Objectives

- One dimensional steady conduction in plane wall, composite wall and cylinder is introduced. The approach is to reduce the heat diffusion equation for the case chosen.

- Using the appropriate boundary conditions, the heat diffusion equation is solved for temperature distribution and heat transfer rate is computed

- Analogy between thermal and electrical systems is drawn in order to aid the solving of conduction problems on the basis of electrical circuits

ONE-DIMENSIONAL STEADY STATE CONDUCTION

- We treat situations for which heat is transferred by diffusion under one dimensional, steady state conditions.

- In a one-dimensional system, temperature gradients exist along a single coordinate direction, and heat transfer occurs exclusively in that direction.

- The system is characterized by steady state conditions if the temperature at each point is independent of time.

ONE-DIMENSIONAL, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION

THE PLANE WALL

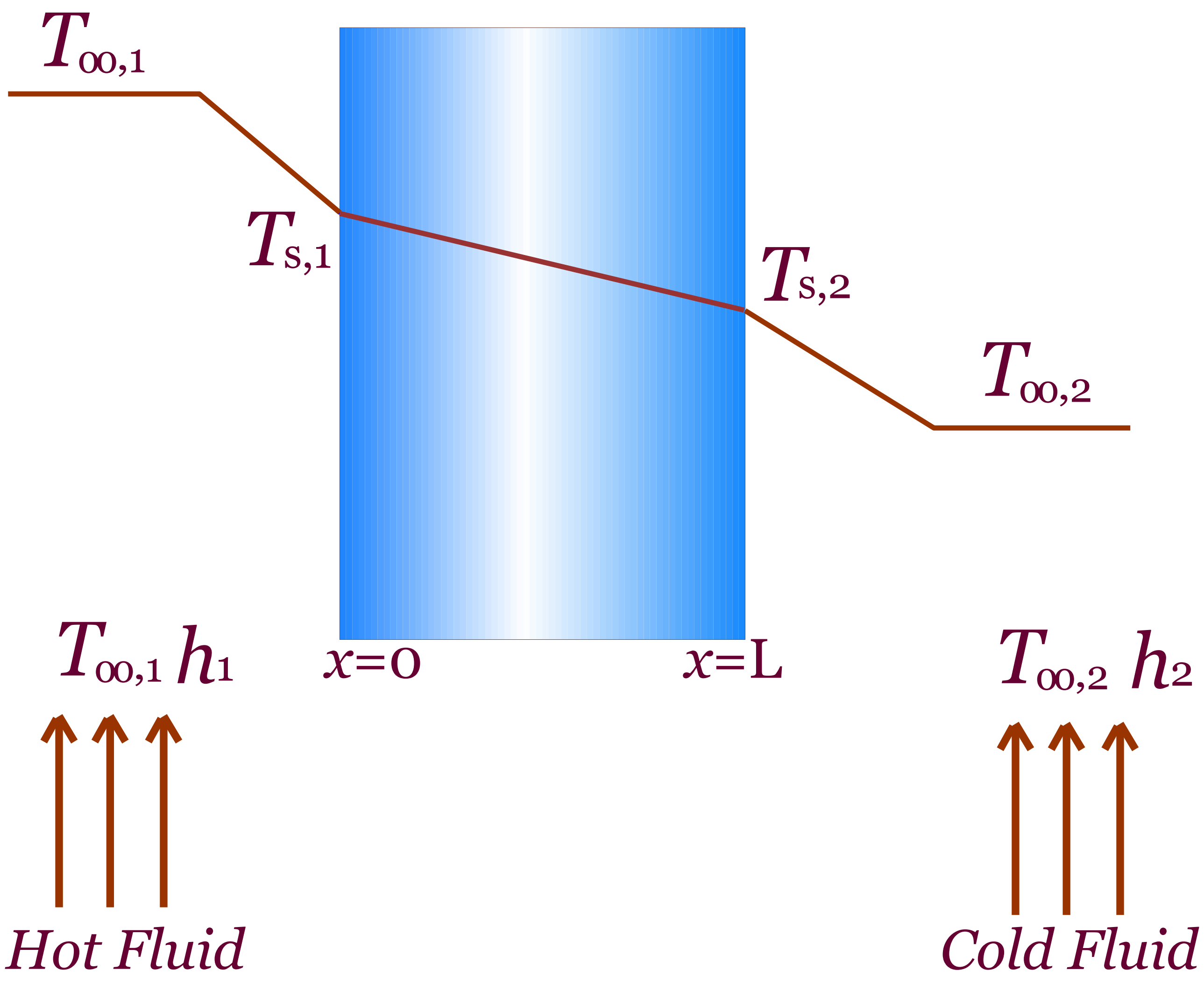
For one dimensional conduction in a plane wall, temperature is a function of the *x* coordinate only and heat is transferred exclusively in this direction.

In Figure, a plane wall separates two fluids of different temperatures.

Heat transfer occurs,

- by convection from the hot fluid at $T_{\infty,1}$ to one surface of the wall at $T_{s,1}$ by conduction through the wall, and
- by convection from the other surface of the wall at $T_{s,2}$ to the cold fluid at $T_{\infty,2}$.

Let us first determine the temperature distribution, from which we can then obtain the conduction heat transfer rate.

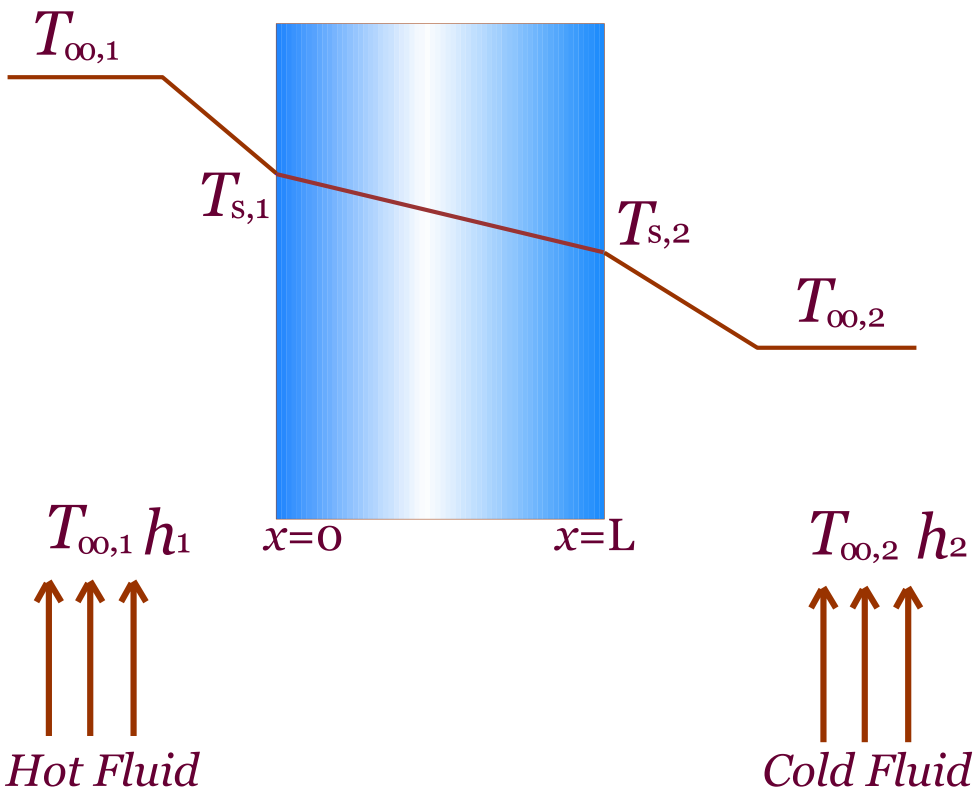


1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE PLANE WALL

Temperature Distribution

The temperature distribution in the wall can be determined by solving the heat equation using proper boundary conditions. For steady state conditions with no energy source within the wall, the appropriate form of the heat equation is

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t}$$



$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=0$$

$$x=0 \quad T=T_{s,1}$$
$$x=L \quad T=T_{s,2}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=C$$

$$\frac{\partial T}{\partial x}=C_1$$

$$T=C_1x+C_2$$

$$x=0 \quad T=T_{s,1}$$

$$T_{s,1}=C_1(0)+C_2$$

$$C_2=T_{s,1}$$

$$x=L \quad T=T_{s,2}$$

$$T_{s,2}=C_1(L)+T_{s,1}$$

$$C_1=\frac{T_{s,2}-T_{s,1}}{L}$$

$$T=T_{s,1}+\left(T_{s,2}-T_{s,1}\right)\left(\frac{x}{L}\right)$$

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE PLANE WALL

Temperature Distribution

$$T = T_{s,1} + (T_{s,2} - T_{s,1})\left(\frac{x}{L}\right)$$

Heat Transfer Rate

$$q_x = -k\,A\frac{\partial T}{\partial x} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

Heat Flux

$$q''_x = \frac{q_x}{A} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

The temperature varies linearly with x for one dimensional, steady state conduction in a plane wall with no heat generation and constant thermal conductivity,

Note that A is the area of the wall normal to the direction of heat transfer and for the plane wall, A is a constant independent of x.

Heat rate q_x and heat flux q''_x are constants.
Hence, both of them are independent of *x*.

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE PLANE WALL

Thermal Resistance

There exists an analogy between the diffusion of heat and electrical charge.
Thermal resistance may be associated with the conduction of heat in the same fashion as an electrical resistance is associated with the conduction of electricity.
Defining resistance as the ratio of a driving potential to the corresponding transfer rate
The thermal resistance for conduction is given by

$$R_{t,cond} = \frac{(T_{s,1} - T_{s,2})}{q_x} = \frac{L}{kA}$$

$$q_x = -k\,A\frac{\partial T}{\partial x} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

Similarly, for electrical conduction, Ohm's law provides an electrical resistance of the form

$$R_e = \frac{(E_{s,1} - E_{s,2})}{i} = \frac{L}{\sigma A}$$

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE PLANE WALL

Thermal Resistance

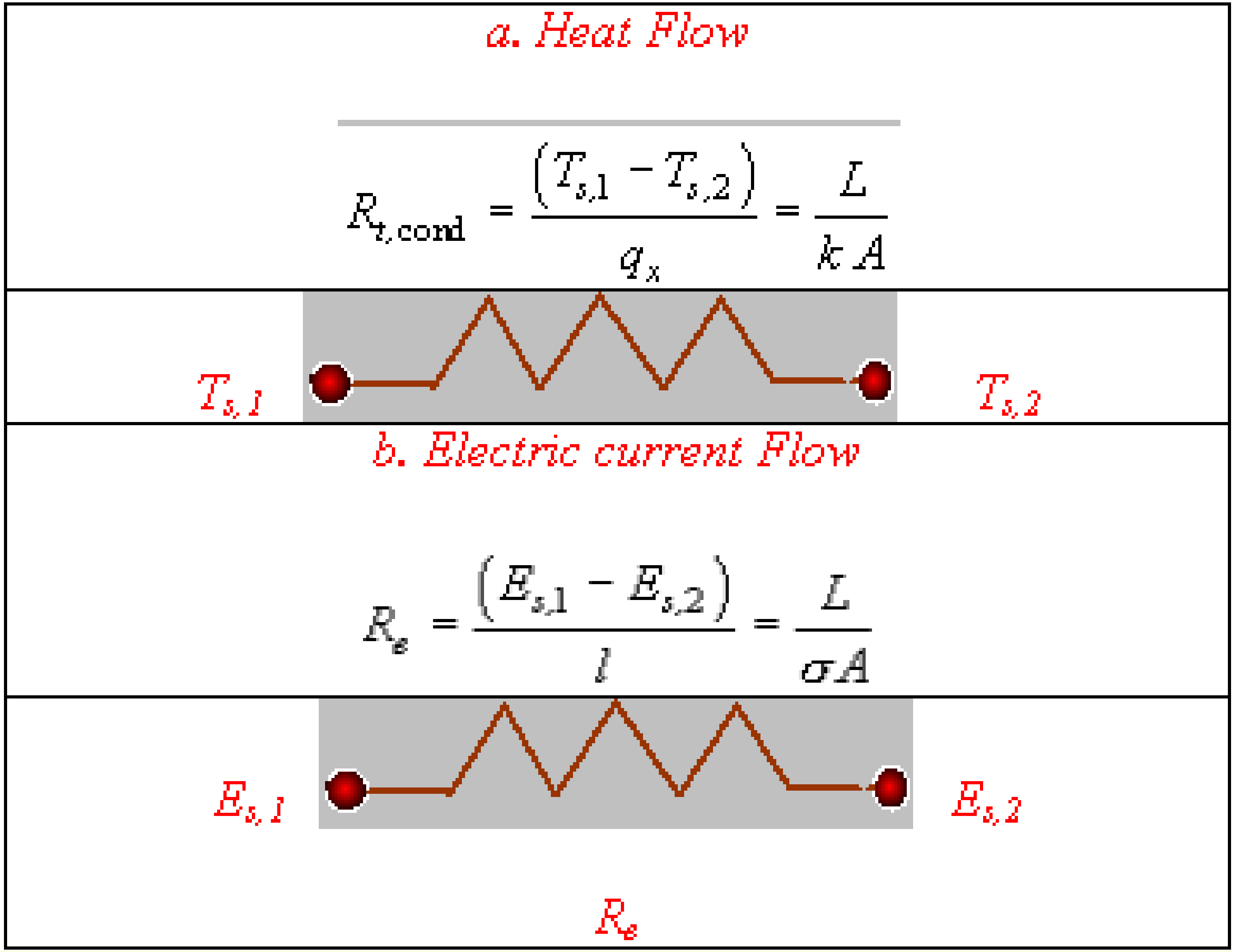
The rate of heat transfer through a plane wall corresponds to the electric current

The thermal resistance corresponds to electrical resistance and

The temperature difference corresponds to voltage difference across the plane wall.

$$R_{t,cond} = \frac{(T_{s,1} - T_{s,2})}{q_x} = \frac{L}{kA}$$

$$R_e = \frac{(E_{s,1} - E_{s,2})}{i} = \frac{L}{\sigma A}$$

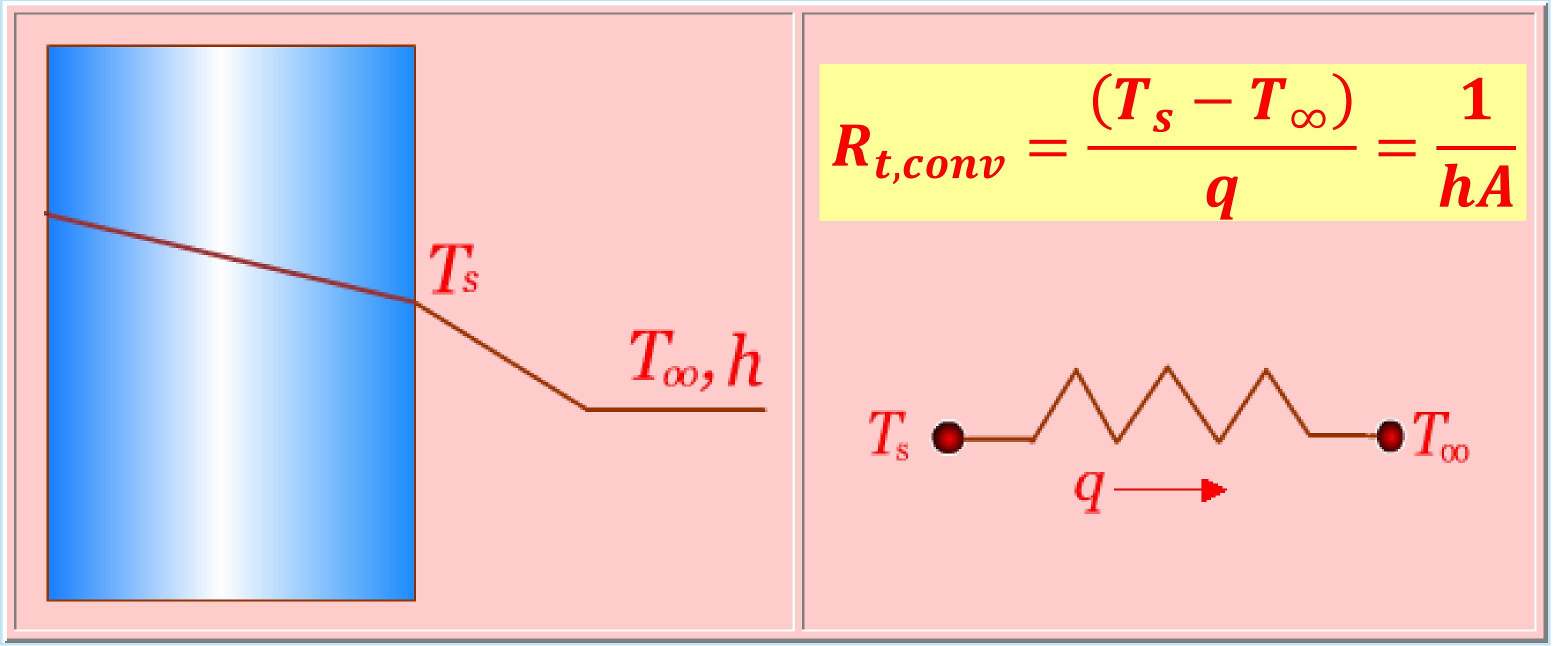


Analogy Between Thermal And Electrical Resistance Concepts

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE PLANE WALL

Thermal Resistance

Consider convection heat transfer from a solid surface of area A and temperature T_s to a fluid whose temperature sufficiently far from the surface is T_∞, with a convection heat transfer coefficient h. Newton's law of cooling for convection heat transfer rate

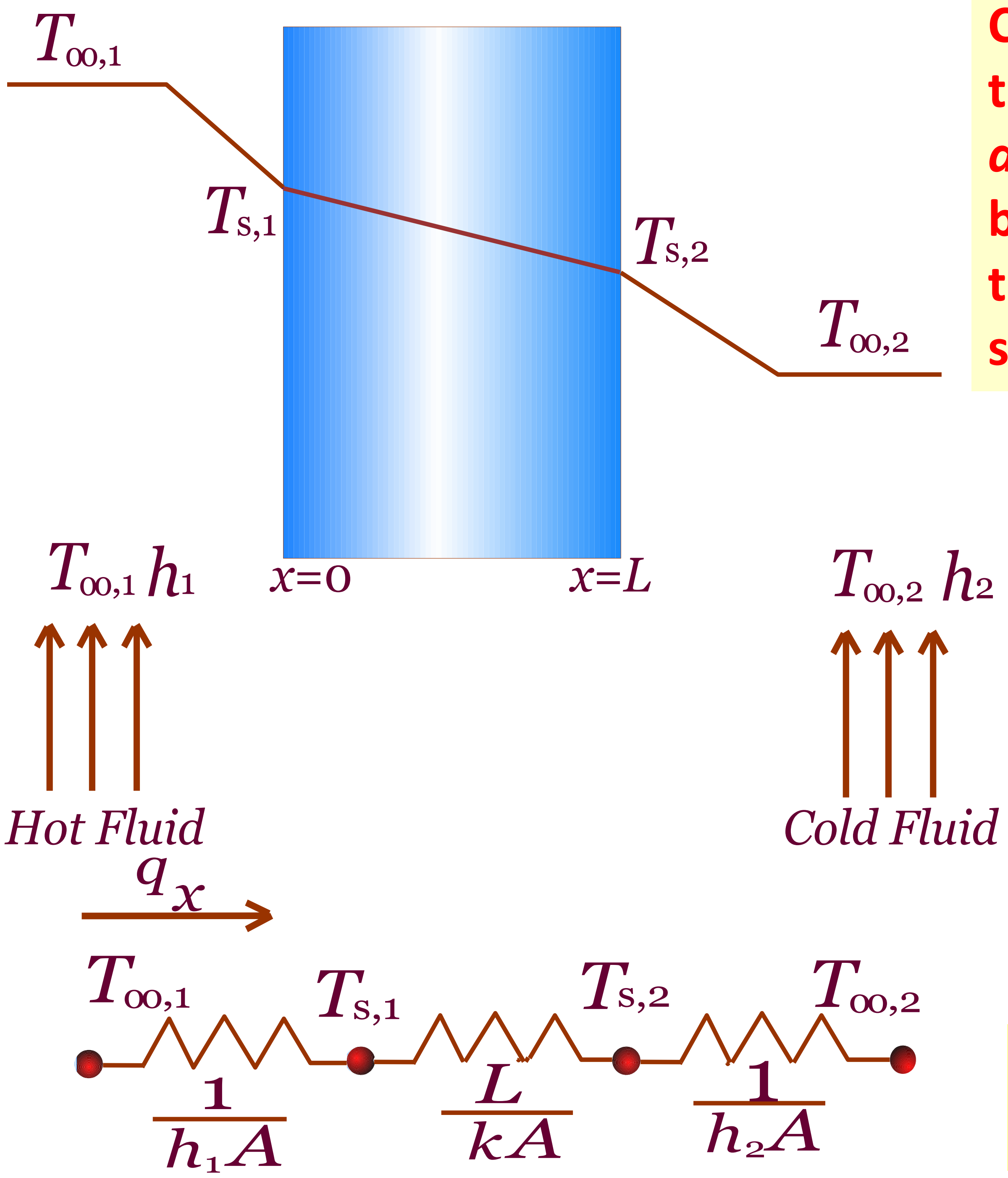


Schematic For Convection Resistance at a Surface

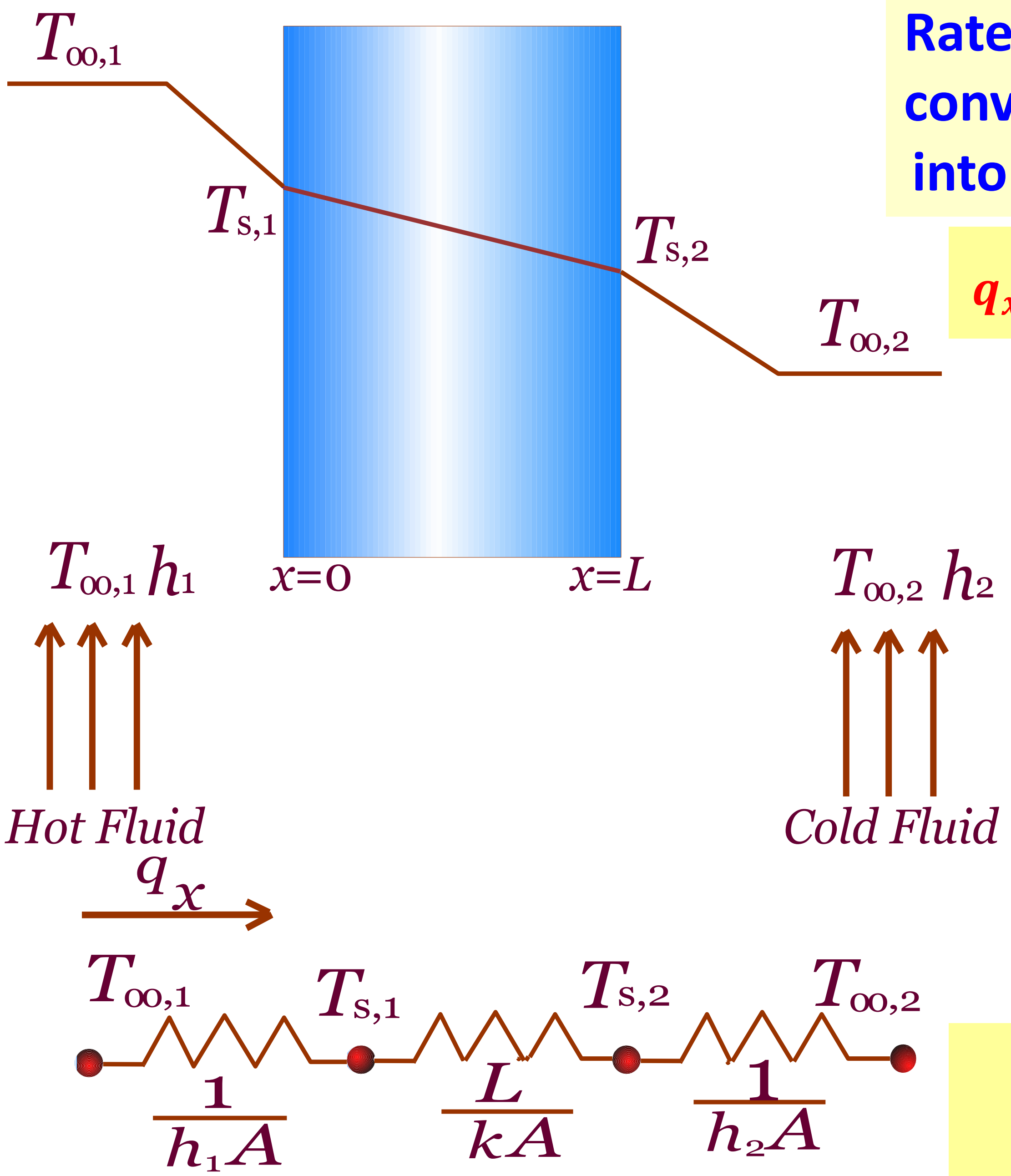
$$q = hA(T_s - T_\infty)$$

$$R_{t,conv} = \frac{(T_s - T_\infty)}{q} = \frac{1}{hA}$$

Consider steady one-dimensional heat flow through a plane wall of thermal conductivity k *and* thickness L that is exposed to convection on both sides to fluids at temperatures and with heat transfer coefficients h_1 and h_2 respectively as shown in Figure.



