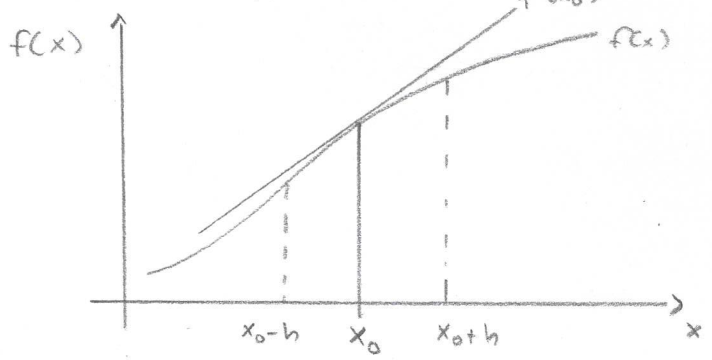
Review Vectors / Linear Algebra

(1) Numerical Differentiation



$$f'(x) = \frac{f(x+h) - f(x)}{h}$$ forward difference

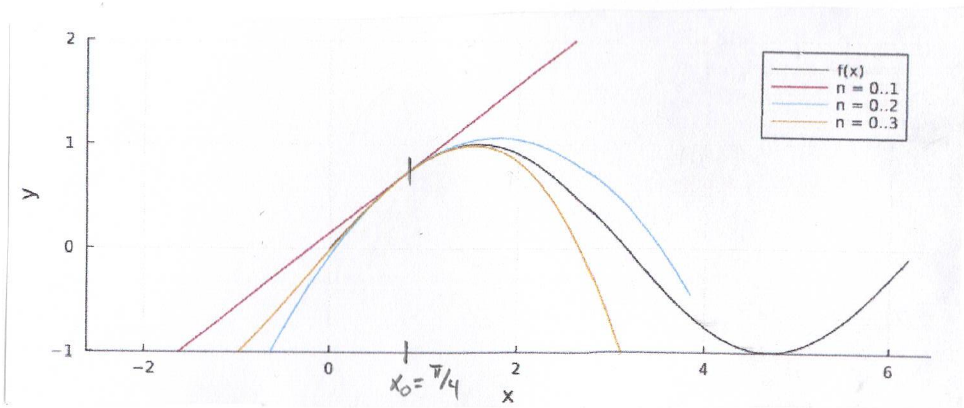
$$f'(x) = \frac{f(x) - f(x-h)}{h}$$ backward difference

$$f'(x) = \frac{f(x+h) - f(x-h)}{h}$$ central difference

Taylor Series : approximate $f(x)$ near x_0

$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n$$

$$= f(x) + \underbrace{\frac{f'(x)}{1!} h}_{n=1} + \underbrace{\frac{f''(x)}{2!} h^2}_{n=2} + \underbrace{\frac{f'''(x)}{3!} h^3}_{n=3} + \underbrace{\dots}_{O(h^4)}$$



$f(x) = \sin(x)$
$f'(x) = \cos(x)$
$f''(x) = -\sin(x)$

Finite Difference from Taylor Series

(1) $f(x+h) \approx f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} + O(h^2)$$

(2) $f(x-h) \approx f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 - \frac{f'''(v)}{3!}h^3$

$$\Rightarrow f'(x) = \frac{f(x) - f(x+h)}{h} + O(h^2)$$

(1) − (2) $f(x+h) - f(x-h) \approx 2f'(x)h + \underbrace{O(h^3) + O(h^5)}_{\text{even terms are missing}}$

$\Rightarrow$ center difference more precise

Higher Order Derivative

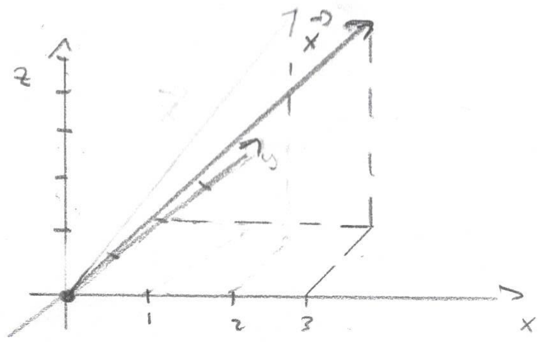
(1) + (2) $f(x+h) + f(x-h) \approx 2f(x) + 2\frac{f''(x)}{2!}h^2 + O(h^4)$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Vectors and Matrices
(array)

