VMO 2024 - Day 1

P_2: $\forall a$: $P(a)$ is a root of

$$x^{2023} + Q(a)x^2 + (a^{2024}+a)x + a^3 + 2025a = 0$$

Find all $P(x), Q(x) \subset \mathbb{R}(x)$

consider function:

$$P(x)^{2023} + Q(x)P(x)^2 + (x^{2024}+x)P(x) + x^3 + 2025x = 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow x^3 + 2025x \ \vdots \ P(x)$

$$\Rightarrow \left[\begin{array}{l} P(x) = K \\ P(x) = Kx \\ P(x) = K(x^2+2025) \\ P(x) = Kx(x^2+2025) \end{array}\right.$$

Consider:

$$R(x) = P(x)^{2023} + Q(x)P(x)^2 + (x^{2024}+x)P(x)$$
$$F(x) = x^3 + 2025x$$

$\Rightarrow R(x) = -F(x)$

Case 1: $P(x) = Kx$ or $P(x) = Kx(x^2+2025) \Rightarrow P(x) \ \vdots \ x^2$

However, $F(x) \not\vdots\ x^2$, hence (X)

Case 2: $P(x) = K(x^2+2025)$

$\Rightarrow K^{2023}.(x^2+2025)^{2022} + Q(x)K^2(x^2+2025) + (x^{2024}+x)K = -x$

$$\Rightarrow Q(x) = \frac{-x - (x^{2024}+x)K - K^{2023}(x^2+2025)^{2022}}{K^2(x^2+2025)}$$

$$\frac{x + (x^{2024} + x)k}{(x^2 + 2025)} = \frac{x + xk + x^{2024}k}{x^2 + 2025}$$

$$x^{2024} = x^{2024} + 2025 \cdot x^{2022} - 2025 \cdot x^{2022} - 2025^2 \cdot x^{2020} + \dots$$

$$= \sum_{i=0}^{1011} \left(2025^i x^{2024-2i} + 2025^{i+1} x^{2024-2(i+1)}\right)(-1)^i$$

$$+ 2025^{1012}$$

$$= \sum_{i=0}^{1011} \left(2025^i x^{2022-2i} (x^2 + 2025)\right) + 2025^{1012}$$

$$\Rightarrow \frac{x + (x^{2024} + x)k}{x^2 + 2025} = \frac{x + xk}{x^2 + 2025} + k\left(\sum_{i=0}^{1011} \left(2025^i x^{2022-2i}\right)\right)$$

$$+ \frac{k \cdot 2025^{1012}}{x^2 + 2025}$$

$$\Rightarrow x + xk + k \cdot 2025^{1012} \vdots x^2 + 2025$$

$$(\Rightarrow) \deg(x + xk + k \cdot 2025^{1012}) \geq \deg(x^2 + 2025) \; (X)$$

Case 3: $P(x) = k$

$$\Rightarrow k^{2023} + Q(x) \cdot k^2 + (x^{2024} + x) \cdot k + x^3 + 2025x = 0$$

$$\Rightarrow Q(x) = \frac{-\left[(x^{2024} + x)k + x^3 + 2025x + k^{2023}\right]}{k^2}$$

and $P(x) = k \quad \forall k \in \mathbb{R}^*$