1. SYSTEM OF PARTICLES: CENTRE OF MASS

Until now we have dealt mainly with single particle. Bodies like block, man, car etc. are also treated as particles while describing its motion. The particle model was adequate since we were concerned only with translational motion. When the motion of a body involves rotation and vibration, we must treat it as a system of particles. In spite of complex motion of which a system is capable, there is a single point, the centre of mass (CM), whose translational motion is characteristic of the system as a whole. Here we shall discuss about location of centre of mass and its motion of a system of particles.

1.1 LOCATION OF CENTRE OF MASS

Consider a set of *n* particles whose masses are m_1, m_2, m_3 …m_i … m_n and whose position vectors relative to an origin O are $\vec{r}_1, \vec{r}_2, \vec{r}_3 \;\; ...\vec{r}_i \;\; ...\vec{r}_n$ respectively.

The centre of mass of this set of particles is defined as the point with position vector $\vec{r}_{CM}$

where,
$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \qquad ...(1)$$

In component form above equation can be written as

$$X_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{i=n} m_i} = \frac{m_1 x_1 + m_2 x_2 + ... m_n x_n}{m_1 + m_2 + ... + m_n} \qquad ...(2)$$

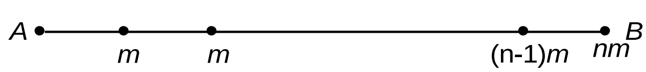
$$Y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{i=n} m_i} = \frac{m_1 y_1 + m_2 y_2 + ... m_n y_n}{m_1 + m_2 + ... + m_n} \qquad ...(3)$$

$$Z_{CM} = \frac{\sum_{i=1}^{i=n} m_i z_i}{\sum_{i=1}^{i=n} m_i} = \frac{m_1 z_1 + m_2 z_2 + ... m_n z_n}{m_1 + m_2 + ... + m_n} \qquad ...(4)$$

Illustration 1

AB is a light rod of length *n* cm. To the rod masses *m*, 2*m*, 3*m*,… *nm* are attached at distances 1, 2, 3, …….. *n* cm respectively from *A*. Find the distance from *A* of the centre of mass of rod.

Solution:



Let us take origin at *A*, then distance of *CM* from *A* (origin) X_{CM} can be written as

$$X_{CM} = \frac{m.1 + 2m.2 + 3m.3 + \dots\dots + nm.n}{m + 2m + 3m + \dots\dots nm} \text{cm} = \frac{m[1^2 + 2^2 + 3^2 + \dots\dots + n^2}{m[1 + 2 + 3 + \dots\dots + n]} \text{cm}$$

$$= \frac{n(n+1)(2n+1)/6}{n(n+1)/2} = \frac{2n+1}{3} \text{cm}$$

Illustration 2

Three particles of masses 2kg, 5kg and 3kg are situated at points with position vector $(\hat{i} + 4\hat{j} - 7\hat{k})$ m, $(3\hat{i} - 2\hat{j} + \hat{k})$ m and $(\hat{i} - 6\hat{j} + 13\hat{k})$ **m** respectively. Find the position vector of centre of mass.

Solution:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

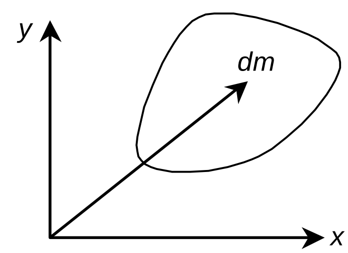
$$= \frac{2(\hat{i} + 4\hat{j} - 7\hat{k}) + 5(3\hat{i} - 2\hat{j} + \hat{k}) + 3(3\hat{i} - 6\hat{j} + 13\hat{k})}{2 + 5 + 3}$$

$$= \mathbf{2\,(2\hat{i} - 2\hat{j} + 3\hat{k})m}$$

1.2 CENTRE OF MASS OF CONTINUOUS BODIES

For calculating centre of mass of a continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice is usually determined by the symmetry of body.

Consider element dm of the body having position vector $\vec{r}$, the quantity $m_i\vec{r}_i$ in equation of CM is replaced by $\vec{r}\ dm$ and the discrete sum over particles $\frac{\Sigma m_i r_i}{M}$, becomes integral over the body:



$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm \qquad \dots(\mathbf{5})$$

In component form this equation can be written as

$$X_{CM} = \frac{1}{M}\int x\,dm \;;\; Y_{CM} = \frac{1}{M}\int y\,dm \text{ and } Z_{CM} = \frac{1}{M}\int Z\,dm \qquad \dots(\mathbf{6})$$

To evaluate the integral we must express the variable m in terms of spatial coordinates x, y, z or $\vec{r}$.

Illustration 3

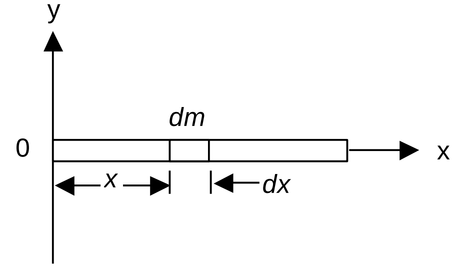
(a) Show that the centre of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

(b) Suppose a rod is non-uniform such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the centre of mass as a fraction of L.

Solution:

(a) By symmetry, we see that $y_{CM} = z_{CM} = 0$ if the rod is placed along the x axis. Furthermore, if we call the mass per unit length λ (the linear mass density), then $\lambda = M/L$ for a uniform rod. If we divide the rod into elements of length dx, then the mass of each element is $dm = \lambda\, dx$. Since an arbitrary element of each element is at a distance x from the origin, equation gives

$$x_{CM} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{\lambda L^2}{2M}$$



Because λ = M/L, this reduces to $x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \boldsymbol{\frac{L}{2}}$

One can also argue that by symmetry, $x_{CM} = L/2$.

(b) In this case, we replace dm by $\lambda\, dx$, where λ is not constant. Therefore, x_{CM} is

$$x_{CM} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\lambda L^3}{3M}$$

We can eliminate α by noting that the total mass of the rod is elated to α through the relationship

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

Substituting this into the expression for x_{CM} gives

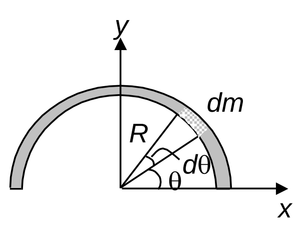
$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boldsymbol{\frac{2}{3}L}$$

Illustration 4

Locate the centre of mass of a uniform semicircular rod of radius R and linear density λ kg/m.

Solution:

From the symmetry of the body we see at once that the CM must lie along the y axis, so $x_{CM} = 0$. In this case it is convenient to express the mass element in terms of the angle θ, measured in radians. The element, which subtends an angle $d\theta$ at the origin, has a length $R\,d\theta$ and a mass $dm = \lambda\, Rd\theta$. Its y coordinate is $y = R\sin\theta$.



Therefore, $y_{CM} = \int \frac{y\,dm}{M}$ takes the

$$y_{CM} = \frac{1}{M}\int_0^\pi \lambda R^2 \sin\theta\,d\theta = \frac{\lambda R^2}{M}\left[-\cos\theta\right]_0^\pi = \frac{2\lambda R^2}{M}$$

The total mass of the ring is $M = \pi R\lambda$; therefore, $y_{CM} = \boldsymbol{\frac{2R}{\pi}}$.

