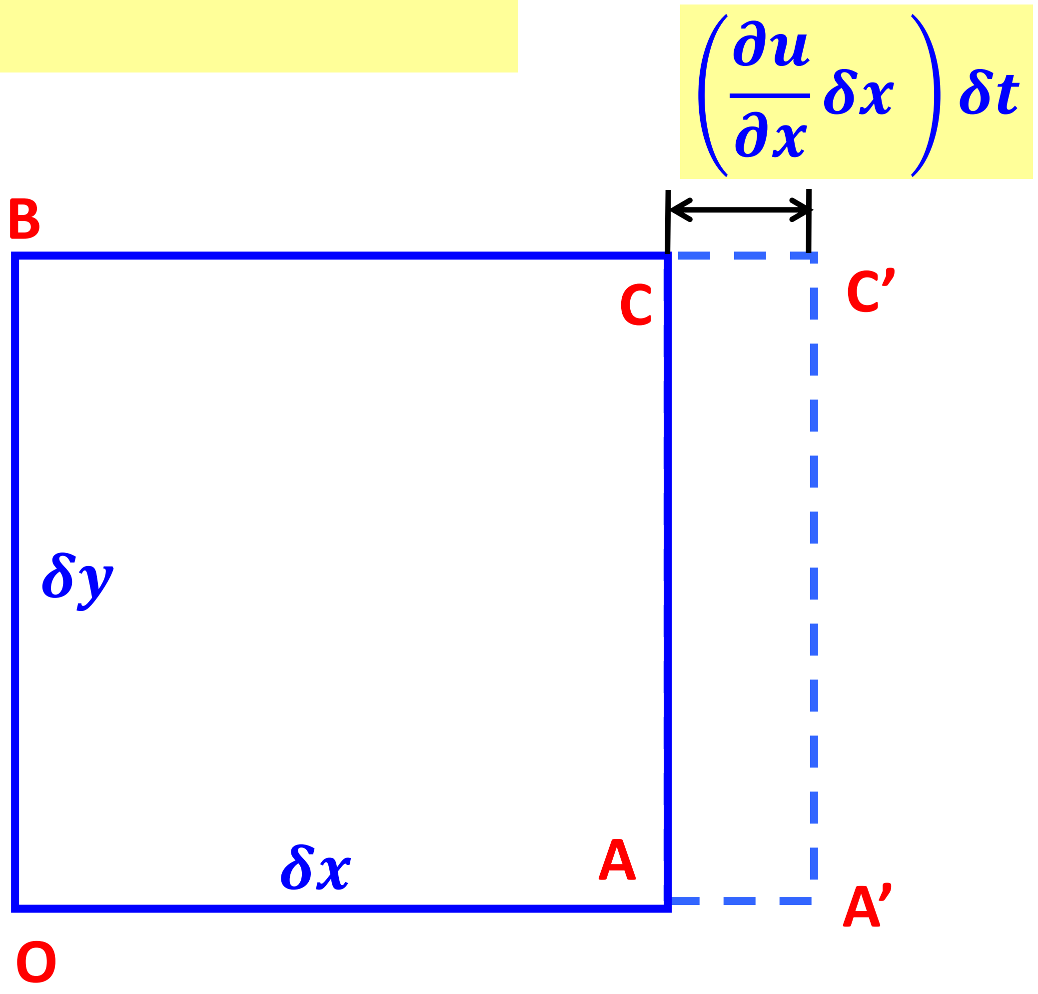
TRANSLATION



If all points in the element have the same velocity which is only true if there are no velocity gradients, then the element will simply TRANSLATE from one position to another.

LINEAR DEFORMATION





Because of the presence of velocity gradients, the element will generally be deformed and rotated as it moves. For example, consider the effect of a single velocity gradient $\frac{\partial u}{\partial x}$ on a small cube having sides δx and δy

x component of velocity of O and B = u

x component of velocity of A and C = $u + \frac{\partial u}{\partial x}$

This difference in the velocity causes a **STRETCHING** of the volume element by a volume

$$\left(\frac{\partial u}{\partial x}\delta x\right)(\delta y \delta z)(\delta t)$$

Rate at which the volume δV is changing per unit volume due the gradient

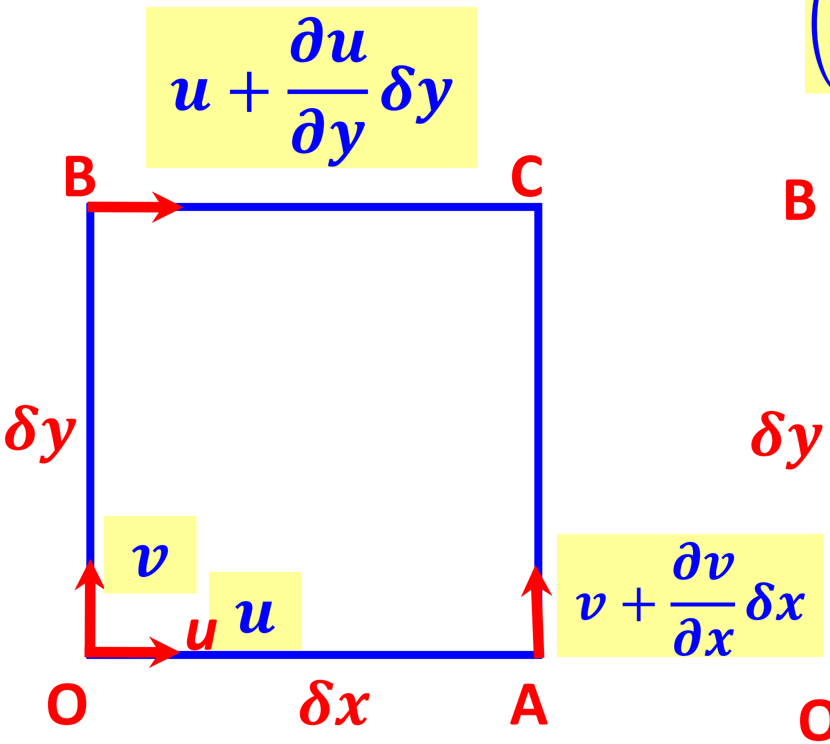
$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \lim_{\delta t \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\delta x\right)}{\delta t}\right] = \frac{\partial u}{\partial x}$$

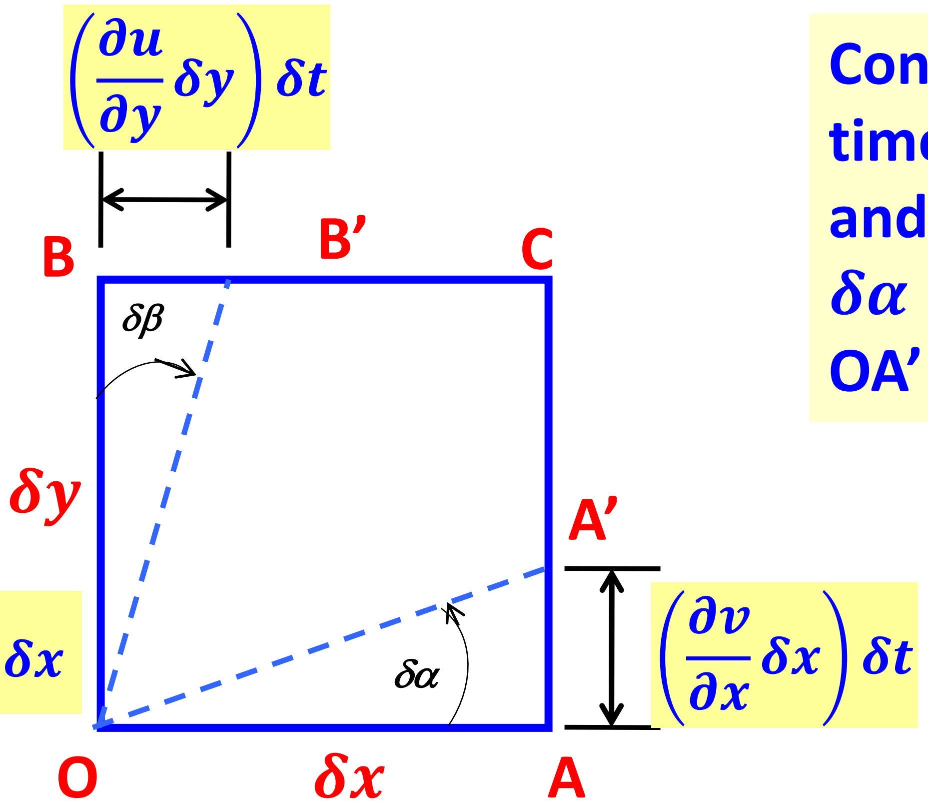
If the velocity gradients $\left(\frac{\partial v}{\partial y}\right)$ and $\left(\frac{\partial w}{\partial z}\right)$ are also present

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

This rate of change of volume per unit volume is called the **VOLUMETRIC DILATION RATE.** Volume of the fluid may change as the element moves from one location to another in the flow field. Incompressible fluid – volumetric dilation rate is zero. Change in volume element = zero; fluid density = constant (The element mass is conserved)

ANGULAR MOTION AND DEFORMATION





Consider xy plane. In a short time interval δt line segment OA and OB will rotate through angles $\delta\alpha$ and $\delta\beta$ to the new positions OA’ and OB’

$$Tan\delta\alpha \approx \delta\alpha = \frac{\left(\frac{\partial v}{\partial x}\delta x\right)\delta t}{\delta x}$$

$$\delta\alpha = \left(\frac{\partial v}{\partial x}\right)\delta t$$

ω_{oA} - Angular velocity of OA

ω_{oB} - Angular velocity of OB

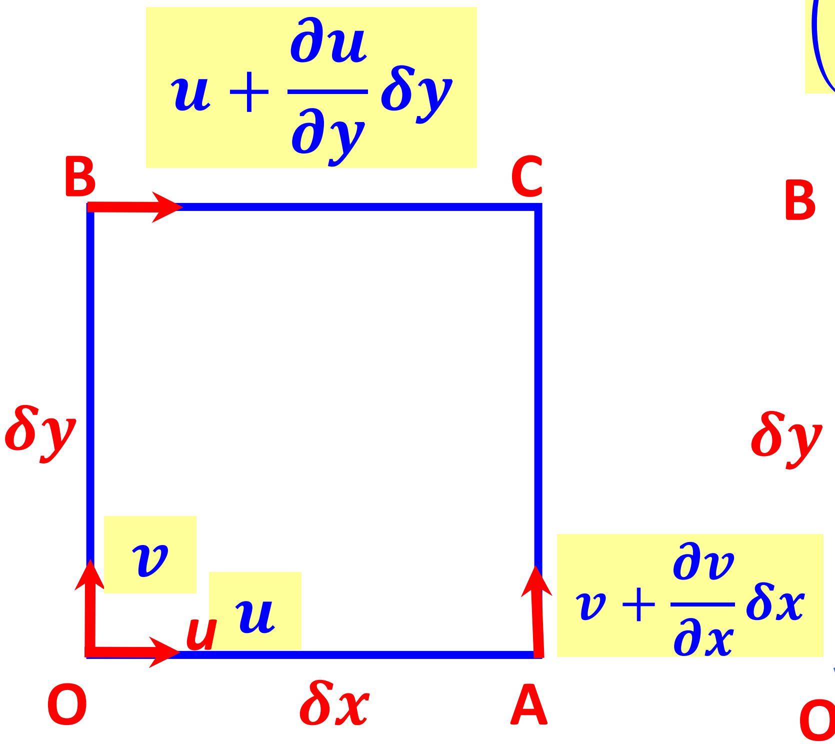
$$\omega_{oA} = \lim_{\delta t \to 0}\frac{\delta\alpha}{\delta t} = \lim_{\delta t \to 0}\frac{\left(\frac{\partial v}{\partial x}\right)\delta t}{\delta t} = \frac{\partial v}{\partial x}$$