Thermal Resistance Network For Heat Transfer Through A Plane Wall



Under steady state conditions, we have

Rate of heat convection into the wall = **Rate of heat conduction through the wall** = **Rate of heat convection from the wall**

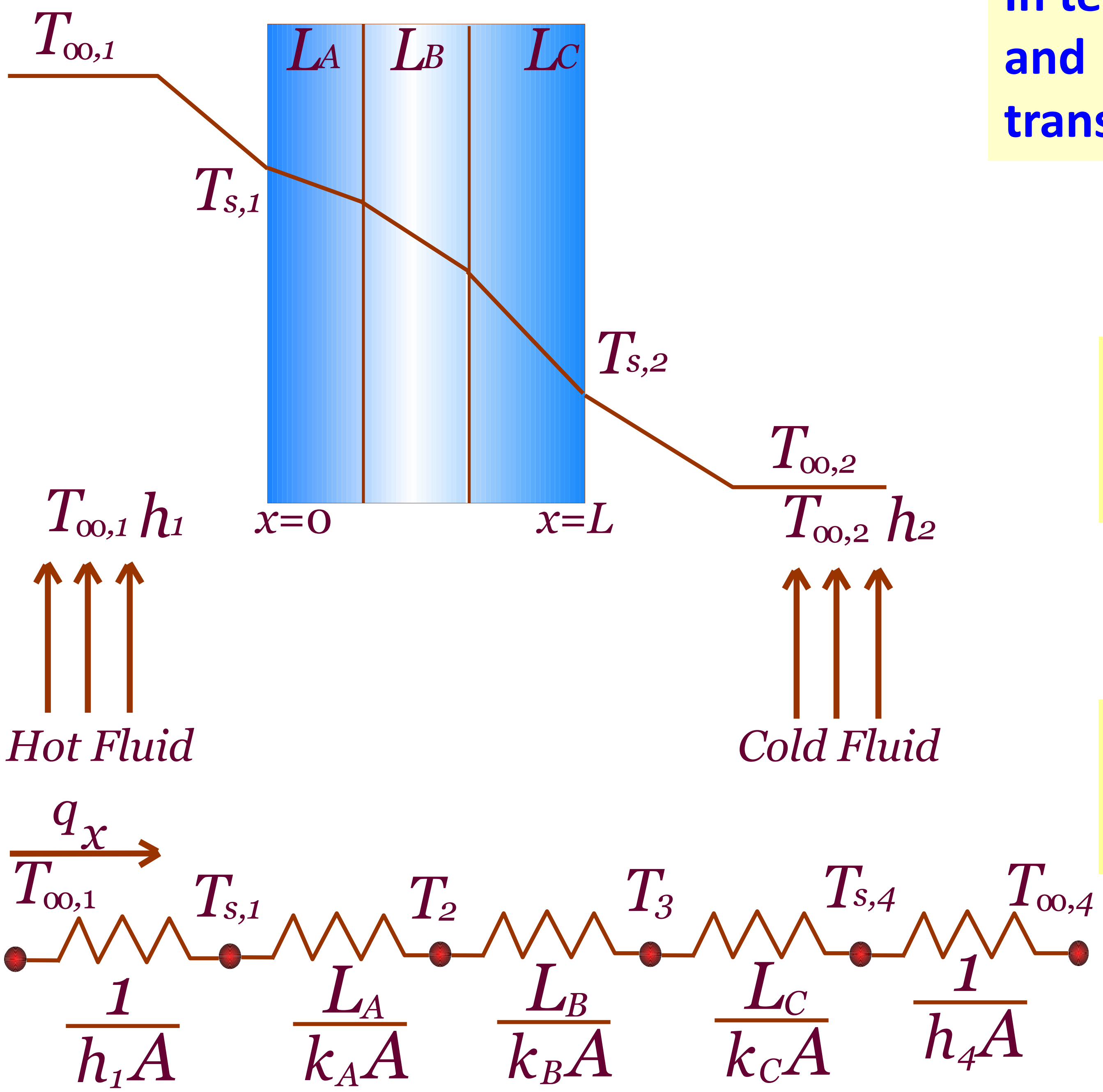
$$q_x = h_1 A\left(T_{\infty,1} - T_{s,1}\right) = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = h_2 A\left(T_{s,2} - T_{\infty,2}\right)$$

$$q_x = \frac{\left(T_{\infty,1} - T_{s,1}\right)}{\frac{1}{h_1 A}} = \frac{\left(T_{s,1} - T_{s,2}\right)}{\frac{L}{kA}} = \frac{\left(T_{s,2} - T_{\infty,2}\right)}{\frac{1}{h_2 A}}$$

In terms of the overall temperature difference, and the total thermal resistance, the heat transfer rate may also be expressed as

$$q_x = \frac{\left(T_{\infty,1} - T_{\infty,2}\right)}{\sum R_t}$$

Thermal Resistance Network For Heat Transfer Through A Plane Wall



In terms of the overall temperature difference, and the total thermal resistance, the heat transfer rate may also be expressed as

$$q_x = \frac{(T_{\infty,1} - T_{\infty,4})}{\sum R_t}$$

$$q_x = \frac{(T_{\infty,1} - T_{\infty,4})}{\frac{1}{h_1A} + \frac{L_A}{k_AA} + \frac{L_B}{k_BA} + \frac{L_C}{k_CA} + \frac{1}{h_4A}}$$

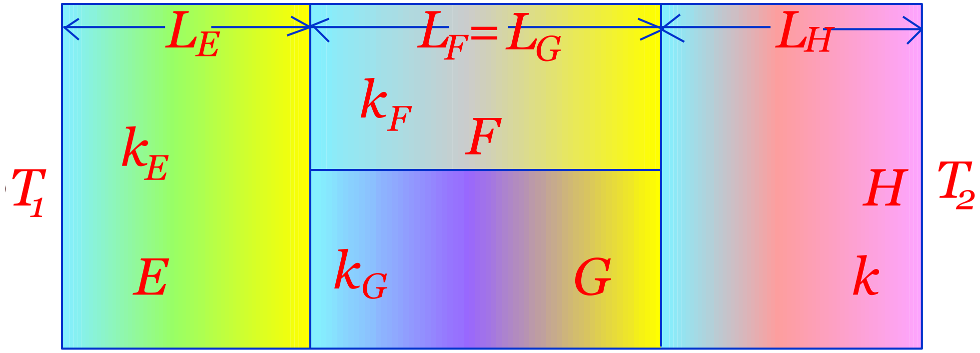
$$q_x = UA(T_{\infty,1} - T_{\infty,4}) = \frac{(T_{\infty,1} - T_{\infty,4})}{\sum R_t}$$

$$U = \frac{1}{A\sum R_t} = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}}$$

U – overall heat transfer coefficient

$$q_x = UA\Delta T$$

$$UA = \frac{1}{\sum R_t} = \frac{q_x}{\Delta T}$$

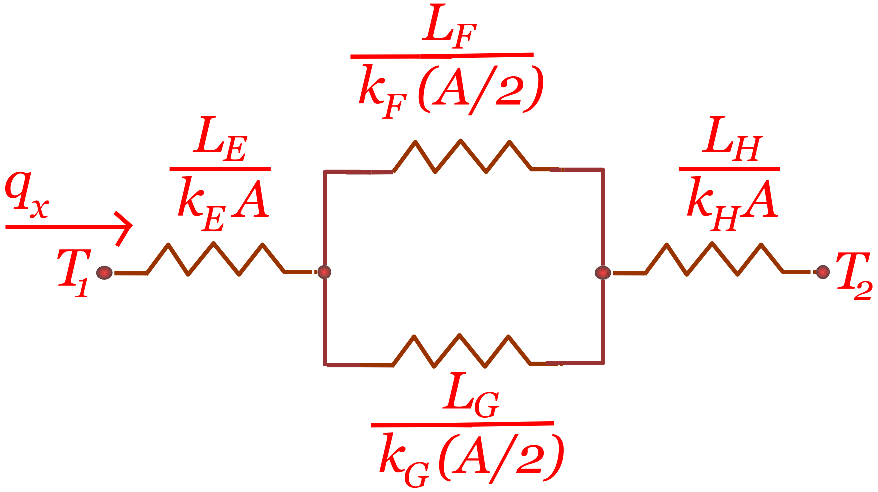


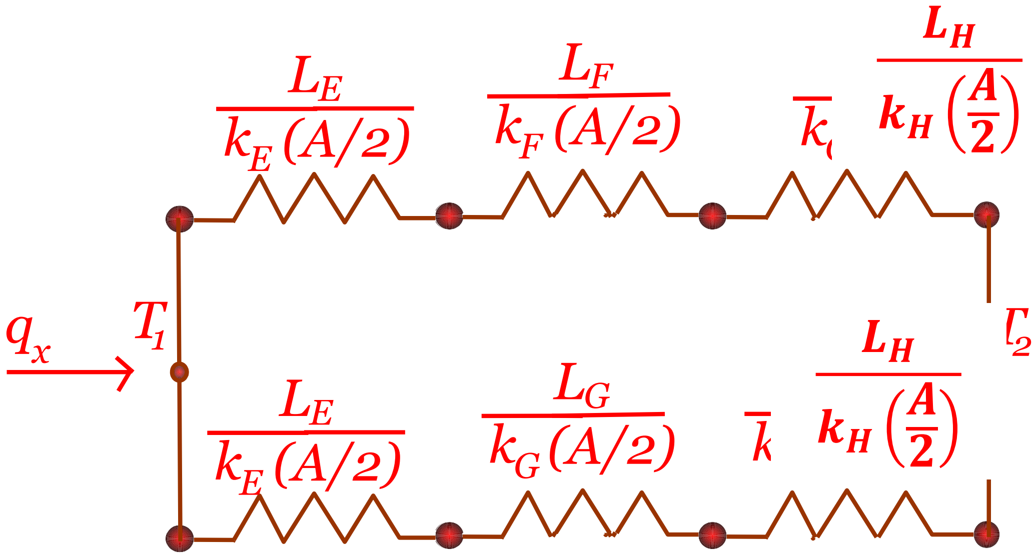
Composite walls may also be characterized by series-parallel configurations

$$q_x = \frac{(T_1 - T_4)}{\sum R_t}$$

$$q_x = q_{x,1} + q_{x,2}$$

$$q_x = \frac{(T_1 - T_4)}{\frac{L_E}{k_E\left(\frac{A}{2}\right)} + \frac{L_F}{k_F\left(\frac{A}{2}\right)} + \frac{L_H}{k_H\left(\frac{A}{2}\right)}} + \frac{(T_1 - T_4)}{\frac{L_E}{k_E\left(\frac{A}{2}\right)} + \frac{L_F}{k_G\left(\frac{A}{2}\right)} + \frac{L_H}{k_H\left(\frac{A}{2}\right)}}$$

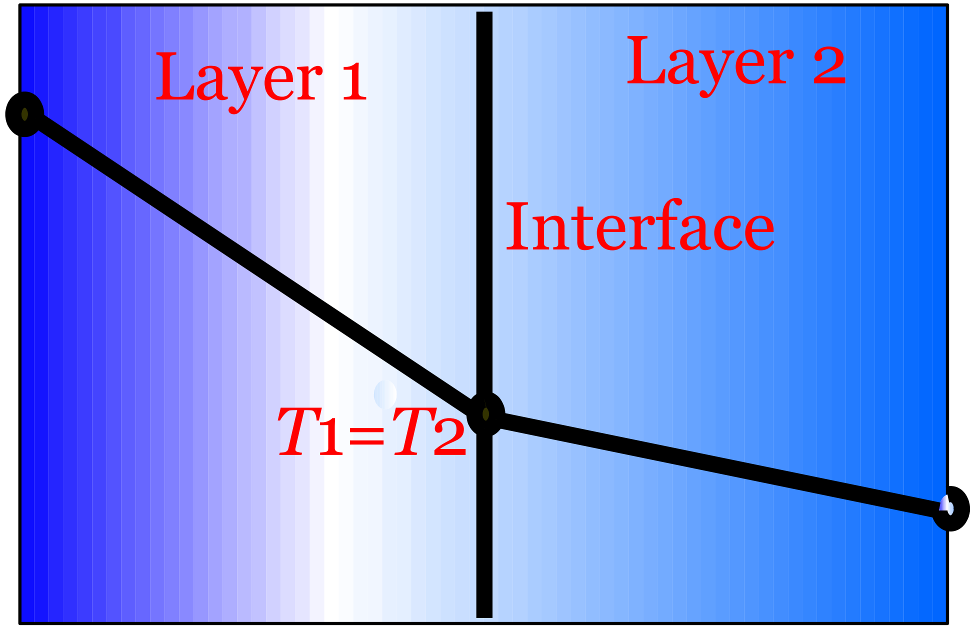




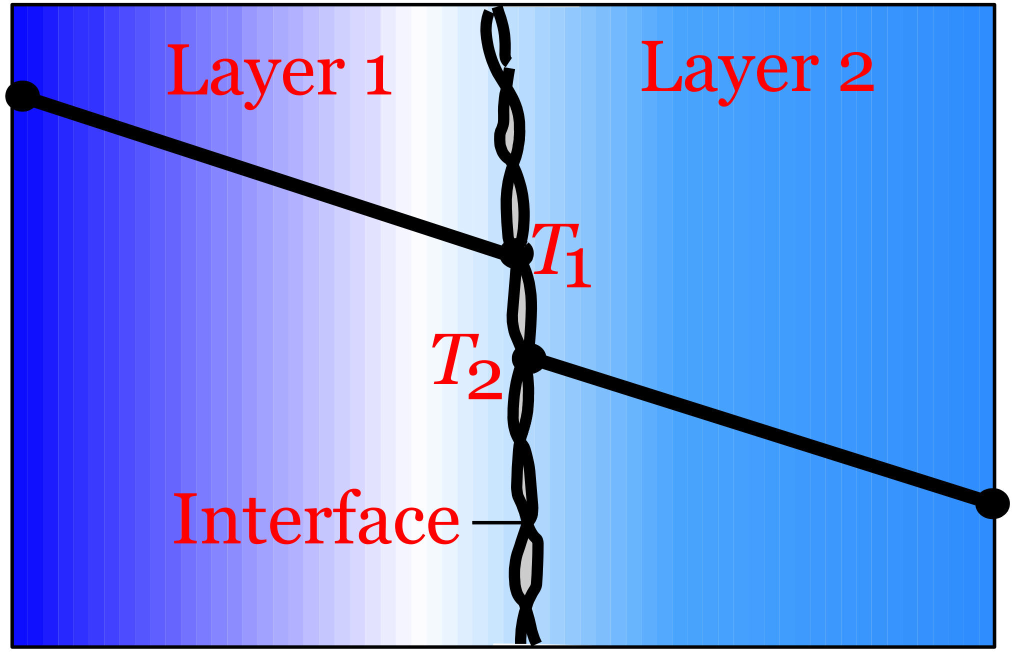
Equivalent Thermal Circuits For A Series-Parallel Composite Wall

CONTACT THERMAL RESISTANCE

In heat conduction analysis through composite walls, we have assumed "perfect contact" at the interface of two layers, and thus no temperature drop at the interface.
This would be the case when the surfaces are perfectly smooth and they produce a perfect contact at each point.
In reality, however, even flat surfaces that appear smooth to the eye turn out to be rather rough when examined under a microscope, as shown in Figure, with numerous peaks and valleys.
That is, a surface is microscopically rough no matter how smooth it appears to be.

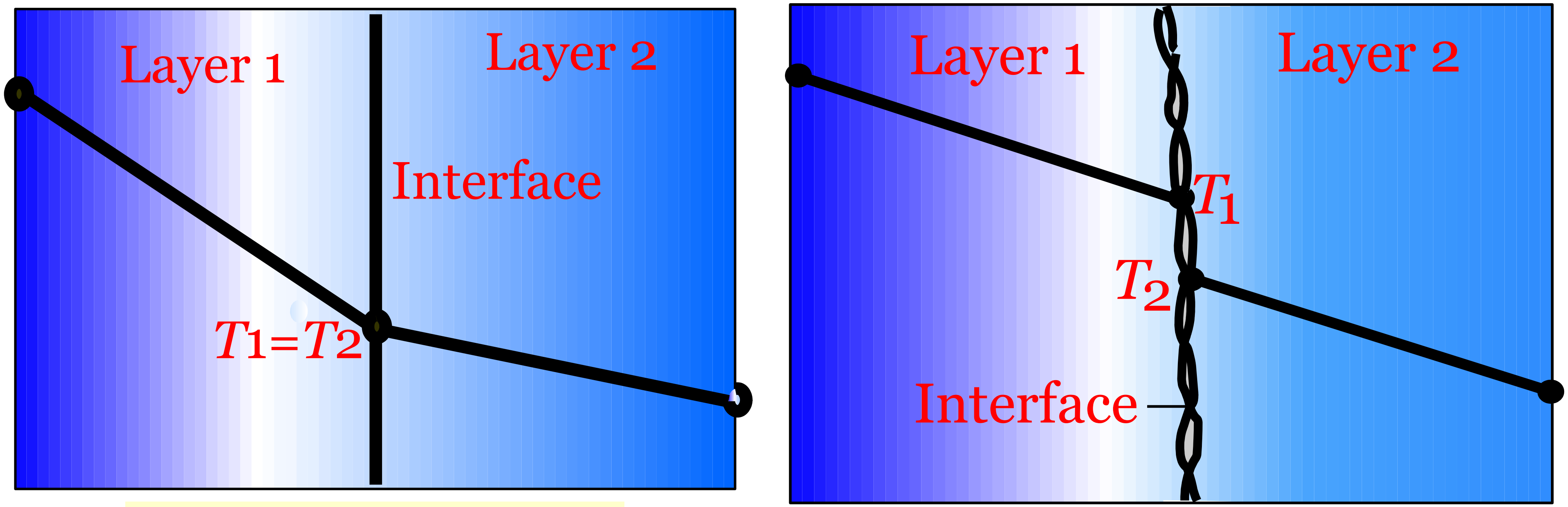


Ideal Thermal Contact



Actual Thermal Contact

Temperature Distribution Along Two Solid Plates Pressed Against Each Other



Ideal Thermal Contact

Actual Thermal Contact

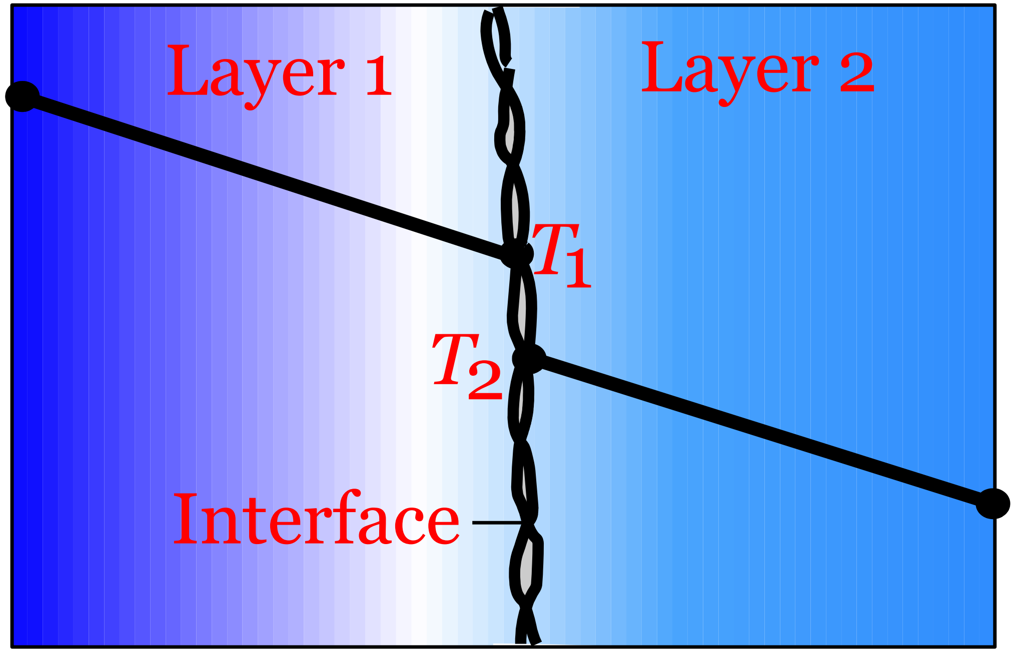
Temperature Distribution Along Two Solid Plates Pressed Against Each Other

When two such surfaces are pressed against each other, the peaks will form good material contact but the valleys will form voids filled with air.

As a result, an interface will contain numerous air gaps of varying sizes that act as insulation because of the low thermal conductivity of air.

Thus, an interface offers some resistance to heat transfer, and this resistance per unit interface area is called thermal contact resistance, $R''_{t,c}$ given by

$$R''_{t,c} = \frac{T_1 - T_2}{q''_x}$$



For solids whose thermal conductivities exceed that of the interfacial fluid, the contact resistance may be reduced by increasing the area of the contact spots.

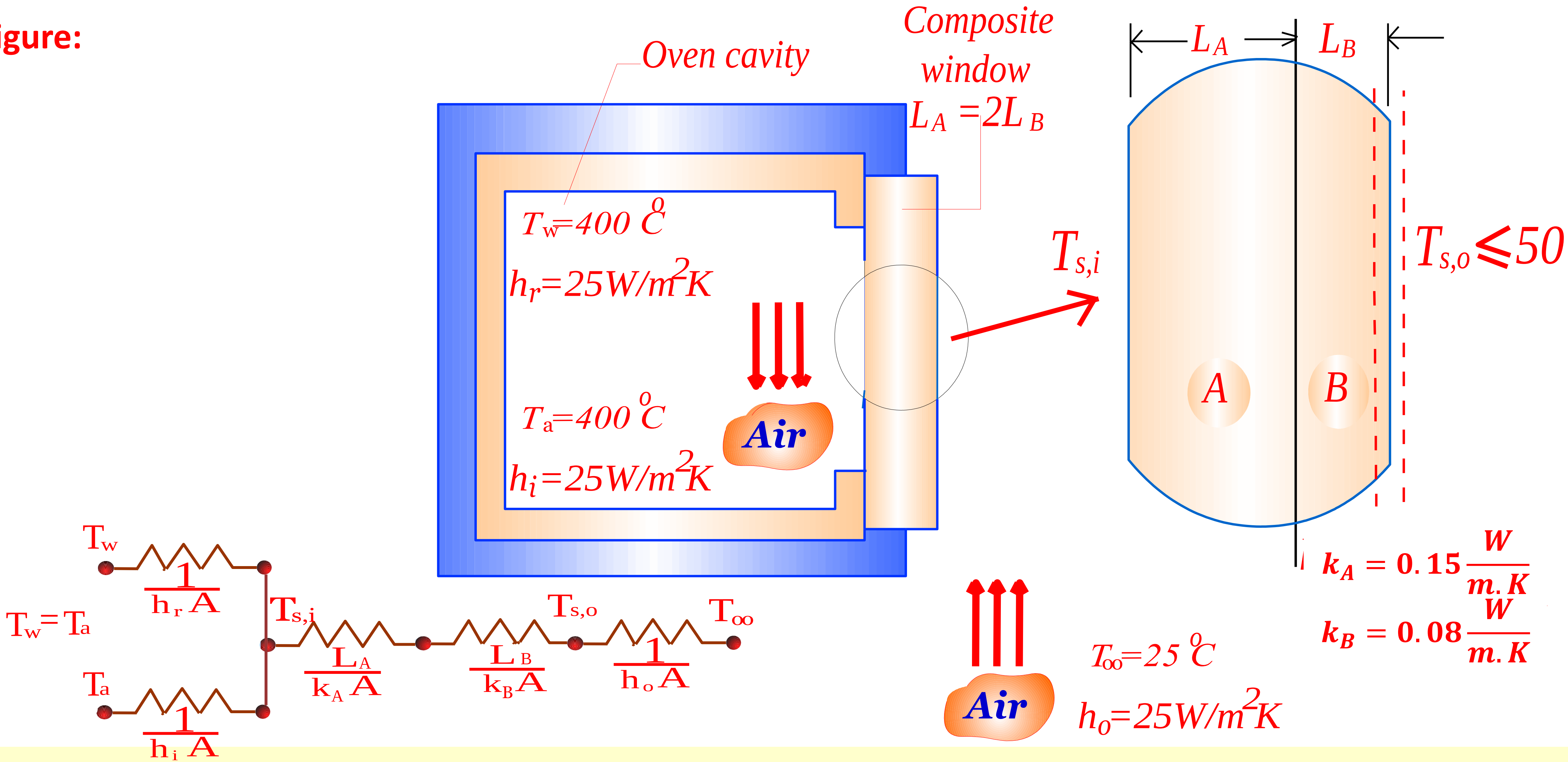
Such an increase may be effected by increasing the joint pressure and/or by reducing the roughness of the mating surfaces.

The contact resistance may also be reduced by selecting an interfacial fluid of large thermal conductivity.

In this respect, no fluid (an evacuated interface) eliminates conduction across the gap, thereby increasing the contact resistance.

A leading manufacturer of household appliances is proposing a self-cleaning oven design that involves use of a composite window separating the oven cavity from the room air. The composite is to consist of two high temperature plastics (A and B) of thicknesses $L_A = 2L_B$ and thermal conductivities $k_A = 0.15\,\frac{W}{m.K}$ and $k_B = 0.08\,\frac{W}{m.K}$. During the self-cleaning process, the oven wall and air temperatures, $T_w = T_a = 400\ ℃$, while the room air temperature is 25 ℃. The inside convection and radiation heat transfer coefficients h_i and h_r, as well as the outside convection coefficient h_o, are each approximately 25 $W/m^2.K$. What is the minimum window thickness, $L = L_A + L_B$, needed to ensure a temperature that is 50 ℃ or less at the outer surface of the window? This temperature must not be exceeded for safety reasons

Figure:



Known: The properties and relative dimensions of plastic materials used for a composite oven window, and conditions associated with self-cleaning operation

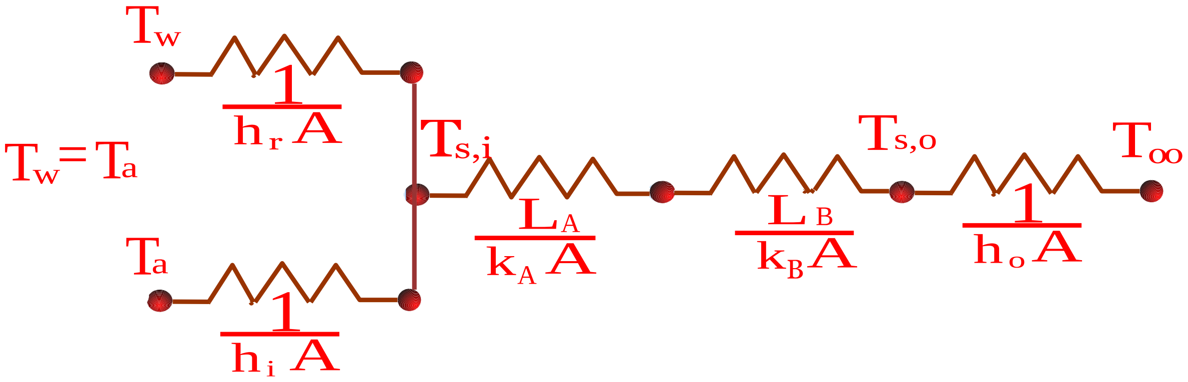
Find : Composite thickness $L_A + L_B$ needed to ensure safe operation

Assumptions:

- Steady state conditions exist
- Conduction through the window is one dimensional
- Contact resistance is negligible
- Radiation absorption within the window is negligible; hence no internal heat generation
- Radiation exchange between window outer surface and surroundings is negligible
- Each plastic is homogeneous with constant properties

Analysis:

The thermal circuit can be constructed by recognizing that resistance to heat flow is associated with convection at the outer surface, conduction in the plastics, and convection and radiation at the inner surface. Accordingly, the circuit and the resistances are of the following form:

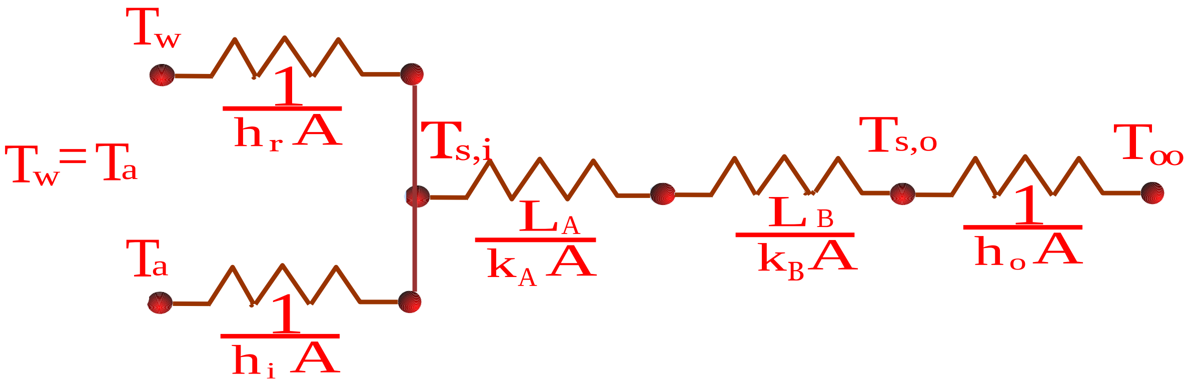


Since the outer surface temperature of the window, $T_{s,o}$ is prescribed, the required window thickness may be obtained by applying an energy balance at this surface.

