# Forecasting I

DS 6030 | Fall 2023

forecasting.pdf

## Contents



# 1 Time Series Data

## 1.1 Chicago Arrest Data

The Chicago Police Department has published arrest data from 2014-2017.

```
library(tidyverse)
url = "https://home.chicagopolice.org/wp-content/uploads/2018/07/PublicReleaseArrestDataUPDATE.csv"
arrests = read_csv(url)
```

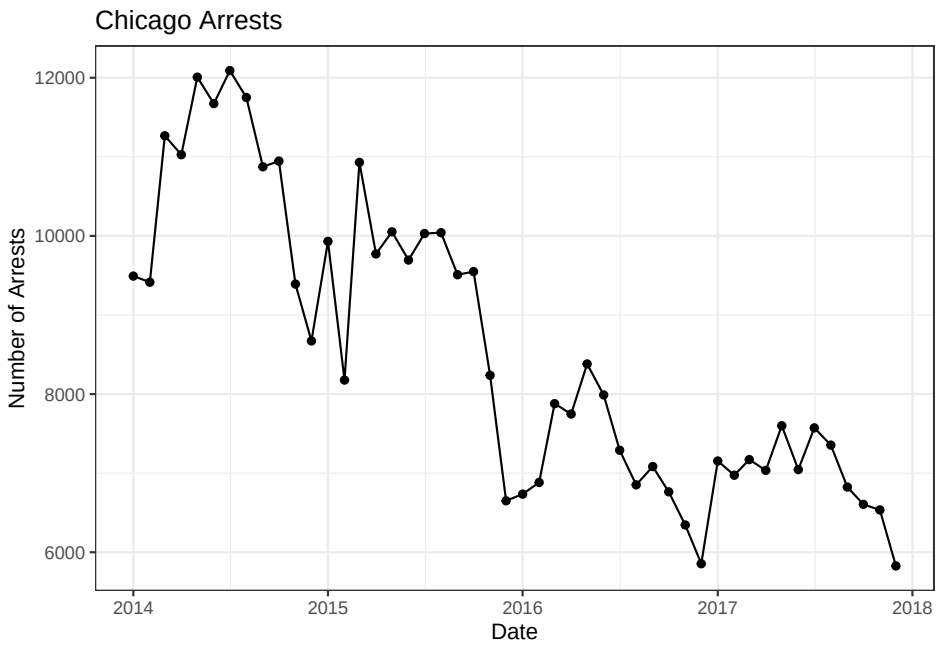
```
#> # A tibble: 6 x 10
#>   ARR_DISTRICT ARR_BEAT ARR_YEAR ARR_MONTH RACE_CODE_CD FBI_CODE STATUTE
#>          <dbl>    <dbl>    <dbl>     <dbl> <chr>        <chr>    <chr>
#> 1           10     1033     2017         8 BLK          18       720 ILCS 570.0~
#> 2            9      923     2017         8 WWH          WRT      725 ILCS 225.0~
#> 3           10     1024     2017         8 BLK          WRT      725 ILCS 5.0/1~
#> 4           11     1112     2017         8 BLK          18       720 ILCS 570.0~
#> 5           25     2524     2017         8 WHI          18       720 ILCS 570.0~
#> 6           11     1122     2017         9 BLK          7        720 ILCS 5.0/2~
#> # i 3 more variables: STAT_DESCR <chr>, CHARGE_CLASS_CD <chr>,
#> #   CHARGE_TYPE_CD <chr>
```

We're going to use the total number of arrest per month as the outcome variable of interest. This calculates the total number of arrests per year-month (`n`) and `date` (using first of month).

```
library(lubridate)  # good package for working with dates and times
arrest_counts = arrests %>%
  count(year = ARR_YEAR, month = ARR_MONTH) %>%   # aggregate number of arrests by month, year
  arrange(year, month) %>%
  mutate(
    date = lubridate::make_date(year, month, day = 1),
    index = 1:n()
  )

head(arrest_counts)
#> # A tibble: 6 x 5
#>    year month     n date       index
#>   <dbl> <dbl> <int> <date>     <int>
#> 1  2014     1  9492 2014-01-01     1
#> 2  2014     2  9415 2014-02-01     2
#> 3  2014     3 11266 2014-03-01     3
#> 4  2014     4 11027 2014-04-01     4
#> 5  2014     5 12007 2014-05-01     5
#> 6  2014     6 11673 2014-06-01     6
```

Time series plot

```
arrest_counts %>%
  ggplot(aes(date, n)) +
  geom_point() + geom_line() +
  labs(title="Chicago Arrests", x = "Date", y = "Number of Arrests")
```




**Your Turn #1**

1. What patterns do you see?

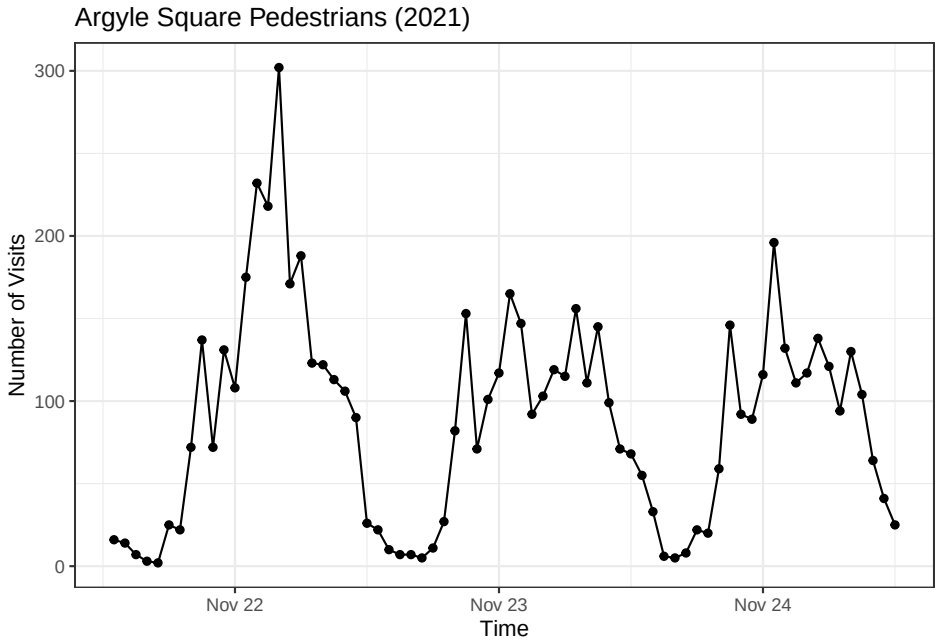
## 1.2 Data Example 2

The City of Melbourne Australia has published pedestrian traffic count data in Argyle Square for a few days in 2021.

```
library(tidyverse)
url = "https://data.melbourne.vic.gov.au/api/explore/v2.1/catalog/datasets/argyle-pedestrian-counting
Argyle = read_csv(url)
```

```
#> # A tibble: 6 x 10
#>   location_id interval_start      walkbys visits unique_visitors
#>         <dbl> <dttm>                <dbl>  <dbl>           <dbl>
#> 1       11675 2021-11-21 14:00:00     264     14              14
#> 2       11675 2021-11-21 16:00:00     189      3               3
#> 3       11675 2021-11-21 17:00:00     148      2               2
#> 4       11675 2021-11-21 20:00:00     591     72              72
#> 5       11675 2021-11-22 00:00:00     947    108             108
#> 6       11675 2021-11-22 02:00:00    1524    232             232
#> # i 5 more variables: returning_visitors <dbl>, bounce_rate <dbl>,
#> #   avg_visit_duration <dbl>, `Sensor Name` <chr>, `Lat Long` <chr>
```

```
Argyle %>%
  ggplot(aes(interval_start, visits)) +
  geom_point() +
  geom_line() +
  labs(x = "Time", y = "Number of Visits", title = "Argyle Square Pedestrians (2021)")
```




**Your Turn #2**

1. What patterns do you see?

### 1.3 Data Example 3

TODO

### 1.4 Time Series Patterns

A time series is data that is recorded at sequentially at *regular intervals*[1] of time. We can write the data as $D = \{(t_i, y_i)\}$ where:

- $t_{i-1} < t_i < t_{i+1}$ is the time of the $i$th observation
- $\delta = t_i - t_{i-1}$ is the fixed interval between times
  - units: years, quarters, months, weeks, days, hours, mins, sec
- The outcome variable $y_i$ can be anything (real valued, integer/count, categorical, graph, image)
- For regular (i.e., fixed interval) time series, it's common to simplify notation by moving time into subscript of $y$ denote the time $\ldots, y_{t-1}, y_t, y_{t+1}, \ldots$

There are some common patterns in time series data that can be exploited to make good forecasts.

1. Trends: observations *close* in time are often similar
   - Observations close in time exhibit simple trends (e.g., linear, quadratic, logistic)
   - e.g., $y_t$ close to $\beta y_{t-1}$ (or $\beta + y_{t-1}$)
2. Seasonality: Observations at *similar* times are often similar.
   - Human regular routines often lead to seasonal patterns
   - E.g., compare this month's outcomes to 12 months ago
   - e.g., $y_t$ close to $y_{t-12}$ (yearly seasonality)
3. Special times
   - Holidays, Events
   - These are short periods of time in which human behavior changes from normal
4. Exogenous dependence/association
   - other time series influencing (or associating with) outcomes
   - e.g., number of arrests may be related to the number of officers
5. Cycles: regular fluctuations, but with non-fixed frequency
   - economic cycles (expansion, peak, contraction, and trough)
   - weather cycles (El Niño and La Niña)
6. Shocks: significance changes to the data generating system
   - The data generating system may experience shocks that temporarily or persistently change the process.
   - E.g., large weather events, change in political leaders
   - These can sometimes be captured in *trends* and *special times*, but there is a field of time series analysis called *intervention analysis* that attempts to properly handle such shocks in presence of temporally correlated errors.

### 1.5 Forecasting / Predicting

The goal of forecasting is predict the outcome for *future* times.

$$\hat{y}(t+h) = \hat{f}_h(y_{1:t}, X_{1:t})$$

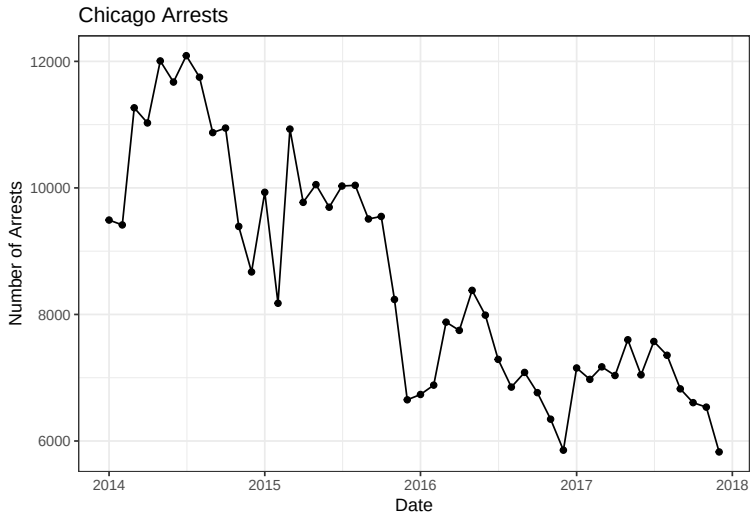
- $h > 0$ is the forecast horizon

[1] Technically, you can have irregularly spaced time series (e.g., stock market closed weekends and holidays).

