An $\mathcal{X}$-cluster character

joint work with Jan E. Grabowski

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The theory of cluster algebras and its applications

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Slides: `https://bit.ly/mdp-abu-dhabi`

General philosophy

- Start with a $\mathbb{K}$-linear, Krull–Schmidt, Frobenius, stably 2-Calabi–Yau, algebraic extriangulated category $\mathcal{C}$, with cluster-tilting subcategories.
- Extract various pieces of data from $\mathcal{C}$ and its cluster-tilting subcategories.

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- Explain how this data transforms under mutation of cluster-tilting subcategories.
- Show that, *under the correct additional assumptions*, we recover cluster-theoretic data in the sense of Fomin–Zelevinsky.
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[1] Fock–Goncharov $\mathcal{A}$ = Fomin–Zelevinsky x, Fock–Goncharov $\mathcal{X}$ = Fomin–Zelevinsky y

Grothendieck groups

- Simplifying assumptions for today:
  - $\mathcal{C}$ is Hom-finite,
  - has cluster-tilting objects ($\rightsquigarrow$ finite rank cluster algebras), and
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- Let $\mathcal{T} \subseteq \mathcal{C}$ be cluster-tilting:
$$\mathcal{T} = \{X \in \mathcal{C} : \mathrm{Ext}^1_{\mathcal{C}}(X, \mathcal{T}) = 0\} = \{X \in \mathcal{C} : \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T}, X)\}$$
- $\rightsquigarrow$ Grothendieck groups $\mathrm{K}_0(\mathcal{T})$ and $\mathrm{K}_0(\mathrm{fd}\,\mathcal{T})$, for $\mathrm{fd}\,\mathcal{T} = \{M \colon \mathcal{T}^{\mathrm{op}} \to \mathrm{fd}\,\mathbb{K}\}$ = finite-dimensional $\mathcal{T}$-modules.

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- Both free of (finite) rank $\#(\mathrm{indec}\,\mathcal{T})$, each $T \in \mathrm{indec}\,T$ indexes dual basis vectors $[T] \in \mathrm{K}_0(\mathcal{T})$ and $[S^{\mathcal{T}}_T] \in \mathrm{K}_0(\mathrm{fd}\,\mathcal{T})$:

$$S^{\mathcal{T}}_T(T') = \begin{cases} \mathbb{K}, & T' = T \\ 0, & \text{otherwise.} \end{cases}$$

Index and coindex

- Fix $\mathcal{T} \subseteq \mathcal{C}$ cluster-tilting, and let $X \in \mathcal{C}$. Then there are conflations

$$\underbrace{K_{\mathcal{T}}X \rightarrowtail R_{\mathcal{T}}}_{\in\mathcal{T}}X \twoheadrightarrow X \dashrightarrow \qquad X \rightarrowtail \underbrace{L_{\mathcal{T}}X \twoheadrightarrow C_{\mathcal{T}}X}_{\in\mathcal{T}} \dashrightarrow .$$

- $\rightsquigarrow \operatorname{ind}^{\mathcal{T}} X = [R_{\mathcal{T}}X]-[K_{\mathcal{T}}X]$, $\operatorname{coind}^{\mathcal{T}} X = [L_{\mathcal{T}}X]-[C_{\mathcal{T}}X] \in \mathrm{K}_0(\mathcal{T})$.

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- For $\mathcal{T},\mathcal{U} \subseteq \mathcal{C}$ cluster-tilting, $\mathrm{ind}^{\mathcal{T}}_{\mathcal{U}}, \mathrm{coind}^{\mathcal{T}}_{\mathcal{U}} \colon \mathrm{K}_0(\mathcal{U}) \to \mathrm{K}_0(\mathcal{T})$.

