Control Systems

Continuous linear time-ivariant systems (LTI)

"easy" $\begin{cases} q'(t) = A\,q(t) + B\,f(t) \\ y(t) = C\,q(t) + D\,f(t) \end{cases}$ ← inhomogeneous system of ODEs

Continuous linear time variant systems

"harder" $\begin{cases} q'(t) = A(t)\,q(t) + B(t)\,f(t) \\ y(t) = C(t)\,q(t) + D(t)\,f(t) \end{cases}$

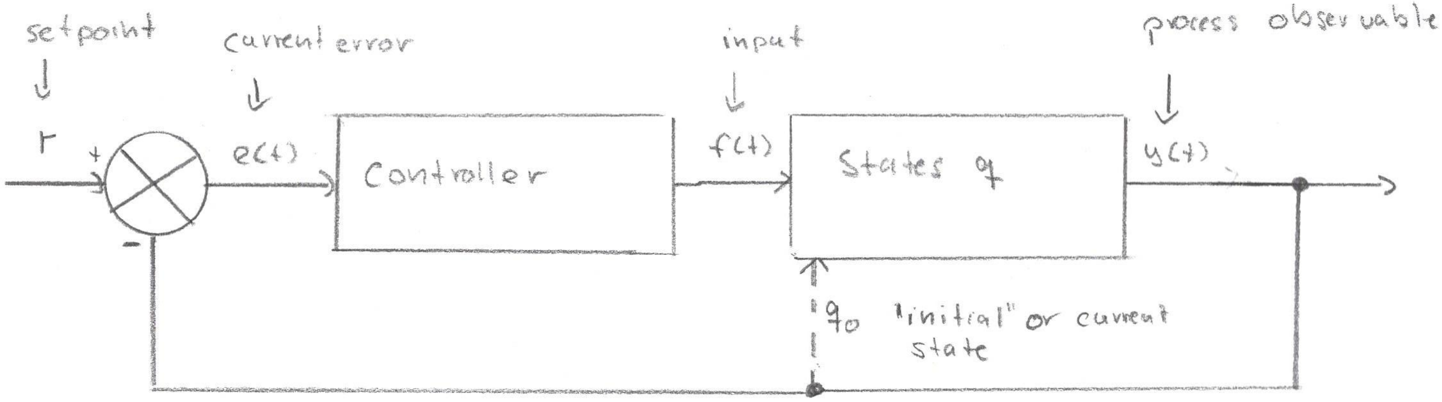
↑ linear but coefficients are time dependent

Non-linear systems

-> no matrix form

"hard" $\begin{cases} q_1' = a\,q_1 + b\,q_2 + c\,q_1 q_2 \\ q_2' = -d\,q_1^3 + e \end{cases}$

$$\left.\begin{aligned} q' &= Aq + Bf \\ y &= Cq + Df \end{aligned}\right\} \text{LTI system}$$

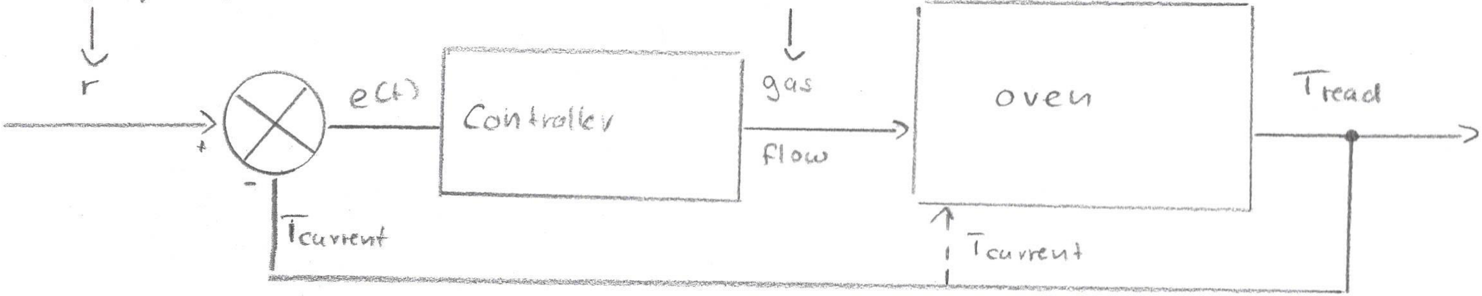


Control problem: how to set input f(t) to drive the system from current state q_0 to desired observable y(t)