- $\hat{f}_h$ is the forecasting model (for horizon $h$)
- $y_{1:t} = y_1, \ldots, y_t$ are the past outcomes
- $X_{1:t} = X_1, \ldots, X_t$ are the other predictive features

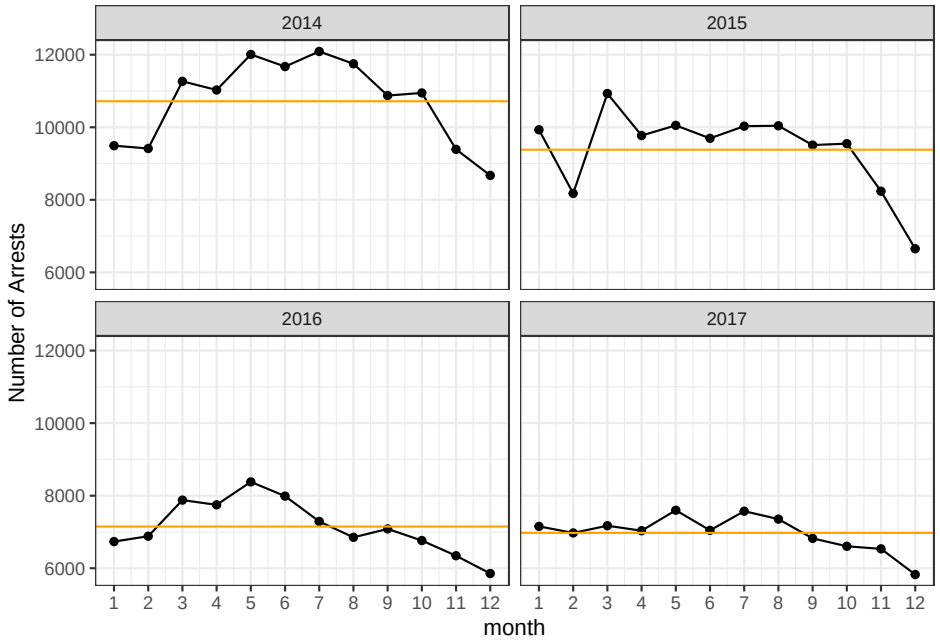
# 2 Chicago Arrest Analysis

## 2.1 Seasonality



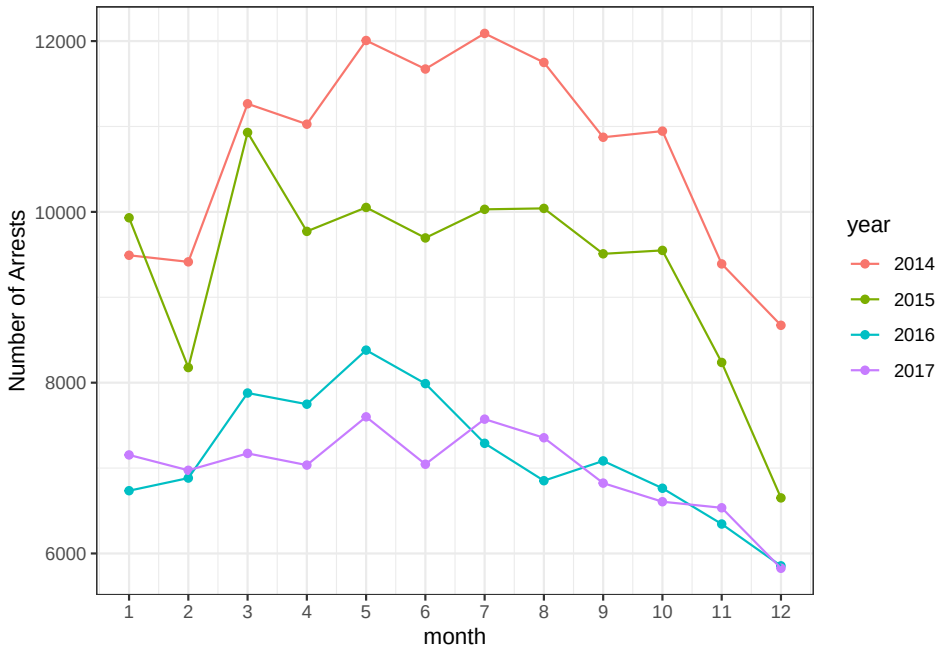

Notice the potential seasonality (e.g., decrease in Oct-Dec, increase in Mar). Let's look at the events for each year separately.

```
arrest_counts %>%
  # add yearly average
  group_by(year) %>% mutate(n_avg = mean(n)) %>% ungroup() %>%
  ggplot(aes(month, n)) +
  geom_point() + geom_line() +
  geom_hline(aes(yintercept = n_avg), color="orange") +
  scale_x_continuous(breaks = 1:12) +
  facet_wrap(~year) +
  labs(y = "Number of Arrests")
```




Or, view on single plot

```
arrest_counts %>% group_by(year) %>% mutate(n_avg = mean(n)) %>% ungroup() %>%
  mutate(year = factor(year)) %>% # make discrete for ggplot
  ggplot(aes(month, n, color=year)) +
  geom_point() + geom_line() +
  scale_x_continuous(breaks = 1:12) +
  labs(y = "Number of Arrests")
```




**Your Turn #3**

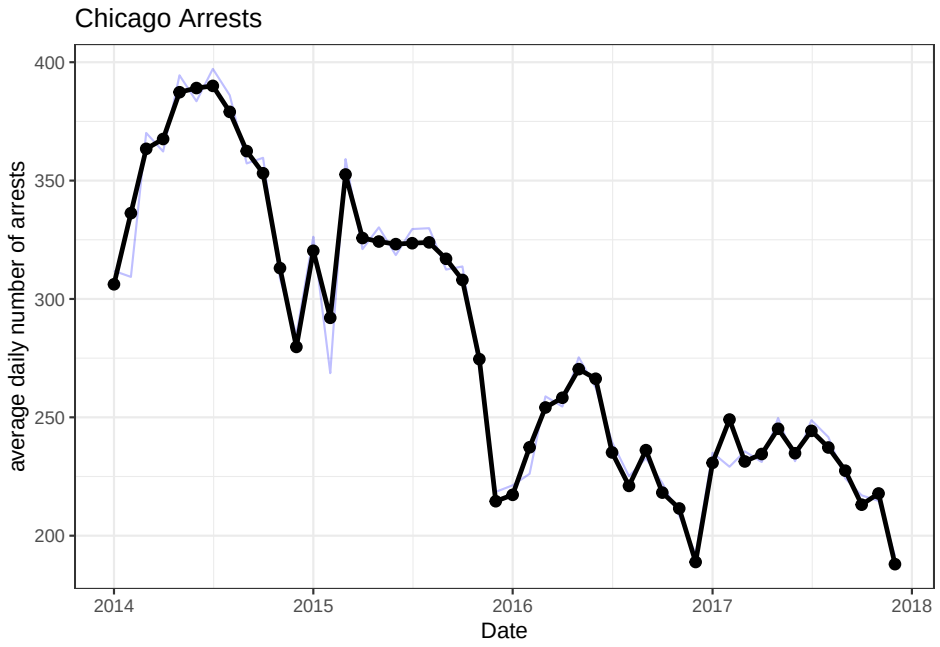
Why are the number of arrests lower in February compared to January and March?

### 2.1.1 Adjustments / Standardization

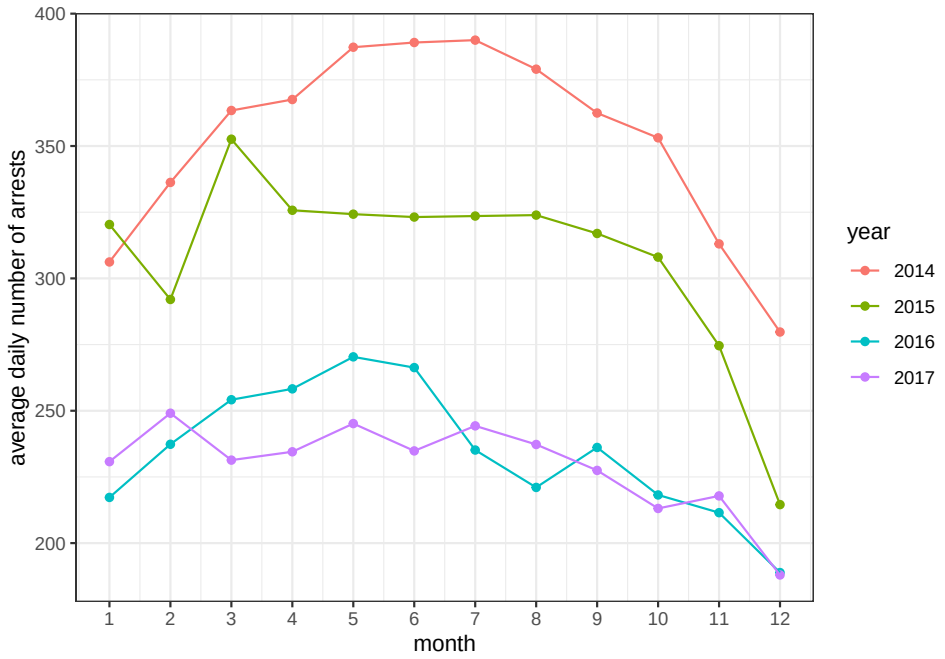
Let's standardize for days per month.

```
library(lubridate)
arrest_rate = arrest_counts %>%
  mutate(
    days_in_month = lubridate::days_in_month(date),
    daily_avg = n / days_in_month
  )
```

```
arrest_rate %>%
  ggplot(aes(date, daily_avg)) +
  geom_line(data = . %>% mutate(daily_avg = n/(365.25/12)),
            linewidth = .5,
            alpha = .25, color="blue") +
  geom_point(size = 2.25) + geom_line(linewidth=1.10) +
  labs(title="Chicago Arrests", x = "Date", y = "average daily number of arrests")
```




```
arrest_rate %>%
  mutate(year = factor(year)) %>%
  ggplot(aes(month, daily_avg, color = year)) +
  geom_point() + geom_line() +
  scale_x_continuous(breaks = 1:12) +
  labs(y = "average daily number of arrests")
```

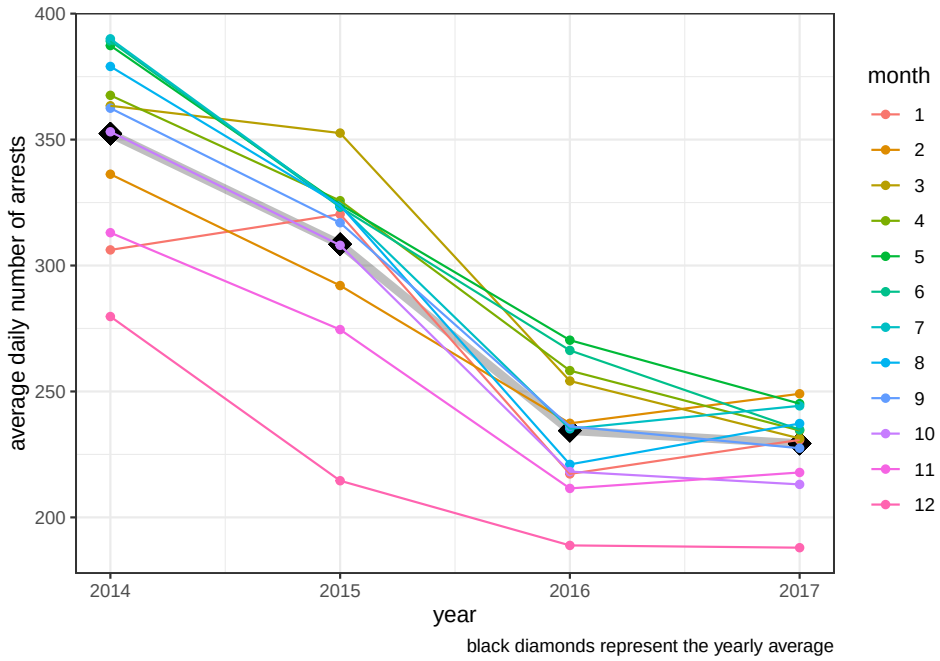



Much smoother path across months and also now a fairer comparison.

