Interactive Awareness of Unawareness*

Burkhard C. Schipper†

Preliminary & Incomplete: May 28, 2021

Abstract

I extend unawareness structures of Heifetz, Meier, and Schipper (2006) to allow for quantification over events. This enables me to model events like "I find it possible that there might exist something that I am unaware of.", "I know that there might exist something that she is aware of but I am unaware of", "I know that there might exist something that I am unaware of but others have common awareness of it even though they do not have common awareness of all the things that I might be unaware of" etc. despite knowledge and unawareness retaining standard properties.

Keywords: Quantification, Aumann structures, Kripke frames, unknown unknowns, partitional information, interactive epistemology.

JEL-Classifications: C70, D83.

*Paper prepared for the invited plenary lecture at TARK 2021, Beijing. I thank Giacomo Bonanno and Aviad Heifetz for comments on the first draft. The responsibility for errors rests with me.

†Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu

"It is not clear how to capture knowledge of unawareness directly in the HMS approach." Halpern and Rêgo (2012)

1 Introduction

After the development of the foundations of modelling unawareness[1], in recent years the focus of the literature began shifting to applications to game theory[2], decision theory[3], contract theory[4], speculation[5], financial markets[6], general equilibrium[7], strategic network formation[8], business strategy[9], and politics[10]. However, one foundational question that is of importance to many potential applications of unawareness remained open: How to model awareness of unawareness in a tractable way? For instance, Grant and Quiggin (2013) argue that an individual should inductively infer from past experience and the unawareness of others that it might be possible that there exists something she herself is unaware of. While models of unawareness allow for non-trivial unawareness, modelling awareness of unawareness proves to be even harder. There has been work on reasoning about knowledge of unawareness by for instance Board and Chung (2011), Halpern and Rêgo (2009, 2012), Sillari (2008a, b), and Halpern and Piermont (2019), some of which inspired the approach presented in this paper. However, these approaches might be a challenge to apply readily to game theory since they either involve a semantics that is not syntax-free or an additional set of "objects" rather than just events. Instead of requiring applied researchers to deal with an sophisticated first-order modal logic of awareness, it would be desirable to model awareness of unawareness in purely event-based unawareness structures that are arguably closer to Aumann structures or type spaces familiar to economists. As Halpern and Rêgo (2012) remark, how to capture awareness of unawareness in event-based unawareness structures remains open. This paper fills the gap.

The goal is to come up with a purely event-based structure capable of modelling non-trivial unawareness of events as well as rich interactive reasoning about knowledge about unawareness such as "I find it possible that there might exists something that I am unaware.", "I know that there might exist events that she is aware of but I am unaware of.", "I know that there might exist events that I am unaware of but both persons 1 and 2 are commonly aware of but person 2 is also are of more events than person 1." and "I know that there might exist different events of persons 1 and 2 are aware of, respectively, but that I am unaware of." etc.[11] All these should be achieved with minimal modifications to existing unawareness structures. In particular, all

[1]See Fagin and Halpern (1988) for the first paper on the topic, Heifetz, Meier and Schipper (2006, 2008, 2013a) and Halpern and Rêgo (2008, 2012) for work closest to the present paper, and Schipper (2015) for a survey of the literature on the foundations of awareness.

[2]See for instance Feinberg (2021), Halpern and Rêgo (2014), Heifetz, Meier, and Schipper (2013b), Meier and Schipper (2014a), and Schipper (2021).

[3]See for instance, Karni and Vierø (2013), Schipper (2014), and Dominiak and Tserenjigmid (2018).

[4]See for instance von Thadden and Zhao (2012), Filiz-Ozbay (2012), Auster (2013), Auster and Pavoni (2021a), and Francetich and Schipper (2021).

[5]See Heifetz, Meier, and Schipper (2013a), Meier and Schipper (2014b) and Galanis (2018).

[6]See for instance Auster and Pavoni (2021b) and Schipper and Zhou (2021).

[7]See for instance Modica, Rustichini, and Tallon (1998) and Teeple (2021).

[8]Schipper (2016)

[9]Bryan, Ryall, and Schipper (2021)

[10]Schipper and Woo (2019)

[11]Modeling such a rich reasoning about unawareness cannot be accommodated with less expressive approaches to knowledge of unawareness by for instance Ågotnes and Alechina (2007), Walker (2014), and Karni and Vierø

standard properties of unawareness should remain intact. Moreover, it should be consistent with the strong notion of knowledge stemming from partitional information structures that have been used extensively in applications throughout economics.

Our starting point are event structures of Heifetz, Meier, and Schipper (2006) which consist of a complete lattice of state-space together with projections. These spaces are partially ordered by their "expressiveness", roughly the richness of the set of events associated with them. For any space (call it the base-space of the event) and subset of the space (call it the base of the event), the union of inverse images of this subset constitutes an event. Negation of the event is defined as the union of the inverse images of the relative complement of the base of the event w.r.t. the base-space. Conjunction of events is the intersection. Disjunction is defined by the DeMorgan Law. Knowledge, awareness, and unawareness of events are defined in a standard way.

In this paper, the first novel addition is a domain correspondence that maps each state to a set of all events of some space. Intuitively, the domain correspondence assigns to each state the domain of discourse. We require that the domain of discourse contains at least all events that are expressible in the space of the state. However, it can even contain more events. Those events remain just "something" at the level of expressiveness associated with the state. As an example, consider the legend that Inuits have a plethora of words for all kinds of snow. Envision an average German listen to an Inuit uttering the proposition "Today we got muruaneq." The German is not able to relate to this proposition in the sense of even recognizing it as a proposition (although he may realize that Intuits recognize it has a proposition). He lacks the background to think about situations in which this statement is true or false. In particular, he does not realize that it is about some kind of soft deep snow. Yet, he must realize that there is "something". So the domain correspondence is precisely our solution to allow reference to "something" that at that state may not be expressible yet but could be expressible at a state in a more expressive state space.

The second novel addition is a set of variables X that eventually stand for events in our structure. The third novel device are general n-ary event operators. These event operators map n events into an event. The final novel aspect is the addition of universally quantified events such as given space S, "for all variables x we have $E(x)$", where $E(x)$ is an event operator. At each state in such a quantified event, the quantifier can range over all events in the domain of discourse at that state. As mentioned earlier, this might contain even events that are not even expressible at that state. We also add existentially quantified events such as given space S, "there might exist x such that $E(x)$". With this additional structure in place, events like "there might exists something that i is unaware of" can be simply expressed by recognizing that the unawareness operator U_i on events is an event operator. All standard properties on knowledge and unawareness of Heifetz, Meier, and Schipper (2006, 2008) remain valid in quantified unawareness structures. Moreover, they extend to quantified events involving knowledge and unawareness of events under the scope of the quantifier. Awareness "inside" the quantifier is related to awareness "outside" the quantifier by a Barcan formula, which is valid in our structures. Yet, the controversial Barcan formula for knowledge is not valid in our structure unless restrictive assumptions on the domain correspondence hold, echoing similar observations made in first-order modal logic (Hughes and Cresswell, 1996, Chapter 15). Generally the approach presented in this paper can be viewed as a purely event-based analogue

(2017) featuring essentially "catch all" propositions.

to first-order modal logic with knowledge and awareness. This is akin to Heifetz, Meier, and Schipper (2006) roughly being the purely event-based analogue to Fagin and Halpern (1988) and Aumann (1999) being the event-based analogue to Kripke structures.

It might be worthwhile considering notions of awareness weaker than the standard notion of (propositional) awareness. However, the point of the paper is to show that modeling awareness of unawareness does not require a tweaking of the existing standard notion of unawareness. This was not obvious ex ante. One of the properties of unawareness is KU-introspection (Dekel, Lipman, Rustichini, 1998) saying that you never know that you are unaware of an event, $K_i U_i(E) = \emptyset^{S(E)}$. Yet, as we show in Example 2 in Section 3 this does not preclude that you could not know that there might exist something that you are unaware of, $K_i(\exists_S x U_i(x)) \neq \emptyset^S$. Another property of unawareness is AU-introspection (Dekel, Lipman, and Rustichini, 1998) saying that if you are unaware of an event then you are unaware that you are unaware of that event, $U_i(E) \subseteq U_i U_i(E)$. Again, this is consistent with being aware that there might exist something that you are unaware of as in the current paper we can have $U_i(E) \cap U_i U_i(E) \cap A_i(\exists_S x U_i(x)) \neq \emptyset^{S \wedge S(E)}$. I.e., you are unaware of the event E, unaware that you are unaware of E, and you are aware that there might exist something that you are unaware of. This is also demonstrated in Example 2 of Section 3. We emphasize again that allowing for a rich notion of awareness of unawareness does not require tweaking of the by now standard notion of unawareness. It does not mean though that it would not be worthwhile to explore weaker or alternative notions of unawareness such as in Fukuda (2021) who gives up AU introspection in order to capture something like awareness of unawareness if structures are infinite. In the present paper, the ability to model non-trivial awareness of unawareness does not depend on the cardinality of the state-space.

The present approach can be generalized in a straight-forward way to weaker notions of knowledge. However, the point of the paper is precisely to show that awareness of unawareness is consistent with the strong "partitional" notion of knowledge used in many applications in economics. This is similar to Heifetz, Meier, and Schipper (2006) who show that modeling unawareness of events does not require a weakening of the notion of knowledge but can be consistent with the strong "partitional" notion of knowledge. It does not mean though that it would not be worthwhile to investigate generalizations of our approach to weaker notions of knowledge, such as giving up the truth property (Halpern and Rêgo, 2008, Board, Chung, and Schipper, 2011), positive and weak negative introspection (Halpern and Rêgo, 2008), or weakening projective properties of the possibility correspondence as in Galanis (2013).

