

Representation theory and positroid varieties

表示论和正拟阵簇

Matthew Pressland

University of Glasgow / Oilthigh Ghlaschu

格拉斯哥大学

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Slides: <https://bit.ly/mdp-icra24>



The totally positive Grassmannian

Definition

$M \in \mathbb{C}^{k \times n}$, $k < n$, is *totally positive* if its maximal minors $\Delta_I(M)$ are positive real numbers.

- ▶ Here $I \in \binom{[n]}{k}$ is a subset of k columns, $\Delta_I(M)$ its determinant.
- ▶ If $\text{rk } M = k$, its row span $[M]$ is in $\text{Gr}_{k,n}$, the *Grassmannian*.
- ▶ Totally positive Grassmannian: $\text{Gr}_{k,n}^{>0} = \{[M] : M \text{ is totally +ve}\}.$

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- ▶ A minimal positivity test needs only $\dim \widehat{\text{Gr}}_{k,n} = k(n-k) + 1$ minors ... chosen carefully!

$$k = 2 : \quad \Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

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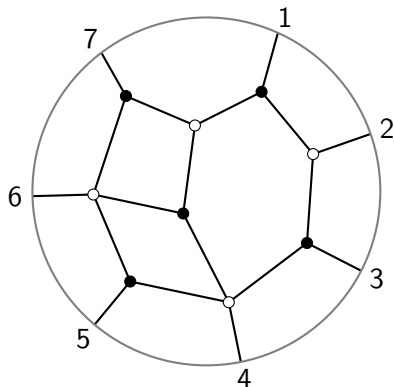
- ▶ $\overline{\text{Gr}_{k,n}^{>0}} = \text{Gr}_{k,n}^{\geq 0}$ decomposes into cells $\Pi_{\mathcal{P}}^{\circ} \cap \text{Gr}_{k,n}^{\geq 0}$, indexed by *positroids* (正拟阵) $\mathcal{P} \subseteq \binom{[n]}{k}$.

Positroid varieties

Postnikov '06⁺, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:

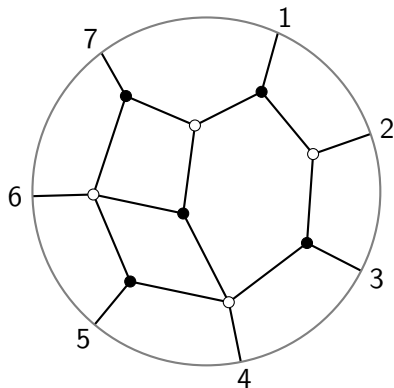
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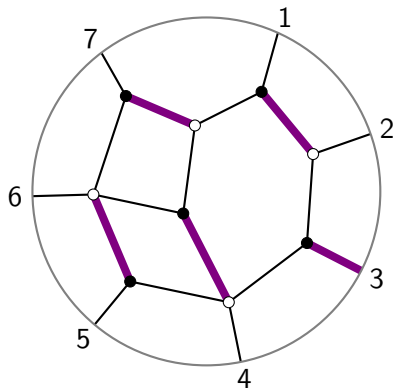
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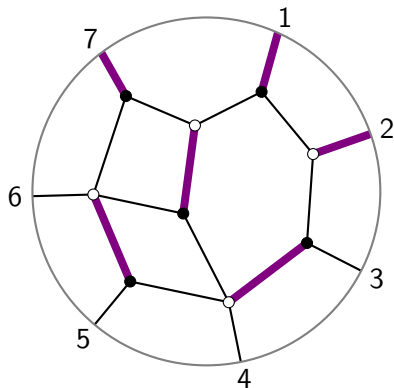
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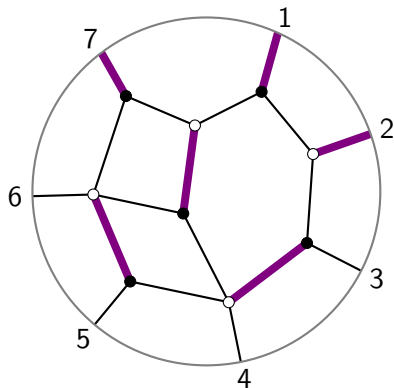
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$$\begin{aligned}\mathcal{P} &= \{\partial\mu : \mu \text{ perfect matching}\} \\ &= \{157, \textcolor{violet}{235}, \dots\}\end{aligned}$$