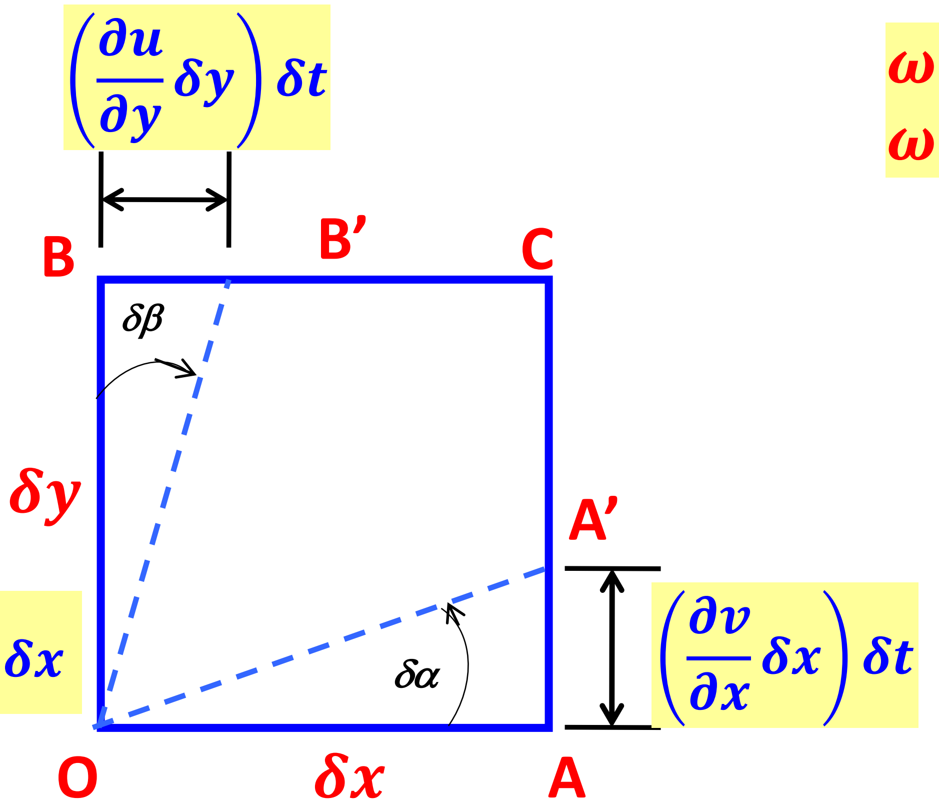
$$Tan\delta\beta \approx \delta\beta = \frac{\left(\frac{\partial u}{\partial y}\delta y\right)\delta t}{\delta y}$$

$$\omega_{oB} = \lim_{\delta t \to 0}\frac{\delta\alpha}{\delta t} = \lim_{\delta t \to 0}\frac{\left(\frac{\partial u}{\partial y}\right)\delta t}{\delta t} = \frac{\partial u}{\partial y}$$

$$\delta\beta = \left(\frac{\partial u}{\partial y}\right)\delta t$$

ANGULAR MOTION AND DEFORMATION





ω_{oA} - Angular velocity of OA

ω_{oB} - Angular velocity of OB

$$\omega_{oA}=\frac{\partial v}{\partial x}$$

$$\omega_{oB}=-\frac{\partial u}{\partial y}$$

Rotation ω_z of the element about the z-axis is defined as the average of the angular velocities ω_{oA} and ω_{oB} of the two mutually perpendicular lines OA and OB. Thus, if counterclockwise rotation is considered positive, it follows that

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

ANGULAR MOTION AND DEFORMATION

Rotation ω_z of the element about the z-axis

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Rotation ω_x of the element about the x-axis

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Rotation ω_x of the element about the y-axis

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}$$

$$\omega = \frac{1}{2}\,Curl\,V = \frac{1}{2}(\nabla \times V)$$

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \hat{\imath}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Vorticity Ω is defined as the vector that is twice the rotation vector

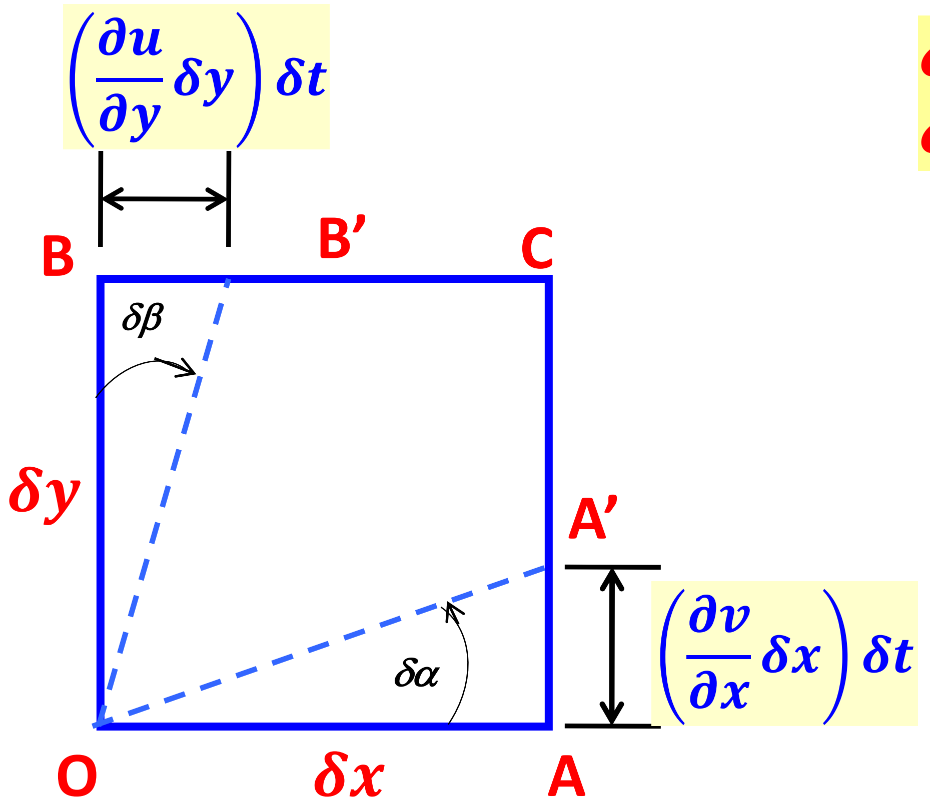
$$\Omega = 2\omega = \nabla \times V$$

Rotation and vorticity are zero;

FLOW FIELD IS IRROTATIONAL

$$\nabla \times V = 0$$

ANGULAR MOTION AND DEFORMATION



ω_{oA} - Angular velocity of OA
ω_{oB} - Angular velocity of OB

$$\omega_{oA} = \frac{\partial v}{\partial x} \qquad \omega_{oB} = -\frac{\partial u}{\partial y}$$

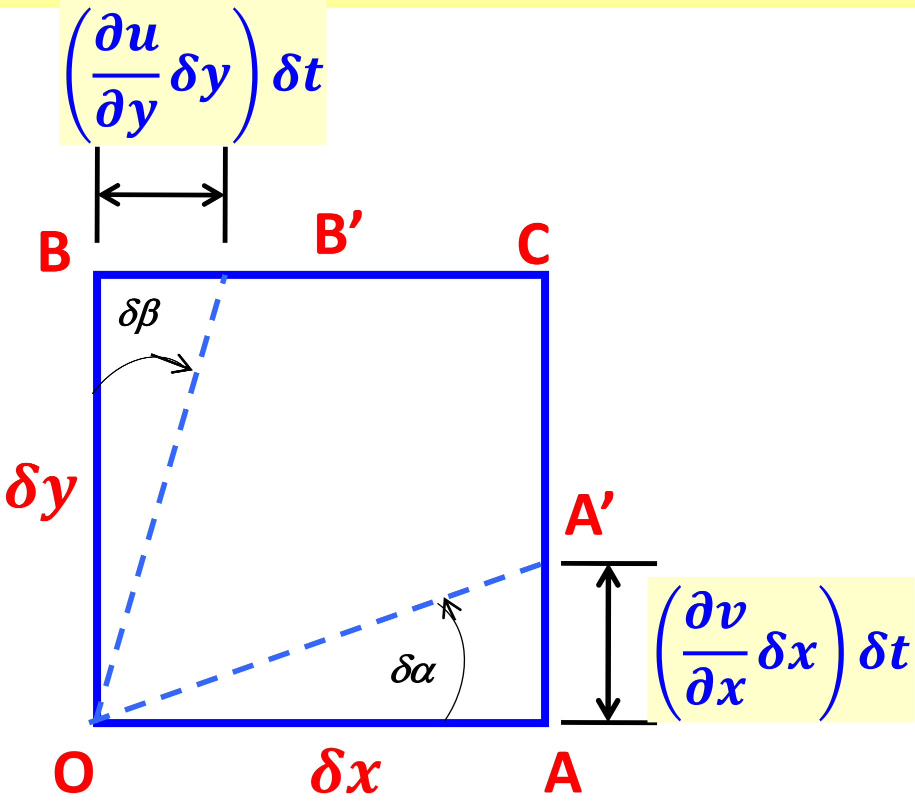
if counterclockwise rotation is considered positive

Fluid element will rotate about the z axis as an undeformed block $(\omega_{oA} = -\omega_{oB})$ only when
Otherwise, the rotation will be associated with an angular deformation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \Rightarrow$ Rotation around the z axis is zero

ANGULAR MOTION AND DEFORMATION



In addition to rotation associated with derivatives $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, these derivatives can cause the fluid element to undergo an angular deformation which results in change of shape

if counterclockwise rotation is considered positive

Change in the original right angle formed by the lines OA and OB is **SHEARING STRAIN** γ

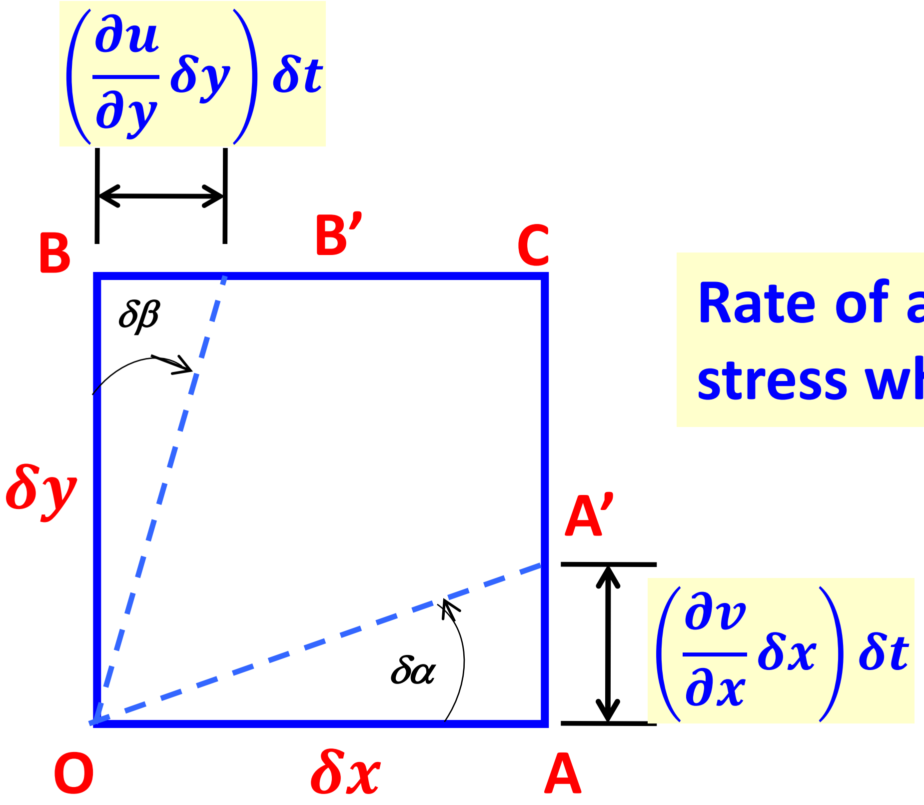
$$\delta\gamma = \delta\alpha + \delta\beta$$

$\delta\gamma$ is positive if the original right angle is decreasing

Rate of Shearing Strain or Rate of Angular Deformation

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0} \frac{\left(\frac{\partial v}{\partial x}\right)\delta t + \frac{\partial u}{\partial y}\delta t}{\delta t}$$

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

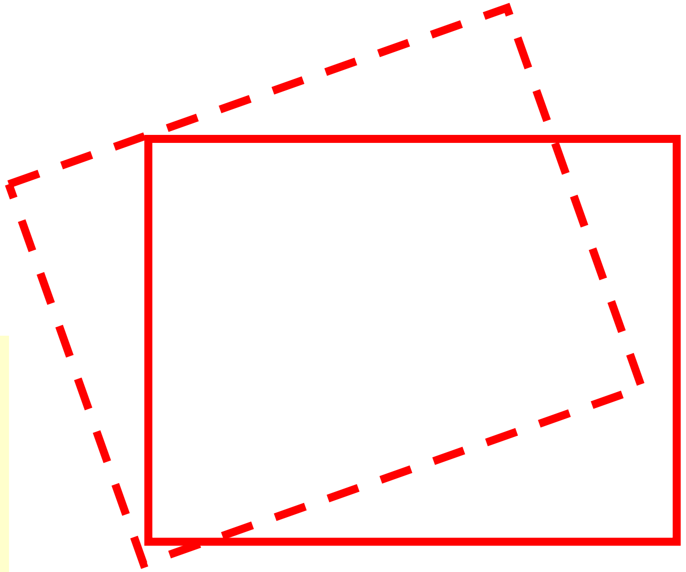


$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Rate of angular deformation is related to a corresponding shearing stress which causes the fluid element to change in shape

if counterclockwise rotation is considered positive

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



Rotation

Rate of angular deformation is zero;

