1. ELASTICITY

In our study of mechanics so far, we have considered the objects as rigid, i.e., whatever force we apply on objects, they remain underformed. In reality, when an external force is applied to a body, the body gets deformed in shape or size or both. Also if we remove the external force the deformed body tends to regain its original shape and size. The property by virtue of which a deformed body tends to regain its original shape and size after removal of deforming force is called elasticity. Let us consider two possibilities:

(i) If the body regains its original size and shape it is said to be perfectly elastic.

(ii) If the body does not have any tendency to regain its original shape or size once deforming forces are removed it is said to be perfectly plastic.

In nature, actual behaviour of bodies lies between these two extreme limits.

1.1 STRESS

When external forces are applied, the body is distorted. i.e. the different portions of the body move relative to each other. Due to these displacements atomic forces (called restoring forces) are set up inside the body to restore the original form. The restoring force per unit area set up inside the body is called stress. This is measured by the magnitude of the deforming force acting per unit area of the body when equilibrium is established. If F is the force applied to an area of cross-section A, then stress = $\frac{F}{A}$ … (**1**)

The unit of stress in S.I system is N/m^2 When the stress is normal to the surface, this is called **normal stress**. The normal stress produces a change in length or a change in volume of a body. The normal stress to a wire or body may be compressive or tensile (expansive) according as it produces a decrease or increase in length of a wire or volume of a body. When the stress is tangential to a surface, it is called **tangential (shearing) stress.**

Illustration 1

A rectangular bar having a cross-sectional area of 70 mm^2 has a tensile force of 14 kN applied to it. Determine the stress in the bar.

Solution:

Cross-sectional area $A = 70\ mm^2 = 70\ (10^{-3} m)^2 = 70 \times 10^{-6}\ m^2$

Tensile force $F = 14\ kN = 14 \times 10^3\ N$

Stress in the bar = $\frac{\text{Force}}{\text{Area}} = \frac{14\times10^3\,N}{70\times10^{-6}\,m^2} = \mathbf{0.20 \times 10^9\ N/m^2}$

1.2 STRAIN

The external forces acting on the body cause a relative displacement of its various parts. A change in length, volume or shape takes place. The body is then said to be strained. The relative change produced in the body under a system of force is called strain.

$$\text{Strain } (e) = \frac{\text{change in dimension}}{\text{original dimension}} \quad \ldots (\mathbf{2})$$

Strain has no dimensions as it is a pure number. The change in length per unit length is called **linear strain**. The change in volume per unit volume is called **volume strain.** If there is a change in shape the strain is called **shearing strain.** This is measured by the angle through which a line originally normal to the fixed surface is turned.

Illustration 2

A wire of length 2.5 m has a percentage strain of 0.012% under a tensile force. Determine the extension in the wire.

Solution:

Original length of wire L = 2.5 m

$$\text{Strain} = 0.012\ \% = \frac{0.012}{100}$$

$$\text{Strain} = \frac{\text{Extension in length}}{\text{Original length}} = \frac{\Delta L}{L} = \frac{0.012}{100}$$

$$\text{Extension} = \Delta L = \frac{0.012}{100} \times L = \frac{0.012}{100} \times 2.5 = \mathbf{0.3 \times 10^{-3}\ m}$$

1.3 HOOKE'S LAW AND MODULI OF ELASTICITY

According to Hooke's law, "within elastic limits stress is proportional to strain".

i.e. $$\frac{\text{stress}}{\text{strain}} = \text{constant} = \lambda \qquad \ldots (\mathbf{3})$$

where λ is called modulus of elasticity. Depending upon different types of strain the following three moduli of elasticity are possible.

(i) Young's modulus: When a wire or rod is stretched by a longitudinal force the ratio of the longitudinal stress to the longitudinal strain within the elastic limits is called Young's modulus.

$$\text{Young's modulus } (Y) = \frac{\text{Longitudinal stress}}{\text{Linear strain}}$$

Consider a wire or rod of length L and radius r under the action of a stretching force F applied normal to its faces. Suppose the wire suffers a change in length l then

$$\text{Longitudinal stress} = \frac{F}{\pi r^2}$$

$$\text{Linear strain} = \frac{l}{L}$$

$$\therefore \text{ Young's modulus } (Y) = \frac{\frac{F}{\pi r^2}}{\frac{l}{L}} = \frac{FL}{\pi r^2 l} \qquad \ldots (\mathbf{4})$$

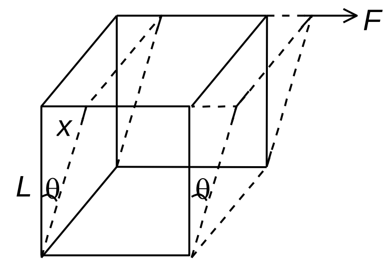
(ii) Bulk modulus: When a solid or fluid (liquid or gas) is subjected to a uniform pressure all over the surface, the shape remains the same, but there is a change in the volume. The force per unit area applied normally and uniformly over the surface is called normal stress. The change in volume per unit volume is called volume or bulk strain. The ratio $\dfrac{\text{Volume stress or normal stress}}{\text{Volume strain}}$ is called bulk modulus (B).

In symbols $B = -\dfrac{\frac{F}{A}}{\frac{\Delta V}{V}} = -\dfrac{FV}{A\Delta V}$ … (**5**)

The reciprocal of bulk modulus is called compressibility.

$\therefore$ compressibility $= \dfrac{1}{\text{bulk modulus}}$ … (**6**)

(iii) Modulus of Rigidity: According to definition, the ratio of shearing stress to shearing strain is called modulus of rigidity (η). In this case the shape of the body changes but its volume remains unchanged. Consider the case of a cube fixed at its lower face and acted upon by a tangential force F on its upper surface of area A as shown in figure.



$\therefore$ shearing stress $= \dfrac{F}{A}$

shearing strain $= \theta = \left(\dfrac{x}{L}\right)$

$\therefore\ \eta = \dfrac{F}{A\theta}$ … (**7**)

Poisson's Ratio: This is the name given to the ratio of lateral strain to the longitudinal strain. For example, consider a force F applied along the length of the wire which elongates the wire along the length while it contracts radially. Then the longitudinal strain $= \dfrac{\Delta L}{L}$ and Lateral strain $= \dfrac{\Delta r}{r}$, where r is the original radius and Δr is the change in radius.

$\therefore$ Poisson's ratio (σ) $= -\dfrac{\frac{\Delta r}{r}}{\frac{\Delta L}{L}}$. … (**8**)

The negative sign indicates that change in length and radius are of opposite sign. The theoretical value of σ lies between 1 and 0.5 while the practical value of σ lies between 0 and 0.5.

Illustration 3

A wire is stretched by 2 mm when a force of 250 N is applied. Determine the force that would stretch the wire by 5 mm assuming that the elastic limit is not exceeded.

Solution:

According to Hooke's law, within elastic limits,

Extension x is proportional to force F

or $x \propto F$

or $x = kF$ where k is a constant.

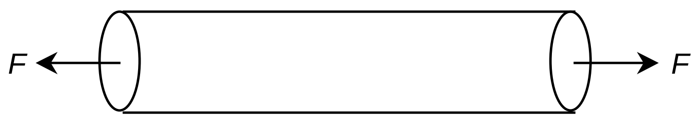
It is given that when $x = 2$ mm, $F = 250$ N

When $x = 5$ mm, $\frac{xmm}{2mm} = \frac{F}{250}$

or the force required $F = \frac{5}{2} \times 250 = \mathbf{625\ N}$

1.4 Stress-strain relationship for a wire subjected to longitudinal stress

Consider a long wire (made of steel) of cross sectional area A and original length L in equilibrium under the action of two equal and opposite variable force F as shown in figure.



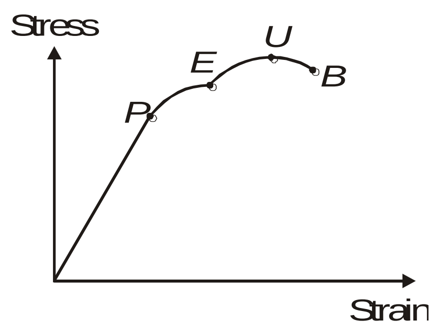
Due to the applied forces the length gets charged to $L + l$.

Then, longitudinal stress = $\frac{F}{A}$ and

longitudinal strain = $\frac{l}{L}$

The graph between the stress and stain, as the elongation gradually increases is shown in figure. Initially for small value of deformation stress is proportional to strain upto point P called as proportional limit. Till this point Hook's law is valid and Young's modulus is defined. If deformation further increases, the curve becomes non-linear but still it has elastic property till point E. This point E is called as elastic limit. After elastic limit yielding of wire starts taking place and body starts gaining some permanent deformation.

That is on the removal of force body remains deformed. If deformation is further increased, the wire breaks at point B called as breaking point. The stress corresponding to this point is called as breaking stress. Before reaching breaking point the curve has a point where tangent to this curve has zero slope. This point corresponds to maximum stress that the body can sustain. This point is called as ultimate point and stress at this point is called as ultimate strength of wire.



1.5 ELASTIC ENERGY STORED IN A DEFORMED BODY

The elastic energy is measured in terms of work done in straining the body up to its elastic limit.

Let F be the force applied across the cross-section A of a wire of length L. Let l be the increase in length. Then

$$Y = \frac{\frac{F}{A}}{\frac{l}{L}} = \frac{FL}{Al} \quad \text{or} \quad F = \frac{YAl}{L}$$

If the wire is stretched through a further distance $d\,l$ the work done = $F \times d\,l = \frac{YAl}{L} dl$

Total work done in stretching the wire from original length L to a length $L + l$

(i.e. from $l = 0$ to $l = l$)

$$W = \int_0^l \frac{YA \cdot l}{L} \cdot dl$$

$$= \frac{YA}{L} \cdot \frac{l^2}{2} = \frac{1}{2}(AL)\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)$$

$\Rightarrow W = \frac{1}{2} \times$ volume $\times$ stress $\times$ strain … (**9**)

Illustration 4

The rubber cord of catapult has a cross-sectional area 1 mm^2 and total unstretched length 10 cm. It is stretched to 12 cm and then released to project a body of mass 5 gm. Taking the Young's modulus of rubber as 5×10^8 N/m^2, calculate the velocity of projection.

Solution:

It can be assumed that the total elastic energy of catapult is converted into kinetic energy of body without any heat loss.

Elastic energy = $\frac{1}{2}$ × load × extension

Extension = 12 – 10 = 2 cm = 2×10^{-2} m

$$\therefore Y = \frac{FL}{A\Delta L}$$

$$F = \frac{5 \times 10^8 \times 1 \times 10^{-6} \times 2 \times 10^{-2}}{10 \times 10^{-2}} = 100 \text{ N}$$

If v is the velocity of projection,

Elastic energy of catapult = Kinetic energy of missile

$$\frac{1}{2} \times \text{load} \times \text{extension} = \frac{1}{2} mv^2$$

$$\frac{1}{2}\times 100\times 2\times 10^{-2} = \frac{1}{2}\times 5\times 10^{-3}\times v^2$$

$$v^2 = \frac{100\times 10^{-2}\times 2}{5\times 10^{-3}} = 400$$

$$v = \mathbf{20\ m/s}$$

2. FLUID STATICS

Fluid statics is the branch of mechanics, which deals with the forces on fluids (liquids and gases) at rest. As far back as 250 B.C. Archimedes, the famous Greek philosopher stated in his works that liquids exert an upward buoyant force on solids immersed in them causing an apparent reduction in their weights. It was about the end of 17th century that Pascal, a French scientist explained the fundamental principles of the subject in a clear manner. The consequences of Pascal's work are far-reaching for it forms the basis of several practical appliances like the fluid pumps, hydraulic presses and brakes, pneumatic drills, etc.

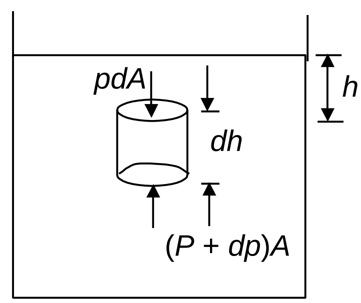
2.1 Thrust and Pressure

A **perfect fluid** resists forces normal to its surface and offers no resistance to forces acting tangential to its surface. A heavy log of wood can be drawn along the surface of water with very little effort because the force applied on the log of wood is horizontal and parallel to the water surface. Thus a fluid is capable of exerting normal stresses on a surface with which it is in contact.

Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure.

2.2 Variation of pressure with Height

In all fluids at rest the pressure is a function of vertical dimension. To determine this consider the forces acting on a vertical column of fluid of cross sectional-area dA as shown in figure. The positive direction of vertical measurement h is taken downward. The pressure on the upper side is P, and that on the lower face is $P + dP$. The weight of the element is $\rho g dh dA$. The normal forces on the vertical surfaces of the column do not affect the balance of forces in the vertical direction. Equilibrium of the fluid element in the vertical direction requires.



$$p dA + \rho g dA dh - (p + dp)\, dA = 0$$

$$\Rightarrow \quad dP = \rho g dh$$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Above equation holds for both liquids and gases and agrees with our common observations of air and water pressure.

Liquids are generally treated as incompressible and we may consider their density ρ constant for every part of the liquid. With ρ as constant equation may be integrated as it stands, and the result is

$$P = P_0 + \rho g h \qquad \ldots (\mathbf{10})$$

The pressure P_0 is the pressure at the surface of the liquid where $h = 0$.

2.3 Characteristics of Fluid Pressure

(i) Pressure at a point acts equally in all directions.

(ii) Liquids at rest exerts lateral pressure, which increases with depth.

(iii) Pressure acts normally on any area in whatever orientation the area may be held.

(iv) Free surface of a liquid at rest remains horizontal.

(v) Pressure at every point in the same horizontal line is the same inside a liquid at rest.

(vi) Liquid at rest stands at the same height in communicating vessels.

Illustration 5

What is the pressure at the bottom of a tank filled with water upto a height of 4 m? Atmospheric pressure is equal to 10 m of water.

Solution:

Pressure at the bottom of the vessel

= atmospheric pressure + pressure due to water column

= 10 m of water + 4 m of water

= 14 m of water

= $h\rho g$

= (14 m)(1000 kg/m^3)(9.8 m/s^2)

= $\mathbf{1.37 \times 10^5}$ **N/m^2**

Illustration 6

If the atmospheric pressure is 76 cm of mercury at what depth of water the pressure will be equal to 2 atmosphere?

Density of mercury = 13600 kg/m^3.

Solution:

Let the pressure be 2 atm at a depth h below the water surface.

Of this pressure, one atmosphere is due to the atmospheric pressure over the surface of water and hence the pressure due to the water column alone = 1 atm.

= 76 cm of mercury

= (0.76)(13600)(9.8) N/m^2

Now a height of h *m* of water column produces this pressure and we are required to find this height h.

$$(h\rho g)_{\text{water}} = (h\rho g)_{\text{mercury}}$$

$$h \times 1000 \times 9.8 = 0.76 \times 13600 \times 9.8$$

$$h = \frac{0.76 \times 13600 \times 9.8}{1000 \times 9.8} = 10.336 \text{ m}$$

Hence the height of water barometer corresponding to standard atmospheric pressure is **10.336 m**.

2.4 Force due to fluid on a plane submerged surface

A surface submerged in liquid such as bottom of a tank or wall of a tank or gate valve in a dam, is subjected to pressure acting normal to its surface and distributed over its area. In problems where the resultant forces are appreciable, we must determine the resultant force due to distribution of pressure on the surface and the position at which resultant force acts.

For system, which is open to earth's atmosphere, the atmospheric pressure P_0 acts over all surfaces and hence, yields a zero resultant. In such cases, then we need to consider only fluid pressure $P = \rho gh$ which is increament above atmospheric pressure.

In general pressure at different point on the submerged surface varies so to calculate resultant force, we divide the surface into number of elementary areas and we calculate force on it first by treating pressure as constant then we integrate it to get net force.

i.e., $F_R = \int PdA$

The point of application of resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the axis.

Illustration 7

Water is filled upto the top in a rectangular tank of square cross section. The sides of cross section is a and height of the tank is H. If density of water is ρ, find force on the bottom of the tank and on one of its wall. Also calculate the position of point of application of the force on the wall.

Solution:

Force on the bottom of tank: pressure at the bottom of tank due to water is uniform of magnitude ρgh.