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

$$\dot{E}_{in} = \dot{E}_{out}$$

$$\frac{T_a - T_{s,o}}{\sum R_t} = h_o A\left(T_{s,o} - T_\infty\right)$$



The total thermal resistance between the oven cavity and the outer surface of the window includes an effective resistance associated with convection and radiation, which act in parallel at the inner surface of the window, and the conduction resistances of the window materials. Hence,

$$\sum R_t = \frac{1}{h_i A + h_r A} + \frac{L_A}{k_A A} + + \frac{L_B}{k_B A}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{\frac{1}{h_i A}} + \frac{1}{\frac{1}{h_r A}}$$

$$\frac{T_a - T_{s,o}}{\frac{1}{h_i A + h_r A} + \frac{L_A}{k_A A} + + \frac{L_B}{k_B A}} = h_o A\left(T_{s,o} - T_\infty\right)$$

$$\frac{1}{R} = h_i A + h_r A$$

$$R = \frac{1}{h_i A + h_r A}$$

$$\frac{T_a - T_{s,o}}{\frac{1}{h_iA + h_rA} + \frac{L_A}{k_AA} + \frac{L_B}{k_BA}} = h_oA\left(T_{s,o} - T_\infty\right)$$

$$\frac{T_a - T_{s,o}}{\frac{1}{h_i + h_r} + \frac{L_A}{k_A} + \frac{L_B}{k_B}} = h_o\left(T_{s,o} - T_\infty\right)$$

$$L_A = 2L_B$$

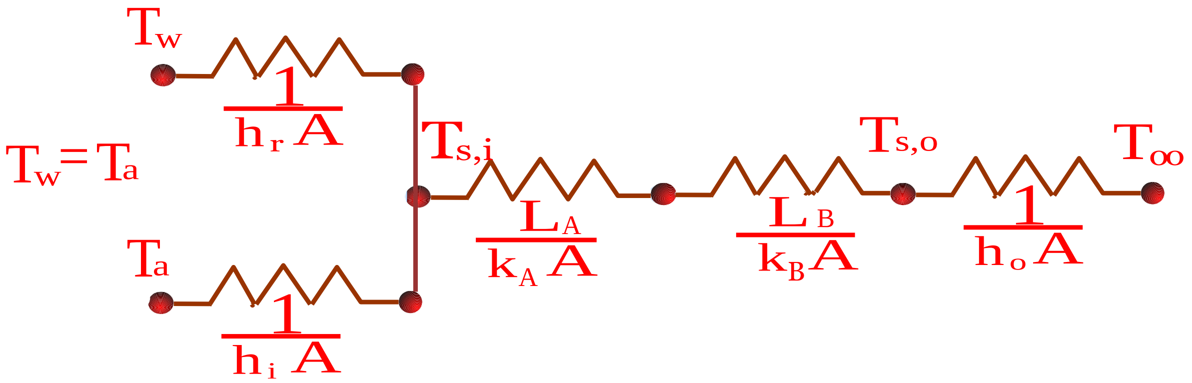
$$\frac{T_a - T_{s,o}}{\frac{1}{h_i + h_r} + \frac{L_A}{k_A} + \frac{L_A}{2k_B}} = h_o\left(T_{s,o} - T_\infty\right)$$

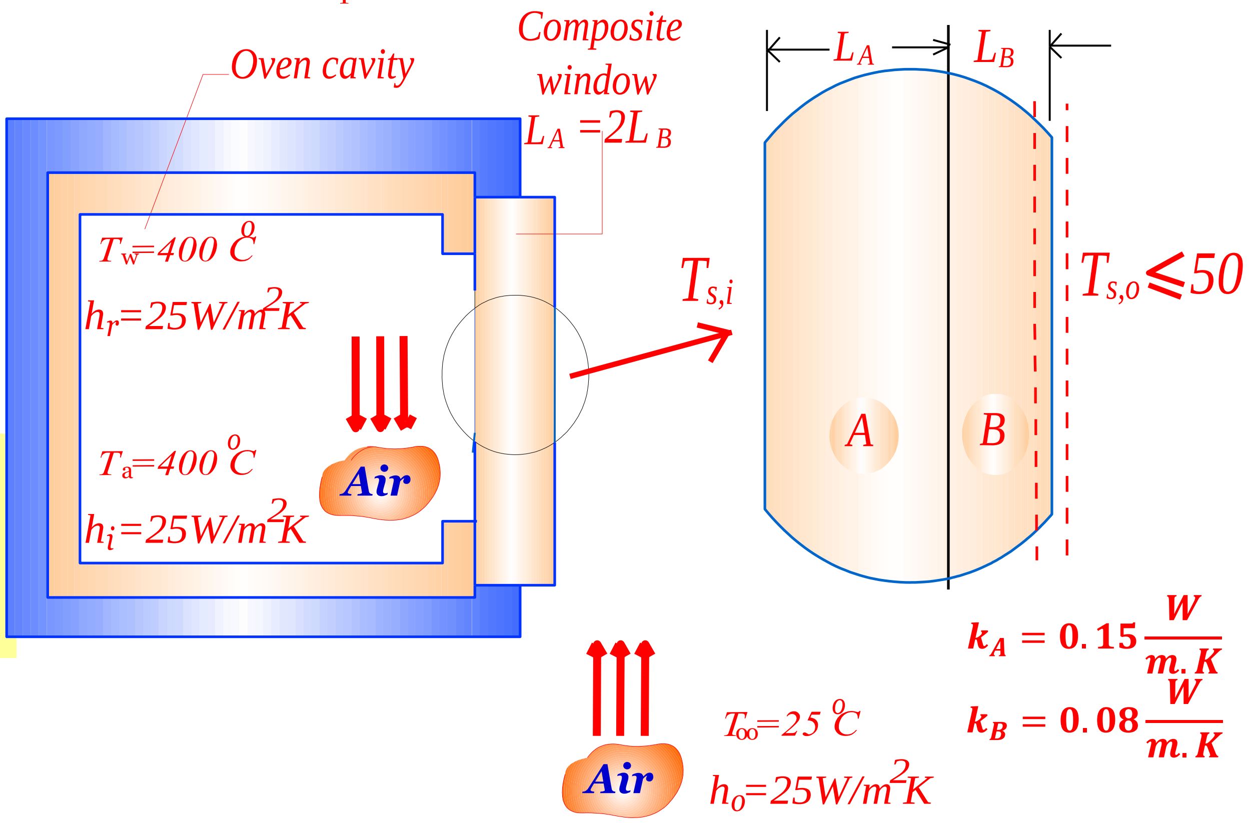
$$\frac{400 - 50}{\frac{1}{25 + 25} + \frac{L_A}{0.15} + \frac{L_A}{2(0.08)}} = 25(50 - 25)$$

$$\frac{400 - 50}{25(50 - 25)} = \frac{1}{50} + \frac{L_A}{0.15} + \frac{L_A}{0.16}$$

$$0.54 = L_A\left(\frac{1}{0.15} + \frac{1}{0.16}\right)$$

$$L_A = 0.0418m$$





$$L_A = 0.0418\ m \qquad L_A = 2L_B$$

$$L_B = 0.0209\ m$$

$$L = L_A + L_B = 0.0418 + 0.0209 = 0.0627\ m$$

$$L = 62.7\ mm$$

Comments:

1. The self cleaning operation is a transient process, as far as the thermal response of the window is concerned, and steady state conditions may not be reached in the time required for cleaning. However, the steady state condition provides the maximum possible value of $T_{s,o}$ and hence is well suited for the design calculation.

2. Radiation exchange between the oven walls and the composite window actually depends on the inner surface temperature $T_{s,i}$, and although it has been neglected, there is radiation exchange between the window and the surroundings, which depends on $T_{s,o}$.

A more complete analysis may be made to concurrently determine $T_{s,i}$ and $T_{s,o}$. Approximating the oven cavity as a large enclosure relative to the window and applying an energy balance, at the inner surface it follows that

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

$$q''_{rad,i} + q''_{conv,i} = q''_{cond}$$

$$\varepsilon\sigma\left(T_{w,i}^4 - T_{s,i}^4\right) + h_i\left(T_a - T_{s,i}\right) = \frac{T_{s,i} - T_{s,o}}{\dfrac{L_A}{k_A} + \dfrac{L_B}{k_B}}$$

Approximating the kitchen walls as a large isothermal enclosure relative to the window, with $T_{w,o} = T_\infty$ and this time applying energy balance at the outer surface, it follows that

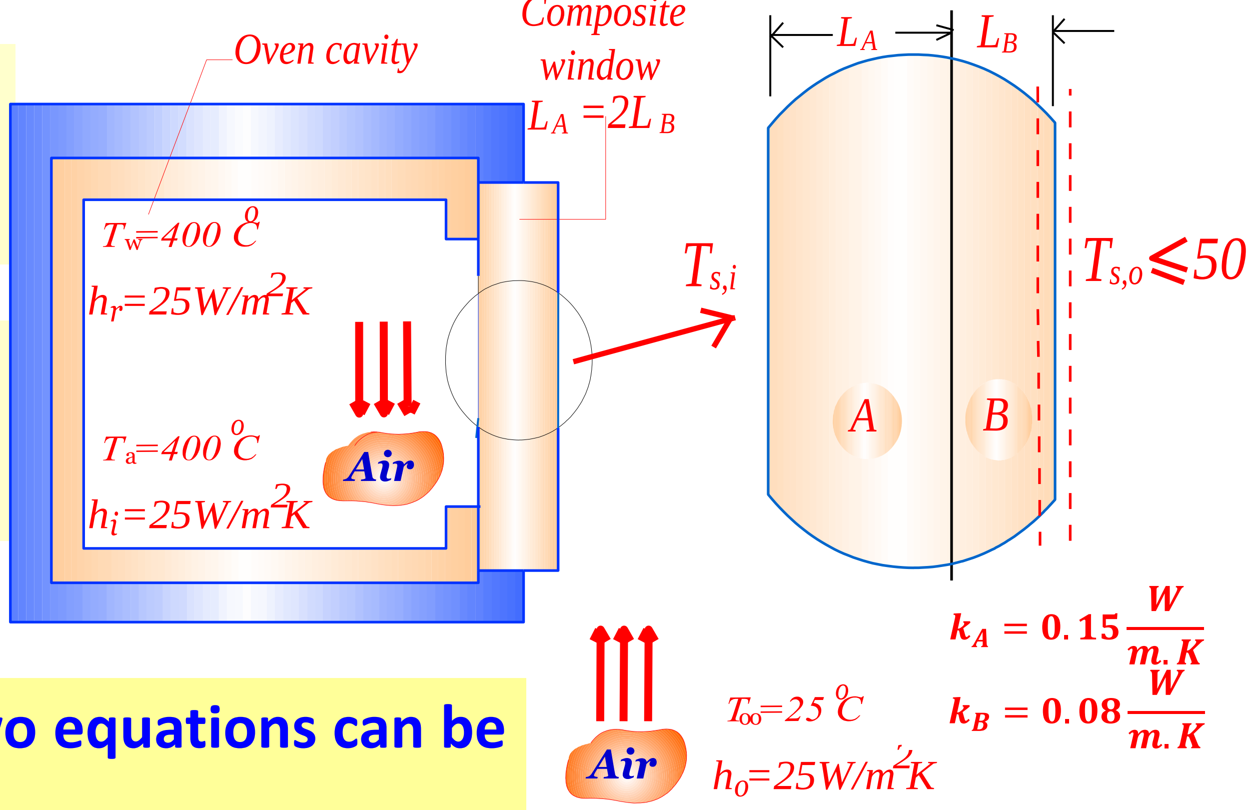
$$q''_{rad,o} + q''_{conv,o} = q''_{cond}$$

$$\varepsilon\sigma\left(T_{s,o}^4 - T_{w,o}^4\right) + h_o\left(T_{s,o} - T_\infty\right) = \frac{T_{s,i} - T_{s,o}}{\dfrac{L_A}{k_A} + \dfrac{L_B}{k_B}}$$

$$\varepsilon\sigma\left(T_{w,i}^4 - T_{s,i}^4\right) + h_i\left(T_a - T_{s,i}\right) = \frac{T_{s,i} - T_{s,o}}{\frac{L_A}{k_A} + \frac{L_B}{k_B}}$$

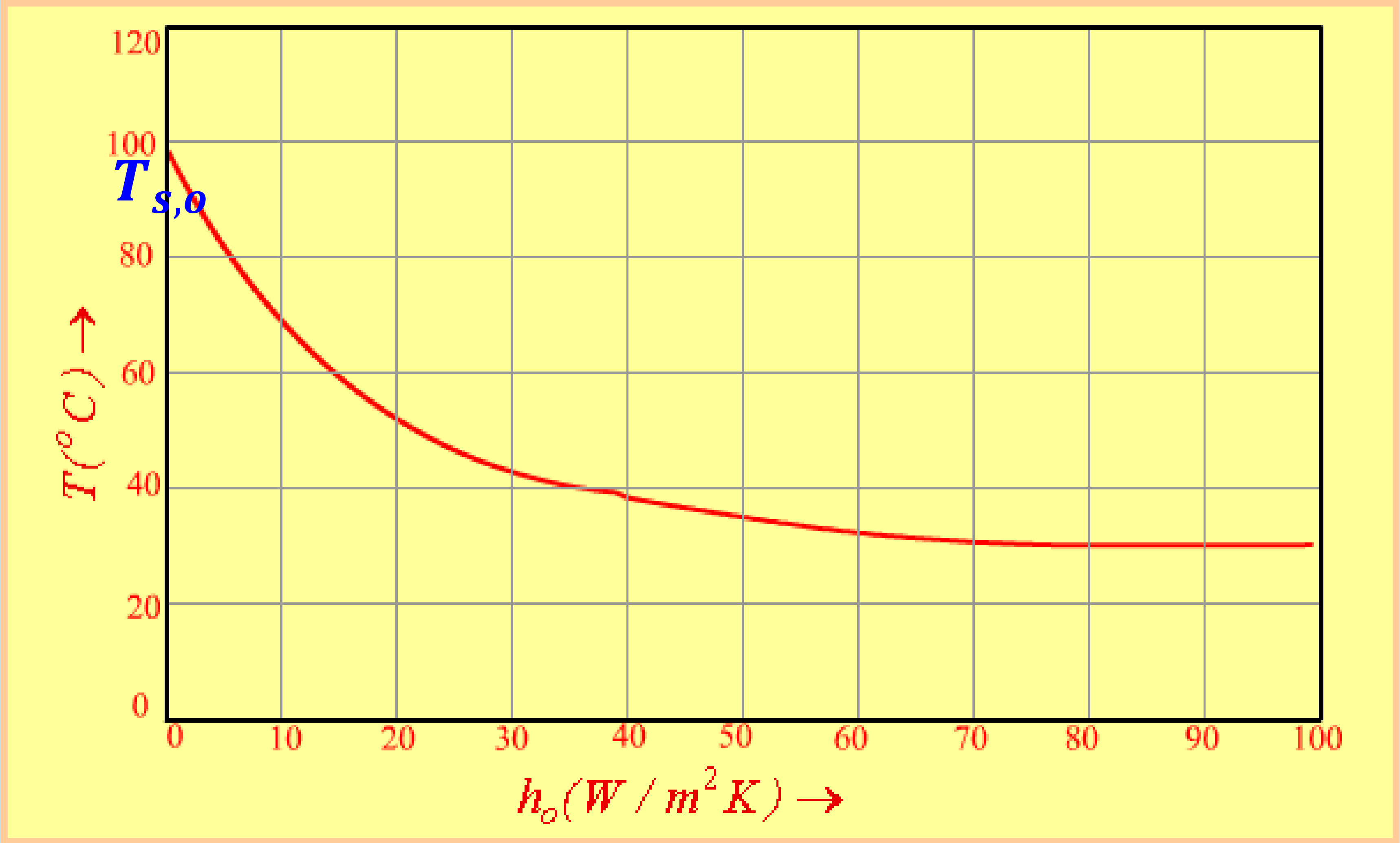
$$\varepsilon\sigma\left(T_{s,o}^4 - T_{w,o}^4\right) + h_o\left(T_{s,o} - T_\infty\right) = \frac{T_{s,i} - T_{s,o}}{\frac{L_A}{k_A} + \frac{L_B}{k_B}}$$

$T_{w,o} = T_\infty$ $\varepsilon = 0.9$



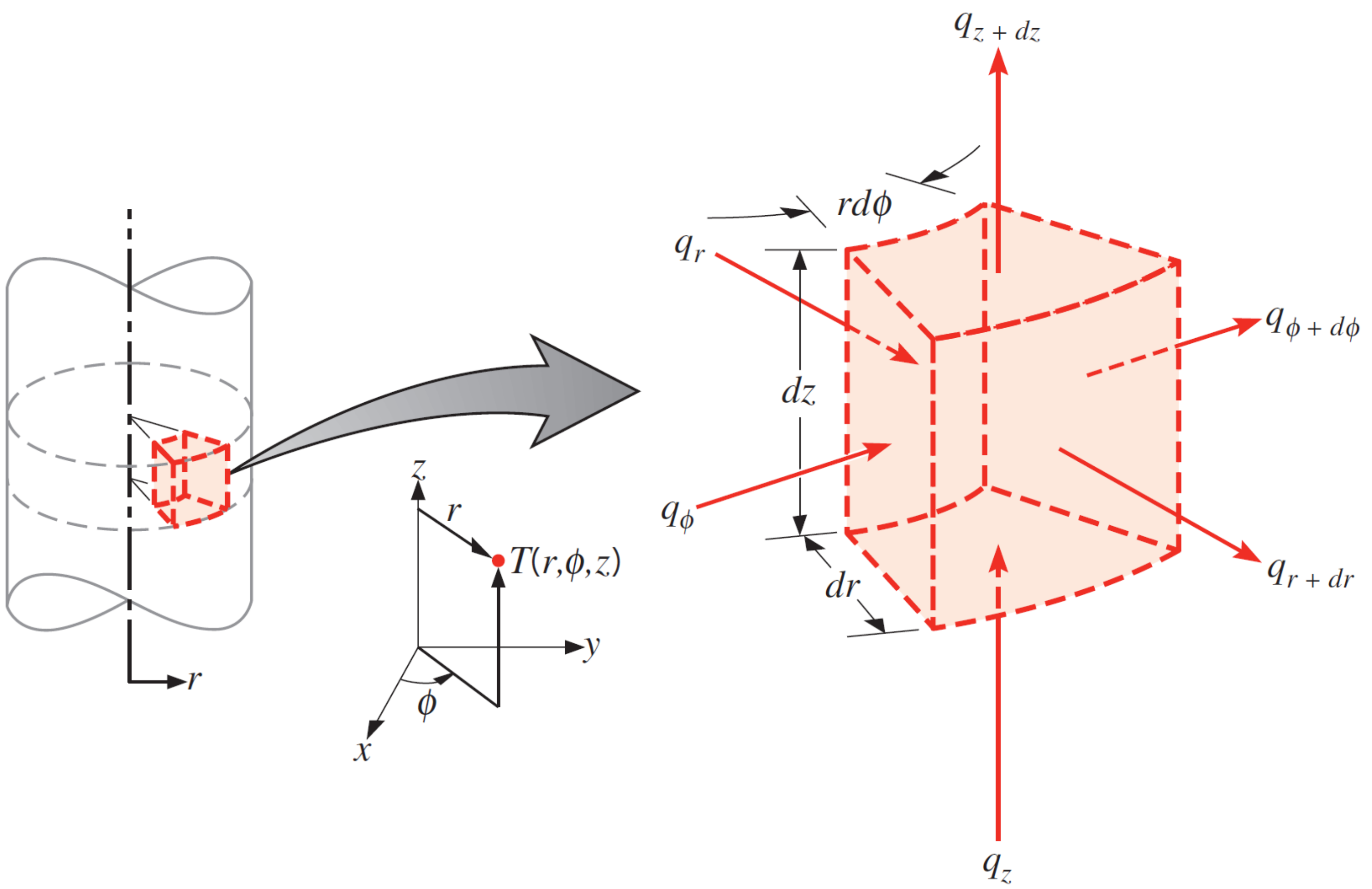
As all other quantities are known, the above two equations can be $T_{s,i}$ and $T_{s,o}$.

We wish to explore the effect on $T_{s,o}$ of varying velocity, and hence the convection coefficient, associated with airflow over the outer surface. With $\varepsilon = 0.9$ and all other conditions remaining the same, the above equations have been solved for values of h_o in the range 0 - 100 $W/m^2\ K$ and the results are represented graphically.



Increasing h_o reduces the corresponding convection resistance, and a value of $h_o = 30\ \frac{W}{m^2K}$ would yield a safe to touch temperature of $T_{s,o} = 43$°C. Further increase in the outer surface heat transfer coefficient would not decrease the temperature. However, beyond $h_o = 70\ \frac{W}{m^2K}$, the temperature of $T_{s,o} = 30$°C would remain almost constant

Heat diffusion equation in cylindrical coordinates



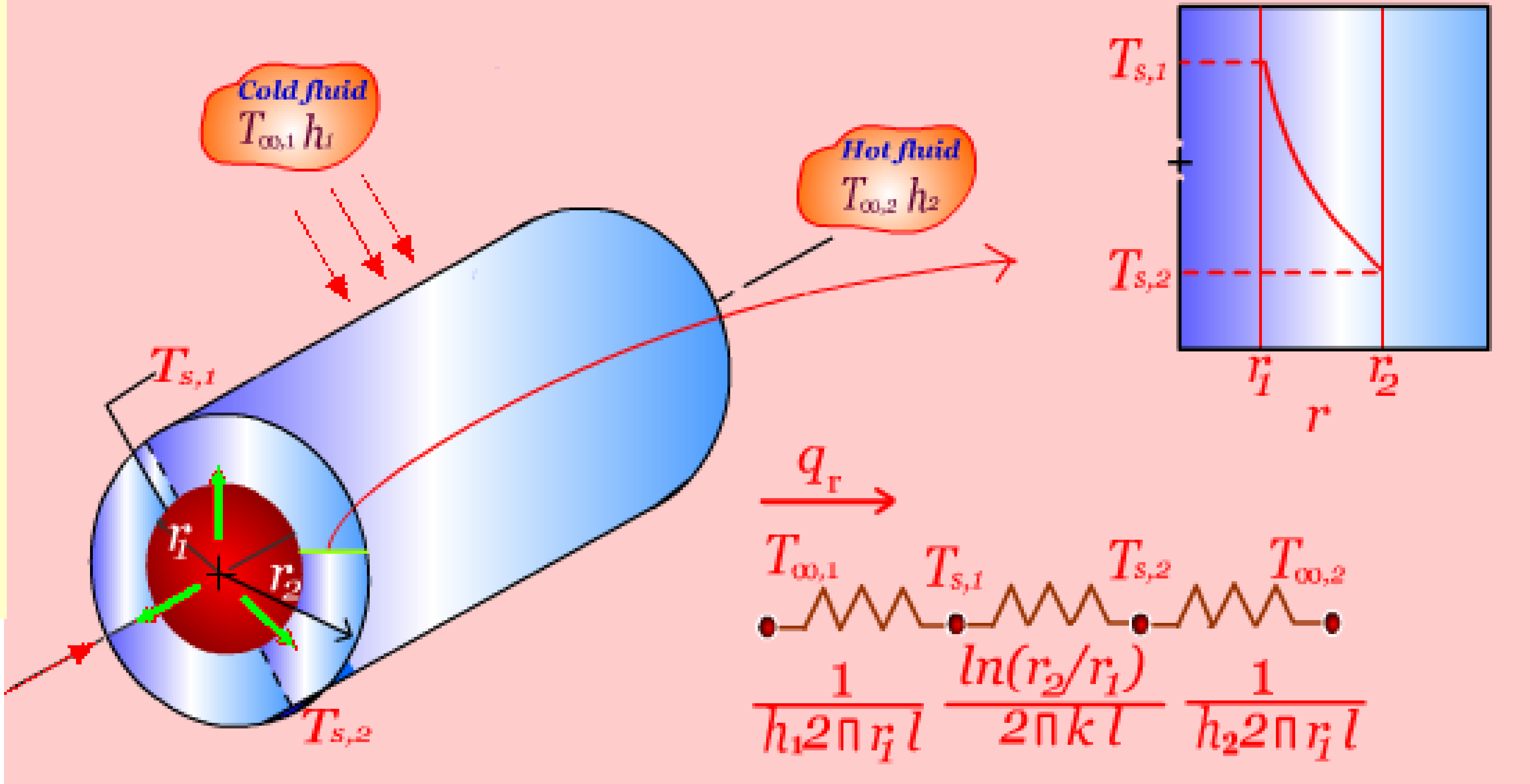
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t}$$

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE CYLINDER

Consider a hollow cylinder, whose inner and outer surfaces are exposed to fluids at different temperatures.

For steady state conditions with no heat generation, the appropriate form of the heat equation

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$



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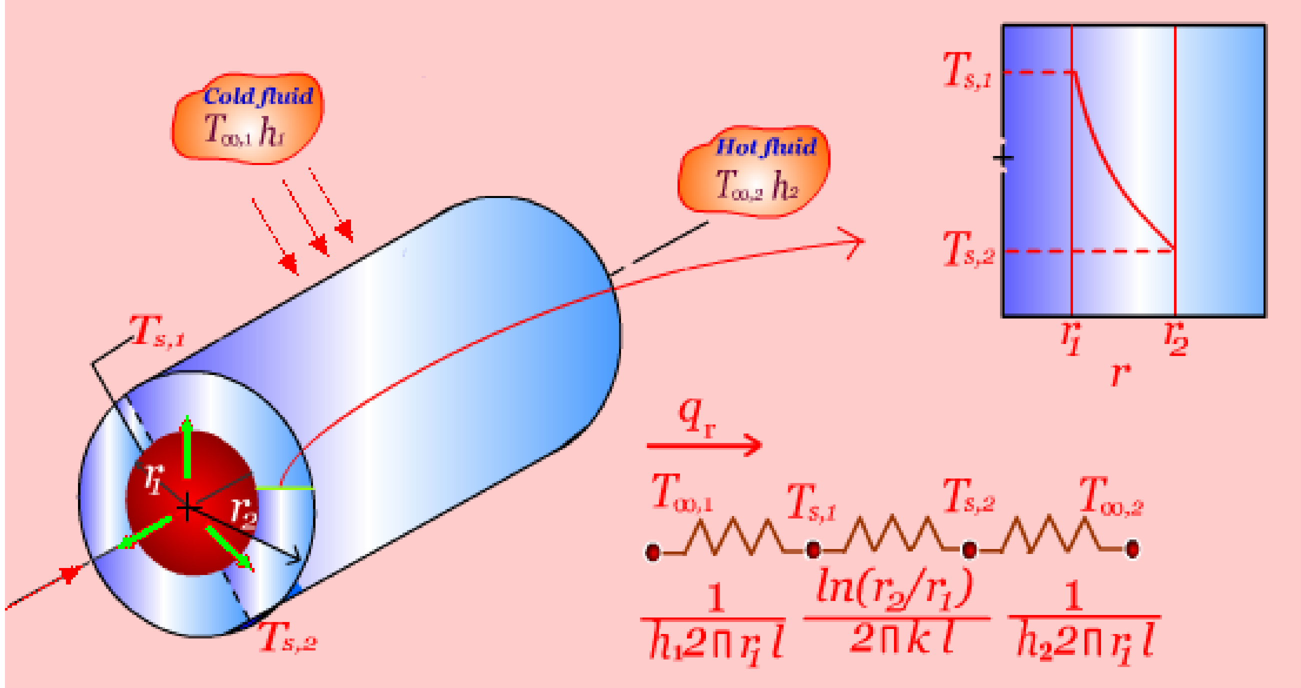


The rate at which energy is conducted across the cylindrical surface in the solid may be expressed as

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

where $A = 2\pi rL$ is the area normal to the direction of heat transfer

Since, the heat equation prescribes that the quantity $kr\frac{dT}{dr}$ is independent of r, it follows from the above equation that the conduction heat transfer rate q_r (not the heat flux q_r'') is a constant in the radial direction

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE CYLINDER

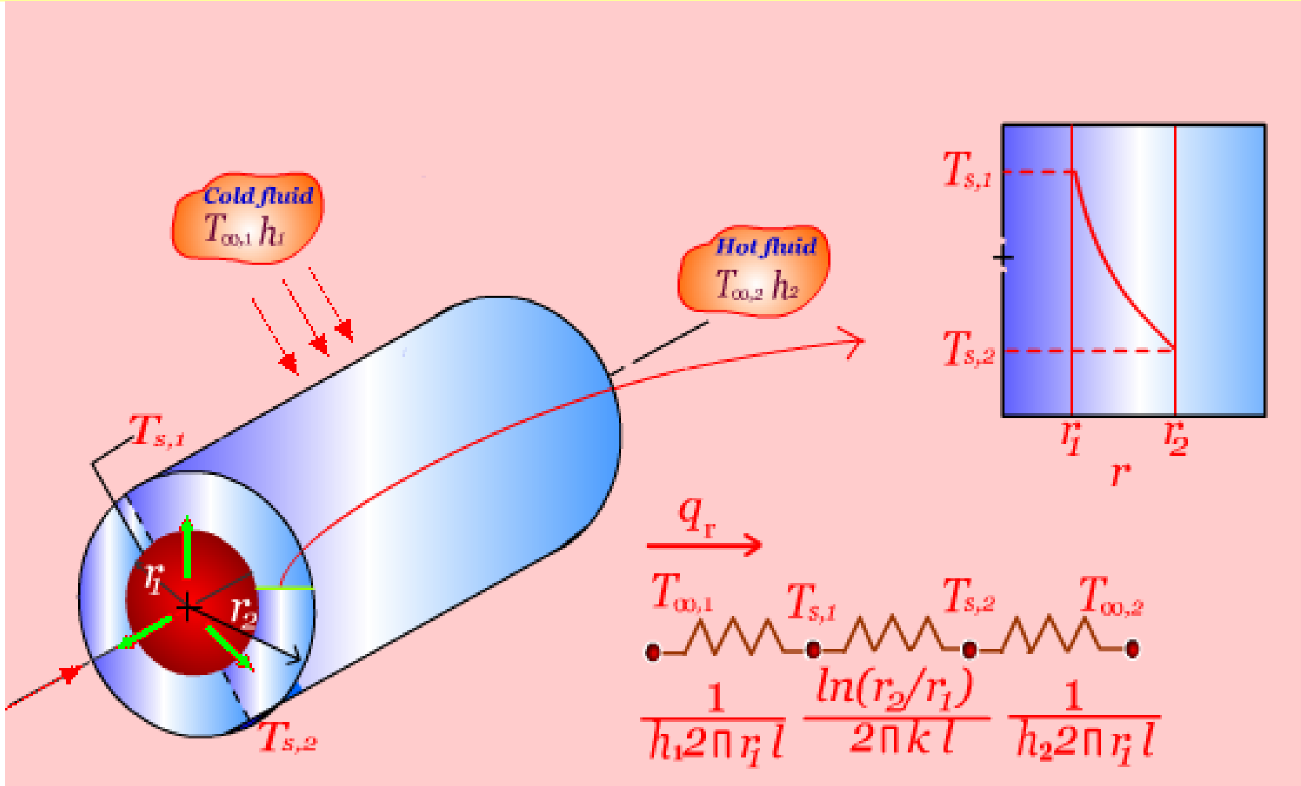
$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

Integrating this eqn twice

$$\left(kr\frac{dT}{dr}\right) = C$$

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T = C_1 lnr + C_2$$



Boundary conditions are

$$r = r_1 \quad T = T_{s,1}$$
$$r = r_2 \quad T = T_{s,2}$$

$$T_{s,1} = C_1 lnr_1 + C_2$$
$$T_{s,2} = C_1 lnr_2 + C_2$$

$$T_{s,1} - T_{s,2} = C_1 lnr_1 - C_1 lnr_2$$

$$T_{s,1} - T_{s,2} = C_1 ln\left(\frac{r_1}{r_2}\right)$$

$$C_1 = \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)}$$

$$T_{s,2} = C_1 lnr_2 + C_2$$

$$T_{s,2} = \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} lnr_2 + C_2$$

$$C_2 = T_{s,2} - \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} lnr_2$$

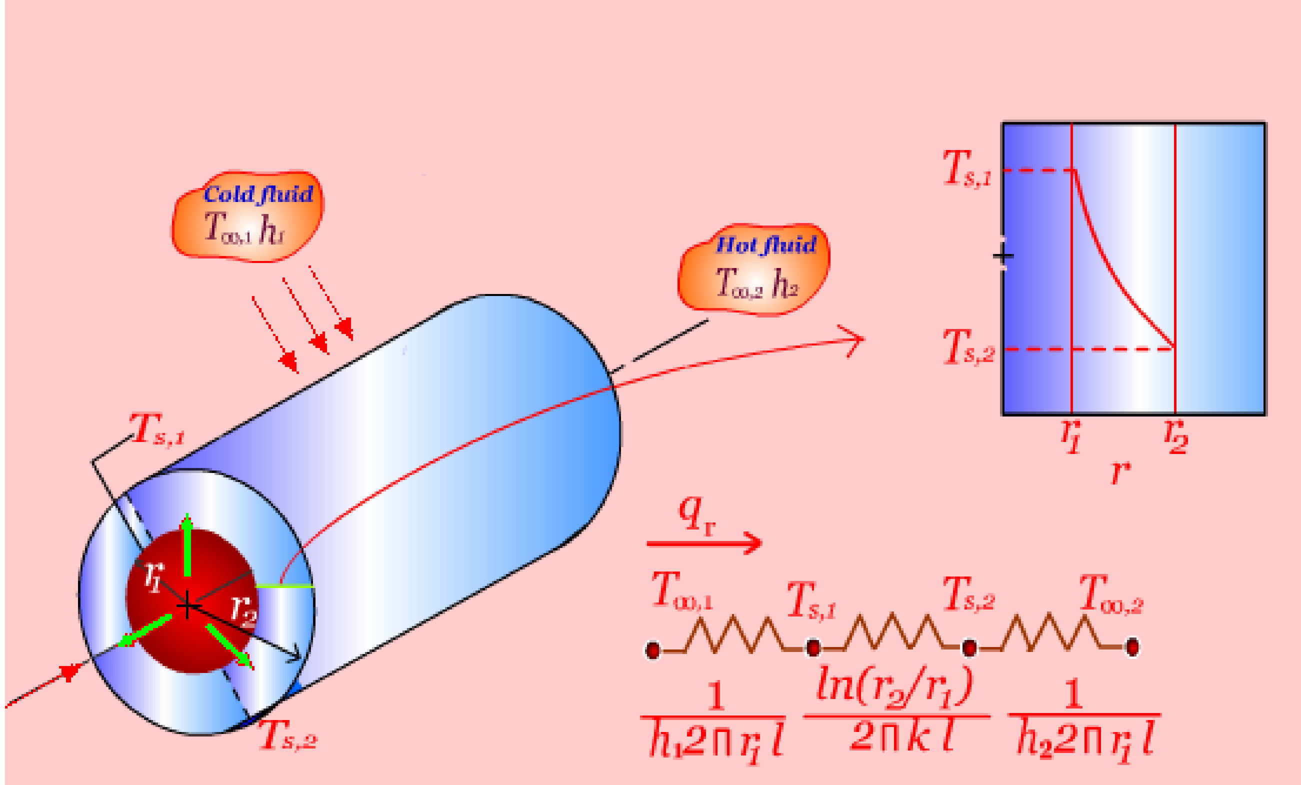
$$T = C_1 lnr + C_2$$

$$C_1 = \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)}$$

$$C_2 = T_{s,2} - \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} lnr_2$$

$$T = \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} lnr + T_{s,2} - \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} lnr_2$$

$$T(r) = \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} ln\left(\frac{r}{r_2}\right) + T_{s,2}$$



The temperature distribution associated with radial conduction through a cylindrical wall is logarithmic, not linear, as it is for the plane wall. The logarithmic distribution is shown in Figure.

