Solution to ODE using Laplace Transforms Concept of Poles, Zeros, Stability

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1

Linear, Constant-Coeff ODEs

Consider the homogeneous equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = 0$$

With initial conditions $y(0) = yo : y^{(1)}(0) = y_0^1; \ldots\ldots; y^{(n-1)}(0) = y_o^{n-1}$

Solution?

$$y_h(t) = Ae^{\lambda t}$$

Remember MA203?☺

λ is a solution to the characteristic eqn

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 = 0$$

2

Linear, Constant-Coeff ODEs contd..

If λi's are distinct:

$$y_H(t) = \sum_{i=1}^{n} A_i e^{\lambda_i t}$$

Ai's are determined from the set of initial conditions

Exercise: what will be the solution if λi's are not distinct?

REVISE MORE OF ODE Solving

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3

First Order Example: Particular solution

$$\dot{y}(t) + ay(t) = bu(t); y(0) = yo$$

What will be the solution?

$$y(t) = \underbrace{e^{-at} yo}_{\text{Homogenous Solution}} + \underbrace{\int_0^t e^{-a(t-r)} bu(\tau) d\tau}_{\text{Particular Solution}}$$

Homogenous Solution: Solution to u(t) = 0. Also termed as the **free response**

Particular Solution: Response to u(t). Initial conditions assumed zero. Also termed as the **forced response**

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4

Solution Using LT

- It is difficult to find inverse Laplace transform of commonly encountered functions in control engineering
- The alternative is to write this functions in combination of easily recognizable forms (which are similar to the Laplace transforms of known functions)
- This can be done taking partial fractions of the given function

Partial-fraction expansion (F(s) having distinct poles)

- Consider F(s) in factored form as

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2).....(s+z_m)}{(s+p_1)(s+p_2).....(s+p_n)}; m < n$$

- Then in partial fraction form it can be written as

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{(s+p_1)} + \frac{a_2}{(s+p_2)} + ... + \frac{a_n}{(s+p_n)}$$

Where a's are residues of the poles.

Partial-fraction expansion (F(s) having distinct poles)

- Now the residues are found as

$$a_k = \left[(s+p_k)\frac{B(s)}{A(s)} \right]_{s=-p_k}$$

- Now applying inverse laplace transform

$$L^{-1}\left[\frac{a_k}{s+p_k} \right] = a_k e^{-p_k t}$$

- Gives f(t) as

$$f(t) = L^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \ldots.. + a_n e^{-p_n t} \; : t \geq 0$$

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7

Partial-fraction expansion (F(s) having repeated poles)

- Consider the function,

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

- Partial fraction of this F(s) has three terms

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

- Where b's are calculated as follows:

$$b_3 = \left[(s+1)^3 \frac{B(s)}{A(s)} \right]_{s=-1}$$

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8

Partial-fraction expansion (F(s) having repeated poles)

$$b_2 = \frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1}$$

$$b_1 = \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1}\right\}$$

- Solving we get $b_1 = 1, b_2 = 0, b_3 = 2$
- So

$$f(t) = L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{0}{(s+1)^2}\right] + L^{-1}\left[\frac{2}{(s+1)^3}\right]$$

- Hence

$$f(t) = (1+t^2)e^{-t}; t \geq 0$$

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When there are multiple roots response has explicit time multiplication term

9

Solving LTI differential equations

1. Convert the given D.E in to an algebraic equation in 's' by applying Laplace transform to each term in the D.E.
2. Obtain the expression for the laplace transform of the dependent variable.
3. By taking inverse laplace transform using partial-fraction method we get time response of the dependent variable.