Element is simply rotating as an undeformed block

Variations in the velocity in the direction of velocity cause LINEAR DEFORMATION

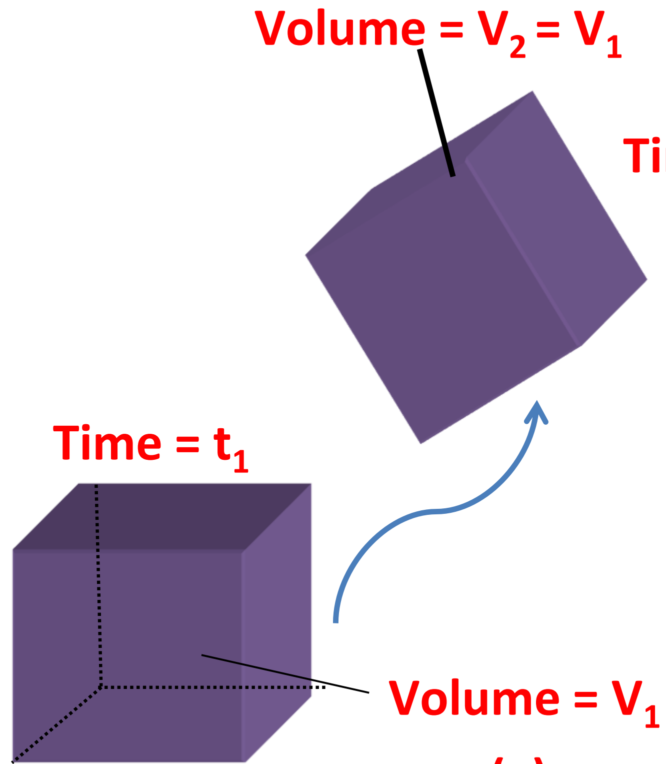
$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$$

Linear deformation of the element does not change the shape of the element

Cross derivates cause the element to ROTATE and undergo ANGULAR DEFORMATION

$$\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$$

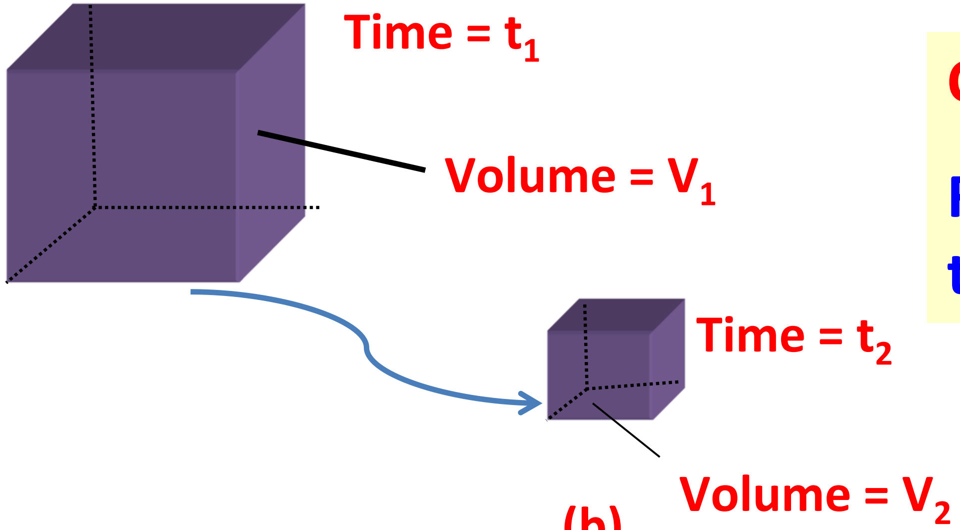
Angular deformation of the element changes the shape of the element



(a)

Incompressible flow field

Fluid elements may translate, distort, and rotate but do not grow or shrink in volume



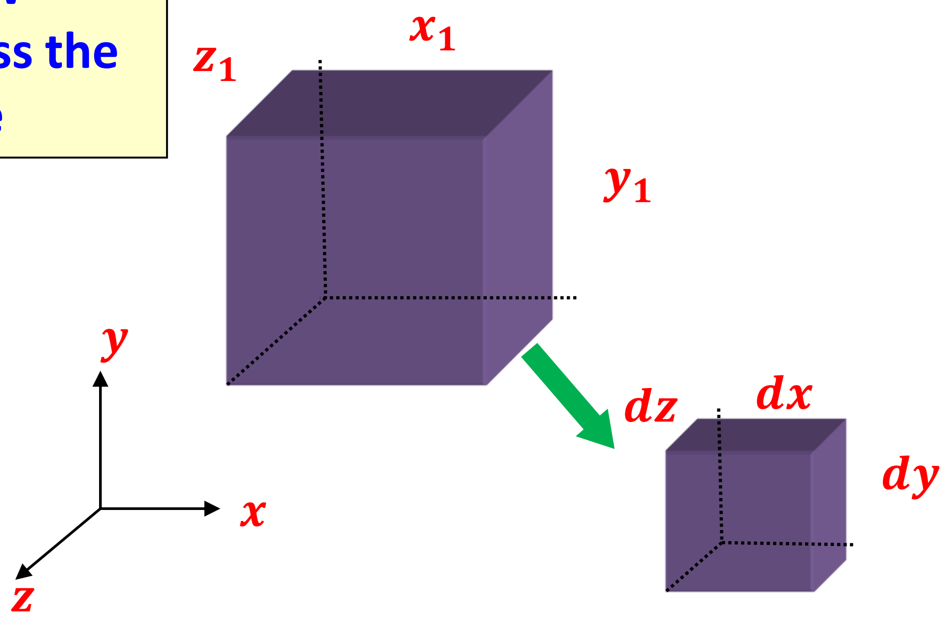
(b)

Compressible flow field

Fluid elements may grow or shrink in volume as they translate, distort or rotate

CONSERVATION OF MASS OR CONTINUITY EQUATION

Time rate of change of the mass of the coincident system = Time rate of change of the mass of the contents of the coincident control volume + Net rate of flow through of mass the control surface



By Reynolds Transport Theorem

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\forall + \int_{cs} \rho b V.\hat{n} dA$$

Conservation of mass $b = 1$

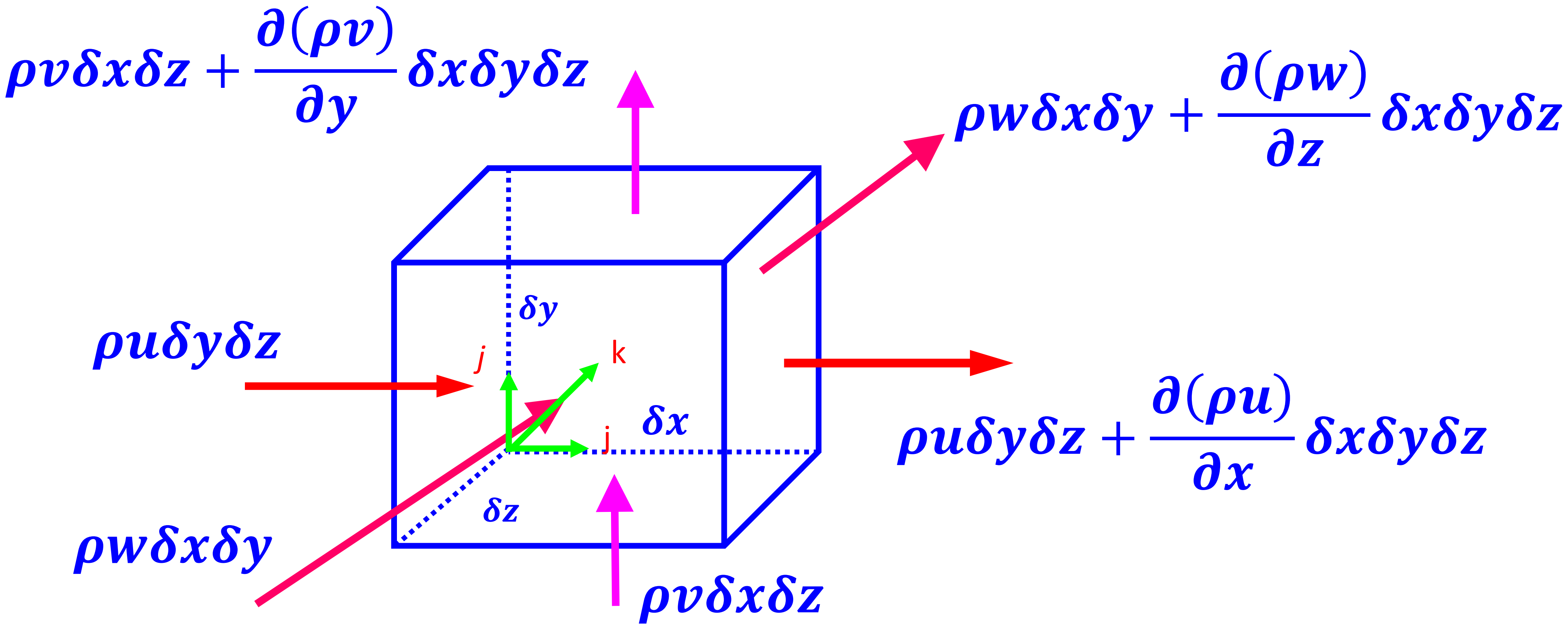
$$0 = \frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho V.\hat{n} dA$$

$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall = \frac{\partial \rho}{\partial t}\delta x\delta y\delta z$$

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z + \int_{cs} \rho V.\hat{n} dA = 0$$

CONSERVATION OF MASS OR CONTINUITY EQUATION

$$\int_{cs} \rho V.\hat{n} dA = -\rho u\delta y\delta z - \rho v\delta x\delta z - \rho w\delta x\delta y + \rho u\delta y\delta z + \frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z + \rho v\delta x\delta z$$
$$+\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z + \rho w\delta x\delta y + \frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$



CONSERVATION OF MASS OR CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z + \int_{cs} \rho V.\hat{n}dA = 0$$

$$\int_{cs} \rho V.\hat{n}dA = -\rho u\delta y\delta z - \rho v\delta x\delta z - \rho w\delta x\delta y + \rho u\delta y\delta z + \frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z + \rho v\delta x\delta z + \frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z + \rho w\delta x\delta y + \frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

$$\int_{cs} \rho V.\hat{n}dA = \frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z + \frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

CONSERVATION OF MASS OR CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla.\vec{V}$$

$$\frac{D\rho}{Dt} + \rho(\nabla.\vec{V}) = 0$$

CONSERVATION OF MOMENTUM

By Reynolds Transport Theorem

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\forall + \int_{cs} \rho b V.\hat{n} dA$$

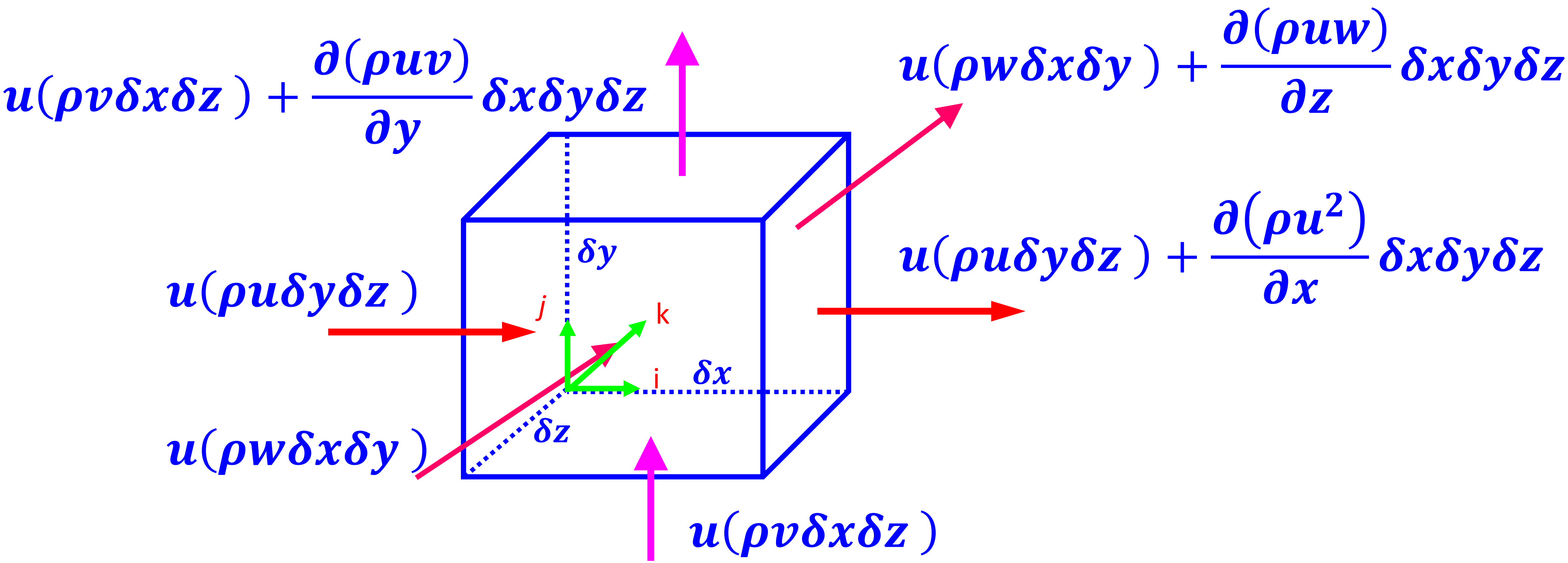
Conservation of momentum $b = V$

$$\frac{\partial}{\partial t}\int_{cv} \rho V d\forall + \int_{cs} V \rho V.\hat{n} dA = \sum F_{\substack{contents\ of \\ Control\ Volume}}$$

