An $\mathcal{X}$-cluster character

joint work with Jan E. Grabowski

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The theory of cluster algebras and its applications

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Slides: `https://bit.ly/mdp-abu-dhabi`

General philosophy

- Start with a $\mathbb{K}$-linear, Krull–Schmidt, Frobenius, stably 2-Calabi–Yau, algebraic extriangulated category $\mathcal{C}$, with cluster-tilting subcategories.
- Extract various pieces of data from $\mathcal{C}$ and its cluster-tilting subcategories.

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- Explain how this data transforms under mutation of cluster-tilting subcategories.
- Show that, *under the correct additional assumptions*, we recover cluster-theoretic data in the sense of Fomin–Zelevinsky.
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[1] Fock–Goncharov $\mathcal{A}$ = Fomin–Zelevinsky x, Fock–Goncharov $\mathcal{X}$ = Fomin–Zelevinsky y

Grothendieck groups

- Simplifying assumptions for today:
  - $\mathcal{C}$ is Hom-finite,
  - has cluster-tilting objects ($\rightsquigarrow$ finite rank cluster algebras), and
  - $\mathbb{K} = \overline{\mathbb{K}}$ ($\rightsquigarrow$ skew-symmetric exchange matrices).

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- Let $\mathcal{T} \subseteq \mathcal{C}$ be cluster-tilting:

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- $\rightsquigarrow$ Grothendieck groups $\mathrm{K}_0(\mathcal{T})$ and $\mathrm{K}_0(\operatorname{fd}\mathcal{T})$, for $\operatorname{fd}\mathcal{T} = \{M \colon \mathcal{T}^{\mathrm{op}} \to \operatorname{fd}\mathbb{K}\}$ = finite-dimensional $\mathcal{T}$-modules.

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- Both free of (finite) rank $\#(\mathrm{indec}\,\mathcal{T})$, each $T \in \mathrm{indec}\,\mathcal{T}$ indexes dual basis vectors $[T] \in \mathrm{K}_0(\mathcal{T})$ and $[S^{\mathcal{T}}_T] \in \mathrm{K}_0(\mathrm{fd}\,\mathcal{T})$:

$$T' \in \mathrm{indec}\,\mathcal{T} \implies S^{\mathcal{T}}_T(T') = \begin{cases} \mathbb{K}, & T' = T \\ 0, & \text{otherwise.} \end{cases}$$

Index and coindex

- Fix $\mathcal{T} \subseteq \mathcal{C}$ cluster-tilting, and let $X \in \mathcal{C}$. Then there are conflations

$$\underbrace{K_{\mathcal{T}}X \rightarrowtail R_{\mathcal{T}}}_{\in\mathcal{T}}X \twoheadrightarrow X \dashrightarrow \qquad X \rightarrowtail \underbrace{L_{\mathcal{T}}X \twoheadrightarrow C_{\mathcal{T}}X}_{\in\mathcal{T}} \dashrightarrow .$$

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- For $\mathcal{T}, \mathcal{U} \subseteq \mathcal{C}$ cluster-tilting, $\operatorname{ind}_{\mathcal{U}}^{\mathcal{T}}, \operatorname{coind}_{\mathcal{U}}^{\mathcal{T}} \colon \mathrm{K}_0(\mathcal{U}) \to \mathrm{K}_0(\mathcal{T})$.

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- Duality (over $\mathbb{Z}$): $\overline{\operatorname{ind}}^{\mathcal{T}}_{\mathcal{U}} := (\operatorname{coind}^{\mathcal{U}}_{\mathcal{T}})^* \colon \mathrm{K}_0(\operatorname{fd}\mathcal{T}) \xrightarrow{\sim} \mathrm{K}_0(\operatorname{fd}\mathcal{U})$, $\overline{\operatorname{coind}}^{\mathcal{T}}_{\mathcal{U}} := (\operatorname{ind}^{\mathcal{U}}_{\mathcal{T}})^*$.
- Cluster dictionary: $\operatorname{ind} \leftrightarrow$ **g**-vector, $\overline{\operatorname{ind}} \leftrightarrow$ **c**-vector.

Exchange matrices

- All of the above applies to the triangulated stable category $\underline{\mathcal{C}}$, with $\{\mathcal{T} \subseteq \mathcal{C} \text{ cluster-tilting}\} = \{\underline{\mathcal{T}} \subseteq \underline{\mathcal{C}} \text{ cluster-tilting}\}$

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Proposition (Keller–Reiten, Koenig–Zhu, Palu, Fu–Keller,...)

