

# On categorification of $g$ -vectors

joint work in progress  
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Slides: <https://bit.ly/3SzZMPG>



## Definition 1: Coindex

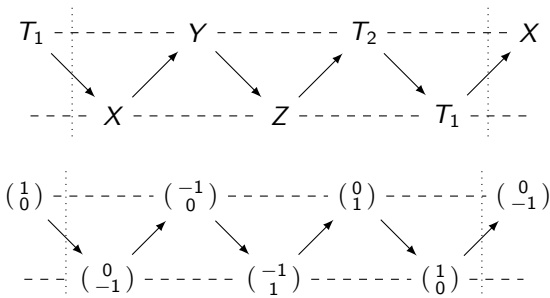
Let  $\mathcal{C}$  be a Hom-finite Krull–Schmidt 2-Calabi–Yau triangulated category.

Let  $T \in \mathcal{C}$  be a cluster-tilting object, meaning

$$\text{add } T = \{X \in \mathcal{C} : \text{Ext}_{\mathcal{C}}^1(T, X) := \text{Hom}_{\mathcal{C}}(T, \Sigma X) = 0\}.$$

Then for all  $X \in \mathcal{C}$  there exists a triangle  $X \rightarrow T_1 \rightarrow T_0 \rightarrow \Sigma X$  with  $T_0, T_1 \in \text{add } T$ .

Define  $\text{coind}_T(X) = [T_1] - [T_0] \in K_0(\text{add } T)$ .



## Definition 2: Projective presentations

Let  $A$  be a finite-dimensional algebra and  $M \in \text{mod } A$ . Take a minimal projective presentation  $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ .

Then the  $g$ -vector of  $M$  is  $[P_1] - [P_0] \in K_0(\text{proj } A)$ .

$$\begin{array}{ccc} & \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \\ \nearrow & & \searrow \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{-----} & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{array}$$

We will now see that for  $\mathcal{C}$  and  $T$  as on the previous slide, and  $A = \text{End}_{\mathcal{C}}(T)^{\text{op}}$ , these definitions are compatible.

## Connection

Take  $X \in \mathcal{C}$ , and choose a triangle  $X \rightarrow T_1 \rightarrow T_0 \rightarrow \Sigma X$  to compute the coindex.

This yields an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(T, T_1) \rightarrow \mathrm{Hom}_{\mathcal{C}}(T, T_0) \rightarrow \mathrm{Ext}_{\mathcal{C}}^1(T, X) \rightarrow 0$$

of  $A = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ -modules.

There are equivalences

$$\mathrm{Hom}_{\mathcal{C}}(T, -): \text{add } T \xrightarrow{\sim} \text{proj } A, \quad \text{Yoneda}$$

$$\mathrm{Ext}_{\mathcal{C}}^1(T, -): \mathcal{C}/(T) \xrightarrow{\sim} \text{mod } A. \quad \text{Buan–Marsh–Reiten, Keller–Reiten, Koenig–Zhu, ...}$$

Thus the g-vector of  $X \in \mathcal{C}$  is equal to the g-vector of  $\mathrm{Ext}_{\mathcal{C}}^1(T, X) \in \text{mod } A$ .

### Aim

Enhance this relationship to an equivalence of ‘categories of g-vectors’.

# Extriangulated categories (Nakaoka–Palu '19)

Idea: additive categories with well-behaved 'extension groups'  $\mathbb{E}(X, Y)$ .

- (0) Exact categories, triangulated categories ( $\mathbb{E} = \text{Ext}_C^1$ ).
- (1) Extension closed subcategories of triangulated categories ( $\mathbb{E} = \text{Ext}_C^1$ ).
- (2) 'Partial stabilisations'  $\mathcal{C}/(P)$  for  $\mathcal{C}$  Frobenius exact,  $P$  projective-injective ( $\mathbb{E} = \text{Ext}_C^1$ ).
- (3) Ex-triangulated categories: Take a triangulated category  $\mathcal{C}$  and choose (carefully) a subfunctor  $\mathbb{E} \leq \text{Ext}_C^1$ .

Carefully = making sure inflations and deflations are closed under composition.  
(Herschend–Liu–Nakaoka)

## Remark

(3) was studied for exact categories by Auslander–Solberg, under the heading of relative homological algebra: the process preserves exactness (but not triangulatedness).

# Harp (The Homotopy ARrow category of Projectives)

Let  $A$  be a finite-dimensional algebra. Then

$$\text{harp } A := \{P_1 \xrightarrow{\varphi} P_0 : P_i \in \text{proj } A\} / \text{homotopy}.$$

We have  $\text{harp } A \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\text{proj } A) \hookrightarrow \mathcal{K}^b(\text{proj } A)$ .

The image is extension-closed, and so  $\text{harp } A$  is naturally extriangulated.

Projective objects are those of the form  $0 \rightarrow P$ , and injectives of the form  $P \rightarrow 0$ . (Objects  $P \xrightarrow{\sim} P$  are projective-injective, but also 0.)

$$\begin{array}{ccccc}
 & |(0 \rightarrow P_1) \cdots (P_2 \rightarrow 0)| & & & \\
 & \nearrow & \searrow & \nearrow & \searrow \\
 |(0 \rightarrow P_2) \cdots (P_2 \rightarrow P_1) \cdots (P_1 \rightarrow 0)| & & & & 
 \end{array}$$

# Relative harp

Choose additionally  $e = e^2 \in A$ , and define

$$\text{harp}_e A := \{P_1 \xrightarrow{\varphi} P_0 \in \text{harp } A : e \cdot \text{coker } \varphi = 0\}.$$

Note that  $\text{harp}_0 A = \text{harp } A$ .