RATE OF INCREASE OF x − MOMENTUM **-** RATE AT WHICH x − MOMENTUM ENTERS **+** RATE AT WHICH x − MOMENTUM LEAVES **=** SUM OF THE x COMP FORCES APPLIED TO FLUID IN CV

CONSERVATION OF MOMENTUM

$$\int_{cs} V\rho V.\hat{n}dA = -u(\rho u\delta y\delta z) + u(\rho u\delta y\delta z) + \frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z - u(\rho v\delta x\delta z) + u(\rho v\delta x\delta z) + \frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z - u(\rho w\delta x\delta y) + u(\rho w\delta x\delta y) + \frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z$$



CONSERVATION OF MOMENTUM

$$\frac{\partial}{\partial t}\int_{cv}\rho V d\forall + \int_{cs} V\rho V.\hat{n}dA = \sum F_{\text{contents of Control Volume}}$$

$$\frac{\partial}{\partial t}\int_{cv}\rho V d\forall = \frac{\partial(\rho u)}{\partial t}\delta x\delta y\delta z$$

$$\int_{cs} V\rho V.\hat{n}dA = -u(\rho u\delta y\delta z) + u(\rho u\delta y\delta z) + \frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z - u(\rho v\delta x\delta z) + u(\rho v\delta x\delta z) + \frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z - u(\rho w\delta x\delta y) + u(\rho w\delta x\delta y) + \frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z$$

$$\int_{cs} V\rho V.\hat{n}dA = \frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z$$

$$\frac{\partial(\rho u)}{\partial t}\delta x\delta y\delta z + \frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z = \sum F_{\text{contents of Control Volume}}$$

CONSERVATION OF MOMENTUM

$$\frac{\partial(\rho u)}{\partial t}\delta x\delta y\delta z+\frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z+\frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z+\frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z=\sum F_{\substack{contents\ of\\ Control\ Volume}}$$

SURFACE FORCES

- NORMAL STRESSES
- SHEAR STRESSES
- PRESSURE

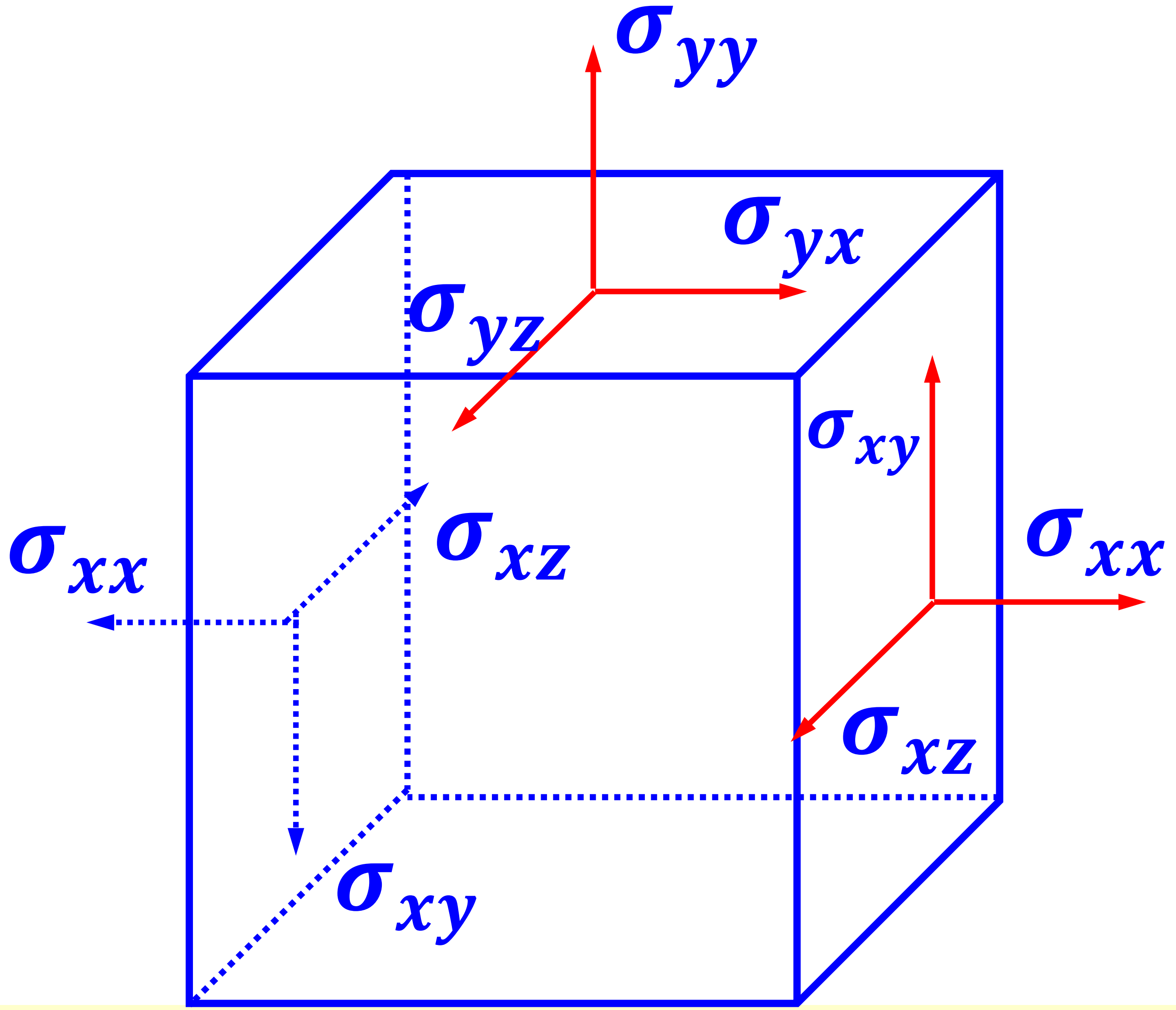
BODY FORCES

- GRAVITY FORCES
- CORIOLIS FORCES
- CENTRIFUGAL FORCES

CONSERVATION OF MOMENTUM

SURFACE FORCES

- **NORMAL STRESSES**
- **SHEAR STRESSES**
- **PRESSURE**



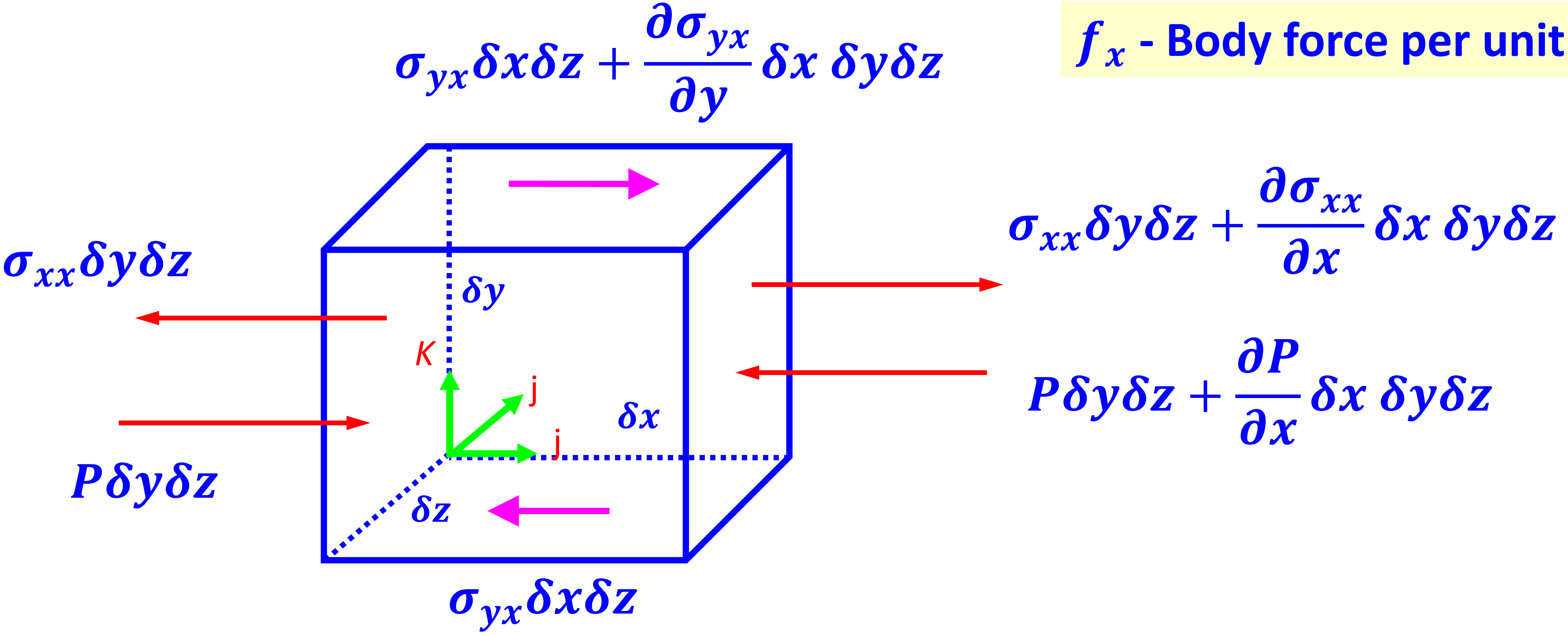
First subscript denotes the direction of the normal to the plane on which the stress acts

Second subscript denotes the direction of the stress

CONSERVATION OF MOMENTUM

$$\sum F_x = -\frac{\partial P}{\partial x}\delta x\,\delta y\delta z + \frac{\partial \sigma_{xx}}{\partial x}\delta x\,\delta y\delta z + \frac{\partial \sigma_{yx}}{\partial y}\delta x\,\delta y\delta z + \frac{\partial \sigma_{zx}}{\partial z}\delta x\,\delta y\delta z + f_x\delta x\delta y\delta z$$

f_x - Body force per unit volume



$$\frac{\partial(\rho u)}{\partial t}\delta x\delta y\delta z + \frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z = \sum F_{\text{contents of Control Volume}}$$

CONSERVATION OF MOMENTUM

$$\sum F_x = -\frac{\partial P}{\partial x}\delta x\,\delta y\delta z + \frac{\partial \sigma_{xx}}{\partial x}\delta x\,\delta y\delta z + \frac{\partial \sigma_{yx}}{\partial y}\delta x\,\delta y\delta z + \frac{\partial \sigma_{zx}}{\partial z}\delta x\,\delta y\delta z + f_x\delta x\delta y\delta z$$

$$\frac{\partial(\rho u)}{\partial t}\delta x\delta y\delta z + \frac{\partial(\rho u^2)}{\partial x}\delta x\delta y\delta z + \frac{\partial(\rho uv)}{\partial y}\delta x\delta y\delta z + \frac{\partial(\rho uw)}{\partial z}\delta x\delta y\delta z = \sum F_{\text{contents of Control Volume}}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial(\rho u)}{\partial x} + \rho v\frac{\partial u}{\partial y} + u\frac{\partial(\rho v)}{\partial y} + \rho w\frac{\partial u}{\partial z} + u\frac{\partial(\rho w)}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

$$u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

$$u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

CONSERVATION OF MOMENTUM

$$u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Continuity equation

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Total Acceleration

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

CAUCHY'S EQN

CONSERVATION OF MOMENTUM

From Stress Strain Relationship

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla.\vec{V}) \qquad \sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \sigma_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x$$

Assuming that viscosity μ is constant

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla.\vec{V})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + f_x$$

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \frac{2}{3}\mu(\nabla.\vec{V})\right] + f_x$$

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\mu(\nabla.\vec{V}) - \frac{2}{3}\mu(\nabla.\vec{V})\right] + f_x$$

CONSERVATION OF MOMENTUM

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\mu(\nabla.\vec{V}) - \frac{2}{3}\mu(\nabla.\vec{V})\right] + f_x$$

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}(\nabla.\vec{V})\right] + f_x$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{\partial}{\partial y}\left[\frac{\mu}{3}(\nabla.\vec{V})\right] + f_y$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{\partial}{\partial z}\left[\frac{\mu}{3}(\nabla.\vec{V})\right] + f_z$$

VISCOUS COMPRESSIBLE FLUID WITH CONSTANT VISCOSITY

$$\rho\frac{D\vec{V}}{Dt} = -\nabla P + \mu\nabla^2\vec{V} + \frac{\mu}{3}\nabla(\nabla.\vec{V}) + f$$

CONSERVATION OF MOMENTUM

VISCOUS COMPRESSIBLE FLUID WITH CONSTANT VISCOSITY

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \mu \nabla^2 \vec{V} + \frac{\mu}{3} \nabla\left(\nabla . \vec{V}\right) + f$$