Example : Oven

MP: Manipulated Variable (value pos / voltage)

SP: Set point



Physical model requires knowledge of V vs. gas flow rate

Discrete Operation ($r = 400$)

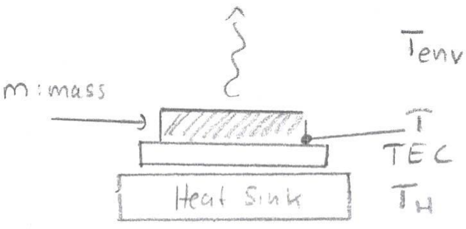
t	Tread	e(t)	MV
0	150	250	10 V
10 s	180	220	9 V
20 s	200	200	?
30 s	¦	¦	¦

Model (LTI) if available or observe

- know e(t) current
- know e(t) past
- no knowledge of future

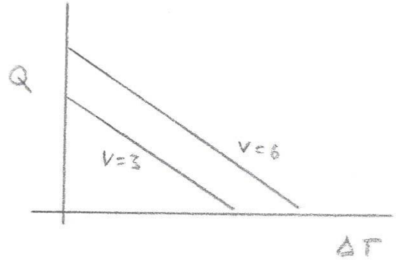
"controller is mapping e(t) to MV"

Concrete Example : Cold Stage



heat flow $[W] = \left[\frac{J}{s}\right]$

$$Q(V, \Delta T) = a_1 \sqrt{V} + a_2 (T_H - T) + a_3$$



How does T change?

$$\left.\frac{dT}{dt} = \underbrace{k(T_{env} - T)}_{\text{Newton's law of cooling}} - \underbrace{\frac{Q}{mc_{\text{heat capacity}}}}_{\text{heat flux + storage}} \right\} \text{physical model of system}$$

Convert model to State space Form

$$\frac{dT}{dt} = k T_{env} - kT - \frac{a_1}{mc}\sqrt{V} + \frac{a_2}{mc} T - \frac{a_3}{mc} - \frac{a_2}{mc} T_H$$

$$\frac{dT}{dt} = \left(-k + \frac{a_2}{mc}\right) T - \frac{a_1}{mc}\sqrt{V} - \frac{a_2}{mc} T_H - \frac{a_3}{mc} + k T_{env}$$

state variable(s) : $\vec{q} = T$

inputs : $\vec{f} = V, T_{env}, T_H$

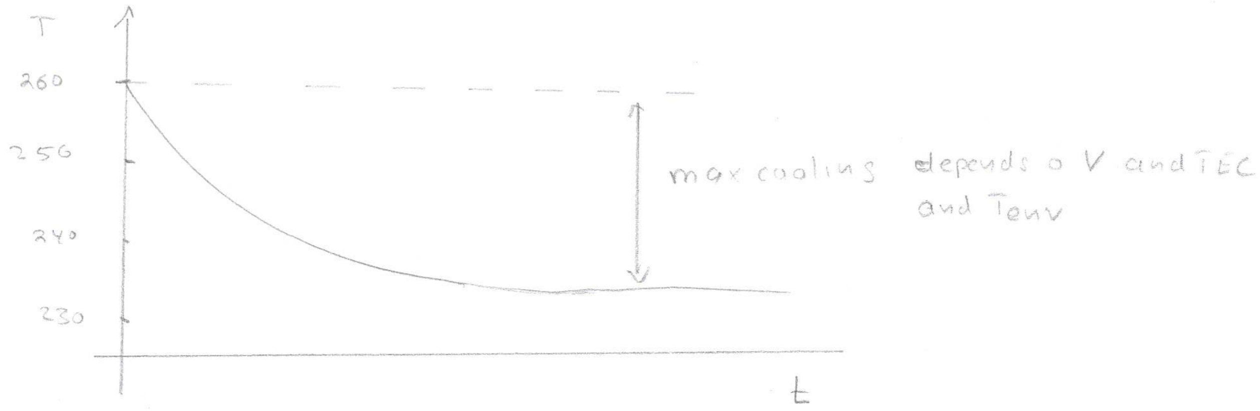
output(s) : $\vec{y} = T$

$q = T \qquad q' = T'$

$$q' = \underbrace{\left[-k + \frac{a_2}{mc}\right]}_{A} \vec{q} + \underbrace{\left[-\frac{a_1}{mc} \quad -\frac{a_2}{mc} \quad -\frac{a_3}{mc} \quad k\right]}_{B} \underbrace{\begin{bmatrix} \sqrt{V} \\ T_H \\ 1 \\ T_{Env} \end{bmatrix}}_{\vec{f}}$$

