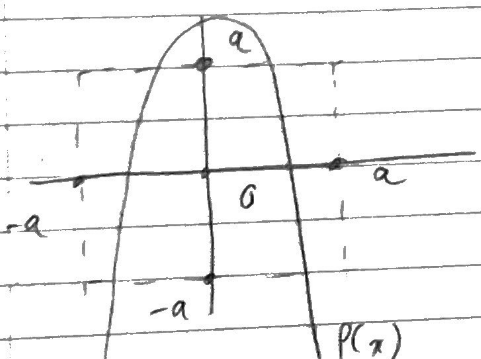
VMO 2024 - Day 2

P_1: $P(x)$

$$P_1(x) = P(x) \ \forall x \in \mathbb{R}$$
$$P_2(x) = P(P_1(x)), \ \forall x \in \mathbb{R}$$
$$\ldots$$
$$P_{2024}(x) = P(P_{2023}(x)) \ \forall x \in \mathbb{R}$$

Consider $a \in \mathbb{R}$, $a > 2$. $\exists$ or $\nexists$ $P(x) \in \mathbb{R}(x)$:
For each $t \in (-a, a)$, $P_{2024}(x) = t$ has 2^{2024} root (exact)

consider this graph.



We will generalize the problem to $P_n(x)$ $\forall x \in \mathbb{R}$, $n \in \mathbb{N}^*$

Assume that there exists $P(x)$ that satisfies the claim.

If $n = 1 \Rightarrow P_1(x) = P(x)$

For each $t \in (-a, a)$, $P_1(x)$ has exact 2 root

which means, we can choose $P(x)$ as shown in the graph above
we will choose $P(x) = bx^2 + c$ $(b < 0)$ fits the graph
The heighest point of $P(x) = \left(0, \frac{4bc}{4b}\right) = (0, c)$

$\Rightarrow c > a$

Consider $P(x) = -a \Leftrightarrow x^2 = \frac{-a-c}{b} \Leftrightarrow \frac{-a-c}{b} < a^2$

Because $a > 2$, if $c = a+1$ and $b = -1$, $\frac{-a-c}{b} = \frac{-a-a-1}{-1} = \frac{2a+}{1}$

Consider $f(x) = x^2 - 2x - 1 \Rightarrow f'(x) = 2x - 2 > 0 \ \forall x \geq 2$

$\Rightarrow a^2 - 2a - 1 > 0 \quad \forall a > 2$

$\Rightarrow \dfrac{2a+1}{1} < a^2 \quad \forall a > 2$

$\Rightarrow P(x) = -x^2 + a + 1$ will fit the graph

which means, fle $t \in (-a, a)$, $\exists x_1, x_2 \in (-a, a)$: $P(x_1) = P(x_2) = t$

$\deg P_1 = 2 \Rightarrow P_1(x) = t$ has exact 2 root fle $t \in (-a, a)$

It's easy to see that $\deg P_n(x) = 2^n$

$\Rightarrow P_n(x) = t$ has at most 2^n root foreach $t \in (-a, a)$

consider $P_2(x)$, $P_2(x) = P(P_1(x))$

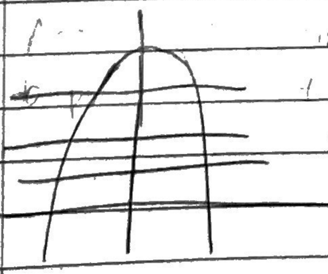
$\Rightarrow \exists x_{1_1}, x_{1_2} \in (-a, a) : P_1(x_{1_1}) = P_1(x_{1_2}) = x_1$

$\exists x_{2_1}, x_{2_2} \in (-a, a) : P_1(x_{2_1}) = P_1(x_{2_2}) = x_2$

$\Rightarrow P_2(x_1) = P(P_1(x_{1_1})) = P(P_1(x_{1_2})) = P(x_1) = t$

$\Rightarrow P_2(x_2) = P(P_1(x_{2_1})) = P(P_1(x_{2_2})) = P(x_2) = t$

$x_1 \neq x_2 \Rightarrow x_{1_1} \neq x_{1_2} \neq x_{2_1} \neq x_{2_2}$



$\Rightarrow$ Roots are always unique

$\Rightarrow$ We apply the same concept to $P_3, P_4, \dots P_n$

We have $P_n(x) = t$ has exact 2^n root fle $t \in (-a, a)$

$\Rightarrow P_{2024}(x) = t$ has 2^{2024} root fle $t \in (-a, a)$