$\vec{x}$ $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ $\vec{x} = [3 \quad 2 \quad 5]$

- collection of points
- mag. + direction



$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$

↑
Euclidian Norm (magnitude)

python / c

x[0] = 3
x[1] = 2
x[2] = 5
↑
array index

julia / R / fortran

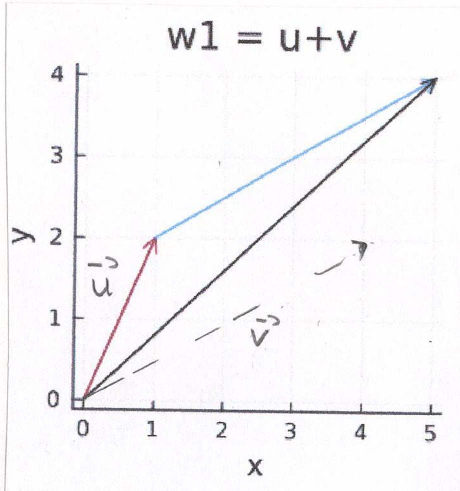
x[1] = 3
x[2] = 2
x[3] = 5

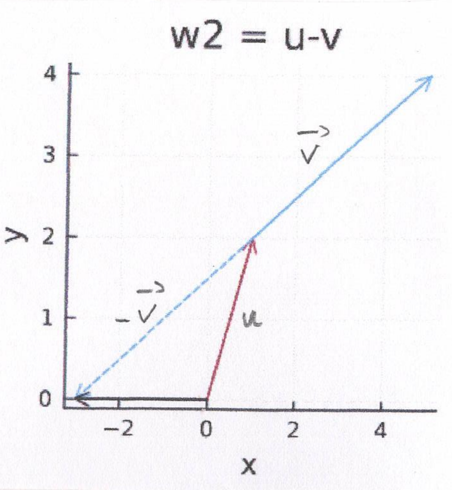
Addition and Subtraction

$\vec{u} = [1,2]$

$\vec{v} = [4,2]$

$\vec{u} + \vec{v} = [u_1+v_1 \quad u_2+v_2]$ $\qquad$ $\vec{u} - \vec{v} = [u_1-v_1 \quad u_2-v_2]$





commutative

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

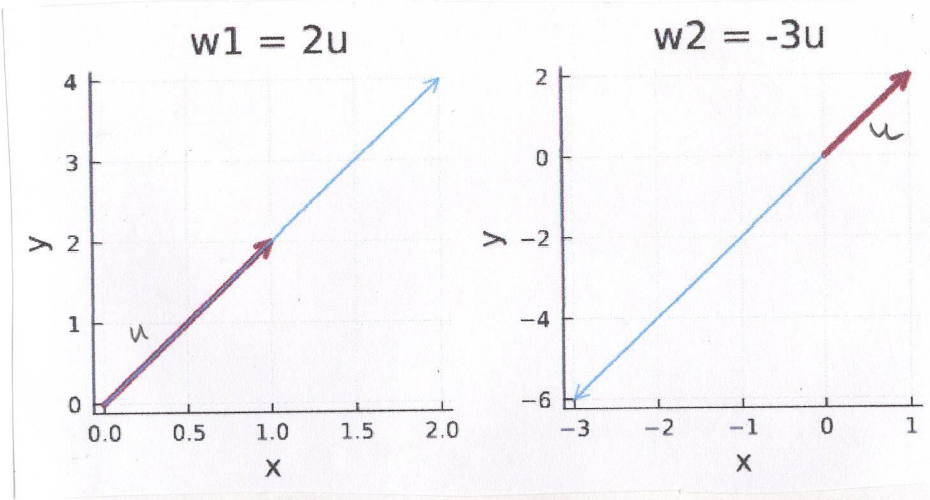
$(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$

Associative

Multiplication by Scalar

$\vec{u} = [2, 4]$

$a \cdot \vec{u} = [au_1 \quad au_2]$



$s\vec{v} = \vec{v}s$

$r(s\vec{v}) = (rs)\vec{v}$

$(q+r+s)\vec{v} = q\vec{v} + r\vec{v} + s\vec{v}$

(distributive)