## 2.2 Trends

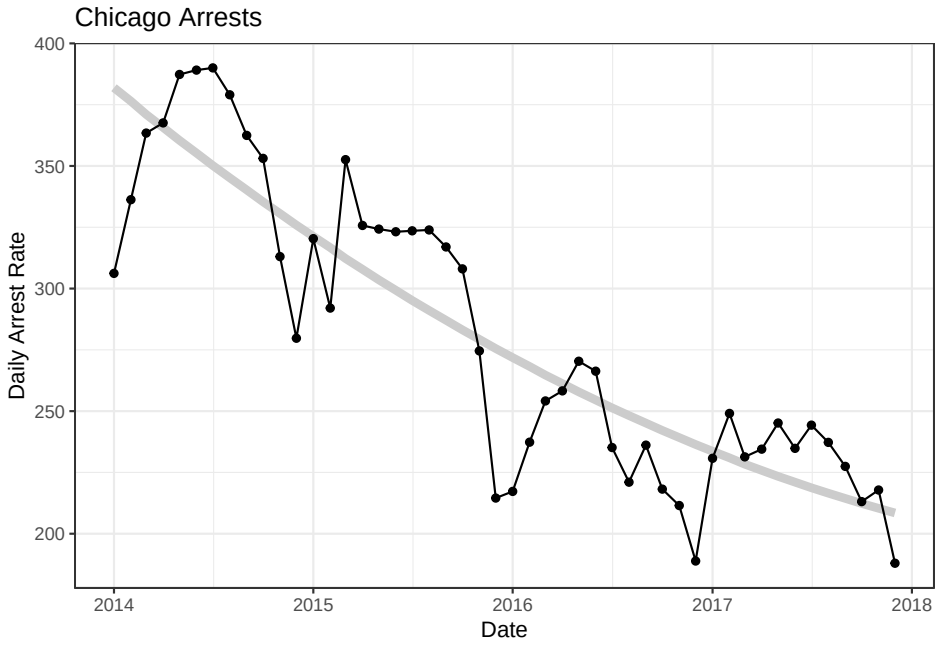
The number of arrests (and arrest rates) appear to be reducing over time. Here we examine each month separately as a function of year:

```
arrest_rate %>%
  mutate(
    days_in_year = ifelse(year %% 4 == 0, 366, 365)
  ) %>%
  group_by(year) %>% mutate(yearly_avg = sum(n) / days_in_year) %>% ungroup() %>%
  ggplot(aes(year, daily_avg, color = factor(month))) +
  geom_point(aes(y = yearly_avg), color="black", shape=18, size = 5) +
  geom_line(aes(y = yearly_avg), color="black", alpha = .25, linewidth = 2) +
  geom_point() + geom_line() +
  labs(y = "average daily number of arrests", color="month",
       caption = "black diamonds represent the yearly average")
```



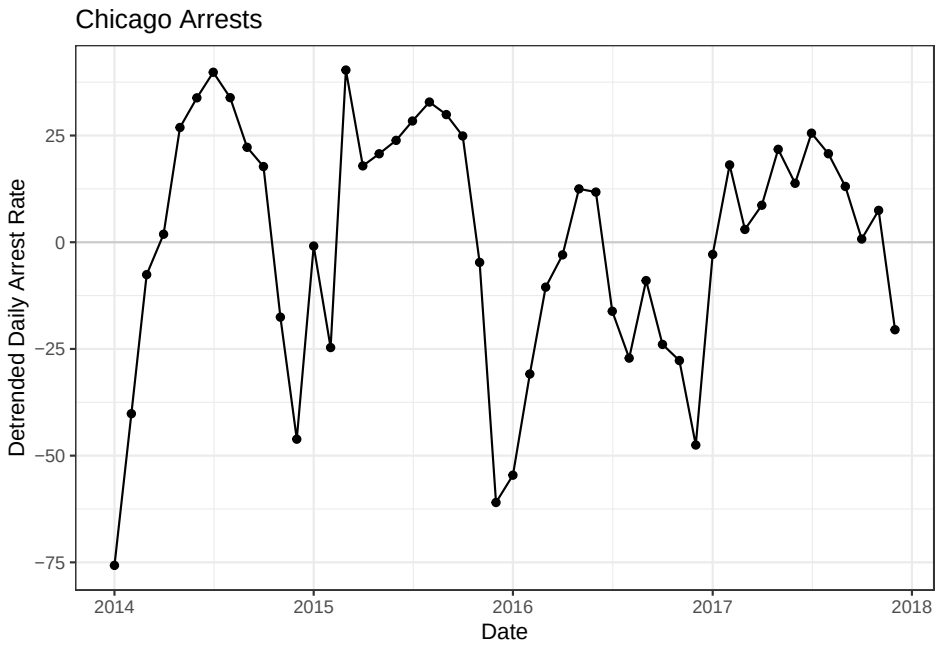

```r
#: fit a quadratic trend
trend_quad = lm(daily_avg ~ poly(index, 2), data = arrest_rate)

#: add trend to data
arrest_rate_trend = arrest_rate %>%
  mutate(trend = predict(trend_quad, .))
```

```r
#: add trend line to time series
arrest_rate_trend %>%
  ggplot(aes(date, daily_avg)) +
  geom_line(aes(y = trend), color = "grey80", linewidth=2) +
  geom_point() + geom_line() +
  labs(title="Chicago Arrests", x = "Date", y = "Daily Arrest Rate")
```



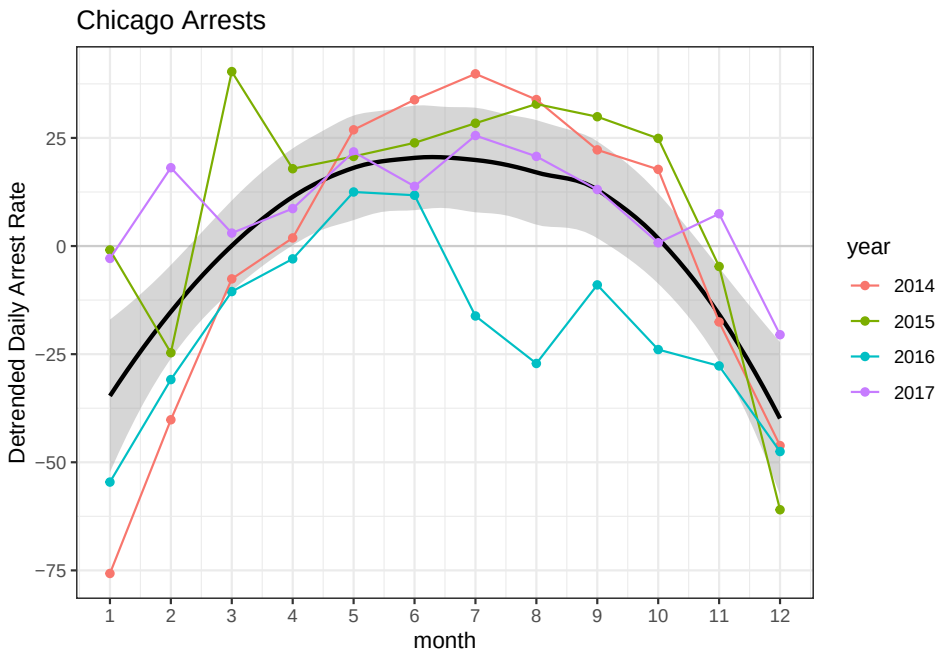
The *detrended* series is the outcome minus the trend

```
arrest_rate_trend %>%
  mutate(detrended = daily_avg - trend) %>%
  ggplot(aes(date, detrended)) +
  geom_hline(yintercept = 0, color="grey80") +
  geom_point() + geom_line() +
  labs(title="Chicago Arrests", x = "Date", y = "Detrended Daily Arrest Rate")
```




Use the detrended residuals to identify seasonality:

```
arrest_rate_trend %>%
  mutate(detrended = daily_avg - trend, year = factor(year)) %>%
  ggplot(aes(month, detrended)) +
  geom_hline(yintercept = 0, color="grey80") +
  geom_smooth(color="black") +
  geom_point(aes(color=year)) + geom_line(aes(color=year)) +
  scale_x_continuous(breaks = 1:12) +
  labs(title = "Chicago Arrests", y = "Detrended Daily Arrest Rate")
```