The paper is organized as follows: The next section introduces quantified unawareness structures. This is followed by an illustration with an application to consulting with expert(s) in Section 3. I conclude with a discussion in Section 4. Proofs are relegated to the appendix.

2 Quantified Unawareness Structures

As in Heifetz, Meier, and Schipper (2006), we consider a complete lattice of nonempty disjoint spaces $\mathcal{S} = \{S_\ell\}_{\ell \in L}$ with a partial order $\succeq$ and surjective projections $r^{S'}_S : S' \longrightarrow S$ for $S', S \in \mathcal{S}$ with $S' \succeq S$ such that for all $S \in \mathcal{S}$, $r^S_S = id_S$. Projections commute, i.e., for all $S'', S', S \in \mathcal{S}$ with $S'' \succeq S' \succeq S$, $r^{S''}_S = r^{S'}_S \circ r^{S''}_{S'}$. We denote by $\Omega := \bigcup_{\ell \in L} S_\ell$ and by $\omega_S := r^{S'}_S(\omega)$ when $\omega \in S'$ with $S' \succeq S$. Sometimes we denote by S_ω the space $S \in \mathcal{S}$ for which $\omega \in S$.

For any $S \in \mathcal{S}$ and $D \subseteq S$, denote by $D^{\uparrow} := \bigcup_{S' \succeq S} \left(r_S^{S'}\right)^{-1}(D)$. An event $E \subseteq \Omega$ is a set of states such that there is a base $D \subseteq S$ for some $S \in \mathcal{S}$ and $E := D^{\uparrow}$. We call S the bases-space of E and denote it by $S(E)$. The definition of event is clear unless the event is empty. Thus, for each $S \in \mathcal{S}$ we denote by $\emptyset^S$ the empty event with base-space S. If E is an event with base $D \subseteq S$, then $\neg E := (S \setminus D)^{\uparrow}$. If E and F are both events, then the conjunction of the events is simply $E \cap F$. Moreover, disjunction of events, denoted by $E \cup F$, is defined by $\neg(\neg E \cap \neg F)$ (using the above definition of negation). For any space S, denote by $\Sigma(S)$ the set of events with base-space $\cancel{S' \preceq}\ S$. Clearly, for any $S, S' \in \mathcal{S}$ with $S' \preceq S$ we have $\Sigma(S') \subseteq \Sigma(S)$. If $\overline{S}$ is the join of the lattice, then we denote by $\Sigma = \Sigma(\bar{S})$ the set of all events in the lattice. The meet of the lattice we denote by $\underline{S}$.

We extend the unawareness structure of Heifetz, Meier, and Schipper (2006) by allowing for quantification. Let X be a countably infinite set of variables. We will allow for quantification over these variables. These variables take values in the domain of events to be specified.

For any space $S \in \mathcal{S}$ and finite n, we denote $E(x_1, ..., x_n)$ the S-based n-ary event operator. If $x_1, ..., x_n$ in $E(x_1, ..., x_n)$ are replaced by events $F_1, ..., F_n \in \Sigma$, then $E(F_1, ..., F_n)$ is an event with base-space $S \wedge \bigwedge_{k=1}^{n} S(F_k)$. For instance, the unary event operator $E(x) = x$ is a $\underline{S}$-based operator. For any event $F \in \Sigma$, $E(F) = F$ is the $\underline{S} \wedge S(F) = S(F)$ based-event. Similarly, $E(x) = \neg x$ is a $\underline{S}$-based operator. For any event $F \in \Sigma$, $E(F) = \neg F$ is the $\underline{S} \wedge S(F) = S(F)$ based-event. Jumping ahead, the knowledge, awareness, unawareness, mutual knowledge, common knowledge, mutual awareness, and common awareness operators defined later in the text are $\underline{S}$-based unary event operators. The importance of the base-space of the operator can be illustrated with the operator defined by: For a fixed event $F \in \Sigma$, define the $S(F)$-based unary operator $E(x) = F \cap x$. For any event $G \in \Sigma$, $E(G) = F \cap G$ is the $S(F) \wedge S(G)$ based event. As a final example, conjunction $E(x_1, x_2) = x_1 \cap x_2$ is an example of the $\underline{S}$-based binary event operator. For any pair of events $F, G \in \Sigma$, $E(F, G) = F \cap G$ is the $\underline{S} \wedge S(F) \wedge S(G) = S(F) \wedge S(G)$ based event.

Define a *domain correspondence* $\mathcal{D} : \Omega \longrightarrow \{\Sigma(S_\ell)\}_{\ell \in L}$ such that[12]

(i) $\omega \in S$ implies $\mathcal{D}(\omega) \supseteq \Sigma(S)$,

(ii) $\mathcal{D}(\omega_S) \subseteq \mathcal{D}(\omega)$ for all $\omega \in \Omega$ and $S \preceq S_\omega$.

The domain correspondence assigns to each state a domain of discourse. This is a set of events that contains at least the set of events that are expressible in the "language" associated with the space of which the state is an element of. That is, at each state the domain of discourse is as least as rich as the language associated with the state. Yet, it may contain more events even though these events cannot be evaluated w.r.t. whether or not they obtain in the space of which the state is an element of. Intuitively, the domain of discourse at a state may contain statements that are not meaningful events expressible in the state space of that state. In the introduction we mentioned as examples of such statements propositions involving the rich

[12] In some applications, one may want to impose that for any $S \in \mathcal{S}$ the set of states $[\Sigma(S)] := \{\omega \in \Omega : \Sigma(S) \subseteq \mathcal{D}(\omega)\}$ forms an event in the unawareness structure. In such a case, $[\Sigma(S)]$ is the event that the domain of discourse contains at the least the events in $\Sigma(S)$. This assumption would allow individuals to reason about the domain of discourse. We will not impose this property in general though. As in real life, the domain of discourse is always present in the background of any thoughts or conversations but rarely itself the object of thoughts and conversations.

vocabulary of Inuits for various kinds of snow that would not be comprehensible to people lacking the background of living in the Arctic.

For each space $S \in \mathcal{S}$, we devise a universal quantifier $\forall_S$ quantifying over variables that can refer to events as well as "things" associated with that space by the domain of discourse. For each event operator and space $S \in \mathcal{S}$,

$$\forall_S x E(x) := \left\{ \omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcap_{F \in \mathcal{D}(\omega)} E(F) \right) \subseteq \bigcap_{F \in \mathcal{D}(\omega)} E(F) \right\}^{\uparrow}.$$

In words, we interpret this event as "given S, for all events x we have $E(x)$". To understand this definition, think of $\forall_S$ as a universal quantifier quantifying over *at least* all events in $\Sigma(S)$ and potentially more "things". What are events and what are just "things" depends on the space S. That's why there is a quantifier for each space S in the lattice of state-spaces $\mathcal{S}$. We have $\omega \in \forall_S x E(x)$ if and only if the event $\{\omega\}^{\uparrow} \cap S^*$ implies $\bigcap_{F \in \mathcal{D}(\omega)} E(F)$ where S^* is the base-space of the event $\bigcap_{F \in \mathcal{D}(\omega)} E(F)$. The richer the domain of discourse, the more expressive must be S^*. When S^* is strictly more expressive than S, there will be events in the domain of discourse that remain just "things" at the level of space S. Note that by definition, $\forall_S x E(x)$ is an S-base event.

The existential quantifier is defined us usual using the universal quantifier and (our notion of) of negation. For each event operator and space $S \in \mathcal{S}$,

$$\exists_S x E(x) := \neg \forall_S x \neg E(x).$$

We interpret it as "given S there might exist an event x for which $E(x)$". In Appendix A we further discuss the definition of quantifiers including characterizations and less appropriate alternatives.

If the lattice of spaces is a singleton, i.e., we just have a standard state-space, then simply $\forall x E(x) = \bigcap_{F \in \Sigma} E(F)$ and $\exists x E(x) = \bigcup_{F \in \Sigma} E(F)$. When the lattice of spaces is non-trivial, then these formulas hold for the join of the lattice, $\overline{S}$.

Lemma 1 *For the join of the lattice, $\overline{S}$,*

$$\begin{aligned} \forall_{\overline{S}} x E(x) &= \bigcap_{F \in \Sigma} E(F) \\ \exists_{\overline{S}} x E(x) &= \bigcup_{F \in \Sigma} E(F) \end{aligned}$$

The proof is immediate upon realizing that for all $\omega \in \overline{S}$ we must have $\mathcal{D}(\omega) = \Sigma$. We show with a counterexample below that the observation does not extend necessarily to other spaces of the lattice.

The short proof of the following standard identities and implications are contained in the appendix.

Lemma 2 *For any space $S \in \mathcal{S}$ any event operators $E(x)$ and $F(x)$,*

$$\begin{aligned} \forall_S x E(x) \cap \forall_S x F(x) &= \forall_S x (E(x) \cap F(x)) \\ \exists_S x E(x) \cup \exists_S x F(x) &= \exists_S x (E(x) \cup F(x)) \end{aligned}$$

Lemma 3 *For any space $S \in \mathcal{S}$ and any event operator $E(x)$,*

$$\begin{aligned} \forall_S x \neg E(x) &\subseteq \neg \forall_S x E(x) \\ \neg \exists_S x E(x) &\subseteq \exists_S x \neg E(x) \end{aligned}$$

The analogue of following properties are typical axioms in some first-order logics:

Lemma 4 *The following properties hold:*

(i) For any space $S \in \mathcal{S}$ and any event operators $E(x)$ and $F(x)$

$$\forall_S x(\neg E(x) \cap F(x)) \subseteq \neg \forall_S x E(x) \cap \forall_S x F(x).$$

(ii) For any S'-based event operator $E(x)$ and $S \succeq S'$

$$\forall_S x E(x) \subseteq E(F)$$

for any event F with base-space $S(F)$ such that $S \succeq S(F)$.

