1. Matrix

A system of m × n numbers arranged in the form of an ordered set of m rows and n columns is called an m × n matrix. It can be read as m by n matrix. It is represented as $A = [a_{ij}]_{m\times n}$ and can be written in expanded form as $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

2. Different Types of Matrices

(i) Horizontal Matrix: Any matrix in which the number of columns is more than the number of rows is called a horizontal matrix.

(ii) Row Matrix: A matrix which has only one row and n columns is called a row matrix of length n.

(iii) Vertical Matrix: Any matrix in which the number of rows is more than the number of columns is called column matrix.

(iv) Column Matrix: A matrix which has only one column and m rows is called a column matrix of length m.

(v) Null or Zero Matrix: The matrix whose all elements are zero is called null matrix or zero matrix. It is usually denoted by O.

(vi) Square Matrix: A matrix for which the number of rows is equal to the number of columns (each equal to n) is called a square matrix of order n.

(vii) Diagonal Matrix: A square matrix of any order with zero elements every-where, except on the main diagonal, is called a diagonal matrix.

(viii) Scalar matrix: A matrix whose diagonal elements are all equal and other entries are zero, is called a scalar matrix

(ix) Identity or Unit Matrix: A square matrix in which all the elements along the main diagonal (elements of the form aij) are unity is called an identity matrix or a unit matrix. An identity matrix of order n is denoted by I_n.

(x) Triangular Matrix: A square matrix whose elements above the main diagonal or below the main diagonal are all zero is called a triangular matrix.

(i) $[a_{ij}]_{n\times n}$ is said to be upper triangular matrix if $i > j \Rightarrow a_{ij} = 0$,

(ii) $[a_{ij}]_{n\times n}$ is said to be lower triangular matrix if $i < j \Rightarrow a_{ij} = 0$.

(xi) Sub Matrix: A matrix obtained by omitting some rows or some columns or both of a given matrix A is called a sub matrix of A.

3. Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of the same order and their corresponding elements are equal. If two matrices A and B are equal, we write A = B.

4. Addition of Matrices

If A and B are two matrices of the same order $m \times n$, then their sum is defined to be the matrix of order $m \times n$ obtained by adding the corresponding elements of A and B.

4. 1 Properties of Matrix Addition

(i) Matrix Addition is Commutative

If A and B are two $m \times n$ matrices, then $A + B = B + A$.

(ii) Matrix addition is associative

If A, B, C are three matrices, each of the order m × n, then (A + B) + C = A + (B + C).

(iii) Existence of additive identity

If O is the m × n null matrix, then A + O = A = O + A for every m × n matrix A. O is called additive identity.

Illustration 1:

If ω is an imaginary cube root of unity, show that

$$\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} + \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} \text{is null matrix.}$$

Solution

Since ω is a cube root of unity is given. We can write

$\omega^3 = 1$

$\Rightarrow \omega^3 = 1^3$

$\Rightarrow \omega^3 - 1^3 = 0$

$\Rightarrow (\omega - 1)(1+\omega+\omega^2) = 0$

$\Rightarrow (1+\omega+\omega^2) = 0$ or $(\omega - 1) = 0$

Now consider,

$$\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} + \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$, which is a null matrix.

5. Multiplication of a Matrix By a Scalar

If A = $[a_{ij}]_{m\times n}$ and k is a scalar, then kA = $[ka_{ij}]_{m\times n}$

6. MULTIPLICATION OF TWO MATRICES

Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{n\times p}$ be two matrices such that the number of columns in A is equal to number of rows in B. Then the m × p matrix C = $[c_{ij}]_{m\times p}$,

where $c_{ik} = \sum_{j=1}^{n} a_{ij}.b_{ik}$ (where i = 1, 2, 3 ... m, k = 1, 2, 3 ... p), is called the product of the matrices A and B. We have

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}_{n\times p} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}_{m\times p}$$

Where, $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Properties of Matrix Multiplication

(i) Matrix multiplication is associative

$(AB)C = A(BC)$ A, B and C are $m \times n$, $n \times p$ and $p \times q$ matrices respectively.

