Set Theory

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Set Theory: Basics

Set is a correlation of mathematical objects taken from a suitable domain of disclosure.

Examples:

- $\mathbf{N} = \{0, 1, 2, 3, 4. . .\}$

  (set of **natural numbers)**

- $\mathbf{Z} = \{. . ., -3, -2, -1, 0, 1, 2, 3,. . .\}$

  (set of **integers)**

- $\mathrm{E} = \{0, 2, 4, 6. . .\}$

  (set of **even natural numbers)**

Set Operators

- *Union* of two sets A and B is the set of all elements in <u>either</u> set A or B.
  - Written $A \cup B$.
  - $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

- *Intersection* of two sets A and B is the set of all elements in both sets A or B.
  - Written $A \cap B$.
  - $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- *Difference* of two sets A and B is the set of all elements in set A which are not in set B.
  - Written $A - B$.
  - $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Set Operators

- ***Symmetric Difference*** of two sets A and B is the set of all elements which are either in "set A and not in B" or in "set B and not in A"
  - Written $A \Delta B$.
  - $A \Delta B = (A \setminus B) \cup (B \setminus A)$

- ***Complement*** of a set is the set of all elements not in the set.
  - Written A^c
  - Depends on the choice of superset/universal set S
  - $A^c = \{x \mid x \in \mathbf{S} / A\}$

Cartesain Product

- ***Cartesian Product***: Given two sets A and B, the set of
  - All *ordered pairs* of the form (a , b) where a is any element of A and b any element of B, is called the *Cartesian product* of A and B.
- Denoted as A x B
  - $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$
  - **Example**: Let A = {1,2,3}; B = {x,y}
    - A x B = $\{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$
    - B x A = $\{(x,1),(y,1),(x,2),(y,2),(x,3),(y,3)\}$
    - B x B = $B^2 = \{(x,x),(x,y),(y,x),(y,y)\}$
- ***In general,***

$$A_1 \times A_2 \times \ldots\ldots \times A_n = \{(a_1,a_2\ldots\ldots\ldots a_n) \mid a_1 \in A_1, \ldots\ldots\ldots\ a_n \in A_n\}$$

Example:

$$\begin{aligned} R^2 &= R\ X\ R\ (2 - D\ Euclidean\ space) \\ &= \{(\boldsymbol{x_1}, \boldsymbol{x_2}): \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{R}\} \\ R^d &= R\ X\ R \ldots\ldots\ldots X\ R(d - D\ Euclidean\ space) \\ &= \{(\boldsymbol{x_1}, \boldsymbol{x_2} \ldots\ldots\ldots \boldsymbol{x_d}): \boldsymbol{x_1}, \boldsymbol{x_2} \ldots\ldots\ldots \boldsymbol{x_d} \in \boldsymbol{R}\} \end{aligned}$$

Topology in Euclidean Space

- Euclidean Metric – A measure of distance between two points

$$||x - y|| = \sqrt{(x_1 - y_1)^2 + \dots\dots\dots (x_d - y_d)^2}$$

- Closed Ball

$$b(a, r) = \{x \in R^d : ||x - a|| \le r\}$$

- Open Ball

$$b^{int}(a, r) = \{x \in R^d : ||x - a|| < r\}$$

- Bounded Set – Set A is bounded if there is a ball $b(a, r)$, such that $A \subset b(a, r)$

- A sequence $\{x_1, x_2 \dots\dots\dots\}$ is said to converge to x if $\lim_{n \to \infty} ||x_n - x|| = 0$

Topology in Euclidean Space

- **Open Set** - A set is said to be open if for each $x \in A$, a positive number ϵ can be found depending on x, such that $b(x, \epsilon) \subset A$
- *Example***:** In case of d = 1, (u,v) is a open set
  - System of open sets of R^d is denoted by O.