The proof of the next lemma is immediate for the case of $\overline{S}$ or a singleton lattice of spaces. More generally, it follows for any $S \in \mathcal{S}$ immediately from the definitions.

Lemma 5 *For any event operators, $E(x)$ and $F(x)$, if $E(G) \subseteq F(G)$ for all $G \in \Sigma$, then for any $S \in \mathcal{S}$,*

$$\begin{aligned} \forall_S x E(x) &\subseteq \forall_S x F(x) \\ \exists_S x E(x) &\subseteq \exists_S x F(x) \end{aligned}$$

In Appendix A we present counterexamples that prove the following remark.

Remark 1 *For any $S, S' \in \mathcal{S}$ with $S' \succ S$ and any event operator $E(x)$, the event $\exists_{S'} x E(x)$ is unrelated to the event $\exists_S x E(x)$ and the event $\forall_{S'} x E(x)$ is unrelated to the event $\forall_S x E(x)$.*

Fix a set of individuals I. As in Heifetz, Meier, and Schipper (2006), there is for each $i \in I$ a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega}$ such that

(0) Confinement: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.

(i) Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.

(ii) Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.

(iii) Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$.

(iv) Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, then $(\Pi_i(\omega))_S = \Pi_i(\omega_S)$.

For each $i \in I$, define the knowledge operator on events by

$$K_i(E) := \{\omega \in \Omega : \Pi_i(\omega) \subseteq E\},$$

if there is a state ω such that $\Pi_i(\omega) \subseteq E$, and by $K_i(E) := \emptyset^{S(E)}$ otherwise. Heifetz, Meier, and Schipper (2006, Proposition 1) show that for any event E, the set $K_i(E)$ is an $S(E)$-based event. For each $i \in I$, define also the awareness and unawareness operator on event, respectively by

$$\begin{array}{rcl} A_i(E) & := & K(S(E)^{\uparrow}) \\ U_i(E) & := & \neg A_i(E) \end{array}$$

We refer to Heifetz, Meier, and Schipper (2006, 2008) for standard properties satisfied by knowledge, awareness, and unawareness in unawareness structures. Note that the knowledge operator, awareness operator, and unawareness operator are all $\underline{S}$- based event operators according to the terminology introduced earlier.

The following observations are immediate corollaries of Lemma 5 and Heifetz, Meier, and Schipper (2006, Propositions 2 and 3). Essentially, properties of knowledge, awareness, and unawareness of unawareness structures are preserved under quantifiers.

Corollary 1 *For any $i \in I$, event operators $E(x), E_1(x_1), E_2(x_2), ...$, and space S,*

1. *Monotonicity under Quantifiers: $\forall_S x E(x) \subseteq \forall_S x F(x)$ implies $\forall_S x K_i(E(x)) \subseteq \forall_S x K_i(F(x))$*
2. *Distribution under Quantifiers: $\forall_S x_1, x_2, ... K_i \left(\bigcap_{n=1,2,...} E_n(x_n) \right) \subseteq \bigcap_{n=1,2,...} \forall_S x_n K_i(E_n(x_n))$*
3. *Truth under Quantifiers: $\forall_S x K_i(E(x)) \subseteq \forall_S x E(x)$*
4. *Positive Introspection under Quantifiers: $\forall_S x K_i(x) \subseteq \forall_S x K_i(K_i(x))$*
5. *Generalized Negative Introspection under Quantifiers I: $\forall_S x(\neg K_i(x) \cap A_i(x)) \subseteq \forall_S x K_i(\neg K_i(x))$*
6. *Generalized Negative Introspection under Quantifiers II: $\forall_S x(\neg K_i(x) \cap \neg K_i \neg K_i(x)) \subseteq \forall_S x \neg K_i \neg K_i \neg K_i(x)$*

All properties above hold also when all $\forall_S$ are replaced by $\exists_S$.

Corollary 2 *For any $i \in I$, event operators $E(x), E_1(x_1), E_2(x_2), ...$, and space S,*

1. *Quantified Plausibility: $\exists_S x U_i(x) = \exists_S x \neg K_i \neg K_i(E(x))$*
2. *Quantified Strong Plausibility: $\exists_S x U_i(x) = \exists_S x \bigcap_{n=1}^{\infty} (\neg K_i)^n \neg K_i(E(x))$*
3. *Quantified KU Introspection: $\exists_S x K_i U_i(x) = \emptyset^S$*
4. *Quantified AU Introspection: $\exists_S x U_i(x) = \exists_S x U_i U_i(x)$*
5. *Quantified Weak Necessitation: $\exists_S x A_i(x) = K_i(S^{\uparrow})$*
6. *Quantified Symmetry: $\exists_S x A_i(E(x)) = \exists_S x A_i(\neg E(x))$*

7. *Quantified A-Conjunction:* $\forall_S x_1, x_2, ...A_i \left(\bigcap_{n=1,2,...} E_n(x_n)\right) \subseteq \bigcap_{n=1,2,...} \forall_S x_n A_i(E_n(x_n))$
8. *Quantified AK-Self-Reflection:* $\exists_S x A_i(E(x)) = \exists_S x A_i K_i(E(x))$
9. *Quantified AA-Self-Reflection:* $\exists_S x A_i(E(x)) = \exists_S x A_i A_i(E(x))$
10. *Quantified A-Introspection:* $\exists_S x A_i(E(x)) = \exists_S x K_i A_i(E(x))$

All properties above hold also when all $\exists_S$ are replaced by $\forall_S$.

While previous properties readily followed from Heifetz, Meier, and Schipper (2006) and Lemma 5, we can derive a number of new properties. In quantified modal logic, the so called Barcan formulas establish a connection between modal operators inside and outside the quantifier. We can show that the Barcan formula holds for awareness. The proof is contained in the appendix.

Proposition 1 (Barcan Formula for Awareness) *For any individual $i \in I$, space $S \in \mathcal{S}$, and event operator $E(x)$,*

$$\forall_S x A_i(E(x)) \subseteq A_i \forall_S x E(x).$$

That is, if at space S for all x individual i is aware of $E(x)$ then the individual is aware that for all x we have $E(x)$.

The Barcan formula for knowledge does not hold unless the discourse correspondence is measurable w.r.t. the possibility correspondence.

Proposition 2 (Barcan Formula for Knowledge) *For any individual $i \in I$ if for any $\omega \in \Omega$, $\omega' \in \Pi_i(\omega)$ implies that $\mathcal{D}(\omega') \subseteq \mathcal{D}(\omega)$, then for any space $S \in \mathcal{S}$ and event operator $E(x)$,*

$$\forall_S x K_i(E(x)) \subseteq K_i \forall_S x E(x).$$

Under the measurability condition, given the space S, if for all x the individual i knows $E(x)$ then she knows that for all x we have $E(x)$. The measurability is not necessary. This is demonstrated in Example 1 below. The proof of Proposition 2 is contained in the appendix. It relies on generalized reflexivity (i.e., truth) and stationarity (i.e., positive introspection). Note that the condition $\omega' \in \Pi_i(\omega)$ implies that $\mathcal{D}(\omega') \subseteq \mathcal{D}(\omega)$ can be replaced by $\omega' \in \Pi_i(\omega)$ implies that $\mathcal{D}(\omega') = \mathcal{D}(\omega)$ since we assume stationarity anyway. This condition is undesirable as it precludes uncertainty about the domain of discourse. We will *not* assume this condition for the remainder of the paper. Note though that in the most expressive space, $\overline{S}$, $\mathcal{D}(\omega) = \Sigma$ for all $\omega \in \overline{S}$. Thus, the Barcan formula for knowledge holds in the join space of the lattice.

Corollary 3 (Barcan Formula for Knowledge in the Most Expressive Space) *For any individual $i \in I$ and event operator $E(x)$,*

$$\forall_{\overline{S}} x K_i(E(x)) \subseteq K_i \forall_{\overline{S}} x E(x).$$

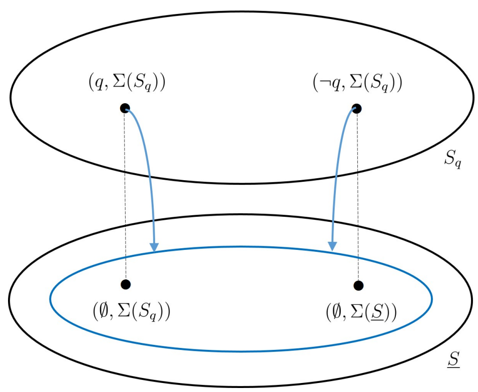
The previous observations suggest that uncertainty about the domain of discourse is required to model the possibility of being unaware of something. We like to allow for nonempty events like

$$\neg K_i \exists_S x U_i(x) \cap \neg K_i \forall_S x A_i(x).$$

Individual i does not know whether there might exist something that she is unaware of and she does not know whether she is aware of all things. That is, she considers it possible that she is unaware of something but she is not sure about it. The following example illustrates such a state of mind.

Example 1 (Possibility of Being Unaware of Something) Consider the unawareness structure of Figure 1. There are two spaces. In the upper space, S_q, there are two states, one for q and one for $\neg q$. The domain of discourse is $\Sigma(S_q)$ in both states. The lower space, $\underline{S}$, features no proposition, i.e., an empty description. The two states just differ by the domain of discourse. Projections are indicated by dashed lines. There is just one individual whose possibility correspondence is depicted by the blue ovals and arrows. No matter the state, the possibility set is on $\underline{S}$ and comprises of both states in $\underline{S}$. Let $[q]$ be the event that q. I.e., $[q] = \{(q, \Sigma(S_q))\}$. It is easy to verify that $U_i([q]) = S_q$, $A_i([q]) = \emptyset^{S_q}$, and $A_i(\Omega) = \Omega$.