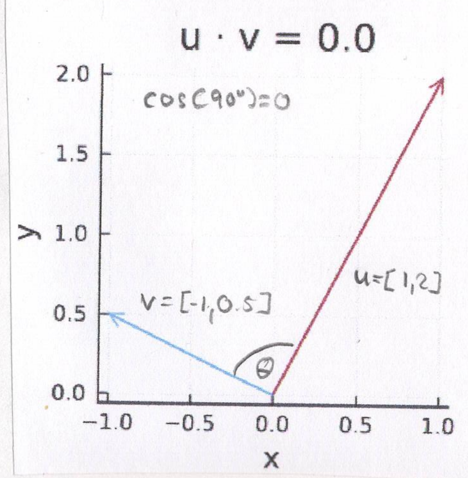
Scalar Product (Dot Product)

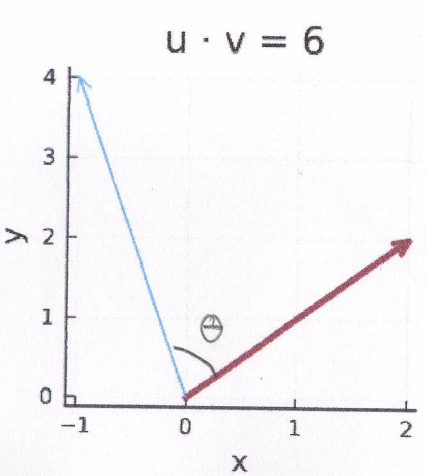
Test orthogonality (right angle)

$$\vec{u} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta)$$

$$\vec{u} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

$$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$$





$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$(\vec{u} \cdot \vec{v})\vec{w} \neq \vec{u}(\vec{v} \cdot \vec{w})$$

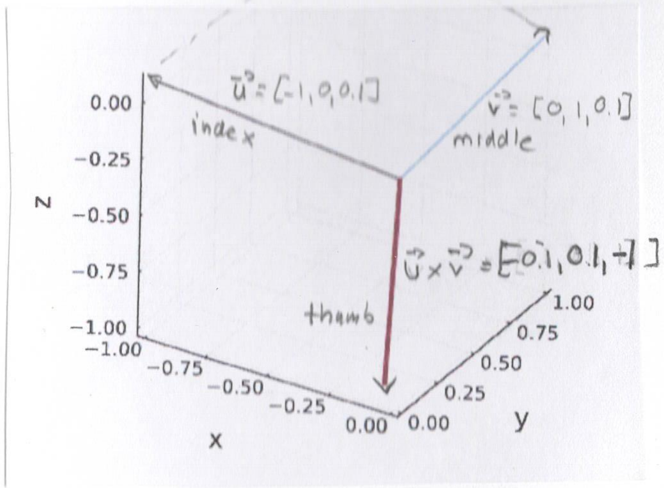
$$\vec{u} \cdot [\vec{v} + \vec{w}] = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Vector Product (Cross Product)

- defined on 3-dim vectors
- $\vec{v} \times \vec{w} = \|\vec{v}\| \|\vec{w}\| \sin(\theta)\, \hat{n}$ → zero for parallel vectors $\Rightarrow \vec{v} \times \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$ is a vector

$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin(\theta)$



$(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$

$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$

- right-hand rule

Matrix

$$3x + 4y - 5z = 10 \qquad [3 \;\; 4 \;\; -5] \cdot [x \;\; y \;\; z]$$
$$3x - 5y + 7z = 11 \qquad [3 \;\; -5 \;\; 7] \cdot [x \;\; y \;\; z]$$
$$-3x + 6y + 9z = 12 \qquad [-3 \;\; 6 \;\; 9] \cdot [x \;\; y \;\; z]$$

$$\begin{array}{cc} & \xrightarrow{\text{column}} \\ \text{row} \downarrow & \begin{bmatrix} 3 & 4 & -5 \\ 3 & -5 & 7 \\ -3 & 6 & 9 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

$$A \qquad \vec{x} = \vec{b}$$

Matrix A with m rows and n colums

A_{ij} or a_{ij} entry of i^{th} row and j^{th} column

Scalar times matrix

$cA = c\,a_{ij}$

Matrix times Matrix

$A * B = A_{i,:} \cdot B_{:,j}$ ← all rows colum j

↑ dot product

row i, all colums :

