Homotopy arrow categories for finite-dimensional algebras

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Extriangulated categories (Nakaoka–Palu '19)

Formalism of additive categories with "extensions".

Declared class of *conflations*:

$$X \rightarrowtail Y \twoheadrightarrow Z$$

in which the maps compose to 0.

$$\rightarrowtail = \textit{inflation} \qquad\qquad \twoheadrightarrow = \textit{deflation}$$

Conflations are parametrised by abelian groups $\mathbb{E}(Z, X)$, functorial in both arguments:

$$\delta \in \mathbb{E}(Z, X) \rightsquigarrow \text{ conflation } X \rightarrowtail Y \twoheadrightarrow Z \overset{\delta}{\dashrightarrow}$$

Warning: Inflation $\not\Rightarrow$ monic, and deflation $\not\Rightarrow$ epic.

Examples / etymology

(0) Exact and triangulated categories.

	Exact	Triangulated
Conflations	Exact sequences	Distinguished triangles[1]
Inflations	Admissible monos	All morphisms
Deflations	Admissible epis	All morphisms
$\mathbb{E}$	Ext^1	$\mathrm{Hom}(-, \Sigma -)$

(1) Extension-closed subcategories of triangulated categories: $\mathcal{C} \subseteq \mathcal{T}$ such that if

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{\delta} \Sigma X$$

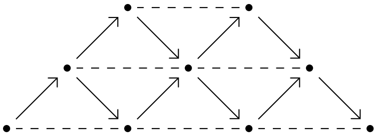
is a triangle with $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.

This gives the conflation $X \rightarrowtail Y \twoheadrightarrow Z \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$, $\mathbb{E}(-,-) = \mathrm{Hom}_{\mathcal{T}}(-, \Sigma -)|_{\mathcal{C} \times \mathcal{C}}$.

[1]Truncated, i.e. forgetting the third morphism.

Examples / etymology

(1) Special case: $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \{P_1 \to P_0\} \subseteq \mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$



(2) Ex-triangulated categories: start with a triangulated category $\mathcal{T}$, and throw out some distinguished triangles.

This has to be done carefully (Herschend–Liu–Nakaoka '21):

- the remaining triangles should be parametrised by a subfunctor $\mathbb{E} \leqslant \operatorname{Hom}_{\mathcal{T}}(-, \Sigma-)$, and
- inflations should remain closed under composition (equivalently, deflations remain closed under composition).

Remark

The constructions in (1) and (2) apply to arbitrary extriangulated categories.

The class of exact categories is closed under these operations (Auslander–Solberg '93), but the class of triangulated categories is not.

Homological algebra

For $\mathcal{C}$ extriangulated, say $P \in \mathcal{C}$ is *projective* if $\mathbb{E}(P, -) \equiv 0$, or equivalently:

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted, "\exists"'] \\ X \arrow[r, two heads, "\forall"] & Y \end{tikzcd}$$

Say $\mathcal{C}$ has *enough projectives* if for all $X \in \mathcal{C}$, there exists projective $P_X \twoheadrightarrow X$.

In this case we can define projective dimension in the usual way.

Similarly, we can define injectives, global dimension, dominant dimension, etc.

Reminder: for a minimal injective resolution $X \rightarrowtail I_0 \to I_1 \to \cdots$,

$$\operatorname{dom.dim} X = \min\{j : I_j \text{ is not projective}\}$$

Warning: If $\mathcal{T}$ is triangulated, only 0 is projective or injective, but this is enough!

$$X \rightarrowtail 0 \twoheadrightarrow \Sigma X$$

0-Auslander categories

Definition (Gorsky–Nakaoka–Palu '23$^+$)

An extriangulated category $\mathcal{C}$ is 0-Auslander if

- it has enough projectives,
- it is hereditary, i.e. $\operatorname{p.dim} X \leqslant 1$ for all $X \in \mathcal{C}$, and
- $\operatorname{dom.dim} P \geqslant 1$ for all projectives $P \in \mathcal{C}$.