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10

Example

- What is the solution of the D.E

$$\ddot{x} + 3\dot{x} + 2x = 0, \qquad x(0) = 1, \dot{x}(0) = 2$$

- The Laplace transform of dependent variables with initial conditions are given by

$$L[x(t)] = X(s)$$

$$L\left[\dot{x}(t)\right] = sX(s) - x(0)$$

$$L\left[\ddot{x}(t)\right] = s^2X(s) - sx(0) - \dot{x}(0)$$

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11

Example (contd)

- Now using these relations and solving for X(s) from the D.E we get

$$X(s) = \frac{s+5}{s^2+3s+2}$$

- By partial-fractions we get

$$X(s) = \frac{4}{s+1} - \frac{3}{s+2}$$

- By taking inverse laplace transform

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{4}{s+1}\right] - L^{-1}\left[\frac{3}{s+2}\right]$$

$$x(t) = 4e^{-t} - 3e^{-2t}$$

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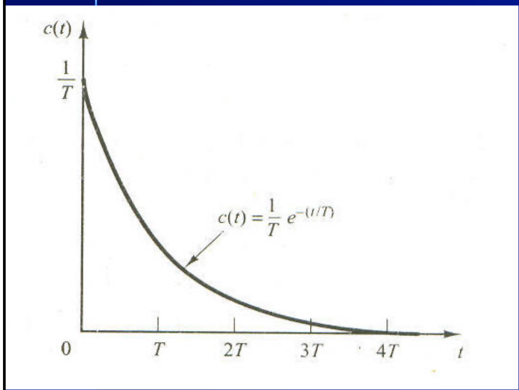
12

Standard Behavior

First order system: impulse



- First order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ts+1}$$

- Unit impulse response

$$U(s) = 1$$

- Using Laplace inverse

$$y(t) = \frac{1}{T} e^{-t/T}$$

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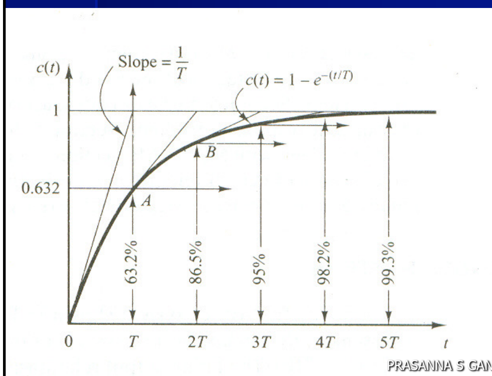
13

Step Response of First Order System

$$\dot{y}(t) + ay(t) = bu(t); y(0) = yo$$

Heat transfer system

Liquid level dynamics in tank system



with u(t) being the unit step input

Assuming a>0, sketch y(t) as a function of t

Looking at physical system with mathematical eye

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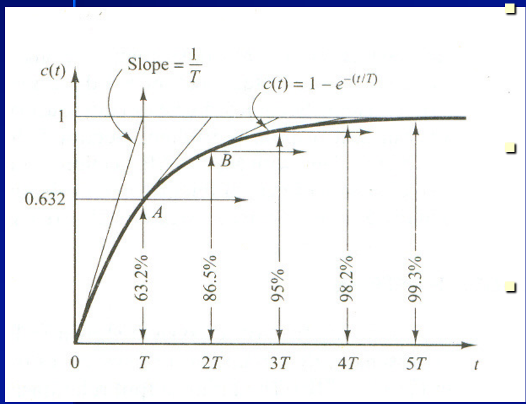
14

Standard Behavior

First order system: Step



- First order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ts+1}$$

- Unit step response

$$U(s) = \frac{1}{s}$$

- Using Laplace inverse

$$y(t) = 1 - e^{-t/T}$$

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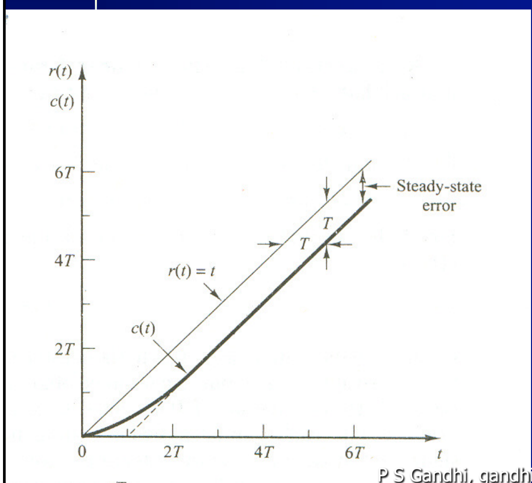
15

