Economics 1011B
Section 6

Spring 2023

Today's Outline

- What is Asset Pricing?
- Risk-Neutral Asset Pricing
  - Buy-and-Sell
  - Buy-and-Hold
  - Gordon Growth Formula
- Asset Pricing with Risk Aversion
  - Buy-and-Sell
  - Stochastic Discount Factor

What is Asset Pricing?

- Assets: something you can own that generates economic returns in the future (for example: stocks, bonds, housing, bitcoin...).

- Asset pricing: What should the price of an asset be? What are the fundamental principles governing asset markets? When is an asset a 'good deal' to purchase, given its price?

- The basics of asset pricing will be important background information for a theory of investment (next week) and, in turn, businesss cycles (two weeks).

- Big-picture asset pricing questions:
  - Why is the expected return on stocks so high relative to bonds?
  - Why does the stock market exhibit so much volatility day-to-day?
  - How does risk (uncertain asset returns) and household risk aversion shape asset prices?

Two Frameworks

- Today, we'll walk through two complementary frameworks to 'price' a simple asset with stochastic returns.

- **Approach #1**: Risk-neutral asset pricing.
    - Asset price = expected present discounted value of asset return.

- **Approach #2**: Consumption-based asset pricing.
    - Asset price = expected present discounted value of asset return, discounted by risk.
    - Looks a lot like consumption-savings model!
    - Key new idea: stochastic discount factor.

Expectations: Handy Properties

- Expectations (as mathematical operators) satisfy a few handy properties:
    1. Distribution across addition: $E[A+B]=E[A]+E[B]$ for any A and B.
    2. Factor out constants: for a constant c and random variable A, $E[cA]=cE[A]$
    3. Expectation of a constant: for any constant c, $E[c]=c$.
    4. Applying to equations: can apply $E[\cdot]$ to both sides of an equation.
    5. Can 'pass through' derivatives: $\frac{d}{dx}E[f(x)]=E[\frac{d}{dx}f(x)]$

- We can 'apply expectations' to both sides of an equation (like other operations you know: e.g. addition/subtraction, multiplication/division, integration, and differentiation).

- When we subscript our expectations with t, it is a notational device to remind ourselves that we are conditioning on the information available at date t. For instance, $E_t[D_{t+1}]$ (in words) is the expectation (or expected value) at time t of D next period $(t+1)$.

- Generally speaking, the expectation at time t of a contemporaneous variable is the variable itself: no uncertainty about the present, just the future! $E_t[x_t]=x_t$.

Risk-Neutral Asset Pricing: Key Assumptions and Notation

- A few assumptions we need for risk-neutral asset pricing:
    1. Stochastic returns: asset returns are (at least a little) random.
    2. Safe asset: there exists an asset that pays a riskless (non-stochastic) return of $1 + r$.
    3. Risk neutrality; \$1 in period $t + k$ is worth $\$\frac{1}{(1+r)^k}$ in period t.

- We also need a few new terms / jargon:
    - Dividends: Denote the dividend realized by an asset in period t as D_t. At date t, any future dividend D_{t+k} $(k > 0)$ is uncertain.
    - States of the world: The future is uncertain. Suppose that there are a bunch of possible futures (states of the world) indexed by j (potentially infinite set, but for simplicity, could even just be two possibilities). Denote the probability of state j as $\pi(j)$.

States of the World and Uncertainty

- Uncertainty and risk are critical to asset pricing: many real-world assets that generate economic returns are risky / uncertain (e.g. stocks).

- One common way to represent uncertainty or randomness is to express objects that are uncertain as being functions of j - 'states of the world'.

- When we write $x(j)$ for any variable x, this notation is meant to remind us that x exhibits randomness, and its value depends on the 'state' j. This is a notational choice - its purpose is just to make it clear that the value of x is random/uncertain.

- Example: Suppose y is either a high value y_H or a low value y_L with equal probability.
  - Two states: let's call them $j = 0$ and $j = 1$.
  - Writing y as a function of the state: $y(0) = y_L$ and $y(1) = y_H$
  - Probability of each state is the equal: $\pi(0) = \pi(1) = 0.5$
  - Expected value of y: $E[y(j)] = \pi(0)y(0) + \pi(1)y(1) = 0.5 \times y_L + 0.5 \times y_H$.