$$y = \underbrace{[1]}_{C} \vec{q} + \underbrace{[0 \quad 0 \quad 0 \quad 0]}_{D} \vec{f}$$

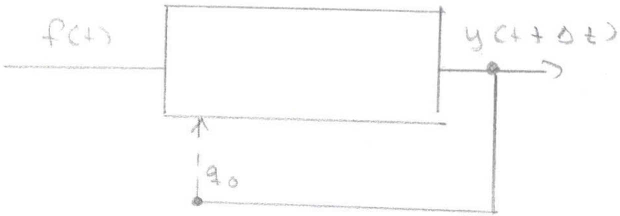
Simple Solution : ODE solver




```
function ode(q, p, t)
    A = p[1]
    B = p[2]
    f = p[3]
    return Aq + Bf
end
```

Discrete Event Simulation

$$q' = Aq + Bf$$
$$y = Cq + Df$$



- initialize ODE at $t = t_0$ with IVP q_0, f
- integrate to $t + \Delta t$
- evaluate solution $\rightarrow$ build time array, possibly update f

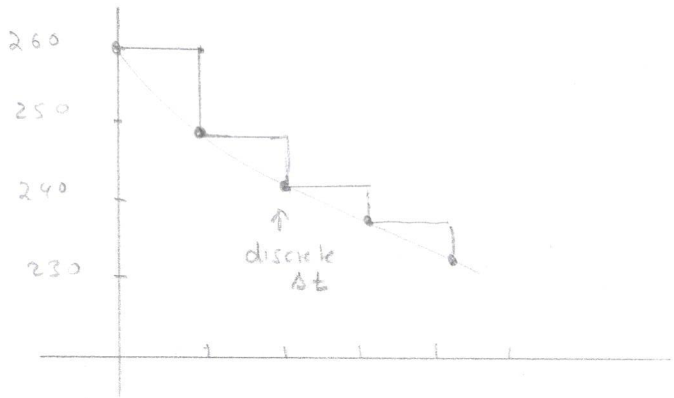
loop

t	q	f
0		
20		
40		
60		

$\Delta t = 20s$

f may change each Δt

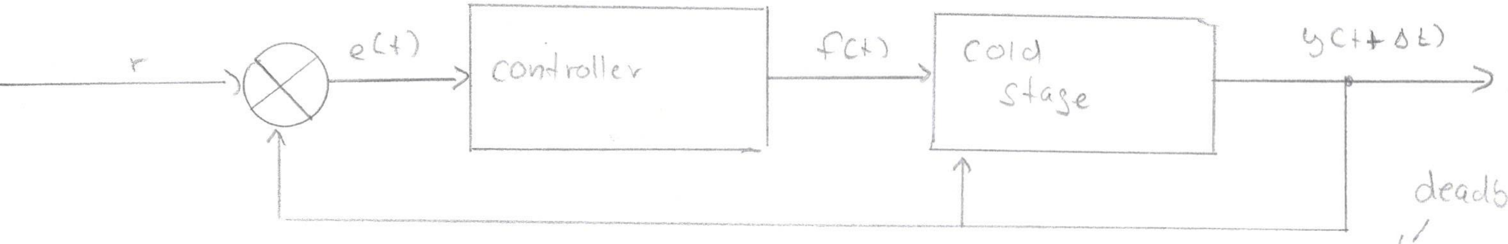
Discrete Event Example



On/Off Control

$+1$ = cooling $\rightarrow$ TEC $= +12V$
-1 = heating $= -12V$

$r = 300$



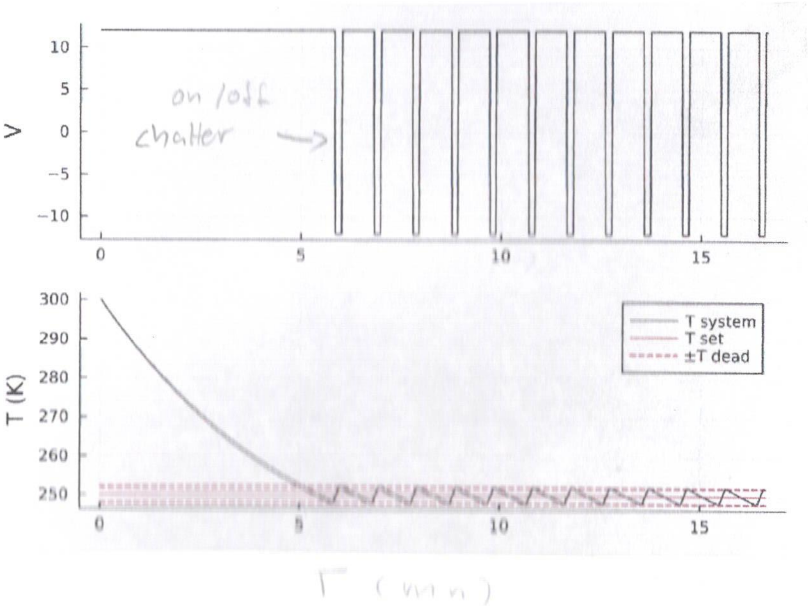
$$e(t) = r - y(t)$$

deadband