Standard Behavior

First order system: Ramp



- First order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ts+1}$$

- Unit ramp response

$$U(s) = \frac{1}{s^2}$$

- Using Laplace inverse

$$y(t) = t - [T - Te^{-t/T}]$$

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16

Step Response of Second Order System

Consider the ODE:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = bu(t);\ \dot{y}(0) = y(0) = 0$$

Spring mass system

Motor system with PD control L=0

with u(t) being the unit step input

Often useful to view this system as a mass-spring-damper system

$$\ddot{y}(t) + 2\zeta w_n \dot{y}(t) + w_n^2 y(t) = bu(t)$$

Sketch step response for cases:

1. $0< \zeta<1$
2. $\zeta=1$
3. $\zeta>1$

Looking at physical system with mathematical eye

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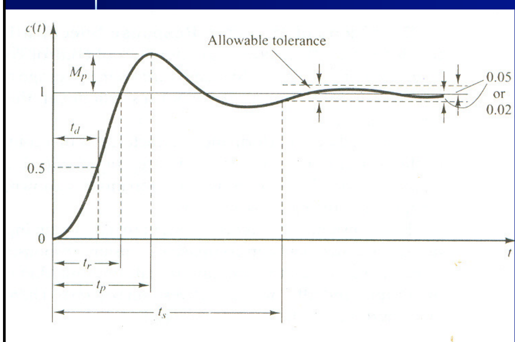
17

Standard Behavior

Second order system: Transient



- Second order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{Js^2 + Bs + K}$$
$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Unit step response

$$U(s) = \frac{1}{s}$$

1. Underdamped
2. Critically damped
3. Overdamped

- Solution using Laplace inverse for different cases

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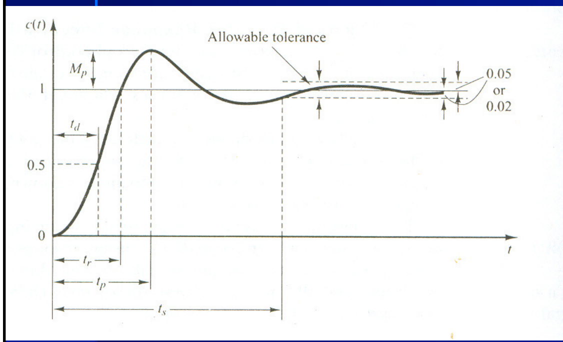
18

Standard Behavior

Second order system: Transient



Response terms

- Mp = Maximum overshoot
- ts = settling time
- tr = rise time
- Tp = peak time

These can be found by using the response

Standard formulae available in book

Use: for designing system Parameters/ control satisfying Certain constraints in response terms

Standard Behavior

Second order system: Transient

- Roots of characteristics equation

For step response

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s - D_1} + \frac{C}{s - D_2}$$

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For underdamped system

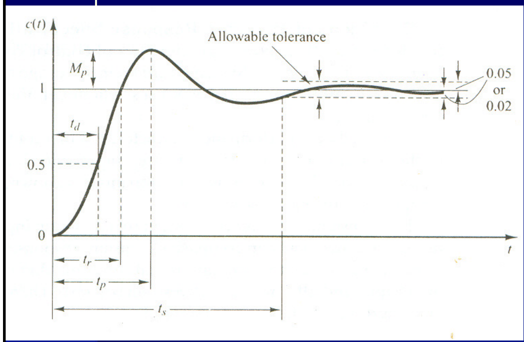
$$y(t) = 1 - \frac{e^{-2\xi\omega_n t}}{\sqrt{1-\xi^2}}\cos(\omega_n\sqrt{1-\xi^2}t + \phi)$$

$$\phi = tan^{-1}(\frac{\xi}{\sqrt{1-\xi^2}})$$

- **Overdamped system** $\zeta > 1$
  $D_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n$
- **Critically damped system** $\zeta = 1$
  $D_{1,2} = -\omega_n$ both values are same
- **Underdamped system** $\zeta < 1$
  $D_{1,2} = -\zeta\omega_n \pm \sqrt{1 - \zeta^2}\omega_n j$

