

An \mathcal{X} -cluster character

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The theory of cluster algebras and its applications

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Slides: <https://bit.ly/mdp-abu-dhabi>



General philosophy

- ▶ Start with a \mathbb{K} -linear, Krull–Schmidt, Frobenius, stably 2-Calabi–Yau, algebraic extriangulated category \mathcal{C} , with cluster-tilting subcategories.
- ▶ Extract various pieces of data from \mathcal{C} and its cluster-tilting subcategories.
- ▶ Explain how this data transforms under mutation of cluster-tilting subcategories.
- ▶ Show that, *under the correct additional assumptions*, we recover cluster-theoretic data in the sense of Fomin–Zelevinsky.
- ▶ Covers \mathbf{g} -vectors, \mathbf{c} -vectors, B -matrices, \mathcal{F} -polynomials, \mathcal{A} -cluster variables, \mathcal{X} -cluster variables¹ and L -matrices.
- ▶ **Disclaimer:** Many results are due to other authors under more restrictive assumptions.

¹Fock–Goncharov \mathcal{A} = Fomin–Zelevinsky \mathbf{x} , Fock–Goncharov \mathcal{X} = Fomin–Zelevinsky \mathbf{y}

Grothendieck groups

- ▶ Simplifying assumptions for today:
 - ▶ \mathcal{C} is Hom-finite,
 - ▶ has cluster-tilting objects (\leadsto finite rank cluster algebras), and
 - ▶ $\mathbb{K} = \overline{\mathbb{K}}$ (\leadsto skew-symmetric exchange matrices).

- ▶ Let $\mathcal{T} \subseteq \mathcal{C}$ be cluster-tilting:

$$\mathcal{T} = \{X \in \mathcal{C} : \text{Ext}_{\mathcal{C}}^1(X, \mathcal{T}) = 0\} = \{X \in \mathcal{C} : \text{Ext}_{\mathcal{C}}^1(\mathcal{T}, X) = 0\}$$

- ▶ \leadsto Grothendieck groups $K_0(\mathcal{T})$ and $K_0(\text{fd } \mathcal{T})$, for $\text{fd } \mathcal{T} = \{M : \mathcal{T}^{\text{op}} \rightarrow \text{fd } \mathbb{K}\} = \text{finite-dimensional } \mathcal{T}\text{-modules}$.
- ▶ Both free of (finite) rank $\#(\text{indec } \mathcal{T})$, each $T \in \text{indec } \mathcal{T}$ indexes dual basis vectors $[T] \in K_0(\mathcal{T})$ and $[S_T^T] \in K_0(\text{fd } \mathcal{T})$:

$$S_T^T(T') = \begin{cases} \mathbb{K}, & T' = T \\ 0, & \text{otherwise.} \end{cases}$$

Index and coindex

- Fix $\mathcal{T} \subseteq \mathcal{C}$ cluster-tilting, and let $X \in \mathcal{C}$. Then there are conflations

$$\underbrace{K_{\mathcal{T}}X \twoheadrightarrow R_{\mathcal{T}}X \twoheadrightarrow X}_{\in \mathcal{T}} \dashrightarrow X \twoheadrightarrow \underbrace{L_{\mathcal{T}}X \twoheadrightarrow C_{\mathcal{T}}X}_{\in \mathcal{T}} \dashrightarrow .$$

- $\leadsto \text{ind}^{\mathcal{T}} X = [R_{\mathcal{T}}X] - [K_{\mathcal{T}}X]$, $\text{coind}^{\mathcal{T}} X = [L_{\mathcal{T}}X] - [C_{\mathcal{T}}X] \in K_0(\mathcal{T})$.
- For $\mathcal{T}, \mathcal{U} \subseteq \mathcal{C}$ cluster-tilting, $\text{ind}_{\mathcal{U}}^{\mathcal{T}}, \text{coind}_{\mathcal{U}}^{\mathcal{T}}: K_0(\mathcal{U}) \rightarrow K_0(\mathcal{T})$.

Theorem (Dehy–Keller '08)

$\text{ind}_{\mathcal{U}}^{\mathcal{T}}$ and $\text{coind}_{\mathcal{T}}^{\mathcal{U}}$ are inverse isomorphisms.

- Duality (over \mathbb{Z}): $\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}} := (\text{coind}_{\mathcal{T}}^{\mathcal{U}})^*: K_0(\text{fd } \mathcal{T}) \xrightarrow{\sim} K_0(\text{fd } \mathcal{U})$,
 $\overline{\text{coind}}_{\mathcal{U}}^{\mathcal{T}} := (\text{ind}_{\mathcal{T}}^{\mathcal{U}})^*$.
- Cluster dictionary: $\text{ind} \leftrightarrow \mathbf{g}\text{-vector}$, $\overline{\text{ind}} \leftrightarrow \mathbf{c}\text{-vector}$.

