Representation theory and positroid varieties
表征理论和正拟阵簇

Matthew Pressland

University of Glasgow / Oilthigh Ghlaschu
格拉斯哥大学

ICRA 2024
上海交通大学
09.08.2024

Slides: `https://bit.ly/mdp-icra24`

The totally positive Grassmannian

Definition

$M \in \mathbb{C}^{k \times n}$, $k < n$, is *totally positive* if its maximal minors $\Delta_I(M)$ are positive real numbers.

- Here $I \in \binom{[n]}{k}$ is a subset of k columns, $\Delta_I(M)$ its determinant.
- If $\operatorname{rk} M = k$, its row span $[M]$ is in $\mathrm{Gr}_{k,n}$, the *Grassmannian*.
- Totally positive Grassmannian: $\mathrm{Gr}_{k,n}^{>0} = \{[M] : M \text{ is totally +ve}\}$.

The totally positive Grassmannian

Definition

$M \in \mathbb{C}^{k \times n}$, $k < n$, is *totally positive* if its maximal minors $\Delta_I(M)$ are positive real numbers.

- Here $I \in \binom{[n]}{k}$ is a subset of k columns, $\Delta_I(M)$ its determinant.
- If $\operatorname{rk} M = k$, its row span $[M]$ is in $\mathrm{Gr}_{k,n}$, the *Grassmannian*.
- Totally positive Grassmannian: $\mathrm{Gr}_{k,n}^{>0} = \{[M] : M \text{ is totally +ve}\}$.
- A minimal positivity test needs only $\dim \widehat{\mathrm{Gr}}_{k,n} = k(n-k) + 1$ minors ... chosen carefully!

$$k = 2 : \quad \Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

The totally positive Grassmannian

Definition

$M \in \mathbb{C}^{k\times n}$, $k < n$, is *totally positive* if its maximal minors $\Delta_I(M)$ are positive real numbers.

- Here $I \in \binom{[n]}{k}$ is a subset of k columns, $\Delta_I(M)$ its determinant.
- If $\mathrm{rk}\, M = k$, its row span $[M]$ is in $\mathrm{Gr}_{k,n}$, the *Grassmannian*.
- Totally positive Grassmannian: $\mathrm{Gr}_{k,n}^{>0} = \{[M] : M \text{ is totally +ve}\}$.
- A minimal positivity test needs only $\dim \widehat{\mathrm{Gr}}_{k,n} = k(n-k)+1$ minors ... chosen carefully!

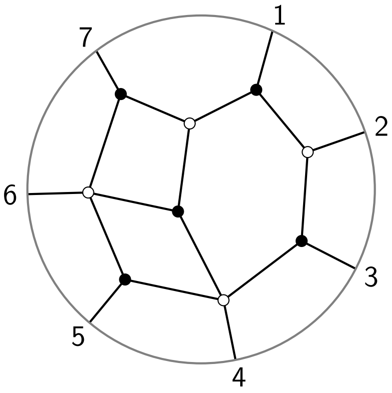
$$k = 2: \quad \Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

- $\overline{\mathrm{Gr}_{k,n}^{>0}} = \mathrm{Gr}_{k,n}^{\geqslant 0}$ decomposes into cells $\Pi_{\mathcal{P}}^{\circ} \cap \mathrm{Gr}_{k,n}^{\geqslant 0}$, indexed by *positroids* (正拟阵) $\mathcal{P} \subseteq \binom{[n]}{k}$.

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:

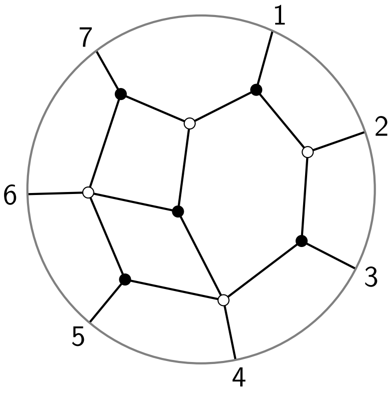


$n = 7$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



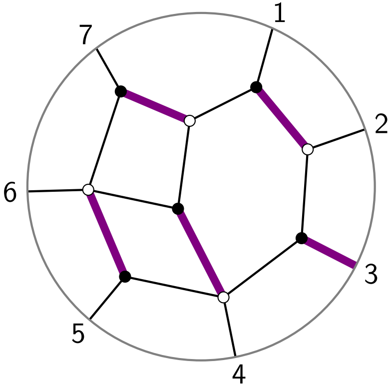
$n = 7$

$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



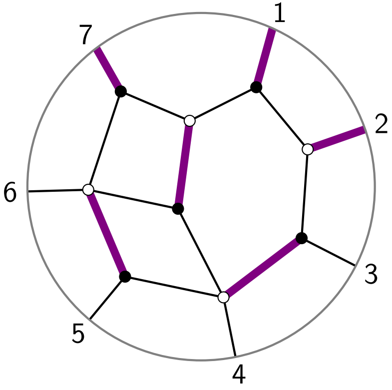
$n = 7$

$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157,$$

Positroid varieties

Postnikov '06^{+}, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



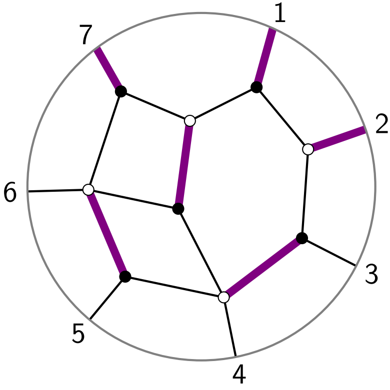
$n = 7$

$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \dots\}$$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$n = 7$

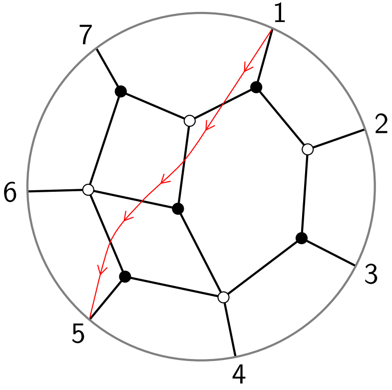
$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \ldots\}$$

$k = |\partial\mu| = 3$

Positroid varieties

Postnikov '06^{+}, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$n = 7$

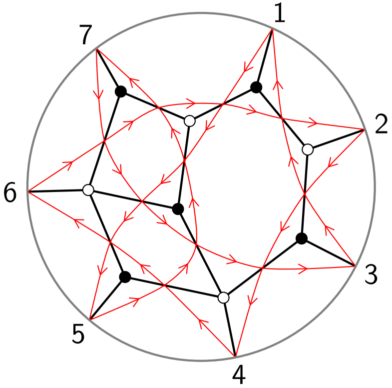
$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$
$= \{157, 235, \ldots\}$

$k = |\partial\mu| = 3$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$n = 7$

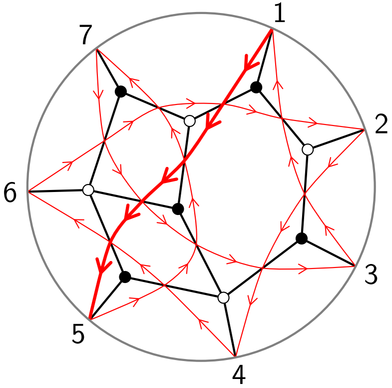
$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \dots\}$$

$k = |\partial\mu| = 3$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$n = 7$

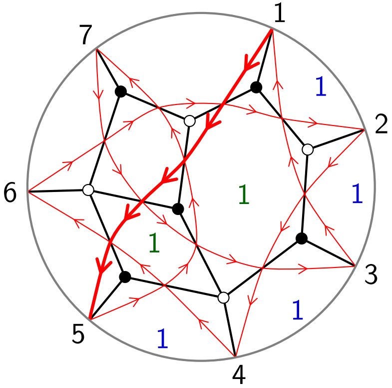
$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \ldots\}$$

$k = |\partial\mu| = 3$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$$n = 7$$

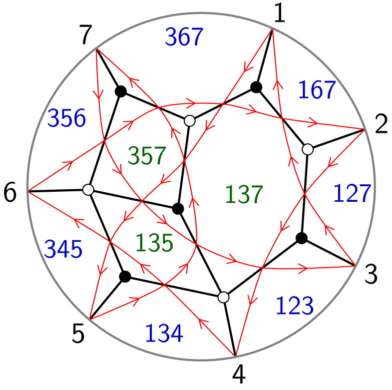
$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \dots\}$$

$$k = |\partial\mu| = 3$$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$$n = 7$$

