1. Introduction to the Fourier Transform

Review: Complex Numbers

$i = \sqrt{-1}$

$i^2 = -1$

$z = \underbrace{a}_{\text{Real}} + \underbrace{bi}_{\text{Imaginary Part}}$

$|z| = \sqrt{a^2 + b^2}$

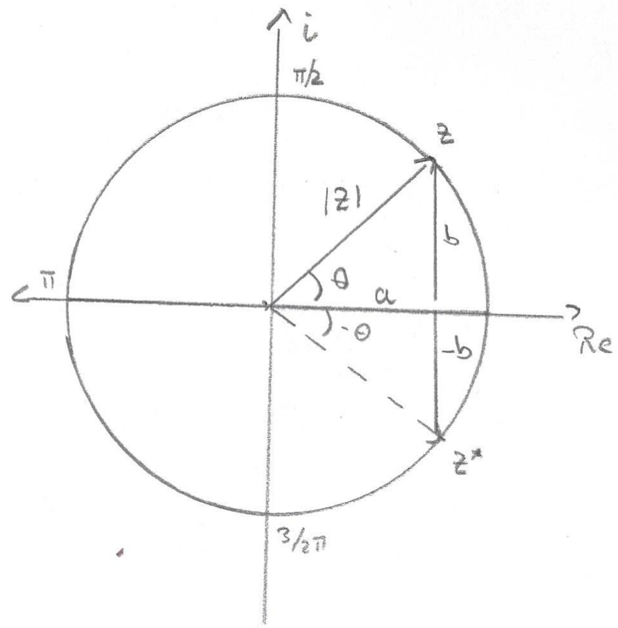
$z^* = a - bi$ "complex conjugate"

$a = \cos(\theta)$

$b = \sin(\theta)$

$\tan(\theta) = \frac{b}{a}$

$\Rightarrow \theta = \underset{\tan^{-1}}{\underset{\uparrow}{\arctan}}\left(\frac{b}{a}\right)$



Euler's Identity

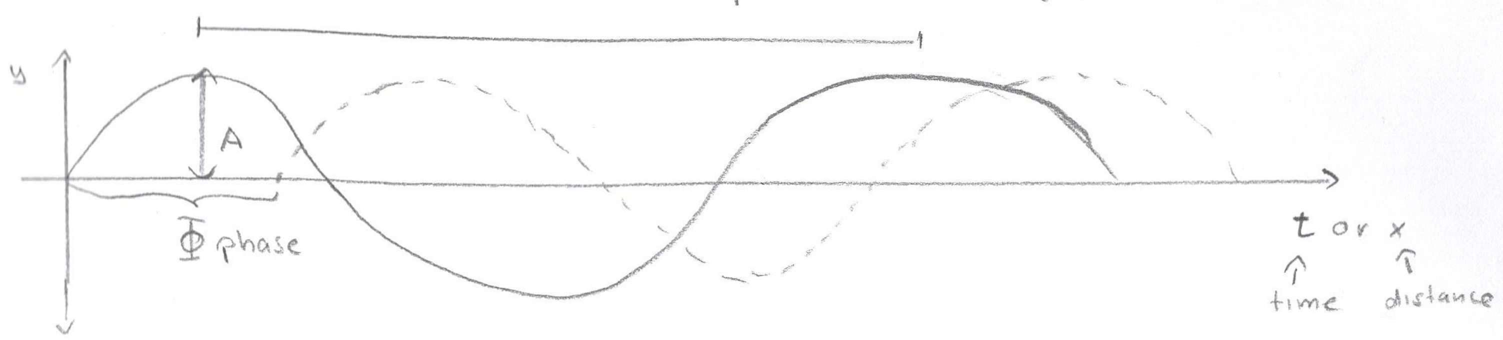
$$\exp(i\theta) = \cos(\theta) + i\sin(\theta)$$

$$\exp(i\pi) = \underbrace{\cos(\pi)}_{-1} + \underbrace{i\sin(\pi)}_{0} = -1$$

$$z = a + bi = |z|\exp(i\theta)$$

$$z^* = a - bi = |z|\exp(-i\theta)$$

Properties of waves



A: Amplitude

T: period [s]

f: frequency $= \frac{1}{T}$ $[\frac{1}{s}]$ [Hz]

λ: wavelength [m]

ν: wavelength $\frac{1}{\lambda}$ $[\frac{1}{m}]$

$y(t) = A\cos(2\pi f t + \phi)$ Time domain

$y(x) = A\cos(2\pi \nu x + \phi)$ space domain

Sum of angels

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$y(t) = A\cos(2\pi f t + \phi)$