(ii) Multiplication of matrices is distributive over addition of matrices

$A(B + C) = AB + AC$

(iii) Existence of multiplicative identity of square matrices.

If A is a square matrix of order n and I_n is the identity matrix of order n, then $A I_n = I_n A = A$.

(iv) Whenever AB and BA both exist, it is not necessary that $AB = BA$.

(v) The product of two matrices can be a zero matrix while neither of them is a zero matrix.

(vi) In the case of matrix multiplication of $AB = 0$, then it doesn't necessarily imply that $A = 0$ or

$B = 0$ or $BA = 0$.

Illustration 2:

Show that product of two upper (lower) triangular matrices is an upper (lower) triangular matrix.

Solution

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ be two upper triangular matrices.

we know that $a_{ij} = 0$, when $i > j$ and $b_{jk} = 0$ when $j > k$.

Now $C = AB$ is of the type $m \times p$, where $C = [c_{ik}]_{m \times p}$, $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

For $i > k$, $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{ik} b_{kk} + a_{ik+1} b_{k+1k} + \ldots + b_{nk} = 0$.

Hence AB is an upper triangular matrix.

7. Trace of a Matrix

Let A be a square matrix of order n. The sum of the diagonal elements of A is called the trace of A. Therefore

$\therefore \quad \text{Trace (A)} = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{mn}$

8. Transpose of a Matrix

The matrix obtained from any given matrix A, by interchanging rows and columns, is called the transpose of A and is denoted by A′ or A^T.

Properties of Transpose of a matrix

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(\alpha A)' = \alpha \bar{A}$, α being any scalar

(iv) $(AB)' = B'A'$

9. SPECIAL MATRICES

1. Symmetric Matrix

A matrix which is unchanged by transposition is called a symmetric matrix. Such a matrix is necessarily square.

Thus if $A = [a_{ij}]_{m\times n}$ is a symmetric matrix then m = n, $a_{ij} = a_{ji}$ i.e., A′ = A.

2. Skew Symmetric Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be skew symmetric, if $a_{ij} = - a_{ji}$ for all i and j.

Thus if $A = [a_{ij}]_{m\times n}$ is a skew symmetric matrix, then m = n, $a_{ij} = - a_{ji}$ i.e., A′ = A.

Obviously diagonal elements of a skew symmetric matrix are zero.

Note: Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric matrix. i.e., $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A)'$, where $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$ are symmetric and skew symmetric parts of A respectively.

Illustration 3:

Express A as the sum of a symmetric and a skew symmetric matrix, where

$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$.

Solution

We have

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

$$\therefore \quad A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

Then $A + A' = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & 6 \\ -8 & -6 & 0 \end{bmatrix} \qquad \ldots(i)$$

and $A - A' = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 14 \end{bmatrix} \qquad \ldots\text{(ii)}$$

Adding (i) and (ii) we get

$$2A = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & 6 \\ -8 & -6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 14 \end{bmatrix}$$

$$\therefore \qquad A = \underbrace{\begin{bmatrix} 4 & 3/2 & -4 \\ 3/2 & 3 & -3 \\ -4 & -3 & 0 \end{bmatrix}}_{\text{Symmetric matrix}} + \underbrace{\begin{bmatrix} 0 & 1/2 & 1 \\ -1/2 & 0 & -3 \\ -1 & 3 & 7 \end{bmatrix}}_{\text{Skew symmetric matrix}}$$

3. Orthogonal Matrix

A square matrix A is said to be orthogonal, if $AA' = A'A = I$, where I is a unit matrix.

Note:

(i) If A is orthogonal, then A' is also orthogonal.

(ii) If A and B are orthogonal matrices then AB and BA are also orthogonal matrices.

Illustration 4:

If $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$, where $< l_1, m_1, n_1 >$, $< l_2, m_2, n_2 >$ and $< l_3, m_3, n_3 >$ are the direction cosines of three mutually perpendicular straight lines, then prove that $AA' = \quad I$.