Heat transfer rate

$$q_r = -k(2\pi rL)\frac{dT}{dr} = -k(2\pi rL)\frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)}\frac{1}{r}$$

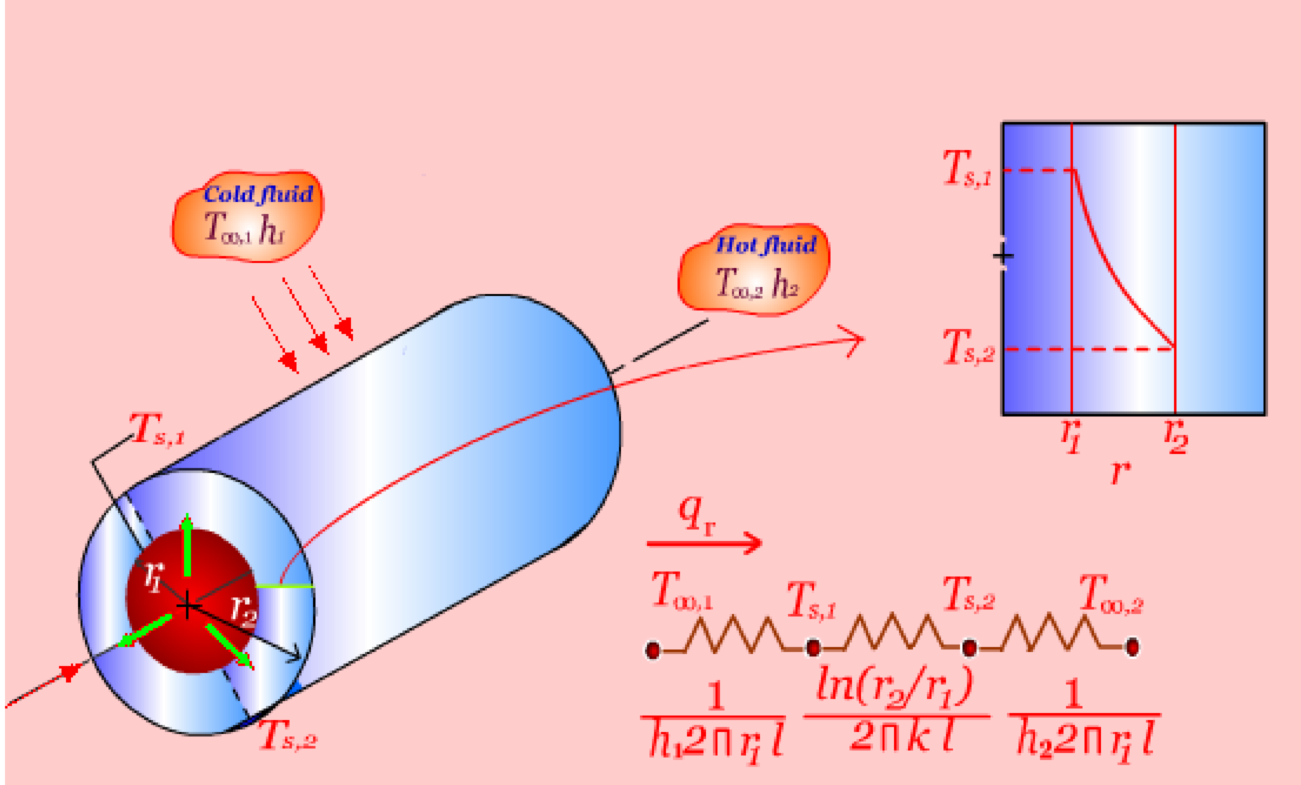
$$q_r = 2\pi Lk\frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_2}{r_1}\right)}$$

q_r is independent of r

$$T(r) = \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_1}{r_2}\right)} ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

$$q_r = 2\pi Lk \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_2}{r_1}\right)}$$

$$R_{t,cond} = \frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi Lk}$$



$$q_r = 2\pi Lk \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_2}{r_1}\right)}$$

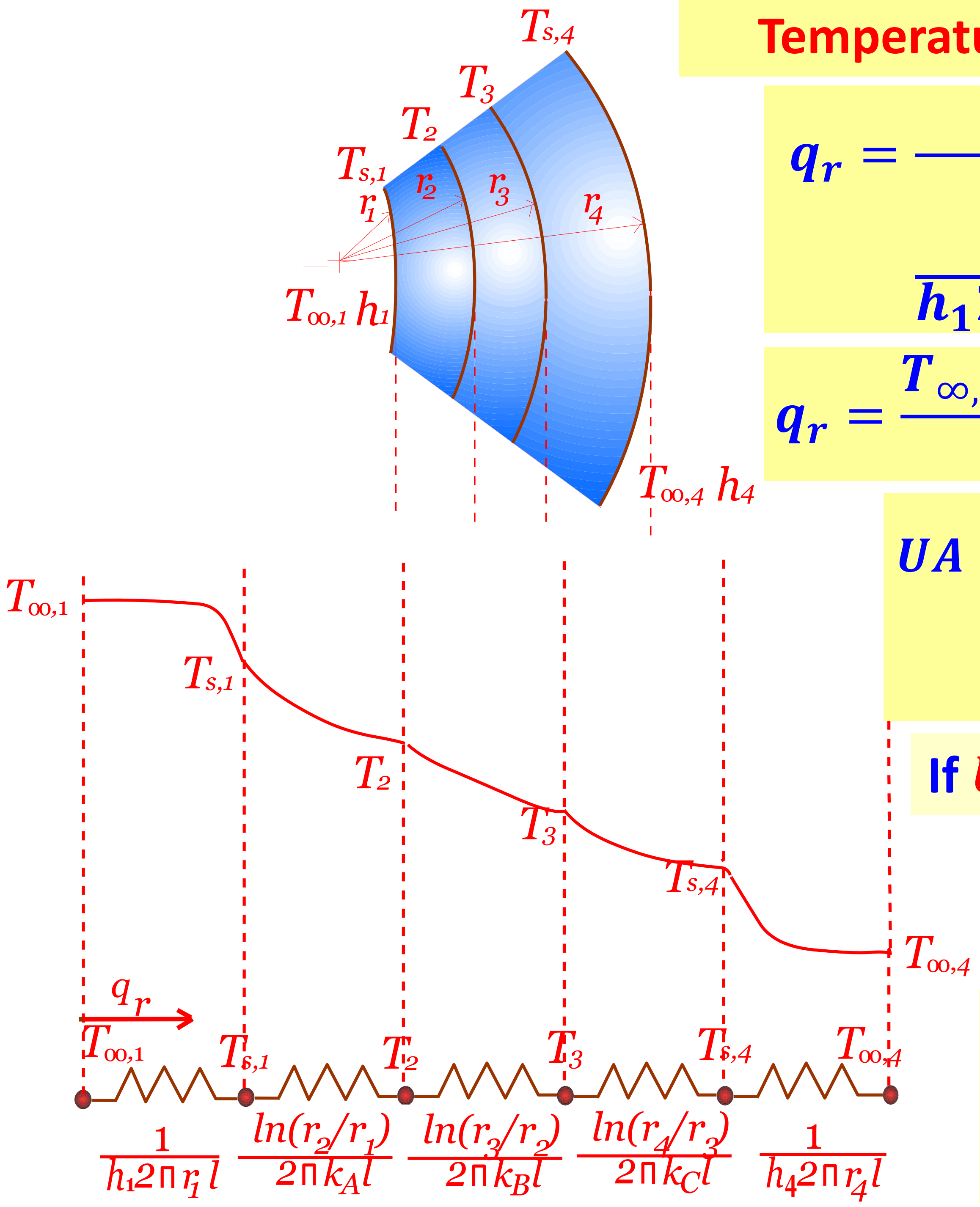
q_r is independent of r

$$q_r'' = \frac{q_r}{2\pi rL} = \frac{2\pi Lk}{2\pi rL} \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_2}{r_1}\right)}$$

$$q_r'' = \frac{k}{r} \frac{T_{s,1} - T_{s,2}}{ln\left(\frac{r_2}{r_1}\right)}$$

q_r'' is dependent of r

Temperature Distribution For A Composite Cylindrical Wall



$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{h_1 2\pi r_1 L} + \frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi L k_A} + \frac{ln\left(\frac{r_3}{r_2}\right)}{2\pi L k_B} + \frac{ln\left(\frac{r_4}{r_3}\right)}{2\pi L k_C} + \frac{1}{h_4 2\pi r_4 L}}$$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t} = UA(T_{\infty,1} - T_{\infty,4})$$

U is overall heat transfer coefficient

$$UA = \frac{1}{\frac{1}{h_1 2\pi r_1 L} + \frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi L k_A} + \frac{ln\left(\frac{r_3}{r_2}\right)}{2\pi L k_B} + \frac{ln\left(\frac{r_4}{r_3}\right)}{2\pi L k_C} + \frac{1}{h_4 2\pi r_4 L}}$$

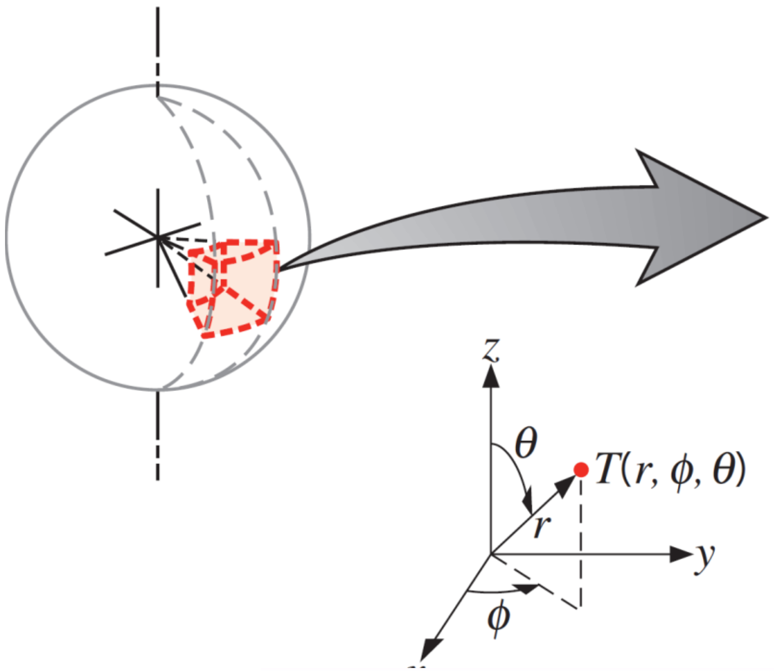
If U is defined in terms of the inside area, $A_1 = 2\pi r_1 L$

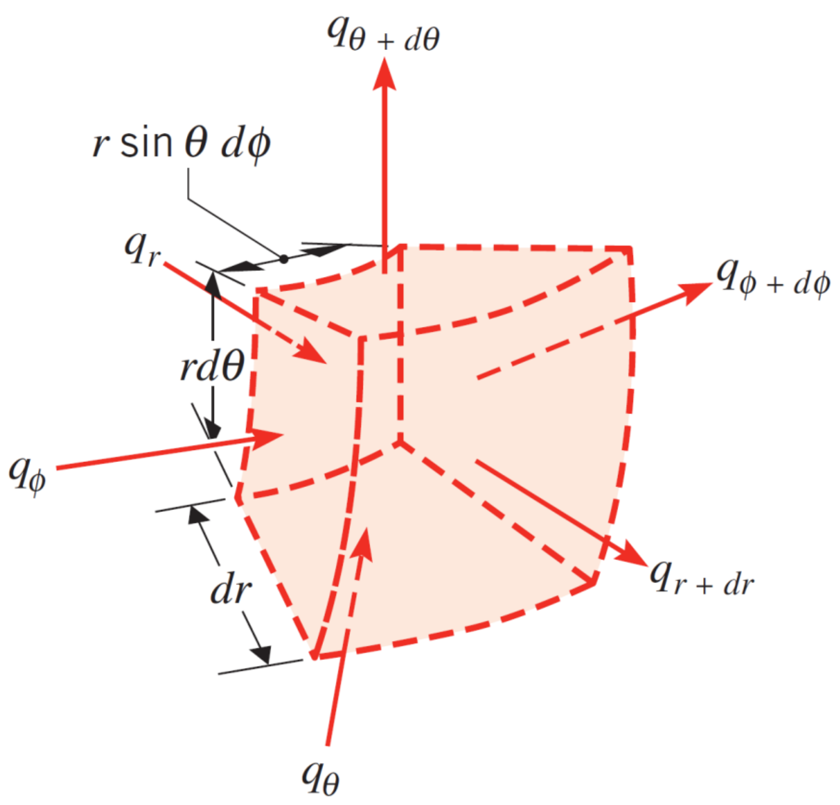
$$U = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A} ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_B} ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{k_C} ln\left(\frac{r_4}{r_3}\right) + \frac{r_1}{r_4}\frac{1}{h_4}}$$

UA is constant, while U is not

In radial system q_r'' is dependent of r, but q_r is independent of r

Heat diffusion equation in spherical coordinates





$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right)$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE SPHERE

Consider a hollow sphere, whose inner and outer surfaces are exposed to fluids at different temperatures

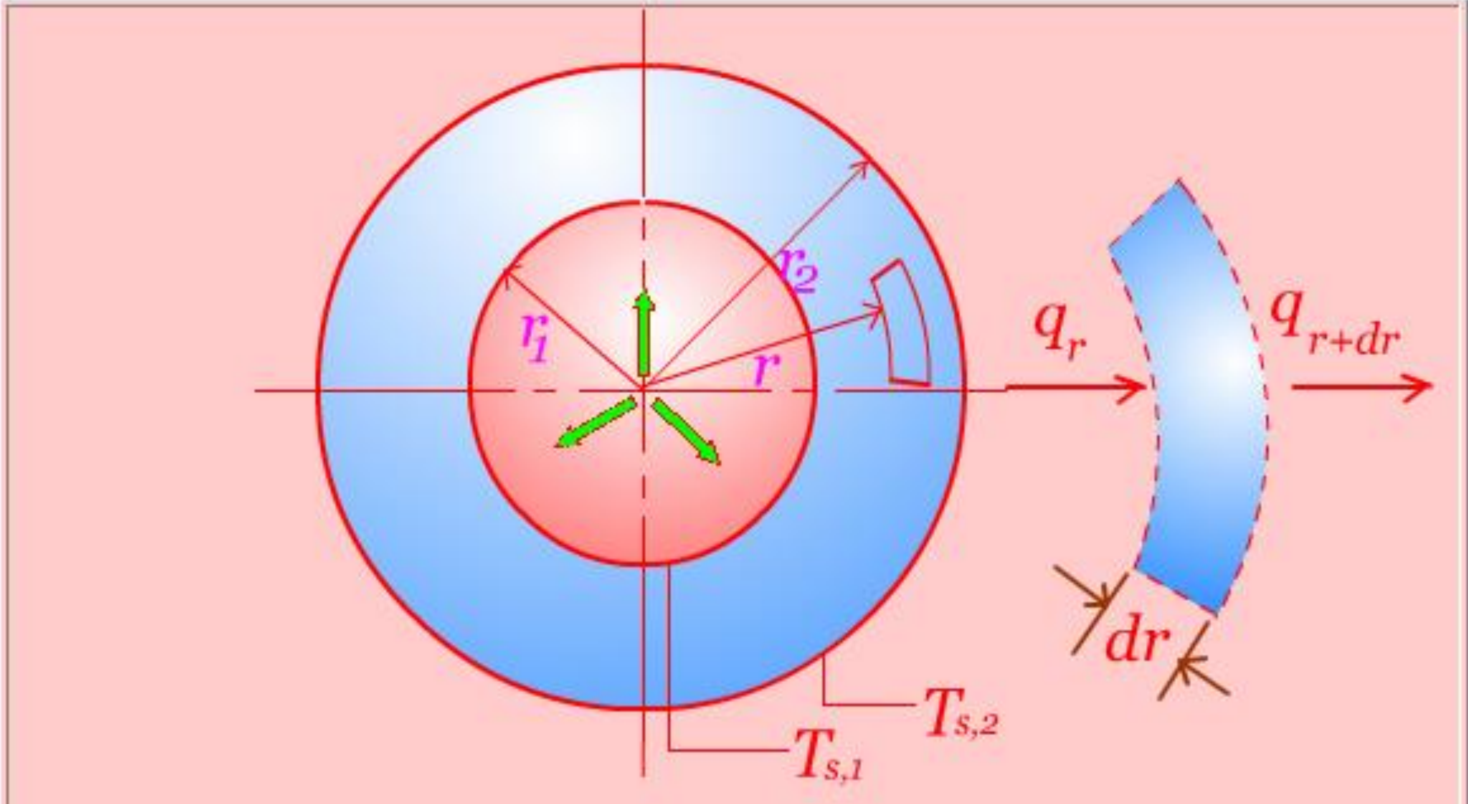
$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$$

Integrating this eqn twice

$$\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$$

$$\frac{dT}{dr} = \frac{C}{kr^2} = \frac{C_1}{r^2}$$

$$T = -\frac{C_1}{r} + C_2$$



Conduction in a spherical shell

Boundary conditions are

$$r = r_1 \quad T = T_{s,1}$$
$$r = r_2 \quad T = T_{s,2}$$

$$T_{s,1} = -\frac{C_1}{r_1} + C_2$$

$$T_{s,2} = -\frac{C_1}{r_2} + C_2$$

$$T_{s,1} - T_{s,2} = -\frac{C_1}{r_1} + \frac{C_1}{r_2}$$

$$C_1 = \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE SPHERE

TEMPERATURE DISTRIBUTION

$$T_{s,1} = -\frac{C_1}{r_1} + C_2$$

$$C_1 = \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$$T_{s,2} = -\frac{C_1}{r_2} + C_2$$

$$T = -\frac{C_1}{r} + C_2$$

$$T_{s,1} = -\frac{1}{r_1}\left[\frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\right] + C_2$$

$$C_2 = T_{s,1} + \frac{1}{r_1}\left[\frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\right]$$

$$T = -\frac{1}{r}\frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} + T_{s,1} + \frac{1}{r_1}\left[\frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\right]$$

$$T = T_{s,1} + \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\left(\frac{1}{r_1} - \frac{1}{r}\right)$$

Note that the temperature distribution associated with radial conduction through a spherical wall is not linear, as it is for the plane wall under the same conditions.

1D, STEADY STATE CONDUCTION WITH NO INTERNAL GENERATION - THE SPHERE

HEAT TRANSFER RATE

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$q_r = -k(4\pi r^2)\frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\left(\frac{1}{r^2}\right)$$

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

q_r is independent of r

$$q_r'' = \frac{q_r}{4\pi r^2} = \frac{4\pi k}{4\pi r^2}\frac{(T_{s,1} - T_{s,2})}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$q_r'' = \frac{k}{r^2}\frac{(T_{s,1} - T_{s,2})}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

q_r'' is dependent of r

TEMPERATURE DISTRIBUTION

$$T = T_{s,1} + \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\left(\frac{1}{r_1} - \frac{1}{r}\right)$$

THERMAL RESISTANCE

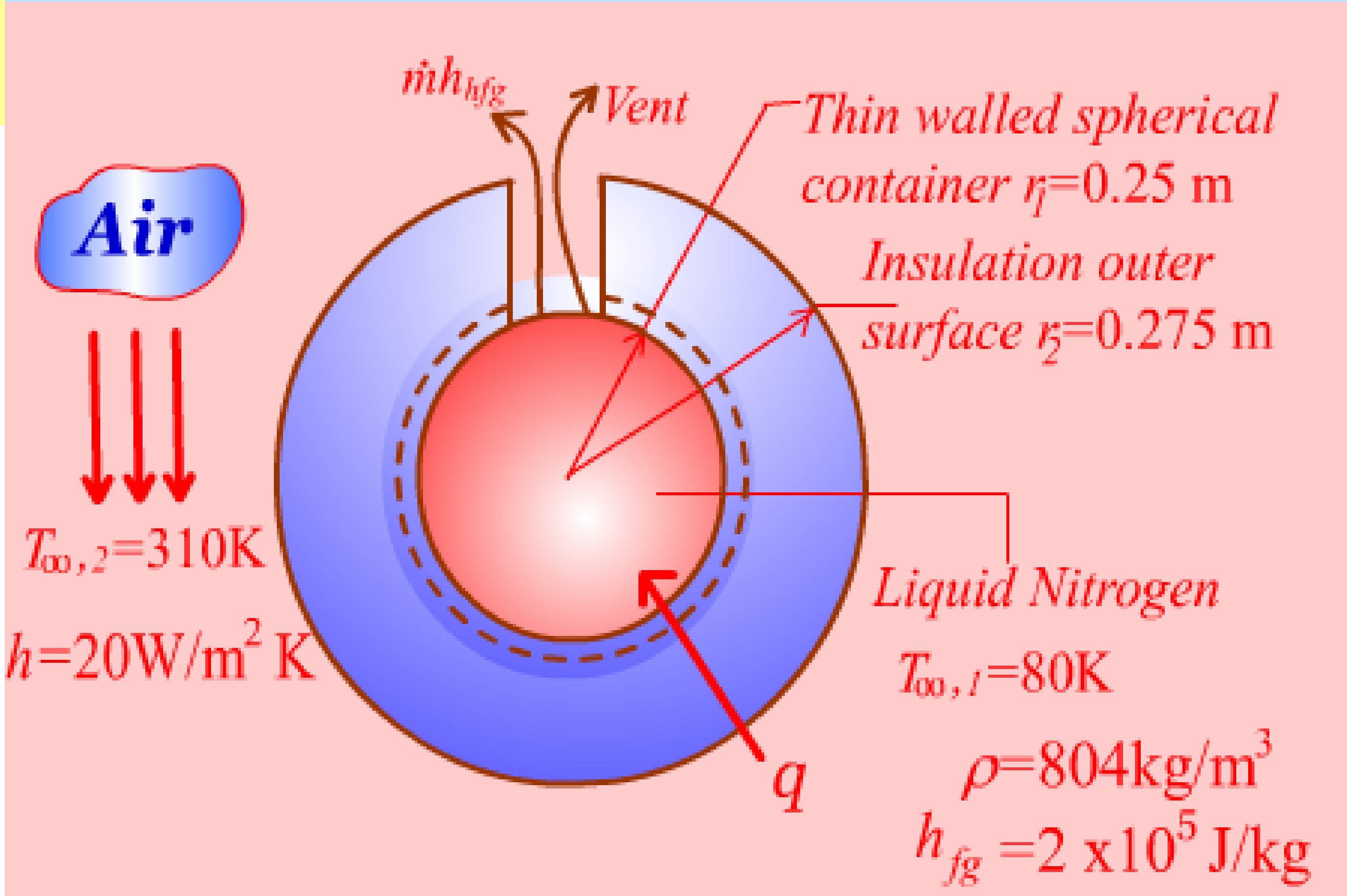
$$R_t = \frac{(T_{s,1} - T_{s,2})}{q_r} = \frac{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{4\pi k}$$

Spherical composites may be treated in much the same way as composite walls and cylinders, where approximate forms of the total resistance and overall heat transfer coefficient may be determined

A spherical thin walled metallic container is used to store liquid nitrogen at 80 K. The container has a diameter of 0.5 *m* and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 310K. The convection coefficient is known to be 20 $W/m^2\ K$. The latent heat of vaporization and the density of the liquid nitrogen are 2 x $10^5 J/kg$ and 804 kg/m^3, respectively. Thermal conductivity of evacuated silica powder (300 K) is 0.0017 $W/m.K$

what is the rate of heat transfer to the liquid nitrogen ? what is the rate of liquid boil-off ?

Figure



Known: Liquid nitrogen is stored in spherical container that is insulated and exposed to ambient air.