Figure 1: Possibility of Being Unaware of Something



It is also easy to verify that $\exists_{\underline{S}} x U_i(x) = \{(\emptyset, \Sigma(S_q))\}^{\uparrow}$ and $\forall_{\underline{S}} x A_i(x) = \{(\emptyset, \Sigma(\underline{S}))\}^{\uparrow}$. Both, $K_i \exists_{\underline{S}} x U_i(x) = \emptyset^{\underline{S}}$ and $K_i \forall_{\underline{S}} x A_i(x) = \emptyset^{\underline{S}}$, and thus both $\neg K_i \exists_{\underline{S}} x U_i(x) = \Omega$ and $\neg K_i \forall_{\underline{S}} x A_i(x) = \Omega$. Hence, $\neg K_i \exists_{\underline{S}} x U_i(x) \cap \neg K_i \forall_{\underline{S}} x A_i(x) = \Omega$. That is, in this simple structure it is always the case that individual i considers it possible that she is unaware of something. Moreover, she is introspective about it, i.e., $K_i(\neg K_i \exists_{\underline{S}} x U_i(x) \cap \neg K_i \forall_{\underline{S}} x A_i(x)) = \Omega$.

In fact, the only event that is known by individual i in this unaware structure is the simplest tautology, Ω. Thus, the Barcan formula for knowledge is trivially satisfied in this unawareness structure despite allowing for uncertainty about the domain of discourse in the lowest space, $\underline{S}$. This demonstrates that the condition of the domain correspondence being measurable w.r.t. the possibility sets in Proposition 2 is not a necessary condition for the Barcan formula of knowledge. $\square$

The Barcan formula for awareness implies as a Corollary a property on unawareness inside and outside the existential quantifiers. The short proof is contained in the appendix.

Corollary 4 *For any individual $i \in I$, space $S \in \mathcal{S}$, and event operator $E(x)$,*

$$U_i \exists_S x E(x) \subseteq \exists_S x U_i(E(x))$$

Typically the converse Barcan formula is valid in first-order modal logic. However, in Appendix B we show by example the following:

Remark 2 *The converse Barcan formula for awareness, $A_i \forall_S x E(x) \subseteq \forall_S x A_i(E(x))$ does not hold. Equivalently, $\exists_S x U_i(E(x)) \subseteq U_i \exists_S x E(x)$ does not hold.*

The fact that the converse Barcan formula for awareness does not hold is not a defect but a blessing! Suppose to the contrary that it would hold, i.e., $A_i \forall_S x E(x) \subseteq \forall_S x A_i(E(x))$. Then an implication is the following: $\exists_S x U_i(E(x)) \subseteq U_i \exists_S x U_i(E(x))$ (see the proof of Remark 2 in the appendix). That is, if there might be an event of which the individual is unaware that she could never be aware of that fact! This is certainly not a desirable property and it does not hold in our structure.

One may wonder whether Barcan style formulas hold when the universal quantifier is replaced by the existential quantifier. This is not the case. The following remark is proved in Appendix B.

Remark 3 *Property $\exists_S x A_i(E(x)) \subseteq A_i \exists_S x E(x)$ does not hold. Equivalently, $U_i \forall_S x E(x) \subseteq \forall_S x U_i(E(x))$ does not hold.*

Yet, the converse properties mentioned in Remark 3 hold. The proof follows from arguments made in the proof of Remark 3 contained in Appendix B.

Remark 4 *For any individual i, space $S \in \mathcal{S}$, and event operator $E(x)$,*

$$A_i \exists_S x E(x) \subseteq \exists_S x A_i(E(x)),$$

and, equivalently,

$$\forall_S x U_i(E(x)) \subseteq U_i \forall_S x E(x).$$

We record some further properties of quantified awareness and unawareness, which are proved in the in Appendix B.

Proposition 3 *Recall that $\underline{S}$ and $\overline{S}$ denote the meet and join of lattice $\mathcal{S}$, respectively.*

1. *Never unaware of everything: For any space $S \in \mathcal{S}$,*

$$\forall_S x U_i(x) = \emptyset^S \text{ or, alternatively, } \exists_S x A_i(x) = S^{\uparrow}$$

2. *FA: For all spaces $S \in \mathcal{S}$,*

$$\forall_S x U_i(x) \subseteq K_i(\forall_S x U_i(x))$$

3. In the least expressive space, awareness of unawareness always obtains:

$$A_i\left(\exists_{\underline{S}} x U_i(x)\right) = \Omega$$

4. For each space, awareness of unawareness is awareness of tautologies: For all $S \in \mathcal{S}$,

$$A_i\left(\exists_S x U_i(x)\right) = A_i(S^{\uparrow})$$

5. In the most expressive space, knowledge of full awareness is knowledge of tautologies:

$$K_i\left(\forall_{\overline{S}} x A_i(x)\right) = K_i(\overline{S})$$

The first property says that an individual can not be unaware of all events. That's intuitive because she should be aware of at least the simplest tautology, Ω. Equivalently, it is always the case that there exists an event of which she is aware of, namely, the simplest tautology.

Property FA has been imposed by Halpern and Rêgo (2012). It says that given any space S, if for all x individual i is unaware of x then she knows that for all x she is unaware of x. It holds trivially in our structures since both sides of the inclusion relation are vacuous events.[13]

Property 4 says that you are aware that you are unaware of something possibly beyond the language $\Sigma(S)$ exactly in the states in which you are aware of all events in $\Sigma(S)$. It does not mean that you are unaware of something beyond the language $\Sigma(S)$ or that you know that you are unaware of something beyond the language $\Sigma(S)$. It just means that you can reason about being unaware of something beyond the language $\Sigma(S)$. Recall that in unawareness structures, an individual may not be aware of all tautologies.

Property 3 is a direct implication of Property 4. In the least expressive space you are always aware that there could be something that you are unaware. This is because you are always aware of all events in the lowest space. Again, it does not mean that you are indeed unaware of something or that you know that you are unaware of something.

Finally, property 5 says that in the upmost space, knowing that you are aware of everything is knowing most complicated tautologies. That is, if you know every theorem there is to know then you know that you are aware of everything and vice versa. Recall that in unawareness structures an individual typically will not know always all tautologies $K_i(S^{\uparrow}) \subsetneqq S^{\uparrow}$.

Before we finish this section, we turn to some multi-agent notions such as mutual and common knowledge and awareness.

Define the mutual knowledge operator on events by for any $E \in \Sigma$,

$$K(E) := \bigcap_{i \in I} K_i(E)$$

and the common knowledge operator by

$$CK(E) := \bigcap_{n=1}^{\infty} K^n(E).$$

[13] Halpern and Rêgo (2012) also propose a weakening of the Barcan formula. In our setting, this weakening reads $A_i(\forall_S x E(x)) \cap \forall_S x(U_i(x) \cap K_i(E(x))) \subseteq K_i(\neg \forall_S x A_i(x) \cap \forall_S x E(x))$. This property holds trivially in our structure as $U_i(F) \cap K_i(E(F)) = \emptyset^{S(F)}$ and thus the antecedent reduces to $\emptyset^S$.

These operator are like in Aumann (1999). Analogously, define the mutual awareness operator on events by for any $E \in \Sigma$,

$$A(E) := \bigcap_{i \in I} A_i(E)$$

and the common knowledge operator by

$$CA(E) := \bigcap_{n=1}^{\infty} A^n(E).$$

We record some multi-agent properties.

The following observations are corollaries of Lemma 5 and Heifetz, Meier, and Schipper (2008, Proposition 11).

Corollary 5 *For any individuals $i, j \in I$, space $S \in \mathcal{S}$ and event operator $E(x)$,*

1. $\exists_S x A_i(x) = \exists_S x A_i A_j(x)$
2. $\exists_S x A_i(x) = \exists_S x A_i K_j(x)$
3. $\exists_S x K_i(x) \subseteq \exists_S x A_i K_j(x)$
4. $\exists_S x A(x) = \exists_S x K(S(x)^{\uparrow})$
5. $\exists_S x A(x) = \exists_S x CA(x)$
6. $\exists_S x K(x) \subseteq \exists_S x A(x)$
7. $\exists_S x CK(x) \subseteq \exists_S x CA(x)$
8. $\exists_S x CK(S(x)^{\uparrow}) \subseteq \exists_S x CA(x)$

These properties hold also when $\exists_S$ is replaced by $\forall_S$.

The proof of following proposition on the Barcan formula for mutual and common awareness is analogous to the proof of Proposition 1.

Proposition 4 (Barcan formula for mutual and common awareness) *For any space $S \in \mathcal{S}$ and event operator $E(x)$,*

$$\begin{aligned} \forall_S x A(E(x)) &\subseteq A \forall_S x E(x) \\ \forall_S x CA(E(x)) &\subseteq CA \forall_S x E(x) \end{aligned}$$

The proof of following proposition on the Barcan formula for mutual and common knowledge is analogous to the proof of Proposition 2. Note that the domain of discourse must now be constant within the possibility set of *each* individual. This is a much stronger assumption than the assumption in Proposition 2.