Solution:

Since $<l_1, m_1, n_1>$, $<l_2, m_2, n_2>$ and $<l_3, m_3, n_3>$ are the direction cosines of three mutually perpendicular straight lines

$l_1^2 + m_1^2 + n_1^2 = 1, l_2^2 + m_2^2 + n_2^2 = 1 \text{ and } l_3^2 + n_3^2 + m_3^2 = 1$

and $l_1 l_2 + m_1 m_2 + n_1 n_2 = l_2 l_3 + m_2 m_3 + n_2 n_3 = l_3 l_1 + m_3 m_1 + n_3 n_1 = 0$

We have $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$.

$$AA' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \Sigma l_1^2 & \Sigma l_1 m_1 & \Sigma l_1 n_1 \\ \Sigma l_1 n_1 & \Sigma m_1^2 & \Sigma m_1 n_1 \\ \Sigma n_1 l_1 & \Sigma m_1 n_1 & \Sigma n_1^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence proved.

4. Idempotent Matrix:

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

5. Involuntary Matrix

A matrix A such that $A^2 = I$, is called involuntary matrix.

Illustration 5:

Show that of matrix A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$satisfies $A^2 = -I$. Hence or otherwise find the 16th power of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Solution

$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

Let B = $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A + I$

$$B^2 = (A + I)(A + I) = A^2 + 2A + I$$

Since $A^2 = -I$, $B^2 = 2A$

$B^{16} = (B^2)^8 = (2A)^8$

$= 2^8 (A^2)^4 = 2^8 (-I)^4 = 2^8 (-I)^4$

$= 2^8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$.

6. Periodic Matrix

A square matrix A is called periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be period of A. For k = 1, we get $A^2 = A$ and we called it to be idempotent matrix.

7. Nilpotent Matrix

A square matrix A is called a nilpotent matrix, if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.

Illustration 6:

Show that the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3.

Solution

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\therefore\ A^3 = A^2 . A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow A^3 = 0$ i.e., A is a nilpotent matrix of index 3.

10. DETERMINANT

Equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ in x and y have a unique solution if and only if $a_1b_2 - a_2b_1 \neq 0$. We write $a_1b_2 - a_2b_1$ as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and call it a **determinant** of order 2.

Similarly the equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ have a unique solution if

$a_1 (b_2 c_3 - b_3c_2) + b_1 (a_3c_2 - a_2c_3) + c_1 (a_2 b_3 - a_3 b_2) \neq 0$

i.e., $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

The number a_i, b_i, c_i (i = 1, 2, 3) are called the elements of the determinant.

The determinant obtained by deleting the i^{th} row and j^{th} column is called the minor of the element at the i^{th} row and j^{th} column. We shall denote it by M_{ij}. The cofactor of this element is $(-1)^{i+j}$ M_{ij}, denoted by C_{ij}.

Let $A = [a_{ij}]_{3\times3}$be a matrix, then the corresponding determinant denoted by det A or | A | is $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

It is easy to see that $| A | = a_{11}C_{11}+ a_{12} C_{12} + a_{13}C_{13}$ (we say that we have expanded the determinant | A | along first row).

In fact value of | A | can be obtained by expanding it along any row or along any column. Further note that if elements of a row (column) are multiplied to the cofactors of other row (column) and then added, then the result is zero: $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$

10.1 Properties of Determinants

(i) The value of a determinant remains unaltered, if its rows are changed into columns and the columns into rows.

(ii) If all the elements of a row (or column) of a determinant are zero, then the value of the determinant is zero.

(iii) If any two rows (columns) of a determinant are identical, then the value of the determinant is zero.