Standard Behavior

Second order system: Transient



Response terms

- Mp = Maximum overshoot

$$M_p = e^{\frac{\pi\xi}{\sqrt{1-\xi^2}}} x100\ \%$$

- ts = settling time

$$t_s = \frac{4}{\xi\omega_n}$$

- tr = rise time

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

Poles and Zeros

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_o s^m + b_1 s^{m-1} + \ldots.. + b_m}{a_o s^n + a_1 s^{n-1} + \ldots.. + a_n} ; m < n$$

- Transfer function G(s)
- Poles: roots of denominator polynomial
  - More formal: a function is said to have a pole of order r at s=si if the $\lim_{s \to s_i} [(s - s_i)^r G(s)]$ has a finite nonzero value.
- Zeros: roots of numerator polynomial
- Characteristic equation is obtained by making denominator of the transfer function equal to zero

Poles and zeros Definition

$$G(s) = \frac{N(s)}{D(s)} = \frac{Y(s)}{U(s)} = A\frac{\prod_{m=0}^{M}(s - s_m)}{\prod_{n=0}^{N}(s - s_n)}$$

zeros

poles

Pole zero plot and animation

http://www-es.fernuni-hagen.de/JAVA/PolZero/polzero.html

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23

Effect of Poles on system response

- The nature of impulse response g(t) depends on the poles of the transfer function G(s) which are the roots of the characteristic equation
- These roots may be both real and complex conjugate and may have multiplicity of various orders.
- The nature of responses contributed by all possible types of poles are shown in the following slides.
- HW: In each case find whether the system is stable or not with the help of BIBO condition (i.e check whether the area under the absolute-valued impulse curve is finite or not).

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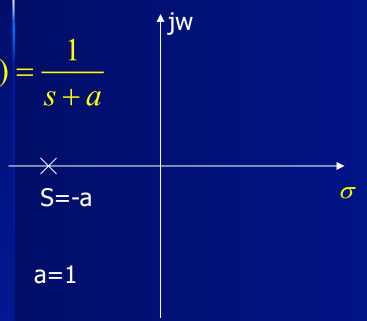
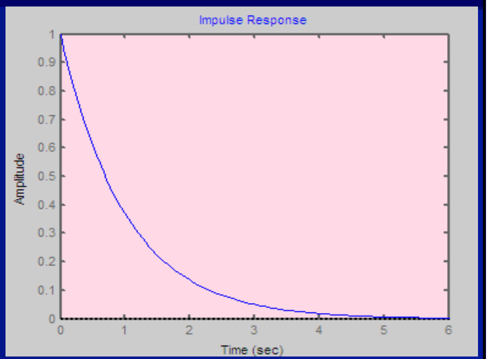
24

System with single real pole in left half s-plane.

$$G(s) = \frac{1}{s+a}$$



a=1



- Impulse response is exponentially decaying (bounded output)
- As BIBO condition satisfied (can you see how?), system is stable

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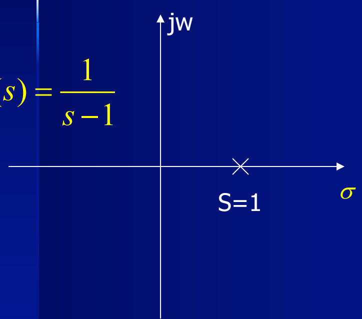
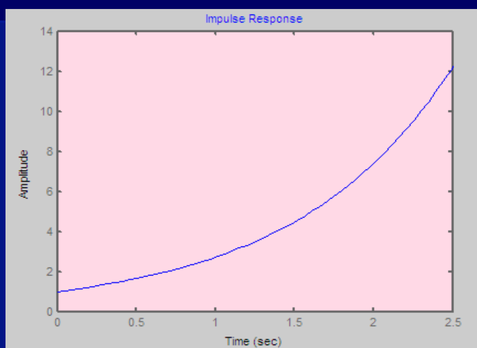
25

System with single real pole in right half s-plane.