Risk-Neutral Asset Pricing: Buy-and-Sell

- Suppose I want to buy an asset at a price P_t and sell it in the next period for price P_{t+1}.

- Since I hold onto the asset for one period, I get one dividend payment, D_{t+1}. I also get money from selling the asset back at the end of period $t+1$. In present discounted value terms, both the dividend and sale of the asset get discounted by $1+r$ (Q: why?)

- Asset price is equal to its expected present discounted value:

$$P_t = \frac{E_t[D_{t+1} + P_{t+1}]}{1+r}$$

- Interpretation: Asset return has two components, D_{t+1} and P_{t+1} (uncertain at date t). Expected return at time t is $E_t[D_{t+1} + P_{t+1}]$, discounted by constant interest rate r.

- Why is P_t equal to this under risk-neutrality? If you buy this asset for less than P_t, it's free money (in expectation). If you buy this asset for more than P_t, you'll lose money (in expectation).

Risk-Neutral Asset Pricing: Buy-and-Sell

- Note this formula relates the price P_t to the expected price next period P_{t+1}. This is clearly recursive!

- Useful exercise for you: 'plug in' P_{t+1} to write P_t in terms of D_{t+1}, D_{t+2}, P_{t+2}. Continue one or two more times, then write it compactly as the sum we saw in class:

$$P_t = \sum_{h=1}^{T} \frac{E_t[D_{t+h}]}{(1+r)^h} + \frac{E_t[P_{t+T}]}{(1+r)^T}$$

- Note this relies on the fact that the expectations operator $E_t[\cdot]$ is linear, which means we can always 'split up' the expectation across additive terms: $E[A+B] = E[A] + E[B]$. In addition, we have $E_t[E_{t+h}[X]] = E_t[X]$ (Q: why is this intuitive?).

Risk-Neutral Asset Pricing: Buy-and-Hold

- What should the price be if people plan to hold on to an asset forever ('buy and hold')?

- If you purchase asset in period t, you receive a sequence of dividends $D_{t+1}, D_{t+2}, \ldots$ from $t+1$ on. Asset price is equal to the dividend stream's expected present discounted value:

$$P_t = \sum_{h=1}^{\infty} \frac{E_t[D_{t+h}]}{(1+r)^h}$$

- Represents an idea that most economists carry around in their heads: an asset's price should reflect the present discounted value of expected future dividends.

- Review question: Some large companies (e.g. Amazon, Google) do not currently pay dividends to shareholders - but people buy their stock for a positive price. Why?

Stochastic Dividends: Random Walk with Drift

- We can say more concrete things if we place some structure on D_t over time. We have assumed D_{t+1} is uncertain at date t, but assuming a particular stochastic process for D_{t+1} gives us more concrete predictions for our asset pricing formulas.

- For instance, one handy form is that D_t is a random walk with drift:

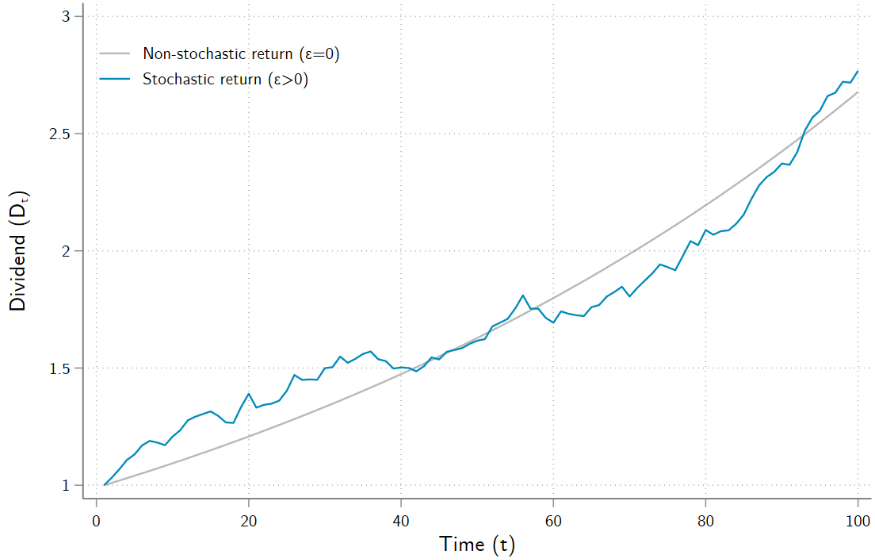
$$D_{t+1} = (1+g)D_t + \epsilon_{t+1}$$

  where ϵ is a random variable with zero mean: $E_t[\epsilon_{t+1}] = 0$.

- The term g is the trend growth rate of the dividend: its average growth rate.

- The dividend is growing over time: average growth rate of g; but period-to-period is subject to unpredictable fluctuations around the average growth rate g.

Random Walk with Drift: Simulated Example (g=0.01)



Expected Dividends: Random Walk with Drift

- At time t, what is the expected dividend at time $t+1$, $E_t[D_{t+1}$?

- Start with process for D, assumed to be a random walk with drift:

$$D_{t+1} = (1+g)D_t + \epsilon_{t+1}$$

- Apply expectations to both sides, use properties of expectations:

$$\begin{aligned}
E_t[D_{t+1}] &= E_t[(1+g)D_t + \epsilon_{t+1}] && \text{(Definition of } D_{t+1}\text{)} \\
&= E_t[(1+g)D_t + E_t[\epsilon_{t+1}] && \text{(Expectations distribute across addition)} \\
&= (1+g)E_t[D_t] + E_t[\epsilon_{t+1}] && \text{(Factoring out constant 1+g)} \\
&= (1+g)D_t + E_t[\epsilon_{t+1}] && (E_t[D_t] = D_t) \\
&= (1+g)D_t && (E_t[\epsilon_{t+1}] = 0)
\end{aligned}$$

- If dividends follow a random walk with drift, expected dividend next period, $E_t[D_{t+1}]$, is just the dividend this period times $(1+g)$. The 'shock' component ϵ is unpredictable.

Expected Dividends: Random Walk with Drift

- What about $E_t[D_{t+h}]$ for any $h > 0$? Note our expression for D_t is recursive:

$$\begin{aligned} D_{t+h} &= (1+g)D_{t+h-1} + \epsilon_{t+h} && \text{(Definition of } D_{t+h}\text{)} \\ &= (1+g)\Big[(1+g)D_{t+h-2} + \epsilon_{t+h-1}\Big] + \epsilon_{t+h} && \text{(Sub. in definition of } D_{t+h-1}\text{)} \\ &\vdots && \text{(Subbing in } D_{t+h-2}, D_{t+h-3}, \ldots\text{)} \\ &= (1+g)^h D_t + \sum_{j=1}^{h} (1+g)^{h-j} \epsilon_{t+j} \end{aligned}$$

- Applying expectations to both sides yields $E_t[D_{t+h}] = (1+g)^h D_t$.

- Intuition: Given D_t, best guess of future dividend, $E_t[D_{t+k}]$ is just the deterministic component: $(1+g)^k D_t$.

Risk-Neutral Asset Pricing: Gordon Growth Formula

- Recall the buy-and-hold formula:

$$P_t = \sum_{h=1}^{\infty} \frac{E_t[D_{t+h}]}{(1+r)^h}$$

- If D_t follows a random walk with drift, we can use our result from the prior slide to give us the Gordon growth formula:

$$P_t = \frac{E_t[D_{t+1}]}{r-g}$$

  Interpretation: In the simple case where dividends follow a random walk with drift (trend growth rate g), asset price just depends on expected dividend next period, interest rate, and growth rate.

- Review question: Why does it make sense that P_t is high when g is high or r is low?

- For your review: Make sure you can derive this! Useful practice (also in Kurlat).

Asset Pricing: Price-Dividend Ratios

- Rearranging the Gordon growth formula gives us an equation that relates the price-dividend ratio to r and g:

$$\frac{P_t}{E_t[D_{t+1}]} = \frac{1}{r-g}$$

- This is a testable prediction of the risk-neutral asset pricing framework: price-dividend ratios fluctuate because of fluctuations in r and g. Is this true?