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- Duality (over $\mathbb{Z}$): $\overline{\operatorname{ind}}^{\mathcal{T}}_{\mathcal{U}} \coloneqq (\operatorname{coind}^{\mathcal{U}}_{\mathcal{T}})^* \colon \mathrm{K}_0(\operatorname{fd}\mathcal{T}) \xrightarrow{\sim} \mathrm{K}_0(\operatorname{fd}\mathcal{U})$, $\overline{\operatorname{coind}}^{\mathcal{T}}_{\mathcal{U}} \coloneqq (\operatorname{ind}^{\mathcal{U}}_{\mathcal{T}})^*$.
- Cluster dictionary: $\operatorname{ind} \leftrightarrow$ $\mathbf{g}$-vector, $\overline{\operatorname{ind}} \leftrightarrow$ $\mathbf{c}$-vector.

Exchange matrices

- All of the above applies to the triangulated stable category $\underline{\mathcal{C}}$, with $\{\mathcal{T} \subseteq \mathcal{C} \text{ cluster-tilting}\} = \{\underline{\mathcal{T}} \subseteq \underline{\mathcal{C}} \text{ cluster-tilting}\}$

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Let $\mathcal{T} \subseteq \mathcal{C}$ be cluster-tilting. Then $\mathrm{E}^{\mathcal{T}} = \mathsf{Ext}^1_{\mathcal{C}}(-,\mathcal{T}) \colon \mathcal{C}/\mathcal{T} \xrightarrow{\sim} \mathsf{fd}\,\underline{\mathcal{T}}$, and there is a linear map $\beta_{\mathcal{T}} \colon \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}}) \to \mathrm{K}_0(\mathcal{T})$ such that

$$\beta_{\mathcal{T}}[\mathrm{E}^{\mathcal{T}} X] = \mathsf{coind}^{\mathcal{T}}(X) - \mathsf{ind}^{\mathcal{T}}(X).$$

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Theorem (Palu '09, ..., Grabowski–P '24$^+$)

For any cluster-tilting $\mathcal{T}, \mathcal{U} \subseteq \mathcal{C}$, there are commutative diagrams

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{fd}\,\underline{\mathcal{U}}) \arrow[r, "\beta_{\mathcal{U}}"] \arrow[d, "\overline{\mathrm{ind}}^{\mathcal{T}}_{\mathcal{U}}"'] & \mathrm{K}_0(\mathcal{U}) \arrow[d, "\mathrm{ind}^{\mathcal{T}}_{\mathcal{U}}"] \\
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Corollary

For $\mathcal{T}$ and $\mathcal{U}$ related by mutation (away from loops or 2-cycles), the matrices of $\beta_{\mathcal{T}}$ and $\beta_{\mathcal{U}}$ with respect to the standard bases are related by Fomin–Zelevinsky matrix mutation.

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Warning: $\mathrm{ind}_{\mathcal{U}}^{\mathcal{T}} \circ \mathrm{ind}_{\mathcal{V}}^{\mathcal{U}} \neq \mathrm{ind}_{\mathcal{V}}^{\mathcal{T}}$ etc. (but $\beta_{\mathcal{T}} \circ \mathrm{ind}_{\mathcal{U}}^{\mathcal{T}} \circ \mathrm{ind}_{\mathcal{V}}^{\mathcal{U}} = \beta_{\mathcal{T}} \circ \mathrm{ind}_{\mathcal{V}}^{\mathcal{T}}$).

Example

Corollary

For $\mathcal{T}$ and $\mathcal{U}$ related by mutation (away from loops or 2-cycles), the matrices of $\beta_{\mathcal{T}}$ and $\beta_{\mathcal{U}}$ with respect to the standard bases are related by Fomin–Zelevinsky matrix mutation.

$$\mathcal{U}: \quad 1 \to 2 \to 3$$

$$\mathcal{T} = \mu_2\mathcal{U}: \quad \begin{tikzcd} 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{U}}) \arrow[r, "\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"'] & \mathrm{K}_0(\mathcal{U}) \arrow[d, "\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] \\ \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}}) \arrow[r, "\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}"'] & \mathrm{K}_0(\mathcal{T}) \end{tikzcd}$$

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Definition

Say $(\mathcal{C}, \mathcal{T})$ has a cluster structure if the quiver of $\mathcal{U}$ has no loops or 2-cycles for any $\mathcal{U} \overset{\text{mut}}{\sim} \mathcal{T}$.