Let $\mathcal{T} \subseteq \mathcal{C}$ be cluster-tilting. Then $\mathrm{E}^{\mathcal{T}} = \operatorname{Ext}^1_{\mathcal{C}}(-,\mathcal{T})\colon \mathcal{C}/\mathcal{T} \xrightarrow{\sim} \operatorname{fd}\underline{\mathcal{T}}$, and there is a linear map $\beta_{\mathcal{T}}\colon \mathrm{K}_0(\operatorname{fd}\underline{\mathcal{T}}) \to \mathrm{K}_0(\mathcal{T})$ such that

$$\beta_{\mathcal{T}}[\mathrm{E}^{\mathcal{T}}X] = \operatorname{coind}^{\mathcal{T}}(X) - \operatorname{ind}^{\mathcal{T}}(X).$$

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For any cluster-tilting $\mathcal{T},\mathcal{U} \subseteq \mathcal{C}$, there are commutative diagrams

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{fd}\,\underline{\mathcal{U}}) \arrow[r, "\beta_{\mathcal{U}}"] \arrow[d, "\overline{\mathrm{ind}}^{\mathcal{T}}_{\mathcal{U}}"'] & \mathrm{K}_0(\mathcal{U}) \arrow[d, "\mathrm{ind}^{\mathcal{T}}_{\mathcal{U}}"] \\
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Corollary

For $\mathcal{T}$ and $\mathcal{U}$ related by mutation (away from loops or 2-cycles), the matrices of $\beta_{\mathcal{T}}$ and $\beta_{\mathcal{U}}$ with respect to the standard bases are related by Fomin–Zelevinsky matrix mutation.

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$$\mathcal{U}: \ 1 \to 2 \to 3$$

$$\mathcal{T} = \mu_2\mathcal{U}: \begin{tikzcd} 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{U}}) \arrow[r, "\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"'] & \mathrm{K}_0(\mathcal{U}) \arrow[d, "\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] \\ \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}}) \arrow[r, "\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}"'] & \mathrm{K}_0(\mathcal{T}) \end{tikzcd}$$

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Definition

Say $(\mathcal{C}, \mathcal{T})$ has a cluster structure if the quiver of $\mathcal{U}$ has no loops or 2-cycles for any $\mathcal{U} \overset{\text{mut}}{\sim} \mathcal{T}$.

$\mathcal{A}$-cluster character reminder

- $M \in \operatorname{fd} \underline{\mathcal{T}}$ has $\mathcal{F}$-polynomial

$$\mathcal{F}(M) = \sum_{[L] \in \mathrm{K}_0(\operatorname{fd} \underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(M)) x^{[L]} \in \mathbb{K}\mathrm{K}_0(\operatorname{fd} \underline{\mathcal{T}}).$$

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Example

$$\mathcal{T}: \begin{tikzcd} 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd} \qquad M = \begin{tikzcd} \mathbb{K} \arrow[dr, "1"'] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd}$$

$$\mathcal{F}(M) = 1 + 2x_2 + x_2^2 + x_1x_2 + x_1x_2^2 = (1 + x_2 + x_1x_2)(1 + x_2)$$

- $X \in \mathcal{C}$ has $\mathcal{A}$-cluster character

$$\begin{aligned} \mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(X) &= a^{\operatorname{ind}^{\mathcal{T}}(X)} (\beta_{\mathcal{T}})_* \mathcal{F}(\mathrm{E}^{\mathcal{T}} X) \\ &= a^{\operatorname{ind}^{\mathcal{T}}(X)} \sum_{[L] \in \mathrm{K}_0(\operatorname{fd} \underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(\mathrm{E}^{\mathcal{T}} X)) a^{\beta_{\mathcal{T}}[L]} \in \mathbb{K}\mathrm{K}_0(\mathcal{T}). \end{aligned}$$

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- $X \in \mathcal{C}$ has $\mathcal{A}$-cluster character

$$\begin{aligned}\mathsf{CC}^{\mathcal{T}}_{\mathcal{A}}(X) &= a^{\operatorname{ind}^{\mathcal{T}}(X)} (\beta_{\mathcal{T}})_* \mathcal{F}(\mathrm{E}^{\mathcal{T}} X) \\ &= a^{\operatorname{ind}^{\mathcal{T}}(X)} \sum_{[L] \in \mathrm{K}_0(\operatorname{fd} \underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(\mathrm{E}^{\mathcal{T}} X)) a^{\beta_{\mathcal{T}}[L]} \in \mathbb{K}\mathrm{K}_0(\mathcal{T}).\end{aligned}$$