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$[1 \ \ 2] \cdot [1 \ \ 3] = 7$ $[1 \ \ 2] \cdot [2 \ \ 4] = 10$

$[3 \ \ 4] \cdot [1 \ \ 3] = 15$ $[3 \ \ 4] \cdot [2 \ \ 4] = 22$

Matrix Transpose

$A^{m\times n} \Rightarrow (A^T)^{n\times m}$ i,j entry becomes j,i entry.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

Rules

$(A^T)^T = A$ (self inverse)

$(A+B)^T = A^T + B^T$

$(cA)^T = cA^T$

$(AB)^T = B^T A^T$ (order reversal)

$a \cdot b = a^T b$ (dot product to matrix product)

Matrix Determinant (n x n matrix)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det \begin{bmatrix} \boxed{a} & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

(1) a square matrix is invertible if and only if $\det(A) \neq 0$

(2) if the rows of colums are linearly dependent, then $\det(A) = 0$

(3) $\det(AB) = \det(A)\det(B)$

(4) $\det(A) = \det(A^T)$

(5) $\det(A^{-1}) = \frac{1}{\det(A)}$

Matrix Inverse

$A A^{-1} = I$

inverse matrix

I = Identity Matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_{i,j} = \delta_{i,j} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

Kronecker delta

Fourier Transform

Einstein Notation

$A I = A$

$I A = A$

Methods of matrix inversion

- Gaussian Elimination
- Numerical linear algebra

\>> inv(A) ; if $\det(A) \neq 0$

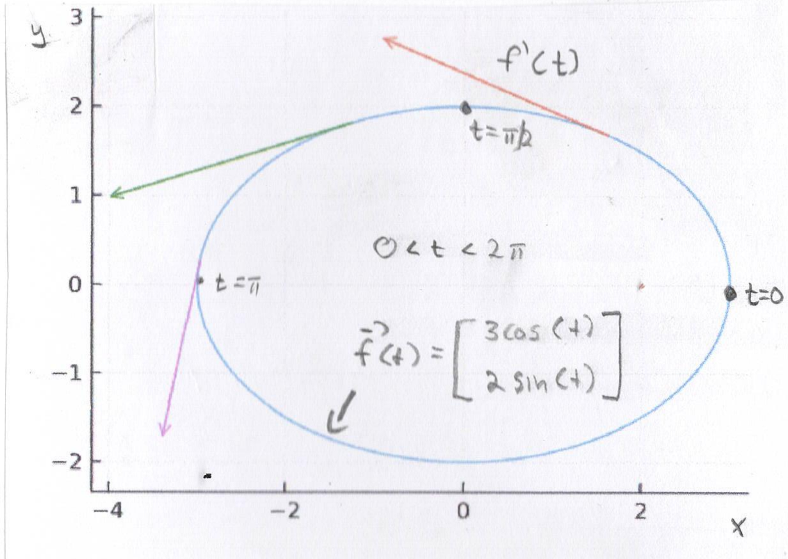
Vector - Valued Functions

$\vec{f} : \mathbb{R} \to \mathbb{R}^n \quad n > 1$

$\vec{f}(t) = \begin{bmatrix} \sin(t) \\ 2\cos(t) \end{bmatrix}$ maps scalar "t" to vector

$\vec{f}'(t) = \lim_{\Delta t \to 0} \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

$f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$



$f'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ = tangent vector

$(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$

$(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

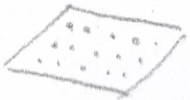
product rule

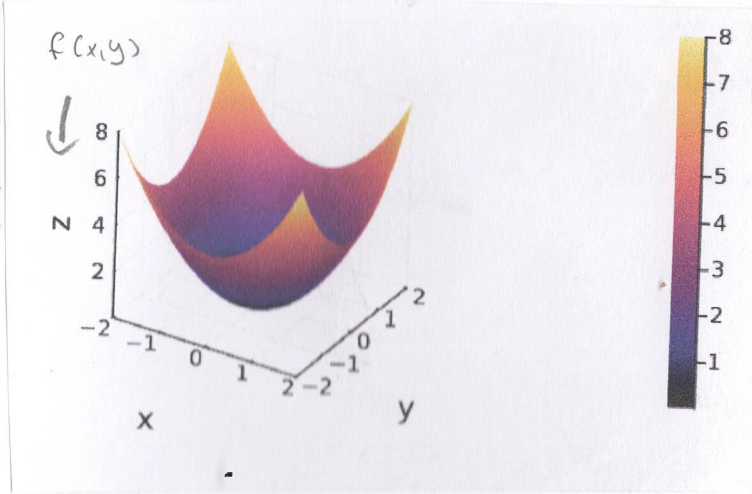
Multivariate Scalar Functions

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ multiple inputs -> scalar output