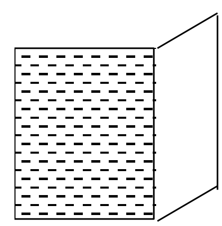
Area of the bottom of the tank = a^2

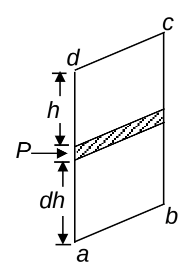
$\therefore$ force on the bottom = pressure × area = ρgHa^2

Force on the wall and its points of application:

Force on the wall of the tank can't be calculated using pressure × area as pressure is not uniform over the surface. The problem can be solved by finding force dF on a thin strip of thickness dh at a depth h below the free surface and then taking its integral.

Pressure (P) of liquid at depth (h) = ρgh





Area of thin strip = adh

$\therefore$ Force of the strip, $Df = P\,adh = \rho gahdh$

$\therefore$ Total force on the wall = $\int_O^H \rho gahdh$

$$F = \rho ga\frac{H^2}{2}$$

The point of application of force on the wall can be calculated by equating the moment of resultant force about any line say *dc* to the moment of distributed force about the same line *dc*.

Moment of *dF* about line *cd* = $dFH = \rho gh.\ adh.\ h$

$\therefore$ Net moment of distributed forces = $\rho ga \int_O^H h^2 dh = \rho ga \dfrac{H^3}{3}$

Let the point of application of net force is at a depth *x* from the line *cd*

Then torque of resultant force about line *cd* = $Fx = \rho ga \dfrac{H^2}{2} x$

From the above discussion,

$$\rho ga \frac{H^2}{2} x = \rho ga \frac{H^3}{3}$$

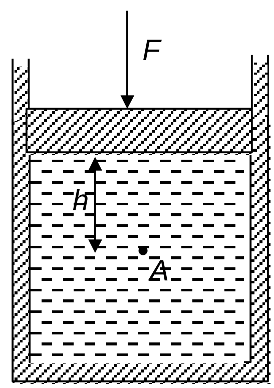
$$\Rightarrow \qquad x = \frac{2H}{3}$$

Hence, resultant force on the vertical wall of the tank will act at a depth $\dfrac{2H}{3}$ from the free surface of water or at a height of $\dfrac{H}{3}$ from the bottom of tank.

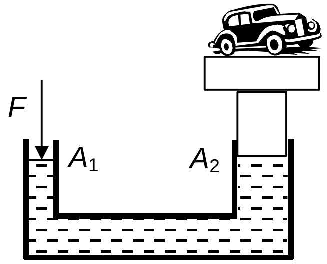
2.5 Pascal’s Law

When we squeeze a tube of toothpaste, the toothpaste comes out of the tube from its opening. This is an example of Pascal’s law. When we squeeze the tube, pressure is applied on the tube and this pressure is transmitted everywhere in the tube and forces the toothpaste out of the tube. According to Pascal, “Pressure applied to a fluid confined in a vassel is transmitted to the entire fluid without being diminished in magnitude”.

To prove Pascal’s law consider a fluid having density ρ in a cylinder fitted with a movable piston. A point *A* is at the depth *h* in the fluid. If no force is applied to the piston pressure just below the piston will have same value say P_0. So according to previous discussion pressure at point *A* will be $P_1 = P_0 + \rho gh$. Now if a force *F* is applied on the piston, pressure just below the piston will have its value initial pressure plus the increased pressure say $P_0 + P_{ext}$. So new pressure at *A* can be written as $P_2 = P_0 + P_{ext} + \rho gh$. Therefore increase in the pressure at *A* will be $P_2 - P_1 = P_{ext}$. That is the increased pressure at the top gets transmitted to *A* also.



Pascal’s law has many important applications and one of the interesting application is in Hydraulic lever. Hydraulic lever is used to lift heavy bodies like automobile by using less effort. It is made of two interconnected vertical cylinders of different cross sectional area A_1 and A_2 and a force *F* is applied to smaller piston causes much longer force on the larger piston, which can lift the body.



The pressure increased below the smaller piston $= \dfrac{F}{A_1}$

The some increment of pressure will take place below the larger piston also.

So force transmitted to larger piston = $A_2 \dfrac{F}{A_1}$.

$\therefore$ If the body lifts due to application of F

$$A_2 \frac{F}{A_1} = Mg$$

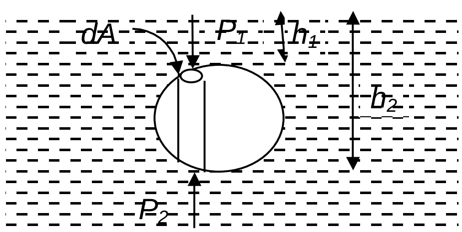
$$\therefore \qquad F = Mg\frac{A_1}{A_2}$$

The ratio $\dfrac{A_1}{A_2}$ is smaller than 1 and thus applied force can be much smaller than the weight Mg that is lifted.

2.6 BUOYANCY AND ARCHIMEDE'S PRINCIPLE

If an object is immersed in or floating on the surface of a liquid, it experiences a net vertically upward force due to liquid pressure. This force is called as buoyant force or force of Buoyancy and it acts from the center of gravity of the displaced liquid. According to Archimede's the magnitude of force of buoyancy is equal to the weight of the displaced liquid.

To prove Archimede's principle, consider a body totally immersed in a liquid as shown in the figure.



The vertical force on the body due to liquid pressure may be found most easily by considering a cylindrical volume similar to the one shown in figure.

The net vertical force an the element is $dF = (P_2 - P_2)\, dA$

$$= \left[\left(P_0 + \rho\int gh_2\right) - \left(P_0 + \rho\int gh_1\right)\right] dA$$

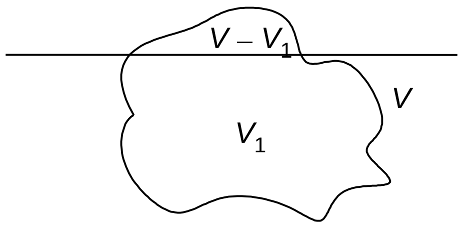
$$= \rho\, g \int g\left(h_2 - h_1\right) dA$$

but $(h_2 - h_1)\, dA = dV$, the volume of element. Thus

Net vertical force on the body $F = \int \rho g dV = V\rho g$

$\therefore$ Force of Buoyancy = $V\rho g$

2.7 EXPRESSION FOR THE IMMERSED VOLUME OF FLOATING BODY



Let a solid of volume V and density p floats in a liquid of density σ with a volume V_1 immersed inside the liquid.

The weight of the floating body = $V\rho g$

The weight of the displaced liquid = $V_1\sigma g$

For the equilibrium of the floating body

$$V_1\sigma g = V\rho g$$

or $$\frac{V_1}{V} = \frac{\rho}{\sigma}$$

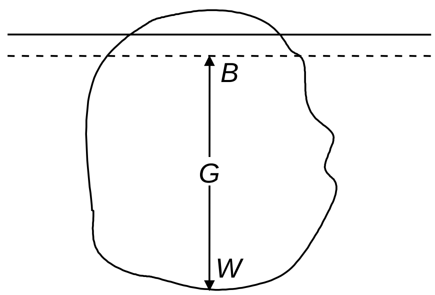
or $$V_1 = \frac{\rho}{\sigma}\times V$$

$$\text{Immersed volume} = \frac{\text{density of solid}}{\text{density of liquid}} \times \text{volume of solid}$$

$$= \frac{\text{mass of solid}}{\text{density of liquid}}$$

From above relations it is clear that the density of the solid must be less than the density of the liquid to enable it to float freely in the liquid. However a metal vessel may float in water though the density of metal is much higher than that of water. The reason in that the floating bodies are hollow inside and hence they have a large displaced volume. When they float in water the weight of the displaced water is equal to the weight of the body.

2.8 LAWS OF FLOATION



The principle of Archimedes may be applied to floating bodies to give the laws of floatation as follows:

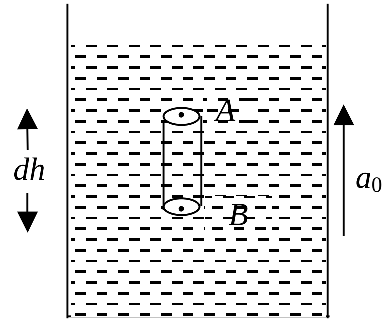
(i) When a body floats freely in a liquid the weight of the body is equal to the weight of the liquid displaced.

(ii)The centre of gravity of the displaced liquid (called the centre of buoyancy) lies vertically above or below the centre of gravity of the body.

2.9 LIQUID IN ACCELERATED VESSEL

A Liquid in accelerated vessel can be considered as in the rigid body motion i.e. motion without deformation as though it were a solid body. As in case of static liquid to determine the pressure variation we apply Newton's law, the same is applicable in case of liquid in accelerated vessel also.

2.9.1 Variation of pressure and force of buoyancy in a liquid kept in vertically accelerated vessel

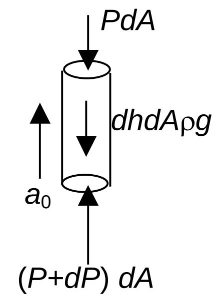


Consider a liquid of density ρ kept in a vessel moving with acceleration a_0 in upward direction.

Let A and B are two points separated vertically by a distance dh. The forces acting an a vertical liquid column of cross sectional area dA are shown in the figure. For the vertical motion of this liquid column,

$$(P + dP)\, dA - PdA - (dh)dA\rho g = (dh)dA\rho a_0$$

$$\Rightarrow \qquad dp = \rho\,(g + a_0)\, dh$$



If pressure at the free surface of liquid is P_0 them pressure P at a depth h from the free surface is given by

$$\int_{P_0}^{P} dp = \int_0^h s\left(g + a_0\right) dh$$

$$\Rightarrow P = P_0 + \rho\,(g + a_0)\, h$$

On the basis of similar calculation, pressure at any depth h from free surface in case of liquid in downward accelerated vessel can be written as $P = P_0 + \rho\,(g - a_0)\, h$

We can generalize the above results and conclude that in liquid for a vertically accelerated vessel the pressure at any depth h below the free surface, $P = P_0 + \rho g_{\text{eff}}\, h$

Where $g_{\text{eff}} = g + a_0$ in case of upward acceleration and

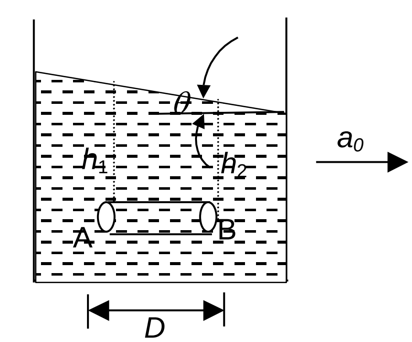
$g_{\text{eff}} = g - a_0$ in case of downward acceleration

Also we can say the force of buoyancy F_B on a body in liquid in vertically accelerated vessel is given by

$$F_{\mathbf{B}} = V\,\rho g_{\text{eff}}$$

Where V is volume of liquid displaced by the body.

2.9.2 Shape of free surface of liquid in horizontally accelerated vessel



When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal. To find angle θ made by free surface with horizontal consider a horizontal liquid column including two points, A and B at depths h_1 and h_2 from the inclined free surface of liquid as shown in figure. D is length of liquid column. Horizontal forces acting on the liquid column are as shown in diagram, for horizontal motion of column we can write,

$$P_1 dA - P_2\, dA = \rho(dA)\, D(a_0)$$

$$(h_1 - h_2)\, g = P\, a_0$$

$$\Rightarrow \qquad \frac{h_1 - h_2}{D} = \frac{a_0}{g} \Rightarrow \tan\theta = \frac{a_0}{g} \;\Rightarrow\; \theta = \tan^{-1}\frac{a_0}{g}$$

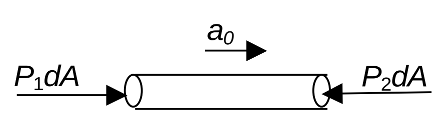


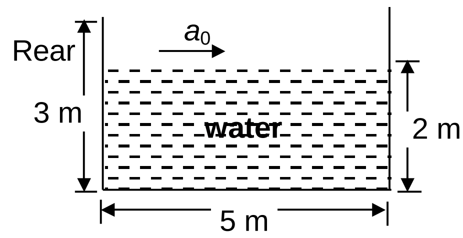
Illustration 8

An open rectangular tank 5 m × 4 m × 3 m high containing water upto a height of 2 m is accelerated horizontally along the longer side.

(a) Determine the maximum acceleration that can be given without spilling the water.

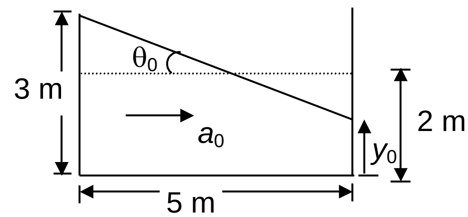
(b) Calculate the percentage of water spilt over, if this acceleration is increased by 20%

(c) If initially, the tank is closed at the top and is accelerated horizontally by 9 m/s^2, find the gauge pressure at the bottom of the front and rear walls of the tank. (Take g = 10 m/s^2).



Solution:

(a) Volume of water inside the tank remains constant



$$\frac{3+y_0}{2} 5\times 4 = 5 \times 2 \times 4$$

or $\quad y_0 = 1$ m

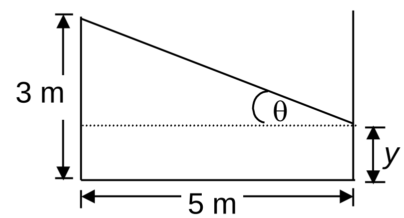
$\therefore \quad \tan\theta_0 = \dfrac{3-1}{5} = \mathbf{4.0}$

since, $\tan\theta_0 = \dfrac{a_0}{g}$, therefore $a_0 = 0.4\, g = \mathbf{4\ m/s^2}$

(b) When acceleration increased by 20%

$a = 1.2\, a_0 = 0.48\, g$

$\therefore \quad \tan\theta = \dfrac{a}{g} = 0.48$



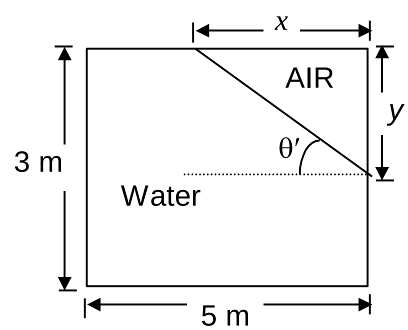
Now, $\quad y = 3 - 5\tan\theta = 3 - 5\,(0.48) = 0.6$ m

$$= \frac{4\times 2\times 5 - \frac{(3+0.6)}{2}\times 5\times 4}{2\times 5\times 4} = 0.1$$

percentage of water spilt over = **10%**

(c) $a' = 0.9\, g$

$\tan\theta' = \dfrac{a'}{g} = 0.9$



volume of air remains constant

$$4 \times \frac{1}{2} yx = (5)\,(1) \times 4$$

since $y = x\tan\theta'$

$\therefore \qquad \frac{1}{2}x^2 \tan\theta' = 5$

or $\qquad x = 3.33$ m; $y =$ **3.0 m**

Gauge pressure at the bottom of the

(i) Front wall p_f = zero

(ii) Rear wall $p_r = (5 \tan\theta')\,\rho w g = 5(0.9)\,(10^3)\,(10) =$ **4.5×10^4 Pa**

Illustration 9

What fraction of the total volume of an iceberg floating on sea-water is exposed? The density of ice is 920 kg/m^3 and that of sea-water is 1030 kg/m^3.

Solution:

If V and V_1 be the total and submerged volumes of the iceberg

$$\frac{V_1}{V} = \frac{\text{density of iceberg}}{\text{density of sea-water}} = \frac{920}{1030} = \frac{92}{103}$$

Now the fraction of volume exposed = $\frac{V - V_1}{V} = 1 - \frac{V_1}{V}$.

or $= 1 - \frac{92}{103} = \frac{11}{103}$ = **10.7%**

Illustration 10

When a boulder of mass 240 kg is placed on a floating iceberg it is found that the iceberg just sinks. What is the mass of the iceberg? Take the relative density of ice as 0.9 and that of sea-water as 1.02.

Solution:

Let M be the weight of iceberg in kg. Then its volume

$$V = \frac{M}{0.9\times10^3}m^3 = \frac{M}{900}m^3$$

When this volume just sinks under sea-water, the weight of water displaced = $V \times 1020$ kg.

