

# Homotopy arrow categories for finite-dimensional algebras

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# Extriangulated categories (Nakaoka–Palu '19)

Formalism of additive categories with “extensions”.

Declared class of *conflations*:

$$X \rightharpoonup Y \twoheadrightarrow Z$$

in which the maps compose to 0.

$$\rightharpoonup = \textit{inflation} \qquad \twoheadrightarrow = \textit{deflation}$$

Conflations are parametrised by abelian groups  $\mathbb{E}(Z, X)$ , functorial in both arguments:

$$\delta \in \mathbb{E}(Z, X) \rightsquigarrow \text{conflation } X \rightharpoonup Y \twoheadrightarrow Z \overset{\delta}{\dashrightarrow}$$

**Warning:** Inflation  $\not\Rightarrow$  monic, and deflation  $\not\Rightarrow$  epic.

## Examples / etymology

(0) Exact and triangulated categories.

	Exact	Triangulated
Conflations	Exact sequences	Distinguished triangles <sup>1</sup>
Inflations	Admissible monos	All morphisms
Deflations	Admissible epis	All morphisms
$\mathbb{E}$	$\text{Ext}^1$	$\text{Hom}(-, \Sigma-)$

(1) Extension-closed subcategories of triangulated categories:

$\mathcal{C} \subseteq \mathcal{T}$  such that if

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{\delta} \Sigma X$$

is a triangle with  $X, Z \in \mathcal{C}$ , then  $Y \in \mathcal{C}$ .

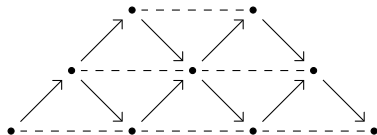
This gives the conflation  $X \rightarrowtail Y \twoheadrightarrow Z \xrightarrow{\delta} \Sigma X$  in  $\mathcal{C}$ ,  
 $\mathbb{E}(-, -) = \text{Hom}_{\mathcal{T}}(-, \Sigma-)|_{\mathcal{C} \times \mathcal{C}}$ .

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<sup>1</sup>Truncated, i.e. forgetting the third morphism.

## Examples / etymology

(1) Special case:  $\mathcal{K}^{[-1,0]}(\text{proj } \Lambda) = \{P_1 \rightarrow P_0\} \subseteq \mathcal{K}^b(\text{proj } \Lambda)$



(2) Ex-triangulated categories: start with a triangulated category  $\mathcal{T}$ , and throw out some distinguished triangles.

This has to be done carefully (Herschend–Liu–Nakaoka '21):

- ▶ the remaining triangles should be parametrised by a subfunctor  $\mathbb{E} \leq \text{Hom}_{\mathcal{T}}(-, \Sigma-)$ , and
- ▶ inflations should remain closed under composition (equivalently, deflations remain closed under composition).

### Remark

The constructions in (1) and (2) apply to arbitrary extriangulated categories.

The class of exact categories is closed under these operations

(Auslander–Solberg '93), but the class of triangulated categories is not.

# Homological algebra

For  $\mathcal{C}$  extriangulated, say  $P \in \mathcal{C}$  is *projective* if  $\mathbb{E}(P, -) \equiv 0$ , or equivalently:

$$\begin{array}{ccc} & & P \\ & \exists & \downarrow \\ X & \xrightarrow{\forall} & Y \end{array}$$

Say  $\mathcal{C}$  has *enough projectives* if for all  $X \in \mathcal{C}$ , there exists projective  $P_X \twoheadrightarrow X$ .

In this case we can define projective dimension in the usual way.

Similarly, we can define injectives, global dimension, dominant dimension, etc.