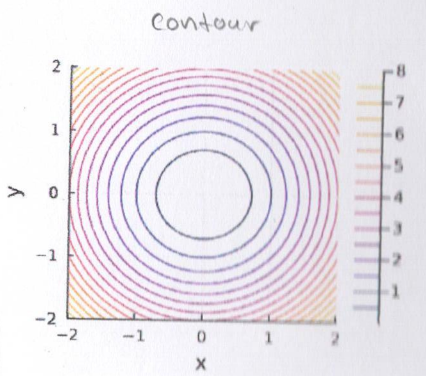
Example: $f(x,y) = x^2 + y^2$

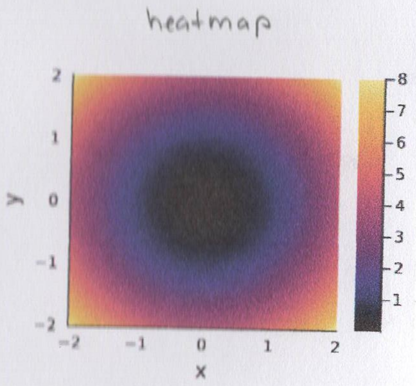
need to evaluate for





top view





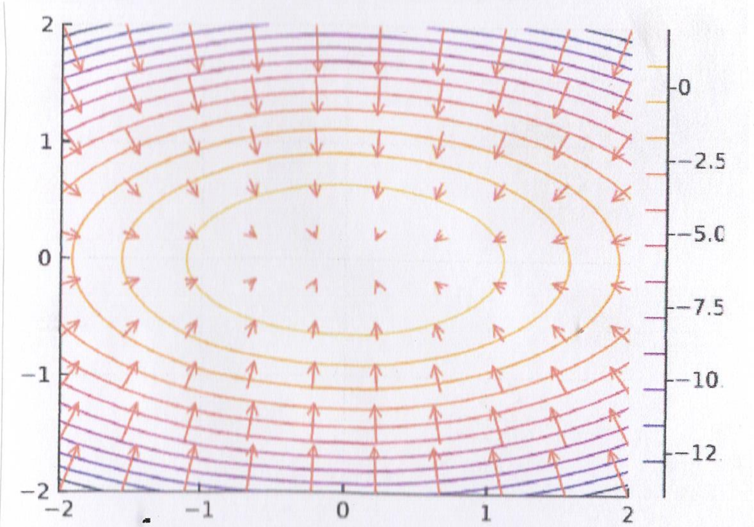
The Gradient

gradient of a scalar function is the direction of steepest ascent

$$\nabla f(x_1, \dots, x_n) = \left[\frac{\partial}{\partial x_1} \ \frac{\partial}{\partial x_2} \ \dots \ \frac{\partial}{\partial x_n}\right] f$$

$$= \left[\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ \dots \ \frac{\partial f}{\partial x_n}\right]$$

$R^n \to R^n$



$\nabla f \neq f \nabla$

$(\nabla f) g \neq \nabla (f g)$

$\nabla (f+g) = \nabla f + \nabla g$

$f(x,y) = 2 - x^2 - 3y^2$

Divergence

Divergence of a vector field $\bar{F} : \mathbb{R}^3 \to \mathbb{R}$

$$\operatorname{div}(\bar{F}) = \nabla \cdot F = \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right] \cdot \left[\bar{F}_x \quad \bar{F}_y \quad \bar{F}_z \right]$$