$$G(s) = \frac{1}{s-1}$$





- Impulse response is exponentially growing (unbounded output).
- As BIBO condition not satisfied, system is unstable

The last line should be seen after
Seeing through concept of stability

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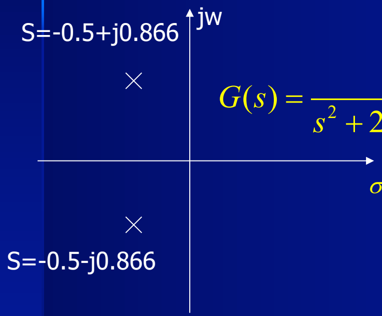
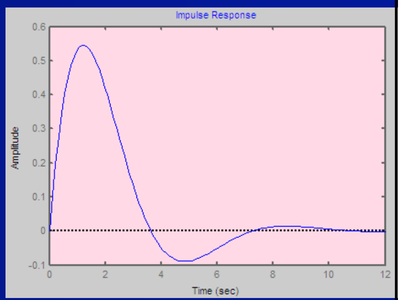
26

System with a pair of complex poles in left half s-plane.



$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n + \omega_n^2}$$



- Impulse response is exponentially decaying sinusoid (bounded output).
- As BIBO condition satisfied, system is stable.

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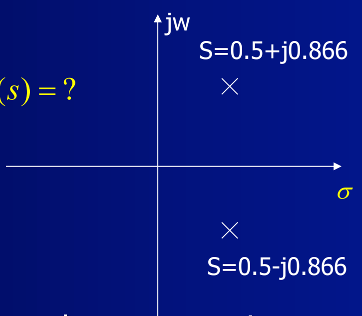
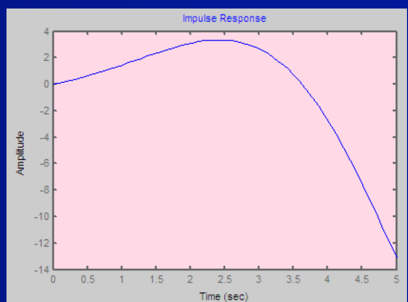
27

System with a pair of complex poles in right half s-plane.

$G(s) = ?$





- Impulse response is exponentially growing sinusoid (unbounded output).
- As BIBO condition not satisfied, system is unstable

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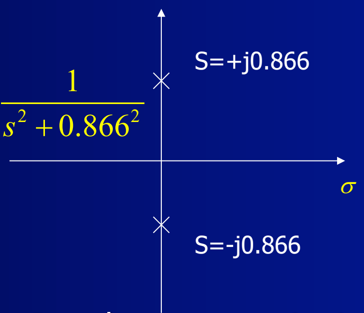
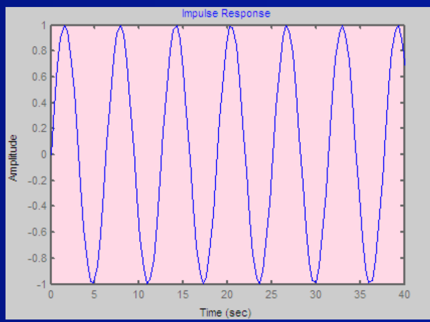
28

System with a pair of complex poles on jw axis.

$$G(s) = \frac{1}{s^2 + 0.866^2}$$





- Impulse response is sinusoidal (bounded output).
- Although BIBO condition not satisfied, because of bounded output these systems are called marginally stable

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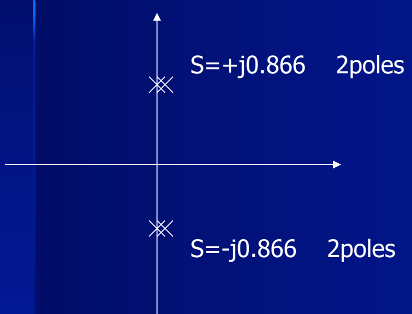
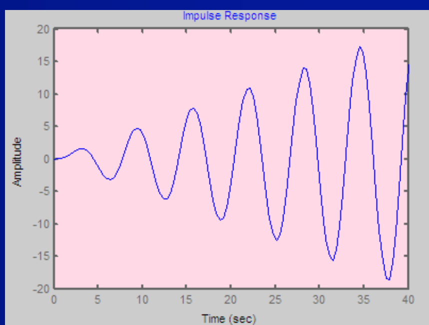
29

System with a double pair of complex poles on jw axis.