$\mathcal{A}$-cluster character reminder

- $M \in \mathsf{fd}\,\mathcal{T}$ has $\mathcal{F}$-polynomial

$$\mathcal{F}(M) = \sum_{[L] \in \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(M)) x^{[L]} \in \mathbb{K}\mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}}).$$

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Example

$$\mathcal{T}: \begin{tikzcd} 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd} \qquad M = \begin{tikzcd} \mathbb{K} \arrow[dr, "1"'] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd}$$

$$\mathcal{F}(M) = 1 + 2x_2 + x_2^2 + x_1x_2 + x_1x_2^2 = (1 + x_2 + x_1x_2)(1 + x_2)$$

- $X \in \mathcal{C}$ has $\mathcal{A}$-cluster character

$$\begin{aligned} \mathrm{CC}^{\mathcal{T}}_{\mathcal{A}}(X) &= x^{\operatorname{ind}^{\mathcal{T}}(X)} (\beta_{\mathcal{T}})_* \mathcal{F}(\mathrm{E}^{\mathcal{T}} X) \\ &= a^{\operatorname{ind}^{\mathcal{T}}(X)} \sum_{[L] \in \mathrm{K}_0(\operatorname{fd} \underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(\mathrm{E}^{\mathcal{T}X})) a^{\beta_{\mathcal{T}}[L]} \in \mathbb{K}\mathrm{K}_0(\mathcal{T}). \end{aligned}$$

$\mathcal{A}$-cluster character reminder

- $M \in \mathsf{fd}\,\mathcal{T}$ has $\mathcal{F}$-polynomial

$$\mathcal{F}(M) = \sum_{[L] \in \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(M)) x^{[L]} \in \mathbb{K}\mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}}).$$

- $X \in \mathcal{C}$ has $\mathcal{A}$-cluster character

$$\begin{aligned} \mathsf{CC}_{\mathcal{A}}^{\mathcal{T}}(X) &= x^{\mathsf{ind}^{\mathcal{T}}(X)} (\beta_{\mathcal{T}})_* \mathcal{F}(\mathrm{E}^{\mathcal{T}} X) \\ &= a^{\mathsf{ind}^{\mathcal{T}}(X)} \sum_{[L] \in \mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(\mathrm{E}^{\mathcal{T} X})) a^{\beta_{\mathcal{T}}[L]} \in \mathbb{K}\mathrm{K}_0(\mathcal{T}). \end{aligned}$$

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- $X \in \mathcal{C}$ has $\mathcal{A}$-cluster character

$$\begin{aligned} \mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(X) &= x^{\operatorname{ind}^{\mathcal{T}}(X)} (\beta_{\mathcal{T}})_* \mathcal{F}(\mathrm{E}^{\mathcal{T}} X) \\ &= a^{\operatorname{ind}^{\mathcal{T}}(X)} \sum_{[L] \in \mathrm{K}_0(\operatorname{fd} \underline{\mathcal{T}})} \chi(\mathrm{Gr}_{[L]}(\mathrm{E}^{\mathcal{T} X})) a^{\beta_{\mathcal{T}}[L]} \in \mathbb{K}\mathrm{K}_0(\mathcal{T}). \end{aligned}$$

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$$\mathrm{E}^{\mathcal{T}} X = \begin{tikzcd} \mathbb{K} \arrow[dr, "1"'] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[ll] \\ & \mathbb{K} \arrow[ur] & \end{tikzcd} \qquad \beta_{\mathcal{T}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\mathcal{F}(\mathrm{E}^{\mathcal{T}} X) = 1 + 2x_2 + x_2^2 + x_1 x_2 + x_1 x_2^2$$

$$\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(X) = a_1 a_2^{-2} a_3 (1 + 2a_1^{-1} a_3 + a_1^{-2} a_3^2 + a_1^{-1} a_2 + a_1^{-2} a_2 a_3)$$

$\mathcal{X}$-cluster character

- Inputs to the $\mathcal{X}$-cluster character are $M \in \mathsf{fd}\,\underline{\mathcal{U}}$ for $\mathcal{U} \subseteq \mathcal{C}$ cluster-tilting.
- $\rightsquigarrow M_{\mathcal{U}}^{\pm} \in \mathcal{U}$ such that $\beta_{\mathcal{U}}[M] = [M_{\mathcal{U}}^{+}] - [M_{\mathcal{U}}^{-}]$.

