Representation theory and positroid varieties
表征理论和正拟阵簇

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Slides: https://bit.ly/mdp-icra24

The totally positive Grassmannian

Definition

$M \in \mathbb{C}^{k \times n}$, $k < n$, is *totally positive* if its maximal minors $\Delta_I(M)$ are positive real numbers.

- Here $I \in \binom{[n]}{k}$ is a subset of k columns, $\Delta_I(M)$ its determinant.
- If $\operatorname{rk} M = k$, its row span $[M]$ is in $\mathrm{Gr}_{k,n}$, the *Grassmannian*.
- Totally positive Grassmannian: $\mathrm{Gr}_{k,n}^{>0} = \{[M] : M \text{ is totally +ve}\}$.

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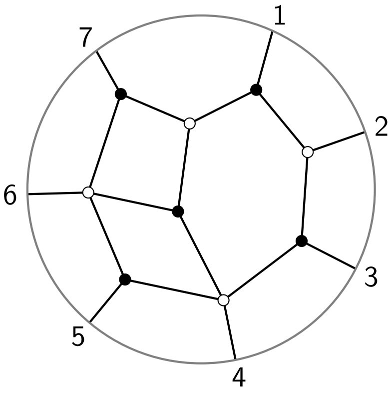
- $\overline{\mathrm{Gr}_{k,n}^{>0}} = \mathrm{Gr}_{k,n}^{\geqslant 0}$ decomposes into cells $\Pi_{\mathcal{P}}^{\circ} \cap \mathrm{Gr}_{k,n}^{\geqslant 0}$, indexed by *positroids* (正拟阵) $\mathcal{P} \subseteq \binom{[n]}{k}$.

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



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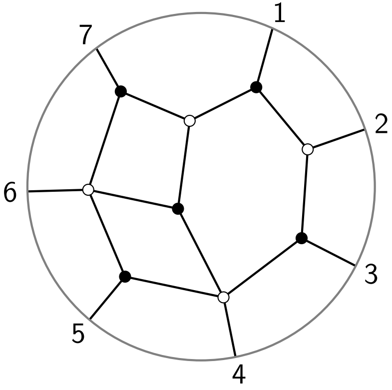

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$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$

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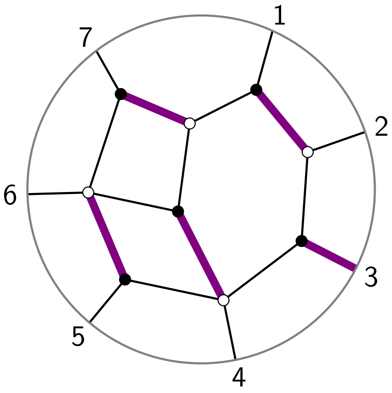

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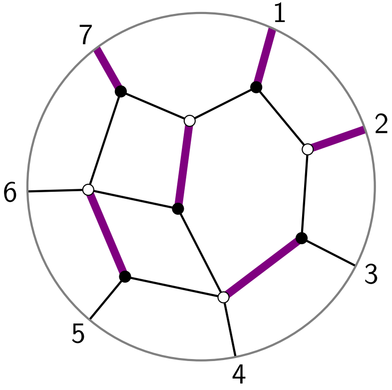

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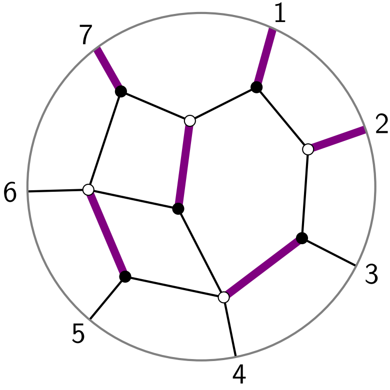

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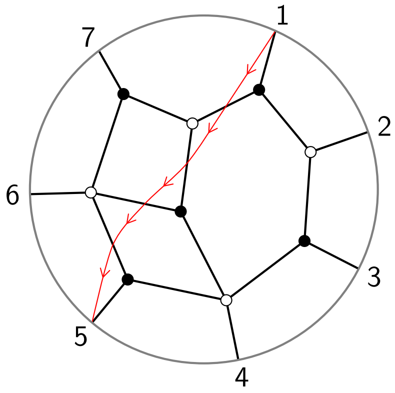

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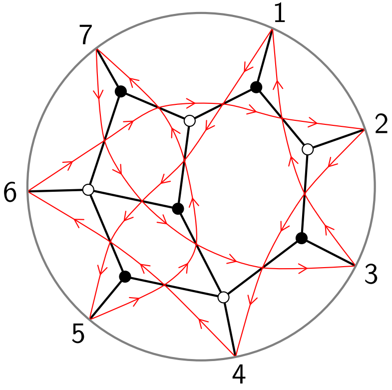