From the principle of buoyancy, this weight must be equal to the weight of iceberg and the boulder on it.

$$M + 240 = \frac{M}{900} \times 1020 = M \times \frac{102}{90}$$

or $\qquad M\left(\frac{102}{90} - 1\right) = 240$

or $\qquad M = \frac{240\times90}{12} = 1800$ kg

The mass of the iceberg = **1800 kg**

3. FLUID DYNAMICS

3.1 RATE OF FLOW

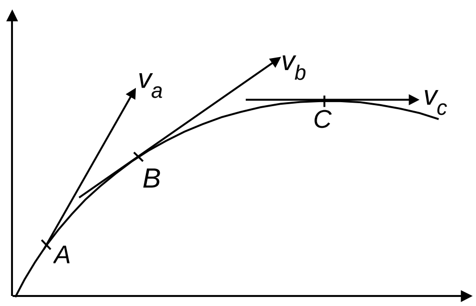
The rate of flow of a liquid is expressed in terms of its volume that flows out from an orifice of known cross-sectional area in unit time. All liquids are practically incompressible and the rate of flow through any section is the same. The rate of flow of a liquid hence is defined as the volume of it that flows across any section in unit time. For example, if the velocity of flow of a liquid is v in a direction perpendicular to two sections A and B of area A and distant l apart and if t be the time taken by the liquid to flow from A to B, we have $vt = l$. Obviously the volume of liquid flowing through section AB is equal to cylindrical column $AB = lA$ or vtA.

$\therefore$ Rate of flow of liquid $= \dfrac{vtA}{t} = vA$

Sometimes the rate of flow of liquid is expressed in terms of mass of liquid flowing across any section in unit time $vA\rho$ where ρ is the density of liquid.

3.2 STREAMLINES

In steady flow the velocity v at a given point does not change with time. Consider the path ABC of a fluid particle in a steady flow and let the velocities be v_a, v_b, v_c at A, B and C respectively as shown in the figure.

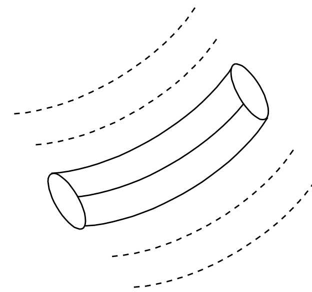


Because v_a does not change with time any particle passing through A will have the same velocity v_a and follow the same path ABC. This curve is called a streamline. A streamline is parallel to the velocity of fluid particle at every point. No two streamlines will ever cross each other. Otherwise any particle at the point of intersection will have two paths to go so that the flow will no more be steady. In non-steady flows the pattern of streamline changes with time. In steady flow if we draw a family of streamlines the tangent to the streamline at any point gives the direction of instantaneous velocity of the fluid at that path of motion. Thus as long as the velocity pattern does not change with time, the streamlines are also the paths along which the particles of the fluid move. In this type of flow the fluid between two surfaces formed by sets of adjacent streamlines is confined to the region between these two surfaces. This is so since the velocity vector has no component perpendicular to any streamline. The flow may be considered to occur in sheets or layers. Hence it is also sometimes called laminar flow. This type of flow is possible only if the velocity is below a certain limiting value called critical velocity. Above this velocity the motion is said to be unsteady or turbulent. The streamlines may be straight or curved depending on to the lateral pressure as it is the same throughout or different.

3.3 TUBE OF FLOW

In a fluid in steady flow, if we select a finite number of streamlines to form a bundle like the streamline pattern shown in figure below the tubular region is called a tube of flow.

The tube of flow is bounded by streamlines so that no fluid can flow across the boundaries of the tube of flow and any fluid that enters at one end must leave at the other end.

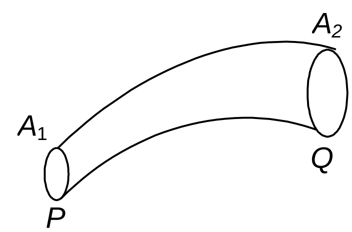


3.4 EQUATION OF CONTINUITY

Consider a thin tube of flow as in the figure. Let A_1 and A_2 be the area of cross-section of the tube perpendicular to the streamlines at P and Q respectively. If v_1 is the velocity of fluid particle at P and v_2 the velocity at Q, then the mass of liquid entering A_1 in a small interval of time Δt is given by $\Delta m_1 = \rho_1 A_1 v_1 \Delta t$.

The mass flux i.e., the mass of fluid flowing per unit time is $\frac{\Delta m_1}{\Delta t} = \rho_1 A_1 v_1$

Let $\Delta t \to 0$ so that v and A does not vary appreciably. The mass flux at P is given by $\rho_1 A_1 v_1$. Similarly the mass flux at $Q = \rho_2 A_2 v_2$ where ρ_1 and ρ_2 are the densities of the fluid at P and Q respectively. Since no fluid crosses the wall of the tube of flow, the mass of fluid crossing each cross-section of the tube per unit time must be the same



$\therefore \quad \rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad$ or $\quad \rho A v =$ constant

This equation expresses the law of conservation of mass in fluid dynamics and is called the **Equation of Continuity**. If the fluid is incompressible as in most liquids ρ = constant

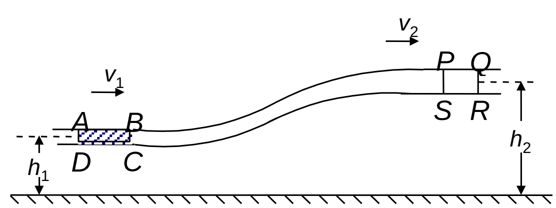
$\therefore \quad A_1 v_1 = A_2 v_2$ or Av = constant. … (**11**)

The product Av represents the rate of flow of fluid. Hence, v is inversely proportional to the area of cross-section along a tube of flow. Therefore, widely spaced streamlines indicate regions of low speed and closely spaced streamlines indicate regions of high speed.

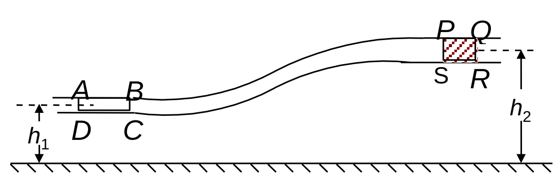
3.5 BERNOULLI'S EQUATION

It is a fundamental equation in fluid mechanics and can be derived from the work-energy theorem. The theorem states that the work done by the resultant force acting on a system is equal to the change in kinetic energy of the system.

Consider a steady, irrotational, non-viscous, incompressible flow of a fluid through a pipe or tube of flow as shown in the figure. A portion of the fluid flowing through a section of pipe line from the position shown in figure (a) to that shown in figure (b).



(a)



(b)

The pipe line is horizontal at A and P and are at heights h_1 and h_2 respectively from a reference plane. The pipe line gradually widens from left to right. Let P_1, A_1, v_1 and P_2, A_2, v_2 be the pressures, areas of cross-section and velocities at A and P respectively.

Suppose the fluid which escapes the volume $ABCD = A_1\Delta l_1$ flows out at P after some time and occupies $PQRS$ which is equal to $A_2\Delta l_2$ as indicated in (a) and figure (b) respectively. The forces acting on the fluid are the pressure forces P_1A_1 and P_2A_2, the force of gravity at A and P. The work done on the system by the resultant force is as follows:

(i) The work done on the system by pressure force P_1A_1 which is $P_1A_1\Delta l_1$

(ii) The work done on the system by pressure force P_2A_2 that is $-P_2A_2\Delta l_2$, the negative sign implying that positive work is done by the system.

(iii) The gravitational work done by the system in lifting the mass m inside from the height h_1 to height $h_2$is given by $mg\ (h_2 - h_1)$ or the work done on the system will be $-\ mg(h_2 - h_1)$

Hence the work done by the resultant force on the system is the sum of all the above terms.

$$W = P_1A_1\Delta l_1 - P_2A_2\Delta l_2 - mg\ (h_2 - h_1)$$

Since $A_1\Delta l_1 = A_2\Delta l_2$ we can put $A_1\Delta l_1 = A_2 l_2 = \frac{m}{\rho}$ where ρ is the density of the fluid

$$\therefore \qquad W = (P_1 - P_2)\ \frac{m}{\rho} - mg\ (h_2 - h_1)$$

The change in kinetic energy of the fluid is $\Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

From the work-energy theorem we can put the work done on the system is equal to the change in kinetic energy of the system.

$$(P_1 - P_2)\frac{m}{\rho} - mg(h_2 - h_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

On multiplying both sides of equation by ρ/m and rearranging we get

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Since the subscripts 1 and 2 refer to any two locations on the pipeline, we can write in general

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \qquad \ldots\ (\mathbf{12})$$

The above is called **Bernoulli's equation** for steady non-viscous incompressible flow. Dividing the above equation by ρg we can rewrite the above as

$$h + \frac{v^2}{2g} + \frac{P}{\rho g} = \text{constant, which is called total head.}$$

Out of the three components of the total head on the left hand side of the above equation the first term '*h*' is called elevation head or gravitational head, the second term $\frac{'v^2'}{2g}$ is called velocity head and the third term $\frac{'P'}{\rho g}$ is called pressure head.

Now if the liquid flows horizontally its potential energy remains constant so that the Bernoulli's equation becomes $P+\frac{1}{2}\rho v^2$ = constant. Further if the fluid is at rest or v = 0, P + ρgh = constant.

The pressure ($P + \rho gh$) which would be present also when there is flow is called the static pressure while the term $\frac{1}{2}\rho v^2$ is called the dynamic pressure. The equation shows that for an ideal liquid the velocity increases when pressure decreases and vice-versa.

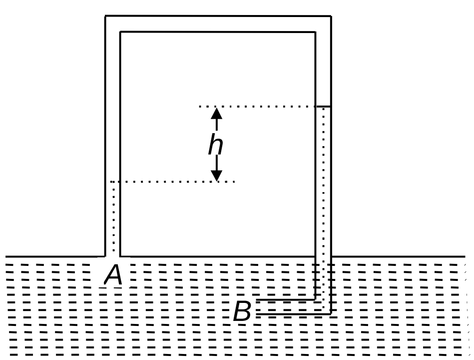
3.6 APPLICATIONS OF BERNOULLI'S THEOREM

We shall consider a few examples wherein Bernoulli's principle is employed.

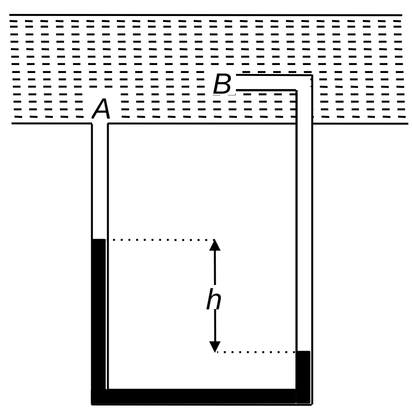
Filter pump: In a filter pump water flows suddenly from a wide tube into a narrow one. The velocity is increased and pressure reduced far below the atmospheric pressure. The receiver to be exhausted is connected by a side tube with this region of low pressure. Air is carried along with water and in a short time air in the receiver is reduced to a pressure, which is slightly greater than the vapour pressure of water.

Atomizer or sprayer: In flowing out of a narrow tube into the atmosphere or a wider tube, air produces a suction effect. In a sprayer a strong jet of air on issuing from a nozzle of a tube lowers the pressure over another tube, which dips in a liquid. The liquid is sucked up and mixed with air stream and thus a fine mist is produced.

Pitot tube: This device is used to measure the flow speed of liquids and gases in pipes. It consists of a manometer tube connected into pipeline as shown in figure. One of the ends of the manometer tube *A* is connected with its plane of aperture parallel to the direction of flow of fluid while the other end *B* is perpendicular to it.



(a) For liquid



(b) For Gases

The pressure and velocity on the left arm of the manometer opening *A* remains the same as they are elsewhere and is equal to static pressure P_a. Since the velocity of fluid at *B* is reduced to zero the pressure is the full arm pressure. Applying Bernoulli's principle, $P_a+\frac{1}{2}\rho v^2 = P_b$ which shows $P_b > P_a$. If *h* is the difference in height of liquid in the manometer and ρ′ the density of manometric liquid,

$P_a + \rho' gh = P_b$

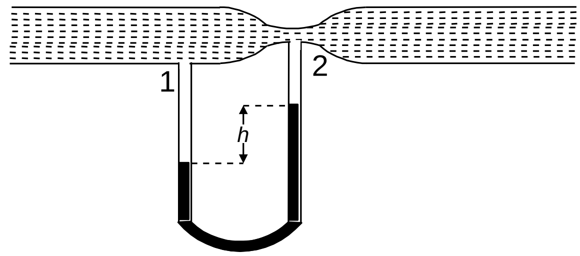
$\therefore \quad \frac{1}{2}\rho v^2 = \rho' gh$

In case of Pitot tube $\rho = \rho'$

$\therefore \quad v^2 = 2gh$

In case of gas Pitot tube $v = \sqrt{\frac{2gh\rho'}{\rho}}$ … (**13**)

Venturimeter: This is a device based on Bernoulli's principle used for measuring the flow of a liquid in pipes. A liquid of density ρ flows through a pipe of cross sectional area A. At the constricted part the cross sectional area is a. A manometer tube with a liquid say mercury having a density ρ' is attached to the tube as shown in figure. If P_1 is the pressure at point 1 and P_2 the pressure at point 2 we have



$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$ where v_1 and v_2 are the velocities at these points respectively.

We have $Av_1 = av_2$

$\therefore \quad v_1 = \frac{av_2}{A}$

$\therefore \quad \frac{1}{2}v_1^2 - \frac{1}{2}v_2^2 = \frac{P_2}{\rho} - \frac{P_1}{\rho} = \frac{P_2 - P_1}{\rho}$

$\therefore \quad v_1^2 - \frac{A^2 v_1^2}{a^2} = \frac{2(P_2 - P_1)}{\rho}$,

$$v_1^2\left(1 - \frac{A^2}{a^2}\right) = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2 = \frac{2(P_2 - P_1)}{\left(1 - \frac{A^2}{a^2}\right)\rho} = \frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}$$

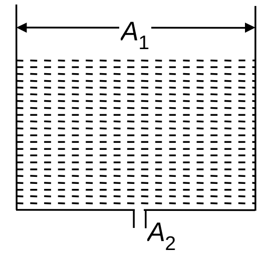
If $a < A$,

$$v_1^2 = \frac{2a^2(P_1 - P_2)}{(A^2 - a^2)\rho} \quad \Rightarrow \quad v_1 = \sqrt{\frac{2(P_1 - P_2)a^2}{(A^2 - a^2)\rho}}$$

Volume of liquid flowing through the pipe per sec. = $Av_1 = Q$

$$Q = Aa\sqrt{\frac{2(P_1 - P_2)}{(A^2 - a^2)\rho}} \quad \dots (\mathbf{14})$$

Speed of Efflux



As shown in figure represents a tank of cross-sectional area A_1, filled to a depth h with a liquid of density ρ. There is a hole of cross-sectional area A_2 at the bottom and the liquid flows out of the tank through the hole $A_2 << A_1$

Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2.

As both the cross-sections are open to the atmosphere, the pressures there equals the atmospheric pressure P_0. If the height of the free surface above the hole is h,

Bernoulli's equation gives

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho gh = P_0 + \frac{1}{2}\rho v_2^2$$

By the equation of continuity,

$$A_2 v_1 = A_2 v_2$$

$$P_0 + \frac{1}{2}\rho\left[\frac{A_2}{A_1}\right]^2 v_2^2 + \rho gh = P_0 + \frac{1}{2}\rho v_2^2$$

$$\left[1-\left(\frac{A_2}{A_1}\right)^2\right]v_2^2 = 2gh \quad \Rightarrow v_2 = \sqrt{\frac{2gh}{1-\left(\frac{A_2}{A_1}\right)^2}} \qquad \ldots \textbf{(15)}$$

If $A_2 <<< A_1$, the equation reduces to $v_2 = \sqrt{2gh}$ … (**16**)

The speed of efflux is the same as the speed a body would acquire in falling freely through a height h. This is known as Torricelli's theorem.

Illustration 11

A tube having its two limbs bent at right angles to each other is held with one end dipping in a stream and opposite to the direction of the flow. If the speed of stream is 2.68 m/see, find the height to which the water rises in the vertical limb of the tube.