$$\mathsf{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(M) = x^{\underline{\overline{\mathsf{ind}}}_{\mathcal{U}}^{\mathcal{T}}[M]} \frac{\mathcal{F}(\mathrm{E}^{\mathcal{T}} M_{\mathcal{U}}^{+})}{\mathcal{F}(\mathrm{E}^{\mathcal{T}} M_{\mathcal{U}}^{-})} \in \mathsf{Frac}(\mathbb{K}\mathrm{K}_0(\mathsf{fd}\,\underline{\mathcal{T}})).$$

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$$\mathrm{CC}^{\mathcal{T},\mathcal{U}}_{\mathcal{X}}(M) = x^{\overline{\mathrm{ind}}^{\mathcal{T}}_{\mathcal{U}}[M]} \frac{\mathcal{F}(\mathrm{E}^{\mathcal{T}} M^{+}_{\mathcal{U}})}{\mathcal{F}(\mathrm{E}^{\mathcal{T}} M^{-}_{\mathcal{U}})} \in \mathrm{Frac}(\mathbb{K}\mathrm{K}_0(\mathrm{fd}\,\underline{\mathcal{T}})).$$

Proposition

$[M] = [L] + [N] \implies \mathrm{CC}^{\mathcal{T},\mathcal{U}}_{\mathcal{X}}(M) = \mathrm{CC}^{\mathcal{T},\mathcal{U}}_{\mathcal{X}}(L)\,\mathrm{CC}^{\mathcal{T},\mathcal{U}}_{\mathcal{X}}(N)$.

- $\rightsquigarrow$ consider the values of $\mathrm{CC}^{\mathcal{T},\mathcal{U}}_{\mathcal{X}}$ on simple U-modules.

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Theorem (Grabowski–P '24^{+})

Assume $(\mathcal{C},\mathcal{T})$ has a cluster structure. Then the $\mathrm{CC}^{\mathcal{T},\mathcal{U}}_{\mathcal{X}}(S^{\mathcal{U}}_{U})$, for all $\mathcal{U} \overset{mut}{\sim} \mathcal{T}$ and $U \in \mathrm{indec}\,\mathcal{U}$, are the $\mathcal{X}$-cluster variables of the cluster algebra associated to the quiver of $\underline{\mathcal{T}}$.

Remarks

Theorem (Grabowski–P '24$^+$)

Assume $(\mathcal{C}, \mathcal{T})$ has a cluster structure. Then the $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_U^{\mathcal{U}})$, for all $\mathcal{U} \overset{mut}{\sim} \mathcal{T}$ and $U \in \mathrm{indec}\,\mathcal{U}$, are the $\mathcal{X}$-cluster variables of the cluster algebra associated to the quiver of $\underline{\mathcal{T}}$.

- The map implicit in the theorem is surjective but not injective, but induces a bijection between exchange pairs for $(\mathcal{C}, \mathcal{T})$ and $\mathcal{X}$-cluster variables (thanks to Cao–Keller–Qin '24).
- For $S = S_U^{\mathcal{U}}$, the objects $S_{\mathcal{U}}^{\pm}$ are the middle terms of exchange conflations $U^* \rightarrowtail S_{\mathcal{U}}^{+} \twoheadrightarrow U$, $U \rightarrowtail S_{\mathcal{U}}^{-} \twoheadrightarrow U^*$.

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- To include $\mathcal{X}$-variables at frozen vertices, we give an ad hoc definition of $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}$ on simple modules at these vertices.

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Assume $(\mathcal{C}, \mathcal{T})$ has a cluster structure. Then the $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_U^{\mathcal{U}})$, for all $\mathcal{U} \overset{mut}{\sim} \mathcal{T}$ and $U \in \operatorname{indec}\mathcal{U}$, are the $\mathcal{X}$-cluster variables of the cluster algebra associated to the quiver of $\underline{\mathcal{T}}$.

