

On categorification of g -vectors

joint work in progress
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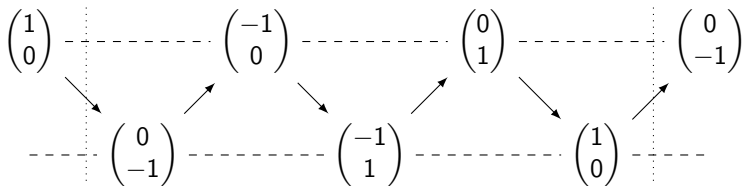
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ICRA 2022: Universidad de Buenos Aires

g-vectors

Cluster algebras: in the cluster algebra \mathcal{A} attached to a quiver Q , each cluster variable has a *g-vector* in \mathbb{Z}^{Q_0} .

This vector was originally defined (Fomin–Zelevinsky) in terms of a grading of the principal coefficient cluster algebra $\mathcal{A}^{\text{prin}}$.



These vectors can be computed categorically in two ways.

Option 1: Coindex

Let \mathcal{C} be a Hom-finite Krull–Schmidt 2-Calabi–Yau triangulated category.

Let $\mathcal{T} \subseteq \mathcal{C}$ be a cluster-tilting subcategory, meaning \mathcal{T} is functorially finite and

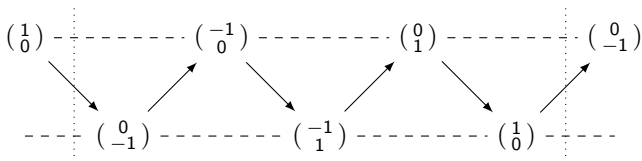
$$\mathcal{T} = \{X \in \mathcal{C} : \text{Ext}_{\mathcal{C}}^1(\mathcal{T}, X) := \text{Hom}_{\mathcal{C}}(\mathcal{T}, \Sigma X) = 0\}.$$

Then for all $X \in \mathcal{C}$ there exists a triangle $X \rightarrow T_1 \rightarrow T_0 \rightarrow \Sigma X$ with $T_0, T_1 \in \mathcal{T}$.

Define $\text{coind}_{\mathcal{T}}(X) = [T_1] - [T_0] \in K_0(\mathcal{T})$.

Theorem (Dehy–Keller, Fu–Keller)

$\text{coind}_{\mathcal{T}}(X)$ is the g -vector of the cluster variable $CC^{\mathcal{T}}(X)$ (when this makes sense).



Option 2: Projective presentations

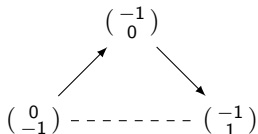
Let A be a finite-dimensional algebra and $M \in \text{mod } A$. Take a minimal projective presentation $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$.

Then the g -vector of M is $[P_1] - [P_0] \in K_0(\text{proj } A)$.

Say A is 2-CY-tilted if it is isomorphic to $\text{End}_{\mathcal{C}}(T)^{\text{op}}$ for \mathcal{C} as above and add T cluster-tilting.

Theorem

When A is 2-CY-tilted and $M \in \text{mod } A$, the g -vector of M is that of the corresponding (non-initial) cluster variable (when this makes sense).



Connection

Take $X \in \mathcal{C}$, and choose a triangle $X \rightarrow T_1 \rightarrow T_0 \rightarrow \Sigma X$ to compute the coindex.

This yields an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, T_1) \rightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, T_0) \rightarrow \mathrm{Ext}_{\mathcal{C}}^1(\mathcal{T}, X) \rightarrow 0$$

of finitely-presented functors on \mathcal{T} .

There are equivalences

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, -): \mathcal{T} \xrightarrow{\sim} \mathrm{proj} \mathcal{T}, \quad \text{Yoneda}$$

$$\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{T}, -): \mathcal{C}/(\mathcal{T}) \xrightarrow{\sim} \mathrm{mod} \mathcal{T}. \quad \text{Buan–Marsh–Reiten, Keller–Reiten, Koenig–Zhu, ...}$$

If $\mathcal{T} = \mathrm{add} T$ then $\mathrm{proj} \mathcal{T} = \mathrm{proj} A$ and $\mathrm{mod} \mathcal{T} = \mathrm{mod} A$ for $A = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.