Solution:

Clearly the flow of water will be stopped by the tube dipping in the stream and facing the flow so that loss of kinetic energy per unit mass of water is $\frac{v^2}{2}$. This will therefore be the gain in pressure energy i.e., $\frac{P}{\rho}$.

$$\therefore \frac{P}{\rho} = \frac{v^2}{2} \qquad \text{or} \qquad P = \frac{v^2\rho}{2}$$

If h is the height to which water rises in the tube then $P = h\rho g$ or $h\rho g = \frac{v^2\rho}{2}$

$\therefore \qquad h = \dfrac{v^2}{2g} = \dfrac{(2.68)^2}{2\times 9.8}$ = 0.3664 metre or **36.64 cm**

Illustration 12

If the diameters of a pipe are 10 cm and 6 cm at points where a venturimeter is connected and the pressures at these points are found to differ by 5 cm of water column, find the volume of water flowing through the pipe per second.

Solution:

We know $Av_1 = Aa\sqrt{\dfrac{2(P_1 - P_2)}{(A^2 - a^2)\rho}}$

$$A = \pi r_1^2 = \pi\left(\frac{10}{2}\right)^2 = 25\pi cm^2 = 25\pi \times 10^{-4}\ \text{m}^2$$

$$a = \pi r_2^2 = \pi\left(\frac{6}{2}\right)^2 = 9\pi cm^2 = 9\pi \times 10^{-4} m^2$$

$P_1 - P_2$ = 5 cm of water column = $5 \times 10^{-2} \times 10^3 \times 9.8 = 5 \times 98$ N/m^2

$$\therefore \quad Av_1 = 25\pi \times 9\pi \times 10^{-4}\sqrt{\frac{2\times 5\times 98\times 10^{-3}}{(25\pi)^2 - (9\pi)^2}}$$

$$= 225\pi^2\sqrt{\frac{0.98}{34\pi \times 16\pi}} \times 10^{-4}$$

$= 30.00 \times 10^{-4}$ m^3/sec

= **3 litres/sec**

Illustration 13

A Pitot tube is fixed in a main of diameter 15 cm and the difference in pressure indicated by gauge is 4 cm of water column. Find the volume of water passing through the main in one minute.

Solution:

Radius of main = 7.5 cm

Area of cross-section = $\pi (7.5)^2$ cm^2

Loss of kinetic energy = $\dfrac{1}{2}v^2 ergs$

Gain of pressure energy = $\dfrac{P}{\rho} = p = hg$ $= 4 \times 981$ ergs

$\therefore \quad \dfrac{1}{2}v^2 = 4 \times 981$

$v = \sqrt{7848}$ = **88.59** cm/sec

Illustration 14

Water stands at a depth H in a tank whose side walls are vertical. A hole is made at one of the walls at a depth h below the water surface. Find at what distance from the foot of the wall does the emerging stream of water strike the floor. What is the maximum possible range?

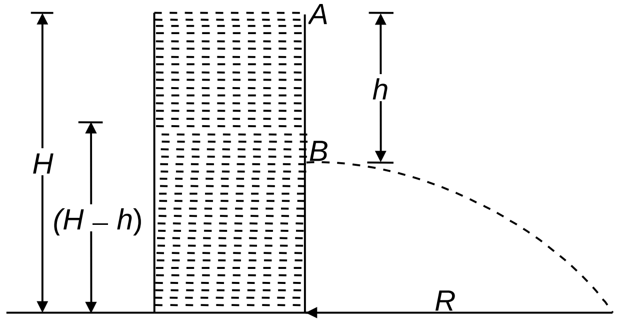
Solution:

Applying Bernoulli's theorem at points A and B,

$$P_A + \frac{1}{2}\rho v_A^2 + \rho g H_A = P_B + \frac{1}{2}\rho v_B^2 + \rho g H_B$$

$$P + 0 + \rho g H = P + \frac{1}{2}\rho v^2 + \rho g(H - h)$$

$\therefore \quad v^2 = 2gh$



The vertical component of velocity of water emerging from hole B is zero. Therefore time taken (t) by the water to fall through a distance $(H - h)$ is given by

$$H - h = \frac{1}{2}gt^2 \text{ or } t = \sqrt{\frac{2(H-h)}{g}}$$

Required horizontal range $R = vt$

$$= \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}}$$

$$= \boldsymbol{2\sqrt{h(H-h)}}$$

The range R is maximum when $\dfrac{dR}{dh} = 0$

$$2 \times \frac{1}{2}(Hh - h^2)^{-\frac{1}{2}}(H - 2h) = 0$$

This gives $h = \dfrac{H}{2}$s

$\therefore$ Maximum possible range $= 2\sqrt{\dfrac{H}{2} \times \left(H - \dfrac{H}{2}\right)}$

$= \boldsymbol{H}$

4. SURFACE TENSION

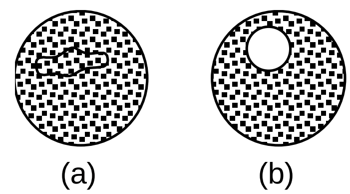
4.1 INTRODUCTION

The surface of a liquid behaves somewhat like a stretched elastic membrane. Just as a stretched elastic membrane has a natural tendency to contract and occupy a minimum area, so also the surface of a liquid has got the natural tendency to contract and occupy a minimum possible area as permitted by the circumstances of the liquid mass. We have many evidences in support of this fact, which we discuss now

(i) We very often find that a small quantity of a liquid spontaneously takes a spherical shape, e.g., rain drops, small quantities of mercury placed on a clean glass plate etc. Now, for a given volume, a sphere has the least surface area. Thus, this fact shows that a liquid always tends to have the least surface area.

(ii) If we immerse a camel-hair brush in water, its hairs spread out, but the moment it is taken out of water, they all cling together as though bound by some sort of elastic thread.

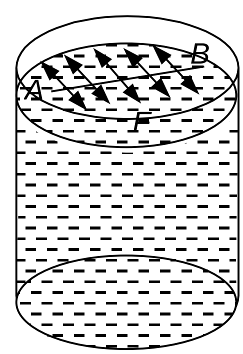
Experimental demonstration: Make a circular loop of wire and dip it in a soap solution. Take it out of the solution and find a thin film of soap solution across the loop. Place a moistened cotton loop gently on the film. The thread will lie on the film in an irregular manner as shown in figure (a).



Now prick the film inside the cotton thread loop by a pin. At once the thread will spontaneously take the shape of a circle as shown in figure (b). The thread has a fixed perimeter. For a given perimeter a circle has got the maximum area. Hence the remaining portion of the film occupies the minimum area.

4.2 DEFINITION

The above evidences and experiment prove beyond doubt that the surface of a liquid behaves like a stretched elastic membrane having a natural tendency to contract and occupy a minimum possible area as permitted by the circumstances of the liquid mass. It is defined as "*the property of the surface of a liquid by virtue of which it tends to contract and occupy the minimum possible area is called surface tension and is measured by the force per unit length of a line drawn on the liquid surface, acting perpendicular to it and tangentially to the surface of the liquid*".



Let an imaginary line *AB* on the surface of liquid of length *L* as shown in figure and force on this line is *F*, So surface tension

$$T = \frac{F}{L} \qquad \ldots \textbf{(17)}$$

Surface tension has dimensions [MT^{-2}] and Its unit is Newton per metre (Nm^{-1}).

It depends on temperature. The surface tension of all liquids decreases linearly with temperature

It is a scalar quantity and become zero at a critical temperature.

Illustration 15

Calculate the force required to take away a flat circular plate of radius 4 cm from the surface of water, surface tension of water being 75 dyne cm^{-1}.

Solution:

Length of the surface = circumference of the circular plate

$= 2\pi r = (8\pi)$ cm

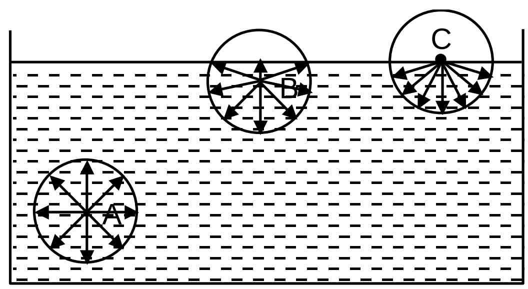
Required force $72 \times 8\pi$ = **1810 dyne**

4.3 MOLECULAR THEORY OF SURFACE TENSION

The surface tension of a liquid arises out of the attraction of its molecules. Molecules of a fluid (liquid and gas) attract one another with a force. It depends on the distance between molecules. The distance up to which the force of attraction between two molecules is appreciable is called the molecular range, and is generally of the order of 10^{-9} metre. A sphere of radius equal to the molecular range drawn around a molecule is called sphere of influence of the molecules lying within the sphere of influence.

Consider three molecules *A*, *B* and *C* having their spheres of influence as shown in the figure. The sphere of influence of *A* is well inside the liquid, that of *B* partly outside and that of *C* exactly half of total.

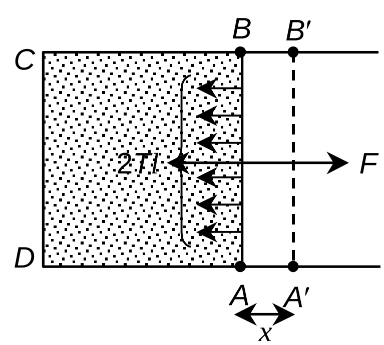
Molecules like *A* do not experience any resultant force, as they are attracted equally in all directions. Molecules like *B* or *C* will experience a resultant force directed inward. Thus the molecules well inside the liquid will have only kinetic energy but the molecules near the surface will have kinetic energy as well as potential energy which is equal to the work done in placing them near the surface against the force of attraction directed inward.



4.4 SURFACE ENERGY

Any strained body possesses potential energy, which is equal to the work done in bringing it to the present state from its initial unstrained state. The surface of a liquid is also a strained system and hence the surface of a liquid also has potential energy, which is equal to the work done in creating the surface. This energy per unit area of the surface is called surface energy.

To derive an expression for surface energy consider a wire frame equipped with a sliding wire *AB* as shown in figure. A film of soap solution is formed across *ABCD* of the frame. The side *AB* is pulled to the left due to surface tension. To keep the wire in position a force F, has to be applied to the right. If *T* is the surface tension and *l* is the length of *AB*, then the force due to surface tension over AB is $2lT$ to the left because the film has two surfaces (upper and lower).



Since the film is in equilibrium, $F = 2lT$

Now, if the wire *AB* is pulled to the right, energy will flow from the agent to the film and this energy is stored as potential energy of the surface created just now. Let the wire be pulled slowly through *x*.

Then the work done = energy added to the film from the agent

$= F.x = 2lTx$

Potential energy per unit area (Surface energy) of the film = $\dfrac{2lTx}{2lx} = T$ … (**18**)

Thus surface energy of the film is numerically equal to its surface tension.

Its unit is joule per square metre (Jm^{-2}).

Illustration 16

Calculate the work done in blowing a soap bubble of radius 10 cm, surface tension being 0.06 Nm^{-1}. What additional work will be done in further blowing it so that its radius is doubled?

Solution:

In case of a soap bubble, there are two free surfaces.

$\therefore$ Work done in blowing a soap bubble of radius R is given by,

$W = 2 \times 4\pi R^2 \times T$, where T is the surface tension of the soap solution.

where $R = 0.1$ m, $T = 0.06\ Nm^{-1}$

$\therefore$ $W' = 8\pi\,(0.1)^2 \times 0.06$ J = 1.51 J

Similarly, work done in forming a bubble of radius 0.2 m is,

$W = 8\pi\,(0.2)^2 \times 0.06$ J = 6.03 J

$\therefore$ Additional work done in doubling the radius of the bubble is given by,

$W' - W$ = 6.03 J – 1.51 J = **4.52 J**

Illustration 17

A film of soap is formed on a rectangular frame of length 10 cm dipping into a soap solution. The frame hangs from the arm of a balance. An extra weight of 0.42 g must be placed in the opposite pan to balance the pull of the frame. Calculate the surface tension of the soap solution.

Solution:

Since the film has two surface, the force of surface tension will be given by

$F = 2 \times 10 \times T = 20\,T$,

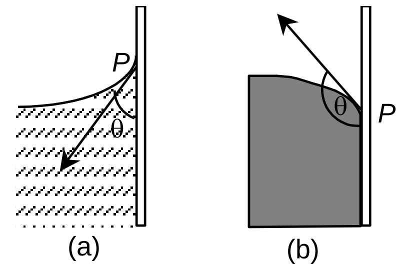
where T is surface tension of soap solution.

The force is balanced by 0.42 g *wt*.

$\therefore$ $0.42 \times 980 = 20\,T$ or $T = \dfrac{0.42 \times 980}{20}$ = **20.6 dyne/cm**

4.5 ANGLE OF CONTACT

When a solid body in the form of a tube or plate is immersed in a liquid, the surface of the liquid near the solid is, in general, curved. It is defined as the angle between the tangents to the liquid surface and the solid surface at the point of contact, for that pair of solid and liquid is called **angle of contact**.

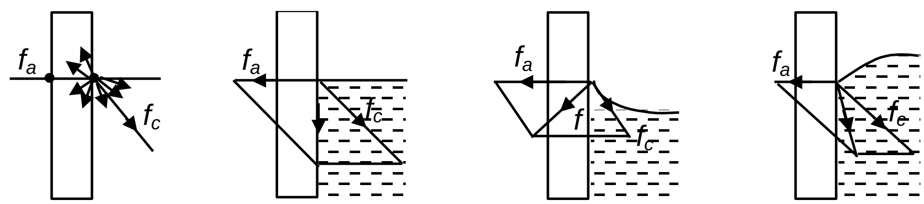


For example when glass strip is dipped in water and mercury as shown in figure (a) and (b) respectively, the angle θ is angle of contact, which is acute in case of water and obtuse in case of mercury.

4.6 ADHESIVE FORCE AND COHESIVE FORCE

The angle of contact arises due to adhesive and cohesive forces on the molecules of the liquid which lie near the solid surface. Forces of attraction between molecules of different substances are called **adhesive forces** and the forces of attraction between molecules of the same substance are called **cohesive forces**.

Consider a liquid molecule near the solid surface. The molecules of the solid wall attract this molecule.



These forces are adhesive and are distributed over 180° and hence their resultant acts at right angles to the solid wall and the forces of cohesion, i.e., attraction by liquid molecules, are distributed over 90° and hence their resultant will be inclined at 45° to the solid wall as shown in figure (a) and (b). Let f_a be the resultant adhesive force and f_c be the resultant cohesive force. Then the angle between f_a and f_c is 135°. Let f be their resultant making angle θ with f_a. Then

$$\tan\theta = \frac{f_c \sin 135°}{f_a + f_c \cos 135°} = \frac{f_c}{\sqrt{2}\, f_a - f_c} \qquad \ldots (\mathbf{19})$$

Case I: If $\sqrt{2}\, f_a = f_c$, then tan θ = ∝ or θ = 90°, i.e., resultant will lie along the solid surface.

Case II: If $\sqrt{2}\, f_a > f_c$, tanθ is positive and hence θ < 90°, i.e., the resultant lies inside the solid as in figure(c).

Case III: If $\sqrt{2}\, f_a < f_c$, tanθ is negative and hence θ > 90°, i.e., the resultant lies inside the liquid as shown in figure (d).

A liquid cannot permanently withstand a shearing force, as it has no rigidity. Hence the free surface of a liquid will be at right angles to the resultant force.

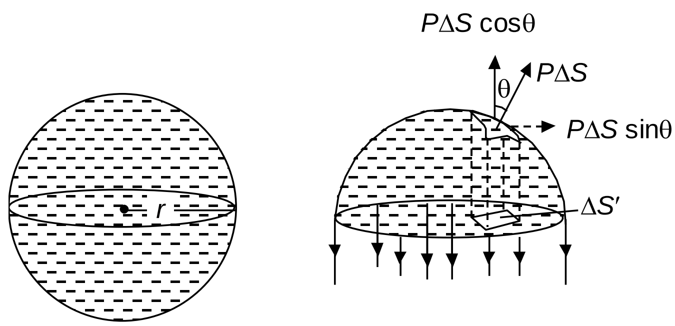
Thus, in the first case, when the resultant force f acts along the solid surface, the liquid surface is at right angles to the solid surface.

In the second case when the cohesive force is smaller than the adhesive force such that $f_c < \sqrt{2}\ f_a$, the resultant is inside the solid and so the liquid surface is inclined to the solid surface at a small angle. The free surface of the liquid is, in consequence, concave upwards near the wall. The concavity of the surface decreases gradually and the free surface becomes horizontal at a large distance from the wall. In this case it is said that the liquid wets the solid, e.g., water wets glass.