Proposition (Gorsky–Nakaoka–Palu '23$^+$)

This definition is self dual, i.e. $\mathcal{C}$ is 0-Auslander if and only if

- *it has enough injectives,*
- $\operatorname{inj.dim} X \leqslant 1$ *for all* $X \in \mathcal{C}$, *and*
- $\operatorname{codom.dim} I \geqslant 1$ *for all injectives* $I \in \mathcal{C}$.

Example

For Λ a finite-dimensional algebra, $\mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$ is 0-Auslander.

The homotopy arrow category

Example

For Λ a finite-dimensional algebra, $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is 0-Auslander.

- Projectives are $P \in \operatorname{proj}\Lambda$.
- Injectives are ΣP, $P \in \operatorname{proj}\Lambda$.
- For all $(P_1 \to P_0) \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, there is a conflation

  $$P_1 \rightarrowtail P_0 \twoheadrightarrow (P_1 \to P_0),$$

  hence enough projectives, $\operatorname{p.dim}(P_1 \to P_0) \leqslant 1$.
- For all $P \in \operatorname{proj}\Lambda$, there is a conflation

  $$P \rightarrowtail 0 \twoheadrightarrow \Sigma P$$

  in which ΣP is injective, and 0 is projective-injective, hence $\operatorname{dom.dim} P \geqslant 1$.

0-Auslander categories

Example

For Λ a finite-dimensional algebra, $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is 0-Auslander.

Question: How many more examples are there?

Note: We have $\operatorname{Hom}_\Lambda(\Sigma Q, P) = 0$ for all $P, Q \in \operatorname{proj}\Lambda$.

That is, in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, there are no non-zero morphisms from injective to projective objects.

Proposition (FGPPP)

$\mathcal{C}$ extriangulated $\implies$ $\mathcal{C}/(\mathit{inj} \to \mathit{proj})$ extriangulated (for the induced $\mathbb{E}$).

Result 1: Cluster categories

An extriangulated category $\mathcal{C}$ is *Frobenius* if it has enough projective and injective objects, and these coincide.

Example

A triangulated category is a Frobenius extriangulated category.

$$X \rightarrowtail 0 \twoheadrightarrow \Sigma X$$

Theorem (Nakaoka–Palu '19)

If $\mathcal{C}$ is a Frobenius extriangulated category, then its stable category $\underline{\mathcal{C}} = \mathcal{C}/(\text{proj-inj})$ is triangulated, and

$$\mathbb{E}_{\mathcal{C}}(X, Y) = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y).$$

Let $\mathcal{C}$ be a Frobenius extriangulated category that is

- weakly idempotent complete, and
- stably 2-Calabi–Yau.

Result 1: Cluster categories

Let $\mathcal{T} \subseteq \mathcal{C}$ be a (2-)cluster-tilting subcategory.

Then $\mathbb{E}_{\mathcal{C}}(X, Y) = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y)$, and so we may define

$$\mathbb{E}_{\mathcal{T}}(X, Y) = (\Sigma\mathcal{T})(X, \Sigma Y) \leqslant \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y).$$

Proposition (Herschend–Liu–Nakaoka '21)

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$ *is an extriangulated category.*

It is 0-Auslander:

- the projectives are $T \in \mathcal{T}$,
- the injectives are U such that $\pi U \in \Sigma\mathcal{T}$ (for $\pi \colon \mathcal{C} \to \underline{\mathcal{C}}$),
- for all $X \in \mathcal{C}$ there exist $T_1, T_0 \in \mathcal{T}$ and $T_1 \rightarrowtail T_0 \twoheadrightarrow X$, (cf. *index*, Palu '08)
- for all $T \in \mathcal{T}$, there is $T \rightarrowtail \Pi_T \twoheadrightarrow \mathbf{\Sigma} T$ with Π_T projective-injective in $\mathcal{C}$, $\pi\mathbf{\Sigma} T = \Sigma T$.