1.3 DISTINCTION BETWEEN CENTRE OF MASS AND CENTRE OF GRAVITY

The position of the centre of mass of a system depends only upon the mass and position of each constituent particles,

i.e., $$\vec{r}_{CM} = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i} \qquad \dots \text{(i)}$$

The location of G, the centre of gravity of the system, depends however upon the moment of the gravitational force acting on each particle in the system (about any point, the sum of the moments for all the constituent particles is equal to the moment for the whole system concentrated at G).

Hence, if g_i is the acceleration vector due to gravity of a particle P, the position vector r_G of the centre of gravity of the system is given by

$$\vec{r}_G \times \Sigma m_i g_i = \Sigma(\vec{r}_i \times m_i g_i) \qquad \dots \text{(ii)}$$

It is only when the system is in a uniform gravitational field, where the acceleration due to gravity (g) is the same for all particles, that equation (ii)

Becomes $$\vec{r}_G = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i} = \vec{r}_{CM}$$

In this case, therefore the centre of gravity and the centre of mass coincide.

If, however the gravitational field is not uniform and g_i is not constant then, in general equation (ii) cannot be simplified and $r_G \neq r_{CM}$.

Thus, for a system of particles in a uniform gravitational field, the centre of mass and the centre of gravity are identical points but in a variable gravitational field, the centre of mass and the centre of gravity are in general, two distinct points.

1.4 VELOCITY AND ACCELERATION OF THE CENTRE OF MASS

By definition position vector of centre of mass,

$$\vec{r}_{CM} = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i}$$

Differentiating once, we will get velocity centre of mass

$$\vec{V}_{CM} = \frac{d\,\vec{r}_{CM}}{dt} = \frac{(\Sigma m_i)\frac{d\vec{r}_i}{dt}}{\Sigma m_i} = \frac{\Sigma m_i\,\vec{v}_i}{\Sigma m_i} \qquad \dots \textbf{(7)}$$

Differentially once most we will get acceleration of centre of mass

$$\vec{a}_{CM} = \frac{d\,\vec{v}_{CM}}{dt} = \frac{(\Sigma m_i)\frac{d\vec{v}_i}{dt}}{\Sigma m_i} = \frac{\Sigma m_i\,\vec{a}_i}{\Sigma m_i} \qquad \dots \textbf{(8)}$$

1.5 EQUATION OF MOTION FOR A SYSTEM OF PARTICLES

Acceleration of centre of mass $\vec{a}_{CM}$ is given by $\vec{a}_{CM} = \frac{\Sigma m_i \vec{a}_i}{\Sigma m_i} = \frac{1}{M} \Sigma m_i . \vec{a}_i$

Rearranging the expression and using Newton's second law, we get

$$M\vec{a}_{CM} = \Sigma m_i \vec{a}_i = \Sigma \vec{F}_i$$

where $\vec{F}_i$ is the force on particle *i*.

The force on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However by Newton's third law, the force exerted by particle 1 on particle 2, for example, is equal to and opposite the force exerted by particle 2 on particle 1. Thus, when we sum over all internal force in above equation they cancel in pairs and the net force is only due to external forces. Thus we can write principle equation in the form.

$$\Sigma \vec{F}_{ext} = M\vec{a}_{CM} \qquad \ldots(\mathbf{9})$$

Thus the acceleration of the centre of mass of a system is the same as that of a particle whose mass is total mass of the system, acted upon by the resultant external forces acting on the system.

If $\Sigma \vec{F}_{ext} = 0$, then centre of mass of system will move with uniform speed and if initially it were at rest it will remains at rest.

Illustration 5

A man weighing 70 kg is standing at the centre of a flat boat of mass 350 kg. The man who is at a distance of 10 m from the shore walks 2 m towards it and stops. How far will he be from the shore? Assume the boat to be of uniform thickness and neglect friction between boat and water.

Solution:

Consider that the boat and the man on it constituting a system. Initially before the man started walking, the centre of mass of the system is at 10 m away from the shore and is at the centre of the boat itself. The centre of mass is also initially at rest.

As no external forces act on this system, the centre of mass will remain stationary at this position. Let us take this point as the origin and the direction towards the shore as x-axis.

If x_1 and x_2 be the position coordinates of man and centre of boat respectively, at any instant, position coordinate of the centre of mass

$$x_c = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

i.e., $$0 = \frac{70x_1 + 350x_2}{70 + 350}$$

$$x_1 + 5x_1 = 0 \qquad \text{.... (i)}$$

Also, $$x_1 - x_2 = 2 \qquad \text{.... (ii)}$$

Solving equations (i) and (ii),

$$x_1 = \frac{5}{3} m$$

Since the centre of mass of the system remains stationary the man will be at a distance $10 - \frac{5}{3} = \mathbf{8.3\ m}$ from the shore.

1.6 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Velocity of centre of mass of a system of particles $\vec{V}_{CM}$ is given by

$$\vec{V}_{CM} = \frac{\Sigma\, m_i \vec{v}_i}{\Sigma\, m_i} = \frac{\Sigma\, m_i \vec{v}_i}{M}$$

Rearranging equation we have,

$$M\vec{V}_{CM} = \Sigma m_i V_i = \Sigma \vec{P}_i = \vec{P}$$

where $\vec{P}$ is total momentum of system.

Thus we conclude that the total linear momentum of the system equals the total mass multiplied by the velocity of centre of mass. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\vec{v}_{CM}$.

Also we get, $\Sigma \vec{F}_{ext} = M \vec{a}_{CM} = M \frac{d}{dt} \vec{V}_{CM}$

$$= \frac{d}{dt}(M\vec{V}_{CM}) = \frac{d\vec{P}}{dt}$$

Also, $(\Sigma \vec{F}_{ext})\, dt = d\vec{P}$

The above equation shows that the resultant impulse acting on the system is equal to the change in the resultant momentum of the set of particles.

Also, in the absence of external force, linear momentum of system of particle will remains conserved.

Illustration 6

A man of mass m is standing over a plank of mass M. The plank is resting on a frictionless surface as shown in figure. If the man starts moving with a velocity v with respect to plank towards right. Find the velocity within which plank will start moving.

Solution:

Coincidently man and plank as a system there is no net external force acting on the system so linear momentum of system will remain conserved.

If planks stats moving with velocity V towards left, then the velocity of man will be $(v - V)$ with respect to surface.

Initial linear momentum of system = 0

Final linear momentum of system = $m(v - V) - MV$.

From conservation of momentum for the system

$m(v - V) - MV = 0$

$$\Rightarrow \quad V = \frac{mv}{m + M}$$

2. IMPULSE AND MOMENTUM

2.1 MOMENTUM

The linear momentum of particle is a vector quantity associated with quantity of motion. It is defined as product of mass of the particle and velocity of particle. i.e, linear momentum $\vec{P}$ of a particle of mass m, moving with velocity $\vec{v}$ is given by

$$\vec{P} = m\vec{v} \qquad \ldots (\mathbf{1})$$

The direction of linear momentum is in the direction of velocity $\vec{v}$ of the particle. The SI unit for linear momentum is kg ms^{-1} and its dimension is [MLT^{-1}].

Using Newton's second law of motion we can relate linear momentum of particle and net force acting on it. The time rate of charge of linear momentum is equal to the resultant force acting on the particle.