**Reminder:** for a minimal injective resolution  $X \hookrightarrow I_0 \rightarrow I_1 \rightarrow \cdots$ ,

$$\text{dom. dim } X = \min\{j : I_j \text{ is not projective}\}$$

**Warning:** If  $\mathcal{T}$  is triangulated, only 0 is projective or injective, but this is enough!

$$X \hookrightarrow 0 \twoheadrightarrow \Sigma X$$

# 0-Auslander categories

## Definition (Gorsky–Nakaoka–Palu '23<sup>+</sup>)

An extriangulated category  $\mathcal{C}$  is 0-Auslander if

- ▶ it has enough projectives,
- ▶ it is hereditary, i.e.  $\text{p. dim } X \leq 1$  for all  $X \in \mathcal{C}$ , and
- ▶  $\text{dom. dim } P \geq 1$  for all projectives  $P \in \mathcal{C}$ .

## Proposition (Gorsky–Nakaoka–Palu '23<sup>+</sup>)

*This definition is self dual, i.e.  $\mathcal{C}$  is 0-Auslander if and only if*

- ▶ *it has enough injectives,*
- ▶  *$\text{inj. dim } X \leq 1$  for all  $X \in \mathcal{C}$ , and*
- ▶  *$\text{codom. dim } I \geq 1$  for all injectives  $I \in \mathcal{C}$ .*

## Example

For  $\Lambda$  a finite-dimensional algebra,  $\mathcal{K}^{[-1,0]}(\text{proj } \Lambda)$  is 0-Auslander.

# The homotopy arrow category

## Example

For  $\Lambda$  a finite-dimensional algebra,  $\mathcal{K}^{[-1,0]}(\text{proj } \Lambda)$  is 0-Auslander.

- ▶ Projectives are  $P \in \text{proj } \Lambda$ .
- ▶ Injectives are  $\Sigma P$ ,  $P \in \text{proj } \Lambda$ .
- ▶ For all  $(P_1 \rightarrow P_0) \in \mathcal{K}^{[-1,0]}(\text{proj } \Lambda)$ , there is a conflation

$$P_1 \twoheadrightarrow P_0 \twoheadrightarrow (P_1 \rightarrow P_0),$$

hence enough projectives,  $\text{p. dim}(P_1 \rightarrow P_0) \leq 1$ .

- ▶ For all  $P \in \text{proj } \Lambda$ , there is a conflation

$$P \twoheadrightarrow 0 \twoheadrightarrow \Sigma P$$

in which  $\Sigma P$  is injective, and 0 is projective-injective, hence  $\text{dom. dim } P \geq 1$ .

# 0-Auslander categories

## Example

For  $\Lambda$  a finite-dimensional algebra,  $\mathcal{K}^{[-1,0]}(\text{proj } \Lambda)$  is 0-Auslander.

**Question:** How many more examples are there?

**Note:** We have  $\text{Hom}_\Lambda(\Sigma Q, P) = 0$  for all  $P, Q \in \text{proj } \Lambda$ .

That is, in  $\mathcal{K}^{[-1,0]}(\text{proj } \Lambda)$ , there are no non-zero morphisms from injective to projective objects.

## Proposition (FGPPP)

$\mathcal{C}$  extriangulated  $\implies \mathcal{C}/(\text{inj} \rightarrow \text{proj})$  extriangulated (for the induced  $\mathbb{E}$ ).



## Result 1: Cluster categories

An extriangulated category  $\mathcal{C}$  is *Frobenius* if it has enough projective and injective objects, and these coincide.

### Example

A triangulated category is a Frobenius extriangulated category.

$$X \rightarrowtail 0 \twoheadrightarrow \Sigma X$$

### Theorem (Nakaoka–Palu '19)

If  $\mathcal{C}$  is a Frobenius extriangulated category, then its stable category  $\underline{\mathcal{C}} = \mathcal{C}/(\text{proj-inj})$  is triangulated, and

$$\mathbb{E}_{\mathcal{C}}(X, Y) = \underline{\text{Hom}}_{\mathcal{C}}(X, \Sigma Y).$$

Let  $\mathcal{C}$  be a Frobenius extriangulated category that is

- ▶ weakly idempotent complete, and
- ▶ stably 2-Calabi–Yau.

## Result 1: Cluster categories

Let  $\mathcal{T} \subseteq \mathcal{C}$  be a (2-)cluster-tilting subcategory.

