VMO 2024 - Day 1

P_1: let $\{a_n\}$: $a_n = \frac{1}{4^{\lfloor -\log_4 n \rfloor}}$,

$\{b_n\}$: $b_n = \frac{1}{n^2}\left(\sum_{k=1}^{n} a_k - \frac{1}{a_1 + a_2} \right)$

a) Find one $P(x)$: $P\left(\frac{a_n}{n}\right) = b_n \quad \forall n \geq 1$

b) Prove exist $\{n_k\}$: $\lim_{k \to \infty} b_{n_k} = \frac{2024}{2025}$

a)

(1) If $n = 4^k \Rightarrow a_n = \frac{1}{4^{-k}} = 4^k \quad \forall k \geq 1$

(2) If $n \in (4^{k-1}, 4^k)$, $k \geq 1$

It can be seen that $\lfloor -\log_4 n \rfloor = -(\lfloor \log_4 n \rfloor + 1)$

$= -(k-1+1) = -k$

$\Rightarrow a_n = \frac{1}{4^{-k}} = 4^k \quad \forall k \geq 1$

(1), (2) $\Rightarrow a_n = 4^k$ when $n \in (4^{k-1}, 4^k] \quad \forall k \geq 1$

let $n = 4^{s-1} + m$

$$\Rightarrow \sum_{k=2}^{n} a_k = \sum_{k=2}^{4^{s-1}} a_k + \sum_{j=4^{s-1}+1}^{n} a_j$$

$$= \sum_{t=1}^{s-1} 3.4^{t-1}.4^t + m.4^s$$

$$= \sum_{t=1}^{s-1} \frac{3}{4} \cdot 16^{t} + m \cdot 4^{s}$$

$$= \frac{3}{4}\left(\sum_{t=0}^{s-1} 16^{t} - 1 \right) + m \cdot 4^{s}$$

$$= \frac{3}{4}\left(\frac{16^{s}-1}{16-1} - 1 \right) + m \cdot 4^{s} = \frac{3}{4}\left(\frac{16^{s}-16}{15} \right) + m \cdot 4^{s}$$

$$= \frac{4(16^{s-1}-1)}{5} + m \cdot 4^{s}$$

$$\Rightarrow b_n = \frac{1}{(4^{s-1}+m)^2}\left(a_1 + \frac{4}{5}(16^{s-1}-1) + m \cdot 4^{s} - \frac{1}{a_1+a_2} \right)$$

$$= \frac{1}{(4^{s-1}+m)^2}\left(\frac{4}{5} \cdot 16^{s-1} + 4 \cdot 4^{s-1} \cdot m \right)$$

$$= \frac{4}{5} \cdot \left(\frac{4^{s-1}}{4^{s-1}+m} \right)^2 + 4\left(1 - \frac{4^{s-1}}{4^{s-1}+m} \right) \cdot \frac{4^{s-1}}{4^{s-1}+m}$$

$$= \frac{4}{5}\left(\frac{1}{4} \cdot \frac{4^{s}}{4^{s-1}+m} \right)^2 + \left(1 - \frac{1}{4} \frac{4^{s}}{4^{s-1}+m} \right) \frac{4^{s}}{4^{s-1}+m}$$

$$= P\left(\frac{a_n}{n} \right) = P\left(\frac{4^{s}}{4^{s-1}+m} \right)$$

$$\Rightarrow P(x) = \frac{1}{20} x^2 + \left(1 - \frac{1}{4} x \right) x$$

$$= \frac{1}{20} x^2 + x - \frac{1}{4} x^2 = \frac{-1}{5} x^2 + x$$

b)

$$b_n = \frac{4}{5}\left(\frac{n-m}{n}\right)^2 + 4\left(1-\frac{n-m}{n}\right)\left(\frac{n-m}{n}\right)$$

$$= \frac{4}{5}\left(\frac{n-m}{n}\right)^2 + 4\left(\frac{n-m}{n}\right) - 4\left(\frac{n-m}{n}\right)^2$$

$$= \frac{-16}{5}\left(\frac{m}{n}-1\right)^2 + 4\left(1-\frac{m}{n}\right)$$

$$= \frac{-16}{5}\left(\left(\frac{m}{n}\right)^2 - \frac{2m}{n} + 1\right) + 4 - \frac{4m}{n}$$

$$= \frac{-16}{5}\left(\frac{m}{n}\right)^2 + \frac{32}{5}\frac{m}{n} - \frac{16}{5} + 4 - \frac{4m}{n}$$

$$= \frac{-16}{5}\left(\frac{m}{n}\right)^2 + \frac{12}{5}\left(\frac{m}{n}\right) + \frac{4}{5}$$

Consider

$$F(x) = \frac{-16}{5}x^2 + \frac{12}{5}x + \frac{4}{5}$$

$$\Rightarrow F'(x) = \frac{-32}{5}x + \frac{12}{5}$$

$\Rightarrow$

	0		$\frac{3}{8}$		$+\infty$
x		$+$		$-$	
$F(x)$		$\nearrow$		$\searrow$	

We have $F(x)$ continuous on $\mathbb{R}$ and $F(0,1) > \frac{2024}{2025} > F(0)$

$\Rightarrow \exists x_0 \in (0, 0{,}1) : F(x_0) = \dfrac{2024}{2025}$

Because $F(x)$ continuous on $\mathbb{R} \Rightarrow \lim\limits_{x \to x_0} F(x) = F(x_0)$

Choose m such that

$$\lim_{s \to \infty} \frac{m}{4^{s-1} + m} = x_0 \Leftrightarrow \lim_{s \to \infty} \frac{1}{\frac{4^{s-1}}{m} + 1} = x_0 \Leftrightarrow \lim_{s \to \infty} \frac{4^{s-1}}{m} = \frac{1}{x_0} - 1$$

We have $m \in [0, 3.4^{s-1})$

$\Rightarrow$ Consider $m = \left\lfloor \dfrac{4^{s-1} x_0}{1 - x_0} \right\rfloor$

$$\Rightarrow \frac{4^{s-1} x_0}{1 - x_0} - 1 < m \leq \frac{4^{s-1} x_0}{1 - x_0}$$

$$\Rightarrow \frac{4^{s-1}(1 - x_0)}{4^{s-1} x_0} = \frac{4^{s-1}}{\frac{4^{s-1} x_0}{1 - x_0}} < \frac{4^{s-1}}{m} \leq \frac{4^{s-1}}{\frac{4^{s-1} x_0 - 1 + x_0}{1 - x_0}} = \frac{4^{s-1}(1 - x_0)}{4^{s-1} x_0 - 1 + x_0}$$

$$\frac{1 - x_0}{x_0}$$

Consider $\{n_s\} : n_s = \dfrac{\left\lfloor \frac{4^{s-1} x_0}{1 - x_0} \right\rfloor}{4^{s-1} + \left\lfloor \frac{4^{s-1} x_0}{1 - x_0} \right\rfloor}$

$\Rightarrow \lim\limits_{s \to \infty} n_s = x_0 \Rightarrow \lim\limits_{s \to \infty} F(n_s) = F(x_0)$

$\Rightarrow \{ b_{n_s} \} = F(n_s)$

$\Rightarrow \lim\limits_{s \to \infty} b_{n_s} = \lim\limits_{s \to \infty} F(n_s) = F(x_0) = \dfrac{2024}{2025}$