Exchange matrices

- All of the above applies to the triangulated stable category $\underline{\mathcal{C}}$, with $\{\mathcal{T} \subseteq \mathcal{C} \text{ cluster-tilting}\} = \{\underline{\mathcal{T}} \subseteq \underline{\mathcal{C}} \text{ cluster-tilting}\}$

Proposition (Keller–Reiten, Koenig–Zhu, Palu, Fu–Keller,...)

Let $\mathcal{T} \subseteq \mathcal{C}$ be cluster-tilting. Then $E^{\mathcal{T}} = \text{Ext}_{\mathcal{C}}^1(-, \mathcal{T}): \mathcal{C}/\mathcal{T} \xrightarrow{\sim} \text{fd } \underline{\mathcal{T}}$, and there is a linear map $\beta_{\mathcal{T}}: K_0(\text{fd } \underline{\mathcal{T}}) \rightarrow K_0(\mathcal{T})$ such that

$$\beta_{\mathcal{T}}[E^{\mathcal{T}} X] = \text{coind}^{\mathcal{T}}(X) - \text{ind}^{\mathcal{T}}(X).$$

Theorem (Palu '09, ..., Grabowski–P '24⁺)

For any cluster-tilting $\mathcal{T}, \mathcal{U} \subseteq \mathcal{C}$, there are commutative diagrams

$$\begin{array}{ccc} K_0(\text{fd } \underline{\mathcal{U}}) & \xrightarrow{\beta_{\mathcal{U}}} & K_0(\mathcal{U}) \\ \text{ind}_{\mathcal{U}}^{\mathcal{T}} \downarrow & & \downarrow \text{ind}_{\mathcal{U}}^{\mathcal{T}} \\ K_0(\text{fd } \underline{\mathcal{T}}) & \xrightarrow{\beta_{\mathcal{T}}} & K_0(\mathcal{T}) \end{array} \qquad \begin{array}{ccc} K_0(\text{fd } \underline{\mathcal{U}}) & \xrightarrow{\beta_{\mathcal{U}}} & K_0(\mathcal{U}) \\ \text{coind}_{\mathcal{U}}^{\mathcal{T}} \downarrow & & \downarrow \text{coind}_{\mathcal{U}}^{\mathcal{T}} \\ K_0(\text{fd } \underline{\mathcal{T}}) & \xrightarrow{\beta_{\mathcal{T}}} & K_0(\mathcal{T}) \end{array}$$

Exchange matrices

Theorem (Palu '09, ..., Grabowski–P '24⁺)

For any cluster-tilting $\mathcal{T}, \mathcal{U} \subseteq \mathcal{C}$, there are commutative diagrams

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Corollary

For \mathcal{T} and \mathcal{U} related by mutation (away from loops or 2-cycles), the matrices of $\beta_{\mathcal{T}}$ and $\beta_{\mathcal{U}}$ with respect to the standard bases are related by Fomin–Zelevinsky matrix mutation.

Warning: $\text{ind}_{\mathcal{U}}^{\mathcal{T}} \circ \text{ind}_{\mathcal{V}}^{\mathcal{U}} \neq \text{ind}_{\mathcal{V}}^{\mathcal{T}}$ etc. (but $\beta_{\mathcal{T}} \circ \text{ind}_{\mathcal{U}}^{\mathcal{T}} \circ \text{ind}_{\mathcal{V}}^{\mathcal{U}} = \beta_{\mathcal{T}} \circ \text{ind}_{\mathcal{V}}^{\mathcal{T}}$).

Example

Corollary

For \mathcal{T} and \mathcal{U} related by mutation (away from loops or 2-cycles), the matrices of $\beta_{\mathcal{T}}$ and $\beta_{\mathcal{U}}$ with respect to the standard bases are related by Fomin–Zelevinsky matrix mutation.

$$\begin{array}{ccc}
 \mathcal{U}: 1 \rightarrow 2 \rightarrow 3 & K_0(\text{fd } \underline{\mathcal{U}}) \xrightarrow{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}} & K_0(\mathcal{U}) \\
 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \downarrow & \downarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 \mathcal{T} = \mu_2 \mathcal{U}: 1 \begin{array}{c} \xrightarrow{\quad} 3 \\ \swarrow \quad \searrow \\ \quad 2 \end{array} & K_0(\text{fd } \underline{\mathcal{T}}) \xrightarrow{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}} & K_0(\mathcal{T})
 \end{array}$$

Definition

Say $(\mathcal{C}, \mathcal{T})$ has a cluster structure if the quiver of \mathcal{U} has no loops or 2-cycles for any $\mathcal{U} \stackrel{\text{mut}}{\sim} \mathcal{T}$.