- Impulse response is linearly increasing sinusoid (unbounded output).
- As BIBO condition not satisfied, system is unstable.

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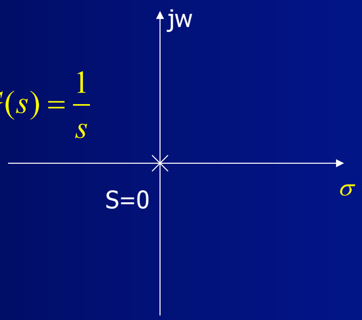
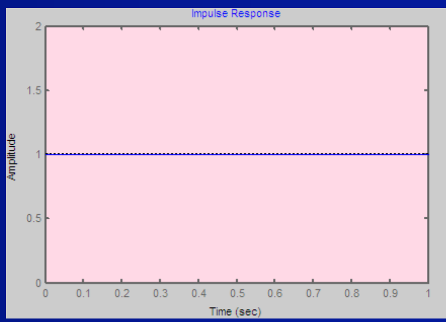
30

System with a pole at origin

$$G(s) = \frac{1}{s}$$





- Impulse response is constant (bounded output).
- Although BIBO condition not satisfied, because of bounded output these systems are called marginally stable

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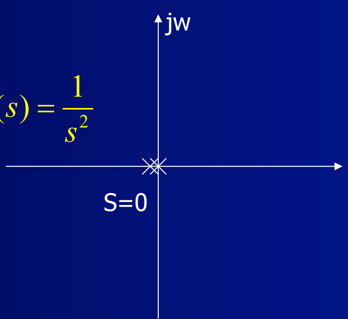
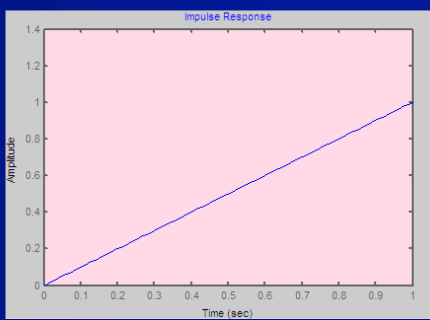
31

System with a double pole at origin

$$G(s) = \frac{1}{s^2}$$





- Impulse response is linearly increasing (unbounded output).
- As BIBO condition not satisfied, system is unstable

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32

Observations

i. If all the roots of the characteristic equation (poles) have negative real parts, the system is *stable*.

ii. If any ONE root of the characteristic equation has a positive real part or if there is a repeated root on the jw-axis, the system is *unstable*.

iii. If the condition (i) satisfied except for the presence of one or more non repeated roots on the jw-axis, the system is *marginally stable*.

Concept of stability

- Very important characteristic of the transient performance of the system.
- An LTI system is stable if the following two notions of system stability are satisfied:

  (i) When the system is excited by a bounded input, the output is bounded (BIBO).

  →(ii) In the absence of input, output tends towards zero (the equilibrium state of the system) irrespective of initial conditions(this is also called as asymptotic stability).

BIBO stability condition

- For a single input, single output system with transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_o s^m + b_1 s^{m-1} + \dots + b_m}{a_o s^n + a_1 s^{n-1} + \dots + a_n}; m < n$$

- With initial conditions assumed zero, the output of the system is given by:

$$y(t) = L^{-1}[G(s).U(s)]$$

- Therefore,

$$y(t) = \int_0^{\infty} g(\tau) u(t-\tau) d\tau$$

- Where $g(t) = L^{-1}(G(s))$ is the impulse response of the system.