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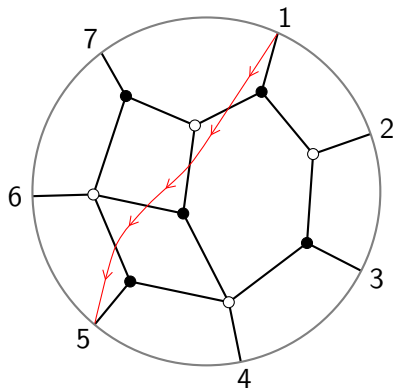
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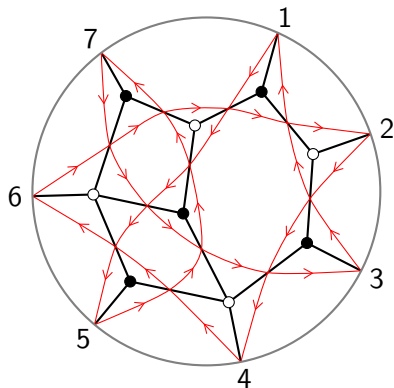
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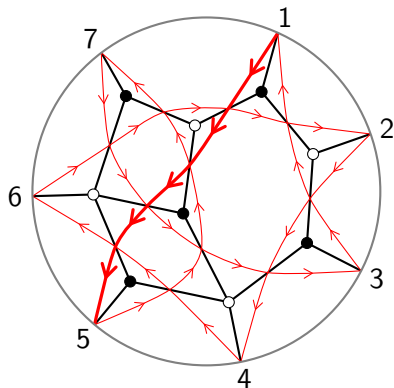
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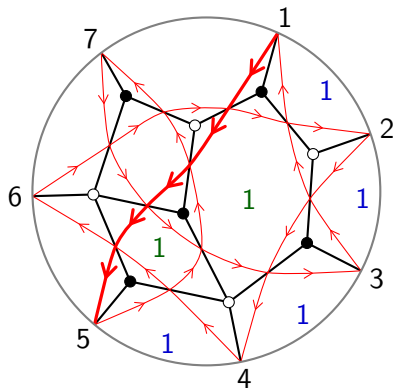
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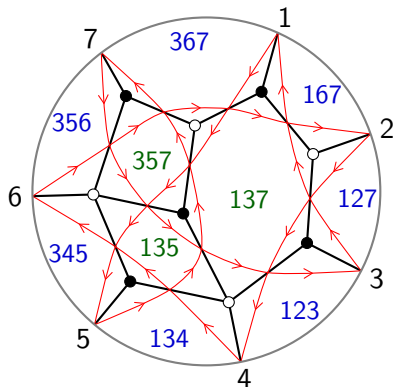
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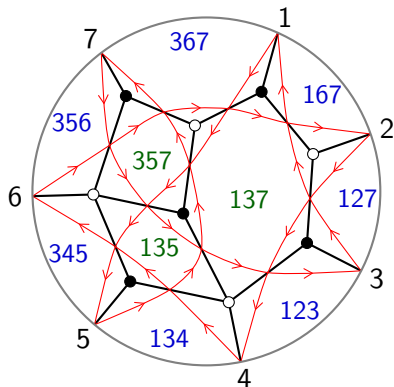
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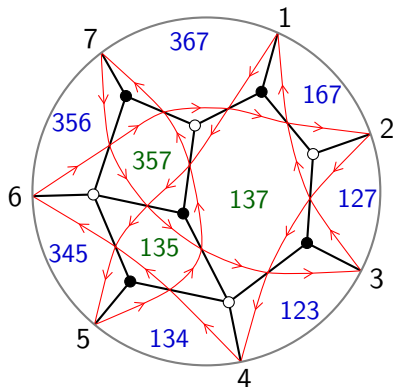
$$\begin{aligned}\mathcal{F}^+ &= \{\text{boundary labels}\} \\ &= \{123, 134, 345, \dots\}\end{aligned}$$

$$\Pi_{\mathcal{P}}^{\circ} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P}, \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^+\}$$