- The map implicit in the theorem is surjective but not injective, but induces a bijection between exchange pairs for $(\mathcal{C}, \mathcal{T})$ and $\mathcal{X}$-cluster variables (thanks to Cao–Keller–Qin '24).
- For $S = S_U^{\mathcal{U}}$, the objects $S_{\mathcal{U}}^{\pm}$ are the middle terms of exchange conflations $U^* \rightarrowtail S_{\mathcal{U}}^+ \twoheadrightarrow U$, $U \rightarrowtail S_{\mathcal{U}}^- \twoheadrightarrow U^*$.
- To include $\mathcal{X}$-variables at frozen vertices, we give an ad hoc definition of $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}$ on simple modules at these vertices.
- For $M \in \operatorname{fd}\underline{\mathcal{U}}$, we have $(\beta_{\mathcal{T}})_* \,\mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(M) = \frac{\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(M_{\mathcal{U}}^+)}{\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(M_{\mathcal{U}}^-)}$.

Example 1

$\mathcal{U}: \; 1 \to 2 \to 3 \qquad\qquad \mathcal{T} = \mu_2\mathcal{U}:$

$$\begin{tikzcd} 1 \arrow[rr] & & 3 \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd}$$

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$$\mathrm{E}^{\mathcal{T}}(U_2) = S_2^{\mathcal{T}} \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1} \left(\tfrac{1}{1}\right) = x_2^{-1}$$

$$\mathrm{E}^{\mathcal{T}}(U_3) = 0 \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_3^{\mathcal{U}}) = x_3 \left(\tfrac{1+x_2}{1}\right) = x_3 (1 + x_2)$$

Example 2

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$$\mathrm{E}^{\mathcal{T}}(U_2) = \ \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0 \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1}\left(\frac{1+x_1}{1+x_3+x_2x_3+x_1x_2x_3}\right)$$

$$\mathrm{E}^{\mathcal{T}}(U_3) = \ \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \qquad \mathrm{CC}_{\mathcal{X}}^{\mathcal{T},\mathcal{U}}(S_3^{\mathcal{U}}) = x_3^{-1}(1 + x_2 + x_1x_2)$$

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Thank you! / !شكراً / 谢谢!

Bonus slide

$$
\begin{tikzcd}[column sep=small, row sep=small]
T_2 \arrow[dr] \arrow[rr, dash, dashed] & & U_4 \arrow[dr] \arrow[rr, dash, dashed] & & U_2 \arrow[dr] \arrow[rr, dash, dashed] & & T_2 \arrow[dr] & \\
& T_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & U_3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & U_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & T_1 \\
Z_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_2 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_1 \arrow[ur] \arrow[dr] & \\
& Z_2 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_2 \\
U_3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & U_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & T_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & U_3 \arrow[ur] \arrow[dr] & \\
& U_2 \arrow[ur] \arrow[rr, dash, dashed] & & T_2 \arrow[ur] \arrow[rr, dash, dashed] & & U_4 \arrow[ur] \arrow[rr, dash, dashed] & & U_2
\end{tikzcd}
$$

- $T_2 \longrightarrow T_1 \circlearrowleft$ $\rightsquigarrow$ no cluster structure!

Bonus slide

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Z_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_2 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & Z_1 \arrow[ur] \arrow[dr] & \\
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& U_2 \arrow[ur] \arrow[rr, dash, dashed] & & T_2 \arrow[ur] \arrow[rr, dash, dashed] & & U_4 \arrow[ur] \arrow[rr, dash, dashed] & & U_2
\end{tikzcd}
$$

- $T_2 \longrightarrow T_1 \circlearrowleft$ $\rightsquigarrow$ no cluster structure!
- $\mathrm{CC}^{\mathcal{T}}_{\mathcal{A}}(U_1) = a_1^{-1}(1 + a_2 + a_2^2)$,
  $\mathrm{CC}^{\mathcal{T}}_{\mathcal{A}}(U_2) = a_2^{-1}(1 + a_1^{-1} + a_1^{-1}a_2 + a_1^{-1}a_2^2)$,...
- $\rightsquigarrow$ generalised cluster variables (Chekhov–Shapiro)