1. Conic Section

The locus of a point, which moves so that its distance from a fixed point is always in a constant ratio to its distance from a fixed straight line, not passing through the fixed point is called a conic section.

- The fixed point is called the focus.
- The fixed straight line is called the directrix.
- The constant ratio is called the eccentricity and is denoted by *e*.
- When the eccentricity is unity i.e., $e = 1$, the conic is called a parabola ; when $e < 1$ the conic is called an ellipse and when $e > 1$, the conic is called a hyperbola.
- The straight line passing through the focus and perpendicular to the directrix is called the axis of the parabola
- A point of intersection of a conic with its axis is called vertex.

2. Standard Equation of a Parabola

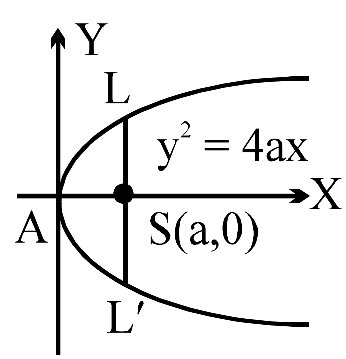
$$\boxed{y^2 = 4ax}$$

2.1 Latus Rectum

The chord of a parabola through the focus and perpendicular to the axis is called the latus rectum.

In the figure LSL′ is the latus rectum.

Also LSL′ = $\sqrt{4a.a} = 4a$ = double ordinate through the focus S.



Note:

- Any chord of the parabola $y^2 = 4ax$ perpendicular to the axis of the parabola is called double ordinate.
- Two parabolas are said to be equal when their latus recta are equal.

2.2 Four Common forms of a Parabola

	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Tangent at the vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$

Illustration 1:

Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola

Solution:

The given equation can be rewritten as $\left(y-\frac{2}{3}\right)^2=\frac{16}{9}\left(x+\frac{61}{16}\right)$

Which is of the form $Y^2 = 4AX$

Hence the vertex is $\left(-\frac{61}{16},\frac{2}{3}\right)$

The axis is $y-\frac{2}{3}=0 \Rightarrow y=\frac{2}{3}$

The directrix is X + A = 0 $\Rightarrow x+\frac{61}{16}=\frac{4}{9} \Rightarrow \quad x=\frac{-485}{144}$

Also $\left(-\frac{485}{144}, \frac{2}{3}\right)$ is the focus.

Length of the latus rectum = 4A = $\frac{16}{19}$

The tangent at the vertex is X = 0 $\Rightarrow x=-\frac{61}{16}$.

Illustration 2:

The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola and the points where it meets the axes.

Solution:

Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is

$$y-4=\frac{0}{5-3}(x-7) \Rightarrow y=4$$

Length of the rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concave rightwards and the other concave leftwards. The vertex of the first parabola is (6.5, 4) and its equation is $(y-4)^2 = 2(x-6.5)$ and it meets the x-axis at (14.5, 0). The equation of the second parabola is $(y-4)^2 = -2(x-7.5)$. It meets the x-axis at (– 0.5, 0) and the y-axis at $(0, 4 \pm \sqrt{15})$.

2.3 Parametric Coordinates:

Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$ and we refer to it as the point 't'. Here, t is a parameter, i.e., it varies from point to point.

Illustration 3:

Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola.

Solution:

Let the two points on the given parabola be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$.

Slope of the line joining these points is = $\dfrac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \dfrac{2}{t_1 + t_2}$

$\Rightarrow t_1 + t_2 = 1$. Hence the two points become $(t_1^2, 2t_1)$ and $((1 - t_1)^2, 2(1 - t_1))$. Let (h, k) be the point which divides these points in the ration 1 : 2.