$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \dots\}$$

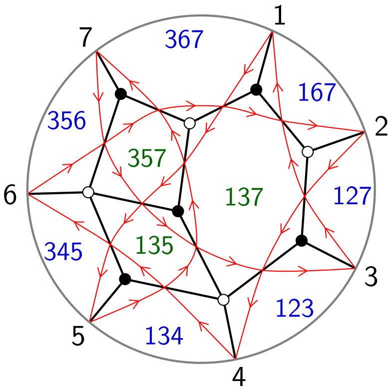
$$k = |\partial\mu| = 3$$

$$\mathcal{C}^+ = \{\text{labels}\}$$
$$= \{137, 167, 135, \dots\}$$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$$n = 7$$

$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \dots\}$$

$$k = |\partial\mu| = 3$$

$$\mathcal{C}^+ = \{\text{labels}\}$$
$$= \{137, 167, 135, \dots\}$$

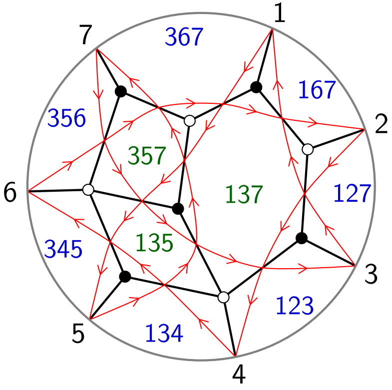
$$\mathcal{F}^+ = \{\text{boundary labels}\}$$
$$= \{123, 134, 345, \dots\}$$

$$\Pi^\circ_{\mathcal{P}} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P},\ \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^+\}$$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$$n = 7$$

$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$
$$= \{157, 235, \dots\}$$

$$k = |\partial\mu| = 3$$

$$\mathcal{C}^+ = \{\text{source labels}\}$$
$$= \{137, 167, 135, \dots\}$$

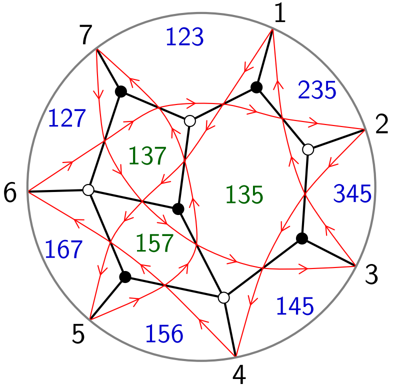
$$\mathcal{F}^+ = \{\text{boundary source labels}\}$$
$$= \{123, 134, 345, \dots\}$$

$$\Pi^\circ_{\mathcal{P}} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P},\ \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^+\}$$

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



$$n = 7$$

$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$$

$$= \{157, 235, \dots\}$$

$$k = |\partial\mu| = 3$$

$$\mathcal{C}^- = \{\text{target labels}\}$$

$$= \{135, 235, 157, \dots\}$$

$$\mathcal{F}^- = \{\text{boundary target labels}\}$$

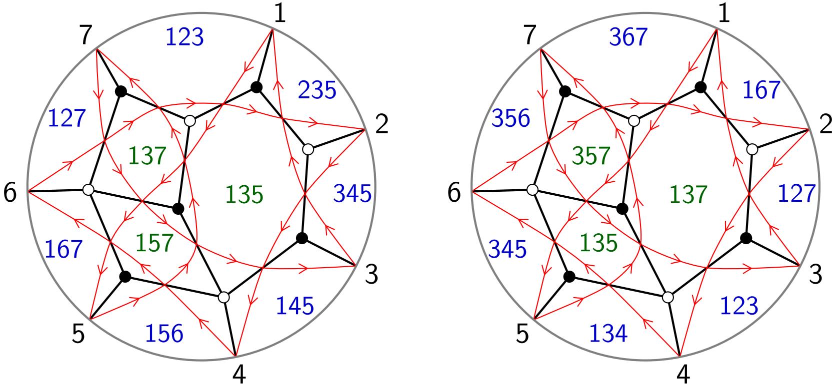
$$= \{145, 156, 167, \dots\}$$

$$\Pi^\circ_{\mathcal{P}} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P},\ \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^-\}$$

Positroid varieties

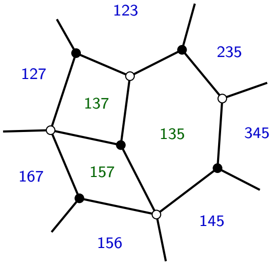
Postnikov '06^{+}, Knutson–Lam–Speyer '13,...