$y(t) = A\cos(2\pi f t)\cos(\phi) - A\sin(2\pi f t)\sin(\phi)$

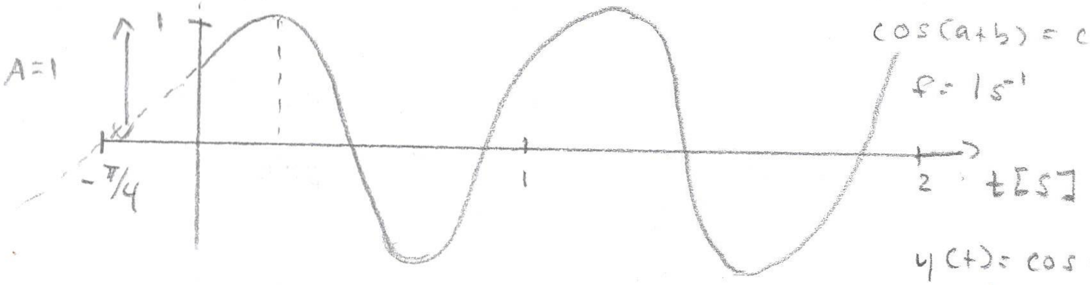
$y(t) = a_k \cos(x) + b_k \sin(x)$

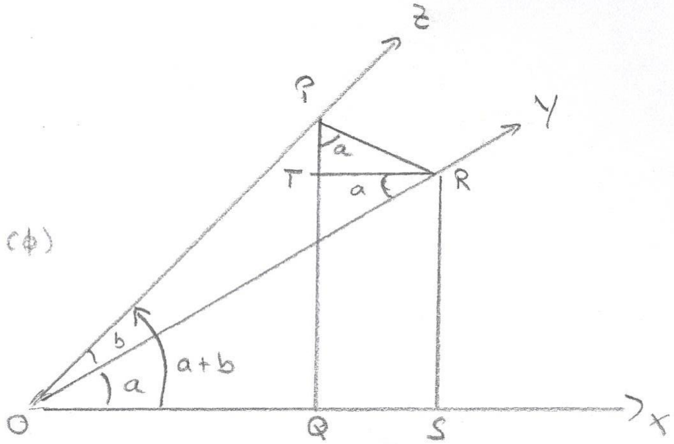
$a_n = A\cos(\phi)$

$b_n = -A\sin(\phi)$

$x = 2\pi f t$

$\phi = -\pi/4$





$\cos(a+b) = \frac{OQ}{OP}$

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$f = 1\,s^{-1}$

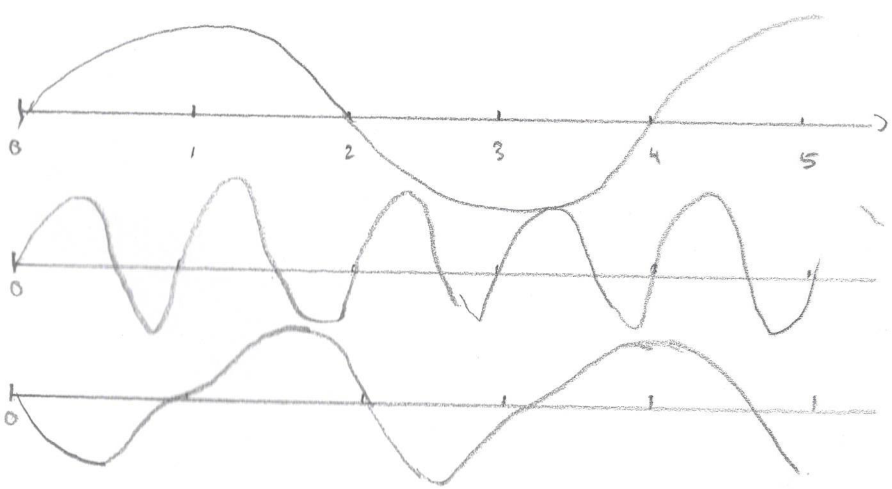
$y(t) = \cos(2\pi t - \pi/4)$

$y(t) = A\cos(\pi/4)\cos(2\pi t)$
$- A\sin(\pi/4)\sin(2\pi f t)$

Fourier series: Expansion of a periodic function into a sum of trigonometric functions

$$y(t) = \text{mean} + \sum_{k=1}^{\infty} A_k \cos\left(2\pi \frac{k}{N} t + \phi_k\right)$$

$$y(t) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos\left(2\pi \frac{k}{N} t\right) + b_k \sin\left(2\pi \frac{k}{N} t\right)$$



$f = 1\,Hz$
$\phi = -90°$

N: Period

$f = 0.5\,Hz$
$\phi = -90°$

$\frac{k}{N}$: Frequency

Σ of waves ⇒ • mean zero
• periodic function

" represent periodic function with period N using Fourier series "

$\frac{1}{N}$ $\left(\frac{1}{5s}\right)$ lowest frequency

$\frac{\infty}{N}$ $\left(\frac{\infty}{5s}\right)$ highest frequency

Fourier Series in the Complex Domain

$$y(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \underbrace{a_k \cos(x) + b_k \sin(x)}$$

$$x = 2\pi \frac{k}{N} t$$

$$\exp(ix) = \cos(x) + i\sin(x)$$

$$\exp(-ix) = \cos(x) - i\sin(x)$$

$$\exp(ix) = \cos(x) + i\sin(x)$$

$$\frac{1}{2}(a_k - ib_k)\exp(ix) = \frac{1}{2}a_k\cos(x) + \frac{1}{2}a_k i\sin(x) - \frac{1}{2}ib_k\cos(x) + \frac{1}{2}b_k\sin(x)$$

$$+ \quad \frac{1}{2}(a_k + ib_k)\exp(-ix) = \frac{1}{2}a_k\cos(x) - \frac{1}{2}a_k i\sin(x) + \frac{1}{2}ib_k\cos(x) + \frac{1}{2}b_k\sin(x)$$

$$X_k^*\exp(ix) + X_k\exp(-ix) = a_k\cos(x) + b_k\sin(x)$$

$$X_k = \frac{1}{2}(a_k + ib_k)$$

$$X_k^* = \frac{1}{2}(a_k - ib_k)$$

$$\frac{1}{2}\sqrt{X \cdot X^*} = A \quad \text{(amplitude)}$$

$$\tan^{-1}\left(\frac{b_k}{a_k}\right) = \tan^{-1}\left(\frac{Im(X_k)}{Re(X_k)}\right) = \phi \quad \text{(phase angle)}$$

$$y(t) = \sum_{-\infty}^{\infty} X_k \exp(-ix)$$

note: $\frac{1}{2}\sqrt{X_0 X_0^*} = \frac{1}{2}a_0$ = mean

each real frequency (e.g. $k=1$) has two X coefficients

Discrete Fourier Transform: how to find X_k?