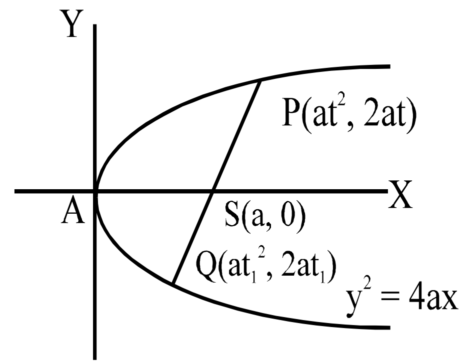
$$\Rightarrow h = \frac{(1-t_1)^2 + 2t_1^2}{3} = \frac{1 - 2t_1 + 3t_1^2}{3} \qquad \ldots(1)$$

$$k = \frac{2(1-t_1) + 4t_1}{3} = \frac{2 + 2t_1}{3} \qquad \ldots(2)$$

Eliminating t_1 from (1) and (2), we find that $4h = 9k^2 - 16k + 8$

Hence locus of (h, k) is $(y - 8/9)^2 = \dfrac{4}{9}\left(x - \dfrac{2}{9}\right)$. This is a parabola with vertex $\left(\dfrac{2}{9}, \dfrac{8}{9}\right)$.

2.4 Focal Chord



Any chord to $y^2 = 4ax$ which passes through the focus is called a focal chord of the parabola $y^2 = 4ax$.

Let $y^2 = 4ax$ be the equation of a parabola and $(at^2, 2at)$ a point P on it. Suppose the coordinates of the other extremity Q of the focal chord through P are $(at_1^2, 2at_1)$.

Then, PS and SQ, where S is the focus $(a, 0)$ have the same slopes.

$$\Rightarrow \quad \frac{2at-0}{at^2-a} = \frac{2at-0}{at_1^2-a} \Rightarrow \qquad tt_1^2 - t = t_1t^2 - t_1 \Rightarrow (tt_1+1)(t_1-t) = 0$$

Hence $\quad t_1 = -1/t$, i.e. the point Q is $(a/t^2, -2a/t)$.

The extremities of a focal chord of the parabola $y^2 = 4ax$ may be taken as the points t and $-1/t$.

Illustration 4:

Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. The locus of the middle point of PQ is:

Solution:

$y^2 = 4x = 4ax$ say $\qquad\qquad$ …(i)

$a = 1$, Let $P = (t_1^2, 2t_1)$; $Q(t_2^2, 2t_2)$,

slope of $OP = \dfrac{2t_1}{t_1^2} = \dfrac{2}{t_1}$, slope of $OQ = \dfrac{2}{t_2}$

Since $OP \perp OQ \Rightarrow m_1 m_2 = -1 \Rightarrow t_1 t_2 = -4$.

Let R (α, β) be the middle point of chord PQ. Then

$$\alpha = \frac{t_1^2 + t_2^2}{2}, \beta = \frac{2t_1 + 2t_2}{2}$$

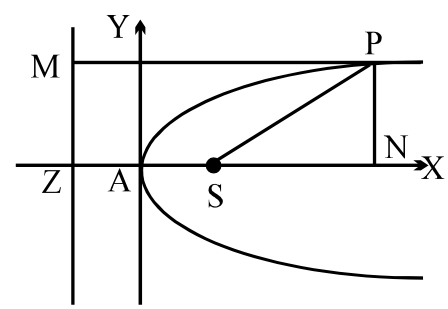
$$\Rightarrow 2\alpha = t_1^2 + t_2^2, \beta = t_1 + t_2$$

$$\Rightarrow (t_1 + t_2)^2 = (t_1^2 + t_2^2) + 2t_1t_2 \Rightarrow \beta^2 = 2\alpha + 2(-4)$$

$$\Rightarrow \text{locus of } (\alpha, \beta) \text{ is } y^2 = 2x - 8$$

2.5 Focal Distance of Any Point

The focal distance of any point P on the parabola $y^2 = 4ax$ is the distance between the point P and the focus S, i.e. PS.



Thus the focal distance is equal to

PS = PM = ZN = ZA + AN = $\boldsymbol{a + x}$

Illustration 5:

The focal distance of a point on the parabola $y^2 = 12x$ is 4. Find the abscissa of this point.

Solution:

On comparing $y^2 = 12x$ with the standard form $y^2 = 4ax$ we get $a = 3$

If the given point on $y^2 = 12x$ is (x, y) then its focal distance is $x + 3$.

$\therefore\ x + 3 = 4 \Rightarrow x = 1$.

Hence the abscissa of the given point is 1.

2.6 Position of a Point Relative to the Parabola:

Consider the parabola: $y^2 = 4ax$. If (x_1, y_1) is a given point and $y_1^2 - 4ax_1 = 0$, then the point lies on the parabola.