This is, $$\vec{F} = \frac{d\vec{P}}{dt} \qquad \ldots (\mathbf{2})$$

2.2 IMPULSE OF FORCE AND CONSERVATION OF LINEAR MOMENTUM

As we have seen, the force is related to momentum as

$$\vec{F} = \frac{d\vec{P}}{dt} \Rightarrow \vec{F}\,dt = d\vec{P}$$

If momentum of particle changes from $\vec{P}_i$ to $\vec{P}_f$ during a time interval of t_i to t_f, we can write

$$\int_{t_i}^{t_f} \vec{F}\,dt = \int_{\vec{P}_i}^{\vec{P}_f} d\vec{P}$$

$$\Rightarrow \qquad \int_{t_i}^{t_f} \vec{F}dt = \vec{P}_f - \vec{P}_i = \Delta\vec{P}$$

The quantity on the left hand side of this equation is called the impulse of force for the time interval $\Delta t = t_f - t_i$. Impulse is represented by $\vec{J}$ and is given by

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\, dt = \Delta\vec{P} \qquad \dots (\mathbf{3})$$

That is, "*The impulse of force equals the change in momentum of the particle*." This statement called '*impulse momentum theorem*,' is equivalent to Newton's second law.

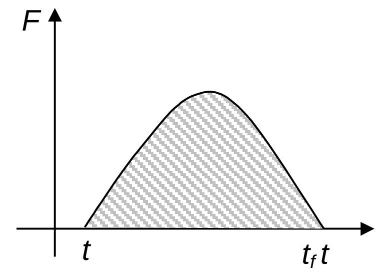
From the equation of impulse, we can see that impulse is a vector quantity having magnitude equal to the area under force-time curve as shown in figure by the shaded area.

Since the force can vary with time, we can define average force $\overline{\vec{F}}$ as

$$\overline{\vec{F}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{F}\, dt \qquad \dots (\mathbf{4})$$

Therefore we can also write,

$$\overline{\vec{F}}\ \Delta t = \Delta\vec{P} \qquad \dots (\mathbf{5})$$



From the equation $\vec{F} = \dfrac{d\vec{P}}{dt}$, we can see that that if the resultant force is zero, the time derivative of the momentum is zero and therefore the linear momentum of a particle is constant. This is called 'conservation of linear momentum. This conservation principle, we apply for a particle as well as system of particles also. Hence we can define conservation of linear momentum as

"If sum total of forces acting on a particle of system of particles is zero. The linear momentum of the system will remain conserved".

Illustration 7

In a particular crash test, an automobile of mass 1500 kg collides with a wall as in figure (a). The initial and final velocities of the automobile are v_i = 15.0 m/s and v_f = 2.6 m/s. If the collision lasts for 0.150 s, find the impulse due to the collision and the average force exerted on the automobile.

Solution:

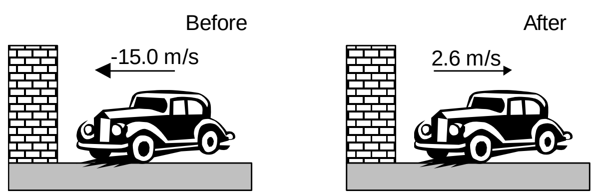


Fig. (a) **Fig. (b)**

The initial and final momenta of the automobile are (taking rightward as positive)

$p_i = mv_i$ = (1500 kg) (–15.0 m/s) = 2.25 × 10^4 kg.m/s

$p_f = mv_f$ = (1500 kg) (2.6 m/s) = 0.39 × 10^4 kg.m/s

Hence, the impulse is

$J = \Delta p = p_f - p_i$ = 0.39 × 10^4 kg.m/s – (–2.25 × 10^4 kg.m/s)

J = 2.64 × 10^4 kg.m/s

The average force exerted on the automobile is given by

$$\overline{F} = \frac{\Delta p}{\Delta t} = \frac{2.64\times10^4\ kg.m/s}{0.150s} = \mathbf{1.76 \times 10^5\ N}$$

Illustration 8

A baseball player uses a pitching machine to help him improve his batting average. He places the 50-kg machine on a frictionless surface as in figure. The machine fires a 0.15 kg baseball horizontally with a velocity of 36 m/s. What is the recoil velocity of the machine?

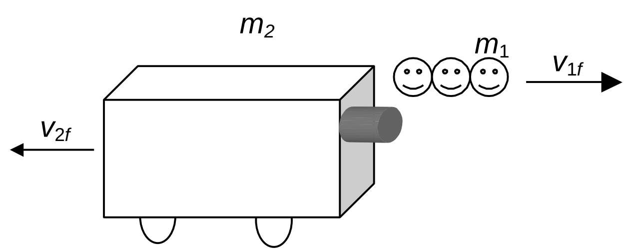
Solution:

We take the system which consists of the baseball and the pitching machine. Because of the force of gravity and the normal force, the system is not really isolated. However, both of these forces are directed perpendicularly to the motion of the system. Therefore, momentum is constant in the x-direction because there are no external forces in this direction (as the surface is frictionless).

The total momentum of the system before firing is zero ($m_1 v_{1i} + m_2 v_2 i = 0$). Therefore, the total momentum after firing must be zero; that is,

$$m_1 v_{1f} + m_2 v_{2f} = 0$$

With m_1 = 0.15 kg, v_{1i} = 36 m/s, and m_2 = 50 kg, solving for v_{2f}, we find the recoil velocity of the pitching machine to be

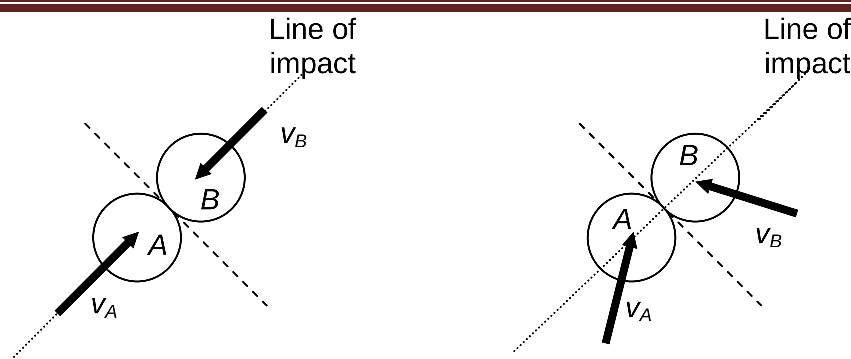


$$v_{2f} = -\frac{m_2}{m_2} v_{1f} = -\left(\frac{0.15kg}{50\ kg}\right)(36\ \text{m/s}) = \mathbf{-0.11\ m/s}$$

The negative sign for v_{2f} indicates that the pitching machine is moving to the left after firing, in the direction opposite the direction of motion of the cannon. In the words of Newton's third law, for every force (to the left) on the pitching machine, there is an equal but opposite force (to the right) on the ball. Because the pitching machine is much more massive than the ball, the acceleration and consequent speed of the pitching machine are much smaller than the acceleration and speed of the ball.

3. IMPACT

A collision between two bodies which occurs in a very small interval of time and during which the two bodies exert relatively large forces on each other is called impact. The common normal to the surfaces in contact during the impact is called the line of impact. If the mass centers of the two colliding bodies are located on this line, the impact is a central impact. Otherwise, the impact is said to be eccentric. Our present study will be limited to the central impact of two particles. The analysis of the eccentric impact of two rigid bodies will be considered later.



Direct central impact   Oblique central impact

If the velocities of the two bodies are directed along the line of impact, the impact is said to be a direct impact as shown in figure given above (left). If either or both bodies move along a line other than the line of impact, the impact is said to be an oblique impact as in figure given above (right).