Fourier Series

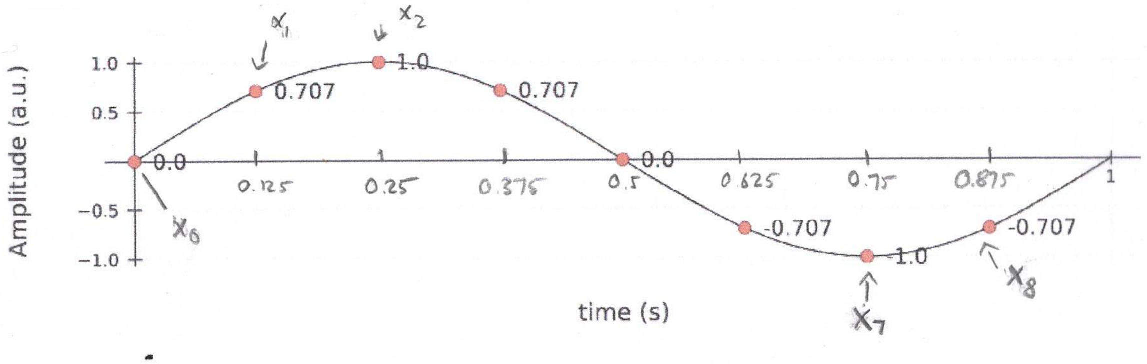
$$y(t) = \sum_{-\infty}^{\infty} X_n \exp(-ix) \qquad x = 2\pi \frac{k}{N} t$$

Infinite series representing an N-periodic function in the time domain $\Rightarrow$ infinite X_k

Discrete Fourier Series

$$X_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp(-ix) \qquad x = 2\pi \frac{k}{N} n$$

Finite sequence of N equally spaced samples of a function $y(t)$



$y(t) = \sin(2\pi t)$

$f = 8\,Hz$ "sample frequency"

$\Delta t = \frac{1}{8} = 0.125\,s$

$N = 8$ points

$y_0 = 0$

$\vdots$

$y_2 = 1.0$

How to find X_k?

$$X_k = \sum_{n=0}^{N-1} x_n \exp(-ix) \qquad x = 2\pi \frac{k}{N} n$$

$\Rightarrow$ N Fourier coefficients X_k for N sample points

$$X_0 = \sum_{n=0}^{N-1} x_n \qquad \Rightarrow \text{sample mean} = \frac{1}{N} \sum_{n=0}^{N-1} x_0 = \frac{1}{N} X_0$$

k	t	f	X_k	A
0	0	0 Hz	0	mean
1	0.125	1 Hz	$0-4i$	1
2	0.25	2	0	0
3		3	0	0
4		4	0	0
5		5	0	unresolvable
6		6	0	
7	0.875	7	$0+4i$	

$f_s = \frac{1}{\Delta t} = 8Hz$

$f = \frac{k}{N} f_s$

$A = \frac{2}{N}\sqrt{X_k X_k^*}$

$f_s/2 = N_{Ny}$

"Nyquist Frequency"

$X_1 = X_7^*$

How to find X_k?

$X_k = \sum_{n=0}^{N-1} x_n \exp(-ix)$ each X_k requires k additions

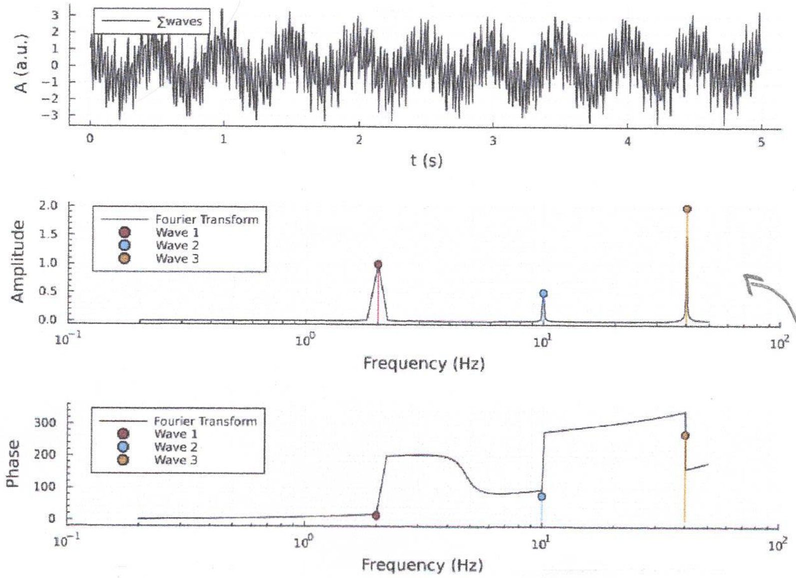
thus need $N \times N = N^2$ operations

DFT = $O(N^2)$ => slow if N is large

FFT: Fast Fourier Transform

- implemented as fft(x) in most languages
- requires 2^N data points 16, 32, 64, ...
- $O(N \log N)$