But when $y_1^2 - 4ax_1 \neq 0$ then, substituting these values in equation of parabola, the condition for P to lie outside the parabola becomes $y_1^2 - 4ax_1 > 0$.

Similarly, the condition for P to lie inside the parabola is $y_1^2 - 4ax_1 < 0$.

Illustration 6:

For what values of 'α' the point P (α, α) lies inside, on or outside the parabola $(y - 2)^2 = 4(x - 3)$.

Solution:

Given equation can be written as $y^2 - 4y - 4x + 16 = 0$

Point P(α, α) lies inside parabola if $\alpha^2 - 8\alpha + 16 < 0$

$\Rightarrow (\alpha - 4)^2 < 0 \Rightarrow$ no such α exist.

Point P (α, α) lies on parabola if $(\alpha - 4)^2 = 0 \Rightarrow \alpha = 4$

Point P(α, α) lies outside parabola if $(\alpha - 4)^2 > 0 \Rightarrow \alpha \in R - \{4\}$

3. The General Equation of a Parabola

The equation of parabola whose focus is any point (h, k) and the equation of directrix is $lx + my + n = 0$ is

$$\boxed{(mx - ny)^2 + 2gx + fy + d = 0}$$

This is the general equation of a parabola. It is clear that second-degree terms in the equation of a parabola form a perfect square. The converse is also true, i.e. if in an equation of the second degree, the second degree terms form a perfect square then the equation represents a parabola, unless it represents two parabolas, unless it represents two parallel straight lines.

Note: The general equation of second degree i.e. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $\Delta \neq 0$ and $h^2 = ab$, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Special case:

Let the vertex be (α, β) and the axis be parallel to the x-axis. Then the equation of parabola is given by $(y - b)^2 = 4a(x - \alpha)$ which is equivalent to $x = Ay^2 + By + C$

If three points are given we can find A, B and C.

Similarly, when the axis is parallel to the y-axis, the equation of parabola is
$y = A'x^2 + B'x + C'$

Illustration 7:

Find the equation of the parabola whose focus is (3, – 4) and directrix x – y + 5 = 0.

Solution:

Let P(x, y) be any point on the parabola. Then

$$\sqrt{(x-3)^2 + (y+4)^2} = \frac{|x-y+5|}{\sqrt{1+1}} \Rightarrow (x-3)^2 + (y+4)^2 = \frac{(x-y+5)^2}{2}$$

$$\Rightarrow x^2 + y^2 + 2xy - 22x + 26y + 25 = 0 \Rightarrow (x + y)^2 = 22x - 26y - 25$$

4. Tangent Drawn At a Point Lying On a Given Parabola

(i) If P (x_1, y_1) be a point on the parabola $y^2 = 4ax$, then the equation of the tangent at P is $yy_1 = 2a(x + x_1)$ i.e. $T(x, y) = 0$

(ii) If $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then , slope of the tangent at P $= \frac{2a}{2at} = \frac{1}{t}$ $\left(\frac{dy}{dx} = \frac{2a}{y}\right)$ and hence its equation is:

$yt = x + at^2$ $[(y - y_1) = \frac{dy}{dx}(x - x_1)]$ or $y = mx + \left(\frac{a}{m}\right)$ where $m = \frac{1}{t}$

4.1 Point of Intersection of Tangents at 't_1' & 't_2'

Equations of the tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are

$yt_1 = x + at_1^2$ and $yt_2 = x + at_2^2$ respectively.

Solution of these equations gives the coordinates of the intersection point of these two tangents as $\{at_1t_2, a(t_1 + t_2)\}$.

Illustration 8:

Tangents at the extremities of a focal chord of a parabola intersect on the line

Solution:

$h = at_1t_2$, $k = a(t_1 + t_2)$

give the points of intersection of tangents at t_1 and t_2.

Since chord is focal therefore $t_1t_2 = -1 \therefore h = -a \therefore x = -a$.

Hence they intersect on directrix.

Again $m_1m_2 = \frac{1}{t_1} \cdot \frac{1}{t_2} = -1$

$\therefore$ The two tangents are perpendicular.