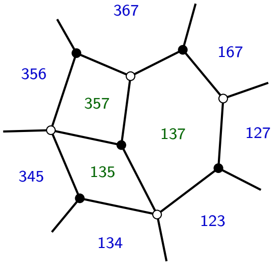
To compute a positroid, use a Postnikov diagram / dimer model:



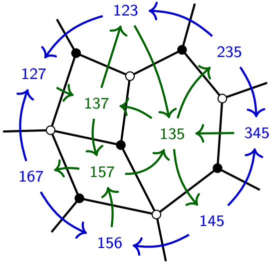
$$\Pi^\circ_{\mathcal{P}} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P},\ \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^\pm\}$$

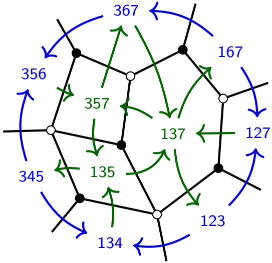
Cluster structures



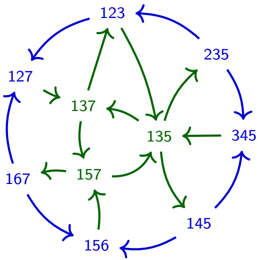


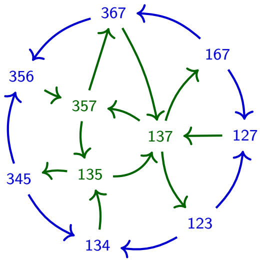
Cluster structures





Cluster structures





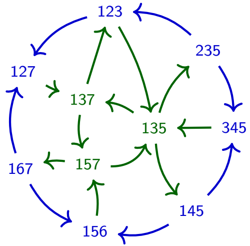
Theorem (Galashin–Lam '23)

$\mathbb{C}[\widehat{\Pi}^\circ_{\mathcal{P}}]$ has two natural cluster algebra structures: one cluster algebra $\mathscr{A}_{\mathcal{P}}$, two isomorphisms $\eta^{\pm}\colon \mathscr{A}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^\circ_{\mathcal{P}}]$.

Theorem (P '23$^+$, conj. Muller–Speyer '16)

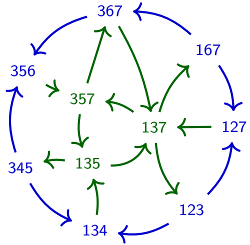
The cluster structures $\eta^{\pm}\colon \mathscr{A}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^\circ_{\mathcal{P}}]$ quasi-coincide.

Example



Target-labelled structure

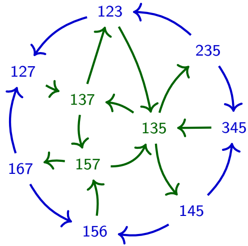
Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$



Source-labelled structure

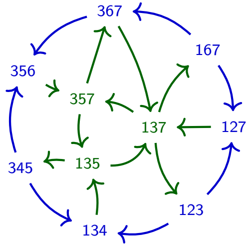
Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
Mutable, degree 2	$\Delta_{125}\Delta_{367}, \Delta_{124}\Delta_{367}$

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$

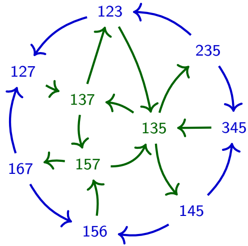


Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
Mutable, degree 2	$\Delta_{125}\Delta_{367}, \Delta_{124}\Delta_{367}$

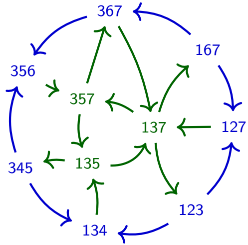
Δ_{157}

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$

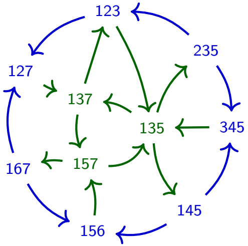


Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
Mutable, degree 2	$\Delta_{125}\Delta_{367}, \Delta_{124}\Delta_{367}$

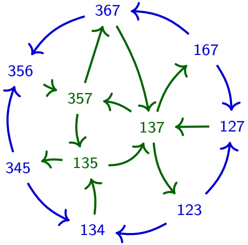
$$\Delta_{157} = \frac{\Delta_{357}\Delta_{167} + \Delta_{137}\Delta_{567}}{\Delta_{367}}$$