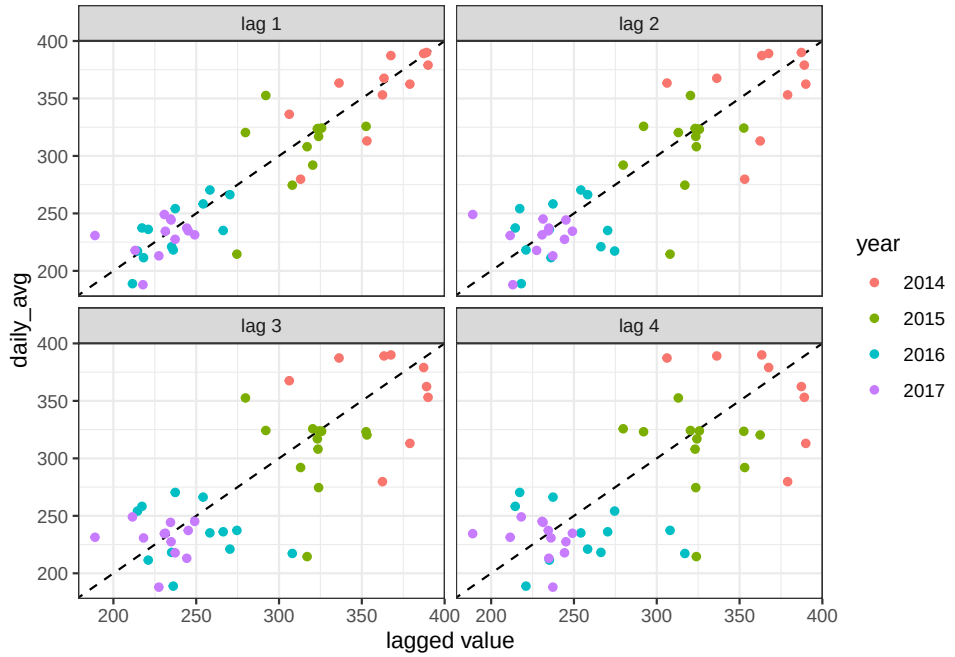



## 2.3 Autocorrelation

How similar are the arrest rates for nearby months? We can plot the lagged arrest rates. The lagged values are $\text{lag}_k(t) = y(t) - y(t-k)$.

```
lags = arrest_rate %>%
  mutate(
    year = factor(year)
  ) %>%
  arrange(date) %>%
  mutate(
    "lag 1" = lag(daily_avg),
    "lag 2" = lag(daily_avg, 2),
    "lag 3" = lag(daily_avg, 3),
    "lag 4" = lag(daily_avg, 4),
  )
head(lags)
#> # A tibble: 6 x 11
#>   year  month     n date       index days_in_month daily_avg `lag 1` `lag 2`
#>   <fct> <dbl> <int> <date>     <int>         <int>     <dbl>   <dbl>   <dbl>
#> 1 2014      1  9492 2014-01-01     1            31      306.     NA      NA
#> 2 2014      2  9415 2014-02-01     2            28      336.    306.     NA
#> 3 2014      3 11266 2014-03-01     3            31      363.    336.    306.
#> 4 2014      4 11027 2014-04-01     4            30      368.    363.    336.
#> 5 2014      5 12007 2014-05-01     5            31      387.    368.    363.
#> 6 2014      6 11673 2014-06-01     6            30      389.    387.    368.
#> # i 2 more variables: `lag 3` <dbl>, `lag 4` <dbl>
```

```
lags %>%
  pivot_longer(cols = starts_with("lag"), names_to="lag") %>% filter(!is.na(value)) %>%
  ggplot(aes(value, daily_avg, color=year)) +
  geom_abline(slope = 1, intercept = 0, linetype = "dashed") +
  geom_point() +
  facet_wrap(~lag) +
  labs(x = "lagged value")
```



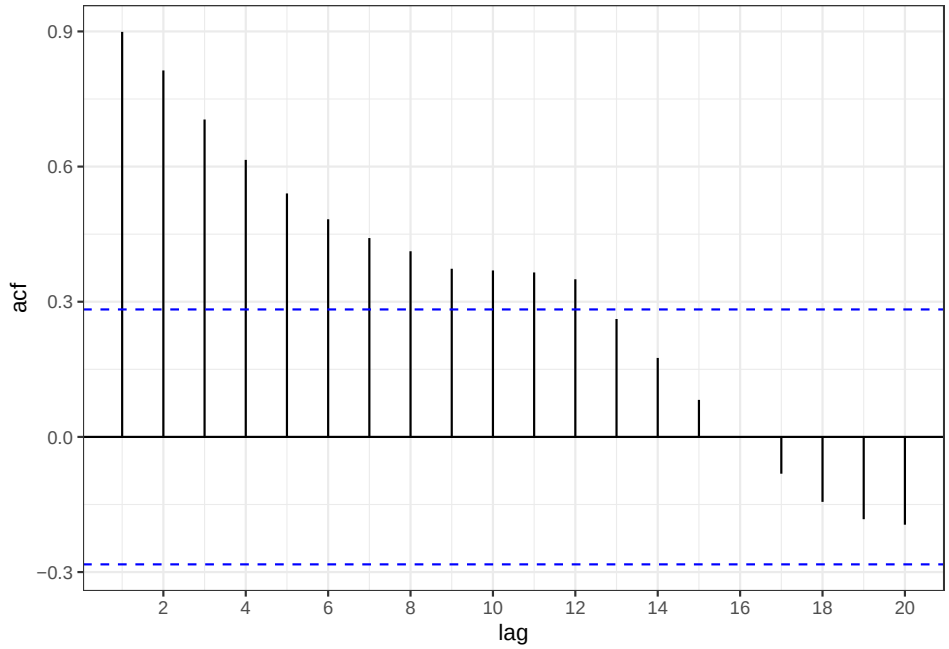

Notice that the lag 1 values have the highest correlation. We can explicitly calculate the (auto) correlation

```
lags %>%
  pivot_longer(cols = starts_with("lag"), names_to="lag") %>%
  filter(!is.na(value)) %>%
  group_by(lag) %>%
    summarize(r = cor(daily_avg, value))
#> # A tibble: 4 x 2
#>   lag       r
#>   <chr> <dbl>
#> 1 lag 1 0.926
#> 2 lag 2 0.860
#> 3 lag 3 0.779
#> 4 lag 4 0.713
```

The *autocorrelation* function (ACF) gives the correlation between $y_t$ and a range of lagged values. To get the ACF, we'll need to convert our data into a proper time series object. I'll use the `tsibble` representation from the `tsibble` package (part of `fpp3`).

```
library(fpp3)
arrest_ts = arrest_rate %>%
  mutate(
    time = make_yearmonth(year, month) # use (month, year) for temporal index
  ) %>%
  as_tsibble(index = time)
```

```
arrest_ts %>%
  ACF(daily_avg, lag_max = 20) %>%
  autoplot() +
  scale_x_continuous(breaks = seq(0, 24, by=2)) +
  labs(x = "lag")
```




## 2.4 Time Series Decomposition

Consider the additive decomposition model:

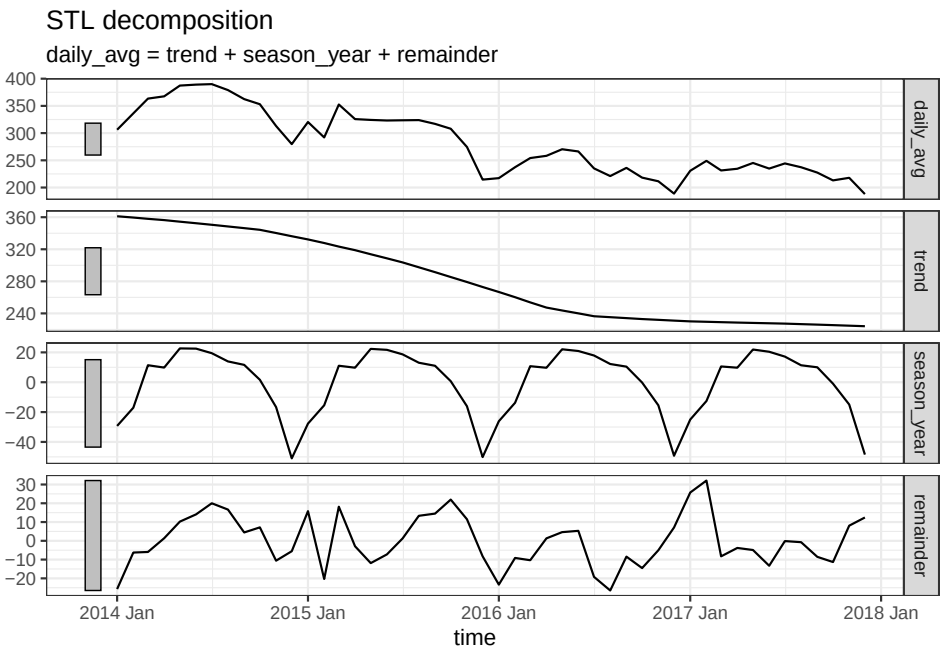
$$y_t = T_t + S_t + R_t$$

- $T_t$ captures the *trend* (and possibly cycles)
- $S_t$ captures the *seasonality* signal
- $R_t$ is the *remainder*. This can also be termed *unexplained noise*.

There are several common approaches to estimate these terms. Later we will see how Facebook Prophet uses a similar additive structure. Here is how to implement the STL approach. STL is an acronym for *Seasonal and Trend decomposition using Loess*[2]:

```
decomp = arrest_ts %>% model(stl = STL(daily_avg))
components(decomp) %>% autoplot()
```

[2]Cleveland, R. B., Cleveland, W. S., McRae, J. E., & Terpenning, I. (1990). STL: A seasonal-trend decomposition. J. Off. Stat, 6(1), 3-73.



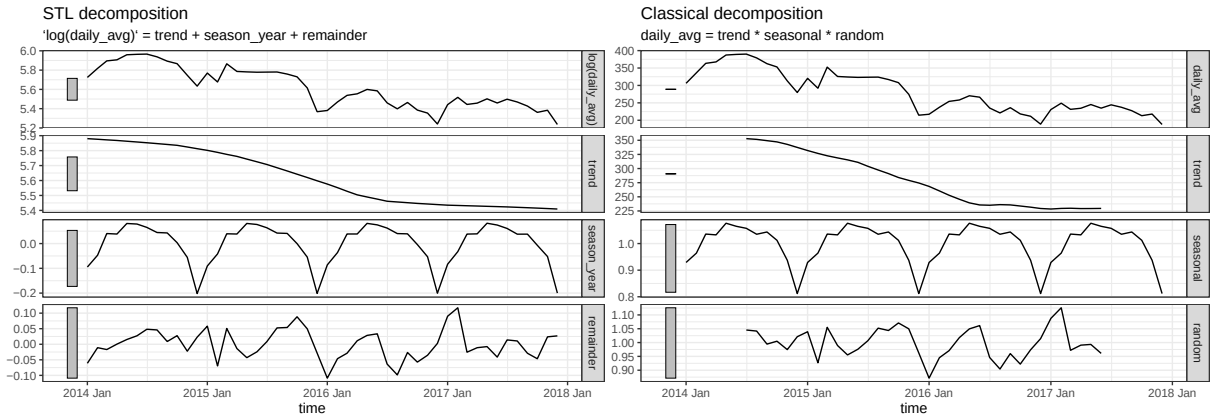

It is also common to consider a *multiplicative* structure:

$$y_t = T_t \cdot S_t \cdot R_t$$

Although this changes the units for performance estimation, it is useful to consider that taking logs converts the multiplicative structure to something additive

$$\log y_t = \log T_t + \log S_t + \log R_t$$
$$y'_t = T'_t + S'_t + R'_t$$

```
#: multiplicative decompose by STL using logged values
decomp_mlog = arrest_ts %>% model(stl = STL(log(daily_avg)))
components(decomp_mlog) %>% autoplot()

#: multiplicative decomposition by "classical" methods
decomp_m = arrest_ts %>% model(
  classical_decomposition(daily_avg, type = "multiplicative")
)
components(decomp_m) %>% autoplot()
#> Warning: Removed 6 rows containing missing values (`geom_line()`).
```



# 3 Models

## 3.1 Regression

Regression with $p$ predictor variables

$$\hat{y}_{t+h} = \hat{\beta}_{0,h} + \hat{\beta}_{1,h}X_{1,t} + \hat{\beta}_{2,h}X_{2,t} + \ldots + \hat{\beta}_{p,h}X_{p,t}$$

Notice that the coefficients $\hat{\beta}_{j,h}$ are a function of $h$; the effects of a predictor may change for different forecast horizons.