Find:

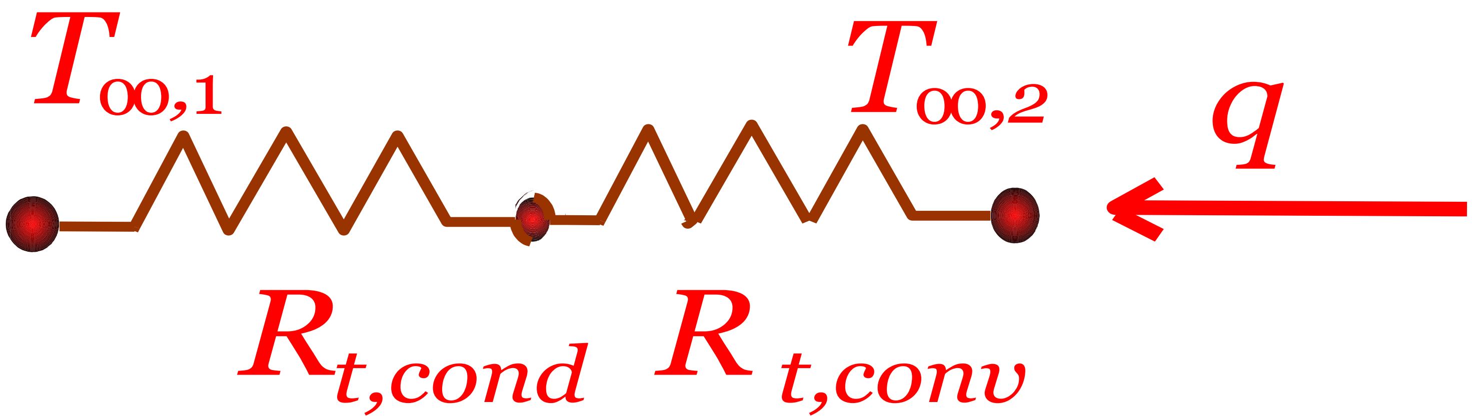
- The rate of heat transfer to the nitrogen.
- The mass rate of nitrogen boil-off.

Assumptions:

1. Steady state conditions and one dimensional transfer in the radial direction
2. Negligible resistance to heat transfer through the container wall and from the container to the nitrogen
3. Constant properties
4. Negligible radiation exchange between outer surface of insulation and surroundings

Analysis:

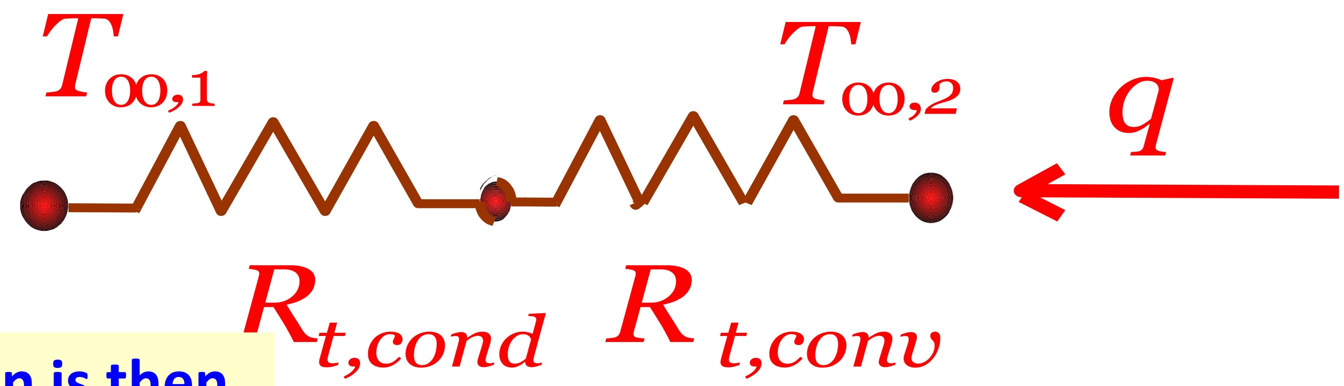
1. The thermal circuit involves a conduction and convection resistance in series and is of the form



$$R_{t,cond}=\frac{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{4\pi k}$$

$$R_{t,conv}=\frac{1}{hA}=\frac{1}{h(4\pi r_2^2)}$$

The rate of heat transfer to the nitrogen



The rate of heat transfer to the liquid nitrogen is then

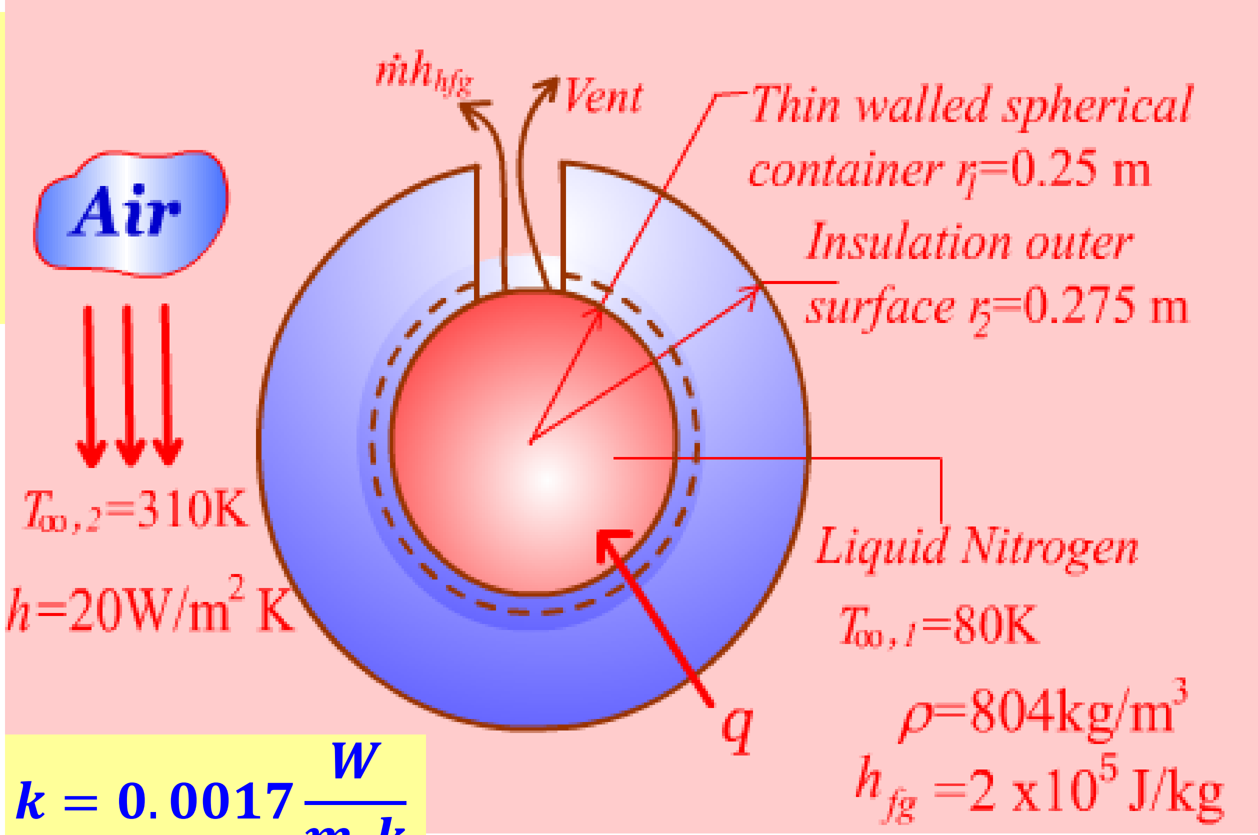
$$q_r=\frac{T_{\infty,2}-T_{\infty,1}}{R_{t,cond}+R_{t,conv}}=\frac{T_{\infty,2}-T_{\infty,1}}{\frac{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{4\pi k}+\frac{1}{h(4\pi r_2^2)}}$$

$$q_r=\frac{310-80}{\frac{\left(\frac{1}{0.25}-\frac{1}{0.275}\right)}{4\pi(0.0017)}+\frac{1}{20(4\pi(0.275)^2)}}$$

$$q_r=\frac{230}{17.02+0.05}$$

$$q_r=13.47\ W$$

$$R_{t,cond}\gg R_{t,conv}$$



$$k=0.0017\frac{W}{m.k}$$

The rate of liquid boil off

Performing an energy balance for a control surface about the nitrogen

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

$$\dot{E}_{in} = \dot{E}_{out}$$

$$q_r = 13.47\ W$$

$$q_r = \dot{m}h_{fg}$$

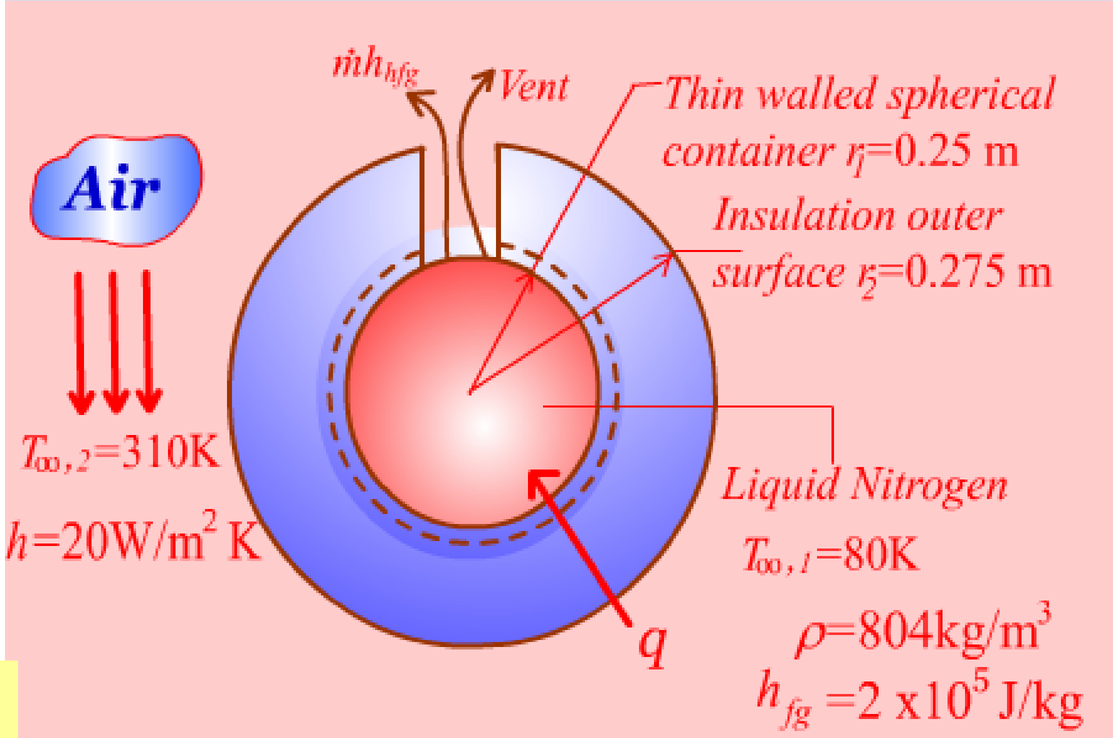
$$13.47 = \dot{m}(2 \times 10^5)$$

$$\dot{m} = 6.74 \times 10^{-5}\ \frac{kg}{s}$$

$$\dot{m} = 6.74 \times 10^{-5}\ \frac{kg}{s} \times 3600\ \frac{s}{hour} \times 24\ \frac{hours}{day}$$

$$\dot{m} = 5.82\ \frac{kg}{day}$$

$$V = \frac{\dot{m}}{\rho} = \frac{5.82\ \frac{kg}{day}}{804\ \frac{kg}{m^3}} = 0.0724\ \frac{m^3}{day} = 7.24\ \frac{litres}{day}$$



With a container volume of $\frac{4}{3}(0.25)^3 = 0.065\ m^3 = 65$ **litres, the daily loss is** $(7.24/$

THE CRITICAL RADIUS OF INSULATION

We know that by adding more insulation to a wall always decreases heat transfer.

This is expected, since the heat transfer area *A* is constant, and adding insulation will always increase the thermal resistance of the wall without affecting the convection resistance.

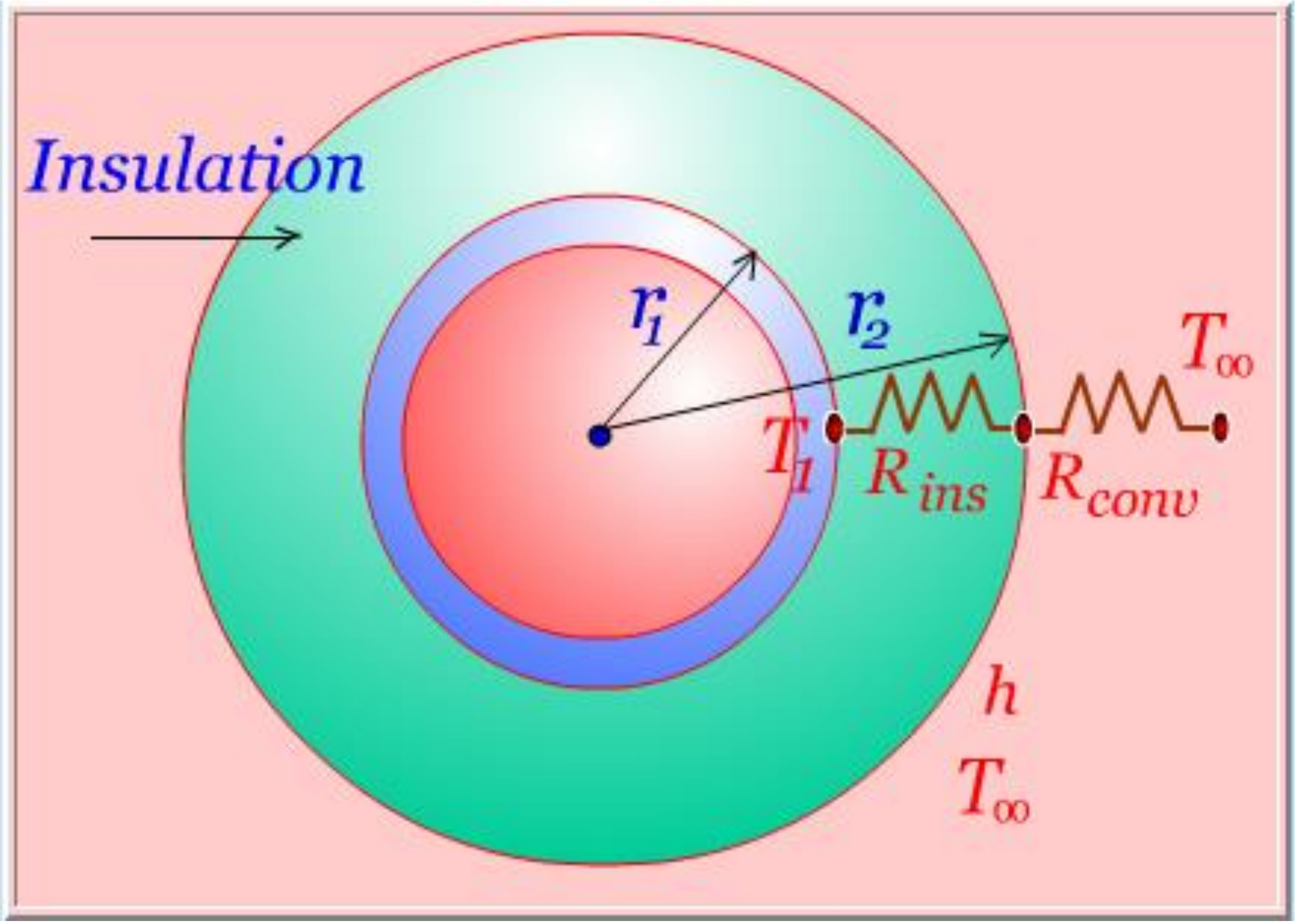
However, adding insulation to a cylindrical piece or a spherical shell, is a different matter.

The additional insulation increases the conduction resistance of the insulation layer but it also decreases the convection resistance of the surface because of the increase in the outer surface area for convection.

Therefore, the heat transfer from the pipe may increase or decrease, depending on which effect dominates.

THE CRITICAL RADIUS OF INSULATION

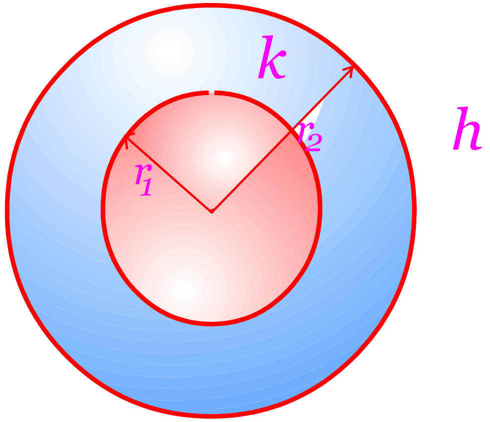
Consider a cylindrical pipe where,
r_1 **- outer radius**
T_1 **- constant outer surface temperature**
k **- thermal conductivity of the insulation**
r_2 **- outer radius**
T_∞ **- temperature of surrounding medium**
h **- convection heat transfer coefficient**

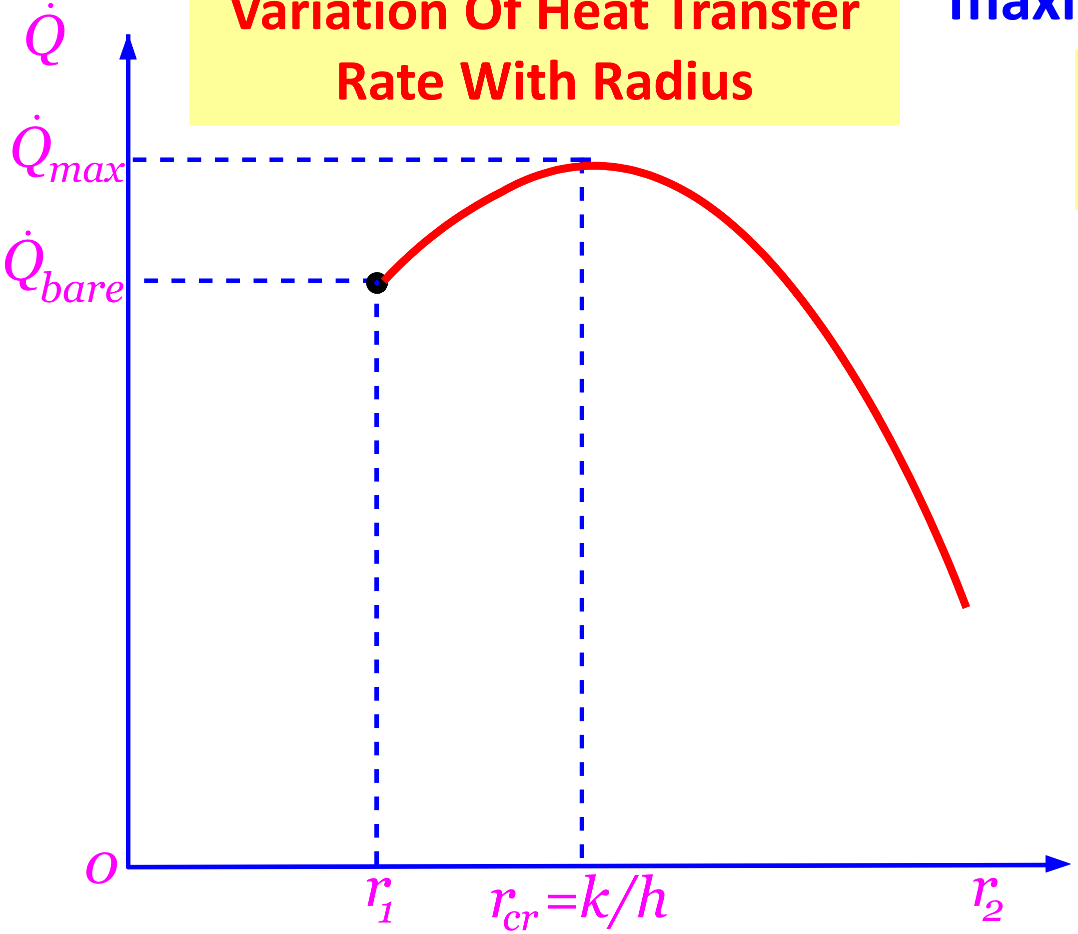


Insulated Cylindrical Pipe

The rate of heat transfer from the insulated pipe to the surrounding air

$$q_r = \frac{T_1 - T_\infty}{\frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi L k} + \frac{1}{h(2\pi r_2 L)}}$$





Variation Of Heat Transfer Rate With Radius

$$q_r = \frac{T_1 - T_\infty}{\dfrac{ln\left(\dfrac{r_2}{r_1}\right)}{2\pi Lk} + \dfrac{1}{h(2\pi r_2 L)}}$$

The value of r_2 at which heat transfer rate reaches maximum is determined from (zero slope).

$$\frac{dq_r}{dr_2} = 0 \qquad \frac{1}{2\pi Lkr_2} - \frac{1}{h\left(2\pi r_2^2 L\right)} = 0$$

$$r_{cr} = \frac{k}{h}$$

The rate of heat transfer from the cylinder increases with the addition of insulation for $r_2 < r_{cr}$, reaches a maximum when $r_2 = r_{cr}$, and starts to decrease for $r_2 > r_{cr}$. Thus, insulating the pipe may actually increase the rate of heat transfer from the pipe instead of decreasing it when $r_2 < r_{cr}$.

THE CRITICAL RADIUS OF INSULATION

The important question to answer at this point is,

- **Whether we need to be concerned about the critical radius of insulation when insulating hot water pipes or even hot water tanks?**
- **Should we always check and make sure that the outer radius of insulation exceeds the critical radius before we install any insulation?**

Probably not, as explained below.

- **The value of the critical radius r_{cr} will be the largest when k is large and h is small.**
- **Noting that the lowest value of h encountered in practice is about 5 W/m^{2}K for the case of natural convection of gases**
- **Also, the thermal conductivity of common insulating materials is 0.05 W/m^{2}K,**
- **The largest value of the critical radius we are likely to encounter is**

$$r_{cr} = \frac{k}{h} = \frac{0.05}{5} = 0.01m = 10\ mm$$

- **This value would be even smaller when the radiation effects are considered.**

- The critical radius would be much less in forced convection, often less than 1 *mm*, because of much larger h values associated with forced convection.
- Therefore, we can insulate hot water or steam pipes freely without worrying about the possibility of increasing the heat transfer by insulating the pipes.
- The radius of electric wires may be smaller than the critical radius.
- Therefore, the plastic electrical insulation may actually enhance the heat transfer from electric wires and thus keep their steady operating temperatures at lower and thus safer levels.

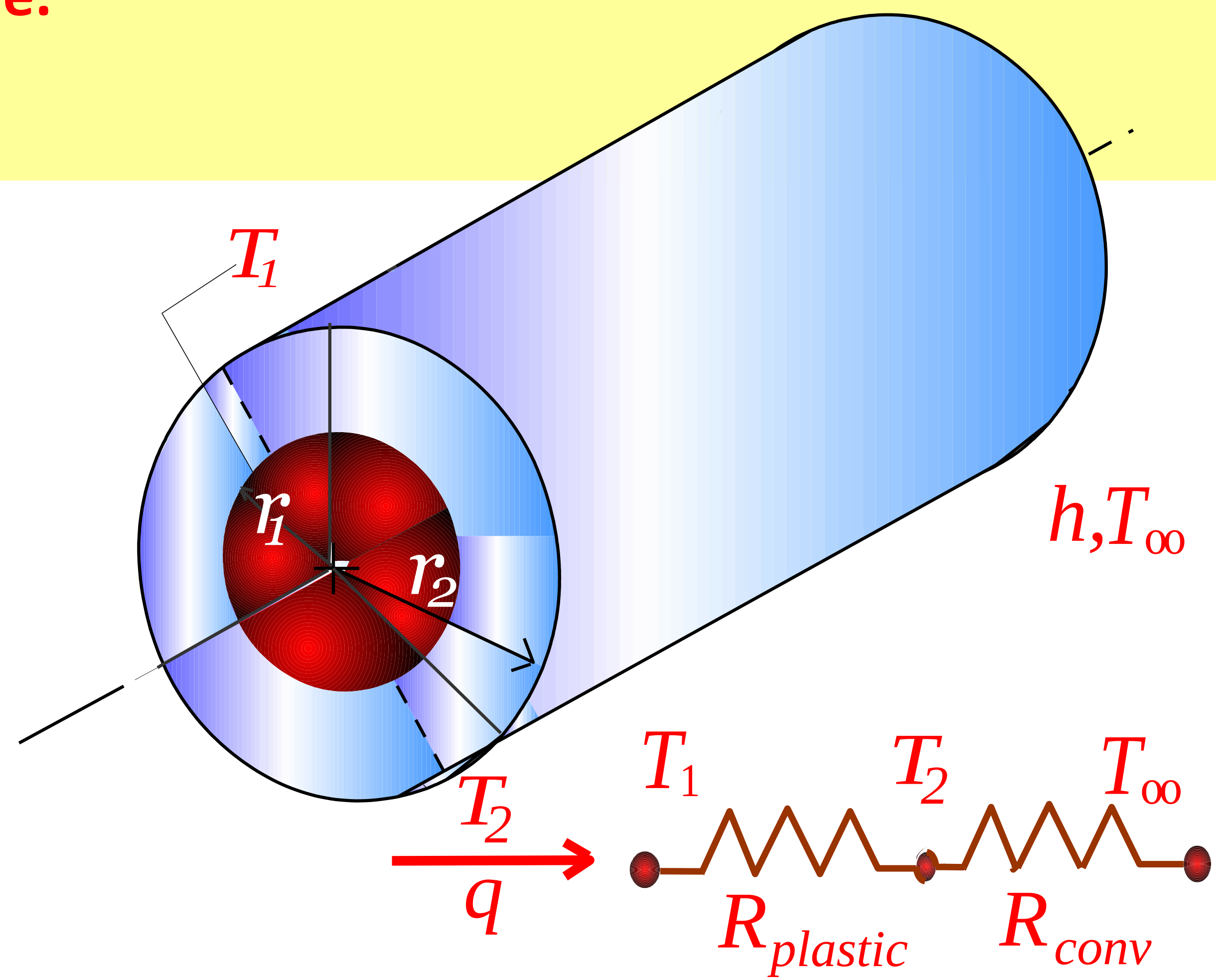
Similarly for a sphere, it can be shown that the critical radius of insulation for a spherical shell is

$$r_{cr} = \frac{2k}{h}$$

where k is the thermal conductivity of the insulation and h is the convection heat transfer coefficient on the outer surface.

A 3 mm diameter and 6 *m* long electric wire is tightly wrapped with a 2 mm thick plastic cover whose thermal conductivity is $k = 0.15\ W/m.^0C$. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at 27°C with a heat transfer coefficient of $h=12\ W/m^2.^oC$, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

Figure



Known : Size of the electric wire, thermal conductivity of the wire, current and voltage supplied to the wire, ambient conditions and heat transfer coefficient.

Find :

Convection heat transfer coefficient between the outer surface of the wire and the air in the room.

1. Heat transfer is steady since there is no indication of any change with time.
2. Heat transfer is one dimensional since there is thermal symmetry about the center line and no variation in the axial direction.
3. Thermal conductivities are constant.
4. The thermal contact resistance at the interface is negligible.
5. Heat transfer coefficient incorporates the radiation effects, if any.

