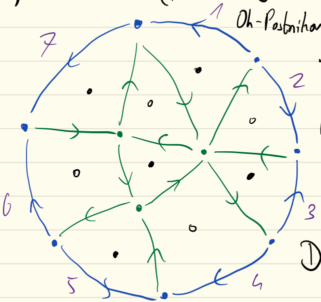
arXiv:2307.13369

Positroid varieties via rep. theory

1) Positroids etc

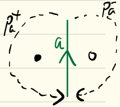
(Çanakçı–King–P remix, see also Postnikov, Oh–Postnikov–Speyer, Knutson–Lam–Speyer, ...)



Input: consistent dimer quiver Q on the disc, with faces $Q_2^\bullet \sqcup Q_2^\circ$.

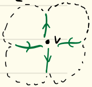
Dimer algebra $A = A_Q$:

$$A = \widehat{\mathbb{C}Q} \Big/ \overline{\langle p_a^+ = p_a^- : a \in Q_1 \text{ internal} \rangle}$$



$$= J(Q, F, W), \quad F = \partial Q, \quad W = \sum_{f \in Q_2^+} \partial f - \sum_{f \in Q_2^-} \partial f$$

For $v \in Q_0$, choose path $t_v : v \to v$ bounding a face $\to$ in A, indep. of choice, get $t = \sum_{v \in Q_0} t_v$.



$\Rightarrow A$ is a Z-algebra for $Z = \mathbb{C}[[t]]$.

Λ Z-algebra $\rightsquigarrow$ $CM(\Lambda) = \{M \in \operatorname{mod}\Lambda : {}_Z M \text{ free + f.g.}\}$ (i.e. ${}_Z M \in CM(Z)$)

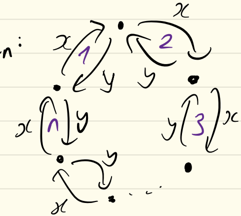
For $M \in CM(A)$, $\operatorname{rk}_Z M(v) =: \operatorname{rk} M$ is indep. of $v \in Q_0$.

Prop (ÇKP) Consistency $\Rightarrow P_v = Ae_v \in CM(A)$, $\operatorname{rk} P_v = 1$ $\forall v \in Q_0$ ($\Leftarrow$ [Berggren–Serhiyenko])

In particular, $A \in CM(A)$.

$$e = \sum_{v \in F_0 = \partial Q_0} e_v \rightsquigarrow B := eAe \quad \text{boundary algebra}$$

Q_n:



$$\Pi_n = \widehat{\mathbb{C}Q_n} \Big/ \overline{(xy - yx)}$$

($\hat{A}$-type preprojective algebra)

For $0 < k < n$:

$$C = C_{k,n} = \Pi_n \Big/ \overline{(y^k - x^{n-k})}$$

$\binom{n}{k} := \{ I \subseteq \{1,\dots,n\} : |I| = k \}$.

Given $I \in \binom{n}{k}$, define $M_I \in CM(C)$ by

$$M_I(v) = Z \ \forall v, \qquad M_I(x_i) = \begin{cases} t, & i \in I \\ 1, & i \notin I \end{cases} \qquad M_I(y_i) = \begin{cases} 1, & i \in I \\ t, & i \notin I \end{cases}$$

Thm (Jensen-King-Su '16)

1) $C \in CM(C)$ (for Z-algebra structure $t = xy = yx$).
2) $M \in CM(C)$ rk 1 $\Leftrightarrow$ $M \cong M_I$ for $I \in \binom{n}{k}$.

(Note: $M_I \cong M_J \Rightarrow I = J$.)

Rem Similar classification of $N \in CM(A)$ rk 1 in terms of perfect matchings of Q $\rightsquigarrow$ critical ingredient in proofs!

Obs $\exists$ canonical map $\Pi_n \to B$.

Prop (CKP) $\exists!\ 0 < k < n$ s.t.

$$\begin{tikzcd} \Pi_n \arrow[rr] \arrow[dr] & & B \\ & C_{k,n} \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

Moreover, res: $CM(B) \xhookrightarrow{\text{f.f.}} CM(C)$.

Def $\mathcal{P} = \{ I \in \binom{n}{k} : M_I \in CM(B) \}$ is a (connected) positroid.