### 3.1.1 AR(p) Autoregression models

Autoregressive model use lagged outcome values as the predictor variables:

$$\begin{aligned}\hat{y}_{t+1} &= \hat{\beta}_0 + \hat{\beta}_1 y_{t-1} + \hat{\beta}_2 y_{t-2} + \ldots + \hat{\beta}_p y_{t-p} \\ &= \hat{\beta}_0 + \sum_{j=1}^{p} \hat{\beta}_j y_{t-j}\end{aligned}$$

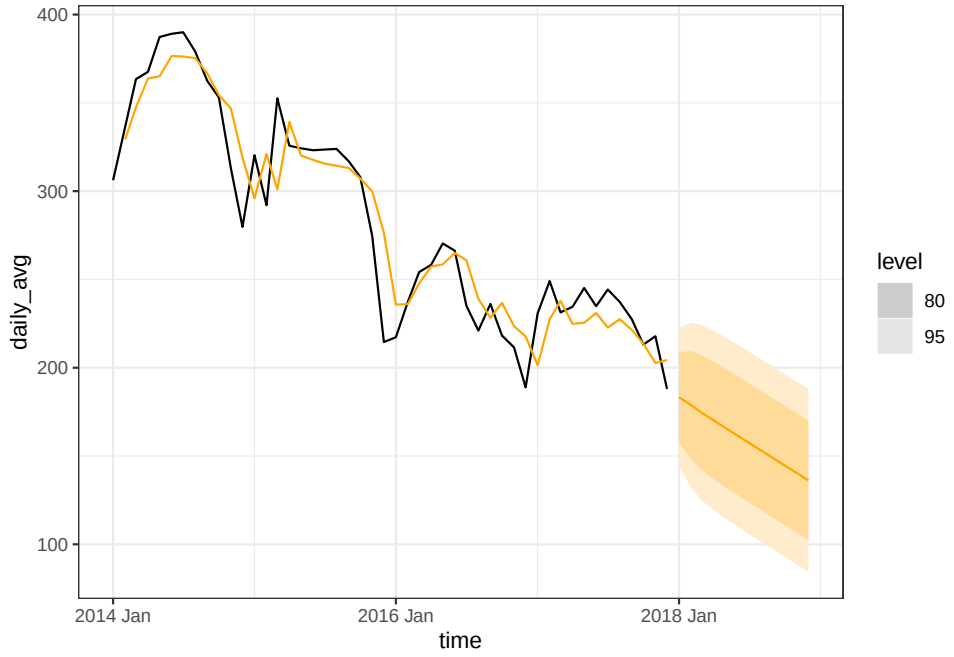
Here we'll mix autoregressive terms with a linear trend. `order()` specifies the number of AR terms to consider and `trend()` creates the linear trend. Note: higher order trends are not common in forecasting because of the possible extreme values when extrapolating.

```
fit1 = arrest_ts %>%
  model(AR(daily_avg ~ order(1:12) + trend()))

fit1 %>% report()
#> Series: daily_avg
#> Model: AR(1) w/ mean
#>
#> Coefficients:
#>   constant  trend()     ar1
#>      131.8   -1.458  0.6545
#>
#> sigma^2 estimated as 398.1
#> AIC = -98.6  AICc = -98.05   BIC = -92.98
```

The `fable::AR()` function uses AICc to select the number of AR terms (in this case only $p = 1$).

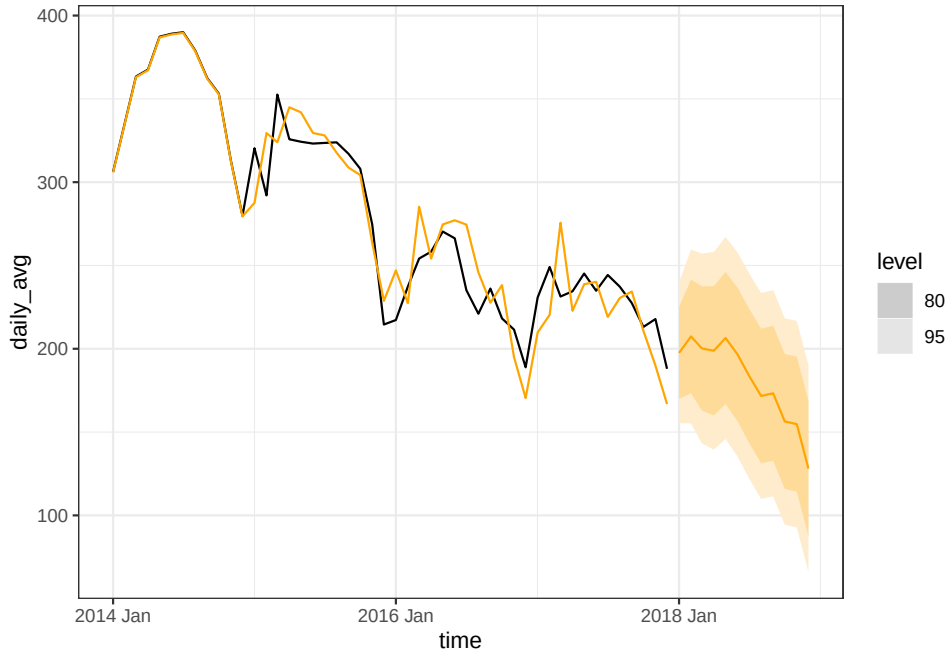
The forecasts can be obtained using the `forecast()` function.

```
fit1 %>%
  forecast(h=12) %>%
  autoplot(arrest_ts, fill="orange") +
  geom_line(data = augment(fit1), aes(time, .fitted), color="orange")
```




A better way to capture trends, autoregressive components is with a (possibly seasonal) ARIMA model. We're not going to cover ARIMA explictly, but it combines autoregressive terms, moving average terms, and differencing.

```
fit2 = arrest_ts %>%
  model(ARIMA(daily_avg))

fit2 %>% report()
#> Series: daily_avg
#> Model: ARIMA(1,0,0)(1,1,0)[12] w/ drift
#>
#> Coefficients:
#>          ar1     sar1  constant
#>       0.7340  -0.5017   -16.352
#> s.e.  0.1224   0.1528     3.753
#>
#> sigma^2 estimated as 461.3:  log likelihood=-162
#> AIC=332.1   AICc=333.4   BIC=338.4

fit2 %>%
  forecast(h=12) %>%
  autoplot(arrest_ts, fill="orange") +
  geom_line(data = augment(fit2), aes(time, .fitted), color="orange")
```




## 3.2 Prophet

The Facebook Prophet specifies an additive decomposition model

$$\hat{y}_t = \hat{T}_t + \hat{S}_t + \hat{H}_t$$

where $T$ is trend, $S$ is seasonality, and $H$ is for holidays or special times.

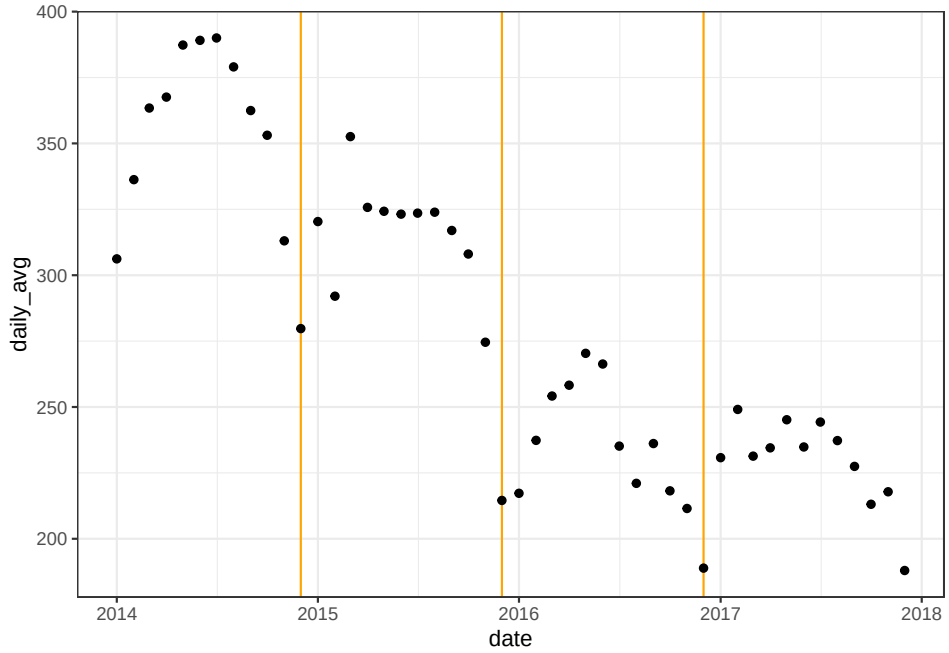
### 3.2.1 Trend Model

There are a few ways to include trends.

1. `growth` is *flat*, *linear*, or *logistic*.
2. The growth rate changes at a set of *change points*

This implies a join-point model for the trend. Here's an example of a linear trend. Suppose we specify change points at January of each year.

```
chgpts = seq(12, 36, by=12) # one at start of each year
arrest_ts %>%
  mutate(chgpt = index %in% !!chgpts) %>%
  ggplot(aes(date, daily_avg)) +
    geom_vline(data = . %>% filter(chgpt),
               aes(xintercept = date), color="orange") +
  geom_point()
```




To get a piecewise linear fit, we need to create a special model matrix that have columns (predictor variables) that are 0 for all times before the change point ($s_j$) and linear after the change point.

$$x_j(t) = (t - s_j)_+$$

This specifies the trend model of:

$$T(t) = \beta_0 + \sum_j x_j(t)\beta_j$$

```
data_train = arrest_ts %>%
  transmute(
    daily_avg,
    index,
    x = 1:n(),
    x1 = ifelse(index < chgpts[1], 0, index - chgpts[1]),
    x2 = ifelse(index < chgpts[2], 0, index - chgpts[2]),
    x3 = ifelse(index < chgpts[3], 0, index - chgpts[3]),
  )

print(data_train, n=25)
#> # A tsibble: 48 x 7 [1M]
#>         time daily_avg index     x    x1    x2    x3
#>        <mth>     <dbl> <int> <int> <dbl> <dbl> <dbl>
#>  1 2014 Jan      306.      1     1     0     0     0
#>  2 2014 Feb      336.      2     2     0     0     0
#>  3 2014 Mar      363.      3     3     0     0     0
#>  4 2014 Apr      368.      4     4     0     0     0
#>  5 2014 May      387.      5     5     0     0     0
#>  6 2014 Jun      389.      6     6     0     0     0
#>  7 2014 Jul      390       7     7     0     0     0
#>  8 2014 Aug      379.      8     8     0     0     0
#>  9 2014 Sep      362.      9     9     0     0     0
#> 10 2014 Oct      353.     10    10     0     0     0
#> 11 2014 Nov      313.     11    11     0     0     0
#> 12 2014 Dec      280.     12    12     0     0     0
```