Analysis:

Heat is generated in the wire and its temperature rises as a result of resistance heating. We assume heating is generated uniformly throughout the wire and is transferred to the surrounding medium in the radial direction. In steady operation, the rate of heat transfer becomes equal to the heat generated within the wire, which is determined from

$$\dot{Q} = \mathrm{VI} = 8 \times 10 = 80W$$

The thermal resistance network for this problem involves a conduction resistance for the plastic cover and a convection resistance for the outer surface in series, as shown in Fig. The values of these two resistances are determined to be

$$A_2 = 2\pi r_2 L = 2\pi \times 3.5 \times 10^{-3}(6) = 0.132\ m^2$$

3 mm diameter and 2 mm thick plastic cover

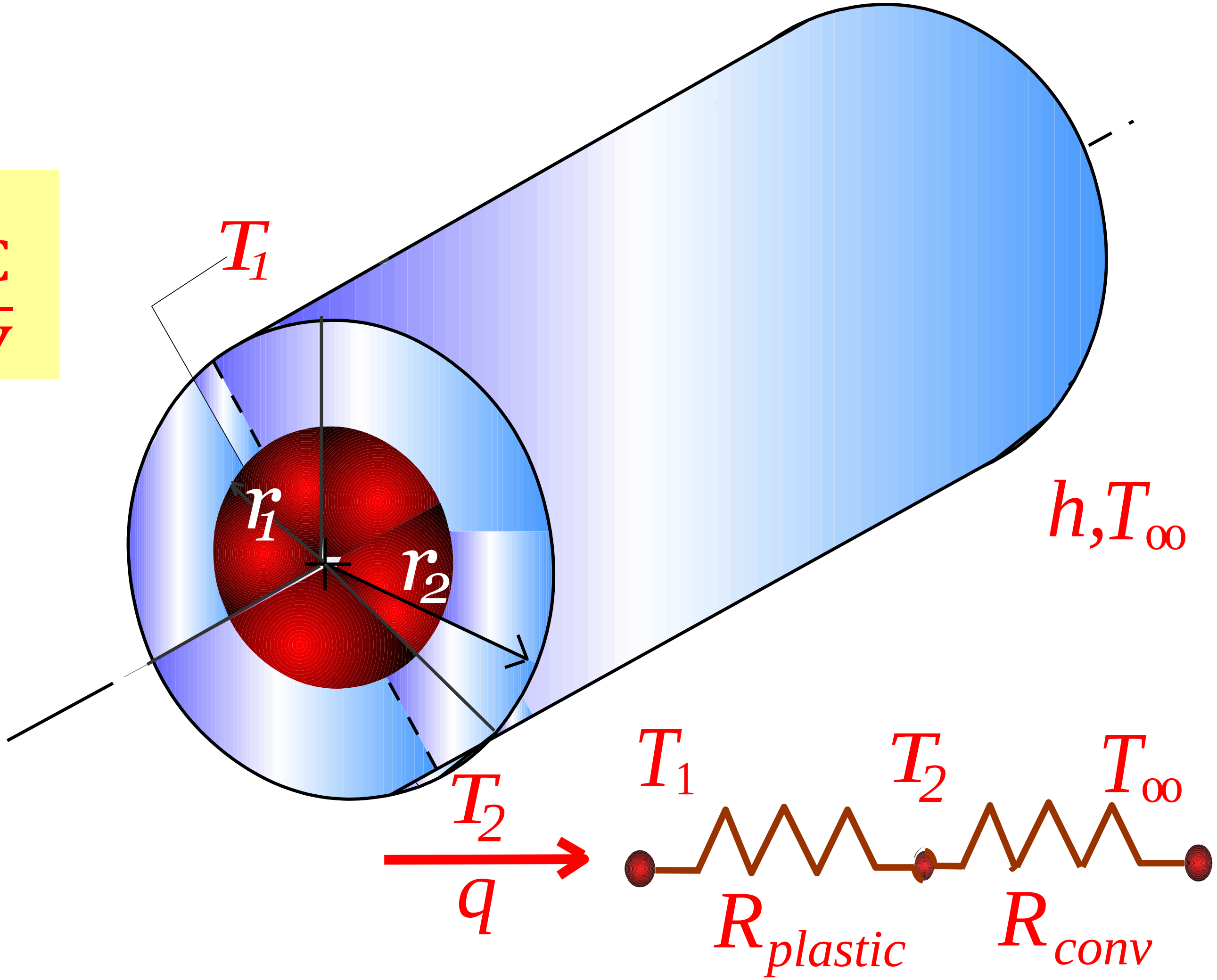
$$R_{conv} = \frac{1}{hA_2} = \frac{1}{12 \times 0.132} = 0.63\,\frac{°C}{W}$$

$$R_{platic} = \frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi Lk} = \frac{ln\left(\frac{3.5}{1.5}\right)}{2\pi(0.15)6} = 0.15\,\frac{°C}{W}$$

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{total}}$$

$$80 = \frac{T_1 - 27}{0.63 + 0.15}$$

$$T_1 = 89.4°C$$

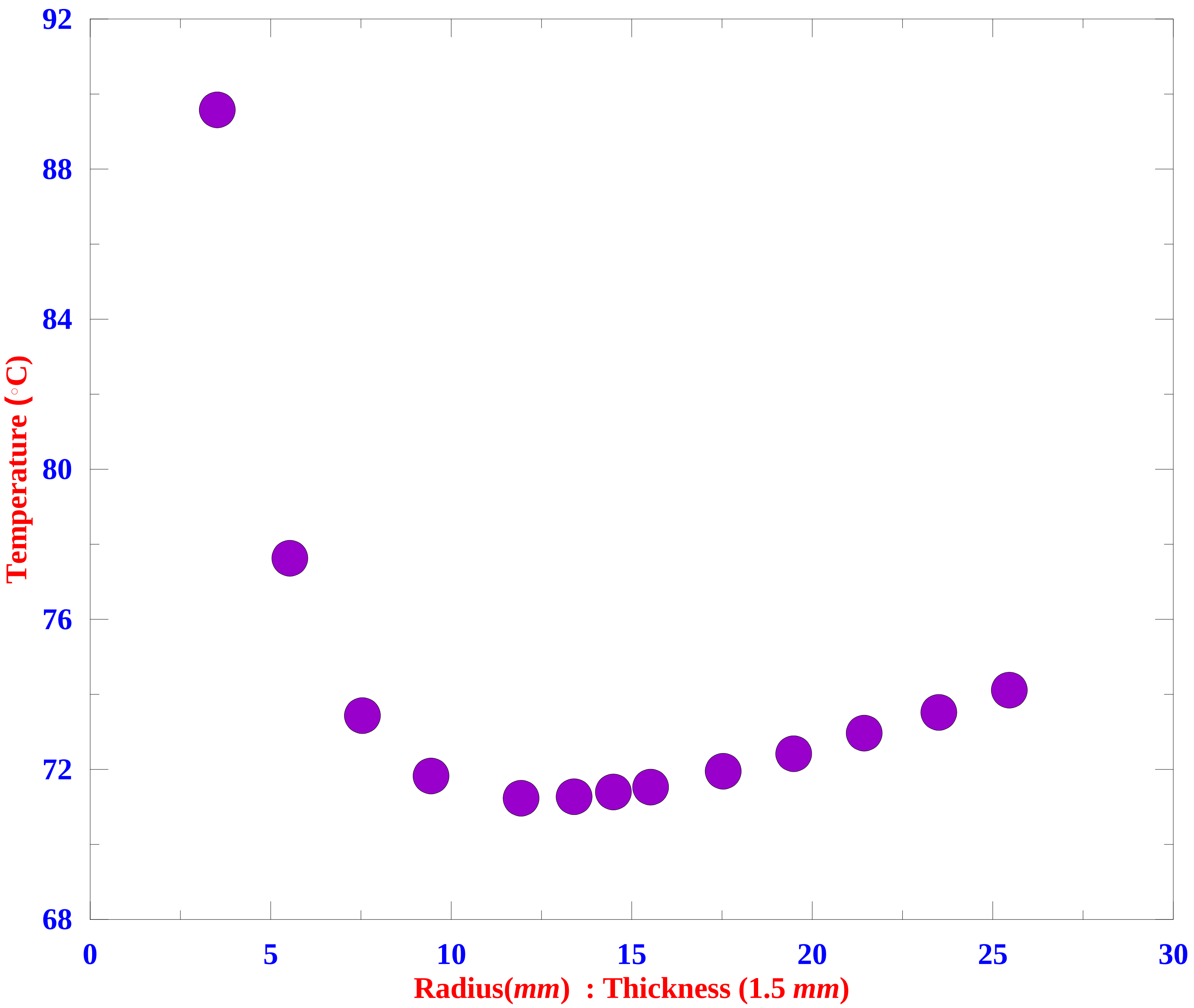


Note that we did not involve the electrical wire directly in the thermal resistance network, since the wire involves heat generation.

To answer the second part of the question, we need to know the critical radius of insulation of the plastic cover.

$$r_{cr} = \frac{k}{h} = \frac{0.15}{12} = 0.0125m = 12.5\,mm$$

which is larger than the radius of the plastic cover. Therefore, increasing the thickness of the plastic cover will enhance heat transfer until the outer radius of the cover reaches 12.5 mm. As a result, the rate of heat transfer will increase when the interface temperature T_1 is held constant, or T_1 will decrease when is held constant, which is the case here.



Comments:

For a 4 mm plastic cover that the interface temperature drops to 77.54°C when the thickness of the plastic cover is doubled.

The interface reaches a minimum temperature of 71.14°C when the outer radius of the plastic cover equals the critical radius.

Summary of 1D steady state heat conduction without heat generation	Plane Wall $\Delta T = T_{s,1} - T_{s,2}$	Cylindrical Wall $\Delta T = T_{s,1} - T_{s,2}$	Spherical Wall $\Delta T = T_{s,1} - T_{s,2}$
Heat Equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$
Temperature Distribution	$T = T_{s,1} + \Delta T\left(\frac{x}{L}\right)$	$\frac{\Delta T}{ln\left(\frac{r_1}{r_2}\right)} ln\left(\frac{r}{r_2}\right) + T_{s,2}$	$T_{s,1} + \frac{\Delta T}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\left(\frac{1}{r_1} - \frac{1}{r}\right)$
Heat Flux (q''_x)	$q''_x = k\frac{\Delta T}{L}$	$\frac{k}{r}\frac{\Delta T}{ln\left(\frac{r_2}{r_1}\right)}$	$\frac{k}{r^2}\frac{\Delta T}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$
Heat Rate (q)	$q_x = kA\frac{\Delta T}{L}$	$2\pi Lk\frac{\Delta T}{ln\left(\frac{r_2}{r_1}\right)}$	$\frac{4\pi k\Delta T}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$
Thermal Resistance ($R_{t,\ cond}$)	$\frac{L}{kA}$	$\frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi Lk}$	$\frac{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{4\pi k}$

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION WITH HEAT GENERATION

We will consider situations for which thermal energy is being generated due to conversion from some other energy form.

A very common thermal energy generation process involves the conversion from electrical to thermal energy in a current carrying medium (resistance heating). The rate at which energy is generated by passing a current I through a medium of electrical resistance R_e is

$$\dot{E}_g = I^2 R_e$$

If this power generation occurs uniformly throughout the medium of volume *V*, the volumetric generation rate (W/m^3) is then

$$\dot{q} = \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V}$$

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION WITH HEAT GENERATION – PLANE WALL

Consider the plane wall of Fig. in which there is uniform energy generation per unit volume (is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$. For constant thermal conductivity k, the appropriate form of the heat equation,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t}$$

$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0$$

$$\frac{dT}{dx}+\frac{\dot{q}}{k}x=C_1$$

$$T+\frac{\dot{q}}{k}\frac{x^2}{2}=C_1x+C_2$$

$$T(-L)=T_{s,1}$$

$$T(L)=T_{s,2}$$

Asymmetrical Boundary Condition

$$T_{s,1}+\frac{\dot{q}}{k}\frac{L^2}{2}=-C_1L+C_2$$

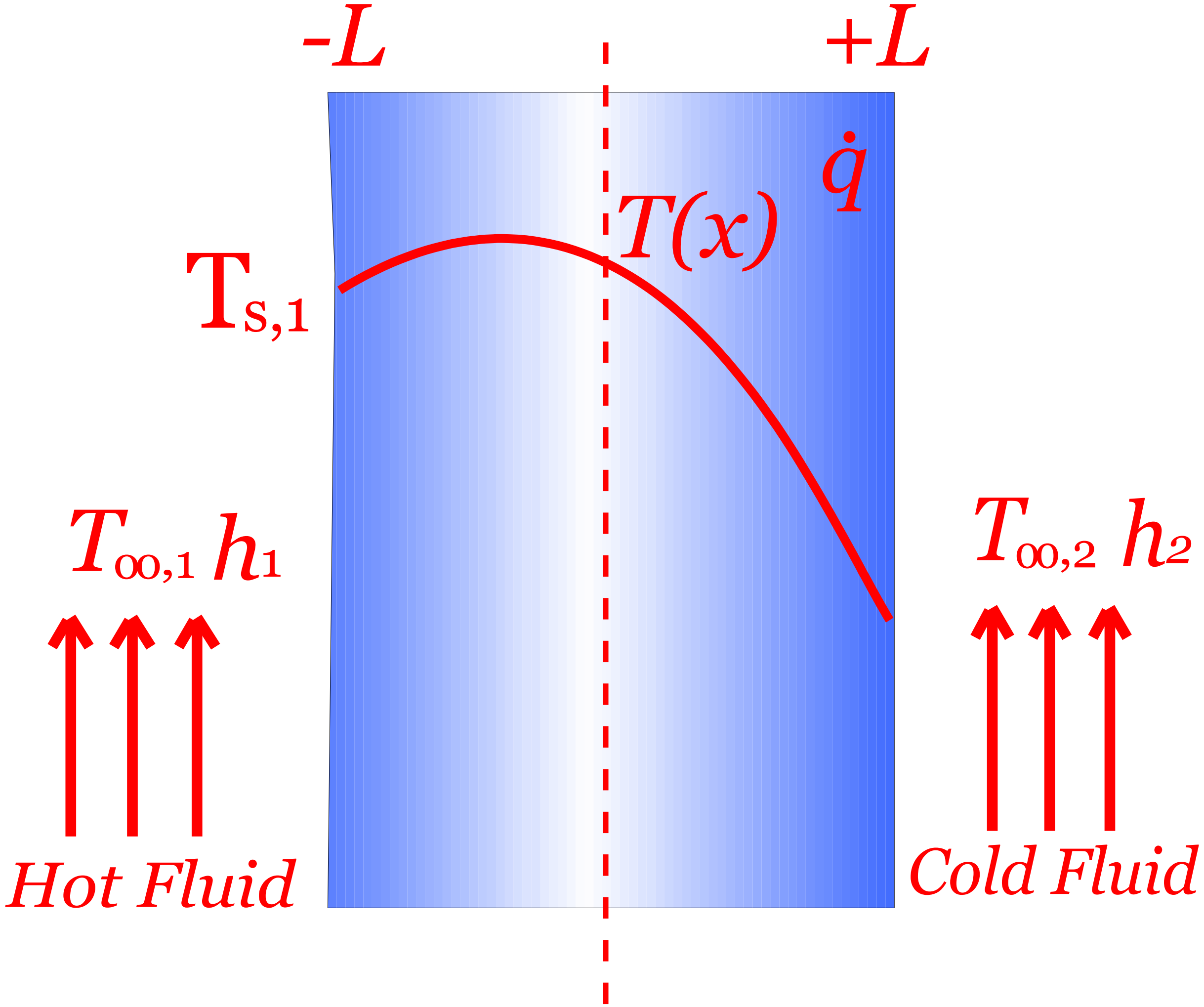
$$T_{s,2}+\frac{\dot{q}}{k}\frac{L^2}{2}=C_1L+C_2$$

$$T_{s,1}-T_{s,2}=-2C_1L$$

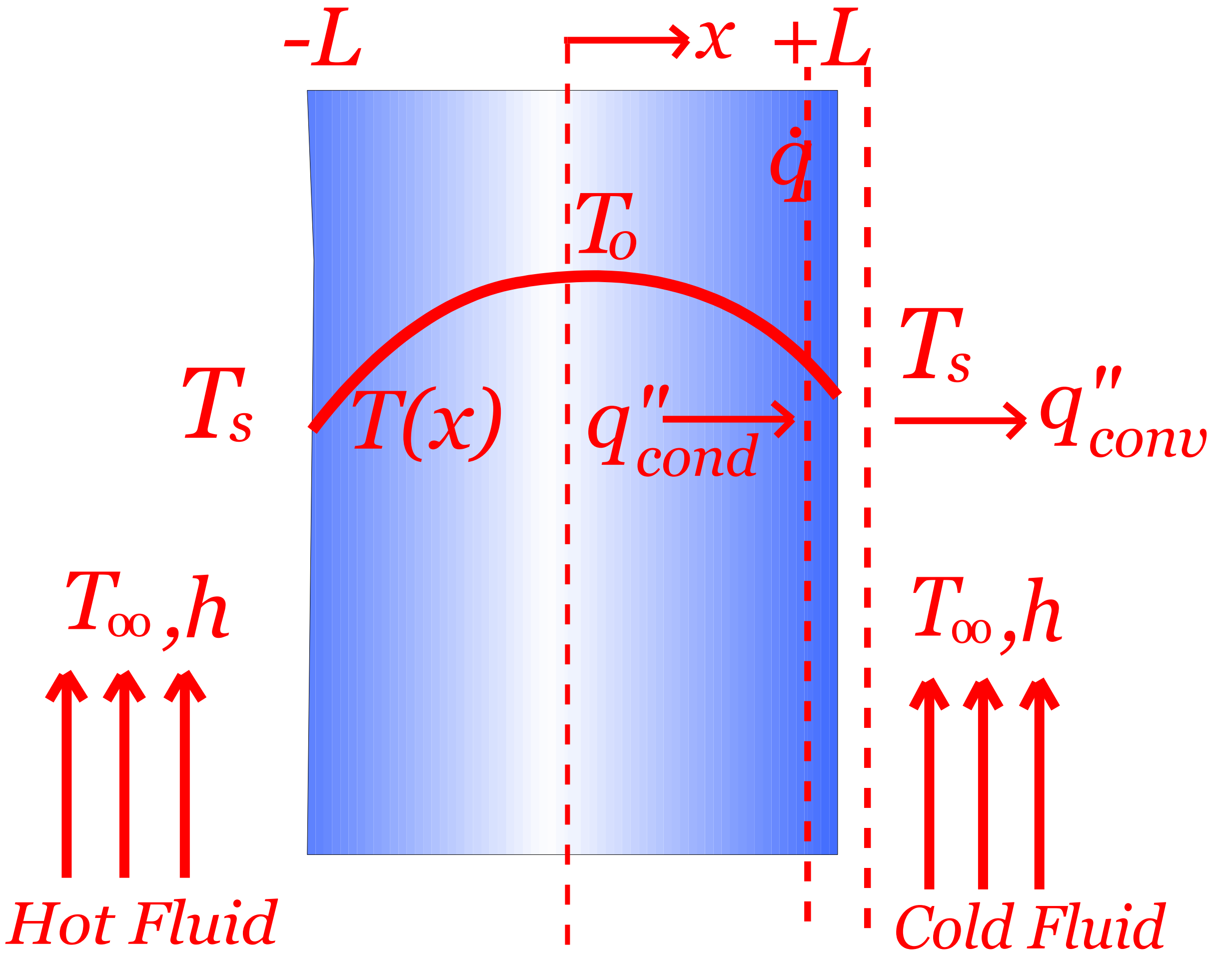
$$C_1=\frac{T_{s,2}-T_{s,1}}{2L}$$

$$T_{s,1}+\frac{\dot{q}}{k}\frac{L^2}{2}+T_{s,2}+\frac{\dot{q}}{k}\frac{L^2}{2}=2C_2$$

$$\frac{T_{s,1}+T_{s,2}}{2}+\frac{\dot{q}}{2k}L^2=C_2$$



Asymmetrical Boundary Condition



Symmetrical Boundary Condition

$$T + \frac{\dot{q}}{k}\frac{x^2}{2} = C_1 x + C_2$$

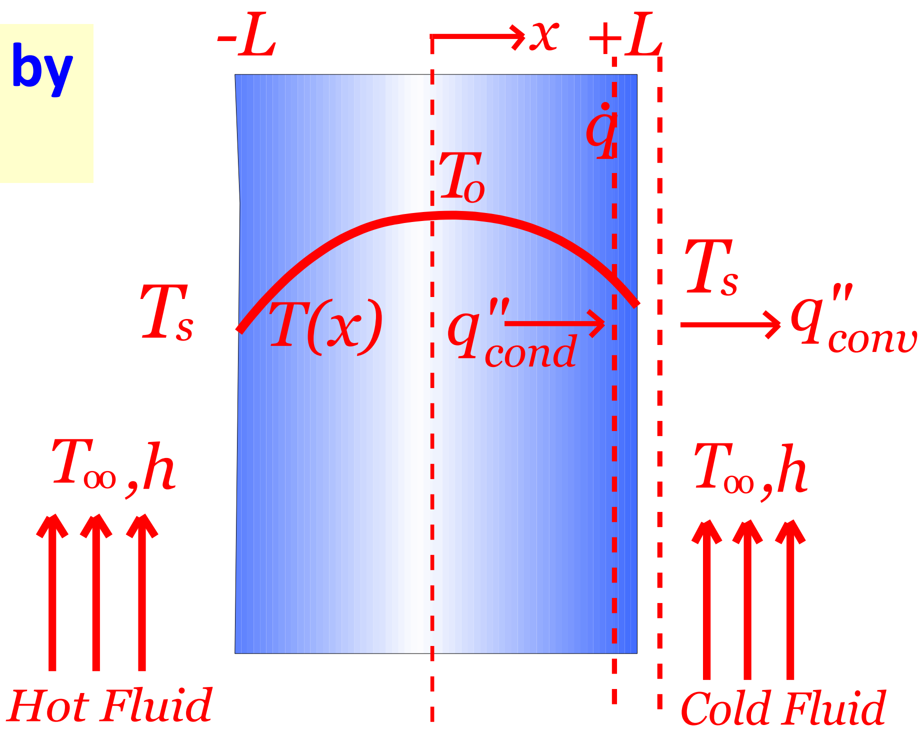
$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L}$$

$$C_2 = \frac{T_{s,1} + T_{s,2}}{2} + \frac{\dot{q}}{2k}L^2$$

$$T + \frac{\dot{q}}{k}\frac{x^2}{2} = \frac{T_{s,2} - T_{s,1}}{2L}x + \frac{T_{s,1} + T_{s,2}}{2} + \frac{\dot{q}}{2k}L^2$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

The heat flux at any point in the wall may, be determined by using the above equation with Fourier's Law.



Symmetrical Boundary Condition

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

$$T_o = \frac{\dot{q}L^2}{2k} + T_s$$

Symmetrical Boundary Condition

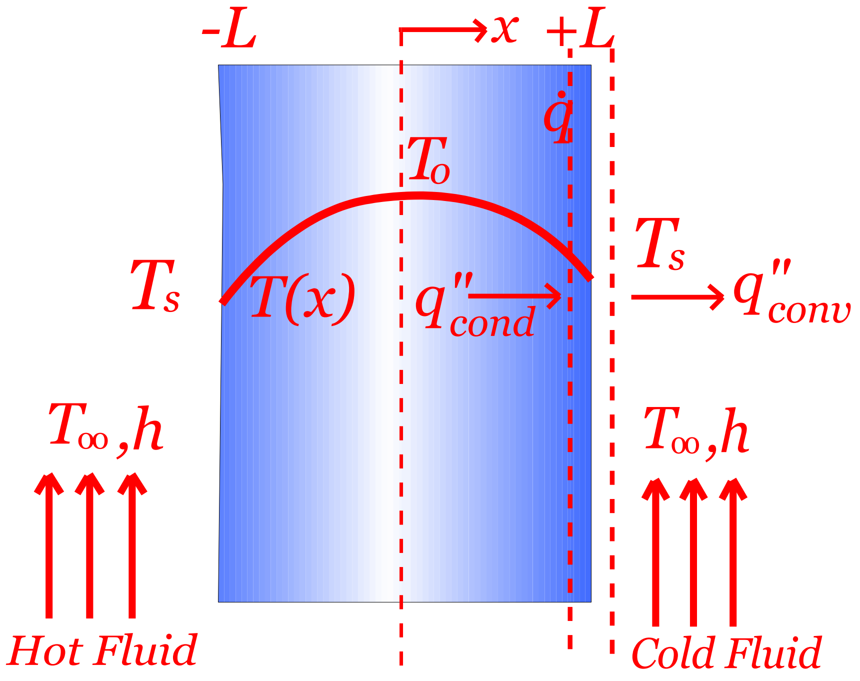
$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

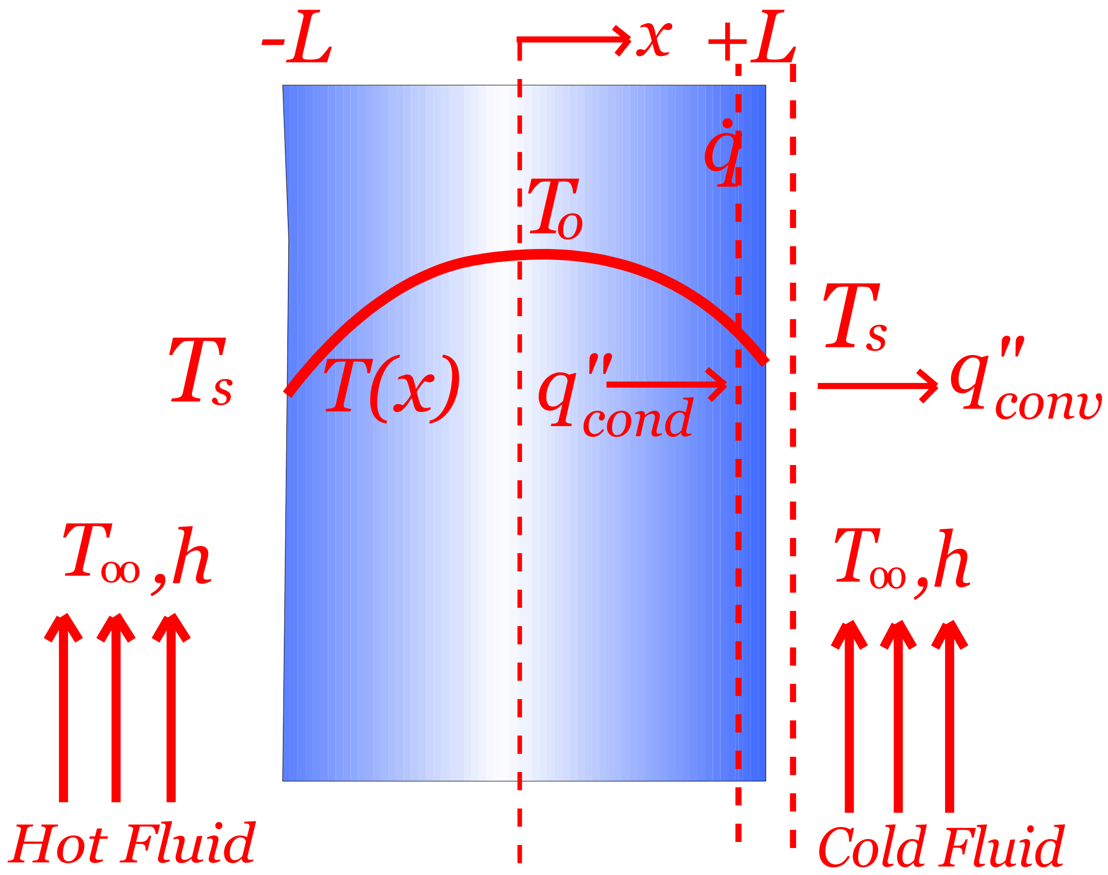
$$T_o = \frac{\dot{q}L^2}{2k} + T_s \qquad T_s = T_o - \frac{\dot{q}L^2}{2k}$$

$$T(x) - T_o + \frac{\dot{q}L^2}{2k} = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right)$$

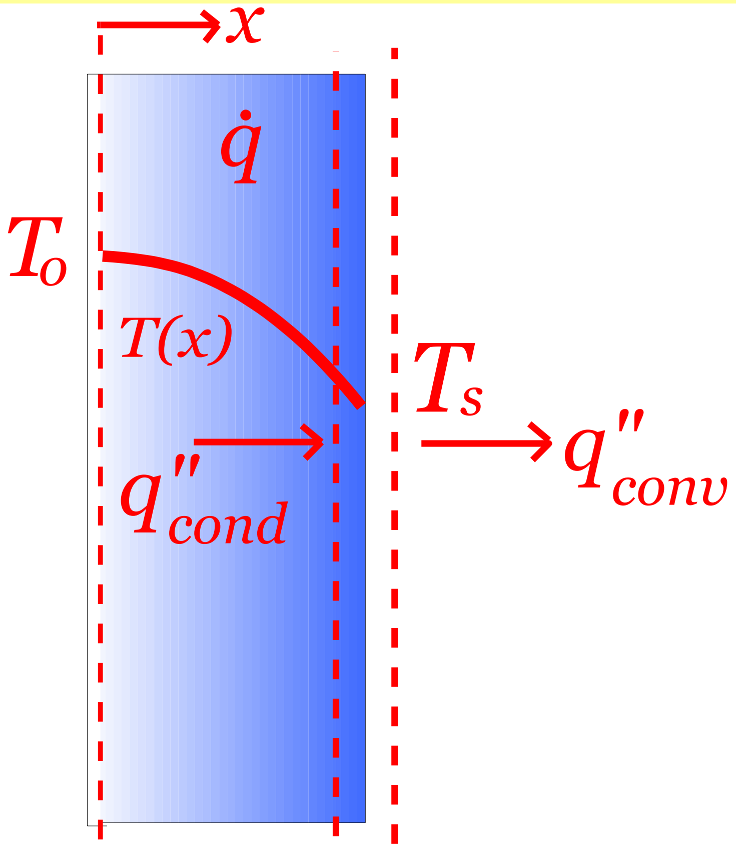
$$T(x) - T_o = \frac{\dot{q}L^2}{2k}\left(\frac{x^2}{L^2}\right)$$

$$T_o = \frac{\dot{q}L^2}{2k} + T_s$$





Adiabatic Surface At Midplane



It is important to note that the plane of symmetry in Figure, the temperature gradient is zero, $(dT/dx)_{x=0}$.
There is no heat transfer across this symmetric plane and it may be represented by the adiabatic surface shown in Figure.
One implication of this result is that present derived equation also applies to plane walls that are perfectly insulated on one side ($x = 0$) and maintained at a fixed temperature T_s on the

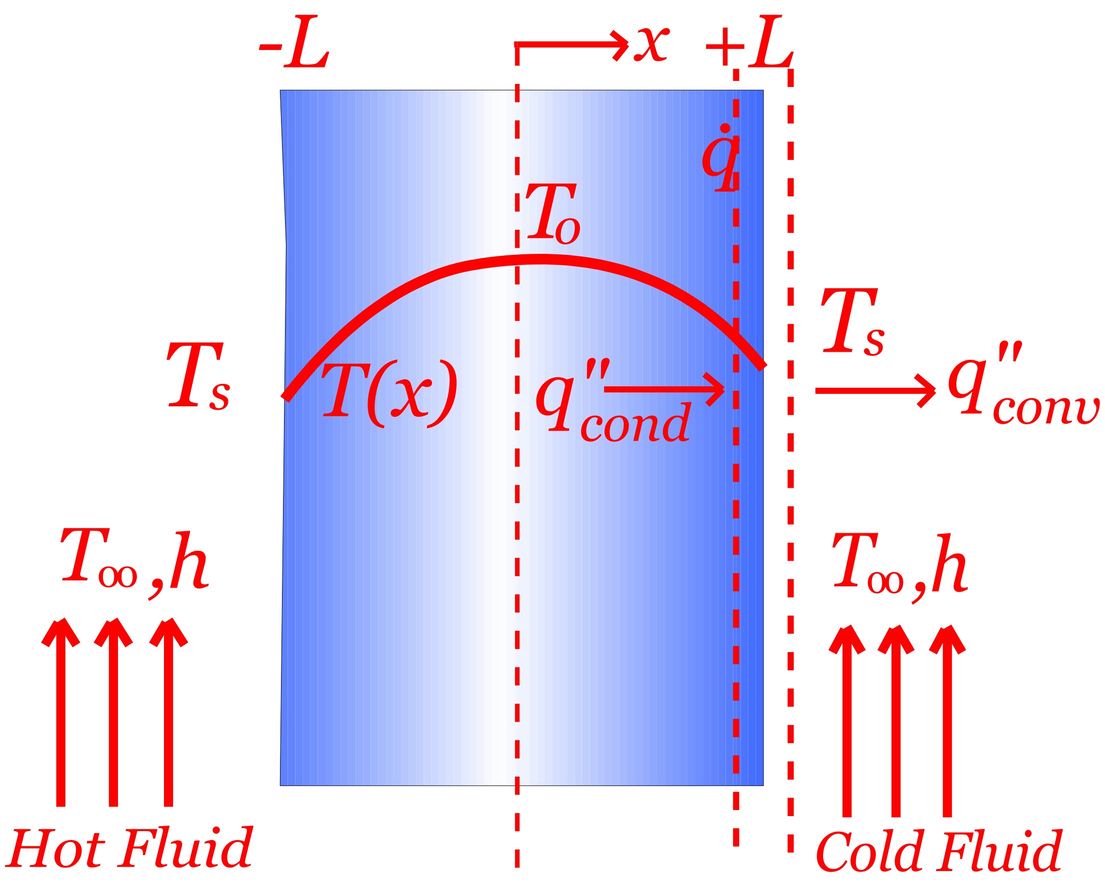
Symmetrical Boundary Condition

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

$$T_o = \frac{\dot{q}L^2}{2k} + T_s$$

To use the above results the surface temperature T_s must be known.
However, a common situation is one for which it is the temperature of an adjoining fluid,, and not T_s, which is known.
It then becomes necessary to relate T_s to this relation may be developed by applying a surface energy balance.
Consider the surface at $x = L$ for the symmetrical plane wall or the insulated plane wall Neglecting the radiation and substituting the appropriate rate equations, the energy balance is given by

Consider the surface at $x = L$ for the symmetrical plane wall or the insulated plane wall Neglecting the radiation and substituting the appropriate rate equations, the energy balance is given by



$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

$$T_o = \frac{\dot{q}L^2}{2k} + T_s$$

$$-k\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty)$$

$$-k\frac{\dot{q}L^2}{2k}\left(-\frac{2L}{L^2}\right) = h(T_s - T_\infty)$$

$$\frac{\dot{q}L}{2h} = (T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}L}{2h}$$

Hence, T_s may be computed from knowledge of T_∞, $\dot{q}$, L and h.