\mathcal{A} -cluster character reminder

- $M \in \text{fd } \mathcal{T}$ has \mathcal{F} -polynomial

$$\mathcal{F}(M) = \sum_{[L] \in K_0(\text{fd } \underline{\mathcal{T}})} \chi(\text{Gr}_{[L]}(M)) x^{[L]} \in \mathbb{K}K_0(\text{fd } \underline{\mathcal{T}}).$$

- $X \in \mathcal{C}$ has \mathcal{A} -cluster character

$$\begin{aligned} \text{CC}_{\mathcal{A}}^{\mathcal{T}}(X) &= x^{\text{ind}^{\mathcal{T}}(X)} (\beta_{\mathcal{T}})_* \mathcal{F}(\text{E}^{\mathcal{T}} X) \\ &= a^{\text{ind}^{\mathcal{T}}(X)} \sum_{[L] \in K_0(\text{fd } \underline{\mathcal{T}})} \chi(\text{Gr}_{[L]}(\text{E}^{\mathcal{T}} X)) a^{\beta_{\mathcal{T}}[L]} \in \mathbb{K}K_0(\mathcal{T}). \end{aligned}$$

Example

$$\text{E}^{\mathcal{T}} X = \begin{array}{ccccc} \mathbb{K} & \xleftarrow{\quad} & 0 & \oplus & 0 & \xleftarrow{\quad} & 0 \\ & \searrow 1 & \nearrow & & \searrow & \nearrow & \\ & \mathbb{K} & & & \mathbb{K} & & \end{array} \quad \beta_{\mathcal{T}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\mathcal{F}(\text{E}^{\mathcal{T}} X) = 1 + 2x_2 + x_2^2 + x_1x_2 + x_1x_2^2$$

$$\text{CC}_{\mathcal{A}}^{\mathcal{T}}(X) = a_1 a_2^{-2} a_3 (1 + 2a_1^{-1} a_3 + a_1^{-2} a_3^2 + a_1^{-1} a_2 + a_1^{-2} a_2 a_3)$$

\mathcal{X} -cluster character

- ▶ Inputs to the \mathcal{X} -cluster character are $M \in \text{fd } \underline{\mathcal{U}}$ for $\mathcal{U} \subseteq \mathcal{C}$ cluster-tilting.
- ▶ $\leadsto M_{\mathcal{U}}^{\pm} \in \mathcal{U}$ such that $\beta_{\mathcal{U}}[M] = [M_{\mathcal{U}}^+] - [M_{\mathcal{U}}^-]$.

$$\text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(M) = x^{\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[M]} \frac{\mathcal{F}(\text{E}^{\mathcal{T}} M_{\mathcal{U}}^+)}{\mathcal{F}(\text{E}^{\mathcal{T}} M_{\mathcal{U}}^-)} \in \text{Frac}(\mathbb{K}\mathbb{K}_0(\text{fd } \underline{\mathcal{T}})).$$

Proposition

$$[M] = [L] + [N] \implies \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(M) = \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(L) \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(N).$$

- ▶ \leadsto consider the values of $\text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}$ on simple U -modules.

Theorem (Grabowski–P '24⁺)

Assume $(\mathcal{C}, \mathcal{T})$ has a cluster structure. Then the $\text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_U^{\mathcal{U}})$, for all $\mathcal{U} \stackrel{\text{mut}}{\sim} \mathcal{T}$ and $U \in \text{indec } \mathcal{U}$, are the \mathcal{X} -cluster variables of the cluster algebra associated to the quiver of $\underline{\mathcal{T}}$.

Remarks

Theorem (Grabowski–P '24⁺)

Assume $(\mathcal{C}, \mathcal{T})$ has a cluster structure. Then the $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_U^{\mathcal{U}})$, for all $\mathcal{U} \stackrel{\text{mut}}{\sim} \mathcal{T}$ and $U \in \mathrm{indec} \mathcal{U}$, are the \mathcal{X} -cluster variables of the cluster algebra associated to the quiver of $\underline{\mathcal{T}}$.

- ▶ The map implicit in the theorem is surjective but not injective, but induces a bijection between exchange pairs for $(\mathcal{C}, \mathcal{T})$ and \mathcal{X} -cluster variables (thanks to Cao–Keller–Qin '24).
- ▶ For $S = S_U^{\mathcal{U}}$, the objects $S_{\mathcal{U}}^{\pm}$ are the middle terms of exchange conflations $U^* \twoheadrightarrow S_{\mathcal{U}}^+ \twoheadrightarrow U$, $U \twoheadrightarrow S_{\mathcal{U}}^- \twoheadrightarrow U^*$.
- ▶ To include \mathcal{X} -variables at frozen vertices, we give an ad hoc definition of $\mathrm{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}$ on simple modules at these vertices.
- ▶ For $M \in \mathrm{fd} \underline{\mathcal{U}}$, we have $(\beta_{\mathcal{T}})_* \mathrm{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(M) = \frac{\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(M_{\mathcal{U}}^+)}{\mathrm{CC}_{\mathcal{A}}^{\mathcal{T}}(M_{\mathcal{U}}^-)}$.