Result 1: Cluster categories

$$\mathbb{E}_{\mathcal{T}}(X, Y) = (\Sigma\mathcal{T})(X, \Sigma Y) \leqslant \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y)$$

Theorem (FGPPP)

Assume $\mathcal{T} = \operatorname{add} T$, and that $\mathcal{C}$ is either

(1) *exact, or*

(2) *a Higgs category (Yilin Wu '23).*

Then there is an equivalence of extriangulated categories

$$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})/(\mathit{inj} \to \mathit{proj}) \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$$

where $\Lambda = \underline{\mathrm{End}}_{\mathcal{C}}(T)^{\mathrm{op}}$.

Remark

Generalised cluster categories (Amiot '09) are precisely the Higgs categories which are triangulated.

Example 1 (A_2 preprojective algebra)

$\mathcal{C} = \operatorname{mod} \Pi$ for Π the preprojective algebra of type A_2:

$$
\begin{tikzcd}
{|T_2|} \arrow[dr] & & {|T_3|} \arrow[dr] & & {|T_2|} \arrow[dr] & \\
& \boldsymbol{\Sigma} T_1 \arrow[ur] \arrow[rr, dash, dashed] & & T_1 \arrow[ur] \arrow[rr, dash, dashed] & & \boldsymbol{\Sigma} T_1
\end{tikzcd}
$$

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$:

$$
\begin{tikzcd}
{|T_2|} \arrow[dr] & & {|T_3|} \arrow[dr] & & {|T_2|} \arrow[dr] & \\
& \boldsymbol{\Sigma} T_1| \arrow[ur] & & |T_1 \arrow[ur] \arrow[rr, dash, dashed] & & \boldsymbol{\Sigma} T_1|
\end{tikzcd}
$$

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})/(\text{inj} \to \text{proj})$:

$$
\begin{tikzcd}
|T_1 \arrow[r, dash, dashed] & \boldsymbol{\Sigma} T_1|
\end{tikzcd}
$$

Example 2 (A_3 cluster category)

$\mathcal{C} = \mathcal{C}(A_3)$:

$$
\begin{tikzcd}[column sep=small]
\Sigma T_3 \arrow[dr] \arrow[rr, dashed, no head] & & T_3 \arrow[dr] \arrow[rr, dashed, no head] & & \Sigma T_1 \arrow[dr] \arrow[rr, dashed, no head] & & T_1 \arrow[dr] \arrow[r, dashed, no head] & {} \\
{} \arrow[r, dashed, no head] & T_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \Sigma T_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & T_2 \\
T_1 \arrow[ur] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[rr, dashed, no head] & & \Sigma T_3 \arrow[ur] \arrow[r, dashed, no head] & {}
\end{tikzcd}
$$

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$:

$$
\begin{tikzcd}[column sep=small]
\Sigma T_3| \arrow[dr, red] & & |T_3 \arrow[dr] \arrow[rr, dashed, no head] & & \Sigma T_1| \arrow[dr] & & |T_1 \arrow[dr] \arrow[r, dashed, no head] & {} \\
& |T_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \Sigma T_2| \arrow[ur, red] \arrow[dr] & & |T_2 \\
|T_1 \arrow[ur] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[rr, dashed, no head] & & \Sigma T_3| \arrow[ur, red] &
\end{tikzcd}
$$

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})/(\mathrm{inj} \to \mathrm{proj})$:

$$
\begin{tikzcd}[column sep=small]
& & |T_3 \arrow[dr] \arrow[rr, dashed, no head] & & \Sigma T_1| \arrow[dr] & & \\
& |T_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \Sigma T_2| \arrow[dr] & \\
|T_1 \arrow[ur] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[rr, dashed, no head] & & \bullet \arrow[ur] \arrow[rr, dashed, no head] & & \Sigma T_3|
\end{tikzcd}
$$

Example 2 (A_3 cluster category again)