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$

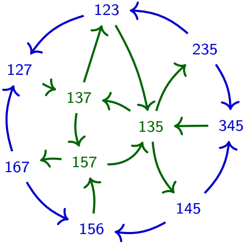


Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
Mutable, degree 2	$\Delta_{125}\Delta_{367}, \Delta_{124}\Delta_{367}$

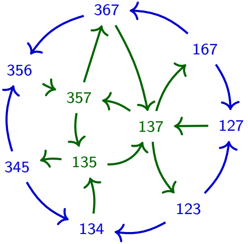
$$\Delta_{157} = \frac{\Delta_{357}\Delta_{167} + \Delta_{137}\cancelto{0}{\Delta_{567}}}{\Delta_{367}}$$

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \color{red}{\Delta_{157}}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$

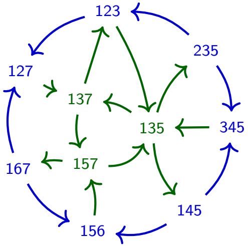


Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
Mutable, degree 2	$\Delta_{125}\Delta_{367}, \Delta_{124}\Delta_{367}$

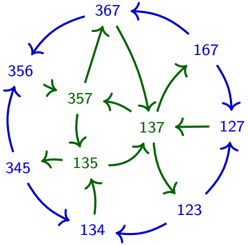
$$\color{red}{\Delta_{157}} = \frac{\Delta_{357}\Delta_{167} + \Delta_{137}\cancelto{0}{\Delta_{567}}}{\Delta_{367}} = \Delta_{357}\frac{\Delta_{167}}{\Delta_{367}}$$

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$

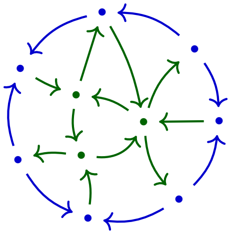


Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
Mutable, degree 2	$\Delta_{125}\Delta_{367}, \Delta_{124}\Delta_{367}$

$$\Delta_{157} = \frac{\Delta_{357}\Delta_{167} + \cancelto{0}{\Delta_{137}\Delta_{567}}}{\Delta_{367}} = \Delta_{357}\frac{\Delta_{167}}{\Delta_{367}}$$

Categorification, part 1: combinatorics



$$A = \mathbb{C}\langle\!\langle Q \rangle\!\rangle / \overline{(\text{dimer relations})}$$

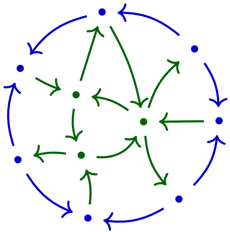
$$B = eAe$$

$$Z = \mathbb{C}[\![t]\!],\ Z \subseteq B \subseteq A$$

$$\operatorname{CM} B = \{X \in \operatorname{mod} B : {}_Z X \text{ free} + \text{f.g.}\}$$

$$\operatorname{gproj} \operatorname{CM} B = \{X \in \operatorname{CM} B : \operatorname{Ext}_B^{>0}(X, B) = 0\}$$

Categorification, part 1: combinatorics



$$A = \mathbb{C}\langle\langle Q \rangle\rangle / \overline{(\text{dimer relations})}$$
$$B = eAe$$
$$Z = \mathbb{C}[\![t]\!],\ Z \subseteq B \subseteq A$$
$$\mathrm{CM}\,B = \{X \in \mathrm{mod}\,B : {}_Z X \text{ free} + \text{f.g.}\}$$
$$\mathrm{gproj}\,\mathrm{CM}\,B = \{X \in \mathrm{CM}\,B : \mathrm{Ext}_B^{>0}(X, B) = 0\}$$

Theorem (P '22)

For each (connected) positroid $\mathcal{P}$, the Frobenius exact category $\mathrm{gproj}\,\mathrm{CM}\,B$ *categorifies the cluster algebra $\mathscr{A}_{\mathcal{P}}$.*

- Key fact: A is internally 3-Calabi–Yau.
- $\mathrm{ginj}\,\mathrm{CM}\,B = \{X \in \mathrm{CM}\,B : \mathrm{Ext}_B^{>0}(B^\vee, X) = 0\}$ also categorifies.
- $\mathrm{gproj}\,\mathrm{CM}\,B \simeq \mathrm{ginj}\,\mathrm{CM}\,B$, but different subcategories of $\mathrm{CM}\,B$!