ONE DIMENSION STEADY STATE HEAT CONDUCTION WITH HEAT GENERATION – RADIAL SYSTEMS

Consider the long, solid cylinder which could represent a current carrying wire.

For steady state conditions the rate at which heat is generated within the cylinder must equal the rate at which heat is convected from the surface of the cylinder to a moving fluid.

This condition allows the surface temperature to be maintained at a fixed value of T_s.

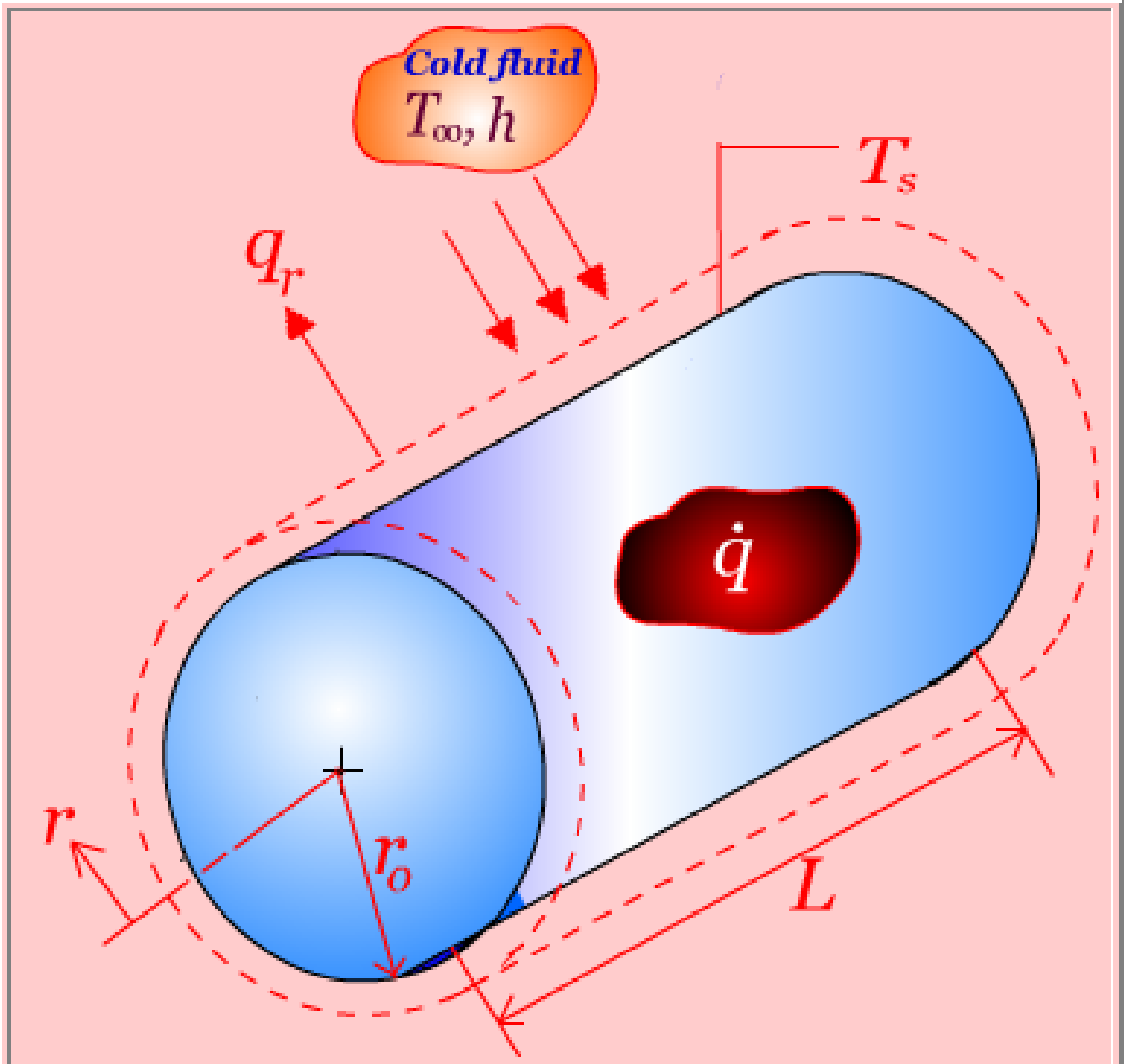
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0 \qquad \frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}r=0$$

$$r\frac{dT}{dr}+\frac{\dot{q}}{2k}r^2=C_1$$

$$\frac{dT}{dr}+\frac{\dot{q}}{2k}r=\frac{C_1}{r}$$

$$T+\frac{\dot{q}}{2k}\frac{r^2}{2}=C_1 lnr+C_2$$

$$T=-\frac{\dot{q}}{4k}r^2+C_1 lnr+C_2$$



Conduction In A Solid Cylinder with Uniform Heat Generation

ONE DIMENSION STEADY STATE HEAT CONDUCTION WITH HEAT GENERATION – RADIAL SYSTEMS

$$T = -\frac{\dot{q}}{4k}r^2 + C_1 lnr + C_2$$

$$\frac{dT}{dr} + \frac{\dot{q}}{2k}r = \frac{C_1}{r}$$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

Symmetric boundary condition

$$T\Big|_{r=r_o} = T_s$$

$$r\frac{dT}{dr} + \frac{\dot{q}}{2k}r^2 = C_1$$

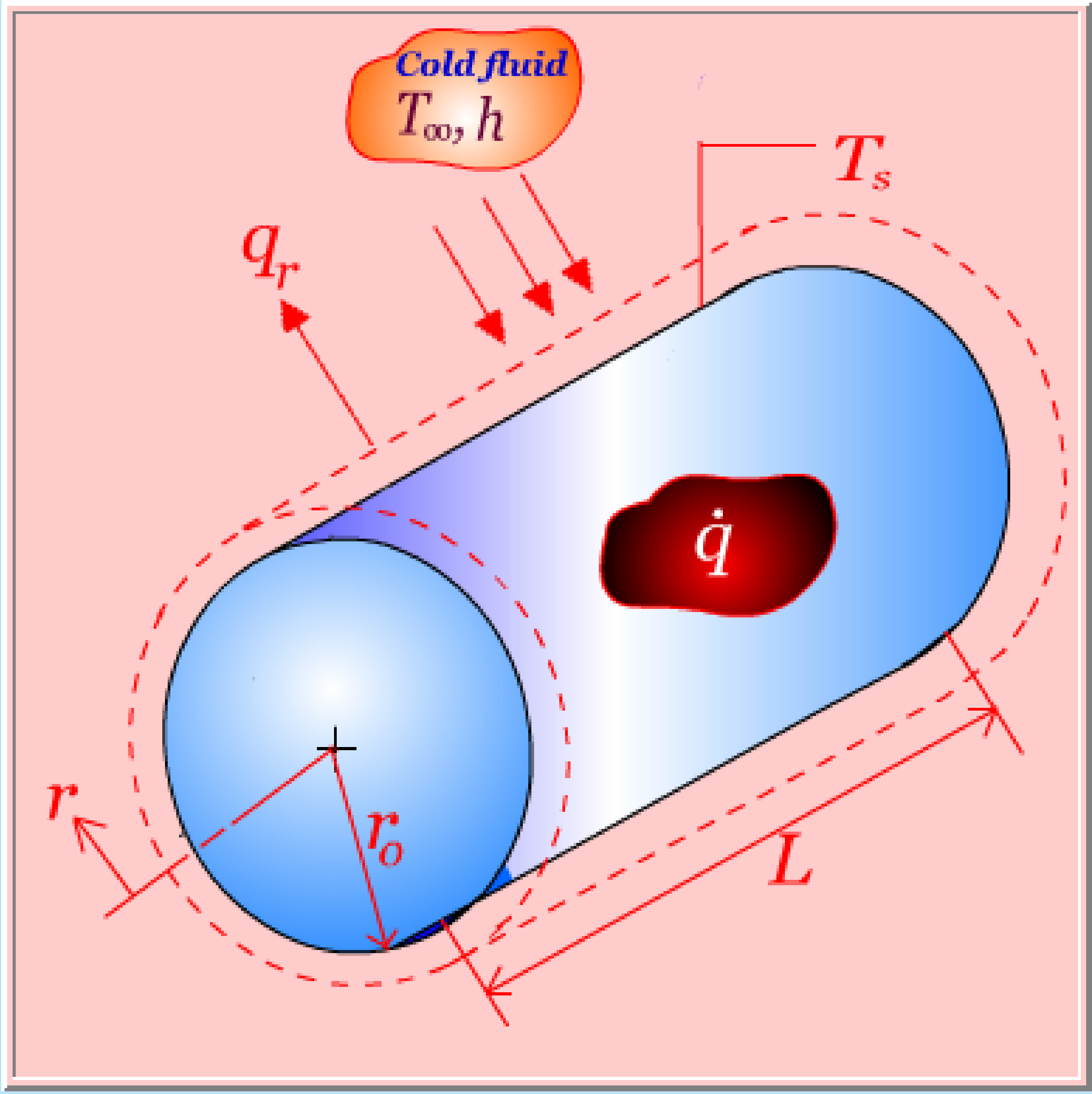
$$C_1 = 0$$

$$T_s = -\frac{\dot{q}}{4k}r_o^2 + (0)lnr + C_2$$

$$C_2 = T_s + \frac{\dot{q}}{4k}r_o^2$$

$$T = -\frac{\dot{q}}{4k}r^2 + T_s + \frac{\dot{q}}{4k}r_o^2$$

$$T = T_s + \frac{\dot{q}r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right)$$



ONE DIMENSION STEADY STATE HEAT CONDUCTION WITH HEAT GENERATION – RADIAL SYSTEMS

$$T = T_s + \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right)$$

$$T_o = T_s + \frac{\dot{q} r_o^2}{4k}$$

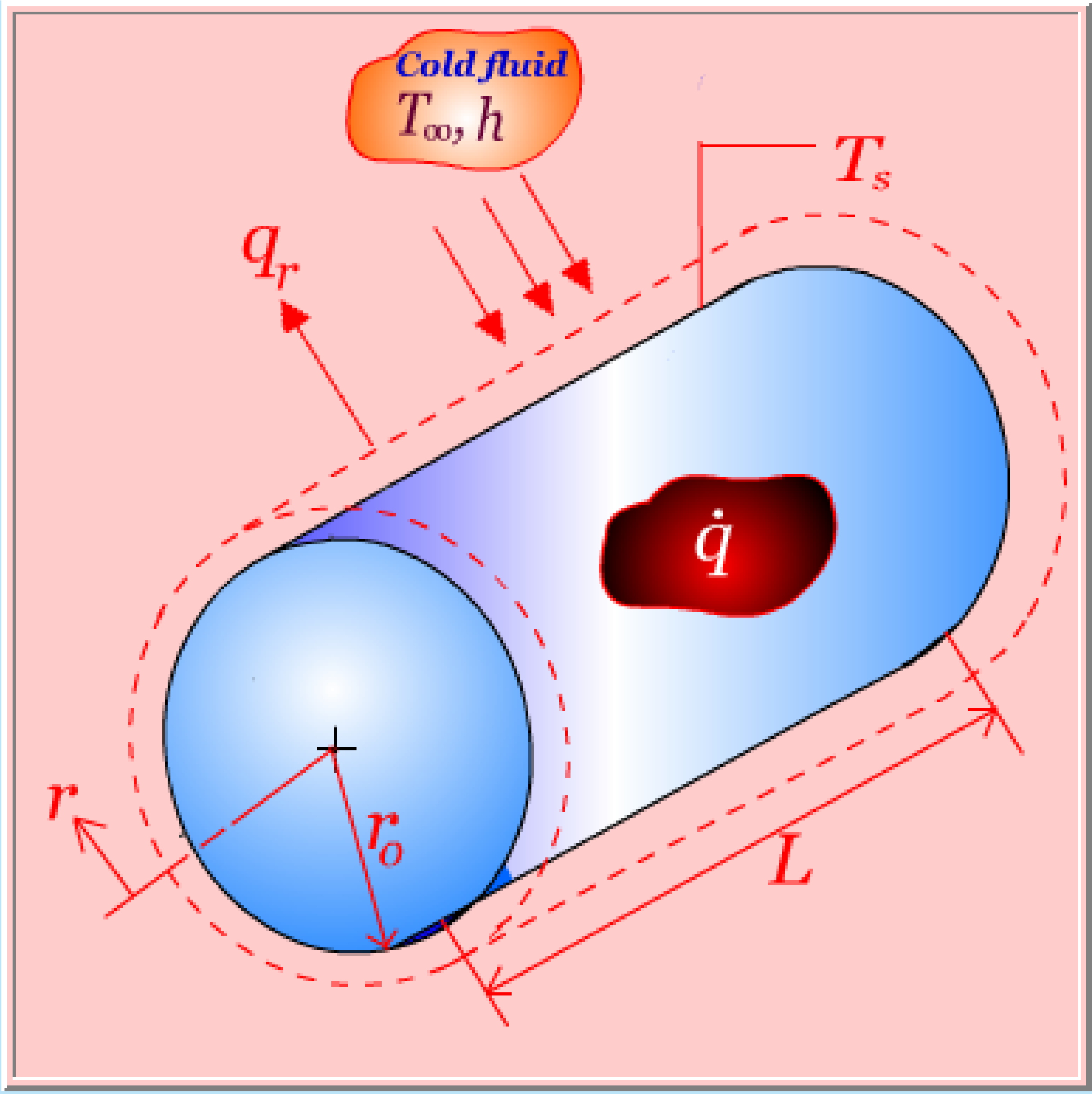
$$\frac{T - T_s}{T_o - T_s} = \left(1 - \frac{r^2}{r_o^2}\right)$$

T_o is the center line temperature.
The heat rate at any radius in the cylinder may be evaluated by using the temperature distribution with Fourier's Law.

$$\dot{E}_g = \dot{E}_{out}$$

$$\dot{q}\pi r_o^2 L = h(2\pi r_o L)(T_s - T_\infty)$$

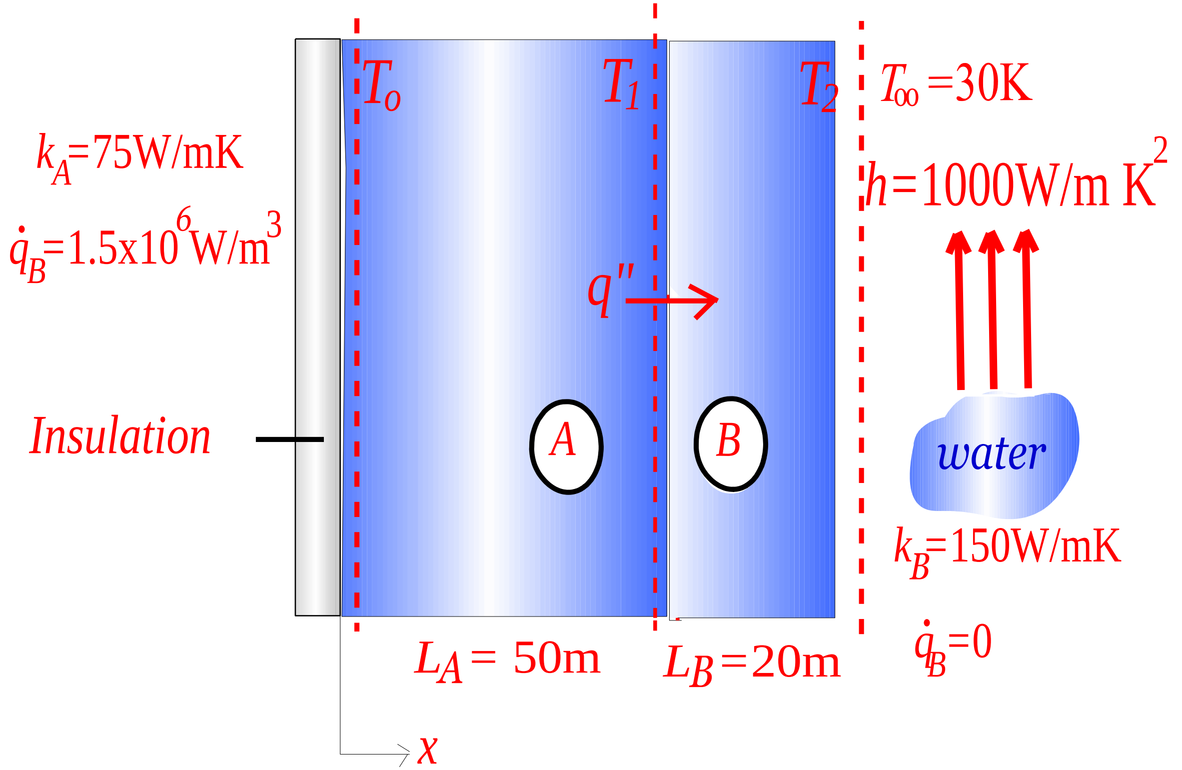
$$T_s = T_\infty + \frac{\dot{q} r_o}{2h}$$



A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6 \ \frac{W}{m^3}$ $k_A = 75 \ \frac{W}{m.K}$ and thickness $L_A = 50\, mm$. The wall material B has no generation with $k_B = 150 \ \frac{W}{m.K}$ and thickness $L_B = 20\, mm$. The inner surface of material A is well insulated, while the outer surface of material B is cooled by water stream $T_\infty = 30℃$ and $h = 1000 \ W/m^2.K$.

Sketch the temperature distribution that exists in the composite under steady state conditions. Determine the temperature T_o of the insulated surface and the temperature T_2 of the cooled surface.

Figure

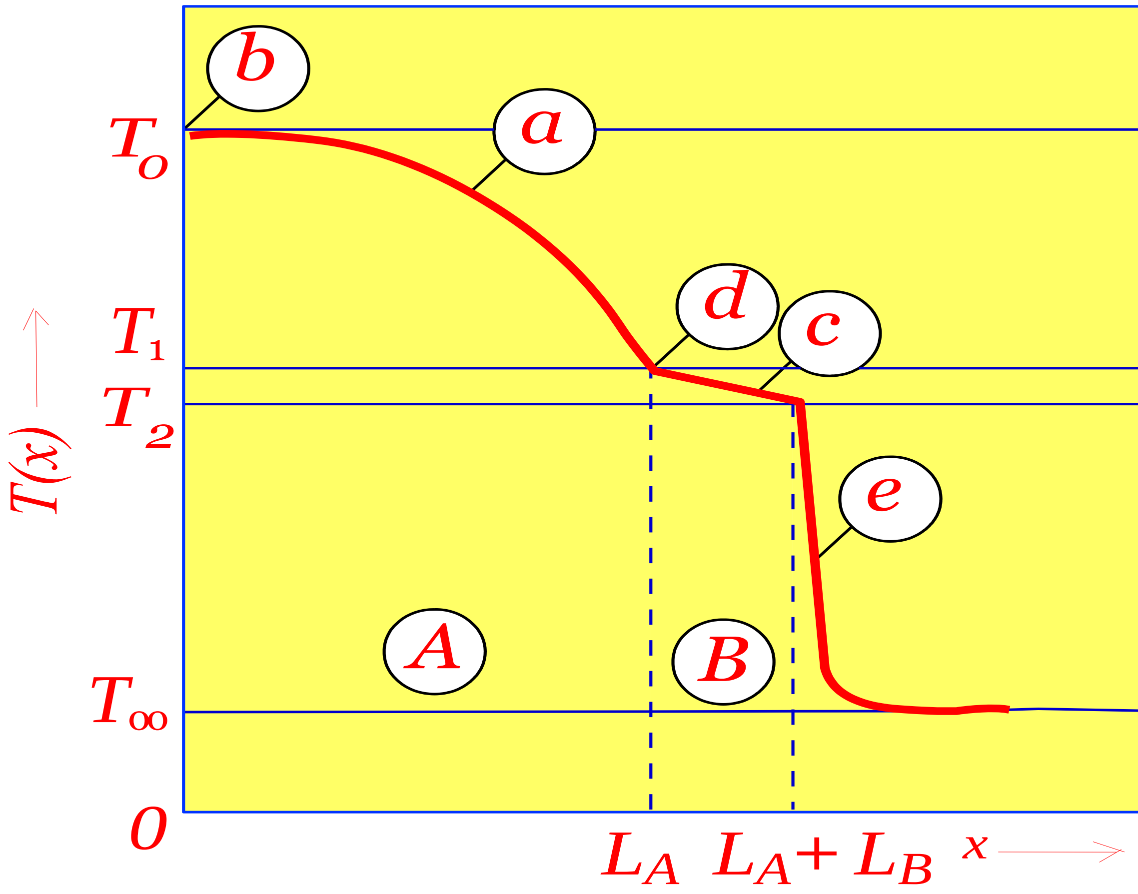


Known : Plane wall of material A with internal heat generation is insulated on one side and bounded by a second wall of material B, which is without heat generation and is subjected to convection cooling

Find : Convection heat transfer coefficient between the outer surface of the wire and the air in the room.

Assumptions:

1. Heat transfer is steady since there is no indication of any change with time.
2. Heat transfer is one dimensional since there is thermal symmetry about the center line and no variation in the axial direction.
3. Thermal conductivities are constant.
4. The thermal contact resistance at the interface is negligible.
5. Heat transfer coefficient incorporates the radiation effects, if any.



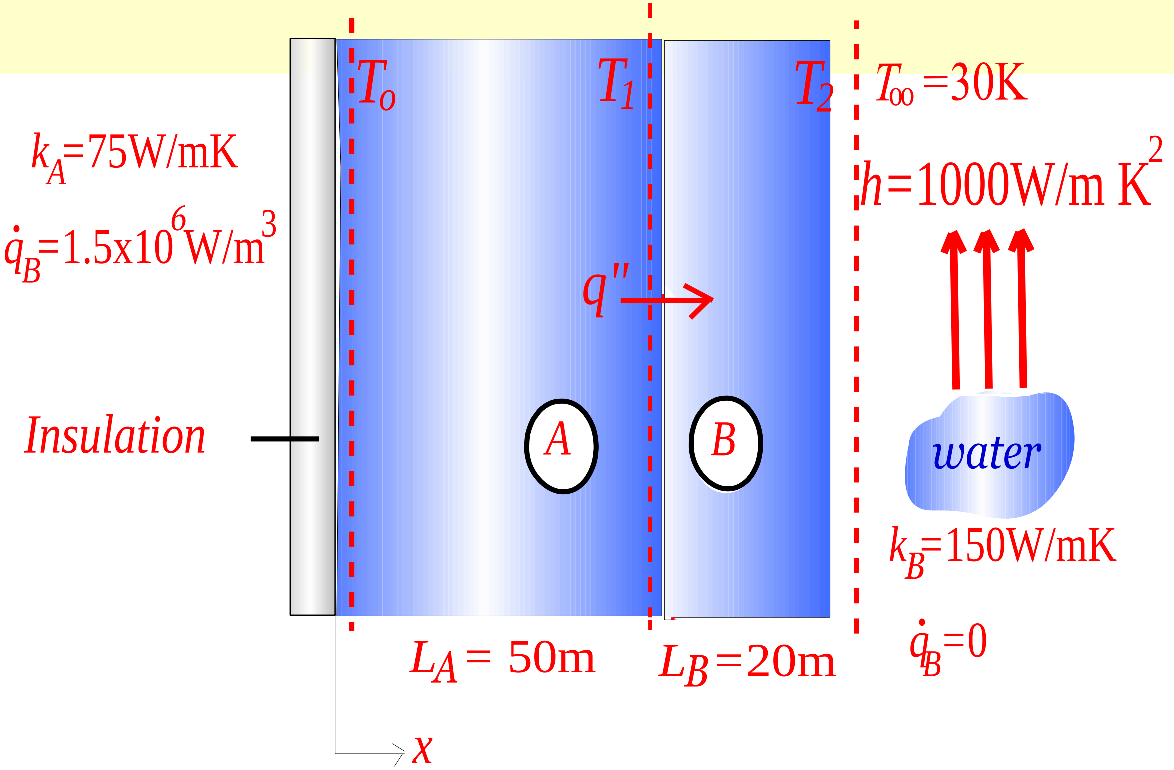
From the prescribed physical conditions, the temperature distribution in the composite is known to have the following features, as shown:

- **Parabolic in material A**
- **Zero slope at insulated boundary**
- **Linear in material A**
- **The temperature distribution in the water is characterized by large gradients near the surface**
- **Slope change = $k_A / k_B = 2$ at interface**

The outer surface temperature T_2 may be obtained by performing an energy balance on the control volume about material B.

Since there is no generation in this material, it follows that, for steady state conditions and a unit surface area, the heat flux into the material at $x = L_A$ must equal the heat flux from the material due to convection at $x = L_A + L_B$

$$q'' = h(T_2 - T_\infty)$$



The heat flux may be determined by performing a second energy balance on a control volume about material A.