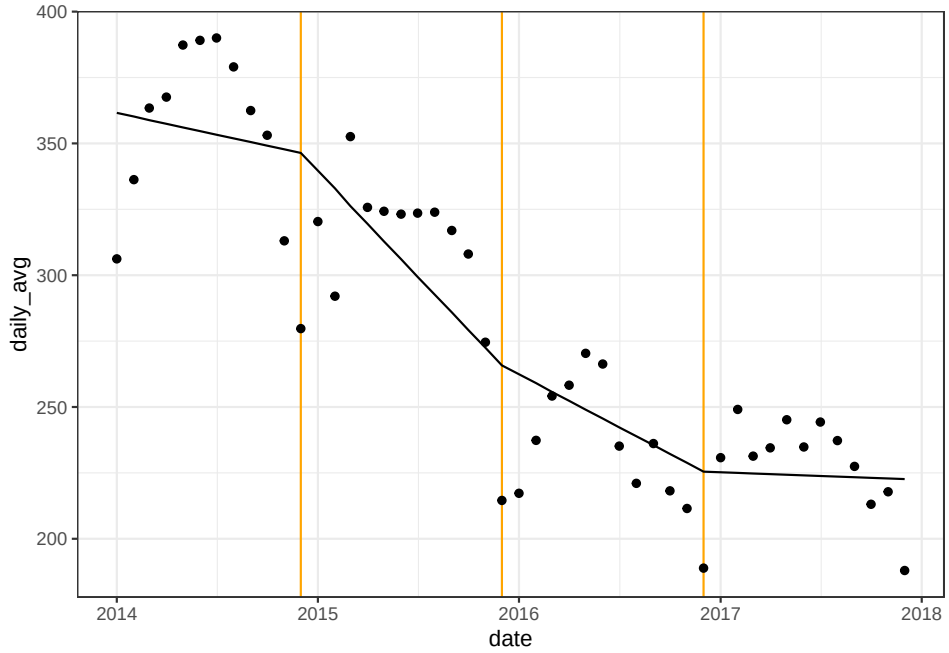
```
#> 13 2015 Jan      320.    13    13     1     0     0
#> 14 2015 Feb      292.    14    14     2     0     0
#> 15 2015 Mar      353.    15    15     3     0     0
#> 16 2015 Apr      326.    16    16     4     0     0
#> 17 2015 May      324.    17    17     5     0     0
#> 18 2015 Jun      323.    18    18     6     0     0
#> 19 2015 Jul      324.    19    19     7     0     0
#> 20 2015 Aug      324.    20    20     8     0     0
#> 21 2015 Sep      317.    21    21     9     0     0
#> 22 2015 Oct      308.    22    22    10     0     0
#> 23 2015 Nov      275.    23    23    11     0     0
#> 24 2015 Dec      215.    24    24    12     0     0
#> 25 2016 Jan      217.    25    25    13     1     0
#> # i 23 more rows
```

Fit a least-squares model using these special predictors:

```
fit_piecewise_linear = lm(daily_avg ~ x + x1 + x2 + x3, data = data_train)
summary(fit_piecewise_linear)
#>
#> Call:
#> lm(formula = daily_avg ~ x + x1 + x2 + x3, data = data_train)
#>
#> Residuals:
#>    Min     1Q Median     3Q    Max
#> -66.64 -17.46   5.73  20.72  36.71
#>
#> Coefficients:
#>             Estimate Std. Error t value Pr(>|t|)
#> (Intercept)   362.97      15.76   23.04   <2e-16 ***
#> x              -1.38       1.86   -0.74    0.460
#> x1             -5.33       2.94   -1.82    0.076 .
#> x2              3.36       2.52    1.33    0.191
#> x3              3.12       2.74    1.14    0.261
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> Residual standard error: 27.2 on 43 degrees of freedom
#> Multiple R-squared:  0.807,  Adjusted R-squared:  0.789
#> F-statistic:   45 on 4 and 43 DF,  p-value: 7.91e-15
```

Which gives the desired piecewise linear fit

```
arrest_ts %>%
  mutate(chgpt = index %in% !!chgpts) %>%
  ggplot(aes(date, daily_avg)) +
  geom_vline(data = . %>% filter(chgpt),
             aes(xintercept = date), color="orange") +
  geom_point() +
  geom_line(
    data = broom::augment(fit_piecewise_linear, arrest_ts),
    aes(y = .fitted)
    )
```




The Prophet model allows automatic detection of change points (and manual specification). The idea is to specify lots of potential change points and fit a lasso model to remove the unnecessary change points.

For this example, I'll specify a change point at each month and fit a lasso model to remove most of the change points.

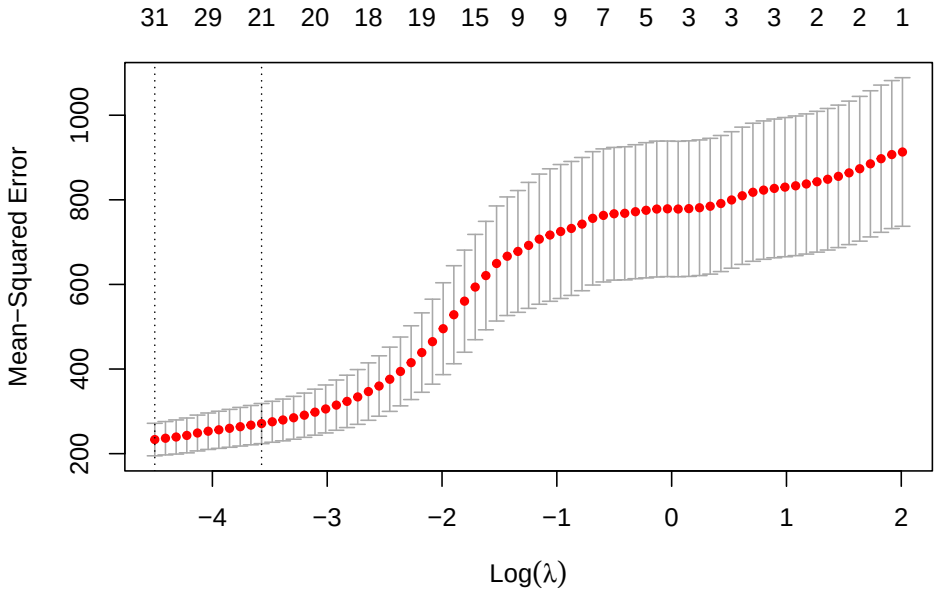
```
#: function to generate df of change point predictor variables
get_chgpt_pred <- function(x, chgpts){
  nm_chgpts = str_c("chgpt_", chgpts)
  map_df(chgpts %>% set_names(nm_chgpts),
      ~ ifelse(x < ., 0, x - .)  )
}

#: add the chgpt predictor variables, convert to matrix for glmnet
chgpts = 5:44  # don't consider changes near edges
X_chgpts = arrest_ts %>% as_tibble() %>%
  mutate(
    x = 1:n(),
    get_chgpt_pred(index, chgpts)
  ) %>%
  select(x, starts_with("chgpt")) %>%
  as.matrix()
Y = arrest_ts$daily_avg
```

Fitting the lasso model.

- It is not best practice to use straight cross-validation with time series data. I used it here to illustrate the general idea without getting into time series cv details.
- Here we have to be especially careful to avoid capturing seasonality (since there are not yet seasonal components added). So I'll manually set lambda to only keep around 3 changepoints.

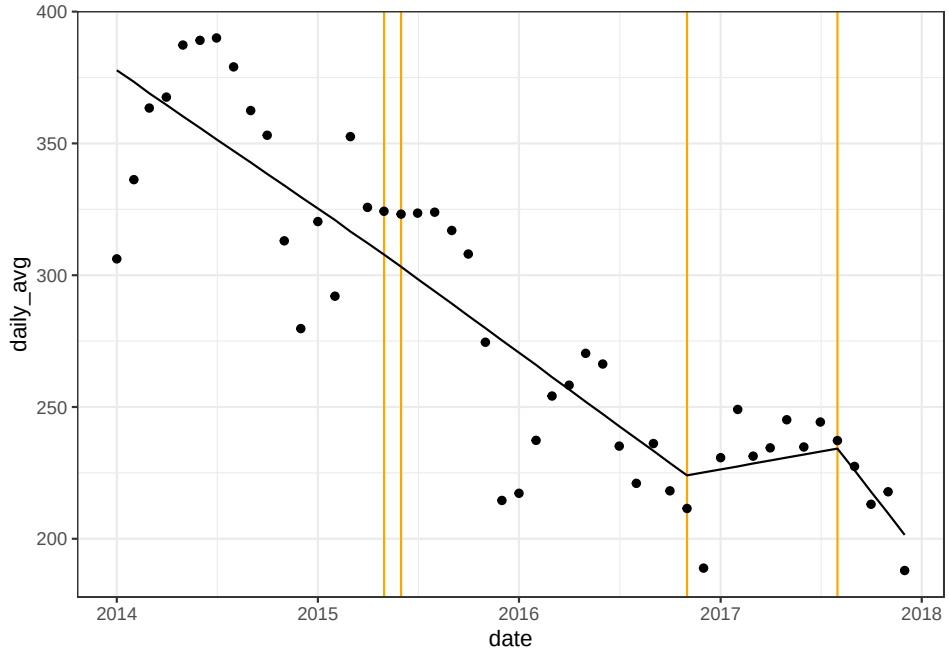
```
library(glmnet)
set.seed(2023)
fit_lasso = cv.glmnet(X_chgpts, Y, alpha = 1, penalty.factor = c(0, rep(1, length(chgpts) )))
plot(fit_lasso)
lam = 1
```

```
coef(fit_lasso, s = lam) %>% as.matrix %>% as_tibble(rownames = "var") %>%
  filter(s1 != 0 )
#> # A tibble: 6 x 2
#>   var                s1
#>   <chr>           <dbl>
#> 1 (Intercept) 382.
#> 2 x            -4.37
#> 3 chgpt_17     -0.247
#> 4 chgpt_18     -0.0395
#> 5 chgpt_35      5.79
#> 6 chgpt_44     -9.31
```




And the plot

```
chgpts_est =
  coef(fit_lasso, s = lam) %>%
  as.matrix %>% as_tibble(rownames = "var") %>%
  filter(s1 != 0, var != "(Intercept)") %>% pull(var) %>%
  str_remove("chgpt_") %>% as.integer

fitted = predict(fit_lasso, s = lam, newx = X_chgpts)

arrest_ts %>%
  mutate(chgpt = index %in% !!chgpts_est) %>%
  ggplot(aes(date, daily_avg)) +
  geom_vline(data = . %>% filter(chgpt),
             aes(xintercept = date), color="orange") +
  geom_point() +
  geom_line(y = fitted)
```




For the logistic growth model the trend (with piecewise change points) is:

$$T(t) = \frac{C(t)}{1 + \exp(\beta_0 + \sum_j x_j(t)\beta_j)}$$

where $C(t)$ is the *carrying capacity* of the system.