Categorification, part 2: geometry

- Jensen–King–Su '16: categorification CM C of $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}]$, with $M_I \in$ CM C for each Δ_I.

Theorem (Çanakçı–King–P '24, P '22)

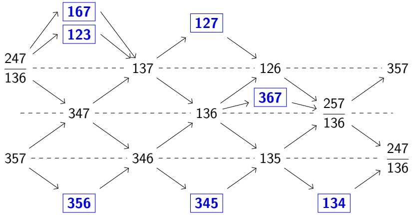
- CM $B \hookrightarrow$ CM C, *with* $M_I \in$ CM $B \iff I \in \mathcal{P}$.
- $M_I \in \operatorname{gproj}$ CM $B \iff \Delta_I$ *is an* η^+*-cluster variable (source labelled).*
- $M_I \in \operatorname{ginj}$ CM $B \iff \Delta_I$ *is an* η^-*-cluster variable (target labelled).*

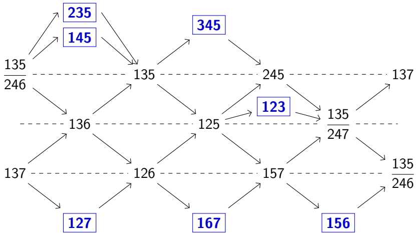
Categorification, part 2: geometry

- Jensen–King–Su '16: categorification CM C of $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}]$, with $M_I \in$ CM C for each Δ_I.

Theorem (Çanakçı–King–P '24, P '22)

- CM $B \hookrightarrow$ CM C, *with* $M_I \in$ CM $B \iff I \in \mathcal{P}$.
- $M_I \in$ gproj CM $B \iff \Delta_I$ *is an* η^+*-cluster variable (source labelled).*
- $M_I \in$ ginj CM $B \iff \Delta_I$ *is an* η^-*-cluster variable (target labelled).*





Proving the Muller–Speyer conjecture

- Reduce (geometrically) to connected positroids, for access to categorifications.
- Key fact: inclusions induce *derived* equivalences

$$\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathsf{CM}\,B) \xhookrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathsf{CM}\,B) \xhookleftarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathsf{CM}\,B)$$

Proving the Muller–Speyer conjecture

- Reduce (geometrically) to connected positroids, for access to categorifications.
- Key fact: inclusions induce *derived* equivalences

$$\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathsf{CM}\,B) \xhookrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathsf{CM}\,B) \xhookleftarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathsf{CM}\,B)$$

- Main step: show that the composition is a quasi-cluster functor (Fraser–Keller '23).
- E.g. induced equivalence $\underline{\operatorname{gproj}}\,\mathsf{CM}\,B \xrightarrow{\sim} \underline{\operatorname{ginj}}\,\mathsf{CM}\,B$ takes initial cluster-tilting object T^+ to reachable cluster-tilting object $\Omega^2 T^-$.

Proving the Muller–Speyer conjecture

- Reduce (geometrically) to connected positroids, for access to categorifications.
- Key fact: inclusions induce *derived* equivalences

$$\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathsf{CM}\,B) \xhookrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathsf{CM}\,B) \xhookleftarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathsf{CM}\,B)$$

- Main step: show that the composition is a quasi-cluster functor (Fraser–Keller '23).
- E.g. induced equivalence $\underline{\operatorname{gproj}}\,\mathsf{CM}\,B \xrightarrow{\sim} \underline{\operatorname{ginj}}\,\mathsf{CM}\,B$ takes initial cluster-tilting object T^+ to reachable cluster-tilting object $\Omega^2 T^-$.

Theorem (P '23$^+$, conj. Muller–Speyer '16)

The cluster structures η^+ and η^- quasi-coincide.

- Independent proof: (Casals–Le–Sherman-Bennett–Weng '23$^+$) Inspired by symplectic topology!

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.

