Harvard Economics
Econometrics Math Camp 2020

1 Break-Out Session #1

The following questions concern deriving basic laws of probability from the setup we discussed during the first hour or so of lecture. It may be helpful to refer to the econometrics math camp slides (or the much beefier econometrics math camp notes) for details on the properties of σ-algebras, measures, etcetera.

These exercises have two purposes. First, I want to demonstrate how you can take a measure-theoretic presentation of probability theory - with a very small list of mathematical axioms - and derive all of the usual laws of probability you're used to. Second, I want you to meet some more folks in your cohort and work through some problems with them. In my view, this second objective is way more important than the first.

1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Please prove the following laws of probability:
   (a) For any $A \in \mathcal{A}$, $P(A^c) = 1 - P(A)$
   (b) $P(\Omega) = 1$
   (c) For all $A \in \mathcal{A}$, $0 \leq P(A) \leq P(\Omega)$
   (d) If $A_1, A_2 \in \mathcal{A}$ with $A_1 \subseteq A_2$, then $P(A_1) \leq P(A_2)$
2. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Consider K disjoint events C_k that partition the sample space Ω, and let $A \in \mathcal{A}$ be an event. Please prove the law of total probability, which states that $P(A) = \sum_{i=1}^{K} P(A|C_i)P(C_i)$.
3. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and consider two events $A, B \in \mathcal{A}$. Recall that Bayes' rule can be stated as $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$. Please derive this rule.

2 Break-Out Session #2

The following questions concern random variables, distributions, and expectations.

1. I earlier said that any function F which satisfies the four 'handy properties' of a cumulative distribution function can be used to construct a random variable, call it Y, whose cumulative distribution function is F. Please detail this construction.

2. We talked a little bit about how transformations of random variables affect their distributions. We can often say something concrete if the transformation is 'nice'. Suppose that X is drawn from a uniform distribution on the unit interval and that $Y = X^2$. What is the cdf and pdf of Y?

3. The following questions concern characterizing the distribution of a variable X that is uniformly distributed on the interval $[-1, 1]$.

   (a) What is the pdf of X? What about the cdf?

   (b) Derive an expression for the q^{th} quantile of X.

   (c) What is the mean of X?

4. Recall the CEF decomposition theorem, which states that a random variable Y can be expressed as $Y = E[Y|X] + \epsilon$, with ϵ uncorrelated with any function of another random variable X. Prove this. Hint: start by showing $E[\epsilon|X] = 0$, and then use the law of iterated expectations with this result to prove that ϵ is uncorrelated with any function of X. This is a little tricky!

5. Let X and Y denote random variables. Recall the law of iterated expectations, $E_Y[Y] = E_X E_{Y|X}[Y]$. Prove it. It may be useful to start from the right-hand side of the equation, write it as a double integral, and rearrange terms until you find that it is equal to $E[Y]$.