VISCOUS INCOMPRESSIBLE FLUID WITH CONSTANT VISCOSITY

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \mu \nabla^2 \vec{V} + f$$

INVISCID INCOMPRESSIBLE FLUID WITH CONSTANT VISCOSITY

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + f$$

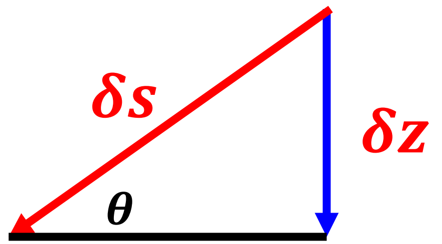
EULER'S EQN

REDUCTION OF EULER'S EQUATION INTO THE BERNOULLIS EQUATION

INVISCID INCOMPRESSIBLE FLUID WITH CONSTANT VISCOSITY

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + f$$

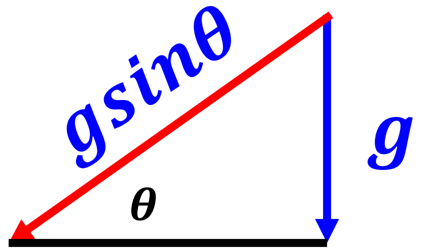
EULER'S EQN



$$sin\theta = \frac{\delta z}{\delta s}$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial P}{\partial x} + f$$

Along a streamline (steady, incompressible and inviscid fluid)



$$\rho \left[u \frac{\partial u}{\partial s} \right] = -\frac{\partial P}{\partial s} - \rho g sin\theta$$

$$\rho u \frac{\partial u}{\partial s} = -\frac{\partial P}{\partial s} - \rho g \frac{\partial z}{\partial s}$$

$$\rho u \frac{\partial u}{\partial s} ds = -\frac{\partial P}{\partial s} ds - \rho g \frac{\partial z}{\partial s} ds$$

$$\rho u \, du = -dP - \rho g dz$$

$$\rho \, u \, du = -dP - \rho g dz$$

$$\frac{\rho u^2}{2} = -P - \rho g z + C$$

$$\frac{P}{\rho} + \frac{u^2}{2} + gz = C$$

Continuity equation

$$\frac{D\rho}{Dt} + \rho(\nabla.\vec{V}) = 0$$

x- momentum

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}(\nabla.\vec{V})\right] + f_x$$

y- momentum

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{\partial}{\partial y}\left[\frac{\mu}{3}(\nabla.\vec{V})\right] + f_y$$

z- momentum

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{\partial}{\partial z}\left[\frac{\mu}{3}(\nabla.\vec{V})\right] + f_z$$

Navier – French mathematician Stokes – English Mechanician

FOUR EQUATION AND FOUR UNKNOWNS – u, v, w and P

Mathematically well posed

Nonlinear, second order partial differential equations

CONSERVATION OF ENERGY AND SIMILARITY ANALYSIS

OBJECTIVES

- Derive the conservation of energy in Cartesian coordinates
- Similarity analysis of Energy equation in order to arrive at dimensionless numbers

CONSERVATION OF ENERGY

First law of thermodynamics

$$dE = dQ + dW$$

$$\frac{DE}{Dt} = \frac{DQ}{Dt} + \frac{DW}{Dt}$$

dE – increment in the (kinetic plus thermal energy) of the system

dQ – heat transfer to the system

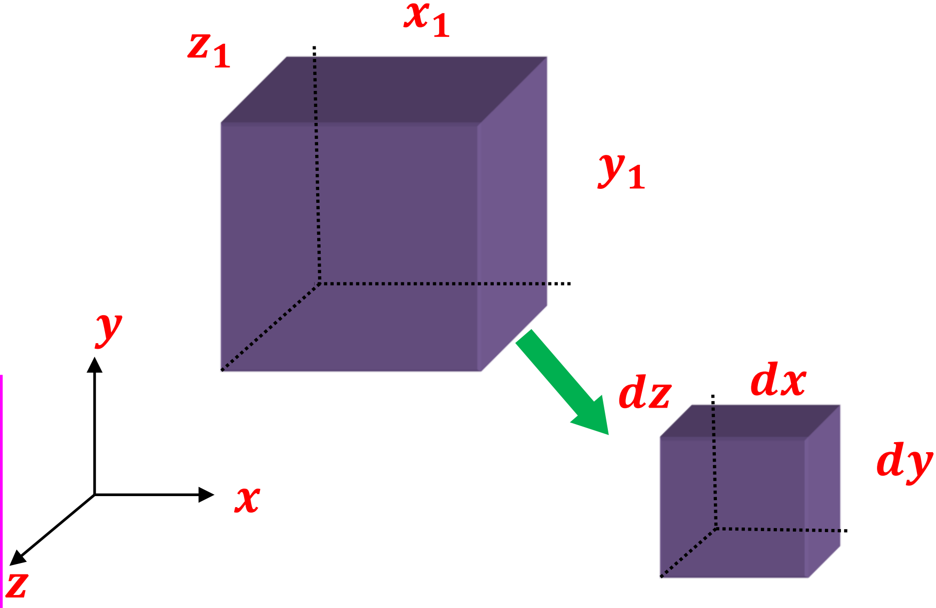
dW – work done on the system

Internal energy per unit mass of the fluid consists of the sum of the kinetic energy $\left(\frac{u^2+v^2}{2}\right)$ and thermal internal energy $e = C_v T$

Writing the above equation in the substantial derivative form, so that it applies to transport of E by a moving system

Rate of increase of E in CV − Rate at which E enters through surface of CV + Rate at which E leaves through surface of CV

= Rate of heat transfer into CV by conduction + Rate of surface and body forces do work on CV



Conservation of energy

Reynolds Transport Theorem Suggests that

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho\, b d\forall + \int_{cs} \rho\, b\, \vec{V}.\hat{n}\, dA$$

$$\frac{DE}{Dt} = \frac{DQ}{Dt} + \frac{DW}{Dt}$$

For energy equation, $b = E = e + \frac{u^2+v^2}{2}$ and $B = mb$

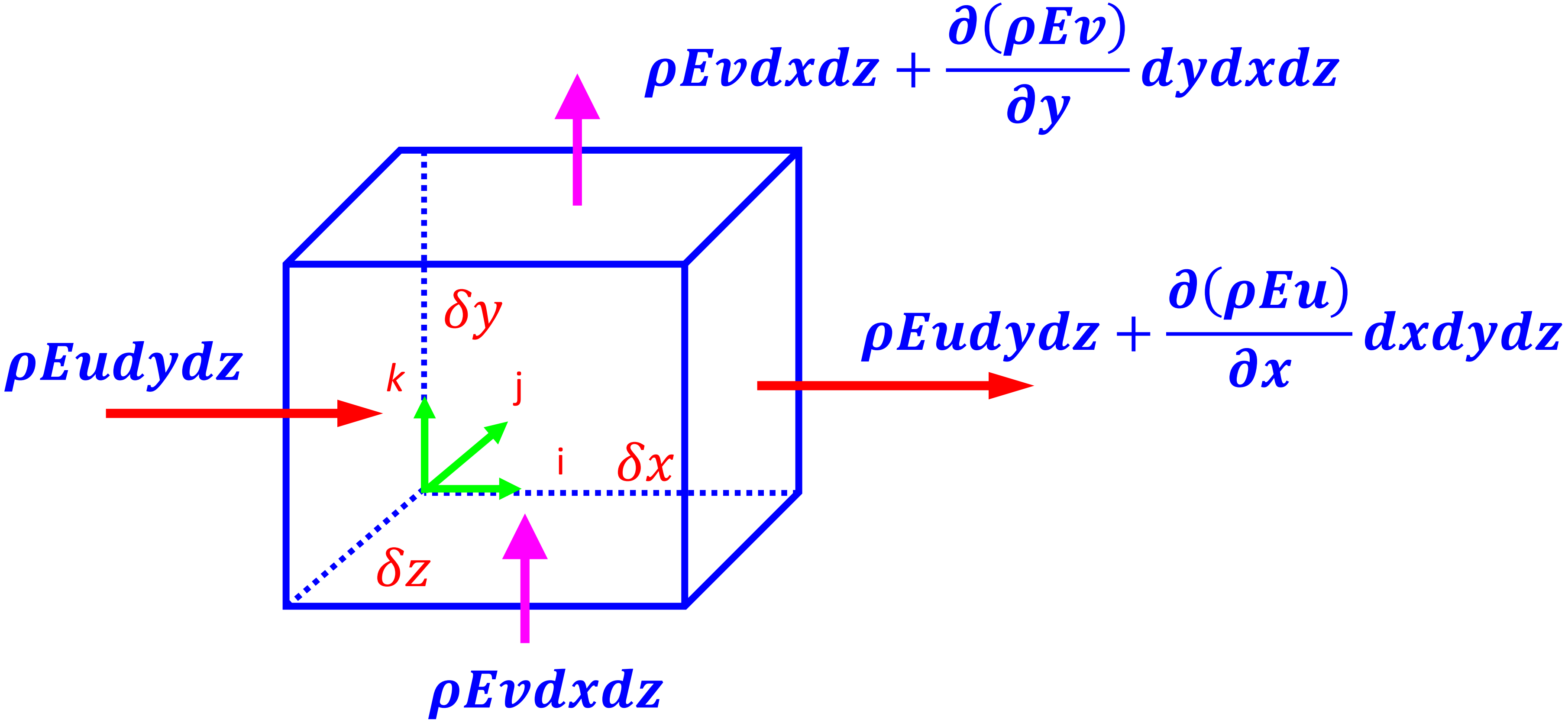
$$\frac{\partial}{\partial t}\int_{cv} \rho\, E dxdydz + \int_{cs} \rho\, E\, \vec{V}.\hat{n}\, dA =$$ Rate of heat transfer into CV by conduction + Rate of surface and body forces do work on CV

Rate of increase of E in CV − Rate at which E enters through surface of CV + Rate at which E leaves through surface of CV

= Rate of heat transfer into CV by conduction + Rate of surface and body forces do work on CV

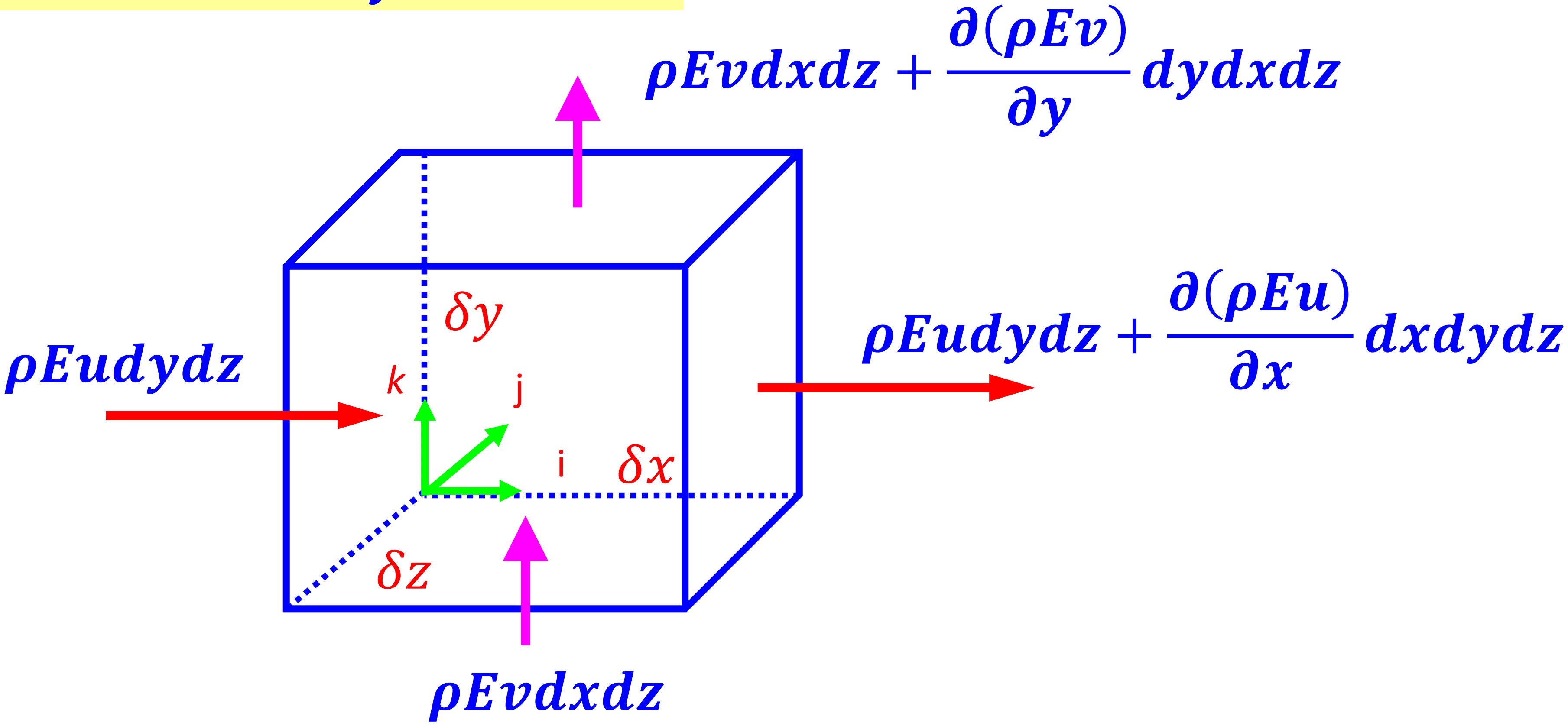
$$LHS = \frac{\partial}{\partial t}\int_{cv} \rho\, E dxdydz + \int_{cs} \rho\, E\, \vec{V}.\hat{n}\, dA$$