Thus the g-vector of $X \in \mathcal{C}$ is equal to the g-vector of $\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{T}, X) \in \mathrm{mod} A$.

Aim

Enhance this relationship to an equivalence of ‘categories of g-vectors’.

Extriangulated categories (Nakaoka–Palu)

Idea: additive categories with well-behaved ‘extension groups’ $\mathbb{E}(X, Y)$.

Example

- (0) Exact categories, triangulated categories ($\mathbb{E} = \text{Ext}_{\mathcal{C}}^1$).
- (1) Extension closed subcategories of triangulated categories ($\mathbb{E} = \text{Ext}_{\mathcal{C}}^1$).
- (2) ‘Partial stabilisations’ $\mathcal{C}/(P)$ for \mathcal{C} Frobenius exact, P projective-injective ($\mathbb{E} = \text{Ext}_{\mathcal{C}}^1$).
- (3) Ex-triangulated categories: Take a triangulated category \mathcal{C} and choose (carefully) a subfunctor $\mathbb{E} \leq \text{Ext}_{\mathcal{C}}^1$.

Carefully = making sure inflations and deflations are closed under composition.
(Herschend–Liu–Nakaoka)

Remark

(3) was studied for exact categories by Auslander–Solberg, under the heading of relative homological algebra: the process preserves exactness (but not triangulatedness).

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- (1) Extension closed subcategories of extriangulated categories.
- (2) ‘Partial stabilisations’ $\mathcal{C}/(P)$ for \mathcal{C} Frobenius extriangulated, P projective-injective.
- (3) Take an extriangulated category \mathcal{C} and choose (carefully) a subfunctor of \mathbb{E} .

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Harp (The Homotopy ARrow category of Projectives)

Let A be a finite dimensional algebra. Then

$$\text{harp } A := \{P_1 \xrightarrow{\varphi} P_0 : P_i \in \text{proj } A\} / \text{homotopy}.$$

We have $\text{harp } A \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\text{proj } A) \hookrightarrow \mathcal{K}^b(\text{proj } A)$.

The image is extension-closed, and so $\text{harp } A$ is naturally extriangulated.

Projective objects are those of the form $0 \rightarrow P$, and injectives of the form $P \rightarrow 0$. (Objects $P \xrightarrow{\sim} P$ are projective-injective, but also 0.)

$$\begin{array}{ccccc}
 & |(0 \rightarrow P_1) \cdots (P_2 \rightarrow 0)| & & & \\
 & \nearrow & \searrow & \nearrow & \searrow \\
 |(0 \rightarrow P_2) \cdots (P_2 \rightarrow P_1) \cdots (P_1 \rightarrow 0)| & & & &
 \end{array}$$

Relative harp

Choose additionally $e = e^2 \in A$, and define

$$\text{harp}_e A := \{P_1 \xrightarrow{\varphi} P_0 \in \text{harp } A : e \cdot \text{coker } \varphi = 0\}.$$

Note that $\text{harp}_0 A = \text{harp } A$.

Proposition (FGPPP)

In $\text{harp}_e(A)$, injectives are $P \rightarrow 0$, while projectives are $P \xrightarrow{\varphi} Q$ such that $P \in \text{add } Ae$. In particular, $Ae \rightarrow 0$ is projective-injective.

$$A = \begin{array}{ccc} \boxed{1} & \xrightarrow{q} & \boxed{2} \\ & \searrow r \quad \swarrow p & \\ & * & \end{array} / (pq, qr)$$

For $e = e_1 + e_2$, the category $\text{harp}_e A$ is

$$\begin{array}{ccccc} |(P_1 \rightarrow 0)| & & |(P_2 \rightarrow 0)| & & |(P_1 \rightarrow 0)| \\ \searrow & & \swarrow \quad \searrow & & \swarrow \quad \searrow \\ \text{---} | (P_* \rightarrow 0)| & & |(P_2 \rightarrow P_*)| \text{---} & & | (P_* \rightarrow 0)| \text{---} \end{array}$$

Main Theorem

Two situations:

- (1) \mathcal{C} is the Amiot cluster category of a Jacobi-finite quiver with potential, with initial cluster-tilting subcategory $\text{add } T$.
- (2) \mathcal{C} is a Krull–Schmidt stably 2-Calabi–Yau Frobenius exact category, with cluster-tilting subcategory $\text{add } T$.