In the third case when the cohesive force (f_c) is larger than the adhesive force such that $f_c > \sqrt{2}\, f_a$ the resultant lies inside the liquid and so the liquid surface is inclined to the solid surface at a large angle. Thus in this case the liquid surface is convex upwards near the wall. In this case it is said that the liquid surface does not wet the solid, e.g., mercury and glass.

4.7 EXCESS PRESSURE

The pressure inside a liquid drop or a soap bubble must be in excess of the pressure outside the bubble drop because without such pressure difference a drop or a bubble cannot be in stable equilibrium. Due to surface tension the drop or bubble has got the tendency to contract and disappear altogether. To balance this, there must be an excess of pressure inside the bubble.



To obtain a relation between the excess of pressure and the surface tension, consider a water drop of radius r and surface tension T. Divide the drop into two halves by a horizontal plane passing through its center as shown in figure and consider the equilibrium of one-half, say, the upper half. The forces acting on it are:

(i) forces due to surface tension distributed along the circumference of the section.

(ii) outward thrusts on elementary areas of it due to excess pressure.

Obviously, both the types of forces are distributed. The first type of distributed forces combine into a force of magnitude $2\pi r \times T$. To find the resultant of the other type of distributed forces, consider an elementary area ΔS of the surface. The outward thrust on $\Delta S = p\ \Delta S$ where p is the excess of the pressure inside the bubble. If this thrust makes an angle θ with the vertical, then it is equivalent to $\Delta Sp \cos\theta$ along the vertical and $\Delta Sp \sin\theta$ along the horizontal. The resolved component $\Delta Sp \sin\theta$ is ineffective as it is perpendicular to the resultant force due to surface tension. The resolved component $\Delta Sp \cos\theta$ contributes to balancing the force due to surface tension.

The resultant outward thrust $= \Sigma\Delta Sp \cos\theta$

$$= p\Sigma\Delta S \cos\theta$$

$$= p\Sigma\Delta S' \text{ where } \Delta S' = \Delta S \cos\theta$$

= area of the projection of ΔS on the horizontal dividing plane

$$= p \times \pi r^2 \quad (\because \Sigma\Delta S' = \pi r^2)$$

For equilibrium of the bubble we have

$$\pi r^2 p = 2\pi r.T$$

or, $$p = \frac{2T}{r} \qquad \ldots (\mathbf{20})$$

If it is a **soap bubble**, the resultant force due to surface tension is $2\pi r.2T$, because a bubble has two surfaces. Hence for the equilibrium of a bubble we have

$$\pi r^2 p = 4\pi r T$$

or, $$p = \frac{4T}{r} \qquad \ldots (\mathbf{21})$$

Illustration 18

Find the difference in the air pressure between the inside and outside of a soap bubble, 5 mm in diameter. Assume surface tension to be 1.6 Nm^{-1}.

Solution:

Excess pressure = $\frac{4T}{r}$, where r is radius of curvature of the surface.

Required pressure difference = $\frac{4\times 1.6}{2.5\times 10^{-3}}$

$= \mathbf{2560\ Nm^{-2}}$

Illustration 19

If a number of little droplets of water, all of the same radius r coalesce to form a single drop of radius R, Show that the rise in temperature of the water is given by, $\frac{3\sigma}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$ where J is the mechanical equivalent of heat, and σ is the surface tension of water.

Solution:

Let n be the number of droplets, each of radius r cm that coalesce to form a single drop of radius R cm.

$\therefore$ Decrease in the surface area = $4\pi r^2 n - 4\pi R^2$

Decrease in the surface energy = $(4\pi r^2 n - 4\pi R^2)\,\sigma$

$\therefore$ Heat energy produced in the drop = $\frac{4\pi\sigma}{J}(nr^2 - R^2)$

Suppose the whole of heat energy is used to raise the temperature of the resultant drop by θ, therefore,

$$mS\theta = \frac{4\pi\sigma}{J}(nr^2 - R^2),$$

where m is the mass of the drop having specific heat S.

$$m = \frac{4}{3}\pi R^3 \times 1 \text{ (density = 1 gm/cc)}$$

S = 1 cal/g°C for water

$$\therefore \qquad \theta = \frac{4\pi\sigma}{J}(nr^2 - R^2) \times \frac{3}{4\pi R^3} = \frac{3\sigma}{J}\left(\frac{nr^2}{R^3} - \frac{1}{R}\right) \qquad \ldots\text{(i)}$$

Since volume remains the same,

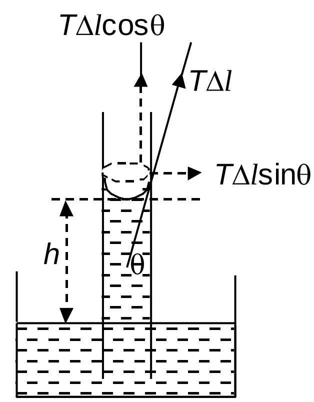
$$n\frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3 \qquad \text{or} \qquad n = \left(\frac{R}{r}\right)^3$$

Putting this value in (i), we have $\theta = \frac{3\sigma}{J}\left(\frac{R^3}{r^3} \times \frac{r^2}{R^3} - \frac{1}{R}\right) = \frac{3\sigma}{J}\left(\frac{1}{r} - \frac{1}{R}\right)$

Thus the rise in temperature is given by, $\theta = \boldsymbol{\frac{3\sigma}{J}\left(\frac{1}{r} - \frac{1}{R}\right)}$

4.8 CAPILLARY ACTION

When a glass tube of very fine bore called a capillary tube is dipped in a liquid (like water,) the liquid immediately rises up into it due to the surface tension. This phenomenon of rise of a liquid in a narrow tube is known as capillarity.



Suppose that a capillary tube of radius r is dipped vertically in a liquid. The liquid surface meets the wall of the tube at some inclination θ called the angle of contact. Due to surface tension a force, ΔlT acts on an element Δl of the circle of contact along which the liquid surface meets the solid surface and it is tangential to the liquid surface at inclination θ to the wall of the tube. (The liquid on the wall of the tube exerts this force. By the third law of motion, the tube exerts the same force on the liquid in the opposite direction.) Resolving this latter force along and perpendicular to the wall of the tube, we have $\Delta lT\cos\theta$ along the tube vertically upward and $\Delta lT\sin\theta$ perpendicular to the wall. The latter component is ineffective. It simply compresses the liquid against the wall of the tube. The vertical component $\Delta lT\cos\theta$ pulls the liquid up the tube.

The total vertical upward force = $\Sigma\Delta lT\cos\theta$

$$= T\cos\theta\Sigma\Delta l = T\cos\theta\, 2\pi r$$

$$(\because \Sigma\Delta l = 2\pi r).$$

Due to this upward pull liquid rises up in the capillary tube till it is balanced by the downward gravitational pull. If h is the height of the liquid column in the tube up to the bottom of the meniscus and v is the volume of the liquid above the horizontal plane touching the meniscus at the bottom, the gravitational pull, i.e., weight of the liquid inside the tube is $(\pi r^2 h + v)\rho g$.

For equilibrium of the liquid column in the tube

$$2\pi rT\cos\theta = (\pi r^2 h + v)\rho g.$$

If volume of the liquid in meniscus is negligible then,

$$2\pi rT\cos\theta = (\pi r^2 h)\rho g.$$

$$h = \frac{2T\cos\theta}{r\rho g} \qquad \dots \textbf{(22)}$$

The small volume of the liquid above the horizontal plane through the lowest point of the meniscus can be calculated if θ is given or known. For pure water and glass θ = 0° and hence the meniscus is hemispherical.

∴ v = volume of the cylinder of height r – volume of hemisphere.

$$= \pi r^3 - \frac{1}{2}\frac{4\pi}{3}r^3 = \pi r^3 - \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

∴ For water and glass

$$2\pi r T = \left(\pi r^2 h + \frac{\pi r^3}{3}\right)\rho g$$

$$2T = r\left(h + \frac{r}{3}\right)\rho g$$

$$h = \frac{2T}{r\rho g} - \frac{r}{3}$$

For a given liquid and solid at a given place as ρ, T, θ and g are constant,

∴ hr = constant

i.e., lesser the radius of capillary greater will be the rise and vice-versa.

Illustration 20

Water rises to a height of 10 cm in a certain capillary tube. The level of mercury in the same tube is depressed by 3.42 cm. Compare the surface tensions of water and mercury. Specific gravity of mercury is 13.6 g/cc and angle of contact for water and mercury are zero and 135° respectively.

Solution:

Using the capillarity relation, $T = \frac{rh\rho g}{2\cos\theta}$

$$T_1 \text{ (for water)} = \frac{r \times 1 \times g \times 10}{2\cos\theta} = 5\text{rg}$$

$$T_2 \text{ (for mercury)} = \frac{r \times 13.6 \times g \times (-3.42)}{2\cos 135^\circ}$$

$$= \frac{r \times 13.6 \times g \times (-3.42)}{2 \times \left(\frac{-1}{\sqrt{2}}\right)}$$

$$= 32.9 \text{ rg}$$

$$\frac{T_1}{T_2} = \frac{5\,rg}{32.9\,rg} = \frac{1}{6.5} = \mathbf{0.15}$$

5. VISCOSITY

If water in a tub is whirled and then left to itself, the motion of the water stops after some time. This is a very common observation. What stops the motion? There is no external force to stop it. A natural conclusion is, therefore, that whenever there is relative motion between parts of a fluid, internal forces are set up in the fluid, which oppose the relative motion between the parts in the same way as forces of friction operate when a block of wood is dragged along the ground. This is why to maintain relative motion between layers of a fluid an external force is needed.

"This property of a fluid by virtue of which it oppose the relative motion between its different layers is known as viscosity and the force that is into play is called the viscous force".

Consider the slow and steady flow of a fluid over a fixed horizontal surface. Let v be the velocity of a thin layer of the fluid at a distance x from the fixed solid surface. Then according to Newton, the viscous force acting tangentially to the layer is proportional to the area of the layer and the velocity gradient at the layer. If F is the viscous force on the layer, then

$F \propto A$ where A is the area of the layer

$$\propto -\frac{dv}{dx}$$

The negative sign is put to account for the fact that the viscous force is opposite to the direction of motion.

$$\Rightarrow \quad F = -\eta A\frac{dv}{dx} \qquad \ldots (\mathbf{23})$$

where η is a constant depending upon the nature of the liquid and is called the coefficient of viscosity and

Velocity gradient = $\dfrac{dv}{dx}$

If $A = 1$ and $\dfrac{dv}{dx} = 1$, we have $F = -\eta$

Thus the coefficient of viscosity of a liquid may be defined as the viscous force per unit area of the layer where velocity gradient is unity.

The coefficient of viscosity has the dimension $[ML^{-1}T^{-1}]$ and its unit is Newton second per square metre (Nsm^{-2}) or kilograme per metre per second ($kgm^{-1}s^{-1}$). In CGS the unit of viscosity is Poise

Illustration 21

A metal plate 100 cm^2 in area rests on a layer of castor oil (η = 15.5 poise) 0.2 cm thick. Calculate the horizontal force required to move the plate with a speed of 3 cm/s.

Solution:

$F = -\eta A\dfrac{dv}{dx}$ where η = 15.5 poise A = 100 cm^2

$$\frac{dv}{dx} = \frac{3}{0.2} = 15s^{-1}$$

F = –15.5 × 100 × 15 = –23250 dyne = **–0.233 N**

5.1 REYNOLDS NUMBER

The stability of laminar flow is maintained by viscous forces. It is observed, however that laminar or steady flow is disrupted if the rate of flow is large. Irregular, unsteady motion, turbulence, sets in at high flow rates.

Reynolds defined a dimensionless number whose value gives one an approximate idea whether the flow rate would be turbulent. This number, called the Reynolds number R_e is defined as,

$$R_e = \frac{\rho v d}{\eta}$$

Where ρ is the density of the fluid flowing with a speed v. The parameter d stands for the typical dimension of the obstacle or boundary to fluid flow. **For a spherical obstacle we may take *d* to be the diameter. For R_e greater than 2000, the flow is often turbulent**. The exact value at which turbulence sets is called the critical Reynolds number.

Illustration 22

The diameter of the tap is 1.25 cm and the flow rate through it is 5×10^{-5} m^3s^{-1}. Is the flow turbulent? Given coefficient of viscosity of water is 10^{-3} Poise.

Solution:

Volume of water flowing out per second is $Q = v\, x \frac{\pi d^2}{4}$

$$\therefore \qquad v = \frac{4Q}{\pi d^2}$$

Reynold's number $R_e = 4\rho Q/\pi d\eta$ = **5100**

The flow will be turbulent

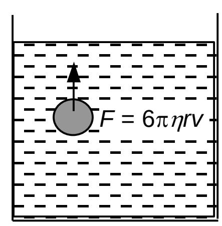
5.2 STOKE'S LAW

When a solid moves through a viscous medium, its motion is opposed by a viscous force depending on the velocity and shape and size of the body. The energy of the body is continually decreases in overcoming the viscous resistance of the medium. This is why cars, aeroplanes etc. are shaped streamline to minimize the viscous resistance on them.

The viscous drag on a spherical body of radius r, moving with velocity v, in a viscous medium of viscosity η is given by

$$F_{\text{viscous}} = 6\pi\eta r v.$$

This relation is called **Stokes' law**



This law can be deduced by the method of dimensions. By experience we guess that the viscous force on a moving spherical body may depend on its velocity, radius and coefficient of viscosity of the medium. We may then write

$$F = k\, v^a r^b \eta^c$$

where k is a constant (dimensionless) and a, b and c are the constants to be determined.

By taking dimensions of both sides, we have

$$MLT^{-2} = (LT^{-1})^a L^b (ML^{-1}T^{-1})^c$$

or, $$MLT^{-2} = M^c L^{a+b-c} T^{-a-c}$$

Equating powers of M, L and T we have

$$c = 1$$

$$a + b - c = 1$$

and $$-a - c = -2$$

Solving we have $a = 1$, $b = 1$ and $c = 1$

$$F = k\eta rv$$

Experimentally k is found to be 6π;

$$F = 6\pi\eta rv. \qquad \ldots \textbf{(24)}$$

Illustration 23

An air bubble of diameter 2 cm rises through a long cylindrical column of a viscous liquid, and travels at 0.21 cms^{-1}. If the density of the liquid is 1.47 g cm^{-1}, find its coefficient of viscosity. Ignore the density of the air.

Solution:

Weight of the bubble is equal to the viscous force.

$$\frac{4}{3}\pi r^3 \rho g = 6\pi\eta rv \quad \text{or} \quad \eta = \frac{2r^2 g\rho}{9v} \qquad \ldots\text{(i)}$$

Given: $r = 10^{-2}$ m ; $\rho = 1.47 \times 10^3$ kg/m^3

$v = 0.21 \times 10^{-2}$ m/s $g = 9.8$ m/s^2

Substituting these values in (i) we have,

$$\eta = \frac{2 \times (10^{-2})^2 \times 1.47 \times 10^3 \times 9.8}{9 \times 0.21 \times 10^{-2}} \approx \mathbf{152.5\ kgm^{-1}\,s^{-1}}$$

Illustration 24

Eight spherical drops of equal size fall vertically through air with a terminal velocity of 0.1 ms^{-1}. What would be the velocity if these eight drops were to combine to form one large spherical drop?

Solution:

Let r_1 be the radius of each small drop, and r_2 that of the bigger drop. As the volume remains constant,

$$\frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi r_1^3 \times 8$$

or, $$\frac{r_1}{r_2} = \frac{1}{2}$$

Since the terminal velocity is proportional to the square of the radius of the drop,

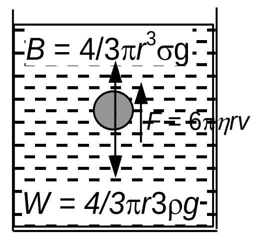
$$\frac{v_1}{v_2} = \left(\frac{r_1}{r_2}\right)^2 = \frac{1}{4}$$

$v_2 = 4 \times 0.1 =$ **0.40 ms^{-1}**

Thus the terminal velocity of the bigger drop is 0.4 m/s.

5.3 TERMINAL VELOCITY

Let the body be driven by a constant force. In the beginning the viscous drag on the body is small. Velocity is small and so the body is accelerated through the medium by the driving force. With the increase of velocity of the body the viscous drag on it will also increase and eventually when it becomes equal to the driving force, the body will acquire a constant velocity. This velocity is called the terminal velocity of the body.