$$\frac{\partial}{\partial t}\int_{cv} \rho\, E dxdydz = \frac{\partial(\rho E)}{\partial t} dxdydz$$



$$\int_{cs} \rho\, E\, \vec{V}.\hat{n}\, dA = -\rho Eudydz + \rho Eudydz + \frac{\partial(\rho Eu)}{\partial x} dxdydz - \rho Evdxdz + \rho Evdxdz + \frac{\partial(\rho Ev)}{\partial y} dydxdz$$

$$\int_{cs} \rho\, E\, \vec{V}.\hat{n}\, dA = \frac{\partial(\rho Eu)}{\partial x} dxdydz + \frac{\partial(\rho Ev)}{\partial y} dydxdz$$



$$LHS = \frac{\partial}{\partial t}\int_{cv} \rho\, E dxdydz + \int_{cs} \rho\, E\, \vec{V}.\hat{n}\, dA$$

$$\frac{\partial}{\partial t}\int_{cv} \rho\, E dxdydz = \frac{\partial(\rho E)}{\partial t} dxdydz$$

$$\int_{cs} \rho\, E\, \vec{V}.\hat{n}\, dA = \frac{\partial(\rho E u)}{\partial x} dxdydz + \frac{\partial(\rho E v)}{\partial y} dydxdz$$

$$\frac{LHS}{dxdydz} = \frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho E u)}{\partial x} + \frac{\partial(\rho E v)}{\partial y}$$

$$\frac{LHS}{dxdydz} = \rho \frac{\partial E}{\partial t} + \rho u \frac{\partial E}{\partial x} + \rho v \frac{\partial E}{\partial y} + E \frac{\partial \rho}{\partial t} + E \frac{\partial(\rho u)}{\partial x} + E \frac{\partial(\rho v)}{\partial y}$$

$$\frac{LHS}{dxdydz} = \rho \left[\frac{\partial E}{\partial t} + u \frac{\partial E}{\partial x} + v \frac{\partial E}{\partial y}\right] + E \cancel{\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right]}$$

By conservation of mass, this term is zero

$$\frac{LHS}{dxdydz} = \rho \frac{DE}{Dt} = \rho \frac{D}{Dt}\left[e + \frac{u^2 + v^2}{2}\right]$$

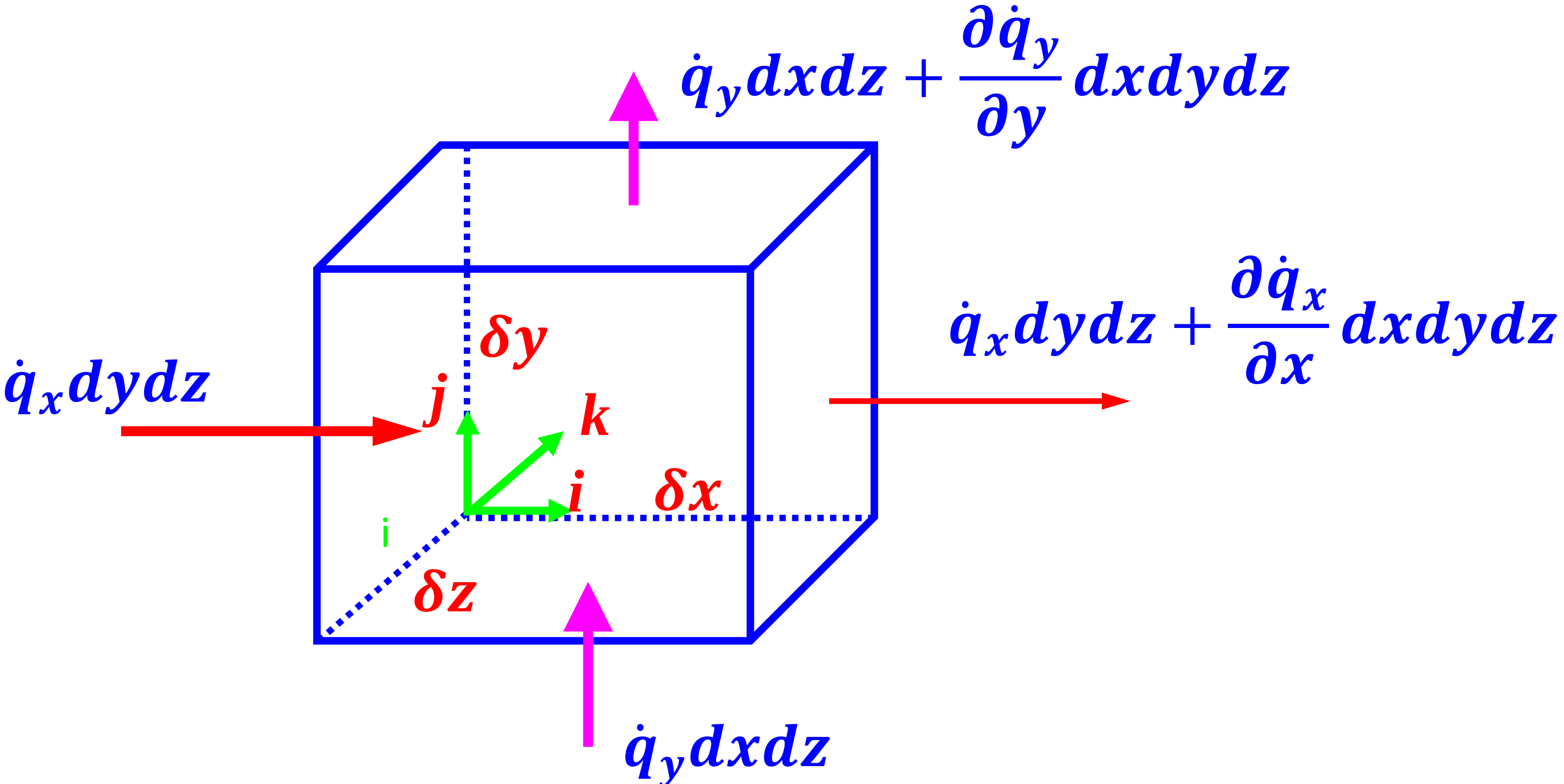
$$\frac{LHS}{dxdydz} = \rho \frac{D}{Dt}\left[e + \frac{u^2 + v^2}{2}\right]$$

Rate of heat transfer into CV by conduction

$$= -\left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y}\right) dxdydz = -\left(\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(-k\frac{\partial T}{\partial y}\right)\right) dxdydz$$

$$= k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) dxdydz$$

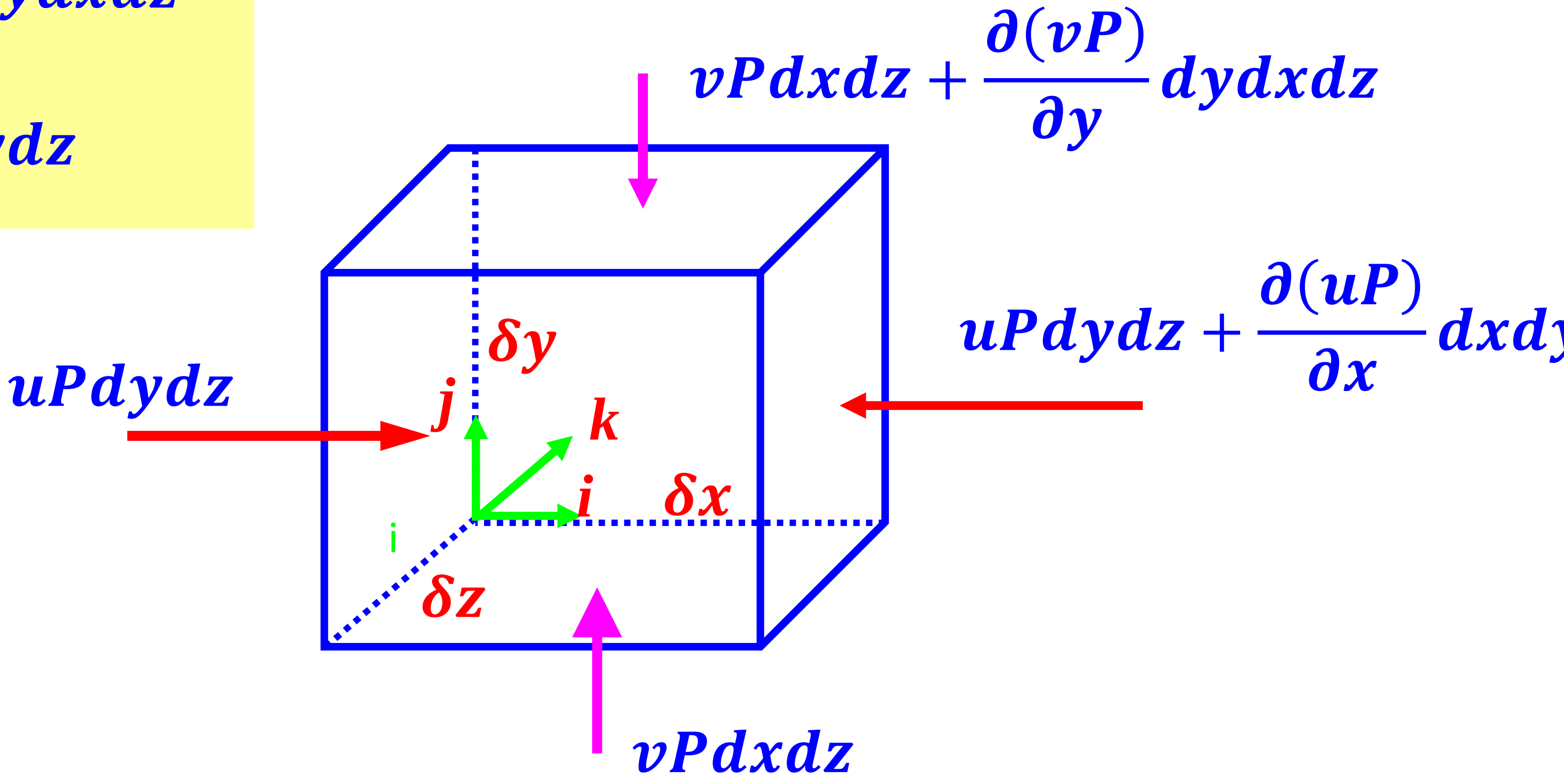
Negative sign arises because heat transfer is counted as positive in the positive coordinate direction



RATE OF WORK DONE BY PRESSURE FORCES

$$uPdydz - uPdydz - \frac{\partial(uP)}{\partial x}dxdydz$$
$$+vPdxdz - vPdxdz - \frac{\partial(vP)}{\partial y}dydxdz =$$
$$-\left(\frac{\partial(uP)}{\partial x} + \frac{\partial(vP)}{\partial y}\right)dxdydz$$

Outward normal stresses are positive. Positive normal stresses are tensile stresses; that is, they tend to stretch the material. Compressive normal stress will give positive value for p



Rate of doing work = force × velocity

The work done on an object by an agent exerting a constant force on the object is the product of the component of the force in the direction of the displacement and the magnitude of the displacement.