- **Closed Set** – A set is said to be closed if its complement A^c is open
- *Important:* For closed set we need a specific superset S to define A^c
- *Example:* Hypercubes

$$[\boldsymbol{u_1}, \boldsymbol{v_1}]\ \boldsymbol{X}\ [\boldsymbol{u_2}, \boldsymbol{v_2}]\ \boldsymbol{X}\ \dots\dots\dots [\boldsymbol{u_d}, \boldsymbol{v_d}]$$

Hyperplanes

$$\boldsymbol{x} = \{(\boldsymbol{x_1}, \dots\dots\dots \boldsymbol{x_d}) \in \boldsymbol{R^d} :\ \textstyle\sum_{i-1}^{d} \boldsymbol{X_i a_i} = \boldsymbol{b}\}$$

where $b, a_1, \dots\dots\dots a_d$ are constants with a_i are not equal to 0

Topology in Euclidean Space

- **The Interior** A^{int} of a general set A is the **union** of all the open sets contained in A
  - $\mathbf{A^{int}}$ is the largest open set contained in A.

- **The Closure** A^{cl} of a general set A is the **intersection** of all the closed sets containing A
  - $\mathbf{A^{cl}}$ is the smallest closed set containing A

- Properties of $\mathbf{A^{cl}}$ and $\mathbf{A^{int}}$
  - $\mathbf{A^{int} \subset A \subset A^{cl}}$
  - Also, $A^{int} = ((A^c)^{cl})^c$
  - A set A is <u>open precisely</u> when $A^{int} = A$
  - A set A is <u>closed precisely</u> when $A^{cl} = A$
  - If $A = (A^{int})^{cl}$, then A is said to be **regular closed**
  - The boundary of a set A, $\partial A = A^{cl} \setminus A^{int}$
  - A set $K \in R^d$ is said to be **Compact**, if it is both closed as well as bounded

Operations on Subsets of Euclidean Space

- **Addition:** $x + y = (x_1 + y_1, x_2 + y_2, \dots \dots \dots x_d + y_d)$

- **Translation:** $A_x = A + x = \{y + x : y \in A\}, for\ x\ and\ A \in R^d$

- **Scalar Multiplication** by $c \in R$,
  $$c.x = cx = (cx_1, cx_2, \dots \dots \dots cx_d)$$

- **Reflection**: Scalar Multiplication by c = -1,
  $$\check{A} = -A = \{-x : x \in A\}\ for\ A \subset R^d$$

Operations on Subsets of Euclidean Space

- **Minkowski Addition:** $A \oplus B = \{x + y : x \in A, y \in B\}$ $for\ A, B$
  - It is both Associative and Commutative
  - $A_x = A \oplus \{x\}$
  - $A \oplus B = \bigcup_{y \in B} A_y = \bigcup_{x \in A} B_x$
  - $B \oplus A = \left\{x : B \cap \check{A}_x\ is\ not\ empty\right\}$
  - $A \oplus (B_1 \cup B_2) = A \oplus B_1 \cup A \oplus B_2$
  - $If\ A_1 \subset A_2, then\ A_1 \oplus B \subset A_2 \oplus B$