2) Positroid varieties

$Gr_{k,n} = \{ V \leq \mathbb{C}^n : \dim V = k \}$ Grassmannian $\rightsquigarrow$ proj. variety.

$\mathbb{C}[\widehat{Gr}_{k,n}] = \mathbb{C}[\Delta_I : I \in \binom{n}{k}] / (\text{Plücker relations})$

$\uparrow$ Plücker coordinate

$\Pi_{\mathcal{P}} = \{ V \in Gr_{k,n} : \Delta_I(V) = 0 \ \forall I \notin \mathcal{P} \}$ (closed) positroid variety.

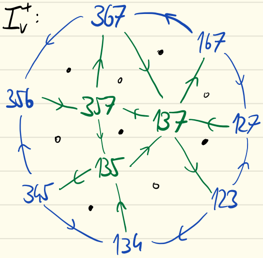
$v \in Q_0 \rightsquigarrow T_v^+ = eAe_v,\ T_v^- = (e_vAe)^\vee \in CM(B) \subseteq CM(C)$

$(-)^\vee = \mathrm{Hom}_Z(-, Z)$.

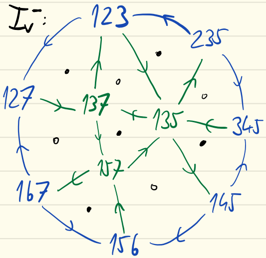
Consistency $\overset{[CKP]}{\Longrightarrow}$ $\mathrm{rk}\, T_v^\pm = 1$ $\overset{[JKS]}{\Longrightarrow}$ $T_v^\pm \cong M(I_v^\pm)$

$\rightsquigarrow$ two labellings of Q_0 by elements of $\binom{n}{k}$.

I_v^+:



I_v^-:



$$\mathcal{F}^\pm = \{ I_v^\pm : v \in F_0 = \partial Q_0 \}$$

$$\Pi_\mathcal{P}^\circ = \{ V \in \Pi_\mathcal{P} : \Delta_I(V) \neq 0 \ \forall I \in \mathcal{F}^\pm \}$$

open positroid variety

A_Q = cluster alg. associated to (Q, F), invertible frozen vars.

Thm (Galashin–Lam) Two isomorphisms

$$\eta^\pm : A_Q \xrightarrow{\sim} \mathbb{C}[\hat{\Pi}_\mathcal{P}^\circ]. \qquad \eta^\pm(x_j) = \Delta(I_j^\pm).$$

Upshot Two cluster algebra structures on $\mathbb{C}[\hat{\Pi}_\mathcal{P}]$.

Rem Special case $\mathcal{P} = \binom{[n]}{k} \Rightarrow \Pi_\mathcal{P} = Gr_{k,n}$.
$\Pi_\mathcal{P}^\circ$ = big cell (dense).

In (only) this case, cluster structures $\eta^\pm$ agree (Scott '06).

Conj (Muller–Speyer '17) $\eta^\pm$ quasi-coincide:

1) $\forall f \in A_Q$ frozen var., $\exists p$ Laurent mon. in frozens with
$$\eta^-(f) = \eta^+(p)$$
and $\forall x \in A_Q$ clust. var. $\exists x'$ clust. var., q Laurent in frozens:
$$\eta^-(x) = \eta^+(x'q)$$

2) $x \longmapsto x'$ permutation of clust. vars, respecting compatibility, mutation.
3) technical 'balancing' condition on monomials p, q.

Thm (P '23+) The conjecture is true.

Rem 1) Both conjecture and theorem apply also to disconnected case.
2) Independent proof by Casals–Le–Sherman-Bennett–Weng, using methods from symplectic geometry.

3) Proof strategy

$$\underline{\text{gproj}}\, CMB = \{X \in CM(B) : \mathrm{Ext}^{>0}_B(X,B)=0\} \subseteq CM(B)$$
$$\text{ginj}\, CMB = \{X \in CM(B) : \mathrm{Ext}^{>0}_B(B^\vee,X)=0\}$$

Thm (P '22)

1) Both are stably 2-CY Frobenius exact categories.

2) $T^+ = Ae \in \text{gproj}\, CMB$, $T^- = (eA)^\vee \in \text{ginj}\, CMB$ are cluster-tilting

3) $\mathrm{End}_B(T^+)^{op} \cong A \cong \mathrm{End}_B(T^-)^{op}$.