Proposition 5 (Barcan formula for mutual and common knowledge) *Assume that for all $i \in I$, $\omega' \in \Pi_i(\omega)$ implies $\mathcal{D}(\omega') \subseteq \mathcal{D}(\omega)$. Then for any space $S \in \mathcal{S}$ and event operator $E(x)$,*

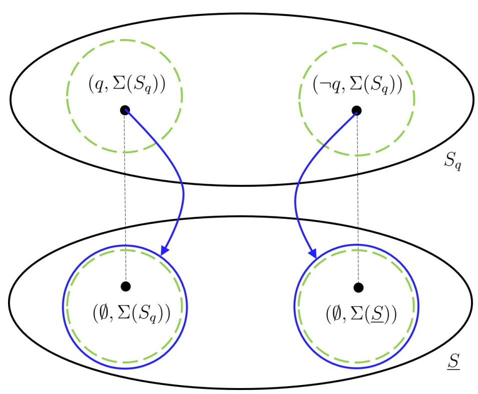
$$\begin{aligned} \forall_S x K(E(x)) &\subseteq K \forall_S x E(x) \\ \forall_S x CK(E(x)) &\subseteq CK \forall_S x E(x) \end{aligned}$$

3 Illustration: Consulting with Expert(s)

In many situations in real life we consult with experts, not just because they may have better information than we have but often because they are aware of things that we are unaware of. For instance, we may consult with a doctor not just because she might know whether or not we have a particular ~~decease~~ but also because she may be aware of certain ~~deceases~~ or treatments that we are not. Even general practitioners may refer us to specialists not just for the latter's better information but also greater awareness. We may consult with a lawyer who might suggest us an action that we have not thought about previously. Governments maintain scientific advisory bodies with an option to harness their wisdom. Music instrument teachers may immediately recognize our bad habits that hold us back that we may not even be aware of. Editors of scientific journals ask referees to access the novelty of the piece of research presumably because editors cannot be aware of all the research out there. Etc.

Example 2 (Individual 1 knows that individual 2 is aware of something that she is unaware of) Suppose individual 1 wants to consult individual 2 but only when she knows that individual 2 is an expert in the sense of being aware of something that individual 1 is unaware of. That is, she will consult individual 2 only in the event $K_1 \exists_{\underline{S}} x (U_1(x) \cap A_2(x))$. The simplest unawareness structure modelling this event in a non-trivial way is depicted in Figure 2.

Figure 2: Unawareness Structure of Example 2

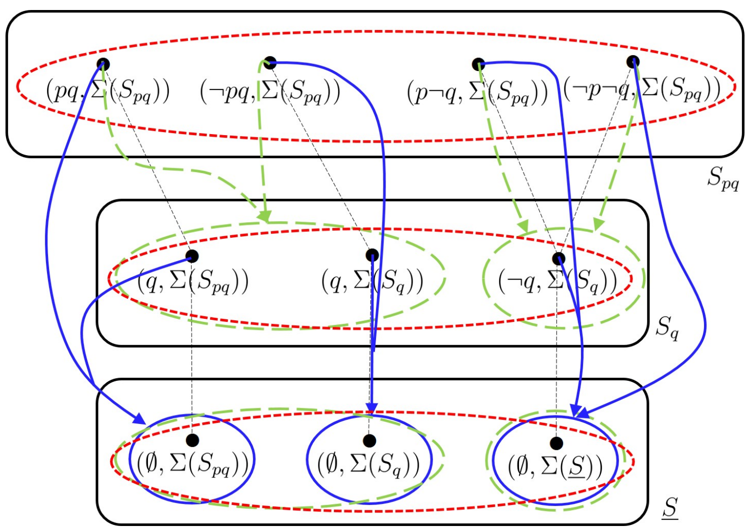


There are two spaces, S_q and $\underline{S}$, each having two states. Projections are indicated by faint dotted lines. The possibility correspondence of person 1 is depicted in blue circles and arrows, the possibility correspondence of person 2 with green dashed circles and arrows. In each state, person 2 knows the state. Person 1 is unaware of all events in space S_q. Her possibility sets at states in S_q are in the space $\underline{S}$. Thus, $U_1([q]) = S_q$ but $A_2([q]) = S_q$. At state $(\emptyset, \Sigma(S_q))$, individual 1 can reason about "things" she is not aware of because her domain of discourse is $\Sigma(S_q)$ with events she is not aware of at $(\emptyset, \Sigma(S_q))$. We got $\exists_{\underline{S}} x (U_1(x) \cap A_2(x)) = \{(\emptyset, \Sigma(S_q))\}$. Since at state $(\emptyset, \Sigma(S_q))$, individual 1 knows $\{(\emptyset, \Sigma(S_q))\}$, we got $K_1(\exists_{\underline{S}} x (U_1(x) \cap A_2(x))) = \{(\emptyset, \Sigma(S_q))\}^{\uparrow}$. We can say more. Person 2 knows that as $K_2 K_1(\exists_{\underline{S}} x (U_1(x) \cap A_2(x))) = \{(\emptyset, \Sigma(S_q))\}^{\uparrow}$. In fact, at that state, it is common knowledge that person 2 is aware of something that person 1 is unaware, i.e., $CK(\exists_{\underline{S}} x (U_1(x) \cap A_2(x))) = \{(\emptyset, \Sigma(S_q))\}^{\uparrow}$.

We noted that in above model, individual 1 is unaware of $[q]$ in all states of S_q, i.e., $U_i([q]) = S_q$. She satisfies KU-introspection, $K_iU_i([q]) = \emptyset^{S_q}$. Yet, we have $K_i(\exists_{\underline{S}} x U_i(x)) = \{(q, \Sigma(S_q))\}^{\uparrow}$. This is an example of our claim in the introduction that KU introspection of an event does not rule out that the individual knows that there might exist something that she is unaware of. Similarly, we have AU-introspection, $U_i([q]) = U_iU_i([q])$ but $U_i([q]) \cap U_iU_i([q]) \cap A_i(\exists_{\underline{S}} x U_i(x)) = \{(q, \Sigma(S_q))\}$. That is, AU-introspection does not rule out that an individual is unaware of an event, unaware that she is unaware of the event, and at the same time is aware that there might exist something that she is unaware of. This demonstrates the claim made in the introduction that AU-introspection does not rule out that an individual is aware that there might exist something she is unaware of. $\square$

Example 3 (Individual 1 knows that both individuals 2 and 3 are aware of something that she is unaware of but only individual 3 is also aware of something that individual 2 is unaware of. Despite that, individual 1 knows that individual 2 knows something that individual 3 does not.) Now we consider three individuals and want to express events like $K_1(\exists_S x(U_1(x) \cap A_2(x) \cap A_3(x)) \cap \exists_S x(U_1(x) \cap U_2(x) \cap A_2(x)))$, i.e., individual 1 knows that there exists something that she is unaware of but individuals 2 and 3 are aware of and that there exists something that she and individual 2 is unaware of but ~~individual 2~~ is aware of. To this end, consider the quantified unawareness structure in Figure 3.

Figure 3: Unawareness Structure of Example 3



There are three spaces totally ordered, $S_{pq} \succ S_q \succ \underline{S}$. Person 1's possibility correspondence is depicted with blue solid-lined arrows and ovals, person 2's with green intermitted arrows and ovals, and person 3's with red dot-lined ovals. Since all possibility sets of person 1 lie in space $\underline{S}$, she is unaware of events $[q] = \{(q, \Sigma(S_{pq})), (q, \Sigma(S_q)), (pq, \Sigma(S_{pq})), (\neg pq, \Sigma(S_{pq}))\}$ and $[p] = \{(pq, \Sigma(S_{pq})), (p\neg q, \Sigma(S_{pq}))\}$. That is, $U_1([q]) = S_q^{\uparrow}$ and $U_1([p]) = S_{pq}$. In contrast, in all states in S_{pq} or S_p, person 2 is aware of the event $[q]$. I.e., $A_2([q]) = S_q^{\uparrow}$. Person 3 is aware of the event $[p]$ in all states in S_{pq} and of the event $[q]$ in all states in $S_q^{\uparrow}$. I.e., $A_3([p]) = S_{pq}$,

$A_3([q]) = S_q^{\uparrow}$.