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$$\mathrm{E}^{\mathcal{T}}X = \begin{tikzcd}[column sep=small] \mathbb{K} \arrow[dr, "1"'] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd} \oplus \begin{tikzcd}[column sep=small] 0 \arrow[dr] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd} \qquad \beta_{\mathcal{T}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\mathcal{F}(\mathrm{E}^{\mathcal{T}}X) = 1 + 2x_2 + x_2^2 + x_1x_2 + x_1x_2^2$$

$$\mathsf{CC}^{\mathcal{T}}_{\mathcal{A}}(X) = a_1a_2^{-2}a_3(1 + 2a_1^{-1}a_3 + a_1^{-2}a_3^2 + a_1^{-1}a_2 + a_1^{-2}a_2a_3)$$

$\mathcal{X}$-cluster character

- Inputs to the $\mathcal{X}$-cluster character are $M \in \mathsf{fd}\,\underline{\mathcal{U}}$ for $\mathcal{U} \subseteq \mathcal{C}$ cluster-tilting.
- $\rightsquigarrow M_{\mathcal{U}}^{\pm} \in \mathcal{U}$ such that $\beta_{\mathcal{U}}[M] = [M_{\mathcal{U}}^{+}] - [M_{\mathcal{U}}^{-}]$.

$$\mathsf{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(M) = x^{\underline{\overline{\mathsf{ind}}}_{\mathcal{U}}^{\mathcal{T}}[M]} \frac{\mathcal{F}(\mathrm{E}^{\mathcal{T}} M_{\mathcal{U}}^{+})}{\mathcal{F}(\mathrm{E}^{\mathcal{T}} M_{\mathcal{U}}^{-})} \in \mathsf{Frac}(\mathbb{K}\mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}})).$$

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Proposition

$$[M] = [L] + [N] \implies \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(M) = \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(L)\,\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(N).$$

- $\rightsquigarrow$ consider the values of $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}$ on simple $\underline{\mathcal{U}}$-modules.

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Theorem (Grabowski–P '24^{+})

Assume $(\mathcal{C}, \mathcal{T})$ has a cluster structure. Then the $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_U^{\mathcal{U}})$, for all $\mathcal{U} \overset{mut}{\sim} \mathcal{T}$ and $U \in \operatorname{indec}\underline{\mathcal{U}}$, are the $\mathcal{X}$-cluster variables of the cluster algebra associated to the quiver of $\underline{\mathcal{T}}$.

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- The map implicit in the theorem is surjective but not injective, but induces a bijection between exchange pairs for $(\mathcal{C}, \mathcal{T})$ and $\mathcal{X}$-cluster variables (thanks to Cao–Keller–Qin '24).
- For $S = S_U^{\mathcal{U}}$, the objects $S_{\mathcal{U}}^{\pm}$ are the middle terms of exchange conflations $U^* \rightarrowtail S_{\mathcal{U}}^{+} \twoheadrightarrow U$, $U \rightarrowtail S_{\mathcal{U}}^{-} \twoheadrightarrow U^*$.

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- To include $\mathcal{X}$-variables at frozen vertices, we give an ad hoc definition of $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}$ on simple modules at these vertices.
- For $M \in \operatorname{fd}\underline{\mathcal{U}}$, we have $(\beta_{\mathcal{T}})_* \, \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(M) = \frac{\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(M_{\mathcal{U}}^{+})}{\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(M_{\mathcal{U}}^{-})}$.

Example 1

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Example 1

$$\mathcal{U}: \; 1 \to 2 \to 3 \qquad\qquad \mathcal{T} = \mu_2 \mathcal{U}: \begin{tikzcd} 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd}$$

$$\begin{aligned} \overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] &= [S_1^{\mathcal{T}}] + [S_2^{\mathcal{T}}] & \qquad (S_1)^+_{\mathcal{U}} &= 0 & \qquad (S_1)^-_{\mathcal{U}} &= U_2 \\ \overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] &= -[S_2^{\mathcal{T}}] & (S_2)^+_{\mathcal{U}} &= U_1 & (S_1)^-_{\mathcal{U}} &= U_3 \\ \overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] &= [S_3^{\mathcal{T}}] & (S_3)^+_{\mathcal{U}} &= U_2 & (S_1)^-_{\mathcal{U}} &= 0 \end{aligned}$$