$$f(t) = \begin{cases} +1 & e(t)+d < 0 \wedge e(t+\Delta t)+d > 0 \\ -1 & e(t)-d > 0 \wedge \underbrace{e(t-\Delta t)-d}_{\text{past error}} < 0 \\ \text{no change} & \text{ele} \end{cases}$$

t	$y(t)$	$e(t)$	$e(t)+d$	$e(t)-d$	$f(t)$	
0	303	-3	-1	-5	$+1$	cooling
1	295	$+5$	$+7$	$+3$	-1	heating
2	299	$+1$	$+3$	-1	-1	(no change) "dead band"
3	304	-4	-2	-6	$+1$	cooling
⋮						

Example Cold Stage



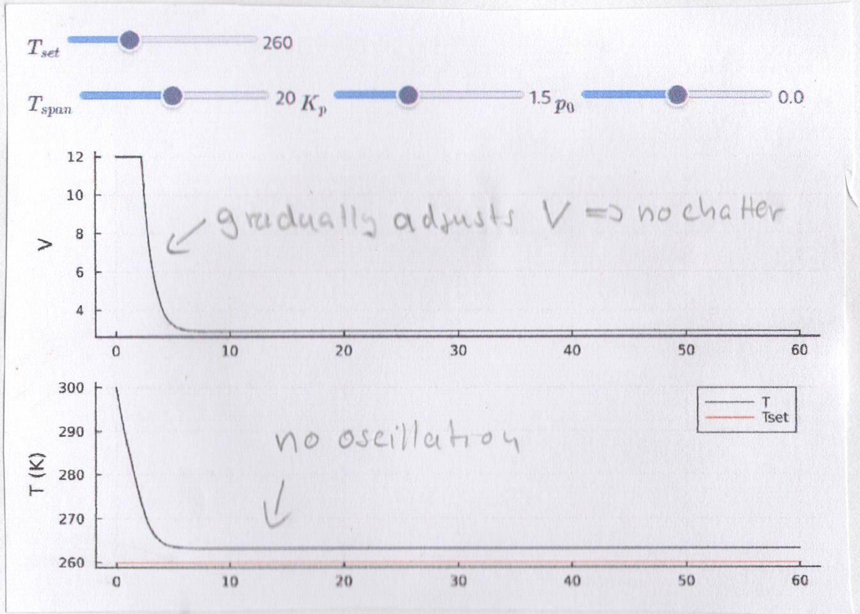
} dead band

- solution oscillates
- control within ± d
- prone to chatter

Proportional Control

$$g(t) = \underset{\substack{\uparrow \\ \text{proportional} \\ \text{gain}}}{K_p} \frac{\overbrace{r - y(t)}^{\text{error}}}{\underset{\substack{\uparrow \\ \text{span}}}{S}} + \underset{\substack{\uparrow \\ \text{offset}}}{p_0}$$

$$f(t) = \begin{cases} -1 & g(t) < -1 \\ 1 & g(t) > 1 \\ g(t) & \text{else} \end{cases}$$