Example Reconstruction : DFT of 3 waves



$y(t) = A\cos(2\pi f t + \phi)$

← sum of three waves

$A_1 = 1$ $f_1 = 2\,Hz$ $\phi_1 = 15°$

$A_2 = 0.5$ $f_2 = 10\,Hz$ $\phi_2 = 77°$

$A_3 = 2$ $f_2 = 40\,Hz$ $\phi_3 = 270°$

$N = 500$ points

$\Delta t = 0.01\,s$

$f_s = 100\,Hz$

Amplitude Spectrum
- plot of f vs. A
- peaks recover f_i
- smeared due to resolution (leakage)
- peaks recover A_i

Phase spectrum
- f vs. θ
- typically noisy

Fourier Table for 3 wave example

fft(x)

	k	f	X	A	Θ
1	0	0.0	1.0784+0.0im	0.00431361	0.0
2	1	0.2	1.08794+0.0382485im	0.00435444	2.01351
3	2	0.4	1.11773+0.0785207im	0.00448195	4.01843
4	3	0.6	1.17173+0.123278im	0.0047128	6.00599
5	4	0.8	1.25807+0.176076im	0.00508134	7.96716
6	5	1.0	1.39262+0.242862im	0.00565457	9.89243
7	6	1.2	1.60812+0.33512im	0.00657065	11.7716
8	7	1.4	1.98165+0.479166im	0.00815503	13.5933
9	8	1.6	2.74408+0.753014im	0.0113821	15.345
10	9	1.8	5.03318+1.53995im	0.021054	17.0121
⋮ more					
251	250	50.0	-6.03357+0.0im	0.0241343	180.0

$f = k \frac{f_s}{N}$

$k=1 \Rightarrow f = \frac{100\,Hz}{500} = 0.2\,Hz$

lowest resolved frequency

$f_{Ny} = \frac{f_s}{2} = 50\,Hz \quad (k = 250)$

$A = \frac{2}{N}\sqrt{X_k X_k^*}$

$\phi = \text{atan}\left(\frac{\text{Im}(X_k)}{\text{Re}(X_k)}\right)$

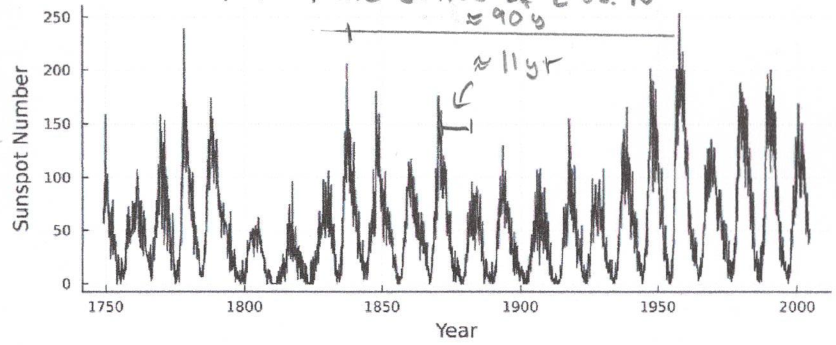
typically in radians

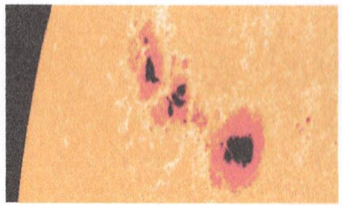
$\hookrightarrow \Theta = \left(\phi \frac{180}{\pi} + 360\right) \text{mod } 360$

modulo is the remainder of division

Example application: Sunspot Cycle

raw time series of t vs. N





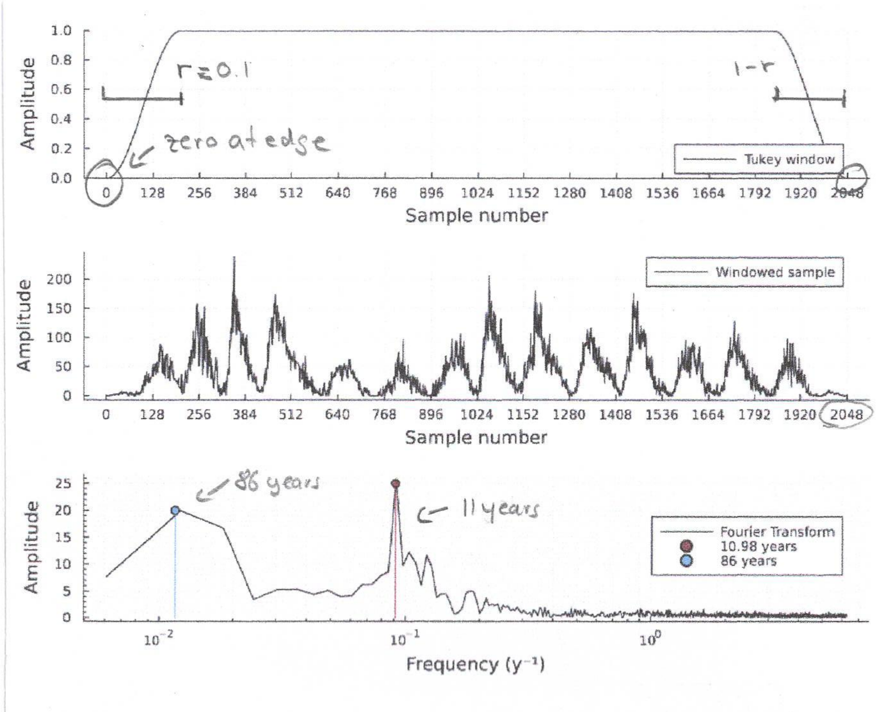
- check if dataset is clean (equal Δt, fix missing data)
- check resolution $\Delta t \approx 0.08\ y$