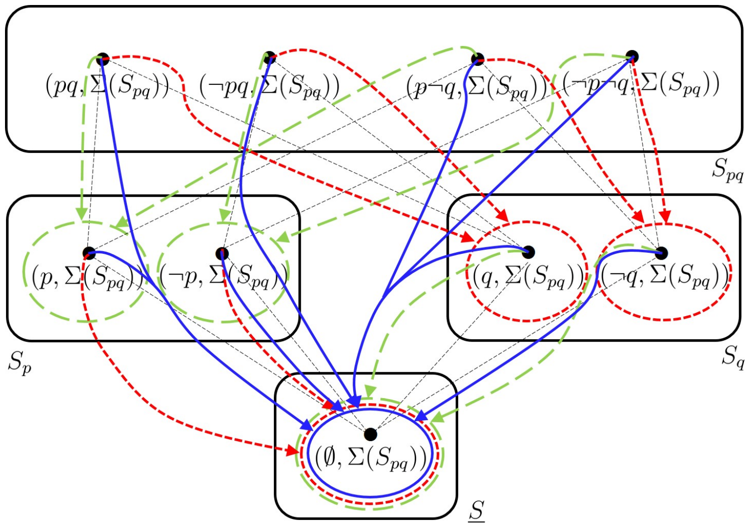
Now consider the least expressive space. We have $\exists_{\underline{S}} x(U_1(x) \cap A_2(x) \cap A_3(x)) = \{(\emptyset, \Sigma(S_{pq}), (\emptyset, \Sigma(S_q)\}^{\uparrow}$. This is, the event that there might exist something that person 1 is unaware of but persons 2 and 3 are aware of. Further, we have $\exists_{\underline{S}} x(U_1(x) \cap U_2(x) \cap A_3(x)) = \{(\emptyset, \Sigma(S_{pq})\}^{\uparrow}$. This is the event that there might exist something that person 1 and 2 are unaware of but person 3 is aware. Thus, $\exists_{\underline{S}} x(U_1(x) \cap A_2(x) \cap A_3(x)) \cap \exists_{\underline{S}} x(U_1(x) \cap U_2(x) \cap A_3(x)) = \{(\emptyset, \Sigma(S_{pq})\}^{\uparrow}$. This is the event that there might exist something that person 1 is unaware of but both persons 2 and 3 are aware of and there might exist something that persons 1 and 2 are unaware of but person 3 is aware of. Since at $(\emptyset, \Sigma(S_{pq})$, person 2 just considers this state possible, we have $K_1(\exists_{\underline{S}} x(U_1(x) \cap A_2(x) \cap A_3(x)) \cap \exists_{\underline{S}} x(U_1(x) \cap U_2(x) \cap A_3(x))) = \{(\emptyset, \Sigma(S_{pq})\}^{\uparrow}$. That is, person 1 knows that person 3 is a better expert in terms of awareness than person 2. (The event that there might exist an event that person 2 is aware of something that person 3 is unaware of is empty.) But is person 3 also a better expert in terms of knowledge? Note that person 2 knows whether the event $[q]$ obtains. That is, if $[q]$ obtains, then person 2 knows that and if $\neg[q]$ obtains, then person 2 knows that too. Thus, person 2 could be really called an expert on $[q]$. In contrast, person 3 has no such knowledge neither w.r.t. to event $[q]$ nor event $[p]$. More formally, $K_2([q]) = [q] = \{(q, \Sigma(S_{pq})), (q, \Sigma(S_q))\}^{\uparrow}$ while $K_3([q]) = \emptyset^{S_q}$. We have $\exists_{\underline{S}} x(K_2(x) \cap \neg K_3(x)) = \{(\emptyset, \Sigma(S_{pq}))\}^{\uparrow}$. Person 1 knows $K_1(\exists_{\underline{S}} x(K_2(x) \cap \neg K_3(x))) = \{(\emptyset, \Sigma(S_{pq}))\}^{\uparrow}$. So even though person 1 realizes that there might exist an event that only person 3 is aware of, she also realizes that person 2 knows something that person 3 does not. Hence, engaging both experts, persons 2 and 3, may be useful to person 1. This example emphasizes that an expert with more awareness does not need to have more knowledge and an expert with better knowledge of an event may not have more awareness, and that someone who reasons about engaging such experts may even know this even though she is unaware of the awareness and knowledge of those experts. In this example, person 3 may be called a generalist, as she is aware of many things without necessarily knowing about them, while person 2 can really be specialist w.r.t. to event $[q]$. Experts may come in both forms, generalists and specialists.[14] □

Example 4 (Person 1 knows that there might exist something that person 2 is aware of but person 3 is unaware of and vice versa.) So far, our examples involved just chains of totally ordered state spaces. The power of the lattice of spaces is that we can model diverse awareness levels including incomparable ones. Consider the unawareness structure of Figure 4. There are four spaces with $S_{pq} \succ S_p \succ \underline{S}$ and $S_{pq} \succ S_q \succ \underline{S}$ but S_p and S_q are not comparable. The obvious projections are indicated by faint grey lines between states. The possibility correspondence of person 1 is solid-lined blue, of person 2 dashed green, and of person 3 dotted-lined red. Person 1 is unaware both of events $[p]$ and $[q]$ since at all states her possibility set is on the least expressive space, $\underline{S}$. Person 2 is aware of event $[p]$ but unaware of $[q]$. Finally, person 3 is aware of event $[q]$ but unaware of event $[p]$.

Now, $\exists_{\underline{S}} x(U_1(x) \cap A_2(x)) \cap \exists_{\underline{S}} x(U_1(x) \cap A_3(x)) \cap \neg \exists_{\underline{S}} x(U_1(x) \cap A_2(x) \cap A_3(x)) = \{(\emptyset, \Sigma(S_{pq})\}^{\uparrow} = \Omega$. That is, at $(\emptyset, \Sigma(S_{pq})$ there might exist an event of which person 1 is unaware of but person 2 is aware of, there might exist also an event of which person 1 is unaware of but person 3 is aware of but it is not the case that there might exist an event of which person 1 is unaware of

[14] Schipper (2016) analyzes how social networks how generalists facilitate social network formation among specialists to create the fabric of "organizational" knowledge and awareness.

Figure 4: Unawareness Structure of Example 4



but both persons 2 and 3 are aware of. Moreover, person 1 knows this. In fact, it is common knowledge, $CK(\exists_{\underline{S}} x(U_1(x) \cap A_2(x)) \cap \exists_{\underline{S}} x(U_1(x) \cap A_3(x)) \cap \neg \exists_{\underline{S}} x(U_1(x) \cap A_2(x) \cap A_3(x))) = \Omega$. $\square$

4 Discussion

I require that at each state $\omega \in \Omega$, the domain of discourse, $\mathcal{D}(\omega)$ is equal to the set of all event $\Sigma(S)$ for some space S (with $S \succeq S_\omega$). So even though the domain of discourse $\mathcal{D}(\omega)$ may contain events that are not expressible at ω, they will be expressible at some state in a more expressive space. Since this holds at every state in the unawareness structure, this is implicitly common knowledge among all individuals. That is, even though a person may not be able to make sense of a statement, she implicitly knows that there is an awareness level at which it becomes meaningful. In this sense, if there exists something that an individual is unaware of, then it is implicitly common knowledge that whatever it is, it is meaningful at some awareness level and not just something that is nonsense no matter what.

Quantified unawareness structures differ from unawareness structures in a particular feature that may be not too obvious at the first glance. In unawareness structures of Heifetz, Meier, and Schipper (2006, 2008), if one picks a space S in the lattice $\mathcal{S}$ and looks at the sublattice consisting of all spaces $S' \preceq S$, $S' \in \mathcal{S}$, then this is just like any other unawareness structure. This differs from quantified unawareness since the domain of discourse at states in S may contain also events expressive only in some space $S'' \prec S$. It follows that the upmost space $\bar{S}$ of $\mathcal{S}$ is really special since at any state in this space the domain of discourse *must* contain *exactly* all events expressible in the upmost space of $\mathcal{S}$. If an individual's possibility set is located in this upmost space, then she is aware of everything and knows that for all events she is aware of them (Proposition 3). If one were to model god (e.g., extending Biblical games of Brams (2003)), this is were all his possibility sets would be located. Compare this to standard states spaces where everybody must be god and there is (literally!) no space for mortals with limited awareness or unawareness structures where every mortal with limited awareness must think he is god.

Quantified event structures avoid some subtle difficulties of first-order logic. For instance, in first-order logic with quantification over formulae it is difficult to give semantics to $\forall xx$. It is difficult to determine the truth when x is substituted by the formula $\forall xx$, creating a circularity. That's why first-order logic restricts to quantification over non-quantified formulae. In the event-based approach, we replace x by events (i.e., subsets of states) in a pre-specified set of all events (given by the respective domain of discourse). Because the set of events is pre-specified, there is no circularity problem. I.e., given S, $\forall_S xx = \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcap_{F \in \mathcal{D}(\omega)} F\right) \subseteq \bigcap_{F \in \mathcal{D}(\omega)} F\right\} = \emptyset^S$. It is never the case that all events obtain simultaneously.

Quantified event structures do not feature a pre-defined separate set of objects and pre-defined properties. Yet, we can models objects and properties as events. Fix a set of objects O and a set of properties P. Define an object correspondence $\mathcal{O} : \Omega \longrightarrow 2^O$ such that for any subset of objects $O' \subseteq O$, $[O'] := \{\omega \in \Omega : O' \subseteq \mathcal{O}(\omega)\}$ is an event in the quantified unawareness structures. $[O']$ is the event that all objects in O' exist. Analogously, define the property correspondence and the event $[P']$ that all properties in P' exist. It is now straightforward to model knowledge and awareness of sets of objects and properties. Moreover, the event $[O'] \cap [P']$ is the event that objects O' have properties P'. Similarly, one may want to model awareness of actions and consequences in the event based approach to capture the spirit of the special state-space structure of Karni and Vierø (2013, 2017).

A On the Definition of Quantifiers

Note that throughout the text we interpreted events like $\exists_S x E(x)$ as "given S there *might* exists an event x for which $E(x)$" rather than "given S there exists an event x for which $E(x)$". The reason is the following characterization whose proof is contained in Appendix B.

Lemma 6 *For every $S \in \mathcal{S}$ and event operator $E(x)$,*

$$\exists_S x E(x) = \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \cap \bigcup_{F \in \mathcal{D}(\omega)} E(F) \neq \emptyset\right\}^{\uparrow}.$$

Proof. By definition

$$\begin{aligned}
\exists_S x E(x) &= \neg\forall_S x \neg E(x) \\
&= \left(S \setminus \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcap_{F \in \mathcal{D}(\omega)} \neg E(F)\right) \subseteq \bigcap_{F \in \mathcal{D}(\omega)} \neg E(F)\right\}\right)^{\uparrow} \\
&= \left(S \setminus \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\neg \bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \subseteq \neg \bigcup_{F \in \mathcal{D}(\omega)} E(F)\right\}\right)^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\neg \bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \nsubseteq \neg \bigcup_{F \in \mathcal{D}(\omega)} E(F)\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\neg \bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \cap \bigcup_{F \in \mathcal{D}(\omega)} E(F) \neq \emptyset\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \cap \bigcup_{F \in \mathcal{D}(\omega)} E(F) \neq \emptyset\right\}^{\uparrow}
\end{aligned}$$

□

Note that the characterization does not claim that $\{\omega\}^{\uparrow} \cap S \left(\bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \subseteq \bigcup_{F \in \mathcal{D}(\omega)} E(F)$. Thus, this characterization suggests that the existential quantifier should verbally be interpreted as "given S there *might exists* an event x for which $E(x)$". We believe that this is the appropriate definition especially since we are mainly motivated to express events like "there might exist an event that I am unaware of". To make the point, consider the following two alternatives.