For  $A = \begin{array}{ccc} \boxed{1} & \xrightarrow{q} & \boxed{2} \\ & \swarrow r \quad \searrow p & \\ & * & \end{array} / (pq, qr)$  and  $e = e_1 + e_2$ ,

$$\text{harp}_e(A) = \begin{array}{ccccc} |(P_1 \rightarrow 0)| & & |(P_2 \rightarrow 0)| & & |(P_1 \rightarrow 0)| \\ & \searrow & \nearrow & \searrow & \searrow \\ \text{---} & |(P_* \rightarrow 0)| & & |(P_2 \rightarrow P_*)| & \text{---} & |(P_* \rightarrow 0)| \end{array}$$

## Proposition (FGPPP)

*In  $\text{harp}_e(A)$ , injectives are  $P \rightarrow 0$ , while projectives are  $P \xrightarrow{\varphi} Q$  such that  $P \in \text{add } Ae$ . In particular,  $Ae \rightarrow 0$  is projective-injective.*

# Main Theorem

Two situations:

- (1)  $\mathcal{C}$  is the Amiot cluster category of a Jacobi-finite quiver with potential, with initial cluster-tilting object  $T$ .
- (2)  $\mathcal{C}$  is a Krull–Schmidt stably 2-Calabi–Yau Frobenius exact category, with cluster-tilting object  $T$ .

Write  $A = \text{End}_{\mathcal{C}}(T)^{\text{op}}$  with  $e$  corresponding to projective summands of  $T$  (so  $e = 0$  in case (1)).

## Theorem (FGPPP)

*In situations (1) and (2), there is a full and dense functor  $G: \mathcal{C} \rightarrow \text{harp}_e A$  given by*

$$GX = (\text{Hom}_{\mathcal{C}}(T, T_1) \rightarrow \text{Hom}_{\mathcal{C}}(T, T_0))$$

*for  $X \rightarrow T_1 \rightarrow T_0$  with  $T_i \in \text{add } T$  either a carefully chosen triangle (1) or arbitrary short exact sequence (2). We have*

$$\ker G = \begin{cases} (T \rightarrow \Sigma^{-1}T), & (1) \\ 0. & (2) \end{cases}$$



## Preservation of structure

Give  $\mathcal{C}$  the relative extriangulated structure  $\mathbb{E}_T$  with extriangles  $X \rightarrow Y \rightarrow Z$  such that

$$\mathrm{coind}_T(Y) = \mathrm{coind}_T(X) + \mathrm{coind}_T(Z).$$

**Proposition (Padrol–Palu–Pilaud–Plamondon, 19<sup>+</sup>, cf. Palu '08)**

*The injectives and projectives in  $(\mathcal{C}, \mathbb{E}_T)$  are given respectively by (the preimage under stabilisation of)  $\mathrm{add}\, T$  and  $\mathrm{add}\, \Sigma^{-1}T$  respectively.*

**Proposition (FGPPP)**

*If  $\mathcal{C}$  is extriangulated and  $\mathcal{I} \subseteq (\mathrm{inj} \rightarrow \mathrm{proj})$  is an ideal, then  $\mathcal{C}/\mathcal{I}$  is naturally extriangulated.*

**Theorem (FGPPP)**

*Using the extriangulated structure induced from  $\mathbb{E}_T$  on  $\mathcal{C}/\ker G$ , we obtain an equivalence*

$$\mathcal{C}/\ker G \xrightarrow{\sim} \mathrm{harp}_e A$$

*of extriangulated categories.*

# Corollaries

## Corollary

*In case (1), if  $A$  is selfinjective then  $(\mathcal{C}, \mathbb{E}_T) \simeq \text{harp } A$ .*

## Proof.

We have  $\Sigma^2 T = T$  because  $A$  is selfinjective (Koenig–Zhu, Iyama–Oppermann) so

$$\text{Hom}_{\mathcal{C}}(T, \Sigma^{-1} T) = \text{Hom}_{\mathcal{C}}(T, \Sigma T) = 0$$

because  $T$  is rigid. □

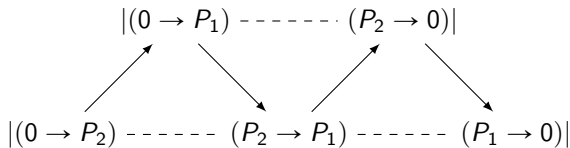
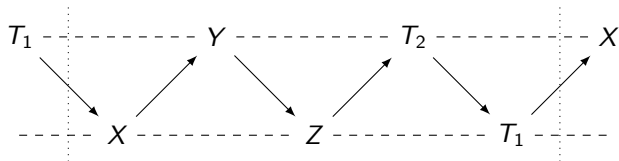
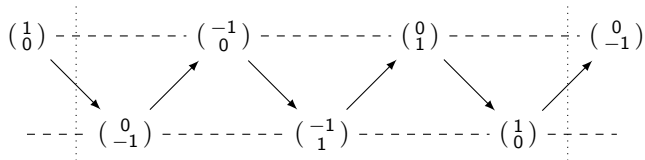
## Corollary

*In case (2),  $\text{harp}_e A$  is exact.*

## Proof.

Since  $\mathcal{C}$  is exact, so is  $(\mathcal{C}, \mathbb{E}_T) \simeq \text{harp}_e(A)$  (Auslander–Solberg). □

# Example 1 ( $A_2$ cluster category)



## Example 2 ( $A_2$ preprojective algebra)