Computation

- Let $X \in \operatorname{ginj} \mathsf{CM}\, B$, compute syzygy ΩX.

$$0 \longrightarrow \Omega X \longrightarrow P \longrightarrow X \longrightarrow 0$$

Computation

- Let $X \in \operatorname{ginj} \mathsf{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathsf{CM}\, B$ (ÇKP '24), so compute cosyzygy here.

$$
\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] \arrow[d, equal] & P \arrow[r] & X \arrow[r] & 0 \\
& \Omega X & & &
\end{tikzcd}
$$

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] \arrow[d, equal] & P \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] & Q \arrow[r] & \Sigma\Omega X \arrow[r] & 0
\end{tikzcd}$$

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] & P \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{CM}\, B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] & P \arrow[r] & {\color{red}M_{157}} \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{CM}\, B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \operatorname{CM} B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \operatorname{CM} B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \operatorname{CM} B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] & M_{167} \oplus M_{345} \arrow[r] & M_{157} \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\operatorname{CM} B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] & M_{167} \oplus M_{345} \arrow[r] & M_{157} \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{CM}\, B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \operatorname{CM} B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \operatorname{CM} B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \operatorname{CM} B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & M_{346} \arrow[r] & \color{blue}{M_{167} \oplus M_{345}} \arrow[r] & \color{red}{M_{157}} \arrow[r] & 0 \\
0 \arrow[r] & M_{346} \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\operatorname{CM} B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \operatorname{CM} B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \operatorname{CM} B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \operatorname{CM} B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & M_{346} \arrow[r] & {\color{blue}M_{167} \oplus M_{345}} \arrow[r] & {\color{red}M_{157}} \arrow[r] & 0 \\
0 \arrow[r] & M_{346} \arrow[r] \arrow[u, equal] & {\color{blue}M_{367} \oplus M_{345}} \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\operatorname{CM} B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & M_{346} \arrow[r] & {\color{blue}M_{167} \oplus M_{345}} \arrow[r] & {\color{red}M_{157}} \arrow[r] & 0 \\
0 \arrow[r] & M_{346} \arrow[r] \arrow[u, equal] & {\color{blue}M_{367} \oplus M_{345}} \arrow[r] \arrow[u] & {\color{green}M_{357}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $X \cong (Q \to P \oplus \Sigma\Omega X)$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{CM}\, B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

- Let $X \in \operatorname{ginj} \mathrm{CM}\, B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \mathrm{CM}\, B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & M_{346} \arrow[r] & M_{167} \oplus M_{345} \arrow[r] & M_{157} \arrow[r] & 0 \\
0 \arrow[r] & M_{346} \arrow[r] \arrow[u, equal] & M_{367} \oplus M_{345} \arrow[r] \arrow[u] & M_{357} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $M_{157} \cong (M_{367} \to M_{167} M_{357})$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{CM}\, B)$, and hence

$$\Psi_X = \Psi_{\Sigma\Omega X} \frac{\Psi_P}{\Psi_Q} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.

Computation

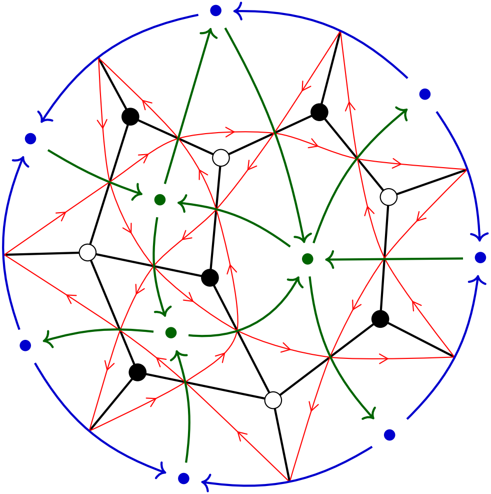
- Let $X \in \operatorname{ginj} \operatorname{CM} B$, compute syzygy ΩX.
- Then $\Omega X \in \operatorname{gproj} \operatorname{CM} B$ (ÇKP '24), so compute cosyzygy here.
- This gives $\Sigma\Omega X \in \operatorname{gproj} \operatorname{CM} B$, and $P, Q \in \operatorname{proj} B$.

$$\begin{tikzcd}
0 \arrow[r] & M_{346} \arrow[r] & M_{167} \oplus M_{345} \arrow[r] & M_{157} \arrow[r] & 0 \\
0 \arrow[r] & M_{346} \arrow[r] \arrow[u, equal] & M_{367} \oplus M_{345} \arrow[r] \arrow[u] & M_{357} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

- Get $M_{157} \cong (M_{367} \to M_{167} M_{357})$ in $\mathcal{D}^{\mathrm{b}}(\operatorname{CM} B)$, and hence

$$\Delta_{357} \frac{\Delta_{167}}{\Delta_{367}} = \Delta_{157} \in \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$$

for Ψ the cluster character.



谢谢!