Then  $\mathbb{E}_{\mathcal{C}} = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y)$ , and so we may define

$$\mathbb{E}_{\mathcal{T}}(X, Y) = (\Sigma \mathcal{T})(X, \Sigma Y) \leq \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y).$$

**Proposition (Herschend–Liu–Nakaoka '21)**

$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$  is an extriangulated category.

It is 0-Auslander:

- ▶ the projectives are  $T \in \mathcal{T}$ ,
- ▶ the injectives are  $U$  such that  $\pi U \in \Sigma \mathcal{T}$  (for  $\pi: \mathcal{C} \rightarrow \underline{\mathcal{C}}$ ),
- ▶ for all  $X \in \mathcal{C}$  there exist  $T_1, T_0 \in \mathcal{T}$  and  $T_1 \twoheadrightarrow T_0 \twoheadrightarrow X$ ,  
(cf. *index*, Palu '08)
- ▶ for all  $T \in \mathcal{T}$ , there is  $T \twoheadrightarrow \Pi_T \twoheadrightarrow \Sigma T$  with  $\Pi_T$  projective-injective in  $\mathcal{C}$ .

## Result 1: Cluster categories

$$\mathbb{E}_{\mathcal{T}}(X, Y) = (\Sigma \mathcal{T})(X, \Sigma Y) \leq \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \Sigma Y)$$

### Theorem (FGPPP)

Assume  $\mathcal{T} = \text{add } T$ , and that  $\mathcal{C}$  is either

- (1) exact, or
- (2) a Higgs category (Yilin Wu '23).

Then there is an equivalence of extriangulated categories

$$(\mathcal{C}, \mathbb{E}_{\mathcal{T}}) / (\text{inj} \rightarrow \text{proj}) \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\text{proj } \Lambda)$$

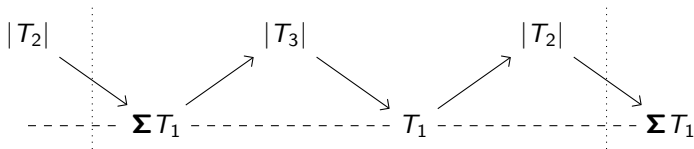
where  $\Lambda = \underline{\mathrm{End}}_{\mathcal{C}}(T)^{\text{op}}$ .

### Remark

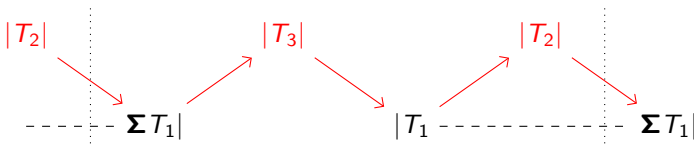
Generalised cluster categories (Amiot '09) are precisely the Higgs categories which are triangulated.

## Example 1 ( $A_2$ preprojective algebra)

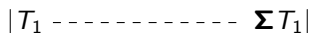
$\mathcal{C} = \text{mod } \Pi$  for  $\Pi$  the preprojective algebra of type  $A_2$ :



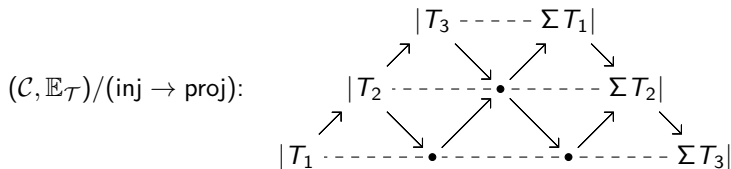
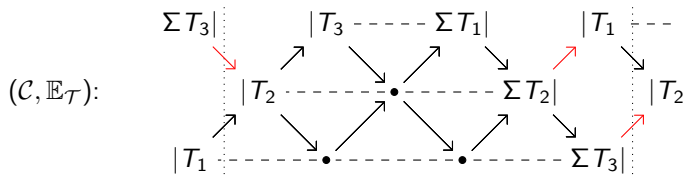
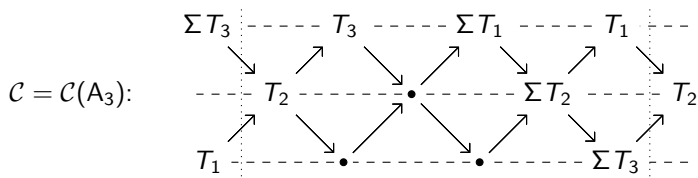
$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$ :