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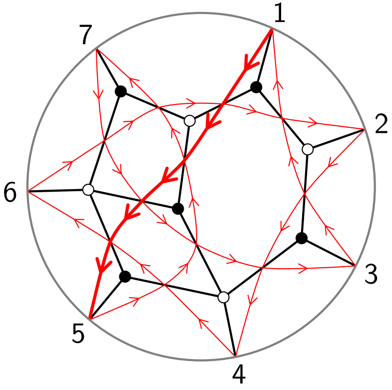

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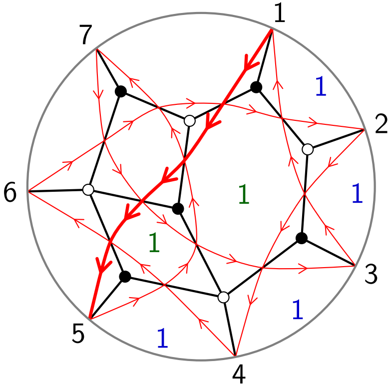

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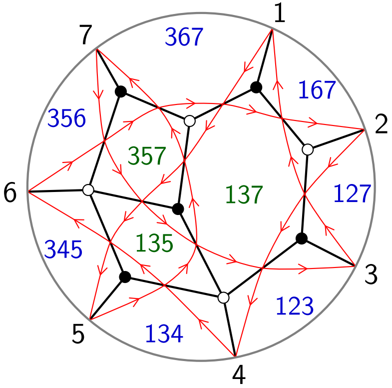

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$$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\} = \{157, 235, \dots\}$$

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$$\mathcal{C}^+ = \{\text{labels}\} = \{137, 167, 135, \dots\}$$

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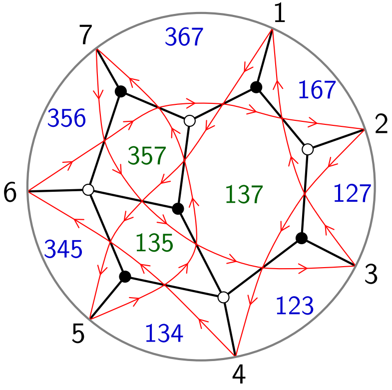

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$$\mathcal{F}^+ = \{\text{boundary labels}\}$$
$$= \{123, 134, 345, \dots\}$$

$$\Pi^\circ_{\mathcal{P}} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P},\ \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^+\}$$

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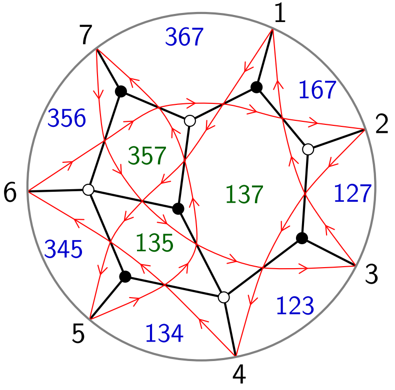

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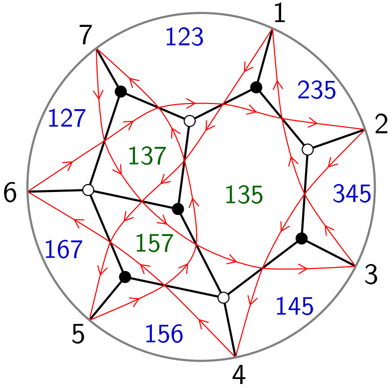

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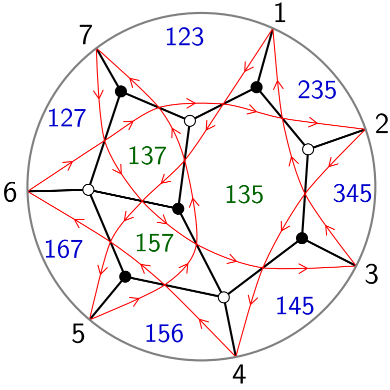

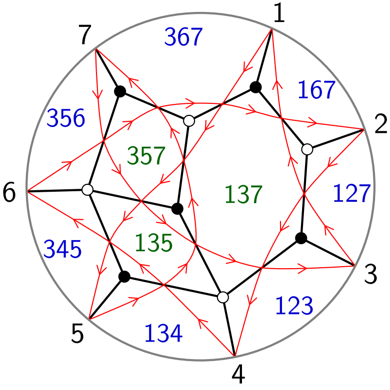

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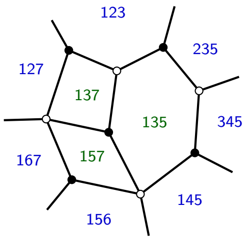
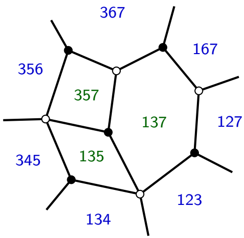
Cluster structures