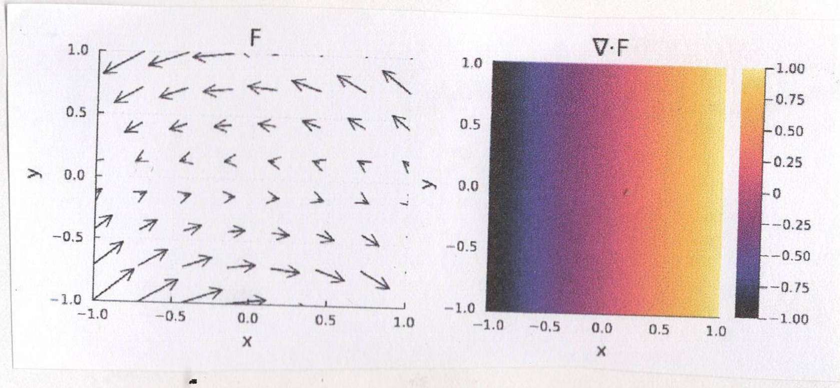
$$\nabla \cdot \bar{F} = \frac{\partial \bar{F}_x}{\partial x} + \frac{\partial \bar{F}_y}{\partial y} + \frac{\partial \bar{F}_z}{\partial z}$$

$\nabla \cdot \vec{u} \neq \vec{u} \cdot \nabla$

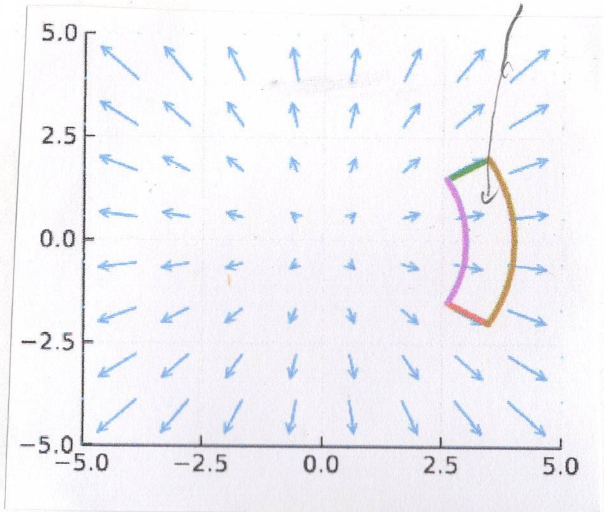
$\nabla \cdot s\vec{u} \neq \nabla s \cdot \vec{u}$

$\nabla \cdot [\vec{u} + \vec{v}] = \nabla \cdot \vec{u} + \nabla \cdot \vec{v}$

$\bar{F} = [-y, xy, 0]$



Source



div > 0 ⇒ outflow > inflow

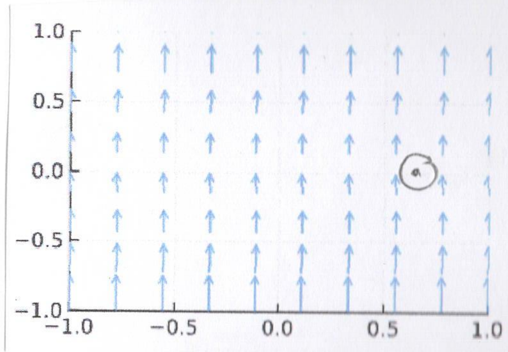
Curl

defined for a 3D vector field $R^3 \to R^3$

$$\text{curl } \bar{F} = \nabla \times F = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{F}_x & \bar{F}_y & \bar{F}_z \end{vmatrix}$$

$$= \left[\frac{\partial \bar{F}_z}{\partial y} - \frac{\partial \bar{F}_y}{\partial z}, -\frac{\partial \bar{F}_z}{\partial \bar{F}_x} + \frac{\partial \bar{F}_x}{\partial z}, \frac{\partial \bar{F}_y}{\partial x} - \frac{\partial \bar{F}_x}{\partial y} \right]$$

no spin ⇒ curl $\bar{F} = 0$

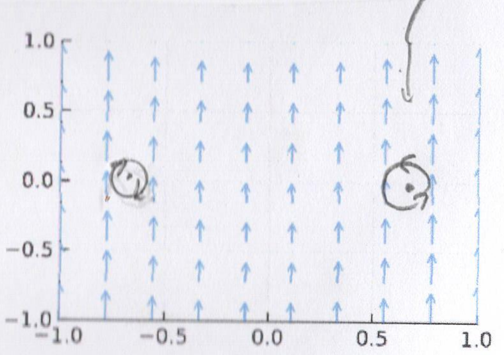


$\bar{F} = [0, 1 + y^2, 0]$

curl downward

right hand rule

→ curl points upward



$\bar{F} = [0, 1 + x^2, 0]$