4.2 Intersection of a Line and a parabola

Let the parabola be $y^2 = 4ax$ ......(1)

and the given line be $y = mx + c$ ......(2)

Eliminating y from (1) and (2), we get

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0 \qquad ......(3)$$

This equation gives two values of x which means that every straight line will cut the parabola in two points may be real, coincident or imaginary according as Discriminant of (3) > , = , < 0

i.e. $4(mc - 2a)^2 - 4m^2c^2 >, =, < 0$

or $4a^2 - 4amc >, =, < 0$

or $a >, =, < mc$ ......(4)

- If m = 0 then equation (3) gives – 4ax + c2 = 0 or $x = \frac{c^2}{4a}$ which gives only one value of x and so every line parallel to x-axis cuts the parabola only in one real point.

4.3 Condition of Tangency

If the line (2) touches the parabola (1), then equation (3) has equal roots

$\therefore$ Discriminant of (3) = 0

$\Rightarrow \quad 4(mc - 2a)^2 - 4m^2c^2 = 0 \quad \Rightarrow -4amc + 4a^2 = 0$

$\Rightarrow \quad c = \frac{a}{m}, m \neq 0$ ......(5)

so, the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ (which is condition of tangency).

Substituting the value of c from (5) in (2) then

$y = mx + \frac{a}{m}$, $m \neq 0$ ......(6)

Hence the line y = mx + $\frac{a}{m}$ will always be a tangent to the parabola y^2 = 4ax

4.4 The point of Contact

The point of contact of the tangents at 't' is $(at^2, 2at)$. In terms of slope 'm' of the tangent the point of contact is, $\left(\frac{a}{m^2}, \frac{2a}{m}\right), (m \neq 0)$.

Illustration 9:

Find the length of the chord of the parabola $y^2 = 4ax$, whose equation is $y = mx + c$.

Solution:

The abscissa of the points common to the straight line $y = mx + c$ and the parabola $y^2 = 4ax$ are given by the equation $m^2 x^2 + (2mc - 4a) x + c^2 = 0$. If (x_1, y_1) and (x_2, y_2) are the points of intersection, then $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$$= \frac{4(mc-2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16a(a-mc)}{m^4} \text{ and } (y_1 - y_2) = m(x_1 - x_2)$$

Hence the required length

$$= \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2} = \sqrt{1+m^2}\,|(x_1 - x_2)| = \frac{4}{m^2}\sqrt{1+m^2}\,\sqrt{a(a-mc)}$$

Illustration 10:

Prove that the straight line $\ell x + my + n = 0$ touches the parabola $y^2 = 4ax$ if $\ell n = am^2$.

Solution:

The given line is

$$lx + my + n = 0$$

or $y = -\frac{\ell}{m}x - \frac{n}{m}$

comparing this line with $y = Mx + c$ …(1)

$\therefore\ M = -\frac{\ell}{m}$ and $c = \frac{-n}{m}$

The line (1) will touch the parabola $y^2 = 4ax$, if $c = \frac{a}{M}$ or $cM = a$

or $\left(-\frac{n}{m}\right)\left(-\frac{\ell}{m}\right) = a$ or $\ell n = am^2$.

5. Equation of Normal to the Parabola

If P(at^2, $2at$) be any point on the parabola $y^2 = 4ax$, then

slope of the tangent at P $= \frac{2a}{2at} = \frac{1}{t}$ $\left[\therefore \frac{dy}{dx} = \frac{2a}{y}\right]$

Therefore, slope of the normal at P = – t and its equation is

i.e. $y = -tx + 2at + at^3$ .....(1)

or $y = mx - 2am - am^3$ (where $m = -t$) ......(2)

is a normal real or imaginary can be drawn from any point to a given parabola and the algebraic sum of the ordinates of the feet of these three normals is zero.

Let equation of a parabola be

$y^2 = 4ax$ ......(3)

and that of a normal to it be

$$y = mx - 2am - am^3 \qquad ......(4)$$

If this normal passes through the point (x_1, y_1),we have

$$y_1 = mx_1 - 2am - am^3$$

i.e. $am^3 + (2a - x_1)\, m + y_1 = 0$

This equation gives three values of m, real or imaginary, If m_1, m_2 and m_3 be the roots of equation (5), then we have $m_1 + m_2 + m_3 = 0$.