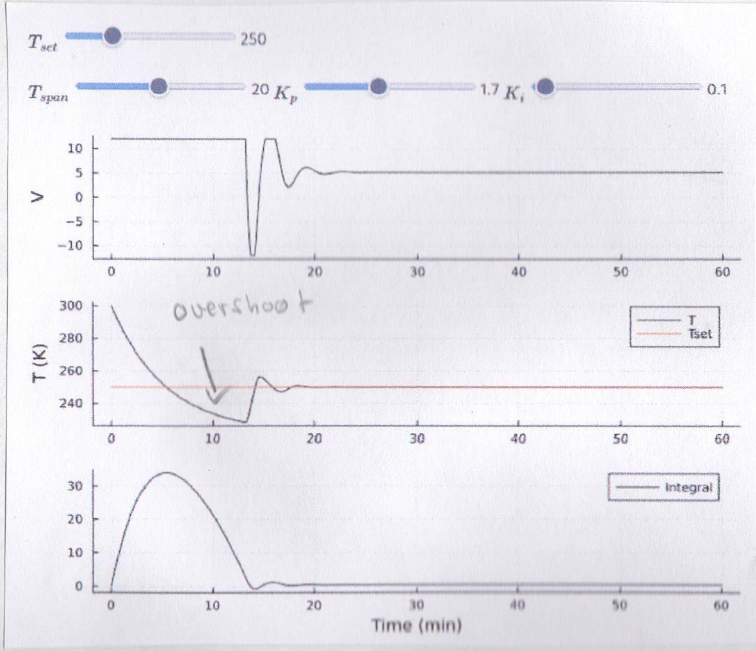
Span: 100% output if not within ± S

} offset error can not be eliminated

Proportional Integral Control (PI)

$$e(t) = \frac{r - y(t)}{g(t)} \qquad g(t) = k_p\, e(t) + k_i \overbrace{\int_0^t e(\tau)\, d\tau}^{\text{Integral over past error}} \qquad f(t) = \begin{cases} -1 & g(t) < -1 \\ 1 & g(t) > 1 \\ g(t) & \text{else} \end{cases}$$

solution is sensitive to integral gain

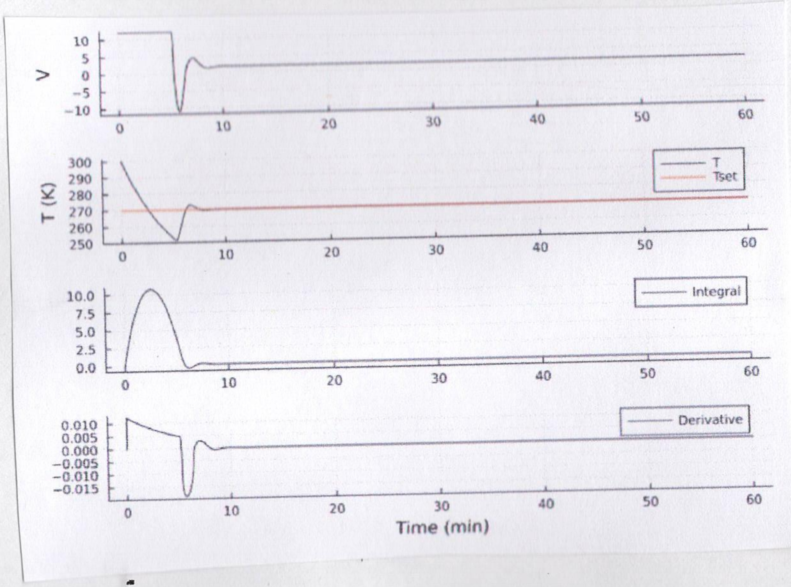


- oscillating solution that generally but not always converges
- no offset error

Proportional - Integral - Derivative (PID) Control

same as PI except $\quad g(t) = k_p e(t) + k_i \int_0^t e(\tau)\,d\tau + k_d \frac{de(t)}{dt}$