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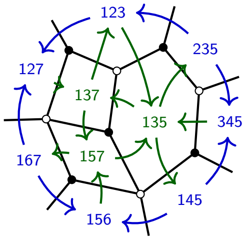

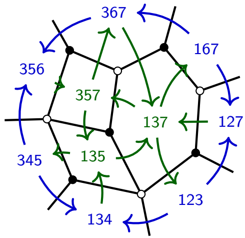

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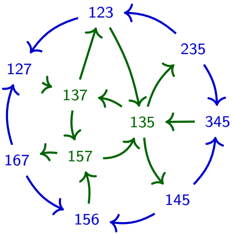
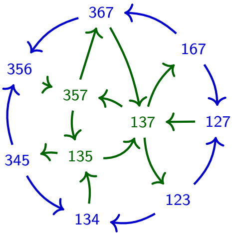




Theorem (Galashin–Lam '23)

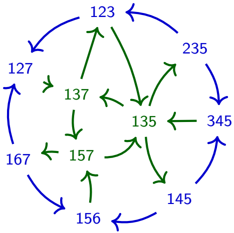
$\mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$ *has two natural cluster algebra structures:*
one cluster algebra $\mathscr{A}_{\mathcal{P}}$*, two isomorphisms* $\eta^{\pm}\colon \mathscr{A}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$*.*

Theorem (P '23$^+$, conj. Muller–Speyer '17)

The cluster structures $\eta^{\pm}\colon \mathscr{A}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$ *quasi-coincide.*

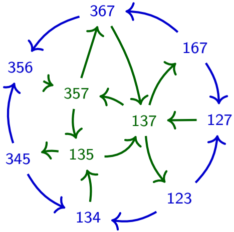
Casals–Le–Sherman-Bennett–Weng '23$^+$: alt. proof

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$



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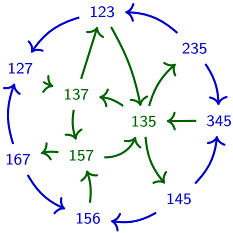

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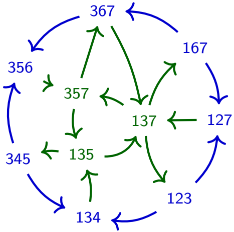

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Δ_{157}

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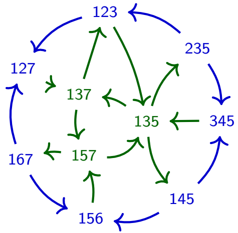

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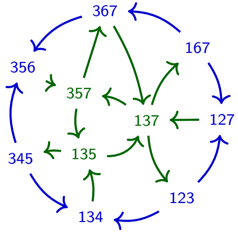

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$$\Delta_{157} = \frac{\Delta_{357}\Delta_{167} + \Delta_{137}\Delta_{567}}{\Delta_{367}}$$

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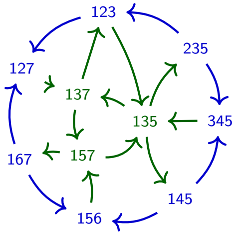

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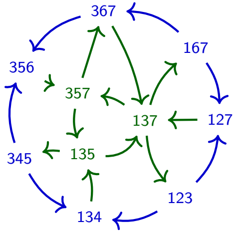

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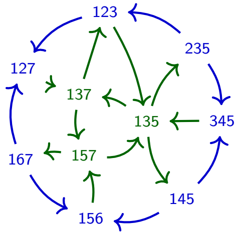

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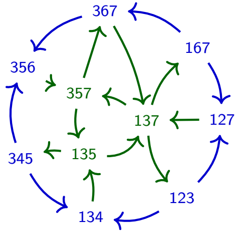

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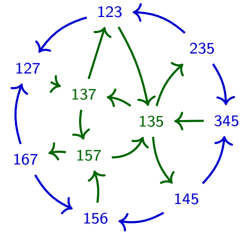

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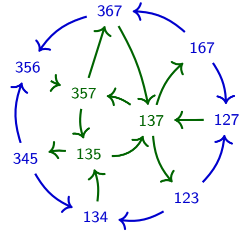