Hence, the sum of the ordinates of the feet of these normals = $-2a(m_1 + m_2 + m_3) = 0$

Note:

(i) If normal at the point 't_1' meets the parabola again at 't_2' `

then $t_2 = -t_1 - \dfrac{2}{t_1}$

(ii) If the normals at t_1 & t_2 meet again on the parabola then $t_1 t_2 = 2$

(iii) The point of intersection of the normals to the parabola $y^2 = 4ax$ at 't_1'and 't_2' is $[2a + a(t_1^2 + t_2^2 + t_1 t_2), -a t_1 t_2 (t_1 + t_2)]$.

6. Rule for Transforming an Equation for the Various Forms of the Parabola

In all the previous articles on the parabola, all the related propositions have been proved and derived for the particular parabola $y^2 = 4ax$.

However, all the results with slight transformations are valid for any parabola. In this Art.

If any equation derived for the parabola $y^2 = 4ax$, $(a > 0)$ is given by

$$E(x, y, a) = 0 \qquad ......(1)$$

then the same equation

for the parabola $y^2 = -4ax$ will be

$$E(x, y, -a) = 0 \qquad ....(2)$$

For the parabola $x^2 = +4ay$ will be

$$E(y, x, a) = 0 \qquad ....(3)$$

and for the parabola $x^2 = -4ay$ will be

$$E(y, x, -a) = 0 \qquad ....(4)$$

Illustration 11:

Find the equation of a line which touches the parabola $9x^2 + 12x + 18y - 14 = 0$ and passes through the point (0, 1).

Solution:

Equation of the given parabola can be written as

$$9x^2 + 12x + 4 + 18y - 18 = 0$$

i.e. $(3x + 2)^2 = -18(y - 1)$

i.e. $$\left(x+\frac{2}{3}\right)^2 = -2(y-1) \qquad \ldots(1)$$

Equation of the tangent to the above parabola can be written as

$$x+\frac{2}{3}=m(y-1)-\frac{1}{2m} \qquad \left[\therefore 4a=-2 \therefore a=-\frac{1}{2}\right] \qquad \ldots(2)$$

If the tangent passes through (0, 1), then we have

$$0.m^2 - 4m - 3 = 0$$

gives $m = -3/4, \infty$

Hence, equation of the required lines, are

$$x+\frac{2}{3}=-\frac{3}{4}(y-1),\ x+\frac{2}{3}=\infty(y-1)$$

i.e. $12x + 9y - 1 = 0$ and $y - 1 = 0$

Illustration 12:

P and Q are the points t_1 and t_2 on the parabola $y^2 = 4ax$. If the normals to the parabola at P and Q meet at R, (a point on the parabola, show that $t_1t_2 = 2$).

Solution:

Let the normals at P and Q meet at R$(at^2, 2at)$.

Hence $t = -t_1 - \dfrac{2}{t_1}$ and $t = -t_2 - \dfrac{2}{t_2}$

Therefore $t_1 + \dfrac{2}{t_1} = t_2 + \dfrac{2}{t_2} \Rightarrow (t_1 - t_2) = \dfrac{2(t_1 - t_2)}{t_1t_2} \Rightarrow t_1t_2 = 2$

Illustration 13:

Find the focal distance of the point (x, y) on the parabola $x^2 - 8x + 16y + 16 = 0$

Solution:

$x^2 - 8x + 16y + 16$

i.e. $(x - 4)^2 = -16y + 16$

i.e. $(x - 4)^2 = -16(y - 1)$ …(1)

The focal distance of any point (x, y) in the parabola $y^2 = 4ax$ is

SP $= |x + a|$

Therefore, using the rules of this Art. The focal distance of any point (x, y) on the parabola $x^2 = 4ay$ will be

SP $= |y + a|$

on the parabola $x^2 = -3ay$ will be

SP $= |y - \beta - a|$

Therefore, the focal distance of any point (x, y) on the given parabola is

$$SP = |y - 1 - a|$$

$$= |y - 1 - 4| \qquad [\therefore\ 4a = 16\ \therefore\ a = 4]$$

$$= |y - 5|$$

Illustration 14:

Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the coordinates of the feet of the normals.