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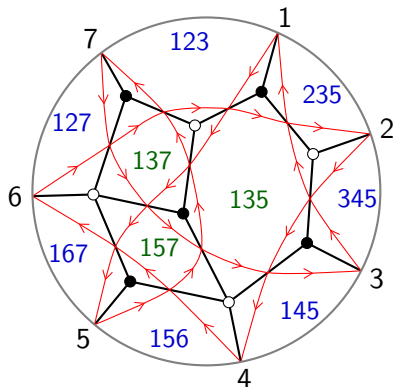
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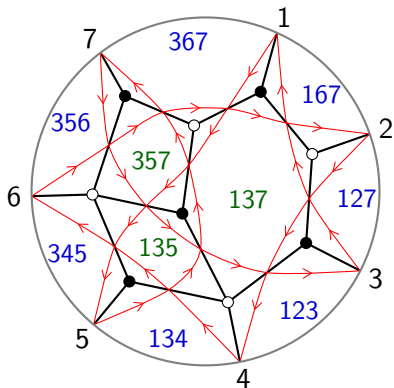
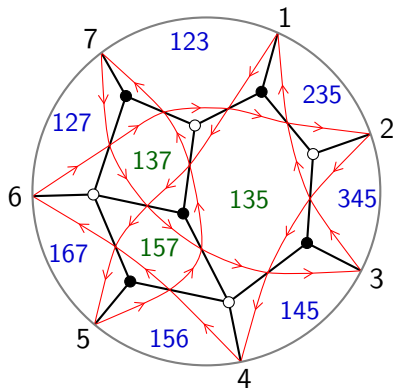
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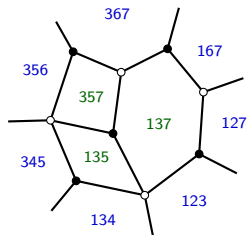
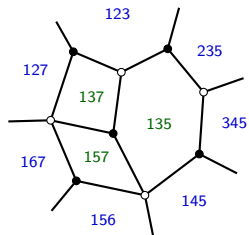
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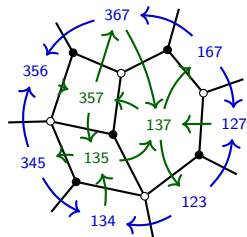
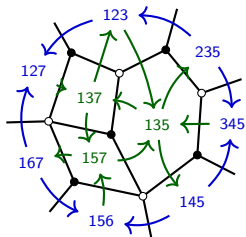


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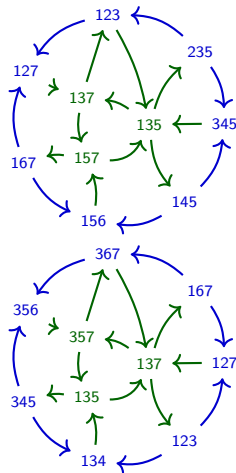
Cluster structures



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Example



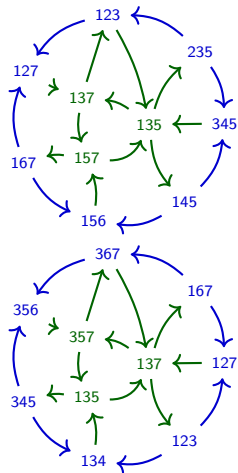
Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
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Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
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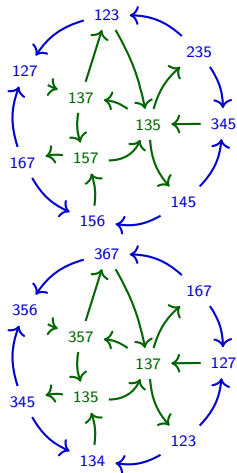
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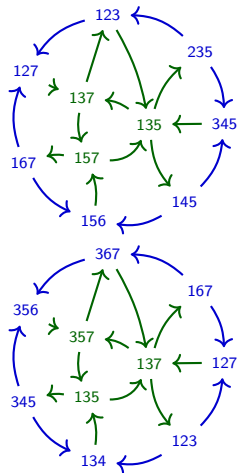
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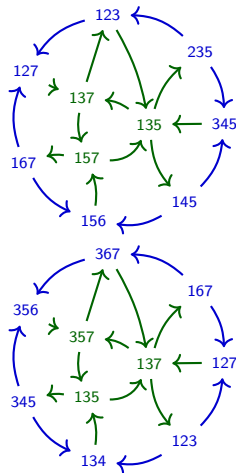
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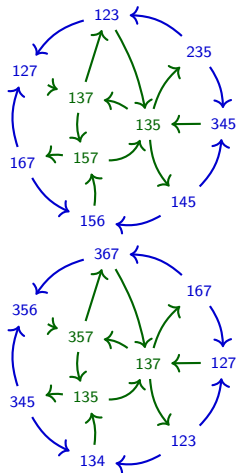
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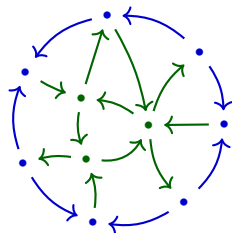
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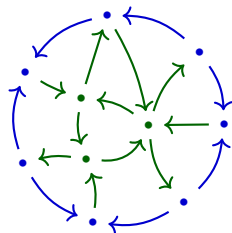
$$A = \mathbb{C}\langle\langle Q \rangle\rangle / \overline{(\text{dimer relations})}$$