Example 1

$$\mathcal{U} : 1 \rightarrow 2 \rightarrow 3 \quad \mathcal{T} = \mu_2 \mathcal{U} : \begin{array}{ccc} 1 & \xrightarrow{\quad} & 3 \\ & \nwarrow \quad \nearrow & \\ & 2 & \end{array}$$

$$\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] = [S_1^{\mathcal{T}}] + [S_2^{\mathcal{T}}] \quad (S_1)_{\mathcal{U}}^+ = 0 \quad (S_1)_{\mathcal{U}}^- = U_2$$

$$\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] = -[S_2^{\mathcal{T}}] \quad (S_2)_{\mathcal{U}}^+ = U_1 \quad (S_1)_{\mathcal{U}}^- = U_3$$

$$\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] = [S_3^{\mathcal{T}}] \quad (S_3)_{\mathcal{U}}^+ = U_2 \quad (S_1)_{\mathcal{U}}^- = 0$$

$$E^{\mathcal{T}}(U_1) = 0 \quad \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_1^{\mathcal{U}}) = x_1 x_2 \left(\frac{1}{1+x_2} \right) = x_1 x_2 (1+x_2)^{-1}$$

$$E^{\mathcal{T}}(U_2) = S_2^{\mathcal{T}} \quad \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1} \left(\frac{1}{1} \right) = x_2^{-1}$$

$$E^{\mathcal{T}}(U_3) = 0 \quad \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_3^{\mathcal{U}}) = x_3 \left(\frac{1+x_2}{1} \right) = x_3 (1+x_2)$$

Example 2

$$\mathcal{U} : 1 \rightarrow 2 \rightarrow 3 \quad \mathcal{T} = \Sigma \mathcal{U} : 1 \rightarrow 2 \rightarrow 3$$

$$\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_1^{\mathcal{U}}] = -[S_1^{\mathcal{T}}] \quad (S_1)_{\mathcal{U}}^+ = 0 \quad (S_1)_{\mathcal{U}}^- = U_2$$

$$\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_2^{\mathcal{U}}] = -[S_2^{\mathcal{T}}] \quad (S_2)_{\mathcal{U}}^+ = U_1 \quad (S_1)_{\mathcal{U}}^- = U_3$$

$$\overline{\text{ind}}_{\mathcal{U}}^{\mathcal{T}}[S_3^{\mathcal{U}}] = -[S_3^{\mathcal{T}}] \quad (S_3)_{\mathcal{U}}^+ = U_2 \quad (S_1)_{\mathcal{U}}^- = 0$$

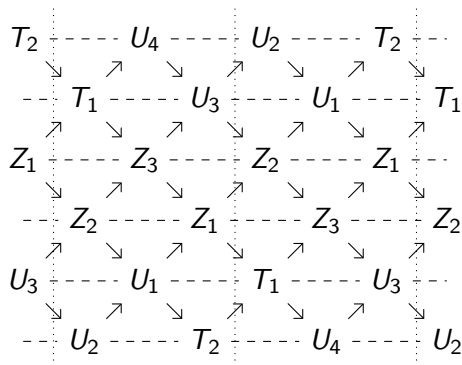
$$E^{\mathcal{T}}(U_1) = \mathbb{K} \rightarrow 0 \rightarrow 0 \quad \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_1^{\mathcal{U}}) = x_1^{-1} \left(\frac{1}{1+x_2+x_1x_2} \right)$$

$$E^{\mathcal{T}}(U_2) = \mathbb{K} \xrightarrow{1} \mathbb{K} \rightarrow 0 \quad \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_2^{\mathcal{U}}) = x_2^{-1} \left(\frac{1+x_1}{1+x_3+x_2x_3+x_1x_2x_3} \right)$$

$$E^{\mathcal{T}}(U_3) = \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \quad \text{CC}_{\mathcal{X}}^{\mathcal{T}, \mathcal{U}}(S_3^{\mathcal{U}}) = x_3^{-1} (1 + x_2 + x_1x_2)$$

Thank you! / شكراً! / 谢谢!

Bonus slide



- ▶ $T_2 \longrightarrow T_1 \curvearrowright \leadsto$ no cluster structure!
- ▶ $\text{CC}_{\mathcal{A}}^T(U_1) = a_1^{-1}(1 + a_2 + a_2^2),$
 $\text{CC}_{\mathcal{A}}^T(U_2) = a_2^{-1}(1 + a_1^{-1} + a_1^{-1}a_2 + a_1^{-1}a_2^2), \dots$
- ▶ \leadsto generalised cluster variables (Chekhov–Shapiro)