On categorification of g-vectors

joint work in progress
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g-vectors

Cluster algebras: in the cluster algebra $\mathscr{A}$ attached to a quiver Q, each cluster variable has a *g-vector* in $\mathbb{Z}^{Q_0}$.

This vector was originally defined (Fomin–Zelevinsky) in terms of a grading of the principal coefficient cluster algebra $\mathscr{A}^{\text{prin}}$.

$$
\begin{tikzcd}[column sep=small]
\begin{pmatrix}1\\0\end{pmatrix} \arrow[dr] \arrow[rr, dash, dashed] & & \begin{pmatrix}-1\\0\end{pmatrix} \arrow[dr] \arrow[rr, dash, dashed] & & \begin{pmatrix}0\\1\end{pmatrix} \arrow[dr] \arrow[rr, dash, dashed] & & \begin{pmatrix}0\\-1\end{pmatrix} \\
& \begin{pmatrix}0\\-1\end{pmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{pmatrix}-1\\1\end{pmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{pmatrix}1\\0\end{pmatrix} \arrow[ur] &
\end{tikzcd}
$$

These vectors can be computed categorically in two ways.

Option 1: Coindex

Let $\mathcal{C}$ be a Hom-finite Krull–Schmidt 2-Calabi–Yau triangulated category.

Let $\mathcal{T} \subseteq \mathcal{C}$ be a cluster-tilting subcategory, meaning $\mathcal{T}$ is functorially finite and

$$\mathcal{T} = \{X \in \mathcal{C} : \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T}, X) := \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, \Sigma X) = 0\}.$$

Then for all $X \in \mathcal{C}$ there exists a triangle $X \to T_1 \to T_0 \to \Sigma X$ with $T_0, T_1 \in \mathcal{T}$.

Define $\mathrm{coind}_{\mathcal{T}}(X) = [T_1] - [T_0] \in \mathrm{K}_0(\mathcal{T})$.

Theorem (Dehy–Keller, Fu–Keller)

$\mathrm{coind}_{\mathcal{T}}(X)$ *is the g-vector of the cluster variable* $\mathrm{CC}^{\mathcal{T}}(X)$ *(when this makes sense).*

$$\begin{tikzcd}
\begin{pmatrix}1\\0\end{pmatrix} \arrow[rr, dash, dashed] \arrow[dr] & & \begin{pmatrix}-1\\0\end{pmatrix} \arrow[rr, dash, dashed] \arrow[dr] & & \begin{pmatrix}0\\1\end{pmatrix} \arrow[rr, dash, dashed] \arrow[dr] & & \begin{pmatrix}0\\-1\end{pmatrix} \\
& \begin{pmatrix}0\\-1\end{pmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{pmatrix}-1\\1\end{pmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{pmatrix}1\\0\end{pmatrix} \arrow[ur] &
\end{tikzcd}$$

Option 2: Projective presentations

Let A be a finite-dimensional algebra and $M \in \operatorname{mod} A$. Take a minimal projective presentation $P_1 \to P_0 \to M \to 0$.

Then the *g-vector* of M is $[P_1] - [P_0] \in \mathrm{K}_0(\operatorname{proj} A)$.

Say A is 2-*CY-tilted* if it is isomorphic to $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ for $\mathcal{C}$ as above and add T cluster-tilting.

Theorem

When A is 2-CY-tilted and $M \in \operatorname{mod} A$, the g-vector of M is that of the corresponding (non-initial) cluster variable (when this makes sense).

$$
\begin{tikzcd}
& \begin{pmatrix} -1 \\ 0 \end{pmatrix} \arrow[dr] & \\
\begin{pmatrix} 0 \\ -1 \end{pmatrix} \arrow[ur] \arrow[rr, dashed, no head] & & \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\end{tikzcd}
$$

Connection

Take $X \in \mathcal{C}$, and choose a triangle $X \to T_1 \to T_0 \to \Sigma X$ to compute the coindex.

This yields an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, T_1) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, T_0) \to \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T}, X) \to 0$$

of finitely-presented functors on $\mathcal{T}$.

There are equivalences

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, -) \colon \mathcal{T} &\xrightarrow{\sim} \mathrm{proj}\,\mathcal{T}, && \text{Yoneda} \\ \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T}, -) \colon \mathcal{C}/(\mathcal{T}) &\xrightarrow{\sim} \mathrm{mod}\,\mathcal{T}. && \text{Buan–Marsh–Reiten, Keller–Reiten, Koenig–Zhu,...} \end{aligned}$$

If $\mathcal{T} = \mathrm{add}\,T$ then $\mathrm{proj}\,\mathcal{T} = \mathrm{proj}\,A$ and $\mathrm{mod}\,\mathcal{T} = \mathrm{mod}\,A$ for $A = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.

Thus the g-vector of $X \in \mathcal{C}$ is equal to the g-vector of $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T}, X) \in \mathrm{mod}\,A$.