Solution:

The equation of the given parabola is

$$x^2 - 8x - 16y = 0$$

Differentiating above equation throughout w.r.t. x, we have

$$2x - 8 - 16\,\frac{dy}{dx} = 0$$

gives $\dfrac{dy}{dx} = \dfrac{x-4}{8}$

If (h, k) be a point on the given parabola, then we have

$$k = \frac{h^2 - 8h}{16}$$

and slope of the normal at (h, k) = $\dfrac{8}{4-h}$ (using result (2))

Therefore, equation of the normal at (h, k) is

$$y - k = \left(\frac{8}{4-h}\right)(x - h)$$

If the normal passes through (7, 14), then we have

$$14-\frac{h^2-8h}{16}=\left(\frac{8}{4-h}\right)(7-h)$$

i.e. $(4-h)(224-h^2+8h) = 128(7-h)$

i.e. $h^3-12h^2-64h=0$

i.e. $h(h^2-12h-64)=0$

gives $h = 0, -4, 16$

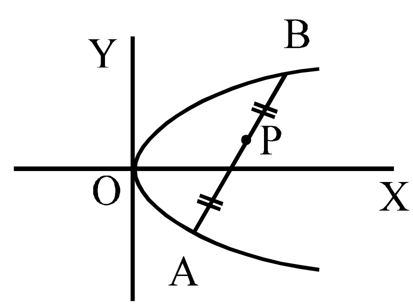
Putting the values of h in equation (3) gives

$k = 0, 3, 8$ respectively.

Therefore, the coordinates of the feet of the normals are

$(0, 0)$, $(-4, 3)$ and $(16, 8)$

7. Equation of the Chord Whose Mid-Point Is Given



Let P(x_1, y_1) be a given point and

$y^2 = 4ax$ ......(1)

be a given parabola

Equation of any line passing through P(x_1, y_1) can be written as

$y - y_1 = m(x - x_1)$ ......(2)

This line meets the given parabola in point A, B whose abscissa are given by the equation

$\{y_1 + m(x - x_1)\}^2 = 4ax$

i.e. $\{mx + y_1 - mx_1\}^2 = 4ax$

i.e. $m^2x^2 + \{2m(y_1 - mx_1) - 4a\}(y_1 - mx_1)^2 = 0$

If x_2, x_3 be the roots of the equation, then according to the given condition, we have

$$x_2 + x_3 = 2x_1 \qquad [\because P(x_1, y_1) \text{ is the mid-point of AB}]$$

ie. $$\frac{4a - 2m(y_1 - mx_1)}{m^2} = 2x_1 \qquad \text{[from equation (3)]}$$

gives $$m = \frac{2a}{y_1}$$

Putting the above value of m in equation (2) gives the equation of the required chord as

$$(y - y_1)y_1 = 2a(x - x_1)$$

which on rearranging reduces to

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 \qquad ....(4)$$

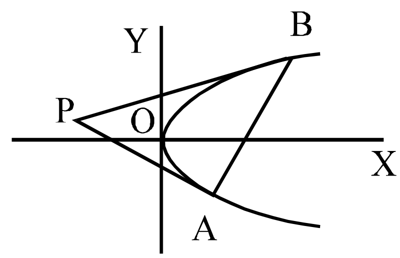
$$\Rightarrow \quad \boxed{\mathbf{T = S_1}}$$

8. Chord of Contact

Let P(x_1, y_1) be a given point and

$$y^2 = 4ax \qquad ....(1)$$

be a given parabola.



Let us assume the coordinate of the point of contact A and B (see fig.) as (x_2, y_2) and (x_3, y_3). Equations of the tangents are A and B are

$$yy_2 - 2a(x + x_2) = 0 \qquad ......(2)$$

and $$yy_3 - 2a(x + x_3) = 0 \qquad ....(3)$$

Since these tangents pass through P(x_1, y_1), hence its coordinates must satisfy the equations (2) and (3)

i.e. $y_1y_2 - 2a(x_1 + x_2) = 0$ ....(4)

and $y_1y_3 - 2a(x_1 + x_3) = 0$ ....(5)

Observing equations (4) and (5) it can be seen that the equation of AB is

$yy_1 - 2a(x + x_1) = 0$ ....(6)

Since equations (4) and (5) are true, it follows that A(x_2, y_2) and B(x_3, y_3) will satisfy equation (6), which proves therefore, that equation (6) represents a straight line passing through A, B and hence the chord of P is

$yy_1 - 2a(x + x_1) = 0$

$\Rightarrow$ $\boxed{\mathbf{T = 0}}$

Illustration 15:

Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ such that tangents at the extremities of the chords are perpendicular.