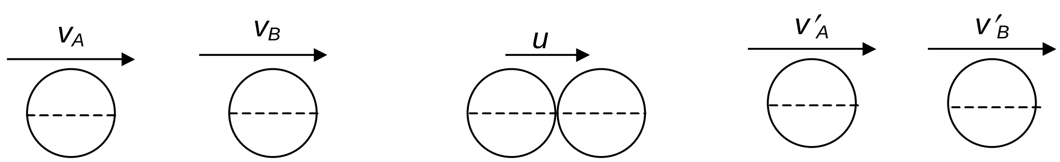
3.1 DIRECT CENTRAL IMPACT OR HEAD ON IMPACT

Consider two spheres A and B of mass m_A and m_B, which are moving in the same straight line and to the right with known velocities $v_{\underline{A}}$ and v_B as shown in figure. If v_A is larger than v_B, particle A will eventually strike the sphere B. Under the impact, the two spheres will deform and at the end of the period of deformation, they will have the same velocity u as shown in figure. A period of restitution will then take place, at the end of which, depending upon the magnitude of the impact forces and upon the materials involved, the two spheres either will have regained their original shape or will stay permanently deformed. Out purpose here is to determine the velocities v'_A and v'_b of the spheres at the end of the period of restitution as shown in figure.

Considering first the two spheres as a single system, we note that there is no impulsive, external force. Thus, the total momentum of the two particles is conserved, and we write

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \qquad \ldots(i)$$

Since all the velocities considered are directed along the same axis, we had written the relation involving only scalar components.



(a) Before impact   (b) At maximum deformation   (c) After impact

A positive value for any of the scalar quantities v_A, v_B, v'_A, or v'_B means that the corresponding vector is directed to the right; a negative value indicates that the corresponding vector is directed to the left.

To obtain the velocities v'_A and v'_B, it is necessary to establish a second relation between the scalars v'_A and v'_B. For this purpose, we use Newton's law of restitution according to which velocity of separation after impact is proportional to the velocity of approach before collisions. In the present situation,

$$(v_B' - v'_A) \propto (v_A - v_B)$$

or, $$(v_B' - v_A') = e\,(v_A - v_B) \qquad \ldots(ii)$$

Here e is a constant called as coefficient of restitution. Its value depends on type of collision. The value of the coefficient e is always between 0 and 1. It depends to a large extent on the two materials involved, but it also varies considerably with the impact velocity and the shape and size of the two colliding bodies.

Two particular cases of impact are of special interest.

(i) $e = 0$, Perfectly Plastic Impact. When $e = 0$, equation (ii) yields $v'_B = v'_A$. There is no period of restitution, and both particles stay together after impact. Substituting $v'_B = v'_A = v'$ into equation (i), which expresses that the total momentum of the particles is conserved,

we write, $m_A v_A + m_B v_B = (m_A + m_B)v'$

This equation can be solved for the common velocity v' of the two particles after impact.

(ii) $e = 1$, Perfectly Elastic Impact. When $e = 1$, equation (ii) reduce to

$$v'_B - v'_A = v_A - v_B$$

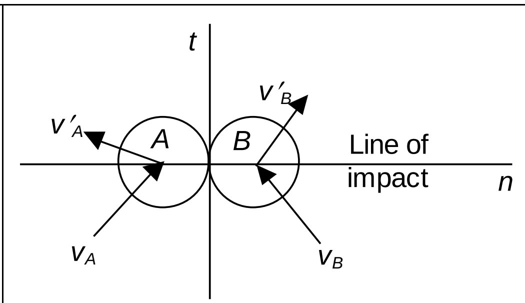
Which expresses that the relative velocities before and after impact are equal. The impulses received by each particle during the period of deformation and during the period of restitution are equal. The particles move away from each other after impact with the same velocity with which they approached each other before impact. The velocities v'_A and v'_B can be obtained by solving equation (i) and (ii) simultaneously.

It is worth noting that in the case of a perfectly elastic impact, the total energy of the two particles, as well as their total momentum, is conserved.

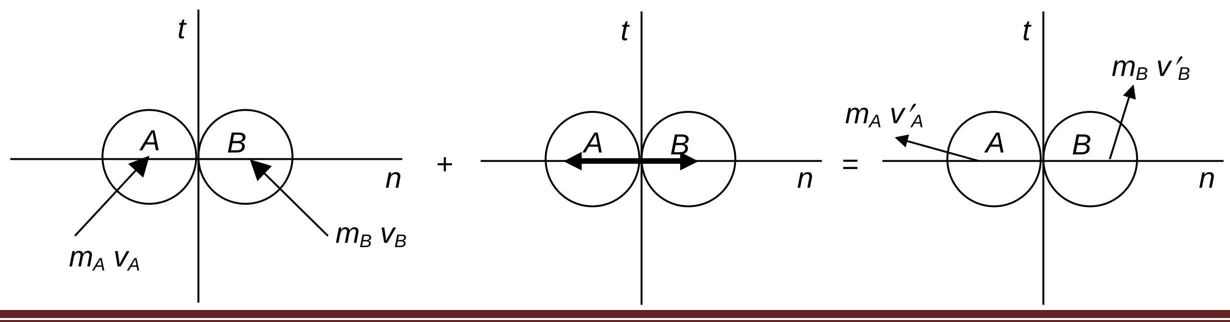
It should be noted, however, that in the general case of impact, i.e., when e is not equal to 1, the total energy of the particles is not conserved. This can be shown in any given case by comparing the kinetic energies before and after impact. The lost kinetic energy is in part transformed into heat and in part spent in generating elastic waves within the two colliding bodies.

3.2 OBLIQUE CENTRAL IMPACT OR INDIRECT IMPACT

Let us now consider the case when the velocities of the two colliding sphere are not directed along the line of impact as shown in figure. As already discussed the impact is said to be oblique. Since velocities v'_A and v'_B of the particles after impact are unknown in direction and magnitude, their determination will require the use of four independent equations. We choose coordinate axes as the n-axis along the line of impact, i.e., along the common normal to the surfaces in contact, and the t-axis along their common tangent.



Assuming that the sphere are perfectly smooth and frictionless, we observe that the only impulses exerted on the sphere during the impact are due to internal forces directed along the line of impact i.e., along the n axis. It follows that



(i) The component along the t axis of the momentum of each particle, considered separately, is conserved; hence the t component of the velocity of each particle remains unchanged. We can write.

$$(v_A)_t = (v'_A)t \;;\quad (v_B)_t = (v'_B)t$$

(ii) The component along the n axis of the total momentum of the two particles is conserved. We write.

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$

(iii) The component along the n axis of the relative velocity of the two particles after impact is obtained by multiplying the n component of their relative velocity before impact by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e[v_A)_n - (v_B)_n]$$

We have thus obtained four independent equations, which can be solved for the components of the velocities of A and B after impact.

Illustration 9

A block of mass 1.2 kg moving at a speed of 20 cm/s collides head on with a similar block kept at rest. The coefficient of restitution is 0.6. Find the loss of kinetic energy during collision.