Consider the downward motion of a spherical body through a viscous medium such as a ball falling through liquid. If r is the radius of the body, ρ the density of the material of the body and σ is the density of the liquid then,

the weight of the body = $\frac{4\pi}{3}r^3\rho g$,downwards

and the buoyancy of the body = $\frac{4\pi}{3}r^3\sigma g$,upwards

The net downward driving force = $\frac{4\pi}{3}r^3(\rho-\sigma)g$

If v is the terminal velocity of the body, then the viscous force on the body is

$$F = 6\pi\eta rv$$

For no acceleration of the body we have

$$6\pi\eta rv = \frac{4\pi}{3}r^3(\rho-\sigma)g \text{ or, } v = \frac{2}{9}.\frac{r^2 g\,(\rho-\sigma)}{\eta} \qquad \dots\textbf{(25)}$$

Illustration 25

A gas bubble of diameter 2 cm rises steadily through a solution of density 1750 kgm^{-3} at the rate of 0.35 cm/s. Calculate the coefficient of viscosity of the solution.

Solution:

We have, $v = \frac{2}{9}.\frac{r^2 g\,(\rho-\sigma)}{\eta}$

Here, ρ = density of air is negligible

$\therefore \quad v = -\frac{2}{9}.\frac{r^2 g\sigma}{\eta}$ The negative sign shows that the velocity is upward

$$\therefore \qquad \eta = \frac{2}{9} \cdot \frac{r^2 g\sigma}{v} = \frac{2}{9} \cdot \frac{0.01^2 \times 9.8 \times 1750}{0.0035}$$

$$= \mathbf{109\ kgm^{-1}\ s^{-1}\ or\ Nsm^{-2}}$$

Illustration 26

A sphere is dropped under gravity through a viscous fluid. Taking the average acceleration to be half the initial acceleration, show that the time required to attain the terminal velocity is independent of the fluid density.

Solution:

Initial acceleration, $\quad a = g\,\dfrac{(\rho - \rho_0)}{\rho}$

Terminal velocity, $\quad v = \dfrac{2r^2}{9\eta} g\,(\rho - \rho_0)$

Let t be the time required to attain the terminal velocity.

$$\therefore \qquad v = 0 + \frac{at}{2} \qquad \text{or} \quad v = \frac{at}{2}$$

Average acceleration = $\dfrac{g}{2}$

Putting the values of 'a' and 'g' in (i) we have,

$$\frac{g}{2\rho}(\rho - \rho_0)\, t = \frac{2r^2}{9\eta} g\,(\rho - \rho_0) \qquad \text{or} \quad t = \boldsymbol{\frac{4r^2\rho}{9\eta}}$$

As the expression for t does not contain ρ_0 it is independent of ρ_0.

Fundamental Solved Examples

Example 1.

A light rod of length 200 cm is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 0.1 sq.cm and the other is made of brass and is of cross-section 0.2 sq.cm. Find the position along the rod at which a weight may be hung to produce.

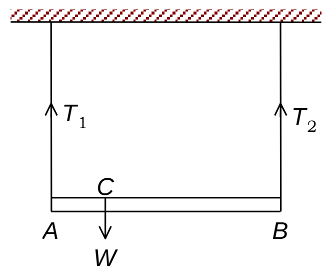
(i) equal stress in both wires

(ii) equal strain in both wires

Y of steel = 20×10^{10} Nm^{-2}

Y of brass = 10×10^{10} Nm^{-2}

Solution:



For equilibrium $W = T_1 + T_2$ ... (i)

where W is the weight, and T_1 and T_2 tensions in the wires.

Taking moments about C

$T_1\,(AC) = T_2\,(BC)$ ... (ii)

(i) If F_1 and F_2 are the elastic stresses in the two wires, their areas of cross-sections are a_1 & a_2 respectively then $T_1 = F_1a_1$ and $T_2 = F_2a_2$

Substituting in equation (2)

$F_1a_1\,(AC) = F_2a_2\,(BC)$

If the stresses in the two wires are equal,

$$F_1 = F_2$$

$$a_1 \times AC = a_2 \times BC$$

$$0.1 \times AC = 0.2 \times BC$$

$$AC = 2\,BC$$

$$AC = \frac{2}{3}AB = \mathbf{\frac{4}{3}\,m}$$

The weight must be suspended at $\frac{4}{3}$ m from steel wire.

(ii) Strain = $\dfrac{\text{increase in length}}{\text{original length}} = \dfrac{\Delta L}{L}$

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{stress}}{\frac{\Delta L}{L}}$$

$\therefore$ stress $= Y\cdot\frac{\Delta L}{L}$

Tension = Stress × area = a Y $\frac{\Delta L}{L}$

Hence for two wires

$T_1 = a_1\, Y_1\, \frac{\Delta L_1}{L}$

$T_2 = a_2\, Y_2\, \frac{\Delta L_2}{L}$

Substituting in equation (2),

$$a_1Y_1\frac{\Delta L_1}{L}\cdot AC \;=\; a_2Y_2,\frac{\Delta L_2}{L}\cdot BC$$

If the strain in the two wires are equal

$$\frac{\Delta L_1}{L}=\frac{\Delta L_2}{L}$$

$a_1\, Y_1\, (AC) \quad = a_2\, Y_2\, (BC)$

$0.1 \times 20 \times 10^{10} \times AC = 0.2 \times 10 \times 10^{10} \times BC$

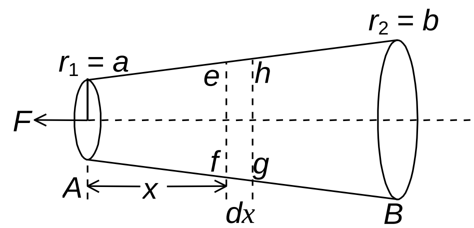
AC = BC

$\therefore$ AC = 1 m

The weight must be suspended at the midpoint of the rod.

Example 2.

A slightly tapering wire of length L and end radii ‘a’ and ‘b’ is subjected to stretching forces F as shown in Figure. If Y is Young’s modulus, calculate the extension produced in the wire.



Solution:

Consider a small element of width dx, radius r situated at a distance x from A. The gradient in radius is uniform. Let it be k.

$\therefore$ $k = \frac{b-a}{L}$ ... (i)

If r is the radius of the elementary portion at ef and (r + dr) at gh, then

$k = \frac{dr}{dx}$ ... (ii)

$\therefore$ $\frac{dr}{dx} = \frac{b-a}{L}$ ... (iii)

$$dx = \frac{dr}{k}$$

If dl is the extension produced in length dx due to stretching force F using the relation

$$Y = \frac{\frac{F}{A}}{\frac{\Delta L}{L}} \text{ we get } \Delta L = \frac{FL}{AY}$$

$$\therefore \quad dl = \frac{F \cdot dx}{\pi r^2 Y}$$

$$= \frac{F \cdot dr}{\pi r^2 Yk} \text{ since } dx = \frac{dr}{k}$$

The variable on R.H.S. is r which varies from a to b.

$$\therefore \text{ Total extension produced} = \int dl = \int_a^b \frac{Fdr}{\pi r^2 Yk}$$

$$l = \frac{F}{\pi Yk}\int_a^b \frac{dr}{r^2} = -\frac{F}{\pi Yk}\left[\frac{1}{r}\right]_a^b$$

$$= \frac{F}{\pi Yk}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= \frac{F(b-a)}{\pi Ykab}$$

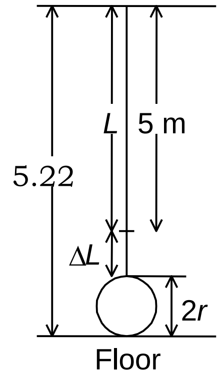
$$\text{But} \quad k = \frac{b-a}{L}$$

$$\therefore \quad l = \frac{F.L}{\pi Yab}$$

Example 3.

A sphere of radius 0.1 m and mass 8π kg is attached to the lower end of a steel wire of length 5 m and diameter 10^{-3} m. The wire is suspended from 5.22 m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Calculate the velocity of the sphere at the lowest position. Young's modulus of steel is 1.994×10^{11} N/m^2.

Solution:



The situation is shown in Figure. Let Δl be the extension of wire at mean position when oscillating and T is the tension.

$$Y = \frac{T/A}{\Delta L/L} \qquad \text{or } T = \frac{Y A \Delta L}{L}$$

$$\Delta L = 5.22 - (L + 2r)$$

$$= 5.22 - (5 + 2 \times 0.1) = 0.02 \text{ m}$$

$$\therefore \ T = \frac{1.994 \times 10^{11} \times \pi (5 \times 10^{-4})^2 \times 0.02}{5} = 199.4\pi \text{ N.}$$

At mean position $T - Mg = \dfrac{Mv^2}{R}$

R, the radius of circular path of oscillating sphere = 5.22 – 0.1 = 5.12 m

$Mg = 8\pi \times 9.8 = 78.4\pi$ N

$$\therefore \ (199.4\pi - 78.4\pi) = \frac{8\pi v^2}{5.12}$$

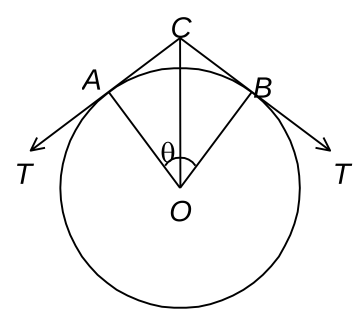
$$v^2 = \frac{121 \times 5.12}{8} = 72.44$$

$$v = 8.8 \text{ m/sec}$$

Example 4.

A thin ring of radius R is made of a material, which has density ρ and Young's modulus Y. If the ring is rotated about its centre in its own plane with an angular velocity ω, find the small increase in its radius.

Solution:



Consider a small element AB of length dl. If a is the area of cross-section the mass will be $ad\ell\rho$. The centrifugal force is equal to $a\, d\ell\, \rho \cdot \omega^2 R$

If consequent tension in the wire is T, the radial component is $2T \sin \dfrac{\theta}{2}$.

For small angle θ this can be written as $2T \cdot \dfrac{\theta}{2} = T\theta$

$$\therefore \ T \cdot \theta = a\, dl\, \rho\, \omega^2 R$$

$$= a R\theta\, \rho\, \omega^2 R$$

$$T = a \rho \omega^2 R^2$$

If the increase in radius is dR, the increase in circumference is $2\pi(R + dR) - 2\pi R = 2\pi dR$

Strain = $\frac{2\pi dR}{2\pi R} = \frac{dR}{R}$

Young's modulus Y = $\dfrac{\frac{T}{a}}{\frac{dR}{R}}$

$T = a\,Y \cdot \frac{dR}{R}$

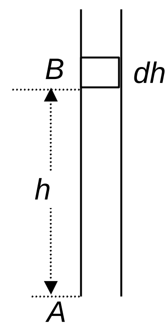
$a\,Y \cdot \frac{dR}{R} = a\,\rho\,\omega^2 R^2$

$\therefore \quad dR = \frac{\rho\omega^2 R^2}{Y}$

Example 5.

A balloon ascends vertically slowly unreeling a long copper wire. Estimate the amount by which the wire has stretched when 1 km of initially unstretched wire has been unreeled. The density of copper is 9×10^3 kg m^{-3} and its Young's modulus Y = 1.2×10^{11} Nm^{-2}.

Solution:



Consider an element of wire of length dh at a height h from the ground. The mass of the part AB = haρ

The load on dh = haρg

$$Y = \frac{stress}{strain} = \frac{ha\rho g}{\frac{e}{dh}}$$

where e is the elongation of element dh.

$$\therefore \quad e = \frac{\rho g h dh}{Y}$$

Stretched length = dh + e = dh + $\frac{\rho g h dh}{Y}$

$$\text{Entire length} = \int_0^{H=1km} dh\left(1+\frac{\rho g h}{Y}\right) = H + \frac{\rho g}{Y}\left[\frac{h^2}{2}\right]_0^H$$

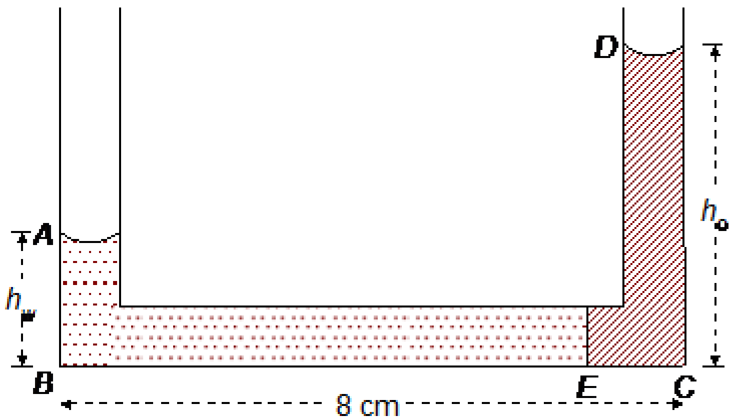
$$= H + \frac{\rho g H^2}{2Y} = 10^3 + \frac{9\times10^3\times9.8}{2\times1.2\times10^{11}}\times(10^3)^2$$

$\therefore$ increase in length = 0.3675 m.

Example 6.

A tube of uniform cross-section consists of two vertical portions with a horizontal portion 8 cm long connecting their lower ends. Enough water to occupy 22 cm of the tube is poured into one branch and enough oil of specific gravity 0.8 to occupy 22 cm is poured into the other. Find the position of the common surface of the two liquids.

Solution:



Let AB be the vertical water column and CD the vertical oil column in the tube.

As the liquids in the tube are at rest, pressure at points B and C at the same horizontal level should be equal.

Let h_W cm be the height of water column AB and h_0 be the height of oil column CD.

Now considering pressures at points B and C

$$h_W d_W g = g_0 d_0 g$$

Where d_W and d_0 are the densities of water and oil respectively and it is given that

$$\frac{d_o}{d_w} = 0.8$$

$$\therefore \qquad \frac{h_w}{h_o} = \frac{d_o}{d_w} = 0.8$$

$$\frac{h_w}{h_o} + 1 = 0.8 + 1$$

or $\quad \dfrac{h_w + h_o}{h_o} = 1.8$ or $\quad \dfrac{22+22-8}{h_o} = 1.8$

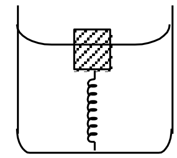
or $\quad h_0 \dfrac{36}{1.8} = 20$ cm

$$h_W = h_0 \times 0.8 = 16 \text{ cm}$$

The common surface E of the two liquids lies at the horizontal portion of the tube and is at a distance 6 cm from B.

Example 7.

A cubical block of wood of edge 3 cm floats in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. Find the maximum weight that can be put on the block without wetting it. Density of wood = 800 kg/m^3 and spring constant of the spring = 50 N/m. Take g = 10 m/s^2.



Solution:

The specific gravity of the block = 0.8. Hence the height inside water = 3 cm × 0.8 = 2.4 cm. The height outside water = 3 cm – 2.4 = 0.6 cm. Suppose the maximum weight that can be put without wetting it is W. The block in this case is completely immersed in the water. The volume of the displaced water

$= \text{volume of the block} = 27 \times 10^{-6}\ \text{m}^3.$

Hence, the force of buoyancy

$= (27 \times 10^{-6}\ \text{m}^3) \times (1000\ \text{kg/m}^3) \times (10\ \text{m/s}^2) = 0.27\ \text{N}$

The spring is compressed by 0.6 cm and hence the upward force exerted by the spring

$= 50\ \text{N/m} \times 0.6\ \text{cm} = 00.3\ \text{N}.$

The force of buoyancy and the spring force taken together balance the weight of the block plus the weight W put on the block. The weight of the block is

$W' = (27 \times 10^{-6}\text{m}) \times I\ (800\ \text{kg/m}^3) \times (10\ \text{m/s}^2) = 0.22\ \text{N}$

Thus, W = 0.27 N + 0.3 N – 0.22 N = 0.335 N

Example 8.

A metal cube floats on mercury with $\frac{1}{8}$th of its volume under mercury. What portion of the cube will remain under mercury if sufficient water is added just to cover the cube. Assume that the top surface of the cube remains horizontal in both cases. Relative density of mercury = 13.6.

Solution:

Let V be the volume and d the density of the cube.

Weight of the floating cube

$= Vdg$

= Weight of the displaced liquid

$= \frac{V}{8}(13.6 \times d_1)g$

where d_1 is the density of water.