Solution:

Let (x_1, y_1) be the mid-point of the chords $y^2 = 4ax$. Its equation is

$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

or $yy_1 - 2ax = y_1^2 - 2ax_1$ …(1)

The tangents at the extremities of the chord are perpendicular to each other. Hence the chord is a focal chord. Hence (I) must pass through the focus (a, 0), so that $-2a^2 = y_1^2 - 2ax_1$

Hence locus of (x_1, y_1) is $y^2 = 2a(x - a)$.

9. Pair of Tangents from a Given Point to a Given Parabola

Let P(x_1, y_1) be a given point and

$y^2 = 4ax$ ....(1)

be a given parabola.

Let M(h, k) be any point on either of the tangents from P on to the given parabola. The equation of the straight line joining P and M is

$$y - y_1 = \frac{(k - y_1)}{(h - x_1)}(x - x_1) \text{ i.e. } y = x\frac{(k - y_1)}{(h - x_1)} + \frac{(hy_1 - kx_1)}{(h - x_1)} \quad ....(2)$$

Since this line is a tangent to the given parabola, therefore it must be of the form

$$y = mx + \left(\frac{a}{m}\right) \quad ....(3)$$

Comparing equations (2) and (3), we have

$$m = \frac{(k - y_1)}{(h - x_1)} \quad ......(4)$$

and $\frac{a}{m} = \frac{(hy_1 - kx_1)}{(h - x_1)}$ ....(5)

Eliminating m by multiplying equations (4) and (5), we have

$a(h - x_1)^2 = (k - y_1)(hy_1 - kx)_1$

Putting (x, y) in place of (h, k) gives the equation of the locus of M (i.e. the required pair of tangents) as

$a(x - x_1)^2 = (y - y_1)(xy_1 - yx_1)$

which on rearranging reduces to

$\{yy_1 - 2a(x + x_1)\}^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$

$\Rightarrow$ $\boxed{\mathbf{T^2 = SS_1}}$

10. Some Standard Properties of the Parabola

(i) The portion of a tangent to a parabola intercepted between the directrix and the curve subtends a right angle at the focus.

(ii) The tangent at any point P of a parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix.

(iii) The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

(iv) If S be the focus of the parabola and tangent and normal at any point P meet its axis in T and G respectively, then ST = SG = SP.

(v) If S be the focus and SH be perpendicular to the tangent at P, then H lies on the tangent at the vertex and SH^2 = OS. SP where O is the vertex of the parabola

(vi) The portion of a tangent to a parabola cut off between the directrix and the curve subtends right angle at the focus.

(vii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P(at2, 2at) as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(viii) Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.

(ix) If the tangents at P and Q meet in T, then:

- TP and TQ subtend equal angles at the focus S.

- $ST^2 = SP.SQ$
- The triangles SPT and STQ are similar.

(x) Tangents and Normal at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ and $(3a, 0)$.

(xi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is; $2a = \dfrac{2bc}{b+c}$ i.e. $\dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a}$

(xii) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(xiii) The orthocenter of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix and has the coordinates $-a$, $a(t_1 + t_2 + t_3 + t_1 t_2 t_3)$.

(xiv) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(xv) If normal drawn to a parabola passes through a point $P(h, k)$ then $k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$. Then gives $m_1 + m_2 + m_3 = 0; m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-h}{a}$; $m_1m_2m_3 = -\dfrac{k}{a}$.

Where, m_1, m_2 and m_3 are the slopes of the three concurrent normal.

Note that the algebraic sum of the:

- Slopes of three concurrent normal is zero.
- Ordinates of the three co-normal points on the parabola are zero.
- Centroid of the Δ formed by three co-normal points lies on the x-axis.

11. Reflection Property of a Parabola

The tangent (PT) and normal (PN) of the parabola $y^2 = 4ax$ at P is the internal and external bisectors of $\angle$SPM and BP is parallel to the axis of the parabola and $\angle$BPN = $\angle$SPN.