First, we could define alternatively

$$\tilde{\exists}_S x E(x) := \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcup_{F \in \mathcal{D}(\omega)} E(F)\right) \subseteq \bigcup_{F \in \mathcal{D}(\omega)} E(F)\right\}^{\uparrow}.$$

This existential quantifier might be interpreted as "given S, there exists and event x for which $E(x)$". Given this stronger notion of existential quantifier, we can let the universal quantifier be defined as usual by

$$\tilde{\forall}_S x E(x) := \neg \tilde{\exists}_S x \neg E(x).$$

This is now a weaker notion of universal quantifier as we have the following characterization. The proof is analogous to the proof of Lemma 6.

Lemma 7 *For every $S \in \mathcal{S}$ and event operator $E(x)$,*

$$\tilde{\forall}_S x E(x) = \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcap_{F \in \mathcal{D}(\omega)} E(F)\right) \cap \bigcap_{F \in \mathcal{D}(\omega)} E(F) \neq \emptyset\right\}^{\uparrow}.$$

If one were to insist on a verbal interpretation, we would have to say something like "given S it might be the case that for all events x we have $E(x)$." We find this notion dissatisfactory especially we awareness in mind. We are less interested in events like "it might be the case that I am unaware of all events".

Second, because of the problem with the weakening of the universal quantifier, we may not insist on defining the universal quantifier with the existential quantifier or vice versa. We could use $\tilde{\exists}_S$ for the existential quantifier and $\forall_S$ (as defined in the main text) as the universal quantifier. Note that we have the following straight-forward implication:

Lemma 8 *For any space S and event operator $E(x)$,*

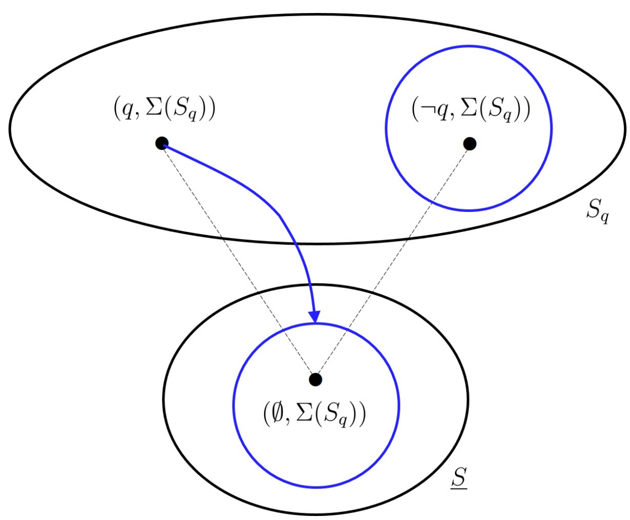
$$\forall_S x E(x) \subseteq \neg\tilde{\exists}_S x \neg E(x).$$

Nevertheless, we belief that the use of $\tilde{\exists}_S$ is highly dissatisfactory especially when modelling events like "there exists an event that I am unaware of". This is easiest explained with an example and pertains exactly to $\exists_S x E(x) \setminus \tilde{\exists}_S x E(x)$.

Example 5 Consider the simple unawareness structure of Figure 6. There are two spaces. In the upper space, S_q, there are two states both of which projecting to the unique state in the lower space, $\underline{S}$. There is one individual whose possibility correspondence is depicted with the blue circles and arrow. Consider the event $[q] = \{(q, \Sigma(S_q))\}$. It is easy to verify that $U_i([q]) = \{(q, \Sigma(S_q))\}$ and $A_i([q]) = \{(\neg q, \Sigma(S_q))\}$. Now verify

$$\begin{aligned} \exists_{\underline{S}} x U_i(x) &= \{(\emptyset, \Sigma(S_q))\}^{\uparrow} = \Omega \\ \tilde{\exists}_{\underline{S}} x U_i(x) &= \emptyset^{\underline{S}} \end{aligned}$$

Figure 5: Differences between $\exists_{\underline{S}} x U_i(x)$ and $\tilde{\exists}_{\underline{S}} x U_i(x)$



$\tilde{\exists}_{\underline{S}} x U_i(x) = \emptyset^{\underline{S}}$ suggests that given S it is not the case that there exists an event of which individual i is unaware of. This is counterintuitive as at $(\emptyset, \Sigma(S_q))$ there is an event in the domain of discourse $\Sigma(S_q)$ (namely $[q]$) for which there is a state, namely $(q, \Sigma(S_q))$, at which the individual i is unaware of $[q]$. There is also a state, namely $(\neg q, \Sigma(S_q))$, at which the individual i is aware of $[q]$. Both of these states project down to $(\emptyset, \Sigma(S_q))$. So at $\underline{S}$, the individual cannot distinguish whether or not really there exist an event of which she is unaware of it. But requiring in that case to treat this situation as if it is not the case that there exist an event of which the individual is unaware of seems to be imprudent. Somehow she does register that there *might* exist an event of which she is unaware of. Thus, we use $\exists_S$ rather than $\tilde{\exists}_S$. $\Box$

The differences between the quantifiers become immaterial under the following sufficient conditions:

Lemma 9 *Given the space $S \in \mathcal{S}$ and any event operator $E(x)$ we have*

$$\forall_S x E(x) = \tilde{\forall}_S x E(x) \textit{ and } \exists_S x E(x) = \tilde{\exists}_S x E(x)$$

if either of the following condition hold

(i) if $S = \bar{S}$ (i.e., the upmost space of the lattice),

(ii) if $\mathcal{S}$ is a singleton consisting just of one space,

(iii) if for all $S, S' \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'}$ is a bijection.

The proof is immediate. Note that lattices of bijective spaces played already a role when showing equal expressiveness of unawareness structures to object-based unawareness structures of Board and Chung (2011). See Board, Chung, and Schipper (2011).

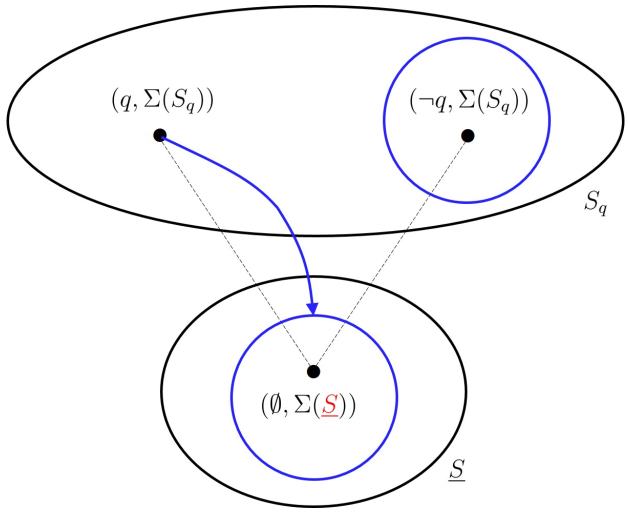
Finally, we want to prove Remark 1. First, suppose for any $S, S' \in \mathcal{S}$ and event operator $E(x)$ we have $\forall_{S'} x E(x) \subseteq \forall_S x E(x)$. Consider Example 5. In this example, we have $\forall_{S_q} x A_i(x) = \{(\neg q, \Sigma(S_q))\}$ but $\forall_{\underline{S}} x A_i(x) = \emptyset^{\underline{S}}$, disproving the conjecture.

Now suppose that for any $S, S' \in \mathcal{S}$ and event operator $E(x)$ we have $\neg \exists_{S'} x E(x) \subseteq \neg \exists_S x E(x)$. Consider Example 5. In this example, we have $\exists_{S_q} x U_i(x) = \{(q, \Sigma(S_q))\}$ and $\exists_{\underline{S}} x U_i(x) = \Omega$. Thus, $\exists_{S_q} x U_i(x) = \{(\neg q, \Sigma(S_q))\}$ and $\neg \exists_{\underline{S}} x U_i(x) = \emptyset^{\underline{S}}$, again disproving the conjecture.

Now suppose that for any $S, S' \in \mathcal{S}$ and event operator $E(x)$ we have $\exists_{S'} x E(x) \subseteq \exists_S x E(x)$. Consider the following example depicted in Figure 1. This is essentially the same as Example 5 except that the description of the unique state in the lowest space, $\underline{S}$, reads $(\emptyset, \Sigma(\underline{S})$. In this example we have $\exists_{S_q} x U_i(x) = \{(q, \Sigma(S_q))\}$ and $\exists_{\underline{S}} x U_i(x) = \emptyset^{\underline{S}}$, disproving the conjecture.

Finally, suppose that for any $S, S' \in \mathcal{S}$ and event operator $E(x)$ we have $\neg \forall_{S'} x E(x) \subseteq \neg \forall_S x E(x)$. Again, consider the example in Figure 1. We have $\forall_{S_q} x A_i(x) = \{(\neg q, \Sigma(S_q))\}$ and $\forall_{\underline{S}} x A_i(x) = \Omega$. Thus, $\neg \forall_{S_q} x A_i(x) = \{(q, \Sigma(S_q))\}$ and $\neg \forall_{\underline{S}} x A_i(x) = \emptyset^{\underline{S}}$, disproving the conjecture. We conclude that quantified events are unrelated across the spaces of the lattice.