$$B = eAe$$

$$R = \mathbb{C}[[t]], \quad R \subseteq B \subseteq A$$

$$\text{CM } B = \{X \in \text{mod } B : {}_R X \text{ free} + \text{f.g.}\}$$

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Theorem (P '22)

For each (connected) positroid \mathcal{P} , the Frobenius exact category $\text{gproj CM } B$ categorifies the cluster algebra $\mathcal{A}_{\mathcal{P}}$.

- ▶ Key fact: A is internally 3-Calabi–Yau.
- ▶ $\text{ginj CM } B = \{X \in \text{CM } B : \text{Ext}_B^{>0}(B^\vee, X) = 0\}$ also categorifies.
- ▶ $\text{gproj CM } B \simeq \text{ginj CM } B$, but different subcategories of $\text{CM } B$!

- ▶ Jensen–King–Su '16: categorification CM \mathcal{C} of $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}]$, with $M_I \in \mathrm{CM} \mathcal{C}$ for each Δ_I .

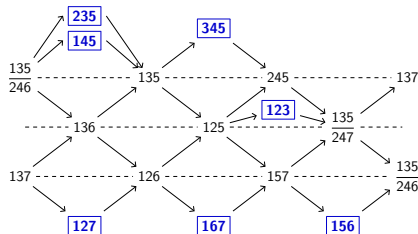
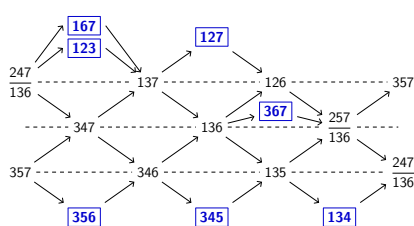
Theorem (Çanakçı–King–P '24, P '22)

- ▶ $\mathrm{CM} B \hookrightarrow \mathrm{CM} \mathcal{C}$, with $M_I \in \mathrm{CM} B \iff I \in \mathcal{P}$.
- ▶ $M_I \in \mathrm{gproj} \mathrm{CM} B \iff \Delta_I$ is an η^+ -cluster variable (source labelled).
- ▶ $M_I \in \mathrm{ginj} \mathrm{CM} B \iff \Delta_I$ is an η^- -cluster variable (target labelled).

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- ▶ Reduce (geometrically) to connected positroids, for access to categorifications.
- ▶ Key fact: inclusions induce *derived* equivalences

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- ▶ Main step: show that the composition is a quasi-cluster functor (Fraser–Keller '23).
- ▶ E.g. induced equivalence $\text{gproj CM } B \xrightarrow{\sim} \text{ginj CM } B$ takes initial cluster-tilting object T^+ to reachable cluster-tilting object $\Omega^2 T^-$.

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Theorem (P '23⁺, conj. Muller–Speyer '16)

The cluster structures η^+ and η^- quasi-coincide.

- ▶ Independent proof: (Casals–Le–Sherman–Bennett–Weng '23⁺)
Inspired by symplectic topology!

Computation

$$0 \longrightarrow \Omega X \longrightarrow P \longrightarrow X \longrightarrow 0$$

- Let $X \in \text{ginj CM } B$, compute syzygy ΩX .

Computation

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- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.