$$f = \frac{1}{\Delta t} = 12.5\ y^{-1}$$

$$f_{Ny} = 6.25\ y^{-1}$$

- evaluate time series by plotting
- evaluate length of data: 3067 data points

DFT analysis



(1) FFT requires 2^n points

$2^{11} = 2048$ $\quad 2^{12} = 4096$

record len 3067

-> pick 2048 points

(2) DFT / FFT is for N periodic functions

-> data may not be for selected subset

window function

$$w = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{r}\left[x - \frac{r}{2}\right]\right)\right] & 0 < x < r/2 \\ 1 & r/2 < x < 1 - r/2 \\ \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{r}\left[x - 1 + \frac{r}{2}\right]\right)\right] & 1 - r/2 < x < 1 \end{cases}$$

r is a fraction

data used in fft(x) is $x[1:2048] \cdot w(x)$ (windowed sample)

DFT table for sunspot example

	k	f	X	A	Θ
1	0	0.0	83709.7+0.0im	40.8739	0.0
2	1	0.00610352	-7833.07-89.1831im	7.64998	180.652
3	2	0.012207	-10183.3-17659.2im	19.9072	240.03
4	3	0.0183105	-15789.9+6492.85im	16.6726	157.647
5	4	0.0244141	183.586+3533.23im	3.45507	87.0256
6	5	0.0305176	-5398.21+399.728im	5.28612	175.765
7	6	0.0366211	-1474.74-5275.08im	5.34897	254.381
8	7	0.0427246	-2306.99+3858.64im	4.39033	120.874
9	8	0.0488281	83.3539+5241.94im	5.11973	89.089
10	9	0.0549316	-2809.03-2938.31im	3.96974	226.289
⋮ more					
1025	1024	6.25	-541.718+0.0im	0.529022	180.0