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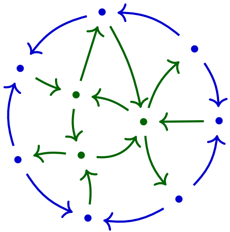
Categorification, part 1: combinatorics



$$A = \mathbb{C}\langle\langle Q \rangle\rangle / \overline{(\text{dimer relations})}$$

$$B = eAe$$

$$Z = \mathbb{C}[[t]], \ Z \subseteq B \subseteq A$$

$$\mathrm{CM}\, B = \{X \in \mathrm{mod}\, B : {}_Z X \text{ free + f.g.}\}$$

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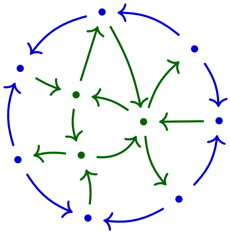

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Theorem (P '22)

For each (connected) positroid $\mathcal{P}$, the Frobenius exact category $\operatorname{gproj} \operatorname{CM} B$ *categorifies the cluster algebra* $\mathscr{A}_{\mathcal{P}}$.

- Key fact: A is internally 3-Calabi–Yau.
- $\operatorname{ginj} \operatorname{CM} B = \{X \in \operatorname{CM} B : \operatorname{Ext}_B^{>0}(B^\vee, X) = 0\}$ also categorifies.
- $\operatorname{gproj} \operatorname{CM} B \simeq \operatorname{ginj} \operatorname{CM} B$, but different subcategories of $\operatorname{CM} B$!

Categorification, part 2: geometry

- Jensen–King–Su '16: categorification CM C of $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}]$, with $M_I \in$ CM C for each Δ_I.

Theorem (Çanakçı–King–P '24, P '22)

- CM $B \hookrightarrow$ CM C, *with* $M_I \in$ CM $B \iff I \in \mathcal{P}$.
- $M_I \in \operatorname{gproj}$ CM $B \iff \Delta_I$ *is an* η^+*-cluster variable (source labelled).*
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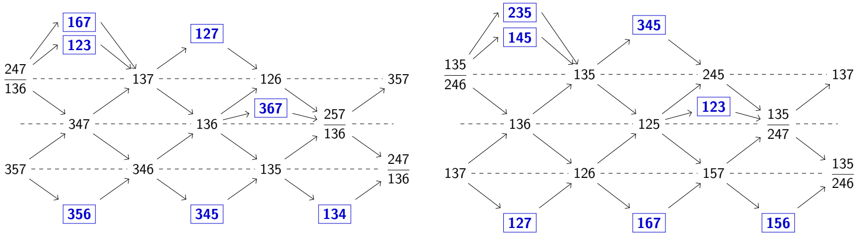

Proving the Muller–Speyer conjecture

- Reduce (geometrically) to connected positroids, for access to categorifications.
- Key fact: inclusions induce *derived* equivalences

$$\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathrm{CM}\,B) \xhookrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathrm{CM}\,B) \xhookleftarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathrm{CM}\,B)$$

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- Main step: show that the composition is a quasi-cluster functor (Fraser–Keller '23).
- E.g. induced equivalence $\underline{\mathsf{gproj}}\,\mathsf{CM}\,B \xrightarrow{\sim} \underline{\mathsf{ginj}}\,\mathsf{CM}\,B$ takes initial cluster-tilting object T^{+} to reachable cluster-tilting object $\Omega^2 T^{-}$.

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Theorem (P '23^{+}, conj. Muller–Speyer '16)

The cluster structures η^{+} and η^{-} quasi-coincide.

- Independent proof: (Casals–Le–Sherman-Bennett–Weng '23^{+}) Inspired by symplectic topology!

Computation

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$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] \arrow[d, equal] & P \arrow[r] & X \arrow[r] & 0 \\
& \Omega X & & &
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$$\begin{tikzcd}
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\begin{tikzcd}
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$$\begin{tikzcd}
0 \arrow[r] & M_{346} \arrow[r] & {\color{blue}M_{167} \oplus M_{345}} \arrow[r] & {\color{red}M_{157}} \arrow[r] & 0 \\
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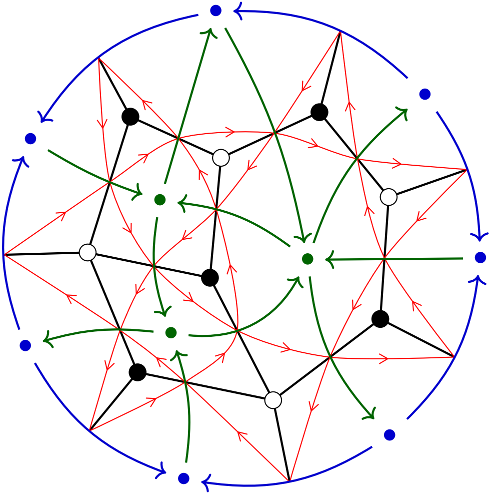

谢谢!