[Fu-Keller] $\Rightarrow$ clust. characters
$$\Psi^+ : \text{gproj}\, CM(B) \to \mathcal{A}_Q \xrightarrow[\sim]{\iota^+} \mathbb{C}[\hat{\Pi}^\circ_P]$$
$$\Psi^- : \text{ginj}\, CM(B) \to \mathcal{A}_Q \xrightarrow[\sim]{\iota^-} \mathbb{C}[\hat{\Pi}^\circ_P]$$

inducing bijections of reachable rigid objects with cluster monomials, ...

Prop (P) $\Psi^\pm(M_I) = \Delta_I$ (when M_I in domain).

[Fraser-Keller] To prove the conjecture, find $\varphi : D^b(\text{gproj}\, CMB) \to D^b(\text{ginj}\, CMB)$ such that:

1)
$$\begin{tikzcd}
K^b(\text{add}\, T^-) \arrow[r] \arrow[d, "\tilde{\varphi}"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b(\text{ginj}\, CMB) \arrow[r] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowleft"] & \underline{\text{ginj}}\, CM(B) \arrow[d, "\underline{\varphi}"] \\
K^b(\text{add}\, T^+) \arrow[r] & D^b(\text{gproj}\, CMB) \arrow[r] & \underline{\text{gproj}}\, CM(B)
\end{tikzcd}$$

2)
$$\begin{tikzcd}
\text{add}\, T^- \arrow[r, "\Psi^-"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}[\hat{\Pi}^\circ_P] \arrow[d, equal] \\
D^b(\text{gproj}\, CMB) \arrow[r, "\Psi^+"] & \mathbb{C}[\hat{\Pi}^\circ_P]
\end{tikzcd}$$

3) $\underline{\varphi}$ is an equivalence

4) $\underline{\varphi}\, T^- \overset{mut}{\sim} T^+$

Sketch Proof $\varphi = D^b(\text{ginj}\, CMB) \xleftrightarrow{\sim} D^b(\text{mod}\, B) \xleftrightarrow{\sim} D^b(\text{gproj}\, CMB)$

$\rightsquigarrow \underline{\varphi} = \underline{\text{ginj}}\, CMB \xrightarrow{\sim} D_{sg}(B) \xleftarrow{\sim} \underline{\text{gproj}}\, CMB \Rightarrow (3)$.

Main step Show $T^+ = \Omega^2 T^- \in D_{sg}(B)$
(g-vector calculation, using ideas from ÇKP)

$\Rightarrow$ (4) by cluster theory (Ford-Serhiyenko)

$\Rightarrow$ (2) by geometry (Muller-Speyer + ÇKP)

1) $0 \to \underbrace{T^+ \to P_1 \to P_0}_{\xi} \to T^- \to 0$

$F = \mathrm{Hom}(T^+, -) : K^b(\mathrm{add}\, T^+) \xrightarrow{\sim} K^b(\mathrm{proj}\, A) = D^b(A)$ (P '22)

[ÇKP+ε]: in $D^b(A)$, $F\xi \cong A^\vee$ is tilting.

$\mathrm{End}_A(A^\vee)^{op} = A$, so:

$$\begin{tikzcd}
T^- \arrow[d, mapsto] & K^b(\mathrm{add}\, T^-) \arrow[d, "\tilde{\varepsilon}"] \arrow[dr, "T^- \mapsto T^-"] & \\
\xi & K^b(\mathrm{add}\, T^+) \arrow[r, "\xi \mapsto \xi \cong T^-"'] \arrow[ur, phantom, "\circlearrowleft"] & D^b(\mathrm{mod}\, B)
\end{tikzcd}$$

$\square$

<u>Rem</u> In practice: take $X \in \mathrm{ginj}\, CM(B)$ reachable rigid, i.e. $\psi^-(X) \in \mathbb{C}[\Pi_J^o]$ is a cluster monomial.

$$\begin{tikzcd}
0 \arrow[r] & \underset{\text{gproj } CMB}{\Omega X} \arrow[r] & P \arrow[r, "\text{proj cover}"] & X \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r, "\text{left proj}B \text{ approx}"'] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \underset{\text{gproj } CMB.}{X'} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Then $\psi^-(X) = \psi^+(Y) \dfrac{\psi^+(P)}{\psi^+(Q)} \in \mathbb{C}[\hat{\Pi}_J^o]$.