Solution:

Suppose the first block moves at a speed v_1 and the second at v_2 after collision. Since the collision is head on, the two blocks move along the original direction of motion of first block. Using the principle of conservation of momentum,

$(1.2 \times 0.2) = 1.2\, v_1 + 1.2\, v_2$

$v_1 + v_2 = 0.2$ … (i)

By Newton's law of restitution,

$V_2 - v_1 = -\, e\,(u_2 - u_1)$

$v_2 - v_1 = -\, 0.6\,(0 - 0.2)$

$v_2 - v_1 = 0.12$ … (ii)

Adding equations (i) and (ii),

$2v_2 = 0.32$

$v_2 = 0.16$ m/s or 16 cm/s

$v_1 = 0.2 - 0.16$

$= 0.04$ m/s

$= 4$ cm/s

Loss of K.E. $= \frac{1}{2}\times 1.2 \times (0.2)^2 - \frac{1}{2}\times 1.2\times(0.16)^2 - \frac{1}{2}\times 1.2\times(0.04)^2$

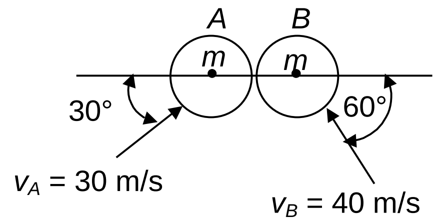
$= 0.6\,[0.04 - 0.0256 - 0.0016]$

$= 0.6 \times 0.0128$

$= \mathbf{7.7 \times 10^{-3}\ J}$

Illustration 10

The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming e = 0.90, determine the magnitude and direction of the velocity of the each ball after the impact.



Solution:

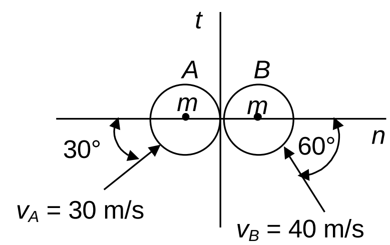
The impulsive force that the balls exert on each other during the impact are directed along a line joining the centers of the balls called the line of impact. Resolving the velocities into components directed, respectively, along the line of impact and along the common tangent to the surfaces in contact, we write

$(v_A)_n = v_A \cos 30° = +26.0$ m/s

$(v_A)_t = v_A \sin 30° = +15.0$ m/s

$(v_B)_n = -v_B \cos 60° = -20.0$ m/s

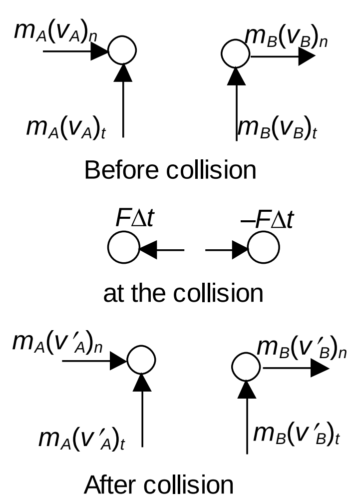
$(v_B)_t = v_B \sin 60° = +34.6$ m/s



Since the impulsive forces are directed along the line of impact, the t component of the momentum, and hence the t component of the velocity of each ball, is unchanged. We have,

$$(v'_A)_t = 15.0 \text{ m/s} \uparrow, \quad (v'_B)_t = 34.6 \text{ m/s} \uparrow$$

In the n direction, we consider the two balls as a single system and not that by Newton's third law, the internal impulses are, respectively, $F\Delta t$ and $-F\Delta t$ and cancel. We thus write that the total momentum of the balls is conserved.



$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$

$$m(26.0) + m(-20.0) = m(v'_A)_n + m(v'_B)_n$$

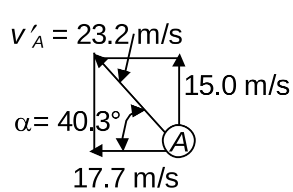
$$(v'_A)_n + (v'_B)_n = 6.0 \qquad \dots \text{(i)}$$

Using law of restitution,

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n]$$

$$(v'_B)_n - (v'_A)_n = (0.90)[26.0 - (-20.0)]$$

$$(v'_A)_n + (v'_A)_n = 41.4 \qquad \dots \text{(ii)}$$



Solving equations (i) and (ii) simultaneously, we obtain

$$(v'_A)_n = -17.7 \text{ m/s} \qquad (v'_B)_n = +23.7 \text{m/s}$$

$$(v'_A)_n = 17.7 \; m/s \leftarrow \quad (v'_B)_n = 23.7 \text{m/s} \rightarrow$$

Resultant Motion: Adding vectorially the velocity components of each ball, we obtain

$v_A' = $ **23.2 m/s** $\angle$ **40.3°** $v_B' = $ **41.9 m/s** $\angle$ **55.6°**

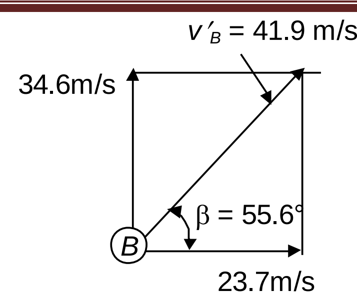
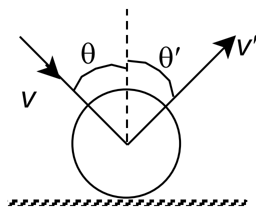


Illustration 11

A ball of mass *m* hits a floor with a speed *v* making an angle of incidence θ with normal. The coefficient of restitution is *e*. Find the speed of reflected ball and the angle of reflection.

Solution:

Suppose the angle of reflection is θ′ and the speed after collision is *v*′. It is an oblique impact. Resolving the velocity *v* along the normal and tangent, the components are *v* cosθ and *v* sinθ. Similarly, resolving the velocity after reflection along the normal and along the tangent the components are – *v*′ cosθ′ and *v*′ sinθ′.



Since there is no tangential action,

$$v \sin\theta = v'\sin\theta' \qquad \ldots \text{(i)}$$

Applying Newton's law for collision,

$$(-v'\cos\theta' - 0) = -e\,(v\cos\theta - 0)$$

$$\therefore \quad v'\cos\theta' = ev\cos\theta \qquad \ldots \text{(ii)}$$

From equations (i) and (ii),

$$v'^2 = v^2\sin^2\theta + e^2 v^2\cos^2\theta$$

$$v' = \sqrt{v^2\sin^2\theta + e^2v^2\cos^2\theta}$$

$$v' = \left(v\sqrt{\sin^2\theta + e^2\cos^2\theta}\right)$$

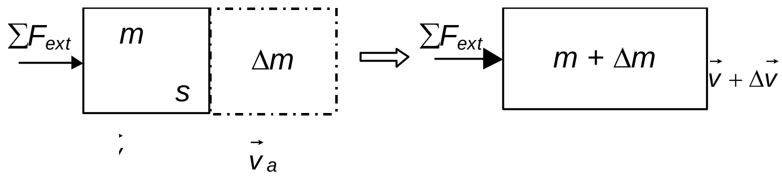
and $$\tan\theta' = \frac{\tan\theta}{e}$$

$$\theta' = \tan^{-1}\left(\frac{\tan\theta}{e}\right)$$

4. SYSTEMS OF VARIABLE MASS

We recall that all the principles established so for were derived for the systems which neither gain nor lose mass. But there are various situations in which system loses or gains mass during its motion. e.g. in case of Rocket propulsion, its motion depends upon the continued ejection of fuel from it.

Let us analyse a system of variable mass. Consider the system S shown in figure. Its mass, equal to m at the instant t increases by Δm in the internal of the Δt. The velocity of S at time t is $\vec{v}$ and the velocity of S at time $t + \Delta t$ becomes $\vec{v} + \Delta\vec{v}$, and the absolute velocity of mass absorbed is $\vec{v}_a$ with respect to stationary frame. $\sum\vec{F}_{ext}$ is net external force acting on it during internal Δt



Applying the Impulse-Momentum theorem,

$$m\vec{v} + \Delta m\,\vec{v}_a + \sum\vec{F}_{ext}\,\Delta t = (m+\Delta m)\,(\vec{v}+\Delta\vec{v})$$

$$\Rightarrow \qquad \sum\vec{F}_{ext}\,\Delta t = m\Delta\vec{v} + \Delta m\,(\vec{v}-\vec{v}_a) + (\Delta m)\,(\Delta v)$$

Here $\vec{v} - \vec{v}_a$ is relative velocity of mass absorbed with respect to system S, let us write it as $\vec{v}_{rel.}$ also last term $\Delta m\ \Delta v$ can be neglected.