$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})/(\text{inj} \rightarrow \text{proj})$ :

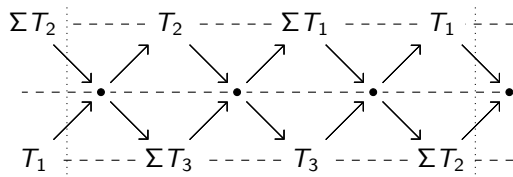


## Example 2 ( $A_3$ cluster category)

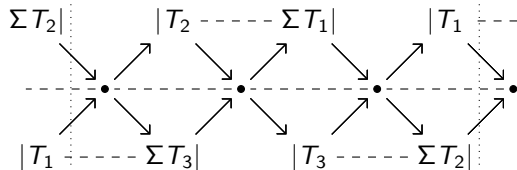


## Example 2 ( $A_3$ cluster category again)

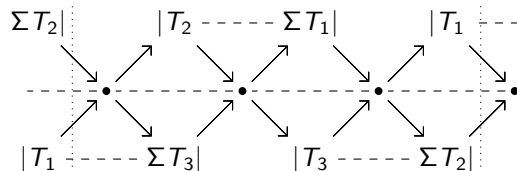
$\mathcal{C} = \mathcal{C}(A_3)$ :



$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})$ :



$(\mathcal{C}, \mathbb{E}_{\mathcal{T}})/(\text{inj} \rightarrow \text{proj})$ :

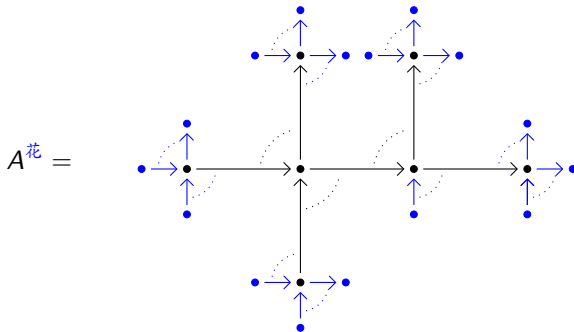


## Result 2: Gentle algebras

Let  $A$  be a gentle algebra.

Write  $A^{\text{花}}$  for the blossoming algebra<sup>2</sup>, with  $A = A^{\text{花}}/(e^{\text{花}})$ .

(Asashiba '12, Brüstle–Douville–Mousavand–Thomas–Yıldırım '20, Palu–Pilaud–Plamondon '21)



The *category of walks* is  $\mathcal{W} = {}^\perp(\Sigma e^{\text{花}} A^{\text{花}}) \subseteq \mathcal{K}^{[-1,0]}(\text{proj } A^{\text{花}})$ .

Related by Palu–Pilaud–Plamondon to combinatorics of non-kissing facets.

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<sup>2</sup>花 = はな = hana (Japanese), or huā (Chinese), meaning flower

## Result 2: Gentle algebras

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The category of walks is  $\mathcal{W} = {}^{\perp}(\Sigma e^{\text{花}} A^{\text{花}}) \subseteq \mathcal{K}^{[-1,0]}(\text{proj } A^{\text{花}})$ .

### Theorem (FGPPP)

For any gentle algebra  $A$ ,

- (1) the category  $\mathcal{W}$  is 0-Auslander,
- (2) there is an equivalence  $\mathcal{W}/(\text{inj} \rightarrow \text{proj}) \xrightarrow{\sim} \mathcal{K}^{[-1,0]}(\text{proj } A)$  of extriangulated categories, and
- (3)  $(\text{inj} \rightarrow \text{proj}) = (e^{\text{花}} A^{\text{花}})$  consists only of maps factoring over a projective-injective object.