- Campbell and Shiller (1982): Price-dividend ratio **far too volatile** to be explained by news about future dividends (g).

- Suggests either: (1) fluctuations in discount rate (r) really important; (2) our model is missing something important. Spawned 1,000 papers and hundreds of jobs in asset pricing.

Asset Pricing with Risk Aversion

- Campbell-Shiller finding suggests that the discount rate is really important. Up until now, we assumed that individuals were risk-neutral, and discount given by exogenous 'safe asset' return r.

- But how does this link to our previous models, where individuals had concave utility functions (implies risk aversion)?

- Asset pricing with risk aversion will be *extremely similar* to our Week 2 consumption-savings framework. Only difference: rather than saving at some constant, known rate of return r, agent saves in an asset that costs P_t to purchase and guarantees an uncertain (stochastic) return.

Asset Pricing with Risk Aversion (in words)

- Timing: For simplicity, two periods, t and $t+1$ (as in lecture slides)

- Household problem: Household chooses consumption each period (c_t, c_{t+1}) and savings (ξ) to maximize expected lifetime discounted utility $u(c_t) + \beta E_t[u(c_{t+1})]$.

- Income: Household receives (exogenous) income y_t and y_{t+1} in each period.

- Assets and savings: Household saves in $t = 0$ by purchasing ξ units of an asset at a per-unit price P_0. In period $t = 1$, household receive a dividend D_1 for each unit of the asset, and sell off their assets at price P_1.

- Uncertainty: Asset dividends D_{t+1} and income in the second period, y_{t+1}, are uncertain at time t: depend on 'state of the world' j that is not revealed until start of period $t+1$.

- Without looking at the next slide, can you take this verbal description and put it into math?

Asset Pricing with Risk Aversion (in math)

- Household maximizes utility by choosing consumption c_t, c_{t+1} and asset holdings ξ to maximize utility (subject to period budget constraints):

$$\max_{c_t, c_{t+1}, \xi} u(c_t) + \beta E_t[u(c_{t+1}(j))] \quad \text{subject to} \quad c_t + P_t \xi = y_t$$

$$c_{t+1}(j) = y_{t+1}(j) + \xi(D_{t+1}(j) + P_{t+1}(j))$$

- We can sub. in both constraints to express this as an unconstrained maximization problem involving just ξ:

$$\max_{\xi} u(Y_t - \xi P_t) + \beta E_t[u(y_{t+1} + \xi(D_{t+1} + P_{t+1}))]$$

- First-order condition ("asset pricing Euler equation"):

$$P_t u'(c_t) = \beta E_t[u'(c_{t+1})(D_{t+1} + P_{t+1})]$$

Stochastic Discount Factor

- Note that we can rewrite our "Euler equation" as:

$$P_t = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \Big(D_{t+1} + P_{t+1} \Big) \right]$$

- Compare with our buy-and-sell formula (using periods 0 and 1):

$$P_t = \frac{E_t[D_{t+1} + P_{t+1}]}{1 + r}$$

- Difference: In the risk-neutral case, the discount was $\frac{1}{1+r}$. Now, it is $\beta E_t[\frac{u'(c_{t+1})}{u'(c_t)}]$.

- The discount $\beta E_t[\frac{u'(c_{t+1})}{u'(c_t)}]$ is often called the stochastic discount factor.

Stochastic Discount Factor

- The stochastic discount factor (sometimes known as the pricing kernel) is crtiical to consumption-based asset pricing. Sometimes, we even give it its own variable!

$$M_{t,t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

  where the subscript indicates that it depends on marginal utility in periods t *and* $t+1$.

- With this in mind, the 'fundamental asset pricing equation' in terms of the expected SDF and returns becomes:

$$P_t = E_t[M_{t,t+1}(D_{t+1} + P_{t+1})]$$

- Notice asset price P_t is increasing in expected SDF!

- Intuition: asset is more valuable when expected marginal utility in period $t+1$ higher than period t, i.e. consumption low, because asset delivers more 'bang for buck'