### 3.2.2 Seasonality

Patterns that repeat at a regular frequency are termed seasonal features.

The Prophet model uses a set of Fourier basis functions to capture seasonality. The Fourier basis functions are represented by a pair of $\sin$ and $\cos$ functions scaled by order $k$ and period $P$.

$$X_{kc}(t) = \cos\left(2\pi t\frac{k}{P}\right)$$
$$X_{ks}(t) = \sin\left(2\pi t\frac{k}{P}\right)$$

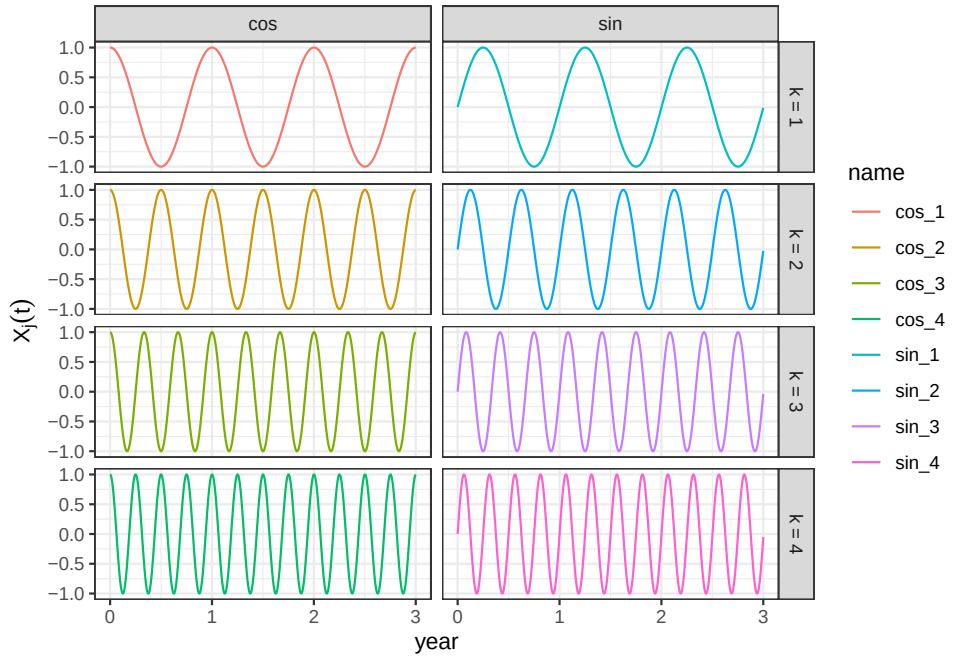
For daily observed data and yearly seasonality, the first few predictors are

$$X_{1c}(t) = \cos\left(2\pi t\frac{1}{365.25}\right)$$
$$X_{1s}(t) = \sin\left(2\pi t\frac{1}{365.25}\right)$$
$$X_{2c}(t) = \cos\left(2\pi t\frac{2}{365.25}\right)$$
$$X_{2s}(t) = \sin\left(2\pi t\frac{2}{365.25}\right)$$
$$X_{3c}(t) = \cos\left(2\pi t\frac{3}{365.25}\right)$$
$$X_{3s}(t) = \sin\left(2\pi t\frac{3}{365.25}\right)$$

Here is a visual:

```
#: function to manually calculate Fourier basis function
fourier_series <- function(x, k = 4, period = 12){
  fourier_k <- function(k){
    tibble(
      !!str_c("cos_",k) := cos(2*pi*x*k/period),
      !!str_c("sin_",k) := sin(2*pi*x*k/period)
    )
  }
  map(1:k, fourier_k) %>% bind_cols()
}
```

```
tibble(x = seq(0, 365*3, length=1000)) %>%
  mutate(fourier_series(x, k = 4, period = 365.25)) %>%
  pivot_longer(-x) %>%
  separate(name, into = c("trig", "k"), sep = "_", remove = FALSE) %>%
  arrange(k, trig) %>% mutate(k = str_c("k = ", k)) %>%
  ggplot(aes(x, value, color=name)) + geom_line() +
  facet_grid(k~trig) +
  scale_x_continuous(breaks = seq(0, 365*5, by=365), labels = 0:5) +
  labs(x = "year", y = expression(X[j](t)))
```




Let $X_P$ be the set of Fourier basis functions for period $P$. Prophet allows multiple seasonal components, e.g.,

$$\hat{S}(t) = X_{P_1}(t)\hat{\beta}_{P_1} + X_{P_2}(t)\hat{\beta}_{P_2}$$

where $P_1 = 365.25$ incorporates yearly seasonality and $P_2 = 7$ weekly. Prophet uses a ridge penalty on the Fourier coefficients.

As a concrete example, here are the the first few Fourier predictors for the Chicago Arrest data (using a period of $P = 12$ months per year).

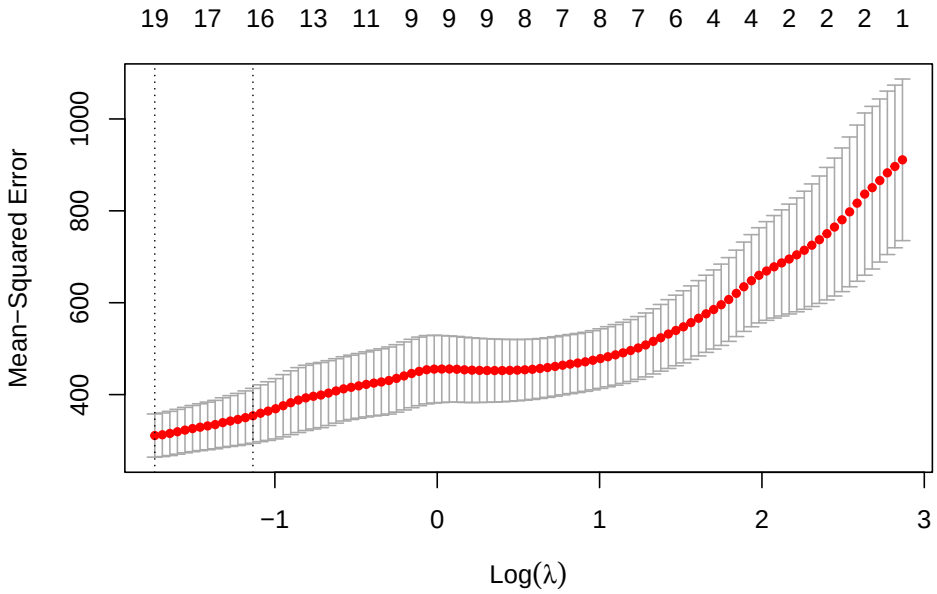
```
arrest_ts %>%
  summarize(fourier_series(index, k = 4, period = 12))
```

Let's merge these seasonal predictors with the changepoint trend model and use lasso to estimate all coefficients.

| time | cos_1 | sin_1 | cos_2 | sin_2 | cos_3 | sin_3 | cos_4 | sin_4 |
|---|---|---|---|---|---|---|---|---|
| 2014 Jan | 0.866 | 0.500 | 0.5 | 0.866 | 0 | 1 | -0.5 | 0.866 |
| 2014 Feb | 0.500 | 0.866 | -0.5 | 0.866 | -1 | 0 | -0.5 | -0.866 |
| 2014 Mar | 0.000 | 1.000 | -1.0 | 0.000 | 0 | -1 | 1.0 | 0.000 |
| 2014 Apr | -0.500 | 0.866 | -0.5 | -0.866 | 1 | 0 | -0.5 | 0.866 |
| 2014 May | -0.866 | 0.500 | 0.5 | -0.866 | 0 | 1 | -0.5 | -0.866 |
| 2014 Jun | -1.000 | 0.000 | 1.0 | 0.000 | -1 | 0 | 1.0 | 0.000 |
| 2014 Jul | -0.866 | -0.500 | 0.5 | 0.866 | 0 | -1 | -0.5 | 0.866 |
| 2014 Aug | -0.500 | -0.866 | -0.5 | 0.866 | 1 | 0 | -0.5 | -0.866 |
| 2014 Sep | 0.000 | -1.000 | -1.0 | 0.000 | 0 | 1 | 1.0 | 0.000 |
| 2014 Oct | 0.500 | -0.866 | -0.5 | -0.866 | -1 | 0 | -0.5 | 0.866 |
| 2014 Nov | 0.866 | -0.500 | 0.5 | -0.866 | 0 | -1 | -0.5 | -0.866 |
| 2014 Dec | 1.000 | 0.000 | 1.0 | 0.000 | 1 | 0 | 1.0 | 0.000 |

```
X_chgpts_season = cbind(
  X_chgpts,
  X_season = arrest_ts %>%
    summarize(fourier_series(index, k = 4, period = 12)) %>%
    as_tibble() %>% select(-time) %>% as.matrix()
)
```

```
library(glmnet)
set.seed(2023)
pen_fac = rep(1, ncol(X_chgpts_season)); pen_fac[1] = 0
fit_lasso2 = cv.glmnet(x=X_chgpts_season, y=Y, alpha = 1, penalty.factor = pen_fac)
plot(fit_lasso2)
```

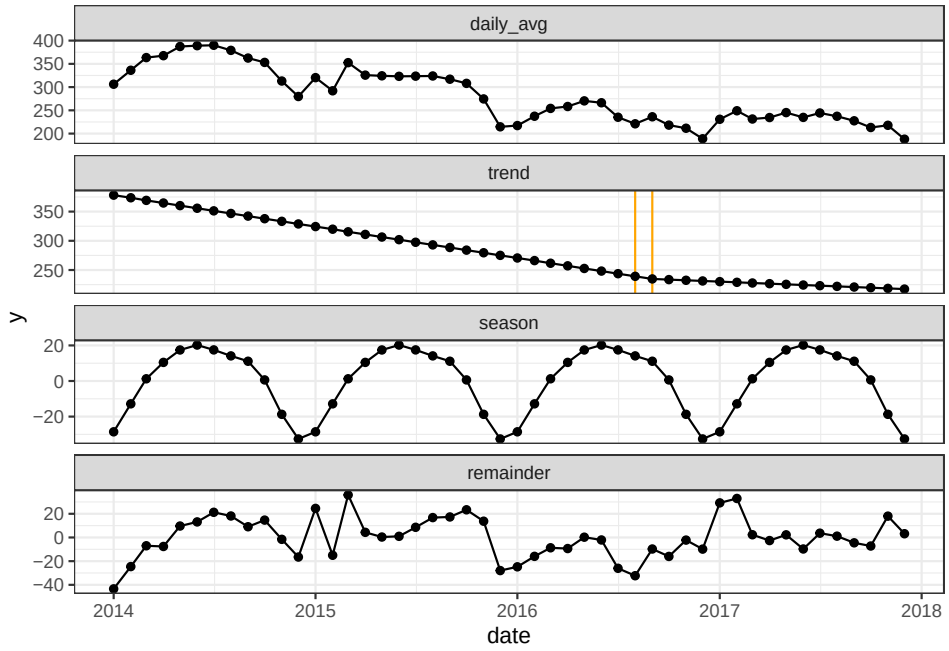



```
# Again, selecting lambda to minimize number of change points
lam = exp(.5)
coef(fit_lasso2, s = lam) %>% as.matrix %>% as_tibble(rownames = "var") %>%
  filter(s1 != 0 )
#> # A tibble: 9 x 2
#>   var                 s1
```

```
#>   <chr>            <dbl>
#> 1 (Intercept) 383.
#> 2 x             -4.48
#> 3 chgpt_32       0.00103
#> 4 chgpt_33       3.32
#> 5 cos_1        -23.7
#> 6 sin_1         -4.94
#> # i 3 more rows
```