We can write, $\sum\vec{F}\,\Delta t = m\Delta\vec{v} - (\Delta m)\,\vec{v}_{rel}$

Dividing both sides by Δt and letting Δt approaches zero, we have $\sum\vec{F} = m\dfrac{d\vec{v}}{dt} - \dfrac{dm}{dt}\vec{v}_{rel}$

Rearranging the terms and recalling $\dfrac{d\vec{v}}{dt} = \vec{a}$ where $\vec{a}$ is acceleration of system,

we can write

$$\sum\vec{F}_{ext} + \frac{dm}{dt}\vec{v}_{rel} = m\vec{a} \qquad \ldots\textbf{(6)}$$

Which shows that the action on S of the mass being absorbed is equivalent to a thrust force $\vec{F}_{th}$ given by,

$$\vec{F}_{th} = \frac{dm}{dt}\vec{v}_{rel} \qquad \ldots\textbf{(7)}$$

Therefore while analyzing systems of variable mass, we need to consider external forces acting on it as well as a thrust force having magnitude equal to the product of rate at which mass of system changes and the relative velocity of mass coming into the system or going out of the system with respect to the system. If mass of system is increasing, then the direction of thrust is same as that of relative velocity $\vec{v}_{rel}$ and vice versa.

Once we consider the thrust force with the net external force, a system of variable mass can be analyzed in the same way as we analyze systems of constant mass by considering external forces only.

Illustration 12

The mass of a rocket is 2.8×10^6 kg at launch time of this 2×10^6 kg is fuel. The exhaust speed is 2500 m/s and the fuel is ejected at the rate of 1.4×10^4 kg/s.

(a) Find thrust on the rocket

(b) What is initial acceleration at launch time? Ignore air resistance.

Solution:

(a) The magnitude of thrust is given by

$$= (1.4 \times 10^4 \text{ kg/s}) \times (2500 \text{ ms}^{-1})$$

$$= \mathbf{3.5 \times 10^7 \text{ N}}$$

$$F_{th} = \frac{dM}{dt} v_{rel}.$$

The direction of thrust will be opposite to the direction of relative velocity as mass is decreasing, i.e., upward

(b) to find acceleration, we can use

$$\sum \vec{F}_{\text{ext}} + \vec{F}_{\text{th}} = M\vec{a}$$

Here external force $\vec{F}$ is weight acting downward and thrust force F_{th} is upward

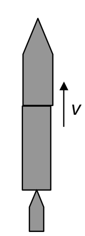
$\therefore$ $-mg + F_{th} = Ma$ (Taking upward as positive)

$$\Rightarrow \quad a = g - \frac{F_{th}}{M}$$

$$= \left(-9.8 + \frac{3.5 \times 10^7}{2.8 \times 10^6}\right) \text{ms}^{-2} = \mathbf{2.7 \text{ ms}^{-2}}$$

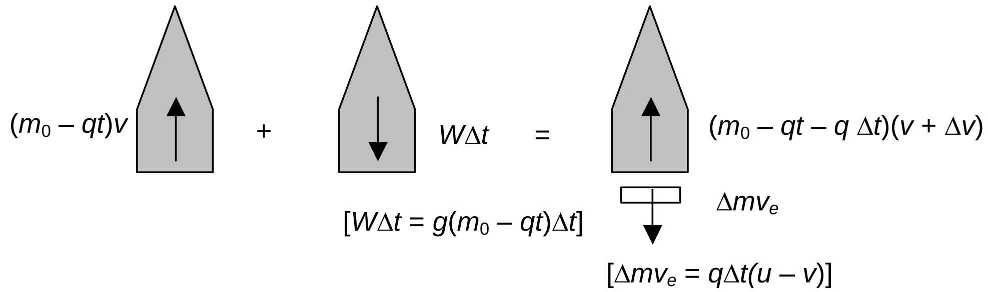
Illustration 13

A rocket of initial mass m_0 (including shell and fuel) is fired vertically at time $t = 0$. The fuel is consumed at a constant rate $q = dm/dt$ and is expelled at a constant speed u relative to the rocket. Derive an expression for the magnitude of the velocity of the rocket at time t, neglecting the resistance of the air and variation of acceleration due to gravity.



Solutions:

At time t, the mass of the rocket shell and remaining fuel is $m = m_0 - qt$, and the velocity is v. During the time interval Δt, a mass of fuel $\Delta m = q\,\Delta t$ is expelled with a speed u relative to the rocket. Denoting by v_e the absolute velocity of expelled fuel, we apply the principle of impulse and momentum between time t and time $t + \Delta t$.



We write

$$(m_0 - qt)v - g(m_0 - qt)\,\Delta t = (m_0 - qt - q\,\Delta t)\,(v + \Delta v) - q\Delta t\,(u - v)$$

Dividing throughout by Δt and letting Δt approach zero, we obtain

$$-g(m_0 - qt) = (m_0 - qt)\frac{dv}{dt} - qu$$

Separating variables and integrating from $t = 0$, $v = 0$ to $t = t$, $v = v$

$$dv = \left(\frac{qu}{m_0 - qt} - g\right)dt$$

$$\int_0^v dv = \int_0^t \left(\frac{qu}{m_0 - qt} - g\right) dt$$

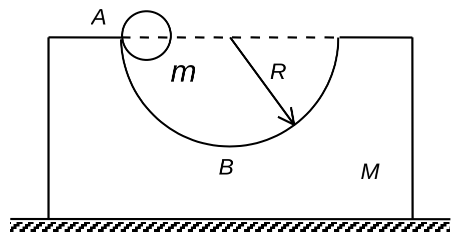
$\Rightarrow$ $$v = [u\, ln\,(m_0 - qt) - gt]_0^t$$

$\therefore$ $$v = \boldsymbol{u ln}\left(\frac{m_0}{m_0 - qt}\right) - gt$$

SOLVED EXAMPLES

Example 1.

A block of mass M with a semicircular track of radius R rests on a horizontal frictionless surface. A uniform cylinder of radius r and mass m is released from rest at the top point A (see Figure). The cylinder slips on the semicircular frictionless track. How far has the block moved when the cylinder reached the bottom (point B) of the track? How fast is the block moving when the cylinder reaches the bottom of the track?



Solution:

The horizontal component of forces acting on M-m system is zero and the centre of mass of the system cannot have any horizontal displacement.

When the cylinder is at B its displacement relative to the block in the horizontal direction is (R – r). Let the consequent displacement of the block to the left be x. The displacement of the cylinder relative to the ground is (R – r – x).

Since the centre of mass has no horizontal displacement

$$M \cdot x = m(R - r - x)$$

$$x(M + m) = (R - r)m$$

$$x = \frac{(R-r)m}{(M+m)}$$

When the cylinder is at A, the total momentum of the system in the horizontal direction is zero. If v is the velocity of the cylinder at B and V, the velocity of the block at the same instant, then

mv + MV = 0, by principle of conservation of momentum.