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BIBO stability condition (contd)

- From this we get,

$$|y(t)| \le \int_0^{\infty} |g(\tau)| |u(t-\tau)| d\tau$$

- If for the bounded input($|u(t)| \le M_1 < \infty$), we need the output to be bounded i.e.

$$|y(t)| \le M_1 \int_0^{\infty} |g(\tau)| d\tau \le M_2 \quad \rightarrow \quad \int_0^{\infty} |g(\tau)| d\tau \le \frac{M_2}{M_1}$$

- Which implies that stability is satisfied if the impulse response $g(t)$ is absolutely integrable, i.e., $\int_0^{\infty} |g(\tau)| d\tau$ is finite.

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Example: stability

- Q: Is the simple spring mass system stable without damper in sense of BIBO stability definition?
- Impulse response is sinusoidal (check why? How?) so $\int_0^\infty |g(\tau)|\,d\tau$ is NOT finite. Hence the system is not BIBO stable
- Q: How do we physically interprete this?

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37

Effect of zeros on stability

- We have seen how the closed loop poles effect the transient response of the system, let us see what is the effect on transient response by the closed loop zeros.
- We see this by adding a zero to the one of the systems that we have just seen.

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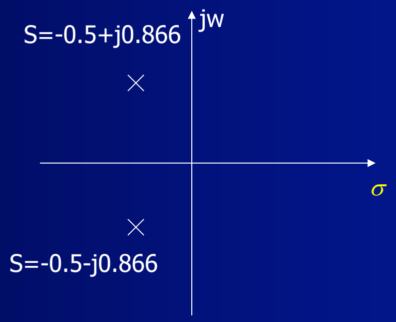
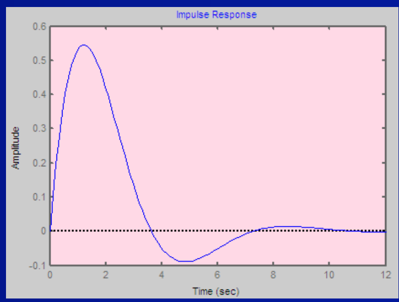
38

Effect of zeros on stability

- Recall the impulse of the system with a pair of complex poles in left half s-plane.





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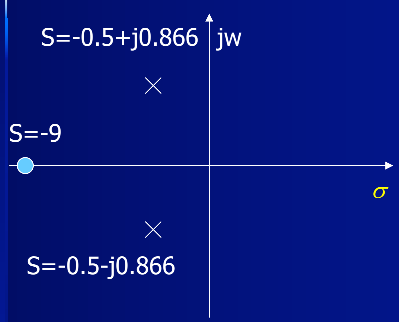
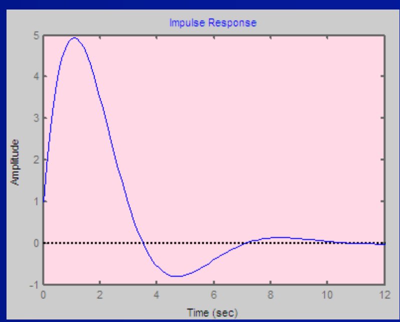
39

Add a left half s-plane zero to this system





- We see no change in stability of the system except a change in the shape of the transient.

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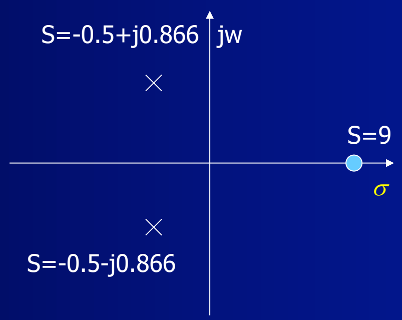
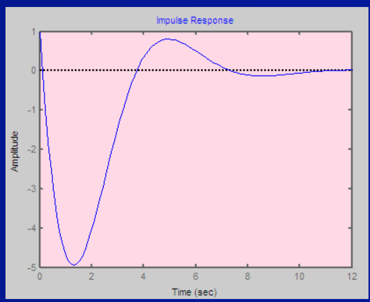
40

Add a right half s-plane zero to this system





- We see no change in stability of the system except a change in the shape of the transient either it is right or left half s-plane zero.

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41

Thank You

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42