$$W = Fdcos\theta$$

P $\rightarrow$ u $\rightarrow$

$\theta = 0 \;\; cos\theta = 1$

$(Pdydz)u \;\; - \;\; Positive$

P $\leftarrow$ u $\rightarrow$

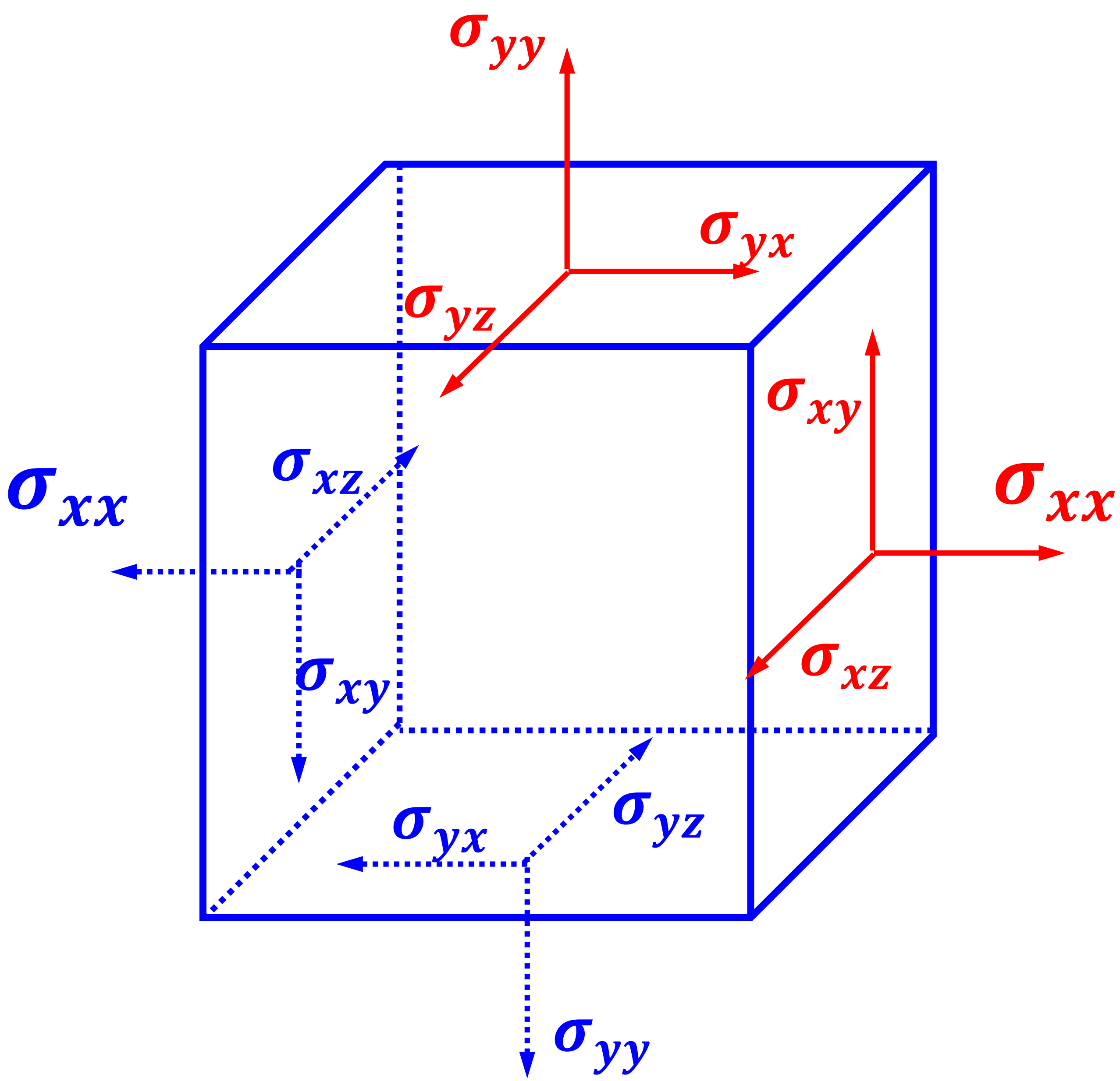
$\theta = 180$

$cos180 = -1$

$(Pdydz)u \;\; - \;\; Negative$

First subscript denotes the direction of the normal to the plane on which the stress acts

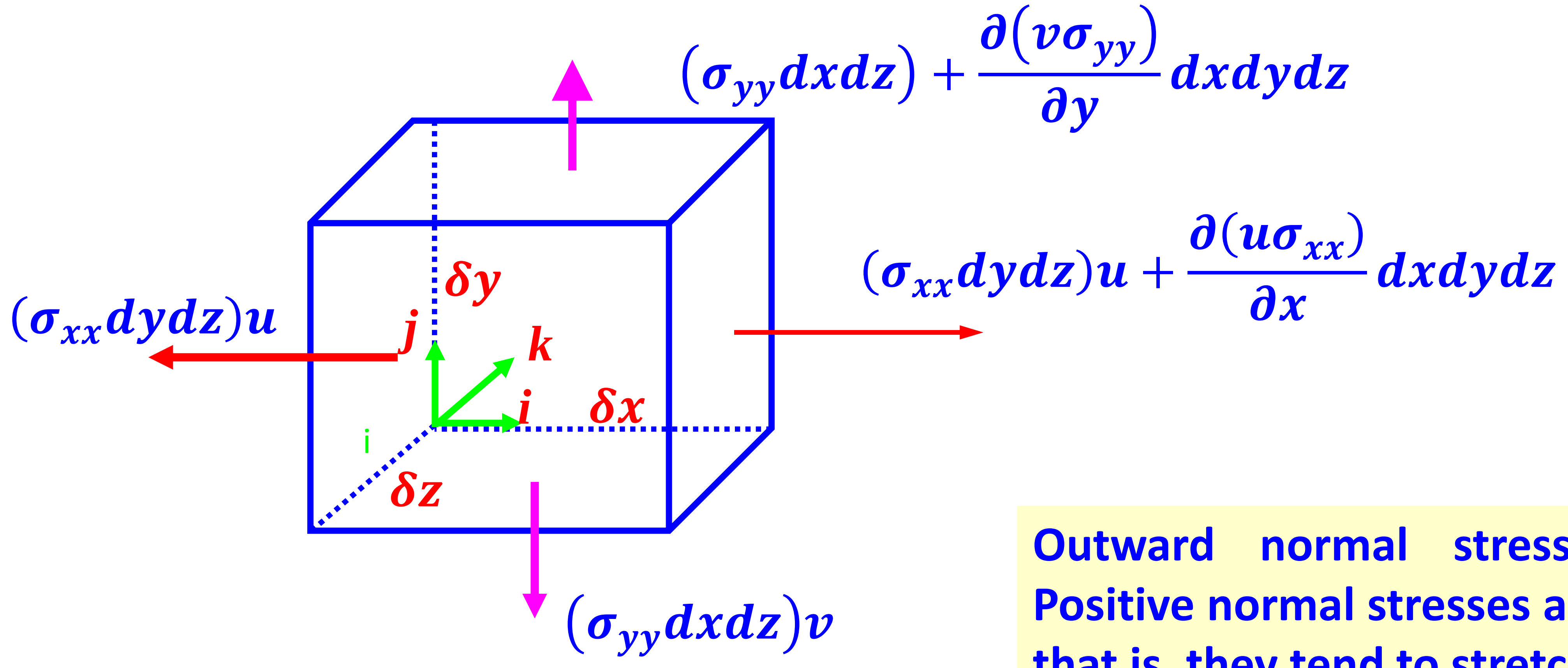
Second subscript denotes the direction of the stress



RATE OF WORK DONE BY NORMAL STRESSES

$$-u\sigma_{xx}dydz + u\sigma_{xx}dydz + \frac{\partial(u\sigma_{xx})}{\partial x}dxdydz - v\sigma_{yy}dxdz + v\sigma_{yy}dxdz + \frac{\partial(v\sigma_{yy})}{\partial y}dydxdz =$$

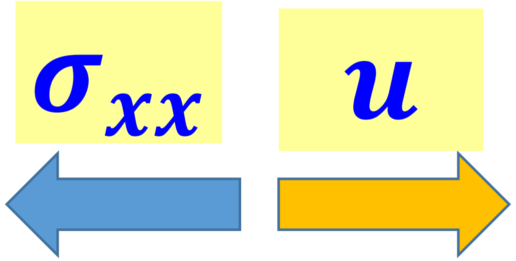
$$\left(\frac{\partial(u\sigma_{xx})}{\partial x} + \frac{\partial(v\sigma_{yy})}{\partial y}\right)dxdydz$$



Outward normal stresses are positive. Positive normal stresses are tensile stresses; that is, they tend to stretch the material.

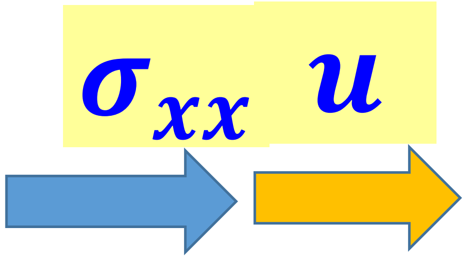
The work done on an object by an agent exerting a constant force on the object is the product of the component of the force in the direction of the displacement and the magnitude of the displacement.

$$\boldsymbol{W = Fd\cos\theta}$$



$$\theta = 180$$
$$\cos 180 = -1$$

$$(\sigma_{xx}dydz)u \;-\; Negative$$



$$\theta = 0 \quad \cos\theta = 1$$

$$(\sigma_{xx}dydz)u \;-\; Positive$$

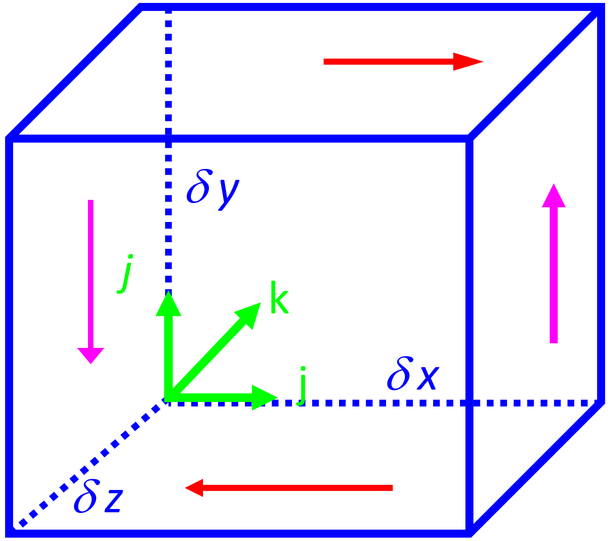
RATE OF WORK DONE BY SHEAR STRESSES

$$-\sigma_{xy}v\,dydz + \sigma_{xy}v\,dydz + \frac{\partial(\sigma_{xy}v)}{\partial x}dxdydz - \sigma_{yx}\,u\,dydz + \sigma_{yx}u\,dydz + \frac{\partial(\sigma_{yx}u)}{\partial x}dxdydz$$

$$\left(\frac{\partial(v\sigma_{xy})}{\partial x} + \frac{\partial(u\sigma_{yx})}{\partial y}\right)dxdydz$$

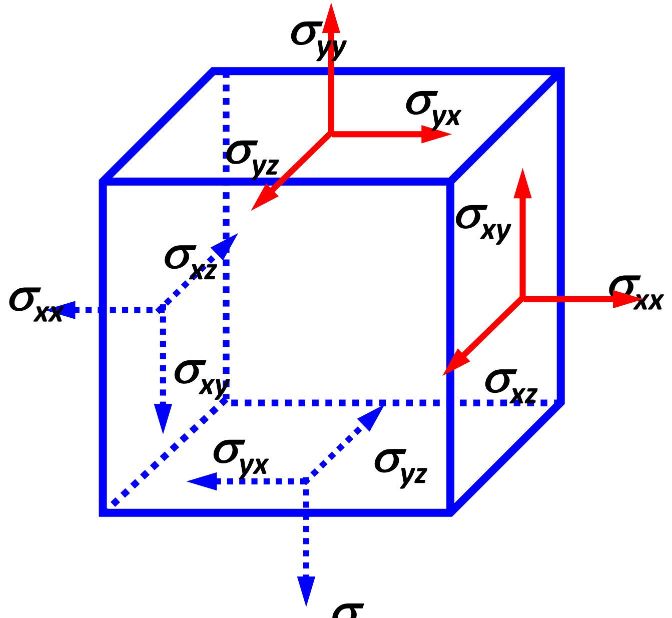
$$\sigma_{yx}u\,dxdz + \frac{\partial(\sigma_{yx}u)}{\partial x}dxdydz$$

$$-\sigma_{xy}v\,dydz$$



$$\sigma_{xy}v\,dydz + \frac{\partial(\sigma_{xy}v)}{\partial x}dxdydz$$

$$-\sigma_{yx}u\,dxdz$$



Rate of increase of E in CV − Rate at which E enters through surface of CV + Rate at which E leaves through surface of CV = Rate of heat transfer into CV by conduction + Rate of surface and body forces do work on CV

$\dot{q}'''$ - Volumetric heat generation

$$\rho \frac{D}{Dt}\left[e + \frac{u^2 + v^2}{2}\right] = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \left(\frac{\partial (uP)}{\partial x} + \frac{\partial (vP)}{\partial y}\right) + \left(\frac{\partial (u\sigma_{xx})}{\partial x} + \frac{\partial (v\sigma_{yy})}{\partial y}\right)$$
$$+ \left(\frac{\partial (v\sigma_{xy})}{\partial x} + \frac{\partial (u\sigma_{yx})}{\partial y}\right) + uf_x + vf_y + \dot{q}'''$$

$$\rho \frac{D}{Dt}\left[e + \frac{u^2 + v^2}{2}\right]$$
$$= k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y}\right)$$
$$+ \left(u\frac{\partial \sigma_{xx}}{\partial x} + \sigma_{xx}\frac{\partial u}{\partial x} + v\frac{\partial \sigma_{yy}}{\partial y} + \sigma_{yy}\frac{\partial v}{\partial y} + v\frac{\partial \sigma_{xy}}{\partial x} + \sigma_{xy}\frac{\partial v}{\partial x} + u\frac{\partial \sigma_{yx}}{\partial y} + \sigma_{yx}\frac{\partial u}{\partial y}\right) + uf_x + vf_y + \dot{q}'''$$