Example 1

$$\mathcal{U}: \ 1 \to 2 \to 3 \qquad \mathcal{T} = \mu_2 \mathcal{U}: \begin{tikzcd}[column sep=small] 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd}$$

$$\overline{\operatorname{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] = [S_1^{\mathcal{T}}] + [S_2^{\mathcal{T}}] \qquad (S_1)_{\mathcal{U}}^{+} = 0 \qquad (S_1)_{\mathcal{U}}^{-} = U_2$$

$$\overline{\operatorname{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] = -[S_2^{\mathcal{T}}] \qquad (S_2)_{\mathcal{U}}^{+} = U_1 \qquad (S_1)_{\mathcal{U}}^{-} = U_3$$

$$\overline{\operatorname{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] = [S_3^{\mathcal{T}}] \qquad (S_3)_{\mathcal{U}}^{+} = U_2 \qquad (S_1)_{\mathcal{U}}^{-} = 0$$

$$\mathrm{E}^{\mathcal{T}}(U_1) = 0 \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_1^{\mathcal{U}}) = x_1 x_2 \left(\tfrac{1}{1+x_2}\right) = x_1 x_2 (1+x_2)^{-1}$$

$$\mathrm{E}^{\mathcal{T}}(U_2) = S_2^{\mathcal{T}} \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1} \left(\tfrac{1}{1}\right) = x_2^{-1}$$

$$\mathrm{E}^{\mathcal{T}}(U_3) = 0 \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_3^{\mathcal{U}}) = x_3 \left(\tfrac{1+x_2}{1}\right) = x_3 (1+x_2)$$

Example 2

$\mathcal{U}: \; 1 \to 2 \to 3 \qquad \mathcal{T} = \Sigma\mathcal{U}: \; 1 \to 2 \to 3$

Example 2

$$\mathcal{U}: \; 1 \to 2 \to 3 \qquad \mathcal{T} = \Sigma\mathcal{U}: \; 1 \to 2 \to 3$$

$$\overline{\mathsf{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] = -[S_1^{\mathcal{T}}] \qquad (S_1)_{\mathcal{U}}^{+} = 0 \qquad (S_1)_{\mathcal{U}}^{-} = U_2$$

$$\overline{\mathsf{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] = -[S_2^{\mathcal{T}}] \qquad (S_2)_{\mathcal{U}}^{+} = U_1 \qquad (S_1)_{\mathcal{U}}^{-} = U_3$$

$$\overline{\mathsf{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] = -[S_3^{\mathcal{T}}] \qquad (S_3)_{\mathcal{U}}^{+} = U_2 \qquad (S_1)_{\mathcal{U}}^{-} = 0$$

Example 2

$$\mathcal{U}: \; 1 \to 2 \to 3 \qquad \mathcal{T} = \Sigma\mathcal{U}: \; 1 \to 2 \to 3$$

$$\begin{array}{lll}
\overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] = -[S_1^{\mathcal{T}}] & (S_1)_{\mathcal{U}}^{+} = 0 & (S_1)_{\mathcal{U}}^{-} = U_2 \\
\overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] = -[S_2^{\mathcal{T}}] & (S_2)_{\mathcal{U}}^{+} = U_1 & (S_1)_{\mathcal{U}}^{-} = U_3 \\
\overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] = -[S_3^{\mathcal{T}}] & (S_3)_{\mathcal{U}}^{+} = U_2 & (S_1)_{\mathcal{U}}^{-} = 0
\end{array}$$

$$\begin{array}{ll}
\mathrm{E}^{\mathcal{T}}(U_1) = \; \mathbb{K} \to 0 \to 0 & CC_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_1^{\mathcal{U}}) = x_1^{-1}\left(\frac{1}{1+x_2+x_1x_2}\right) \\
\mathrm{E}^{\mathcal{T}}(U_2) = \; \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0 & CC_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1}\left(\frac{1+x_1}{1+x_3+x_2x_3+x_1x_2x_3}\right) \\
\mathrm{E}^{\mathcal{T}}(U_3) = \; \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} & CC_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_3^{\mathcal{U}}) = x_3^{-1}(1 + x_2 + x_1x_2)
\end{array}$$