Potential energy of the system at A = mg (R – r)

Kinetic energy of the cylinder at B = $\frac{1}{2}mv^2$

The kinetic energy of the block at that instant $= \frac{1}{2}MV^2$

By principle of conservation of energy,

$$mg(R - r) = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

since $v = -\frac{MV}{m}$

$$mg(R - r) = \frac{1}{2}m\left(-\frac{MV}{m}\right)^2 + \frac{1}{2}MV^2 = \frac{V^2}{2}\left(\frac{M^2}{m} + M\right)$$

$$mg\,(R-r) = \frac{V^2}{2m}\left(M^2+Mm\right)$$

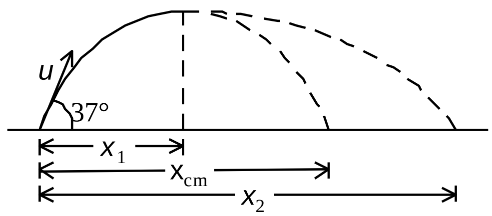
$$V^2 = \frac{2m^2 g(R-r)}{(M^2+Mm)}$$

$$V = \sqrt{\frac{2m^2 g(R-r)}{M(M+m)}}$$

Example 2.

A projectile is fired at a speed of 100 m/s at an angle of 37° above horizontal. At the highest point the projectile breaks into two parts of mass ratio 1 : 3, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

Solution:



Refer the Figure. At the highest point, the projectile has horizontal velocity. The lighter part comes to rest. Hence the heavier part will move with increased velocity in the horizontal direction. In the vertical direction both parts have zero velocity and undergo same acceleration. Hence they will cover equal vertical displacements in a given time. Thus both will hit the ground together. As internal forces do not affect the motion of the centre of mass, the centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is

$$X_m = \frac{2u^2\sin\theta\cos\theta}{g}$$

$$= \frac{2\times(100)^2\times\frac{3}{5}\times\frac{4}{5}}{10} = 960 \text{ m}$$

where $\sin\theta=\frac{3}{5}, \cos\theta=\frac{4}{5}$ and g = 10 m/s^2.

The centre of mass will hit the ground at this position. As the smaller mass comes to rest after breaking it falls down vertically and hits the ground at half the range = 480 m. If the heavier block hits the ground at x_2,

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

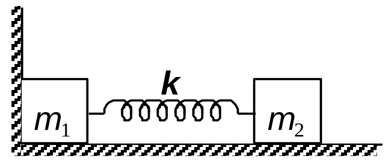
$$960 = \frac{\frac{M}{4}\times 480 + \frac{3M}{4}\times x_2}{M}$$

Solving, $x_2 = 1120$ m

Example 3.

Two blocks of masses m_1 and m_2, connected by a weightless spring of stiffness k, rest on a smooth horizontal plane. Block 2 is shifted a small distance x to the left and released. Find the velocity of the centre of mass of the system after block 1 breaks off the wall.

Solution:



We know that the potential energy of compression

$$= \frac{1}{2}kx^2$$

When the block m_1 breaks off from the wall the spring has its unstretched length and the kinetic energy of the block m_2 is given by

$$\frac{1}{2}m_2 v_2^2 = \frac{1}{2}kx^2$$

$$v_2^2 = \frac{kx^2}{m_2}$$

$$v_2 = x\sqrt{\frac{k}{m_2}}$$

For centre of mass

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The distances x_1 and x_2 are measured from the wall.

$$\frac{dx_{cm}}{dt} = \frac{m_1}{m_1+m_2}\cdot\frac{dx_1}{dt} + \frac{m_2}{m_1+m_2}\cdot\frac{dx_2}{dt}$$

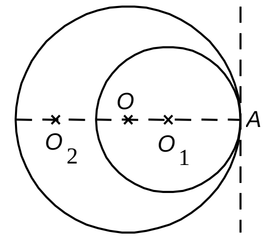
At start $\frac{dx_1}{dt} = 0$

$$\therefore \frac{dx_{cm}}{dt} = \frac{m_2}{m_1+m_2}\cdot v_2 = \frac{m_2 x}{m_1+m_2}\sqrt{\frac{k}{m_2}}$$

Velocity of centre of mass of system = $\mathbf{\frac{x\sqrt{km_2}}{m_1+m_2}}$

Example 4.

A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in Figure. Find the centre of mass of the remaining portion.



Solution:

Let O be the centre of circular plate and O_1, the centre of circular portion removed from the plate. Let O_2 be the centre of mass of the remaining part.

Area of original plate = $\pi R^2 = \pi\left(\frac{56}{2}\right)^2 = 28^2\ \pi\ cm^2$

Area removed from circular part = πr^2

$$= \pi\left(\frac{42}{2}\right)^2 = (21)^2\pi\ cm^2$$

Let σ be the mass per cm^2. Then

mass of original plate, m = $(28)^2\ \pi\sigma$

mass of the removed part, $m_1 = (21)^2\ \pi\sigma$

mass of remaining part, $m_2 = (28)^2\ \pi\sigma - (21)^1\ \pi\sigma = 343\ \pi\sigma$

Now the masses m_1 and m_2 may be supposed to be concentrated at O_1 and O_2 respectively. Their combined centre of mass is at O. Taking O as origin we have from definition of centre of mass,

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$x_1 = OO_1 = OA - O_1A = 28 - 21 = 7$ cm

$x_2 = OO_2 = ?$, $x_{cm} = 0$.

$$\therefore \qquad 0 = \frac{(21)^2\pi\sigma\times 7 + 343\pi\sigma\times x_2}{(m_1+m_2)}$$

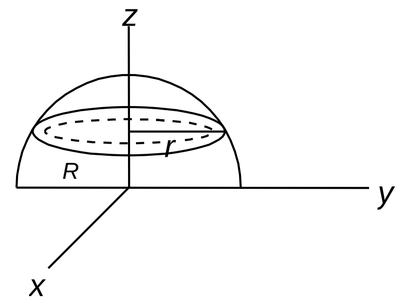
$$x_2 = -\frac{(21)^2\pi\sigma\times 7}{343\pi\sigma} = -\frac{441\times 7}{343} = -9 \text{ cm.}$$

This means that centre of mass of the remaining plate is at a distance 9 cm from the centre of given circular plate opposite to the removed portion.

Example 5.

Find the centre of mass of a uniform solid hemisphere of radius R and mass M with centre of sphere at origin and the flat of the hemisphere in the x, y plane.

Solution:



Let the centre of the sphere be the origin and let the flat of the hemisphere lie in the x, y plane as shown. By symmetry $\bar{x}=\bar{y}=0$. Consider the hemisphere divided into a series of slices parallel to x, y plane. Each slice is of thickness dz.

The slice between z and (z + dz) is a disk of radius, r = $\sqrt{R^2-z^2}$.

Let ρ be the constant density of the uniform sphere.

Mass of the slice, dm = $(\rho\pi r^2)dz$ = $\rho\pi(R^2 - z^2)\,dz$

The $\bar{z}$ value is obtained by $\bar{z} = \dfrac{\int_0^R z\,dm}{M}$

$$= \frac{\int_0^R \pi\rho(R^2z - z^3)dz}{M}$$

$$= \frac{\pi\rho}{M}\left[\left(\frac{R^2z^2}{2} - \frac{z^4}{4}\right)\right]_{z=0}^{z=R}$$

$$\Rightarrow \qquad \bar{z} = \frac{\pi\rho\left(\frac{R^4}{2} - \frac{R^4}{4}\right)}{M}$$

$$\Rightarrow \qquad \bar{z} = \frac{\rho\pi R^4}{4M}$$

Since 2M = $\rho\left(\frac{4}{3}\pi R^3\right)$, we have $\bar{z} = \dfrac{(\rho\pi R^4/4)}{(\rho 2\pi R^3/3)} = \dfrac{3}{8}R$

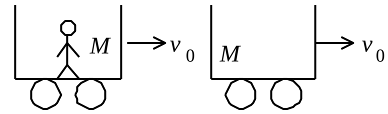
Hence centre of mass has positive coordinates as $\left(\mathbf{0, 0, \frac{3}{8}R}\right)$.