Aim

Enhance this relationship to an equivalence of 'categories of g-vectors'.

Extriangulated categories (Nakaoka–Palu)

Idea: additive categories with well-behaved ‘extension groups’ $\mathbb{E}(X, Y)$.

Example

(0) Exact categories, triangulated categories ($\mathbb{E} = \mathrm{Ext}^1_{\mathcal{C}}$).

(1) Extension closed subcategories of triangulated categories ($\mathbb{E} = \mathrm{Ext}^1_{\mathcal{C}}$).

(2) ‘Partial stabilisations’ $\mathcal{C}/(P)$ for $\mathcal{C}$ Frobenius exact, P projective-injective ($\mathbb{E} = \mathrm{Ext}^1_{\mathcal{C}}$).

(3) Ex-triangulated categories: Take a triangulated category $\mathcal{C}$ and choose (carefully) a subfunctor $\mathbb{E} \leqslant \mathrm{Ext}^1_{\mathcal{C}}$.

Carefully = making sure inflations and deflations are closed under composition. (Herschend–Liu–Nakaoka)

Remark

(3) was studied for exact categories by Auslander–Solberg, under the heading of relative homological algebra: the process preserves exactness (but not triangulatedness).

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(0) Exact categories, triangulated categories ($\mathbb{E} = \mathrm{Ext}^1_{\mathcal{C}}$).

(1) Extension closed subcategories of extriangulated categories.

(2) 'Partial stabilisations' $\mathcal{C}/(P)$ for $\mathcal{C}$ Frobenius extriangulated, P projective-injective.

(3) Take an extriangulated category $\mathcal{C}$ and choose (carefully) a subfunctor of $\mathbb{E}$.

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Harp (The **H**omotopy **AR**row category of **P**rojectives)

Let A be a finite dimensional algebra. Then

$$\operatorname{harp} A := \{P_1 \xrightarrow{\varphi} P_0 : P_i \in \operatorname{proj} A\}/_{\text{homotopy}}.$$

We have $\operatorname{harp} A \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\operatorname{proj} A) \hookrightarrow \mathcal{K}^{\mathrm{b}}(\operatorname{proj} A)$.

The image is extension-closed, and so $\operatorname{harp} A$ is naturally extriangulated.

Projective objects are those of the form $0 \to P$, and injectives of the form $P \to 0$. (Objects $P \xrightarrow{\sim} P$ are projective-injective, but also 0.)

$$\begin{tikzcd}
 & \lvert(0 \to P_1) \arrow[dr] \arrow[rr, dashed, no head] & & (P_2 \to 0)\rvert \arrow[dr] & \\
\lvert(0 \to P_2) \arrow[ur] \arrow[rr, dashed, no head] & & (P_2 \to P_1) \arrow[ur] \arrow[rr, dashed, no head] & & (P_1 \to 0)\rvert
\end{tikzcd}$$

Relative harp

Choose additionally $e = e^2 \in A$, and define

$$\operatorname{harp}_e A := \{P_1 \xrightarrow{\varphi} P_0 \in \operatorname{harp} A : e \cdot \operatorname{coker} \varphi = 0\}.$$

Note that $\operatorname{harp}_0 A = \operatorname{harp} A$.

Proposition (FGPPP)

In $\operatorname{harp}_e(A)$, *injectives are* $P \to 0$, *while projectives are* $P \xrightarrow{\varphi} Q$ *such that* $P \in \operatorname{add} Ae$. *In particular,* $Ae \to 0$ *is projective-injective.*

$$A = \begin{tikzcd} \boxed{1} \arrow[rr, "q"] & & \boxed{2} \arrow[dl, "p"] \\ & * \arrow[ul, "r"] & \end{tikzcd} /(pq, qr)$$

For $e = e_1 + e_2$, the category $\operatorname{harp}_e A$ is

$$\begin{tikzcd}[column sep=small] & |(P_1 \to 0)| \arrow[dr] & & |(P_2 \to 0)| \arrow[dr] & & |(P_1 \to 0)| \arrow[dr] & \\ {} \arrow[rr, dash, dashed] & & (P_* \to 0)| \arrow[ur] & & |(P_2 \to P_*)| \arrow[ur] \arrow[rr, dash, dashed] & & (P_* \to 0)| \end{tikzcd}$$

Main Theorem

Two situations:

(1) $\mathcal{C}$ is the Amiot cluster category of a Jacobi-finite quiver with potential, with initial cluster-tilting subcategory add T.

(2) $\mathcal{C}$ is a Krull–Schmidt stably 2-Calabi–Yau Frobenius exact category, with cluster-tilting subcategory add T.

Write $A = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ with e corresponding to projective summands (so $e = 0$ in case (1)).

