Harvard Econometrics Math Camp
Worksheet #2

The following questions concern random variables, distributions, and expectations.

1. Earlier today, I claimed that any function F which satisfies the four 'handy properties' of a cumulative distribution function can be used to construct a random variable, call it Y, whose cumulative distribution function is F. Please detail this construction. For instance, you may assume you are explaining to an RA how to implement this in their coding language of choice. You may assume that F is invertible for simplicity, and you may assume you can easily "draw from a uniform distribution" (hint hint).

2. We also spent some time discussing the distribution of transformations of random variables. Suppose that X is drawn from a uniform distribution on the unit interval, and the random variable Y is a simple, known function of X: $Y = X^2$. What is the cdf and pdf of Y?

3. The following questions concern characterizing the distribution of a variable X that is uniformly distributed on the interval $[-1, 1]$.

   (a) What is the pdf of X? What about the cdf?

   (b) Derive an expression for the q^{th} quantile of X.

   (c) What is the mean of X?Please compute it using the integral definition of the expectation.

4. Let X and Y denote random variables. The CEF decomposition theorem states that Y can be expressed as $Y = E[Y|X] + \epsilon$, where ϵ is uncorrelated with *any* function of another random variable X. Prove this. You may assume the conditional expectation $E[Y|X]$ exists and is unique.
   Hint: this problem is tricky! You may find it helpful to start by showing $E[\epsilon|X] = 0$, and then use the law of iterated expectations with this result to prove that ϵ is uncorrelated with any function of X.

5. Let X and Y denote random variables. Recall the law of iterated expectations, $E_Y[Y] = E_X E_{Y|X}[Y]$. Prove it. It may be useful to start from the right-hand side of the equation, write it as a double integral, and rearrange terms until you find that it is equal to $E[Y]$.