Figure 6: Counterexample



B Proofs

Proof of Lemma 2

$$
\begin{aligned}
&\forall_S x(E(x) \cap F(x)) \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{G \in D(\omega)} E(G) \cap \bigcap_{G \in D(\omega)} F(G)\right) \subseteq \bigcap_{G \in D(\omega)} E(G) \cap \bigcap_{G \in D(\omega)} F(G)\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{G \in D(\omega)} E(G) \cap \bigcap_{G \in D(\omega)} F(G)\right) \subseteq \bigcap_{G \in D(\omega)} E(G)\right\}^{\uparrow} \\
&\quad \cap \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{G \in D(\omega)} E(G) \cap \bigcap_{G \in D(\omega)} F(G)\right) \subseteq \bigcap_{G \in D(\omega)} F(G)\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{G \in D(\omega)} E(G)\right) \subseteq \bigcap_{G \in D(\omega)} E(G)\right\}^{\uparrow} \\
&\quad \cap \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{G \in D(\omega)} F(G)\right) \subseteq \bigcap_{G \in D(\omega)} F(G)\right\}^{\uparrow} \\
&= \forall_S x E(x) \cap \forall_S x F(x)
\end{aligned}
$$

where the second last equality follows from the fact that $\bigcap_{G \in D(\omega)} E(G)$ and $\bigcap_{G \in D(\omega)} F(G)$ are events. $\square$

Proof of Lemma 3

$$
\begin{aligned}
\neg\forall_S x E(x) &= \left(S \setminus \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} \neg E(F)\right) \subseteq \bigcap_{F\in\mathcal{D}(\omega)} E(F)\right\}\right)^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} \neg E(F)\right) \cap \neg \bigcap_{F\in\mathcal{D}(\omega)} E(F) \neq \emptyset\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} \neg E(F)\right) \cap \bigcup_{F\in\mathcal{D}(\omega)} \neg E(F) \neq \emptyset\right\}^{\uparrow}
\end{aligned}
$$

Since for any $\omega \in S$, $\bigcap_{F\in\mathcal{D}(\omega)} \neg E(F) \subseteq \bigcup_{F\in\mathcal{D}(\omega)} \neg E(F)$ we have by definition of $\forall_S x \neg E(x)$ that $\forall_S x \neg E(x) \subseteq \neg\forall_S x E(x)$. The second implication of Lemma 3 is an equivalent restatement of the first. □

Proof of Lemma 4

Part (i) follows as an immediate corollary from Lemma 3.

For part (ii), note that

$$
\begin{aligned}
\forall_S x E(x) &= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{G\in\mathcal{D}(\omega)} E(G)\right) \subseteq \bigcap_{G\in\mathcal{D}(\omega)} E(G)\right\}^{\uparrow} \\
&\subseteq \{\omega \in S : \omega \in E(F)\}^{\uparrow} \\
&= (E(F) \cap S)^{\uparrow} \\
&\subseteq E(F)
\end{aligned}
$$

where the first subset-relation follows from the fact that $S \succeq S' \wedge S(F)$ and thus $E(F) \in \Sigma(S)$. □

Proof of Proposition 1

$$
\begin{aligned}
\forall_S x A_i(E(x)) &= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} A_i(E(F))\right) \subseteq \bigcap_{F\in\mathcal{D}(\omega)} A_i(E(F))\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} A_i(E(F))\right) \subseteq A_i\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)\right\}^{\uparrow},
\end{aligned}
$$

where last equality follows from A-conjunction (Heifetz, Meier, and Schipper, 2006, Proposition 3). By Heifetz, Meier, and Schipper, 2006, Proposition 1, $S\left(\bigcap_{F\in\mathcal{D}(\omega)} A_i(E(F))\right) = S\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$. It follows now from the definition of the awareness operator that $\{\omega\}^{\uparrow} \cap$

$S\left(\bigcap_{F\in\mathcal{D}(\omega)} A_i(E(F))\right) \subseteq A_i\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$ implies that for all $\omega' \in \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$ we have $\Pi_i(\omega') \subseteq S\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$. Note that $S \preceq S\left(\bigcap_{F\in\mathcal{D}(\omega)} A_i(E(F))\right)$. By Projections Preserve Awareness (Heifetz, Meier, and Schipper, 2006, Remark 3), $\omega \in \Pi_i(\omega)$. Thus, $\Pi_i(\omega) \subseteq S$. Hence $\omega \in K_i(S^{\uparrow})$. By Weak Necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 3), $\omega \in A_i(\forall_S x E(x))$, since $\forall_S x E(x)$ is by definition an S-based event. $\square$

Proof of Proposition 2

$$\begin{aligned}
\forall_S x K_i(E(x)) &= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right) \subseteq \bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right\}^{\uparrow} \\
&= \left\{\omega \in S : \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right) \subseteq K_i\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)\right\}^{\uparrow},
\end{aligned}$$

where last equality follows from Conjunction (Heifetz, Meier, and Schipper, 2006, Proposition 2). By Heifetz, Meier, and Schipper, 2006, Proposition 1, $S\left(\bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right) = S\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$. It follows now from the definition of the knowledge operator that $\{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right) \subseteq K_i\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$ implies that for all $\omega' \in \{\omega\}^{\uparrow} \cap S\left(\bigcap_{F\in\mathcal{D}(\omega)} E(F)\right)$ we have $\Pi_i(\omega') \subseteq \bigcap_{F\in\mathcal{D}(\omega)} E(F)$. Note that $S \preceq S\left(\bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right)$. By Projections Preserve Knowledge, $(\Pi_i(\omega'))_S = \Pi_i(\omega)$. Note that by assumption for all $\omega'' \in \Pi_i(\omega)$, $\mathcal{D}(\omega'') = \mathcal{D}(\omega')$. Thus, $(r_S^{S^*})^{-1}(\Pi_i(\omega'')) = \Pi_i(\omega')$ with $S^* = S\left(\bigcap_{F\in\mathcal{D}(\omega)} K_i(E(F))\right)$. Since $\{\omega''\}^{\uparrow} \cap S^* \subseteq (r_S^{S^*})^{-1}(\Pi_i(\omega''))$, we have $\omega'' \in \forall_S x K_i(E(x))$. Thus, $\omega \in K_i \forall_S x K_i(E(x))$. From Corollary 1 (Truth) follows $\omega \in K_i(\forall_S x E(x))$. Hence, $\forall_S x K_i(E(x)) \subseteq K_i(\forall_S x E(x))$. $\square$

Proof of Corollary 4

$U_i \exists_S x E(x) = U_i \forall_S x E(x) = \neg A_i \forall_S x E(x) \subseteq \neg \forall_S x A_i(E(x)) = \exists_S x U_i(E(x))$. The middle equality follows from Propositions 1. The first equality follows from weak necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 3). $\square$

Proof of Remark 2

Suppose that the converse to the Barcan formula for awareness hold, i.e., for any $i \in I$, $S \in \mathcal{S}$, and event operator $E(x)$, we have $A_i \forall_S x E(x) \subseteq \forall_S x A_i(E(x))$. This is equivalent to

$$\begin{aligned}
\neg \forall_S x A_i(E(x)) &\subseteq \neg A_i \forall_S x E(x) \\
\neg \forall_S x A_i(E(x)) &\subseteq U_i \forall_S x E(x) \\
\exists_S x U_i(E(x)) &\subseteq U_i \forall_S x E(x) \\
\exists_S x U_i(E(x)) &\subseteq U_i \exists_S x E(x)
\end{aligned}$$

Consider as a Example 5. There we have $\exists_{\underline{S}} x U_i U_i(x) = \exists_{\underline{S}} x U_i(x) = \Omega$ (where the first equality follow from AU-introspection but $U_i \exists_{\underline{S}} x U_i(x) = U_i(\underline{S}^{\uparrow}) = \emptyset^{\underline{S}}$, where the first equality follows from weak necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 3). Thus, Example 5 is a counterexample to the converse of the Barcan formula for awareness. $\square$

Proof of Remark 3

Note that by weak necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 3), $A_i \exists_S x E(x) = A_i(S^{\uparrow})$.

By the same arguments as in the proof of 1. of Proposition 3, we have $\exists_S x A_i(E(x)) = S^{\uparrow}$.

Consider Example 5. In this example, $\exists_{S_q} x A_i(x) = S_q \supset A_i(S_q)$. Thus, it is a counterexample. $\square$

Proof of Proposition 3

1. By the properties of the discourse correspondence, for all $\omega \in \Omega$, $\mathcal{D}(\omega) \ni \Omega$. By necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 2), $K_i(\Omega) = \Omega$. Thus, $A_i(\Omega) = \Omega$ and hence $U_i(\Omega) = \emptyset^{\underline{S}}$. Thus, $\forall_{\underline{S}} x U_i(x) = \emptyset^{\underline{S}}$. Since $\Sigma(\underline{S}) \subseteq \Sigma(S)$ for any $S \in \mathcal{S}$, we have $\Sigma(S) \ni \Omega$. Thus, for any $S \in \mathcal{S}$, $\forall_S x U_i(x) = \emptyset^S$.

2. FA holds trivially because for all $S \in \mathcal{S}$, $\emptyset^S = \forall_S x U_i(x) = K_i(\forall_S x U_i(x)) = \emptyset^S$.

4. For any $S \in \mathcal{S}$, $\exists_S x U_i(x)$ is an S-based event. Thus, by Weak Necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 3), $A_i(\exists_S x U_i(x)) = A_i(S^{\uparrow})$.

3. This follows now from 4. and Necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 2).

5. Note that by properties of the discourse correspondence, $\mathcal{D}(\omega) = \Sigma(\overline{S})$ for all $\omega \in \overline{S}$. Thus, $\forall_{\overline{S}} x A_i(x) = A_i(\overline{S})$ by Weak Necessitation (Heifetz, Meier, and Schipper, 2006, Proposition 3). Again, by Weak Necessitation, $A_i(\overline{S}) = K_i(\overline{S})$. Thus, $K_i(\forall_{\overline{S}} x A_i(x)) = K_i K_i(\overline{S}) = K_i(\overline{S})$ by truth (Heifetz, Meier, and Schipper, Proposition 2). $\square$

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