Example 2

$$\mathcal{U}: \; 1 \to 2 \to 3 \qquad \mathcal{T} = \Sigma\mathcal{U}: \; 1 \to 2 \to 3$$

$$\overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] = -[S_1^{\mathcal{T}}] \qquad (S_1)_{\mathcal{U}}^{+} = 0 \qquad (S_1)_{\mathcal{U}}^{-} = U_2$$

$$\overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] = -[S_2^{\mathcal{T}}] \qquad (S_2)_{\mathcal{U}}^{+} = U_1 \qquad (S_1)_{\mathcal{U}}^{-} = U_3$$

$$\overline{\mathrm{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] = -[S_3^{\mathcal{T}}] \qquad (S_3)_{\mathcal{U}}^{+} = U_2 \qquad (S_1)_{\mathcal{U}}^{-} = 0$$

$$\mathrm{E}^{\mathcal{T}}(U_1) = \; \mathbb{K} \to 0 \to 0 \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_1^{\mathcal{U}}) = x_1^{-1}\left(\frac{1}{1+x_2+x_1x_2}\right)$$

$$\mathrm{E}^{\mathcal{T}}(U_2) = \; \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0 \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1}\left(\frac{1+x_1}{1+x_3+x_2x_3+x_1x_2x_3}\right)$$

$$\mathrm{E}^{\mathcal{T}}(U_3) = \; \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_3^{\mathcal{U}}) = x_3^{-1}(1 + x_2 + x_1x_2)$$

Thank you! / شكراً! / 谢谢!

Bonus slide

$$
\begin{tikzcd}[column sep=small]
T_2 \arrow[rr, dashed] \arrow[dr] & & U_4 \arrow[rr, dashed] \arrow[dr] & & U_2 \arrow[rr, dashed] \arrow[dr] & & T_2 \arrow[dr] & \\
& T_1 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & U_3 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & U_1 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & T_1 \\
Z_1 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & Z_3 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & Z_2 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & Z_1 \arrow[ur] \arrow[dr] & \\
& Z_2 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & Z_1 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & Z_3 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & Z_2 \\
U_3 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & U_1 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & T_1 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & U_3 \arrow[ur] \arrow[dr] & \\
& U_2 \arrow[ur] \arrow[rr, dashed] & & T_2 \arrow[ur] \arrow[rr, dashed] & & U_4 \arrow[ur] \arrow[rr, dashed] & & U_2
\end{tikzcd}
$$

- $\begin{tikzcd} T_2 \arrow[r] & T_1 \arrow[loop right] \end{tikzcd}$ $\rightsquigarrow$ no cluster structure!

Bonus slide

$$
\begin{tikzcd}[column sep=small]
T_2 \arrow[rr, dash, dashed] \arrow[dr] & & U_4 \arrow[rr, dash, dashed] \arrow[dr] & & U_2 \arrow[rr, dash, dashed] \arrow[dr] & & T_2 \arrow[dr] & \\
& T_1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & U_3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & U_1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & T_1 \\
Z_1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & Z_3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & Z_2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & Z_1 \arrow[ur] \arrow[dr] & \\
& Z_2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & Z_1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & Z_3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & Z_2 \\
U_3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & U_1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & T_1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & U_3 \arrow[ur] \arrow[dr] & \\
& U_2 \arrow[rr, dash, dashed] \arrow[ur] & & T_2 \arrow[rr, dash, dashed] \arrow[ur] & & U_4 \arrow[rr, dash, dashed] \arrow[ur] & & U_2
\end{tikzcd}
$$

- $T_2 \longrightarrow T_1 \circlearrowleft$ $\rightsquigarrow$ no cluster structure!
- $\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(U_1) = a_1^{-1}(1 + a_2 + a_2^2)$,
  $\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(U_2) = a_2^{-1}(1 + a_1^{-1} + a_1^{-1}a_2 + a_1^{-1}a_2^2)$,...

Bonus slide

$$
\begin{array}{ccccccccccccccc}
T_2 & & & & U_4 & & & & U_2 & & & & T_2 & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & T_1 & & & & U_3 & & & & U_1 & & & & T_1 \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
Z_1 & & & & Z_3 & & & & Z_2 & & & & Z_1 & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & Z_2 & & & & Z_1 & & & & Z_3 & & & & Z_2 \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
U_3 & & & & U_1 & & & & T_1 & & & & U_3 & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & U_2 & & & & T_2 & & & & U_4 & & & & U_2
\end{array}
$$

- $T_2 \longrightarrow T_1 \circlearrowleft$ $\rightsquigarrow$ no cluster structure!
- $\mathrm{CC}^{\mathcal{T}}_{\mathcal{A}}(U_1) = a_1^{-1}(1 + a_2 + a_2^2)$,
  $\mathrm{CC}^{\mathcal{T}}_{\mathcal{A}}(U_2) = a_2^{-1}(1 + a_1^{-1} + a_1^{-1}a_2 + a_1^{-1}a_2^2)$,...
- $\rightsquigarrow$ generalised cluster variables (Chekhov–Shapiro)