(iv) The interchange of any two rows (columns) of a determinant result in change of its sign

e.g., $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} b & a & c \\ e & b & f \\ h & g & i \end{vmatrix}$

(v) If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant multiplied by that constant.

e.g., $\begin{vmatrix} a & kb & c \\ d & ke & f \\ g & kh & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ and $k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix}$

(vi) If each element of a row (column) of a determinant is a sum of two terms, then determinant can be written as sum of two determinants in the following way:

$$\begin{vmatrix} a & b & c+d \\ k & l & m+n \\ p & q & r+s \end{vmatrix} = \begin{vmatrix} a & b & c \\ k & l & m \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & d \\ k & l & n \\ p & q & s \end{vmatrix}$$

$$\sum_{r=1}^{n}\begin{vmatrix} f(r) & g(r) & h(r) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(vii) The value of a determinant remains unaltered under a column operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$ (j, k $\neq$ i) or a row operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ (j, k $\neq$ i).

(viii) Product of two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 l_1 + b_1 m_1 + c_1 n_1 & a_1 l_2 + b_1 m_2 + c_1 n_2 & a_1 l_3 + b_1 m_3 + c_1 n_3 \\ a_2 l_1 + b_2 m_1 + c_2 n_1 & a_2 l_2 + b_2 m_2 + c_2 n_2 & a_2 l_3 + b_2 m_3 + c_2 n_3 \\ a_3 l_1 + b_3 m_1 + c_3 n_1 & a_3 l_1 + b_3 m_2 + c_3 n_2 & a_3 l_3 + b_3 m_3 + c_3 n_3 \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} a_1 l_1 + b_1 l_2 + c_1 l_3 & a_1 m_1 + b_1 m_2 + c_1 m_3 & a_1 n_1 + b_1 n_2 + c_1 n_3 \\ a_2 l_1 + b_2 l_2 + c_2 l_3 & a_2 m_1 + b_2 m_2 + c_2 m_3 & a_2 n_1 + b_2 n_2 + c_2 n_3 \\ a_3 l_1 + b_3 l_2 + c_3 l_3 & a_3 m_1 + b_3 m_2 + c_3 m_3 & a_3 n_1 + b_3 n_2 + c_3 n_3 \end{vmatrix}$$

(row by column multiplication)

We can also multiply determinants column by row or column by column

(ix) Limit of a determinant

$$\text{Let } \Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix},$$

Then $\lim\limits_{x\to a}\Delta(x)=\begin{vmatrix}\lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) & \lim\limits_{x\to a} h(x)\\ \lim\limits_{x\to a} l(x) & \lim\limits_{x\to a} m(x) & \lim\limits_{x\to a} n(x)\\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(x) & \lim\limits_{x\to a} w(x)\end{vmatrix}$

Provided each of nine limiting values exist finitely.

(x) Differentiation of a determinant

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ l(x) & m(x) & n(x)\\ u(x) & v(x) & w(x)\end{vmatrix}$

Then

$$\Delta'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x)\\ l(x) & m(x) & n(x)\\ u(x) & v(x) & w(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x)\\ l'(x) & m'(x) & n'(x)\\ u(x) & v(x) & w(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x)\\ l(x) & m(x) & n(x)\\ u'(x) & v'(x) & w'(x)\end{vmatrix}$$

(xi) Integration of a Determinant

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ a & b & c\\ l & m & n\end{vmatrix}$, where a, b, c, l, m and n are constants.

Then $\int\limits_a^b \Delta(x)dx=\begin{vmatrix} \int\limits_a^b f(x)dx & \int\limits_a^b g(x)dx & \int\limits_a^b h(x)dx\\ a & b & c\\ l & m & n\end{vmatrix}$

Note that if more than one row (column) of Δ(x) are variable, then in order to find $\int_a^b \Delta(x)dx$ first we evaluate the determinant Δ(x) by using the properties of determinants and then we integrate it.

11. Special Determinants

(i) Skew symmetric Determinant

A determinant of a skew symmetric matrix of odd order is zero. e.g., $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) Circulant Determinant

A determinant is called circulant if its rows (columns) are cyclic shifts of the first row (columns).

e.g., $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. It can be show that its value is – ($a^3 + b^3 + c^3$ – 3abc).

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Illustration 7:

If A, B and C are the angles of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

Prove that the ΔABC is isosceles.