Write $A = \text{End}_{\mathcal{C}}(T)^{\text{op}}$ with e corresponding to projective summands (so $e = 0$ in case (1)).

Theorem (FGPPP)

In situations (1) and (2), there is a full and dense functor $G: \mathcal{C} \rightarrow \text{harp}_e A$ given by

$$GX = (\text{Hom}_{\mathcal{C}}(T, T_1) \rightarrow \text{Hom}_{\mathcal{C}}(T, T_0))$$

for $X \rightarrow T_1 \rightarrow T_0$ with $T_i \in \text{add } T$ either a carefully chosen triangle (1) or arbitrary short exact sequence (2). We have

$$\ker G = \begin{cases} (T \rightarrow \Sigma^{-1}T), & (1) \\ 0. & (2) \end{cases}$$

Preservation of structure

Give \mathcal{C} the relative extriangulated structure \mathbb{E}_T with extriangles $X \rightarrow Y \rightarrow Z$ such that

$$\mathrm{coind}_T(Y) = \mathrm{coind}_T(X) + \mathrm{coind}_T(Z).$$

Proposition (Padrol–Palu–Pilaud–Plamondon)

The injectives and projectives in $(\mathcal{C}, \mathbb{E}_T)$ are given respectively by (the preimage under stabilisation of) $\mathrm{add}\, T$ and $\mathrm{add}\, \Sigma^{-1} T$ respectively.

Proposition (FGPPP)

If \mathcal{C} is extriangulated and $\mathcal{I} \subseteq (\mathrm{inj} \rightarrow \mathrm{proj})$ is an ideal, then \mathcal{C}/\mathcal{I} is naturally extriangulated.

Theorem (FGPPP)

Using the extriangulated structure induced from \mathbb{E}_T on $\mathcal{C}/\ker G$, we obtain an equivalence

$$\mathcal{C}/\ker G \xrightarrow{\sim} \mathrm{harp}_e A$$

of extriangulated categories.

Corollary

In case (1), if A is selfinjective then $(\mathcal{C}, \mathbb{E}_T) \simeq \text{harp } A$.

Proof.

We have $\Sigma^2 T = T$ because A is selfinjective (Koenig–Zhu, Iyama–Oppermann)
so

$$\text{Hom}_{\mathcal{C}}(T, \Sigma^{-1} T) = \text{Hom}_{\mathcal{C}}(T, \Sigma T) = 0$$

because T is rigid. □

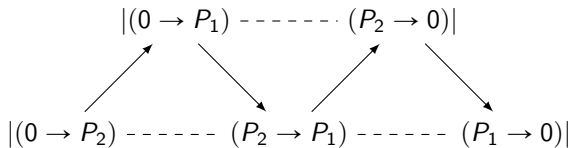
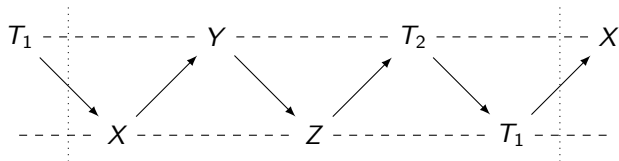
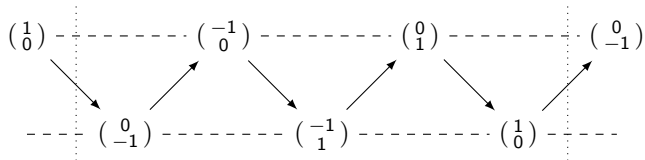
Corollary

In case (2), $\text{harp}_e A$ is exact.

Proof.

Since \mathcal{C} is exact, so is $(\mathcal{C}, \mathbb{E}_T) \simeq \text{harp}_e(A)$ (Auslander–Solberg). □

Example 1 (A_2 cluster category)



Example 2 (A_2 preprojective algebra)