Now $Vdg = \frac{V}{8} \times 13.6 \times d_1 \times g$

$d = 1.7d_1 \qquad \text{... (i)}$

In the second case let a volume V_1 be in mercury and a volume $(V - V_1)$ in water.

Again by the principle of floatation

$$Vdg = V_1 (13.6\, d_1)g + (V - V_1)d_1 g \quad \text{... (ii)}$$

From (1) and (2)

$$V \times 1.7 d_1 g = 13.6\, V_1 \times d_1 g + (V - V_1)d_1 g$$

or $$1.7V = 13.6\, V_1 + (V - V_1)$$

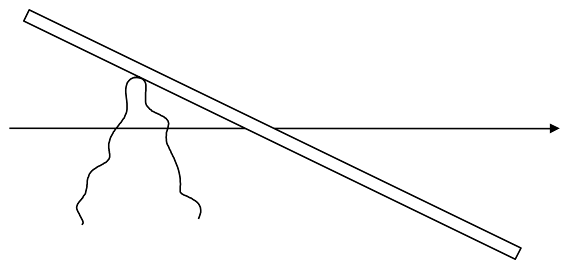
$$0.7\, V = 12.6\, V_1$$

or $$\frac{V_1}{V} = \frac{0.7}{12.6} = \frac{1}{18}$$

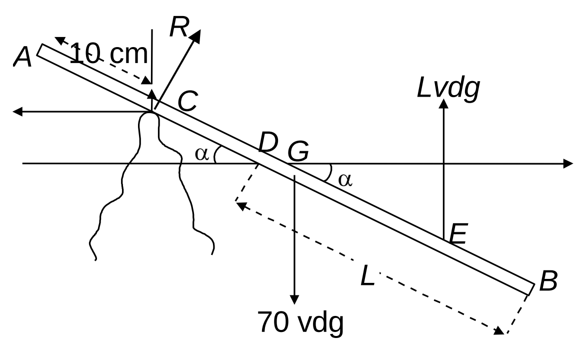
In the second case only $\frac{1}{18}$th of the volume of the cube is under mercury.

Example 9.

One end of a uniform wooden board of length 1 m is placed on the top of a stone protruding from water. 10 cm length of the plank is above the point of support as shown in the figure.. Find the length of the plank immersed in water if the specific gravity of the wood is 0.7.



Solution:



AB is the uniform wooden board with its end A projecting over the point of support C.

Let d be the density of water. Then the density of wood will be 0.7 d.

Let v be the volume per unit length of the plank.

Mass of the plank = 100 v × 0.7 d = 70 vd

Let L be the length of plank under water.

Volume of plank immersed = Lv

Mass of water displaced = Lvd

The forces acting on the plank are

(i) Reaction R at C

(ii) Weight of plank acting down at G.

(iii) Buoyancy Lvdg acting at E, the midpoint of DB.

Taking moments about C, we get, $70\, vdg \times 40 \cos\alpha = Lvdg \left(90 - \frac{L}{2}\right) \cos\alpha$

On solving we get, L = 40 cm or L = 140 cm

Clearly the length under water is 40 cm.

Example 10.

A wooden stick of length L, radius R and density ρ has a small metal piece of mass m (of negligible volume) attached to its one end. Find the minimum value for the mass m (in terms of given parameters) that would make the stick float vertically in equilibrium in a liquid of density σ.

Solution:

For the stick to be vertical for rotational equilibrium, centre of gravity should be below in a vertical line through the centre of buoyancy. For minimum m, the two will coincide.

Let h be the length of immersed portion. For translator equilibrium,

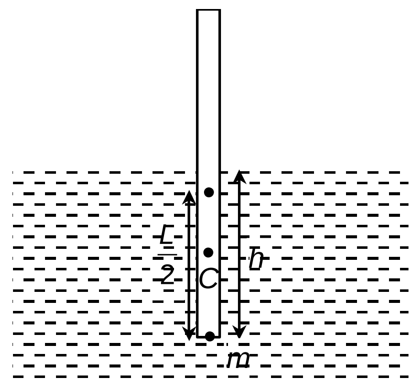
Wt. of rod + mass attached = force of buoyancy

$$(M+m)g = \pi R^2 h\sigma g \qquad \text{... (i)}$$

where $M = \pi R^2 L\rho$.

The height of centre of mass from bottom

$$= \frac{(M)L/2 + m\times 0}{m+M} = \frac{ML}{2(m+M)}$$



For rotatory equilibrium and for minimum m, this should be equal to h/2.

$$\therefore \qquad \frac{h}{2} = \frac{ML}{2(m+M)} \qquad \text{... (ii)}$$

$$\therefore \qquad h = \frac{ML}{(m+M)}$$

Substituting for h in equation (i), we get

$$(M+m)g = \pi R^2 \sigma g.\frac{ML}{(m+M)}$$

$$(M+m)^2 = \pi R^2 \sigma.ML$$

$$(M+m) = \sqrt{M\pi R^2 \sigma L} = \sqrt{\pi R^2 L\rho.\pi R^2 \sigma L}$$

$$m = \pi R^2 L\sqrt{\sigma\rho} - \pi R^2 L\rho$$

$$= \pi R^2 L\rho\left[\sqrt{\frac{\sigma}{\rho}} - 1\right]$$

Example 11.

A tube having its two limbs bent at right angles to each other is held with one end dipping in a stream and opposite to the direction of the flow. If the speed of stream is 2.68 m/see, find the height to which the water rises in the vertical limb of the tube.

Solution:

Clearly the flow of water will be stopped by the tube dipping in the stream and facing the flow so that the loss of kinetic energy per unit mass of water is $\frac{v^2}{2}$. This will therefore be the gain in pressure energy i.e., $\frac{P}{\rho}$.

$$\therefore \qquad \frac{P}{\rho} = \frac{v^2}{2} or\, P = \frac{v^2\rho}{2}.$$

If h is the height to which water rises in the tube

$$P = h\rho g \text{ or } h\rho g = \frac{v^2\rho}{2}$$

$$\therefore \qquad h = \frac{v^2}{2g} = \frac{(2.68)^2}{2\times 9.8} = 0.3664 \text{ metre or } 36.64 \text{ cm}$$

Example 12.

A vertical tube of 4 mm diameter at the bottom has water passing through it. If the pressure be atmospheric at the bottom where the water emerges at the rate of 800 gm per minute, what is the pressure at a point in the tube 25 cm above the bottom where the diameter is 3 mm?

Solution:

Rate of emergence of water = 800 gm/minute = $\frac{40}{3}$ gm/sec

$= \frac{40}{3}$ cm^3/sec

$\because$ density of water, $\rho = 1$ g/cm^3

This will be the same across any section of the tube.

Diameter at bottom = 4 mm = 0.4 cm

Area of cross-section = $\pi(0.2)^2 = 0.04\pi$ cm^2

Volume of water passing through the bottom = cross-sectional area $\times$ velocity

Velocity of water passing through the bottom = $\frac{40}{3\times 0.04\pi}$ say (v_1) cm/sec.

Kinetic energy per unit mass of water at the bottom = $\frac{1}{2}v_1^2 = \frac{1}{2}\left(\frac{40}{3\times 0.04\pi}\right)^2$ ergs

Pressure energy at the bottom = $\frac{P_1}{\rho}$

$= 76 \times 13.6 \times 981$ ergs

$\therefore$ Total energy = kinetic energy + pressure energy

$$= \frac{1}{2}\left(\frac{40}{3\times 0.04\pi}\right)^2 + (76 \times 13.6 \times 981)$$

Again diameter of tube above 25 cm = 3 mm

= 0.3 cm

Area of cross-section = $\pi(0.15)^2$ cm^2

Velocity of flow of water $= \dfrac{40}{3\times\pi(0.15)^2}$

Kinetic energy per unit mass $= \dfrac{1}{2}\left(\dfrac{40}{3\times\pi(0.15)^2}\right)^2$ ergs

Let the pressure be P_2.

$\therefore$ pressure energy per unit mass $= P_2 \times 13.6 \times 981$

Also potential energy per unit mass of water = hg = 25 × 981 ergs

$\therefore$ total energy = kinetic energy + pressure energy + potential energy

$$= \frac{1}{2}\left(\frac{40}{3\times\pi(0.15)^2}\right)^2 + P_2(13.6\times 981) + (25\times 981)$$

Using Bernoulli's theorem,

$$\frac{1}{2}\left(\frac{40}{3\times 0.04\pi}\right)^2 + (76 + 13.6\times 981) = \frac{1}{2}\left(\frac{40}{3\pi\times 0.0225}\right)^2 + (P_2\times 13.6\times 981) + (25\times 981)$$

$$\frac{800}{(0.12\pi)^2} + (76\times 13.6\times 981) = \frac{800}{(0.0675\pi)^2} + (P_2 \times 13.6 \times 981) + (25 \times 981)$$

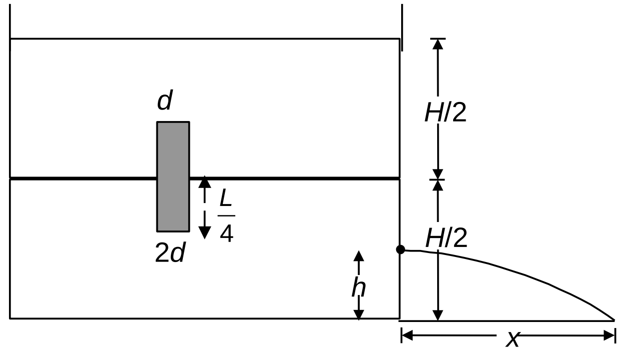
$$5629 + 1013962 = 17790 + 13342P_2 + 24525$$

$$13342P_2 = 5629 + 1013962 - 17790 - 24525 = 977276$$

$\therefore$ $P_2 = 73.25$ cm of Hg

Example 13.

A container of large uniform cross-sectional area A resting on a horizontal surface, holds two immiscible non-viscous and incompressible liquids of densities d and 2d, each of height as shown in Figure. The lower density liquid is open to the atmosphere having pressure P_0.



(a) A homogeneous solid cylinder of length $L\left(L<\frac{H}{2}\right)$, cross-sectional area $\frac{A}{5}$ is immersed such that it floats with its axis vertical at the liquid-liquid interface with length $\frac{L}{4}$ in the denser liquid.

Determine: (I) the density D of the solid and, (II) the total pressure at the bottom of the container.

(b) The cylinder is removed and the original arrangement is restored. A tiny hole of area S(S << A) is punched on the vertical side of the container at a height $h\left(h<\frac{H}{2}\right)$

Determine:

(I) the initial speed of efflux of the liquid at the hole

(II) the horizontal distance x travelled by the liquid initially and

(III) the height h_m at which the hole should be punched so that the liquid travels the maximum distance x_m initially. Also calculate x_m

(Neglect the air resistance in these calculations).

Solution:

(a) The weight of the body = VDg. This must be equal to the weights of the displaced volumes of the two liquids.

i.e., Weight of the cylinder = Net Buoyant force

$$VDg = V_1\,(2d)g + V_2\,(d)\,g \qquad \text{... (i)}$$

$$LADg = \frac{L}{4}A\frac{(2d)g}{5} + \frac{3L}{4}\frac{Adg}{5}$$

Density of the solid D = $\boldsymbol{\frac{5d}{4}}$

Total pressure at the bottom of the container

$$= \frac{\text{weight of liquid} + \text{weight of cylinder}}{A} + P_0$$

$$P = \frac{Ah_1d_1g + Ah_2d_2g + VDg}{A} + P_0 \quad = \frac{d\frac{H}{2}gA + 2d\frac{H}{2}gA + \frac{A5dLg}{5\times 4}}{A} + P_0$$

$$= \boldsymbol{\frac{(6H+L)dg}{4} + P_0}$$

(b) Using Bernoulli's theorem, just inside and just outside orifice,

$$P_0 + dg\frac{H}{2} + 2dg\left(\frac{H}{2} - h\right) = \frac{1}{2}(2d)v^2 + P_0$$

$$v = \sqrt{g\left[\frac{H}{2} + 2\left(\frac{H}{2} - h\right)\right]} = \boldsymbol{\sqrt{g\left[\frac{3H}{2} - 2h\right]}}$$

The horizontal range x = vt $=\sqrt{g\left[\frac{3H}{2}-2h\right]}\sqrt{\frac{2h}{g}} = \sqrt{\left(\frac{3H}{2}-2h\right)(2h)} = \sqrt{\boldsymbol{(3H-4h)h}}$

For x to be maximum $\frac{dx}{dh} = 0$

$$3H = 8h$$

The height at which the hole should be punched so that the liquid travels the maximum distance,

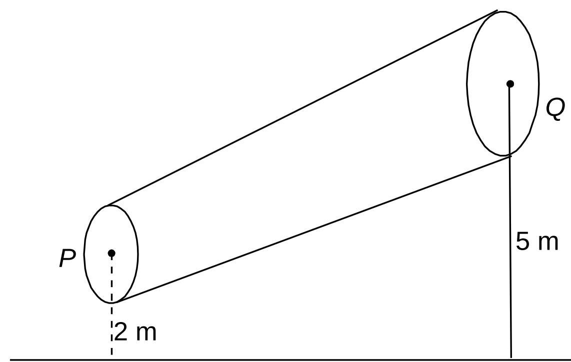
$$h_{max} = \frac{3H}{8}$$

Maximum distance travelled,

$$x_{max} = \frac{3H}{4}$$

Example 14.

A non-viscous liquid of constant density 1000 kg/m^3 flows in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in the Figure. The area of cross-section of the tube at two points P and Q at heights 2 m and 5 m are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at P is 1 m/s. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from P to Q.



Solution:

Work done per unit volume by the gravity force = $\rho g(h_1 - h_2)$

$$= 10^3 \times 9.8 \times (2 - 5)$$

$$= -2.94 \times 10^4 \text{ J/m}^3$$

Work done per unit volume by the pressure force = $\frac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \rho g\left(h_2 - h_1\right)$

$$= -375 + 2.94 \times 10^4$$

$$= 29025 \text{ J/m}^3$$

Example 15.

A tank having cross-sectional area A is filled with water to a height H. If a hole of cross-sectional area 'a' is made at the bottom of the tank, find the time taken by water level to decrease from H_1 to H_2.

Solution:

Let h be the level of water at any instant. Then rate of decrease of water level is $\frac{-dh}{dt}$. Therefore

$$-A\frac{dh}{dt} = av = a\sqrt{2gh}$$

$$-\frac{dh}{dt} = \frac{a}{A}\sqrt{2gh}$$

$$-\int_{H_1}^{H_2}\frac{dh}{\sqrt{h}} = \frac{a}{A}\sqrt{2g}\int_0^t dt$$

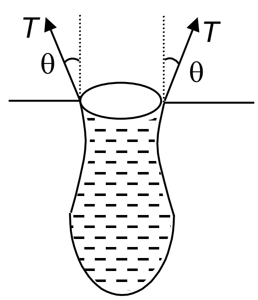
Integrating, $2\left[\sqrt{H_1}-\sqrt{H_2}\right] = \frac{a}{A}\sqrt{2g}\cdot t$

$\therefore$ t = $\frac{A}{a}\sqrt{\frac{2}{g}}\left(\sqrt{H_1}-\sqrt{H_2}\right)$ is the time taken for the level change.

Example 16.

A vessel, whose bottom has round holes with diameter of 1 mm is filled with water. Assuming that surface tension acts only at holes, find the maximum height to which the water can be filled in the vessel without leakage. Given that surface tension of water is 75×10^{-3} N/m and g = 10 m/s^2.

Solution:



As shown in figure here the vertical force due to surface tension at the hole T cosθ × L = T cos θ × 2πr will balance the weight mg, i.e., $\pi r^2 h\rho g$, i.e.,

$T\cos\theta\, 2\pi r = \pi r^2 h\rho g$

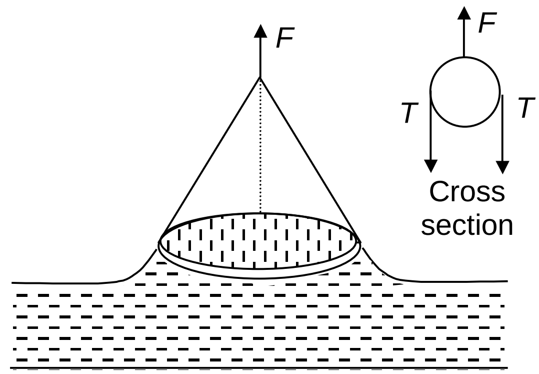
or, $h = (2T\cos\theta/\rho r g)$

This h will be max when cos θ = max = 1

So $(h)_{max} = \frac{2\times75\times10^{-3}}{10^3\times5\times10^{-4}\times10} = 0.03\text{m} = 3$ cm

Example 17.