Theorem (FGPPP)

In situations (1) and (2), there is a full and dense functor $G \colon \mathcal{C} \to \operatorname{harp}_e A$ given by

$$GX = \big(\operatorname{Hom}_{\mathcal{C}}(T, T_1) \to \operatorname{Hom}_{\mathcal{C}}(T, T_0)\big)$$

for $X \to T_1 \to T_0$ with $T_i \in \operatorname{add} T$ either a carefully chosen triangle (1) or arbitrary short exact sequence (2). We have

$$\ker G = \begin{cases} (T \to \Sigma^{-1} T), & (1) \\ 0. & (2) \end{cases}$$

Preservation of structure

Give $\mathcal{C}$ the relative extriangulated structure $\mathbb{E}_T$ with extriangles $X \to Y \to Z$ such that

$$\operatorname{coind}_T(Y) = \operatorname{coind}_T(X) + \operatorname{coind}_T(Z).$$

Proposition (Padrol–Palu–Pilaud–Plamondon)

The injectives and projectives in $(\mathcal{C}, \mathbb{E}_T)$ are given respectively by (the preimage under stabilisation of) $\operatorname{add} T$ and $\operatorname{add} \Sigma^{-1} T$ respectively.

Proposition (FGPPP)

If $\mathcal{C}$ is extriangulated and $\mathcal{I} \subseteq (\mathrm{inj} \to \mathrm{proj})$ is an ideal, then $\mathcal{C}/\mathcal{I}$ is naturally extriangulated.

Theorem (FGPPP)

Using the extriangulated structure induced from $\mathbb{E}_T$ on $\mathcal{C}/\ker G$, we obtain an equivalence

$$\mathcal{C}/\ker G \xrightarrow{\sim} \operatorname{harp}_e A$$

of extriangulated categories.

Corollary

In case (1), if A is selfinjective then $(\mathcal{C}, \mathbb{E}_T) \simeq \operatorname{harp} A$.

Proof.

We have $\Sigma^2 T = T$ because A is selfinjective (Koenig–Zhu, Iyama–Oppermann) so

$$\mathrm{Hom}_{\mathcal{C}}(T, \Sigma^{-1} T) = \mathrm{Hom}_{\mathcal{C}}(T, \Sigma T) = 0$$

because T is rigid. □

Corollary

In case (2), $\operatorname{harp}_e A$ *is exact.*

Proof.

Since $\mathcal{C}$ is exact, so is $(\mathcal{C}, \mathbb{E}_T) \simeq \operatorname{harp}_e(A)$ (Auslander–Solberg). □

Example 1 (A_2 cluster category)

$$
\begin{tikzcd}[column sep=small]
\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dashed, no head] & & \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dashed, no head] & & \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \arrow[dr] \arrow[rr, dashed, no head] & & \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right) \\
& \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right) \arrow[ur] \arrow[rr, dashed, no head] & & \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \arrow[ur] \arrow[rr, dashed, no head] & & \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
T_1 \arrow[dr] \arrow[rr, dashed, no head] & & Y \arrow[dr] \arrow[rr, dashed, no head] & & T_2 \arrow[dr] \arrow[rr, dashed, no head] & & X \\
& X \arrow[ur] \arrow[rr, dashed, no head] & & Z \arrow[ur] \arrow[rr, dashed, no head] & & T_1 \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& |(0 \to P_1) \arrow[dr] \arrow[rr, dashed, no head] & & (P_2 \to 0)| \arrow[dr] & \\
|(0 \to P_2) \arrow[ur] \arrow[rr, dashed, no head] & & (P_2 \to P_1) \arrow[ur] \arrow[rr, dashed, no head] & & (P_1 \to 0)|
\end{tikzcd}
$$

Example 2 (A_2 preprojective algebra)

$$
\begin{tikzcd}[column sep=small]
\begin{pmatrix}0\\1\\0\end{pmatrix} \arrow[dr] & & \begin{pmatrix}0\\0\\1\end{pmatrix} \arrow[dr] & & \begin{pmatrix}0\\1\\0\end{pmatrix} \arrow[dr] & \\
& \begin{pmatrix}1\\0\\0\end{pmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{pmatrix}-1\\1\\0\end{pmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{pmatrix}1\\0\\0\end{pmatrix}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
{|T_1|} \arrow[dr] & & {|T_2|} \arrow[dr] & & {|T_1|} \arrow[dr] & \\
& T_* \arrow[ur] \arrow[rr, dash, dashed] & & X \arrow[ur] \arrow[rr, dash, dashed] & & T_*
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
{|(P_1 \to 0)|} \arrow[dr] & & {|(P_2 \to 0)|} \arrow[dr] & & {|(P_1 \to 0)|} \arrow[dr] & \\
& {(P_* \to 0)|} \arrow[ur] & & {|(P_1 \to P_*)} \arrow[ur] \arrow[rr, dash, dashed] & & {(P_* \to 0)|}
\end{tikzcd}
$$