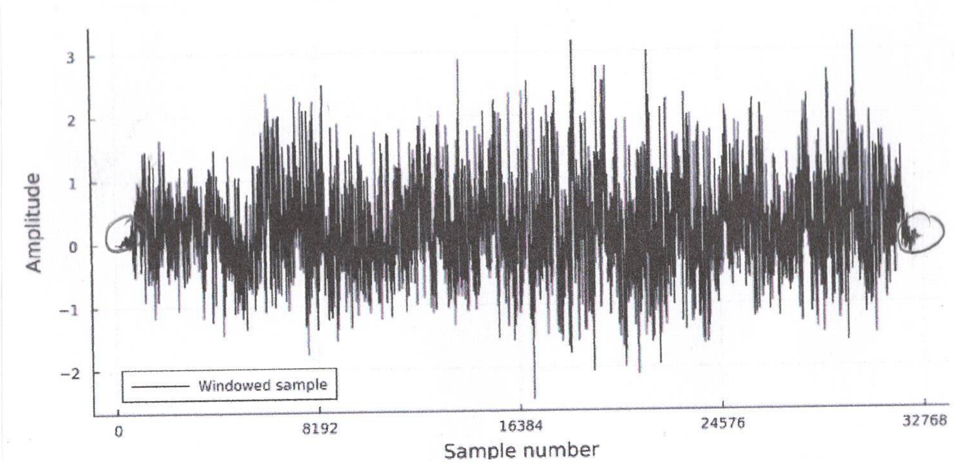
mean of subsetted and windowed data

- check against calculating mean directly

$$\mu = \frac{1}{N} \sum x_i$$

$f_{Ny} = 6.25 \, y^{-1}$

① DFT / FFT of turbulence data



$N = 2^{15} = 32\,768$ points

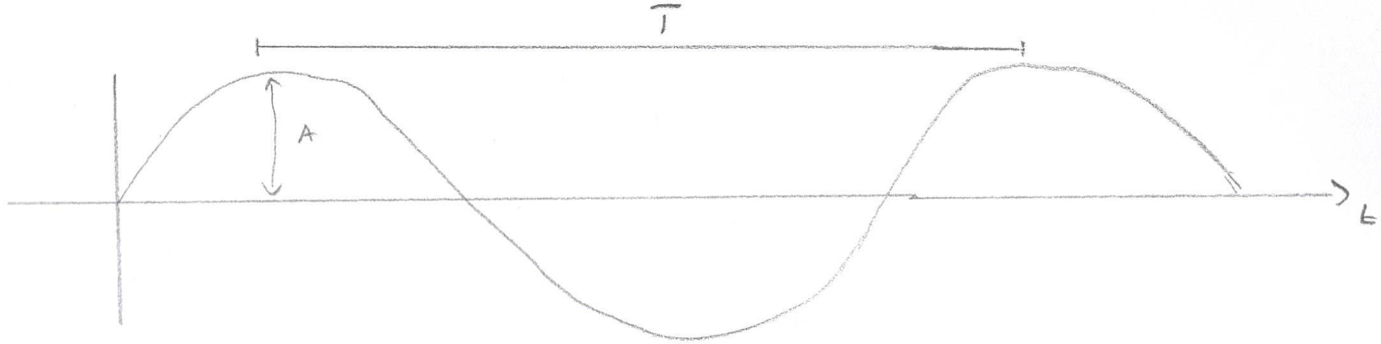
$\Delta t = 0.1\,s$

$f_s = 10\,Hz$ $\quad f_{Nb} = 5\,Hz$

A has units ! m/s

data is windowed

Power of a wave



waves in Physics

- string wave

$$\bar{E} = \frac{1}{2} \mu \omega^2 \lambda A^2$$

Energy, mass, frequency, wavelength, Amplitude

- Ocean wave

$$\bar{E} = \frac{1}{8} g \rho H^2$$

gravity, density, wave height

- Light wave

$$\bar{I} = \frac{1}{2} c \varepsilon_0 E_0^2$$

Intensity, speed of light, electric wave amplitude

Energy of a pulse

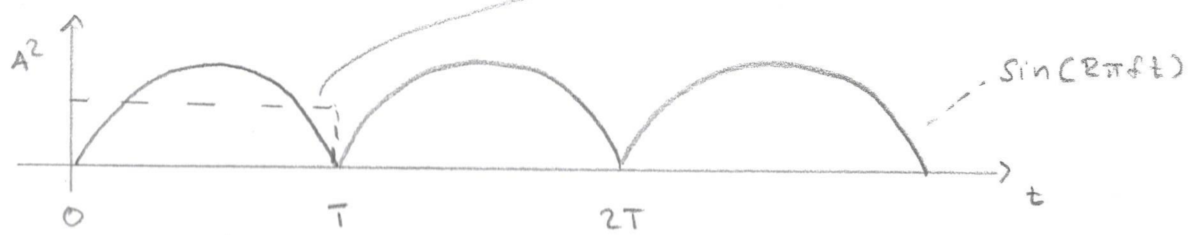
$$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt$$

arbitrary signal

$$\text{Power} = \frac{\text{Energy}}{\text{time}}$$

$$P = \frac{1}{T} \int_0^T g^2(t) \, dt$$ power of a cyclical signal with period T

$$P = \frac{1}{T} \int_0^T [A \sin(2\pi f t)]^2 dt = \frac{1}{2} A^2 = A_{avg}^2$$

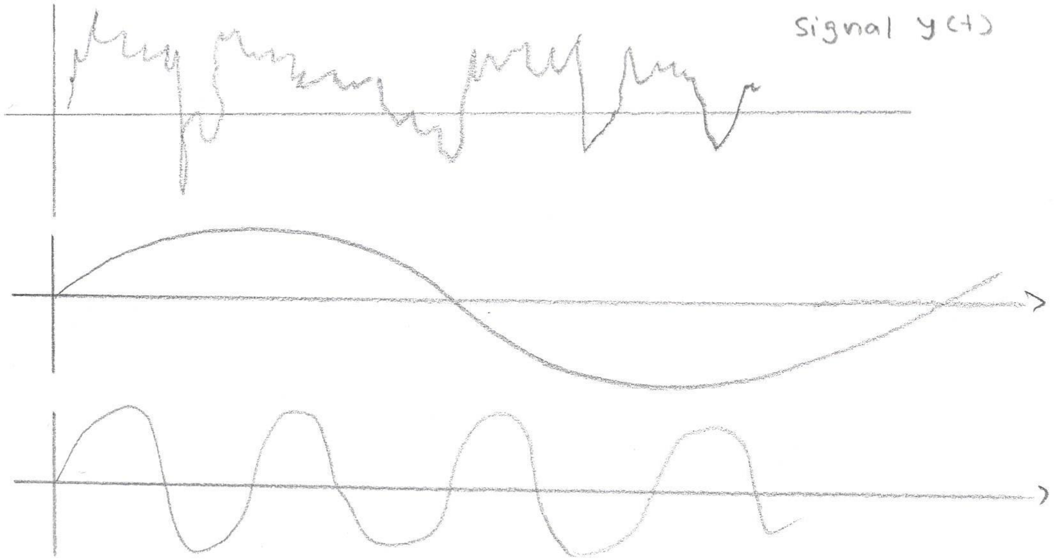


Random Process

$P = E(x^2)$ = variance of signal

expectation value

Random variable



$P_1 = \frac{1}{2} A^2$

$P_2 = \frac{1}{2} A_2$

$\sum_{k=1}^{N} P_i = \sum \frac{1}{2} A_i = Var$

$\sum_{k=1}^{N/2} P_i = Var$

DFT Table of vertical velocity data

	k	f [Hz]	X	A [m/s] mean	P [m^2/s^2]
1	0	0.0 } Δf	6605.28+0.0im	0.201577	0.0812666
2	1	0.000305176	321.645+187.304im	0.0227177	0.000258047
3	2	0.000610352	516.103+887.738im	0.0626746	0.00196405
4	3	0.000915527	-644.831-49.2019im	0.0394718	0.00077901
5	4	0.0012207	168.242+209.318im	0.016391	0.000134332
6	5	0.00152588	772.533+97.5288im	0.047526	0.00112936
7	6	0.00183105	-1675.72-1007.32im	0.119335	0.00712037
8	7	0.00213623	-1114.94-98.3688im	0.0683146	0.00233344
9	8	0.00244141	9.2303+988.143im	0.0603141	0.00181889
10	9	0.00274658	505.846-364.321im	0.0380484	0.000723842
⋮ more					
16385	16384	5.0	-9.86553+0.0im	0.000602144	1.81289e-7