A ring is cut from a platinum tube of 8.5 cm internal and 8.5 cm external diameter. It is supported horizontally from a pan of a balance so that it comes in contact with the water in a glass vessel. What is the surface tension of water if an extra 3.97 g weight is required to pull it away from water? ($g = 980\ cm/s^2$)



Solution:

The ring is in contact with water along its inner and outer circumference; so when pulled out the total force on it due to surface tension will be

$$F = T\ (2\pi r_1 + 2\pi r_2)$$

So $\quad T = \dfrac{mg}{2\pi(r_1 + r_2)}$ (as F = mg)

i.e., $\quad T = \dfrac{3.97 \times 980}{2 \times 3.14 \times (8.5 + 8.7)}$ = 36.02 dyne/cm

Example 18.

The lower end of a capillary tube of diameter 2.00 mm is dropped 8.00 cm below the surface of water in a beaker. What is the pressure required in the tube to blow a bubble at its end in water? Also calculate the excess pressure.

[Surface tension of water = 73×10^{-3} N/m, density of water = $10^3\ kg/m^3$, 1 atmosphere = 1.01×10^5 Pa and $g = 9.8\ m/s^2$].

Solution:

As the bubble is in water, it has only one surface.

So, $\quad p = p_{in} - p_{out} = \dfrac{2T}{r} = \dfrac{2 \times 7.3 \times 10^{-2}}{10^{-3}}$ = 146 Pa

Now as bubble is at a depth of 8 cm in water,

$$p_{out} = p_{at} + h\rho g$$

So that $p_{in} = p + p_{out} = p + p_{at} + g\rho g$

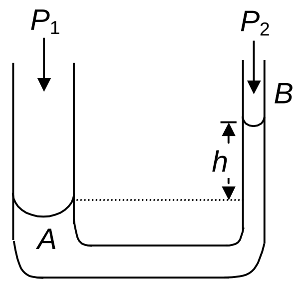
i.e., $\quad p_{in} = 146 + 1.01 \times 10^5 + 8 \times 10^{-2} \times 10^3 \times 9.8 = 1.02 \times 10^5$ Pa

Example 19.

The limbs of a manometer consist of uniform capillary tubes of radii 1.4×10^{-3} m and 7.2×10^{-4} m. Find out the correct pressure difference if the level of the liquid (density 10^3 kg/m^3, surface tension 72×10^{-3} N/m) in narrower tube stands 0.2 m above that in the broader tube.

Solution:

If p_1 and p_2 are the pressures in the broader and narrower tubes of radii r_1 and r_2 respectively, the pressure just below the meniscus in the respective tubes will be



$p_1 - \frac{2T}{r_1}$ and $p_2 - \frac{2T}{r_2}$

So that $\left[p_1 - \frac{2T}{r_1}\right] - \left[p_2 - \frac{2T}{r_2}\right] = h\rho g$

or $\quad p_1 - p_2 = h\rho g - 2T\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$

Assuming the angle of contact to be zero, i.e., radius of meniscus equal to that of capillary,

$$p_1 - p_2 = 0.2 \times 10^3 \times 9.8 - 2 \times 72 \times 10^{-3}\left[\frac{1}{7.2\times10^{-4}} - \frac{1}{14\times10^{-4}}\right]$$

$p_1 - p_2 = 1960 - 97 = 1863$ Pa

Example 20.

Under isothermal condition two soap bubbles of radii a and b coalesce to form a single bubble of radius c. If the external pressure is p_0 show that surface tension.

$$T = \frac{p_0(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}$$

Solution:

As excess pressure for a soap bubble is (4T/r) and external pressure p_0,

$p_i = p_0 + (4T/r)$

So $\quad p_a = \left[p_0 + \frac{4T}{a}\right]$, $p_b = \left[p_0 + \frac{4T}{b}\right]$ and $p_c = \left[p_0 + \frac{4T}{c}\right]$ … (i)

and $\quad V_a = \frac{4}{3}\pi a^3$, $V_b = \frac{4}{3}\pi b^3$ and $V_c = \frac{4}{3}\pi c^3$ … (ii)

Now as mass is conserved, $\mu_a + \mu_b = \mu_c$

i.e., $\quad \frac{p_aV_a}{RT_a} + \frac{p_bV_b}{RT_b} = \frac{p_cV_c}{RT_c}$ [as PV = μRT, i.e., $\mu = \frac{pV}{RT}$]

As temperature is constant, i.e., $T_a = T_b = T_c$, so the above expression reduces to

$p_aV_a + p_bV_b = p_cV_c$

Which in the light of equation (i) and (ii) becomes

$$\left[p_0+\frac{4T}{a}\right]\left[\frac{4}{3}\pi a^3\right]+\left[p_0+\frac{4T}{b}\right]\left[\frac{4}{3}\pi b^3\right]=\left[p_0+\frac{4T}{c}\right]\left[\frac{4}{3}\pi c^3\right]$$

i.e., $4T\,(a^2 + b^2 - c^2) = p_0\,(c^3 - a^3 - b^3)$

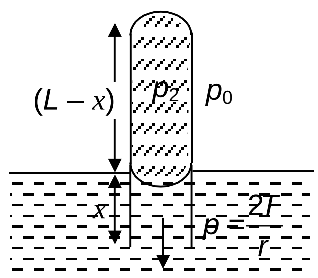
i.e., $T = \dfrac{p_0(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}$

Example 21.

A glass capillary sealed at the upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The tube is immersed vertically into a liquid of surface tension 5.06×10^{-2} N/m. To what length has the capillary to be immersed so that the liquid level inside and outside the capillary becomes the same? What will happen to the water level inside the capillary if the seal is now broken?

Solution:

If A is the cross-sectional area of the tube and L its length, the initial volume of air inside it will be $V_1 = AL$ while pressure $p_1 = p_0$ = atmospheric pressure.



Now when the tube is immersed in water with its length x in water, the level of water inside and outside is same; so the volume of air in the tube will be $V_2 = A(L - x)$. Further if p_2 is the pressure of gas in the tube,

$p_2 - \dfrac{2T}{r} = p_0$, i.e., $p_2 = p_0 + \dfrac{2T}{r}$

Now if temperature is constant, $P_1V_1 = P_2V_2$

$p_0AL = \left[p_0+\dfrac{2T}{r}\right]A(L-x)$ or $x\left[1+\dfrac{rp_0}{2T}\right] = L$

i.e., $x\left[1+\dfrac{1.012\times10^5\times1\times10^{-5}}{2\times5.06\times10^{-2}}\right] = 0.11$ or $x = \dfrac{0.11}{11} = 0.01$ m

If the seal is broken the pressure inside the capillary will become atmospheric, i.e., p_0 while capillarity will take place and the rise will be

$h = \dfrac{2T}{r\rho g} = \dfrac{2\times5.06\times10^{-2}}{10^{-5}\times10^3\times9.8} = 1.03$ m

However, the length of the tube outside the water is 0.11 – 0.01 = 0.1 m; so the tube will be of insufficient length and so the liquid will rise to the top of the tube and will stay there with radius of meniscus,

$r = \dfrac{hR}{L} = \dfrac{1.03\times10^{-5}}{0.1} = 1.03\times10^{-4}$ m

Example 22.

A boat of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. If the river is 1 m deep and the water in contact with the bed is stationary, find the tangential force needed to keep the boat moving with same velocity. Viscosity of water is 0.01 poise.

Solution:

As velocity changes from 2 m/s at the surface to zero at the bed which is at a depth of 1 m,

velocity gradient = $\frac{dv}{dy} = \frac{2-0}{1} = 2\,s^{-1}$

Now from Newton's law of viscous force,

$$|F| = \eta A\,\frac{dv}{dy} = (10^{-2} \times 10^{-1}) \times 10 \times 2 = 0.02\text{ N}$$

So to keep the boat moving at same velocity, force equal to viscous force, i.e., 0.02 N must be applied.

Example 23.

Spherical particles of pollen are shaken up in water and allowed to settle. The depth of the water is 2×10^{-2} m. What is the diameter of largest particles remaining in suspension one hour later?

Density of pollen = 1.8×10^3 kg m^{-3}

Viscosity of water = 1×10^{-2} poise and

Density of water = 1×10^3 kg/m^3

Solution:

For pollen particles not reaching the bottom in 1 hour,

$$v \le \frac{2 \times 10^{-2}}{60 \times 60} = \frac{10^{-4}}{18}\text{ m/s}$$

Due to viscosity effects, the particles will move with terminal velocity v given by

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v$$

$$r^2 = \frac{9\eta v}{2g(\rho - \sigma)}$$

Substituting, $\eta = 1 \times 10^{-2}$ poise = 10^{-3} pl, $\rho = 1.8 \times 10^3$ kg/m^3, $\sigma = 1 \times 10^3$ kg/m^3, g = 10 m/s^2 and

$$v = \frac{10^{-4}}{18}\text{ m/s, we have}$$

$$r^2 = \frac{9 \times 10^{-3}}{2} \times \frac{1}{10\,(1.8-1) \times 10^3}$$

$\therefore$ r = 1.77×10^{-6}m

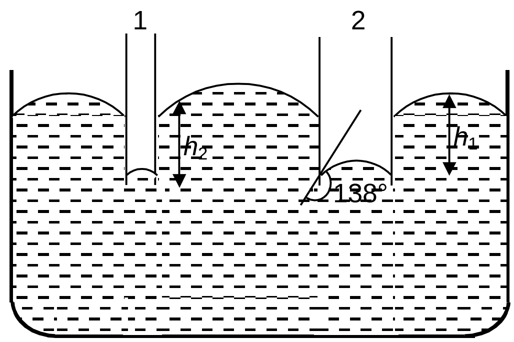
and diameter = 2r = 3.54×10^{-6}m

Example 24.

Find the difference in height of mercury columns in two communicating vertical capillaries whose diameters are $d_1 = 0.50$ mm and $d_2 = 1.00$ mm, if the contact angle $\theta = 138°$.

Solution:

We have for tube number 1 and 2



$$h_1 \rho g = \frac{4T}{d_1}|\cos\theta|$$

$$h_2 \rho g = \frac{4T}{d_2}|\cos\theta|$$

$$\rho g (h_1 - h_2) = 4T|\cos\theta|\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

or $$h_1 - h_2 = 4T|\cos\theta|\left(\frac{d_2 - d_1}{\rho g d_1 d_2}\right)$$

$$= \frac{4\times(490\times10^{-3})|\cos138°|(1-0.5)10^{-3}}{13.6\times10^{+3}\times9.8\times0.5\times10^{-6}} = 11 \text{ mm}$$

Example 25.

When a vertical capillary of length l with the sealed upper end was brought in contact with the surface of a liquid, the level of this liquid rose to the height h. The liquid density is ρ, the inside diameter of the capillary is d, the contact angle is θ, the atmospheric pressure is p_0. Find the surface tension of the liquid.

Solution:

When the liquid rises up in the capillary, it indicates that pressure inside the capillary $p_0 + 4T\frac{\cos\theta}{d}$ is less than the atmospheric pressure by hρg. Applying Boyle's law for the gas inside the capillary

$$p_0 l A = \left(p_0 - \rho g h + \frac{4T\cos\theta}{d}\right)(l-h)A$$

Where A is the area of cross-section of the capillary.

$$\therefore \quad \frac{4T\cos\theta}{d} = \rho g h + \frac{p_0 l}{l-h} - p_0 = \rho g h + \frac{p_o h}{l-h}$$

$$\therefore \quad T = \frac{d}{4\cos\theta}\left(\rho g h + \frac{p_o h}{l-h}\right)$$

Example 26.

A glass rod of diameter d_1 = 1.5 mm is inserted symmetrically into a glass capillary tube with inside diameter d_2 = 2.0 m. Then the whole arrangement is vertically oriented and brought in contact with the surface of water. To what height will the water rise in the capillary?

Solution:

Let h be the height of water inside the capillary. Total upward force tending to pull water is

$$T(2\pi r_1 + 2\pi r_2)$$

this supports the weight of the liquid.

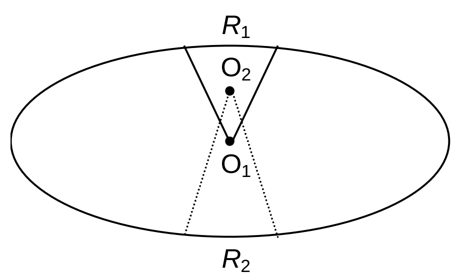
$$h(\pi r_2^2 - \pi r_1^2)\rho g$$

Hence, $$h = \frac{2T(r_1 + r_2)}{(r_2^2 - r_1^2)\rho g} = \frac{2T}{(r_2 - r_1)\rho g} = \frac{4T}{(d_2 - d_1)\rho g}$$

$$= \frac{4 \times 73 \times 10^{-3}}{10^3 \times 9.8(2 - 1.5)10^{-3}} = 6 \text{ cm}$$

Example 27.

A water drop falls in air with uniform velocity. Find the difference between the curvature radii of the drop's surface at the upper and lower points of the drop separated by the distance h = 2.3 mm.



Solution:

The radii of curvatures of the drop are R_1 at the upper end and R_2 at the lower end. Then the pressures inside the surface just below the upper surface is $p_1 + p_0 + \frac{2T}{R_1}$ and just above the bottom is $P_2 = p_0 + \frac{2T}{R_2}$.

As the drop is falling with constant velocity, the net force on the drop is zero in the vertical direction:

$$p_1 = p_2 + \rho gh$$

or $$p_1 - p_2 = \rho gh$$

or $$2T\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \rho g h$$

$$\Rightarrow \quad R_2 - R_1 = \frac{\rho g h R_1 R_2}{2T}$$

h = 2.3 mm and the drop is very small so that approximately $R_1 = R_2 = h/2$. Hence

$$R_2 - R_1 = \frac{\rho g h}{8T} . h^2 = 0.2 \text{ mm}$$

Example 28.

An oil drop falls through air with a terminal velocity of 5×10^{-4} m/s. Calculate

(a) radius of the drop

(b) the terminal velocity of a drop of half of this radius.

Solution:

We know that, terminal velocity

$$v = \frac{2}{9} \times \frac{r^2(\rho - \sigma) g}{\eta} = \frac{2}{9} \frac{r^2 \rho g}{\eta} \qquad \text{(if } \rho >> \sigma\text{)}$$

or $$r = \left[\frac{9}{2} \times \frac{v\eta}{\rho g}\right]^{1/2}$$

Substituting the given values, we have

$$r = \left[\frac{9}{2} \times \frac{(5\times10^{-4})\,(1.8\times10^{-5})}{900\times9.8}\right]^{1/2} = 2.14 \times 10^{-6} \text{ m}$$

It is obvious that the terminal velocity is directly proportional to r^2. When the radius of the drop is half (i.e., $r' = r/2$), let the terminal velocity be v'. Then

$$\frac{v'}{v} = \frac{r'^2}{r^2} \qquad \text{or} \quad v' = \frac{r'^2}{r^2} \times v$$

or $$v' = \frac{(r/2)^2}{r^2} \times (2.14\times10^{-6}) = 1.25 \times 10^{-4} \text{ m/s.}$$

Example 29.

The diameter of a gas bubble formed at the bottom of a pond is d = 4.0 μm. When the bubble rises to the surface its diameter increases n = 1.1 times, find how deep is the pond at that spot. The atmospheric pressure is standard, the gas expansion is assumed isothermal.

Solution:

Apply Boyle's law for the bubble of the bottom and at the top surface.

$$\left(p_0 + \frac{4T}{nd}\right)\frac{4\pi}{3}\left(\frac{nd}{2}\right)^3 = \left(p_0 + h\rho g + \frac{4T}{d}\right)\left(\frac{4\pi}{3}\left(\frac{d}{2}\right)^3\right)$$

or $$p_0 + h\rho g + \frac{4T}{d} = \left(p_0 + \frac{4T}{nd}\right)n^3$$

or $$h = \frac{1}{\rho g}\left[p_0(n^3 - 1) + \frac{4T}{d}(n^2 - 1)\right] = 5\text{m.}$$