And the plot

```
#: estimated change points
chgpts_est =
  coef(fit_lasso2, s = lam) %>%
  as.matrix %>% as_tibble(rownames = "var") %>%
  filter(s1 != 0, var != "(Intercept)") %>% pull(var) %>%
  str_remove("chgpt_") %>% as.integer %>% na.omit()

#: get seasonal and trend components
beta = coef(fit_lasso2, s = lam) %>% as.matrix %>% as_tibble(rownames = "var")

beta_season = beta %>%
  mutate(s1 = ifelse(str_detect(var, "sin|cos"), s1, 0))

beta_trend = beta %>%
  mutate(s1 = ifelse(str_detect(var, "sin|cos"), 0, s1))

components = tibble(
  season = cbind(1, X_chgpts_season) %*% beta_season$s1 %>% as.numeric,
  trend = cbind(1, X_chgpts_season) %*% beta_trend$s1 %>% as.numeric,
  fitted = predict(fit_lasso2, s = lam, newx = X_chgpts_season)[,1]
)

#: plot
arrest_ts %>%
  bind_cols(components) %>%
  mutate(remainder = daily_avg - fitted) %>%
  pivot_longer(
    c(season, trend, remainder, daily_avg), names_to = "component", values_to = "y"
  ) %>%
  mutate(chgpt = component == "trend" & index %in% !!chgpts_est) %>%
  ggplot(aes(date, y)) +
  geom_vline(data = . %>% filter(chgpt),
             aes(xintercept = date), color="orange") +
  geom_point() + geom_line() +
  facet_wrap(~factor(component, c("daily_avg", "trend", "season", "remainder")),
             ncol=1, scales = "free_y")
```




### 3.2.3 Holidays and Special Days

Special days, like holidays, often have observations that don't match the trend and seasonal patterns. In essence, these days produce outliers. But if we have a list of the special days, then we can easily estimate their effects.

Prophet let's the user specify special days with a table, e.g.,

| Holiday | Country | Date |
|---|---|---|
| Thanksgiving | US | 26 Nov 2015 |
| Thanksgiving | US | 24 Nov 2016 |
| Thanksgiving | US | 23 Nov 2017 |
| Thanksgiving | US | 22 Nov 2018 |
| Christmas | * | 25 Dec 2015 |
| Christmas | * | 25 Dec 2016 |
| Christmas | * | 25 Dec 2017 |
| Christmas | * | 25 Dec 2018 |

and a ridge penalized parameter is estimated for each unique holiday. Mathematically, predictor variables are simple indicators $Z(t) = [\mathbb{1}(t \in H_1), \mathbb{1}(t \in H_2), \ldots, \mathbb{1}(t \in H_p)]$ where $H_j$ is the $j$th holiday.

$$\hat{H}(t) = Z(t)\hat{\beta}_Z$$

Some other thoughts:

1. The effects of a holiday may span several time periods. For example, the weekend following Thanksgiving or before Christmas may also have unusual observations. Instead of an indicator function, a function that spans several time periods and is a pre-specified shape can capture such patterns.

2. A technique called intervention analysis is used to model changes or shocks to a time series. Think of policy changes or major events. In the Chicago Arrest data, a new police superintendent started in April 2016 following protests (close to change point), the 2014 shooting of Michael Brown in Ferguson, MO inspired protests across the county, etc.

### 3.2.4 Peyton Manning Wikipedia

The Prophet Quickstart Guide illustrates usage on modeling the *log daily page views for the Wikipedia page of Peyton Manning*.

```
library(tidyverse)
url = 'https://raw.githubusercontent.com/facebook/prophet/main/examples/example_wp_log_peyton_manning
manning = read_csv(url)
head(manning)
```

```
#> # A tibble: 6 x 2
#>   ds              y
#>   <date>      <dbl>
#> 1 2007-12-10  9.59
#> 2 2007-12-11  8.52
#> 3 2007-12-12  8.18
#> 4 2007-12-13  8.07
#> 5 2007-12-14  7.89
#> 6 2007-12-15  7.78
```

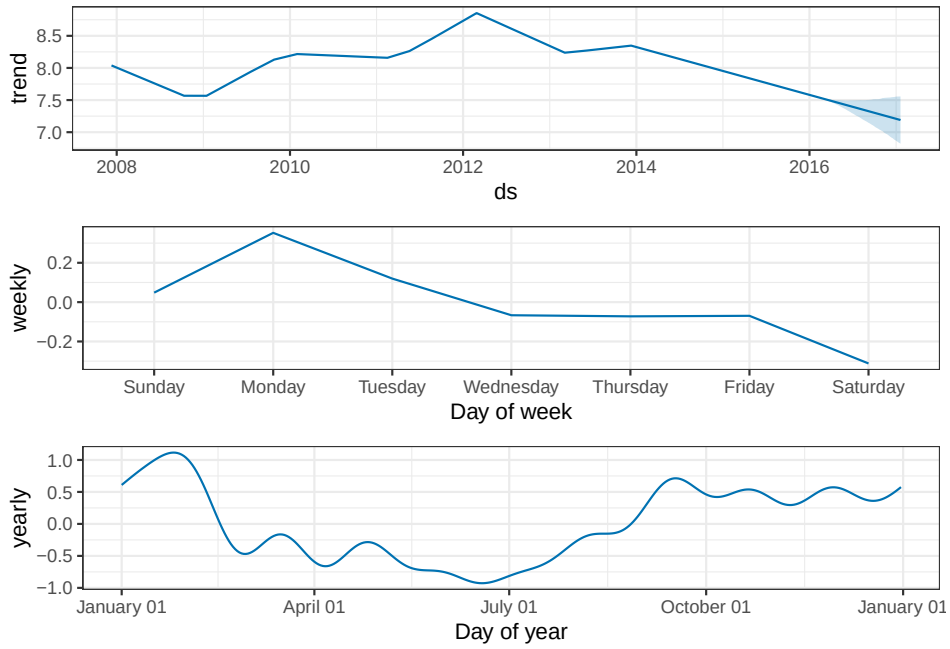
Default fitting

```
library(prophet)
fit = prophet::prophet(manning)          # fit the default prophet model
```

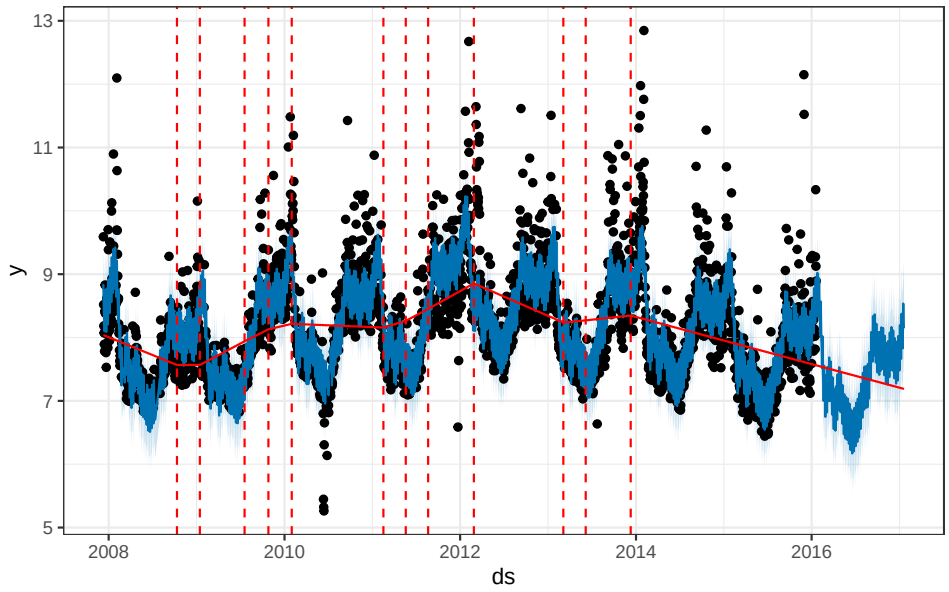
To forecast, we need to extend the data to future periods (here 365 days)

```
manning_future = make_future_dataframe(fit, periods = 365)
forecast = predict(fit, manning_future)
```

```
prophet_plot_components(fit, forecast)
```



```
plot(fit, forecast) + add_changepoints_to_plot(fit)
```




**3.2.4.1 Holidays** Holidays and other special days can be added. Here we add the dates associated with NFL playoffs and Superbowl. And we will add a special seasonality term that allows a different day-of-week effect when the NFL is in season.

```
#: playoffs
playoffs = tibble(
  holiday = 'playoff',
  ds = as.Date(c('2008-01-13', '2009-01-03', '2010-01-16',
                 '2010-01-24', '2010-02-07', '2011-01-08',
                 '2013-01-12', '2014-01-12', '2014-01-19',
                 '2014-02-02', '2015-01-11', '2016-01-17',
                 '2016-01-24', '2016-02-07')),
  lower_window = 0,
  upper_window = 1
)
#: superbowl
superbowls = tibble(
  holiday = 'superbowl',
  ds = as.Date(c('2010-02-07', '2014-02-02', '2016-02-07')),
  lower_window = 0,
  upper_window = 1
)
#: holidays
holidays = bind_rows(playoffs, superbowls)
```

```
#: NFL Season
is_nfl_season <- function(ds) {
  dates <- as.Date(ds)
  month <- as.numeric(format(dates, '%m'))
  return(month > 8 | month < 2)
}
manning$in_season = is_nfl_season(manning$ds)
```

```
#: specify the seasonality details, add holidays, add NFL season
fit_2 = prophet::prophet(fit = FALSE,
  yearly.seasonality = 10,
```

```
  weekly.seasonality = 3,
  holidays = holidays) %>%
  add_country_holidays("US") %>%
  add_seasonality("nfl_season", period = 7, fourier.order = 3, condition.name = "in_season") %>%
  fit.prophet(manning)
```

```
manning_future = make_future_dataframe(fit_2, periods = 365) %>%
  mutate(in_season = is_nfl_season(ds)) # need to add specials to future data
forecast = predict(fit_2, manning_future)
```

```
prophet_plot_components(fit_2, forecast)
```