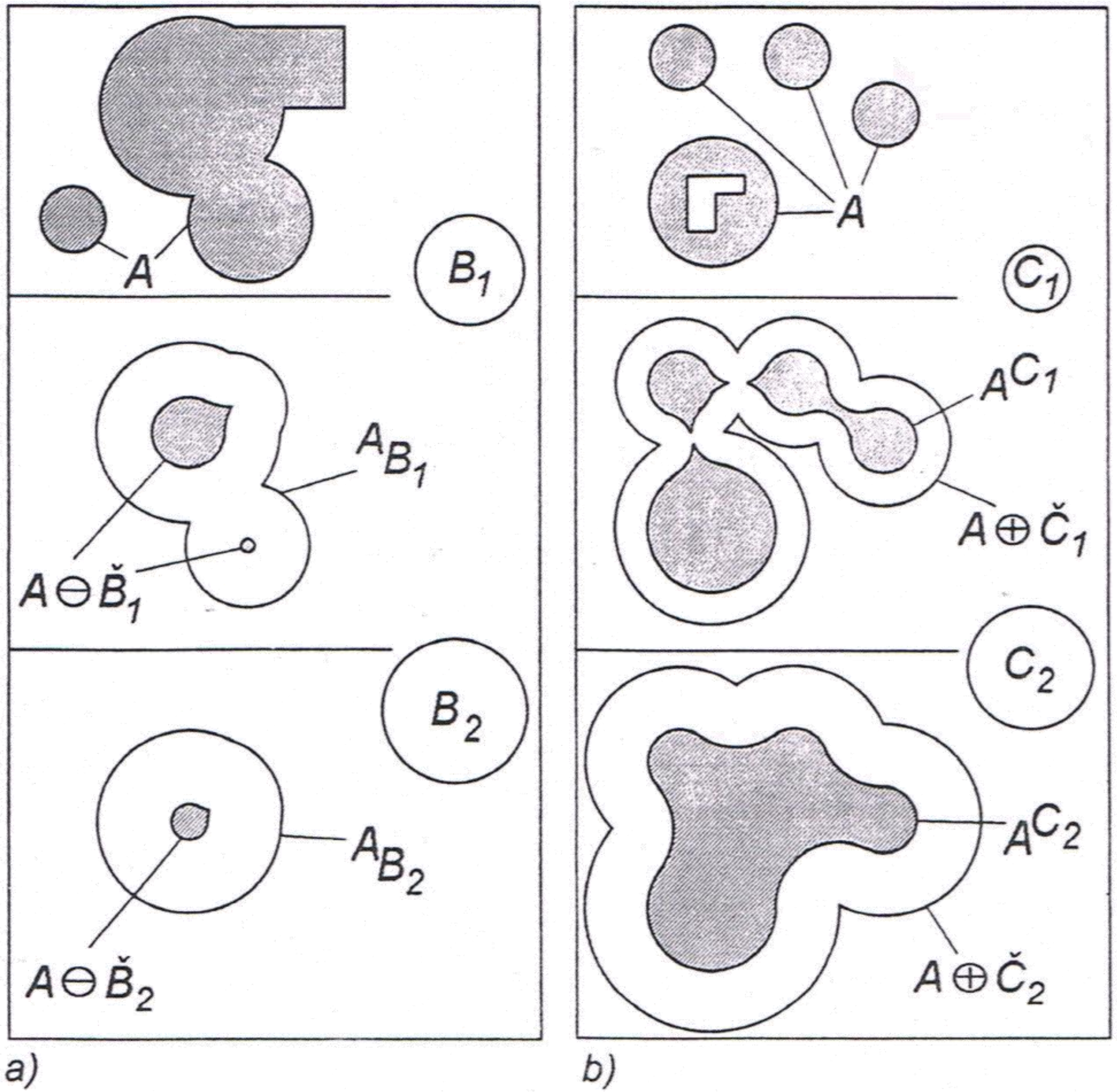
Operations on Subsets of Euclidean Space

- **Minkowski Subtraction:** $A \ominus B = \bigcap_{y \in B} A_y$
  - $or\ (A^c \oplus B)^c$
  - $(\mathrm{A} \ominus \check{\mathrm{C}}) \oplus C \subseteq A \subseteq (\mathrm{A} \oplus \check{\mathrm{C}}) \ominus C$

- **Dilation:** $\boldsymbol{A} \longmapsto \boldsymbol{A} \oplus \check{\mathrm{C}}$

- **Erosion:** $\boldsymbol{A} \longmapsto \boldsymbol{A} \ominus \check{\mathrm{C}}$
  - Opening of A by C (Erosion followed by Dilation)
  - Closing of A by C (Dilation followed by Erosion)

Operations on Subsets of Euclidean Space



(a) The operations of erosion and opening applied to a set. Components that overlap are separated while small components and roughnesses vanish or are reduced.

(b) The operations of dilation and closing applied to a set. Gaps are closed up, concavities vanish or are reduced, and clusters of small particles are merged