A: [m/s]

P: [m^2/s^2]

$P_k = \frac{1}{2} A_k^2$

$\sum_{k=1}^{N/2} P_k = Var(y(t))$

always check

mean, var from data

=== mean, var from DFT

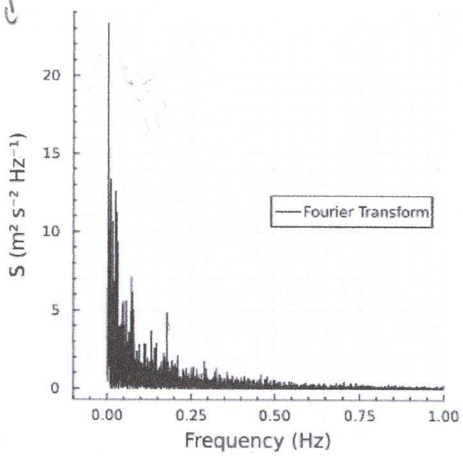
$S = \frac{P}{\Delta f}$ $\left[\frac{m^2}{s^2\ Hz}\right]$

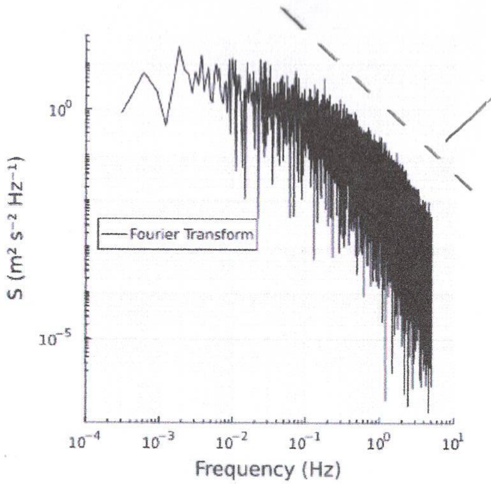
↑ power spectral density

$VaR = \int S\, df$

Power Spectrum

Power spectral density





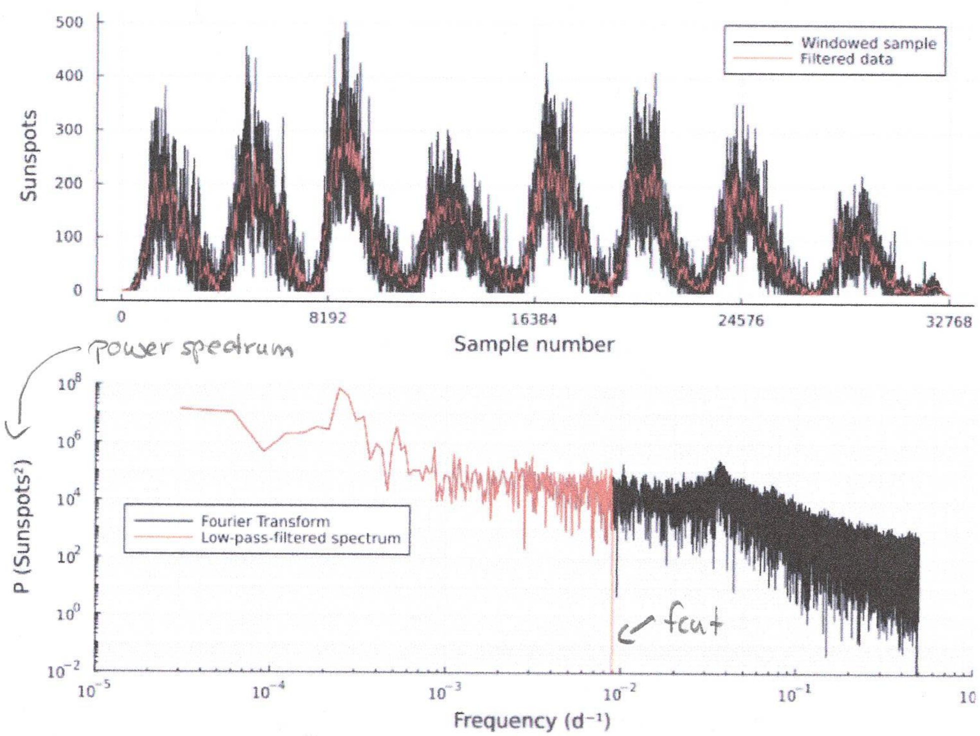
slope of S vs. f
in log-log space
has meaning in
turbulence theory

- same on log-log plot
- area under curve $\neq$ var
- reveals structure of data

plot of S vs. f
reveals spectral contribution
to variance

$$var = \int S\, df$$

Application: Frequency Filtering of Data (High frequency sunspot data)