In particular, since the surface at $x = 0$ is adiabatic, there is no inflow and the rate at which the energy is generated must equal the outflow. Accordingly, for a unit surface area,

$$\dot{q}L_A = q''$$

$$q'' = \dot{q}L_A = 1.5 \times 10^6 (0.05) = 75000\, W/m^2$$

$$q'' = 75000\, W/m^2$$

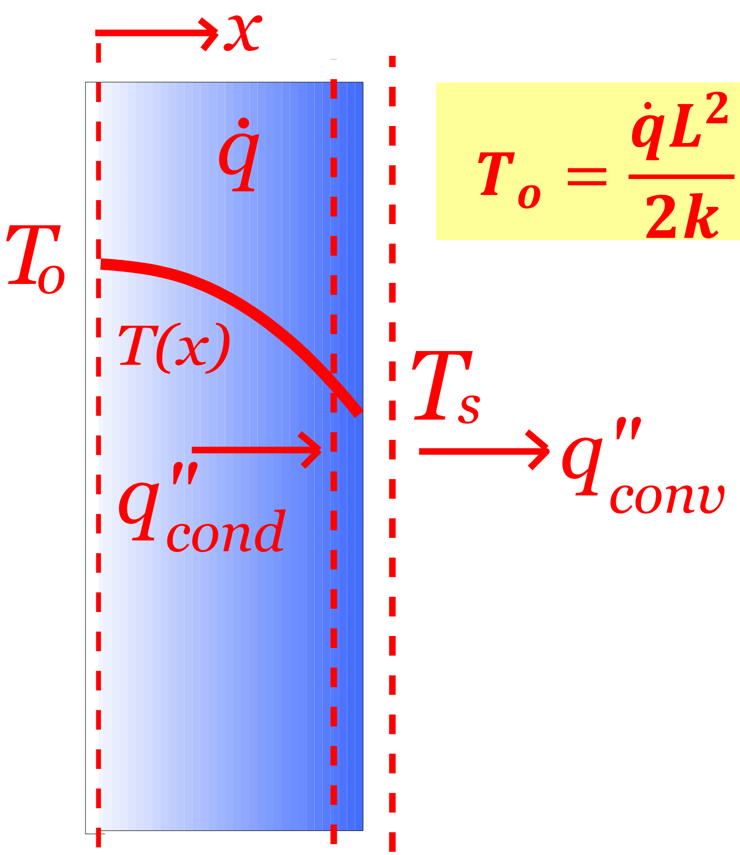
$$\dot{q}L_A = h(T_2 - T_\infty)$$

$$T_2 = T_\infty + \frac{\dot{q}L_A}{h}$$

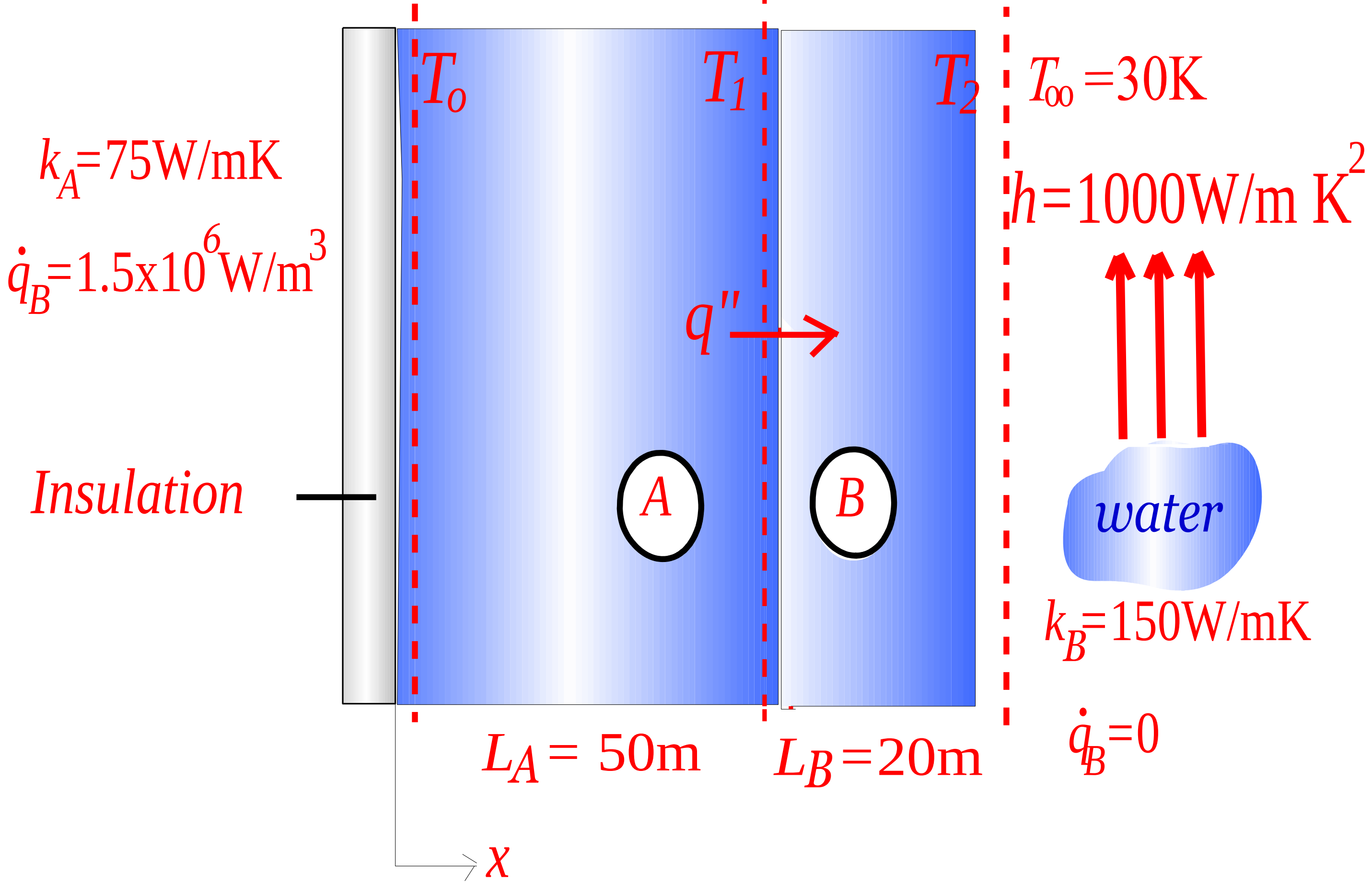
$$T_2 = 30 + \frac{1.5 \times 10^6 (0.05)}{1000} = 105$$

$$T_2 = 105°\text{C}$$

Adiabatic Surface At Midplane



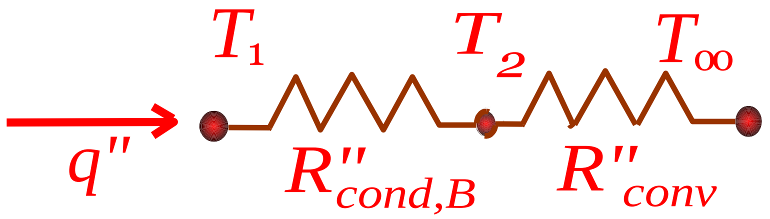
$$T_o = \frac{\dot{q}L^2}{2k} + T_s$$



Temperature at the insulated surface $(x = 0)$

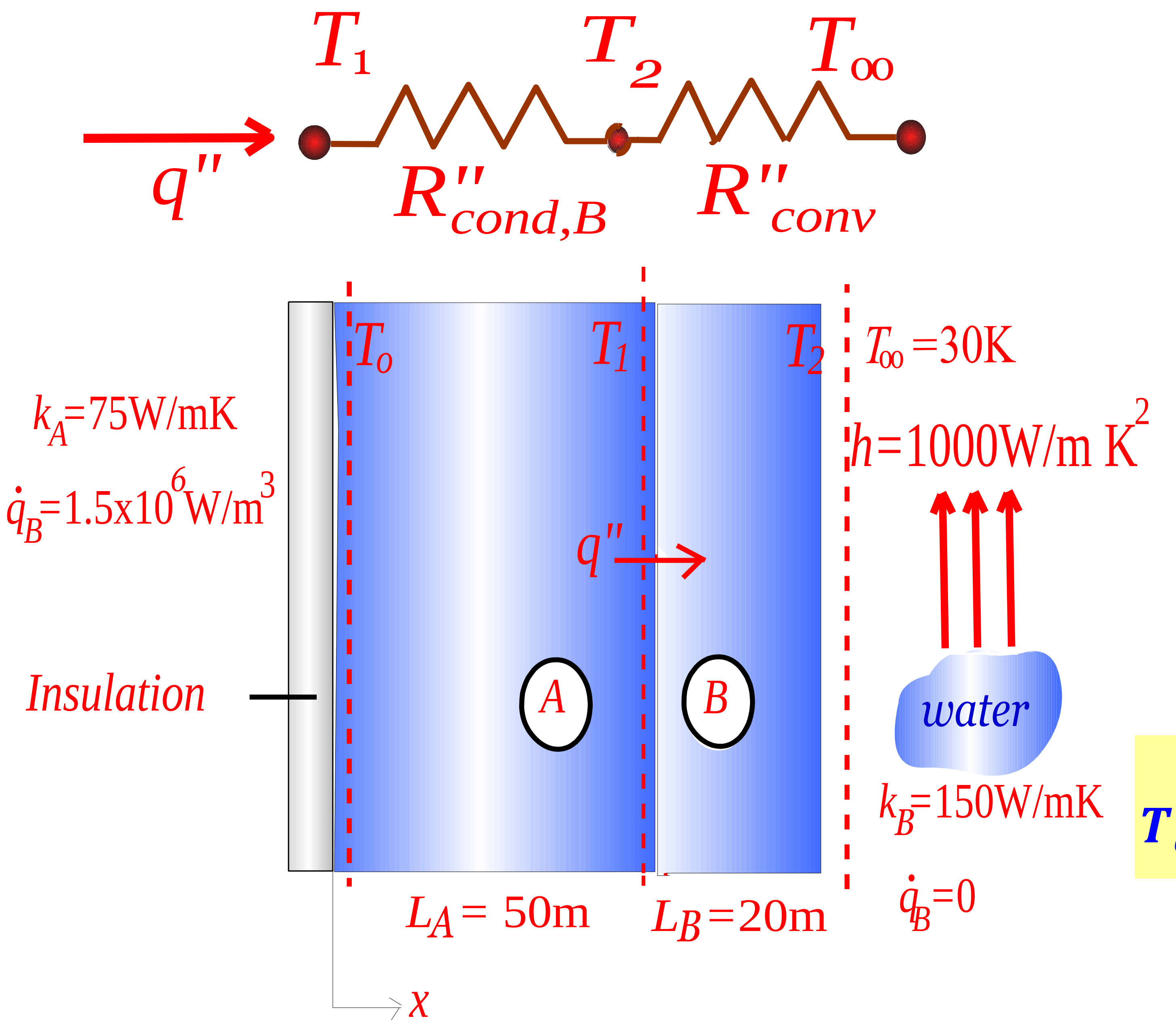
$$T_o = \frac{\dot{q}L^2}{2k} + T_1$$

T_1 may be obtained from the thermal circuit for material B



T_1 may be obtained from the thermal circuit for material B

$$q'' = 75000\,W/m^2$$



$$R''_{cond,B} + R''_{conv,B} = \frac{T_1 - T_\infty}{q''}$$

$$\frac{L_B}{k_B} + \frac{1}{h} = \frac{T_1 - T_\infty}{q''}$$

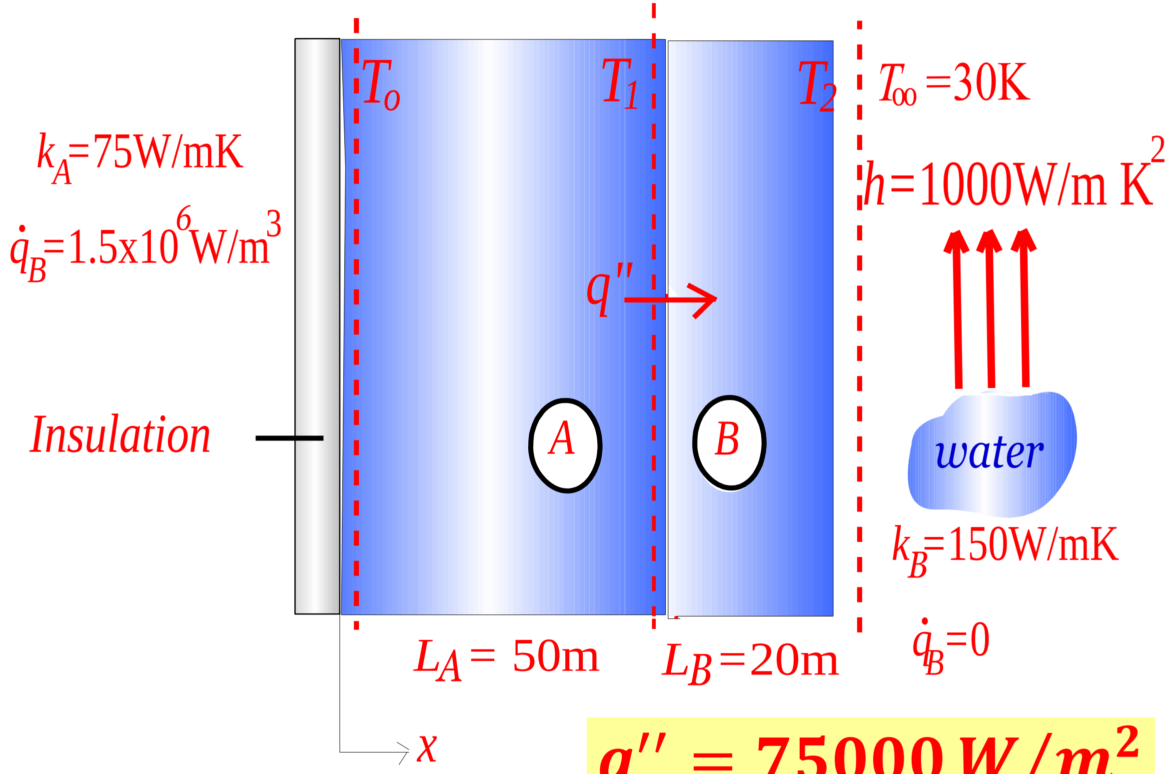
$$\frac{20 \times 10^{-3}}{150} + \frac{1}{1000} = \frac{T_1 - 30}{75000}$$

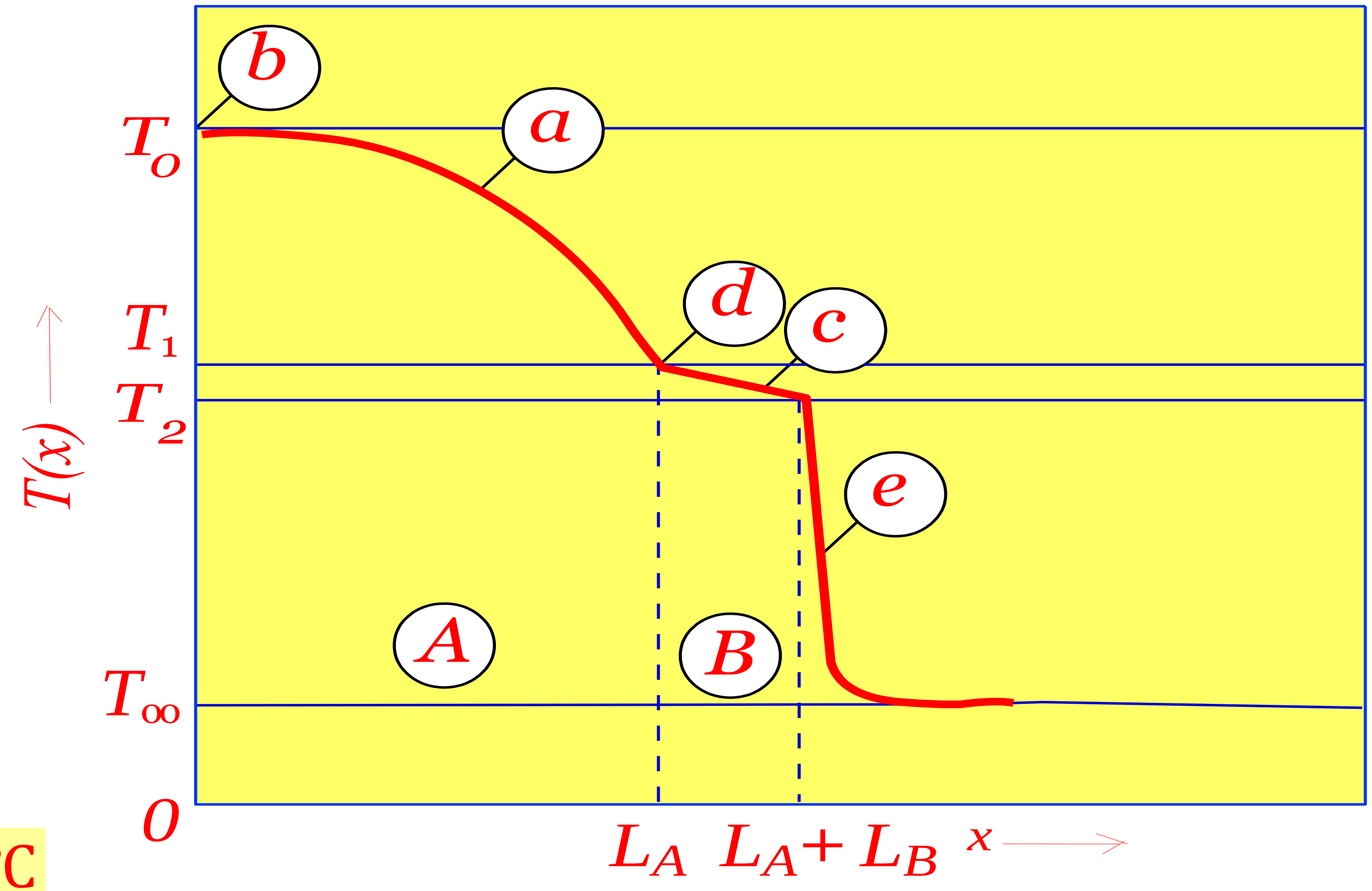
$$T_1 = 115°\text{C}$$

$$T_o = \frac{\dot{q}L^2}{2k} + T_1$$

$$T_o = \frac{1.5 \times 10^6 \times (50 \times 10^{-3})^2}{2 \times 75} + 115$$

$$T_o = 140°\text{C}$$





$$q'' = 75000\,W/m^2$$

$T_o = 140$°C $\quad T_1 = 115$°C $\quad T_2 = 105$°C

Comments:

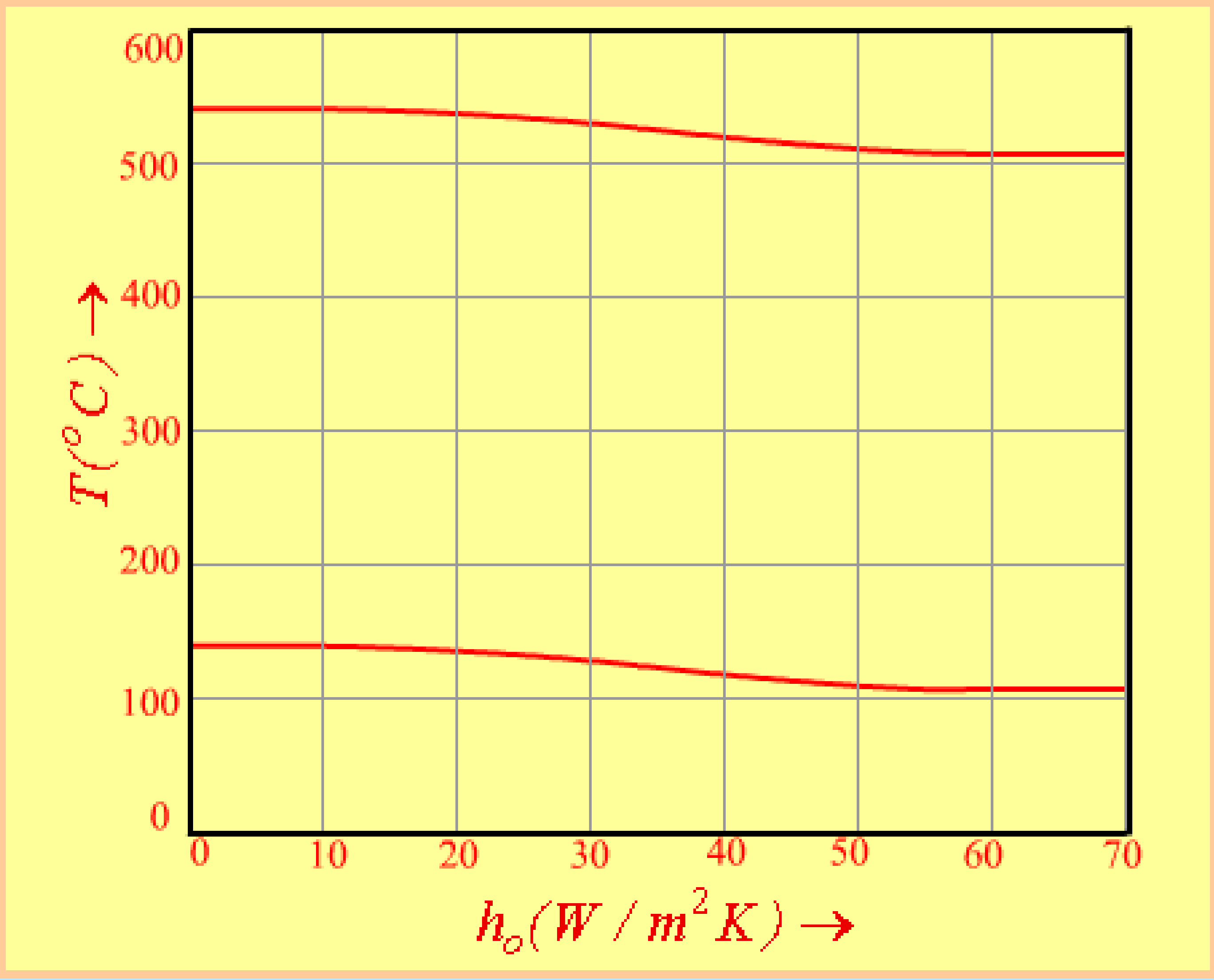
1. **Material A, having heat generation, cannot be represented by a thermal circuit element**
2. **Since the resistance to heat transfer by convection is significantly larger than due to conduction in material B, $R''_{cond,B}/R''_{conv,B} = 7.5$, the surface-to-fluid temperature difference is much larger than the temperature drop across material B,**

   This result is consistent with the temperature distribution plotted in part (1).

The surface and interface temperatures (T_o, T_1, *and* T_2) depend on the generation rate , the thermal conductivities k_A and k_B, and the convection coefficient *h*.

Each material will have a maximum allowable operating temperature, which must not be exceeded if thermal failure of the system is to be avoided.

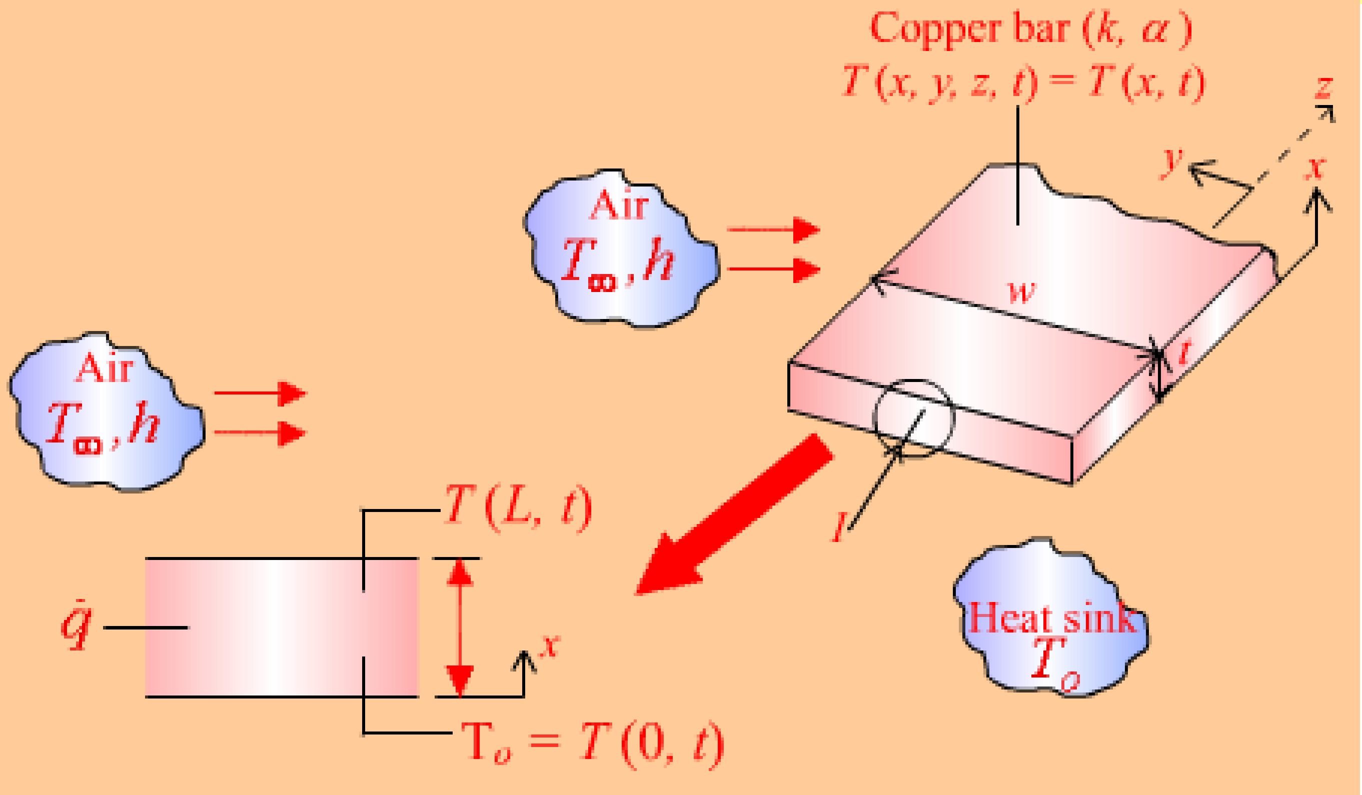
We explore the effect of one of these parameters by computing and plotting the temperature distribution for values of *h* = 200 and 1000 $W/m^2.K$, which would be representative of air and liquid cooling respectively.



For h = 200 W/m2.K, there is a significant increase in temperature throughout the system, and depending on the selection of materials, thermal failure could be a problem. Note the slight discontinuity in the temperature gradient dT/dx, at x = 50 mm.

A long copper bar of rectangular cross-section, whose width w is much greater than its thickness L, is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink, T_o. Suddenly an electric current is passed through the bar and an air stream of temperature T_∞ is passed over the top surface, while the bottom surface continues to be maintained at T_o. Obtain the differential equation and the boundary and initial conditions that could be solved to determine the temperature as a function of position and time in the bar.

Figure



Known : Copper bar initially in thermal equilibrium with heat sink is suddenly heated by passage of an electric current .

Find : Differential equation and boundary and initial conditions needed to determine temperature as a function of position and time within the bar.

Assumptions:

1. Since $w \gg L$, side effects are negligible and heat transfer within the bar is primarily one dimensional in the x-direction.
2. Uniform volumetric heat generation, .
3. Constant properties.

Analysis:

The temperature distribution is governed by the heat equation, which, for the one-dimensional and constant property conditions of the present problem, reduces to

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\,\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}}{k}=\frac{\rho C_p}{k}\frac{\partial T}{\partial t}$$

where the temperature is a function of position of time, $T(x,t)$.

Since this differential equation is second-order in the spatial co-ordinate x and first order in time t , there must be two boundary conditions for the x-direction and one condition termed initial condition, for time.

The boundary condition at the bottom surface corresponds to case 1 of Table 2.1. In particular, since the temperature at the bottom surface is maintained at a value, T_o, which is fixed with time, it follows that

$$T(0,t)=T_o$$

In contrast, the convection surface condition is appropriate for the top surface. Hence

$$-k \left. \frac{\partial T}{\partial x} \right|_{x = L} = h[T(L, t) - T_\infty]$$

The initial condition is inferred from recognition that, the change in conditions, the bar is at uniform temperature T_o. Hence

$$T(x, 0) = T_o$$

If T_o, T_∞ and $\dot{q}$ and h are known, above four equations may be solved to obtain the time-varying temperature distribution $T(x, t)$ following imposition of the electric current.

Comments:

1. The heat sink at $x = 0$ could be maintained by exposing the surface to an ice bath by attaching it to a cold plate. A cold plate contains coolant channels machined in a solid of large thermal conductivity (usually copper). By circulating a liquid (usually water) through the channels, the plates and hence the surface to which it is attached, may be maintained at a nearly uniform temperature.

2. The temperature of the top surface $T(L, t)$ will change with time. This temperature is unknown and may be obtained after finding $T(x, t)$.