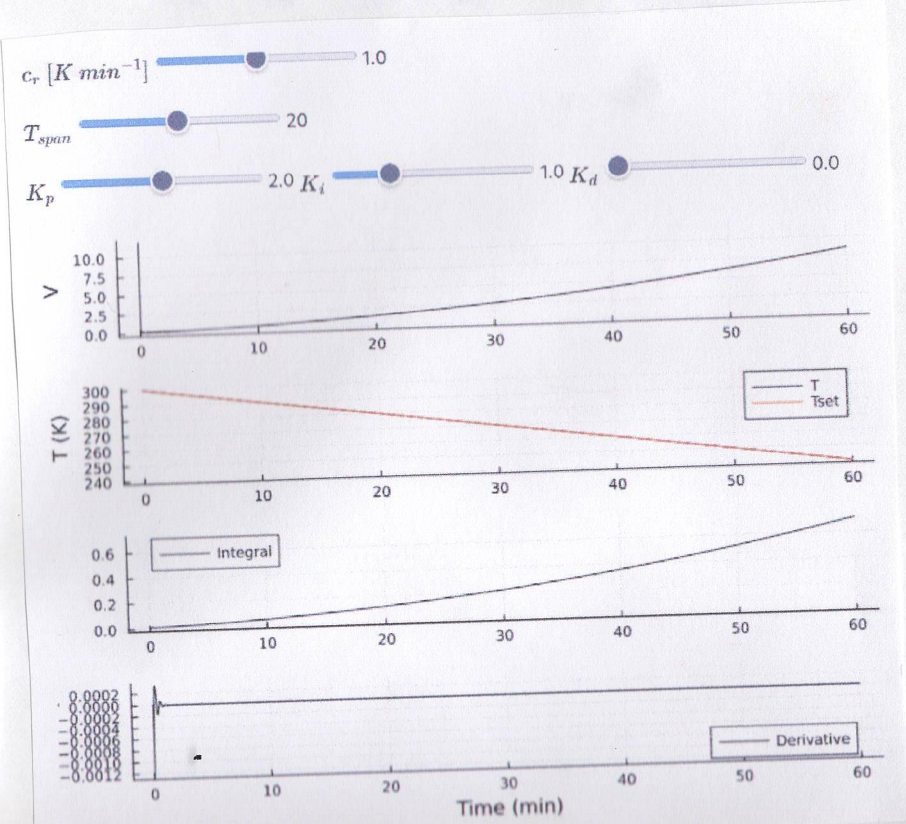
derivative gain



D-term can

- increase convergence rate
- be susceptible to noise amplification

Trajetory Control



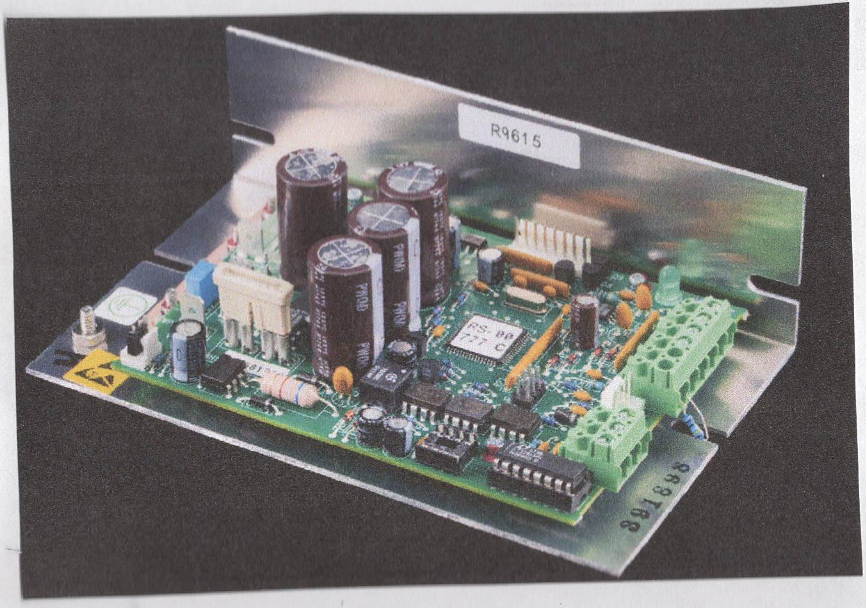
Setpoint r is treated as $r(t)$

e.g. cooling rate $c_r = \frac{dr}{dt} = \text{const}$

- avoids oscillations
- avoids overshoot
- controlled way to steer system

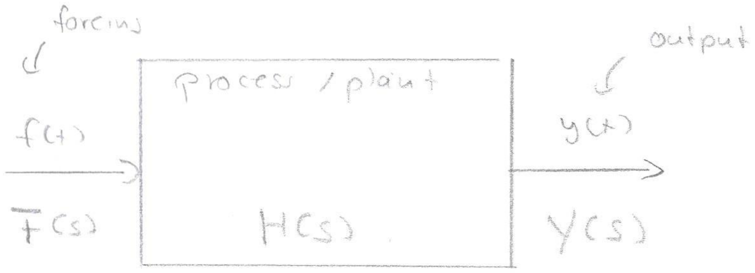
Hardware Controllers (special domain)

PID Temperature Controller



- programable Tset, Tspan, k_p, k_i, k_d and span
- reads temp (thermistor)
- modulates input V to ± output V for thermoelectrics

Formal Reasoning about control systems



$$\boxed{\begin{aligned} \mathcal{L}f'' &= s^2\bar{F} - sf(0) - f'(0) \\ \mathcal{L}f' &= s\bar{F} - sf(0) \\ \mathcal{L}af &= a\bar{F} \end{aligned}}$$

$f(t) = a_n y^{(n)} + \dots a_1 y' + a_0 y$ (zero initial conditions)

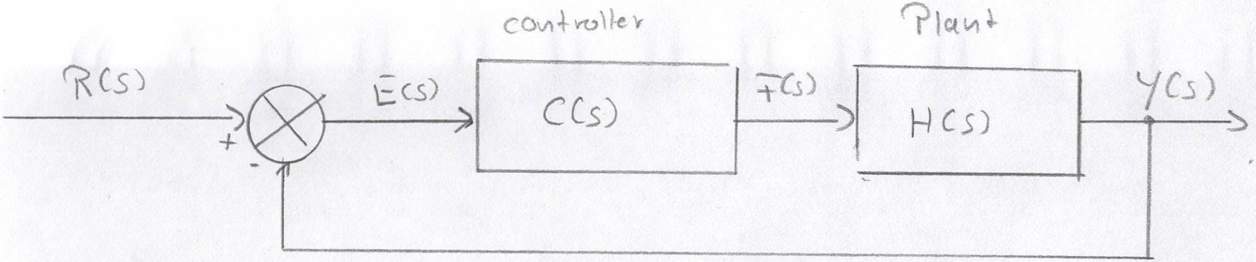
$\mathcal{L}$ $\bar{F}(s) = (a_n s^n + \dots + a_0)\, Y(s)$

$H(s) = \dfrac{Y(s)}{\bar{F}(s)} = \dfrac{1}{a_n s^n + \dots + a_0}$

↑

transferfunction

Closed Loop Control System



$$\frac{Y(s)}{R(s)} = \underbrace{\frac{C(s)H(s)}{1 + C(s)H(s)}}_{\text{use this for stability analysis}}$$

system transfer function

Controller

$$f(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \frac{de(t)}{dt}$$

$$C(s) = K_p + \frac{K_i}{s} + K_d s$$

- roots in numerator => zeros => stabilize system
- roots in denominator => poles => destabilize system
- can add noise filter to C(s)

$\mathcal{L}^{-1}(Y(s)) \Rightarrow y(t)$: analytical analog to simulation

Controllability

$q' = Aq + Bf$

→ reach any state $\vec{q}$ using control system $\vec{q}$ is $n \times 1$

$R = [B \;\; AB \;\; A^2B \; \dots \; A^{n-1}B]$

↑
controllability matrix

→ system is controllable if R is full row rank, i.e $\operatorname{rank}(R) = n$

↳ derive through $\mathcal{L}\, q'$