Eqn A

x- momentum equation

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x$$

$$\rho u \frac{Du}{Dt} = \rho \frac{D\left(\frac{u^2}{2}\right)}{Dt} = -u\frac{\partial P}{\partial x} + u\frac{\partial \sigma_{xx}}{\partial x} + u\frac{\partial \sigma_{yx}}{\partial y} + uf_x$$

$$\rho v \frac{Dv}{Dt} = \rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -v\frac{\partial P}{\partial y} + v\frac{\partial \sigma_{xy}}{\partial x} + v\frac{\partial \sigma_{yy}}{\partial y} + vf_y$$

Eqn B

$$\rho \frac{D\left(\frac{u^2}{2} + \frac{v^2}{2}\right)}{Dt} = -u\frac{\partial P}{\partial x} + u\frac{\partial \sigma_{xx}}{\partial x} + u\frac{\partial \sigma_{yx}}{\partial y} + uf_x - v\frac{\partial P}{\partial y} + v\frac{\partial \sigma_{xy}}{\partial x} + v\frac{\partial \sigma_{yy}}{\partial y} + vf_y$$

Eqn A

$$\rho\frac{D}{Dt}\left[e+\frac{u^2+v^2}{2}\right]$$
$$= k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) - P\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \left(u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}\right)$$
$$+\left(u\frac{\partial \sigma_{xx}}{\partial x}+\sigma_{xx}\frac{\partial u}{\partial x}+v\frac{\partial \sigma_{yy}}{\partial y}+\sigma_{yy}\frac{\partial v}{\partial y}+v\frac{\partial \sigma_{xy}}{\partial x}+\sigma_{xy}\frac{\partial v}{\partial x}+u\frac{\partial \sigma_{yx}}{\partial y}+\sigma_{yx}\frac{\partial u}{\partial y}\right)+uf_x+vf_y+\dot{q}'''$$

Eqn B

$$\rho\frac{D\left(\frac{u^2}{2}+\frac{v^2}{2}\right)}{Dt} = -u\frac{\partial P}{\partial x}+u\frac{\partial \sigma_{xx}}{\partial x}+u\frac{\partial \sigma_{yx}}{\partial y}+uf_x-v\frac{\partial P}{\partial y}+v\frac{\partial \sigma_{xy}}{\partial x}+v\frac{\partial \sigma_{yy}}{\partial y}+vf_y$$

Eqn A - Eqn B

$$\rho\frac{De}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) - P\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \left(\sigma_{xx}\frac{\partial u}{\partial x}+\sigma_{yy}\frac{\partial v}{\partial y}+\sigma_{xy}\frac{\partial v}{\partial x}+\sigma_{yx}\frac{\partial u}{\partial y}\right)+\dot{q}'''$$

$$\rho\frac{De}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{xy}\frac{\partial v}{\partial x} + \sigma_{yx}\frac{\partial u}{\partial y}\right) + \dot{q}'''$$

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\left(\sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{xy}\frac{\partial v}{\partial x} + \sigma_{yx}\frac{\partial u}{\partial y}\right) = \phi$$

$$\phi = 2\mu\left(\frac{\partial u}{\partial x}\right)^2 + 2\mu\left(\frac{\partial v}{\partial y}\right)^2 - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

$$\rho\frac{De}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \phi + \dot{q}'''$$

$$\rho \frac{De}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \phi + \dot{q}'''$$

$$h = e + \frac{P}{\rho}$$

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}$$

$$\frac{D\rho}{Dt} + \rho\left(\nabla . \vec{V}\right) = 0 \quad By \ \ Continuity\ Eqn.$$

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}(-\rho)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\frac{De}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\rho\left(\frac{Dh}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho}\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - P\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \phi + \dot{q}'''$$

$$\rho \frac{Dh}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{DP}{Dt} + \phi + \dot{q}'''$$

$$\rho\,\frac{Dh}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{DP}{Dt} + \phi + \dot{q}'''$$

$$h = C_P T$$

$$\rho C_P \frac{DT}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{DP}{Dt} + \phi + \dot{q}'''$$

Viscous Dissipation

$$\rho C_P\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{DP}{Dt} + \phi + \dot{q}'''$$

Convection | Conduction | Pressure work | Volumetric heat generation

PRINCIPLE OF SIMILARITY

Conservation of momentum

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + f_x$$

Inertia forces	Pressure forces	Viscous forces	Body forces

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

For steady flows and two dimensional flows

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + f_x$$

PRINCIPLE OF SIMILARITY

For steady flows and two dimensional flows

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+\frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+f_x$$

$$x^*=\frac{x}{L};\ y^*=\frac{y}{L};\ u^*=\frac{u}{u_\infty};\ v^*=\frac{v}{u_\infty};\ P^*=\frac{P}{\rho u_\infty^2};$$

u_∞ – Free Stream Velocity or average velocity in pipe
L – Characteristic Length (Length of flat plate or diameter of pipe

Neglecting body force

$$\frac{u_\infty^2}{L}\rho\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{u_\infty^2}{L}\frac{\partial P^*}{\partial x^*}+\mu\frac{u_\infty}{L^2}\left[\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right]+\mu\frac{u_\infty}{L^2}\frac{\partial}{\partial x}\left[\frac{1}{3}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right]$$

$$\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{u_\infty^2}{L}\frac{L}{\rho u_\infty^2}\frac{\partial P^*}{\partial x^*}+\mu\frac{u_\infty}{L^2}\frac{L}{\rho u_\infty^2}\left[\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right]+\mu\frac{u_\infty}{L^2}\frac{L}{\rho u_\infty^2}\frac{\partial}{\partial x}\left[\frac{1}{3}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right]$$

PRINCIPLE OF SIMILARITY

$$\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{u_\infty^2}{L}\frac{L}{\rho u_\infty^2}\frac{\partial P^*}{\partial x^*}+\mu\frac{u_\infty}{L^2}\frac{L}{\rho u_\infty^2}\left[\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right]+\mu\frac{u_\infty}{L^2}\frac{L}{\rho u_\infty^2}\frac{\partial}{\partial x}\left[\frac{1}{3}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right]$$

$$\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{1}{\rho}\frac{\partial P^*}{\partial x^*}+\frac{\mu}{\rho u_\infty L}\left[\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right]+\frac{\mu}{\rho u_\infty L}\frac{\partial}{\partial x}\left[\frac{1}{3}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right]$$

$$\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{1}{\rho}\frac{\partial P^*}{\partial x^*}+\frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right]+\frac{1}{Re}\frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right]$$

$$Re=\frac{\rho_\infty L u_\infty}{\mu}$$

PRINCIPLE OF SIMILARITY

Viscous Dissipation

$$\rho C_P\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{DP}{Dt}+\phi+\dot{q}'''$$

Convection | Conduction | Pressure work | Volumetric heat generation

$$\emptyset=2\mu\left(\frac{\partial u}{\partial x}\right)^2+2\mu\left(\frac{\partial v}{\partial y}\right)^2-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

$$\rho C_P\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\left(\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}\right)+\phi+\dot{q}'''$$

For steady flows

$$\rho C_P\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\left(u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}\right)+\phi+\dot{q}'''$$

For steady flows

PRINCIPLE OF SIMILARITY

$$\rho C_P\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\left(u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}\right)+\phi+\dot{q}'''$$

$$\emptyset=2\mu\left(\frac{\partial u}{\partial x}\right)^2+2\mu\left(\frac{\partial v}{\partial y}\right)^2-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

$$x^*=\frac{x}{L};\,y^*=\frac{y}{L};\,u^*=\frac{u}{u_\infty};\,v^*=\frac{v}{u_\infty};\,P^*=\frac{P}{\rho u_\infty^2};\,T^*=\frac{T-T_S}{T_\infty-T_S}$$

u_∞ – Free Stream Velocity or average velocity in pipe
T_∞- Free Stream temperature or bulk fluid temperature in pipe
T_S - Surface temperature of the plate or pipe
L – Characteristic Length (Length of flat plate or diameter of pipe

For steady flows

PRINCIPLE OF SIMILARITY

$$\rho C_P\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\left(u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}\right)+\phi$$

$$\phi=2\mu\left(\frac{\partial u}{\partial x}\right)^2+2\mu\left(\frac{\partial v}{\partial y}\right)^2-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

$$x^*=\frac{x}{L};\, y^*=\frac{y}{L};\, u^*=\frac{u}{u_\infty};\, v^*=\frac{v}{u_\infty};\, P^*=\frac{P}{\rho u_\infty^2};\, T^*=\frac{T-T_S}{T_\infty-T_S}$$

$$\phi=\mu\frac{u_\infty^2}{L^2}\left[2\left(\frac{\partial u^*}{\partial x^*}\right)^2+2\left(\frac{\partial v}{\partial y^*}\right)^2-\frac{2}{3}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v}{\partial y^*}\right)^2+\left(\frac{\partial u^*}{\partial y^*}+\frac{\partial v}{\partial x^*}\right)^2\right]=\mu\frac{u_\infty^2}{L^2}\phi^*$$

$$\rho C_P\frac{(T_\infty-T_S)u_\infty}{L}\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=k\frac{(T_\infty-T_S)}{L^2}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right)+\frac{\rho u_\infty^2 u_\infty}{L}\left(u\frac{\partial P^*}{\partial x^*}+v\frac{\partial P^*}{\partial y^*}\right)+\mu\frac{u_\infty^2}{L^2}\phi^*$$

$$\rho C_P \frac{(T_\infty - T_S)u_\infty}{L}\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right) = k\frac{(T_\infty - T_S)}{L^2}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right) + \frac{\rho u_\infty^2 u_\infty}{L}\left(u\frac{\partial P^*}{\partial x^*} + v\frac{\partial P^*}{\partial y^*}\right) + \mu\frac{u_\infty^2}{L^2}\phi^*$$

$$\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right)$$
$$= k\frac{(T_\infty - T_S)}{L^2}\frac{L}{\rho C_P(T_\infty - T_S)u_\infty}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right)$$
$$+ \frac{\rho u_\infty^2 u_\infty}{L}\frac{L}{\rho C_P(T_\infty - T_S)u_\infty}\left(u\frac{\partial P^*}{\partial x^*} + v\frac{\partial P^*}{\partial y^*}\right) + \mu\frac{u_\infty^2}{L^2}\frac{L}{\rho C_P(T_\infty - T_S)u_\infty}\phi^*$$

$$\left(u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*}\right)$$
$$= \frac{k}{\rho C_P L u_\infty}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right) + \frac{u_\infty^2}{C_P(T_\infty - T_S)}\left(u\frac{\partial P^*}{\partial x^*} + v\frac{\partial P^*}{\partial y^*}\right) + \frac{u_\infty^2}{C_P(T_\infty - T_S)}\frac{\mu}{\rho u_\infty L}\phi^*$$

$$\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)$$
$$=\frac{k}{\rho C_P L u_\infty}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right)+\frac{u_\infty^2}{C_P(T_\infty-T_S)}\left(u\frac{\partial P^*}{\partial x^*}+v\frac{\partial P^*}{\partial y^*}\right)+\frac{u_\infty^2}{C_P(T_\infty-T_S)}\frac{\mu}{\rho u_\infty L}\phi^*$$

$$\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)$$
$$=\frac{\nu}{L u_\infty}\frac{\alpha}{\nu}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right)+\frac{u_\infty^2}{C_P(T_\infty-T_S)}\left(u\frac{\partial P^*}{\partial x^*}+v\frac{\partial P^*}{\partial y^*}\right)+\frac{u_\infty^2}{C_P(T_\infty-T_S)}\frac{\mu}{\rho u_\infty L}\phi^*$$

$$\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=\frac{1}{Re}\frac{1}{Pr}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right)+Ec\left(u\frac{\partial P^*}{\partial x^*}+v\frac{\partial P^*}{\partial y^*}\right)+\frac{Ec}{Re}\phi^*$$

$$Re=\frac{L u_\infty}{\nu} \qquad Pr=\frac{\nu}{\alpha} \qquad Ec=\frac{u_\infty^2}{C_P(T_\infty-T_S)}$$

Eckert number is measure of the dissipation effects in the flow. Since this grows in proportion to the square of the velocity, it can be neglected for small velocities. In an air flow, $V=10\frac{m}{s}$ $C_p=1050\frac{J}{kg.K}$ and a reference temperature difference of $10\ K$, $Ec\ \approx 0.01$.

$$\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=\frac{1}{RePr}\left(\frac{\partial^2 T^*}{\partial {x^*}^2}+\frac{\partial^2 T^*}{\partial {y^*}^2}\right)+Ec\left(u\frac{\partial P^*}{\partial x^*}+v\frac{\partial P^*}{\partial y^*}\right)+\frac{Ec}{Re}\phi^*$$

$$Re=\frac{Lu_\infty}{\nu}\qquad Pr=\frac{\nu}{\alpha}\qquad Ec=\frac{u_\infty^2}{C_P(T_\infty-T_S)}$$

For subsonic flows

$$\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=\frac{1}{RePr}\left(\frac{\partial^2 T^*}{\partial {x^*}^2}+\frac{\partial^2 T^*}{\partial {y^*}^2}\right)$$