Solution

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

Expanding along R_1

$$\Delta = \begin{vmatrix} \sin B-\sin A & \sin C-A \\ (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$$= (\sin B - \sin A)(\sin C - \sin A) \begin{vmatrix} 1 & 1 \\ \sin B+\sin A+1 & \sin C+\sin A+1 \end{vmatrix}$$

Now $\Delta = 0 \Rightarrow (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B) = 0$

$\Rightarrow \quad \sin B = \sin A$ or $\sin C = \sin A$ or $\sin C = \sin B$

$\Rightarrow \quad B = A$ or $C = A$ or $C = B$

In all the three cases, we will have an isosceles triangle.

12. Adjoint of a Square Matrix

Let $A = [a_{ij}]_{n\times n}$ be an $n \times n$ matrix. The transpose B' of the matrix $B = [A_{ij}]_{n\times n}$, where A_{ij} denotes the cofactor of the element a_{ij} in the determinant $|A|$, is called the adjoint of the matrix A and is denoted by the symbol adj A.

Also, $A\,(\text{adj}A) = (\text{adj } A)\,A = |A|.\,I_n$.

13. Inverse of a Square Matrix

Let A be any n–rowed square matrix. Then a matrix B, if exists, such that $AB = BA = I_n$, is called the inverse of A. Inverse of A is usually denoted by A^{-1} (if exists).

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$ and then $A^{-1} = \dfrac{Adj(A)}{|A|}$. A square matrix A is called non-singular if $|A| \neq 0$.

Hence a square matrix A is invertible if and only if A is non-singular.

Properties of Inverse of a Matrix

(i) Every invertible matrix possesses a unique inverse.

(ii) If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

(iv) If A is a non-singular square matrix of order n, then $|\text{adj}A| = |A|^{n-1}$

(v) If A and B are non-singular square matrices of the same order, then $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

Illustration 8:

If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, then show that $A^2 - 4A - 5I = 0$, where I and 0 are the unit matrix and the null matrix of order 3 respectively. Use this result to find A^{-1}.

Solution

Given $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ $\therefore\ A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$

$$\therefore\ A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ A^2 - 4A - 5I = 0$ or $5I = A^2 - 4A$

On multiplying by A^{-1}, we get

$$5A^{-1} = A - 4I = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$$

14. System of Linear Simultaneous Equations

Consider the system of linear non-homogeneous simultaneous equations in three unknowns *x*, *y* and *z*, given by

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$,

Let $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$,

Let $|A| = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, obtained on replacing first column of Δ by B

Similarly let $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ *and* $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$.

It can be shown that AX = B, $x\Delta = \Delta_x$, $y\Delta = \Delta_y$, $z\Delta = \Delta_z$.

1. Determinant Method of Solution

We have the following two cases**:**

Case I

If $\Delta \neq 0$, then the given system of equations has unique solution, given by

$x = \Delta_x/\Delta$, $y = \Delta_y/\Delta$, $z = \Delta_z/\Delta$.

Case II

If $\Delta = 0$, then two sub cases arise:

(a) At least one of Δ_x, Δ_y and Δ_z is non-zero, say $\Delta_x \neq 0$. Now in $x\Delta = \Delta_x$. L.H.S. is zero and R.H.S. is not equal to zero. Thus we have no value of x satisfying $x\Delta = \Delta_x$. Hence given system of equations has no solution.

(b) $\Delta_x = \Delta_y = \Delta_z = 0$. In the case the given equations are dependent. Delete one or two equation from the given system (as the case may be) to obtain independent equation(s). The remaining equation(s) may have no solution or infinitely many solution(s).

2. Matrix Method of Solution

(a) $\Delta \neq 0$, then A^{-1} exists and hence $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow x = A^{-1}B$ and therefore unique values of x, y and z are obtained.

(b) We have $AX = B \Rightarrow ((\text{adj } A)A)X = (\text{adj } A)B \Rightarrow \Delta X = (\text{adj } A)B$.