$\mathcal{C} = \mathcal{C}(\mathsf{A}_3)$:

$$\begin{tikzcd}
\Sigma T_2 \arrow[dr] \arrow[rr, dash, dashed] & & T_2 \arrow[dr] \arrow[rr, dash, dashed] & & \Sigma T_1 \arrow[dr] \arrow[rr, dash, dashed] & & T_1 \arrow[dr] \arrow[r, dash, dashed] & {} \\
& \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \\
T_1 \arrow[ur] \arrow[rr, dash, dashed] & & \Sigma T_3 \arrow[ur] \arrow[rr, dash, dashed] & & T_3 \arrow[ur] \arrow[rr, dash, dashed] & & \Sigma T_2 \arrow[ur] \arrow[r, dash, dashed] & {}
\end{tikzcd}$$

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$:

$$\begin{tikzcd}
\Sigma T_2| \arrow[dr] & & |T_2 \arrow[dr] \arrow[rr, dash, dashed] & & \Sigma T_1| \arrow[dr] & & |T_1 \arrow[dr] \arrow[r, dash, dashed] & {} \\
& \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \\
|T_1 \arrow[ur] \arrow[rr, dash, dashed] & & \Sigma T_3| \arrow[ur] & & |T_3 \arrow[ur] \arrow[rr, dash, dashed] & & \Sigma T_2| \arrow[ur] &
\end{tikzcd}$$

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})/(\text{inj} \to \text{proj})$:

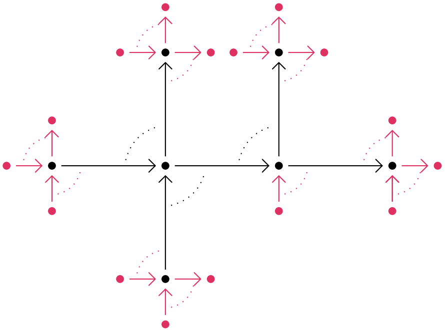
$$\begin{tikzcd}
\Sigma T_2| \arrow[dr] & & |T_2 \arrow[dr] \arrow[rr, dash, dashed] & & \Sigma T_1| \arrow[dr] & & |T_1 \arrow[dr] \arrow[r, dash, dashed] & {} \\
& \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \bullet \\
|T_1 \arrow[ur] \arrow[rr, dash, dashed] & & \Sigma T_3| \arrow[ur] & & |T_3 \arrow[ur] \arrow[rr, dash, dashed] & & \Sigma T_2| \arrow[ur] &
\end{tikzcd}$$

Result 2: Gentle algebras

Let A be a gentle algebra.

Write $A^{花}$ for the blossoming algebra[2], with $A = A^{花}/(e^{花})$.

(Asashiba '12, Brüstle–Douville–Mousavand–Thomas–Yıldırım '20, Palu–Pilaud–Plamondon '21)

$A^{花} =$ 

The *category of walks* is $\mathcal{W} = {}^{\perp}(\Sigma e^{花} A^{花}) \subseteq \mathcal{K}^{[-1,0]}(\operatorname{proj} A^{花})$.

Related by Palu–Pilaud–Plamondon to combinatorics of non-kissing facets.

[2] 花 = はな = hana (Japanese), or huā (Chinese), meaning flower

Result 2: Gentle algebras

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The *category of walks* is $\mathcal{W} = {}^{\perp}(\Sigma e^{花} A^{花}) \subseteq \mathcal{K}^{[-1,0]}(\operatorname{proj} A^{花})$.

Theorem (FGPPP)

For any gentle algebra A,

(1) *the category $\mathcal{W}$ is* 0*-Auslander,*

(2) *there is an equivalence $\mathcal{W}/(\mathit{inj} \to \mathit{proj}) \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\operatorname{proj} A)$ of extriangulated categories, and*

(3) *$(\mathit{inj} \to \mathit{proj}) = (e^{花} A^{花})$ consists only of maps factoring over a projective-injective object.*