(1) window data y_n

(2) $X_k = \sum_{n=0}^{N-1} x_n \exp(-ix)$

(3) $P_k = \frac{1}{2} X_k X_k^*$

(4) remove unwanted frequencies

low-pass: zero X_k for $f < f_{cut}$

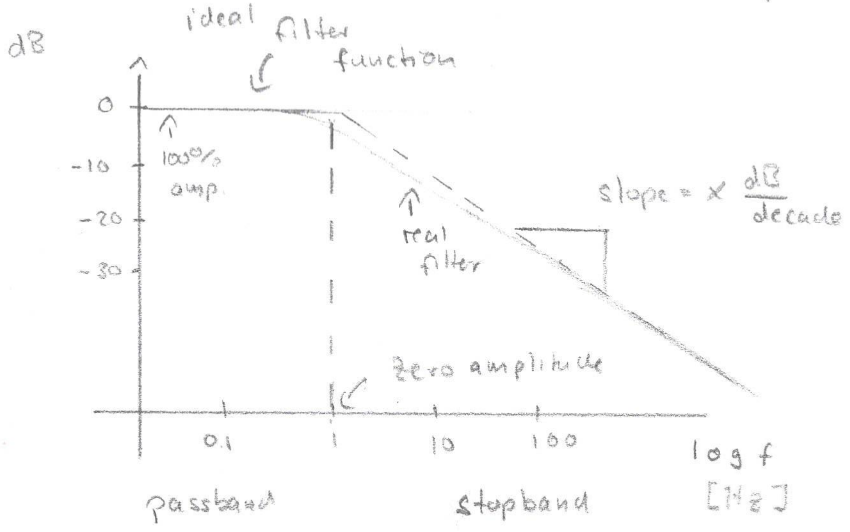
must zero also X_k^* !

(5) $x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp(-ix)$

[use ifft]

Filtering continued

Decibel $= 10\log_{10}(P_1/P_0)$ gain / loss in power relative to reference power P_0



note -10 dB = 1 order of magnitude reduction in power

Fourier Transform of Functions

Discrete

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-i 2\pi \frac{k}{N} n\right)$$

n data $\rightleftharpoons$ n Fourier coefficient

N : period of data

Fourier Series

$$F(k) = \sum_{t=-\infty}^{\infty} f(t) \exp\left(-i 2\pi \frac{k}{N} t\right)$$

continuous $f(t)$ $\rightleftharpoons$ infinite Fourier coefficients

N - periodic function

Fourier Integral : $\lim_{N \to \infty}$ $\quad \frac{k}{N} = \nu$ "oscillation frequency"

$$F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-i 2\pi \nu t)\, dt$$

continuous $f(t)$ $\rightleftharpoons$ continuous ν

non-periodic, converges to zero f1 4 x

$$f(t) = \int_{-\infty}^{\infty} F(\nu) \exp(i 2\pi \nu t)\, d\nu$$

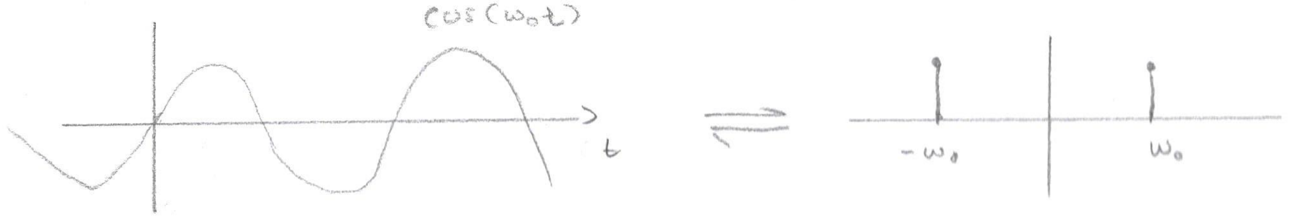
$\omega = 2\pi\nu$ " angular frequency

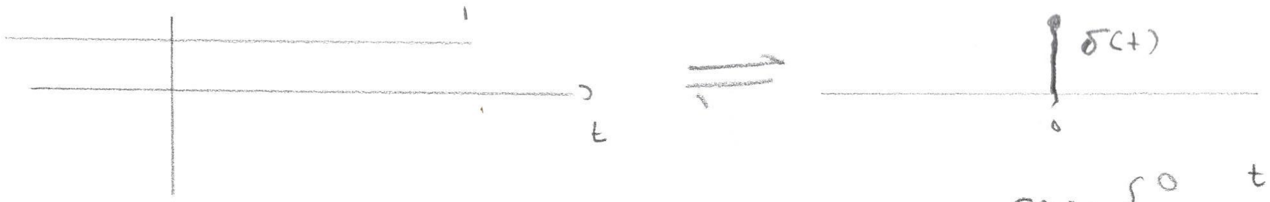
$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega$$

Example Transforms

$f(t)$ $\qquad\qquad$ $F(\omega)$





$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

t - domain $\qquad\qquad$ ω - domain

Applications of Fourier Integral

(1) derivatives become algebraic functions

Fourier transform of $\longrightarrow \mathcal{F}(f'(t)) = -i\omega F(\omega)$

(2) convolutions become multiplication

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$$

$f \circ g = \int f(\tau)\, g(t-\tau)\, d\tau$ $\Rightarrow$ session 3

(3) linear combination

$$\mathcal{F}(a f(t) + b g(t)) = a F(k) + b G(k)$$

Laplace Transform

Fourier $\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt$

Laplace $\mathcal{L}(f(t)) = \int_{-\infty}^{\infty} f(t) \exp(-st)\, dt$

$s = a + bi$ $\Rightarrow$ Fourier transform is special case of Laplace transform

more functions have Laplace transform $\rightarrow$ need not converse at ∞

________ $\rightleftharpoons$ ________

t-domain s-domain

Properties of Fourier transform apply to Laplace transform