Computation

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & P & \longrightarrow & X \longrightarrow 0 \\ & & \parallel & & & & \\ 0 & \longrightarrow & \Omega X & \longrightarrow & Q & \longrightarrow & \Sigma \Omega X \longrightarrow 0 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma \Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.

Computation

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & P & \longrightarrow & X \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Omega X & \longrightarrow & Q & \longrightarrow & \Sigma \Omega X \longrightarrow 0 \end{array}$$

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- ▶ This gives $\Sigma \Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
- ▶ Get $X \cong (Q \rightarrow P \oplus \Sigma \Omega X)$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\psi_X = \psi_{\Sigma \Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\widehat{\Pi}_P^\circ]$$

for ψ the cluster character.

Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega X & \longrightarrow & P & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \Omega X & \longrightarrow & Q & \longrightarrow & \Sigma \Omega X \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma \Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
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$$\psi_X = \psi_{\Sigma \Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for ψ the cluster character.

Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega X & \longrightarrow & M_{167} \oplus M_{345} & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \Omega X & \longrightarrow & Q & \longrightarrow & \Sigma \Omega X \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma \Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
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$$\begin{array}{ccccccc}
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 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
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- ▶ Get $X \cong (Q \rightarrow P \oplus \Sigma \Omega X)$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\psi_X = \psi_{\Sigma \Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for ψ the cluster character.

Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{167} \oplus M_{345} & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & M_{346} & \longrightarrow & Q & \longrightarrow & \Sigma\Omega X \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma\Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
- ▶ Get $X \cong (Q \rightarrow P \oplus \Sigma\Omega X)$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\psi_X = \psi_{\Sigma\Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for ψ the cluster character.

Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{167} \oplus M_{345} & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{367} \oplus M_{345} & \longrightarrow & \Sigma\Omega X \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma\Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
- ▶ Get $X \cong (Q \rightarrow P \oplus \Sigma\Omega X)$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\psi_X = \psi_{\Sigma\Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for ψ the cluster character.

Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{167} \oplus M_{345} & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{367} \oplus M_{345} & \longrightarrow & M_{357} \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma\Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
- ▶ Get $X \cong (Q \rightarrow P \oplus \Sigma\Omega X)$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\psi_X = \psi_{\Sigma\Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for ψ the cluster character.

Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{167} \oplus M_{345} & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{367} \oplus M_{345} & \longrightarrow & M_{357} \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma\Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
- ▶ Get $M_{157} \cong (M_{367} \rightarrow M_{167} M_{357})$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\psi_X = \psi_{\Sigma\Omega X} \frac{\psi_P}{\psi_Q} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for ψ the cluster character.

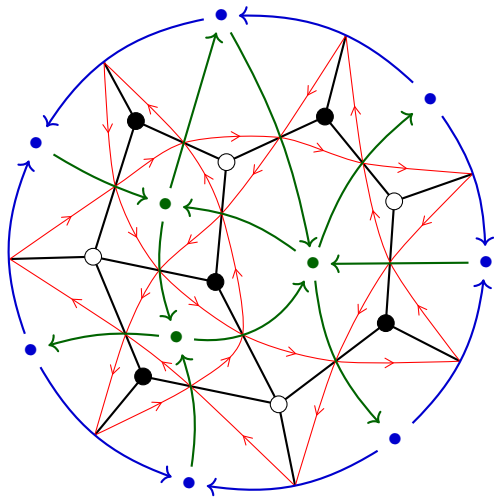
Computation

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{167} \oplus M_{345} & \longrightarrow & M_{157} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & M_{346} & \longrightarrow & M_{367} \oplus M_{345} & \longrightarrow & M_{357} \longrightarrow 0
 \end{array}$$

- ▶ Let $X \in \text{ginj CM } B$, compute syzygy ΩX .
- ▶ Then $\Omega X \in \text{gproj CM } B$ (ÇKP '24), so compute cosyzygy here.
- ▶ This gives $\Sigma \Omega X \in \text{gproj CM } B$, and $P, Q \in \text{proj } B$.
- ▶ Get $M_{157} \cong (M_{367} \rightarrow M_{167} M_{357})$ in $\mathcal{D}^b(\text{CM } B)$, and hence

$$\Delta_{357} \frac{\Delta_{167}}{\Delta_{367}} = \Delta_{157} \in \mathbb{C}[\hat{\Pi}_P^\circ]$$

for Ψ the cluster character.



谢谢!