If $\Delta = 0$, then $\Delta X = 0_{3\times1}$, zero matrix of order 3×1. Now if $(\text{adj } A)B = 0$, then the system $AX = B$ has infinitely many solution, else no solution.

Note: A system of equation is called consistent if it has a least one solution. If the system has no solution, then it is called inconsistent.

Illustration 9:

Solve the system of equations,

$x + 2y + 3z = 1$; $2x + 3y + 2z = 2$; $3x + 3y + 4z = 1$ with the help of matrix inversion.

Solution

The given system of equations in the matrix form can be written as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow AX = B$$

Where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Now $|A| = 1(12 - 6) - 2(8 - 6) + 3(6 - 9) = 6 - 4 - 9 = -7 \neq 0$.

Hence the given system has unique solution.

Let C be the matrix of cofactors of elements in $|A|$.

Then, $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Here $C_{11} = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 6$; $\quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 3$

$C_{12} = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$; $\quad C_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$

$C_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3$; $\quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 4$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1$; $\quad C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$; $\quad C_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -5$

$$\therefore\ C = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix} \Rightarrow \quad \therefore\ \text{Adj } A = C' = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{\text{Adj}A}{|A|} = -\frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -6/7 & -1/7 & 5/7 \\ 2/7 & 5/7 & -4/7 \\ 3/7 & -3/7 & 1/7 \end{bmatrix}$$

$$\therefore\ A^{-1}B = \begin{bmatrix} -6/7 & -1/7 & 5/7 \\ 2/7 & 5/7 & -4/7 \\ 3/7 & -3/7 & 1/7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3/7 \\ 8/7 \\ -2/7 \end{bmatrix} \quad (\because A^{-1}B = X)$$

$\therefore$ x = –3/7, y = 8/7, z = –2/7

15. System of Linear Homogeneous Simultaneous Equations

Consider the system of linear homogeneous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$.

In this case, system of equations is always consistent as $x = y = z = 0$ is always a solution. If the system has unique solution (the case when coefficient determinant $\neq 0$), then $x = y = z = 0$ is the only solution (called trivial solution). However if the system has coefficient determinant = 0, then the system has infinitely many solutions. Hence in this case we get solutions other than trivial solution also and we say that we have non-trivial solutions.

Illustration 10:

Solve: $2x + 3ky + (3k + 4)z = 0$

$x + (k + 4)y + (4k + 2)z = 0$

$x + 2(k + 1)y + (3k + 4)z = 0.$

Solution

The given system of equations is AX = 0,

Where A = $\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$ and X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Now $|A| = 0 \Rightarrow k = \pm 2$

Hence if $k \neq \pm 2$, then $|A| \neq 0 \Rightarrow$ the given system has only trivial solution i.e.,

$x = y = z = 0$.

However if $k = -2$ or 2, then $|A| = 0$ and we consider the cases separately

Case I $k = 2$,

The given equations become

$2x + 6y + 10z = 0 \Rightarrow x + 3y + 5z = 0$ …(i)

$x + 6y + 10z = 0$ …(ii)

$x + 6y + 10z = 0$ …(iii)

Hence independent equations are (i) and (ii). Solving we get,

If $x = 0$, $3y + 5z = 0 \Rightarrow x = 0, y = \lambda, z = -\frac{3}{5}\lambda, \lambda \in R$.

Case II $k = -2$

The given equations becomes

$2x - 6y - 2z = 0 \Rightarrow x - 3y - z = 0$ …(iv)

$x + 2y - 6z = 0$ …(v)

$x - 2y - 2z = 0$ …(vi)

We observe that $\frac{4}{5}$(iv) + $\frac{1}{5}$(v) = (vi). Thus equations are dependent. Consider independent equations $x - 3y - z = 0$ and $x - 2y - 2z = 0$. Let $z = \mu \in R$, then $x - 3y = \mu$ and $x - 2y = 2\mu \Rightarrow y = \mu$, $x = 4\mu$.

Hence $x = 4\mu$, $y = \mu$ and $z = \mu$, $\mu \in R$.