Example 6.

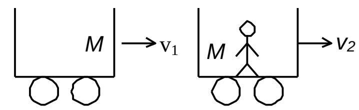
Two identical buggies move one after the other due to inertia (without friction) with the same velocity v_0. A man of mass m rides the rear buggy. At a certain moment the man jumps into the front buggy with a velocity u relative to his buggy. The mass of each buggy is M. Find the velocities with which the buggies will move afterwards.

Solution:

Before jumping



After jumping



Initial momentum of rear buggy = $(M + m)\,v_0$. The momentum of man when he jumps = $m(v_1 + u)$, where v_1 is the velocity of buggy as he jumps.

By the conservation of linear momentum

$(M + m)v_0 = Mv_1 + m(v_1 + u)$

$\Rightarrow\ v_1 (M + m) = (M + m)\,v_0 - mu$

$$v_1 = v_0 - \frac{m}{M+m}u$$

Initial momentum of front buggy = Mv_0

$Mv_0 + m(v_1 + u) = (M + m)\,v_2$

$$Mv_0 + m\left(v_0 - \frac{m}{M+m}u + u\right) = (M+m)v_2$$

$$\Rightarrow\ Mv_0 + m\left(v_0 + \frac{Mu}{M+m}\right) = (M+m)v_2$$

$$\Rightarrow\ (M+m)v_0 + \frac{mMu}{M+m} = (M+m)v_2$$

$$\Rightarrow\ v_2 = v_0 + \frac{mMu}{(M+m)^2}$$

Example 7.

A cannon is mounted on a wagon, which stands on a straight section of the track. The mass of the wagon with the cannon, the shots and the men is M. The mass of the shot is m. When the cannon fires, the shot leaves with a muzzle velocity (relative to the cannon) v_0 in a horizontal direction. Show that after n shots are fired the speed of the wagon is

$$mv_0\left[\frac{1}{M} + \frac{1}{M-m} + \ldots + \frac{1}{M-(n-1)m}\right]$$

Solution:

Before the first shot is fired, the gun is at rest and the momentum of system is zero. Since there are no external forces, the total momentum of the system after the first shot is fired must also be zero.

Let the velocity of the first shot be v_1 and that of gun be V_1 both relative to the ground.

Then, $mv_1 + (M - m)V_1 = 0$

$m(v_1 - V_1) + MV_1 = 0$

The velocity of the shot relative to the gun is $v_1 - V_1$ and this is the muzzle velocity of the shot.

$v_1 - V_1 = v_0$

$\therefore$ $mv_0 + MV_1 = 0$

$$\therefore \quad V_1 = -\frac{mv_0}{M} \qquad \ldots \text{(i)}$$

The second shot is fired when the gun is moving with a velocity V_1. The momentum of the system before the second shot is fired is $(M - m)V_1$. By conservation of momentum the total momentum after second shot is fired is $(M - m)V_1$.

If v_2 and V_2 are the velocities of the second shot and gun respectively relative to the ground,

$mv_2 + (M - 2m)\, V_2 = (M - m)\, V_1$

$mv_2 - mV_2 + (M - m)\, V_2 = (M - m)\, V_1$

$$\therefore \quad m\frac{(v_2 - V_2)}{M - m} + V_2 = V_1$$

Since $v_2 - V_2$ is the velocity of the shot relative to the gun $v_2 - V_2 = v_0$

Thus $$V_2 = V_1 - \frac{mv_0}{M - m} = -\frac{mv_0}{M} - \frac{mv_0}{M - m} = -mv_0\left[\frac{1}{M} + \frac{1}{M - m}\right]$$

The momentum of gun before third shot is fired is $(M - 2m)\, V_2$.

Let the velocity of shot relative to ground be v_3 and that of gun V_3.

Proceeding in the same way the velocity of the wagon after third shot is fired can be seen to be given by

$$V_3 = -mv_0\left[\frac{1}{M} + \frac{1}{M - m} + \frac{1}{M - 2m}\right]$$

Proceeding in the same way the velocity of the wagon after n shots are fired will be given by

$$\boldsymbol{V_n = mv_0\left[\frac{1}{M} + \frac{1}{M - m} + \frac{1}{M - 2m} + \ldots + \frac{1}{M - (n - 1)m}\right]}$$

Example 8.

A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a bob of mass m is suspended from the ceiling by a string of length ℓ. At the initial moment, the wagon and pendulum are at rest and the string is deflected through an angle α from the vertical.

Find:(a) the velocity of wagon, when the string forms an angle β ($\beta < \alpha$) with vertical.

(b) the velocity of wagon, when the pendulum crosses its mean position.

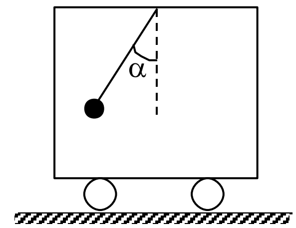
Solution:

(a) Let v be the leftward velocity of wagon (absolute that is relative to earth). Let u be the velocity of pendulum in a frame fixed to the wagon. Then $u \cos \beta$ is the relative horizontal velocity of the bob and $u \sin \beta$ is its vertical velocity. Let v_x and v_y be the absolute horizontal and vertical downward velocities of the bob.

$\Rightarrow \quad v_x = u \cos \beta - v$ and $u \sin \beta = v_y$

There is no external force on the system in the horizontal direction.

Therefore, by the principle of conservation of momentum to the right,



Before releasing the bob

$0 = m\,(u\cos\beta - v) - Mv$

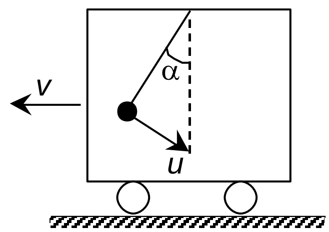
$\Rightarrow \quad u\cos\beta - v = \frac{M}{m}\cdot v$

$\Rightarrow \quad u = \frac{(M+m)v}{m\cos\beta}$

Kinetic energy of bob $= \frac{1}{2}m\left(v_x^2+v_y^2\right)$

By the conservation of energy,

$mg\ell\,(1-\cos\alpha) = mg\ell\,(1-\cos\beta) + \frac{1}{2}Mv^2 + \frac{1}{2}m\,[(u\cos\beta - v)^2 + u^2\sin^2\beta]$



After releasing the bob

or, $\quad 2mgl\,(\cos\beta - \cos\alpha) = Mv^2 + m\frac{M^2v^2}{m^2} + \frac{m\sin^2\beta(M+m)^2v^2}{m^2\cos^2\beta}$

$$= Mv^2\left\{1+\frac{M}{m}\right\}+\frac{(M+m)^2v^2}{m\cos^2\beta}\sin^2\beta = \frac{M(M+m)}{m}v^2+\frac{(M+m)^2v^2\sin^2\beta}{m\cos^2\beta}$$

or, $\quad 2m^2g\ell\,(\cos\beta - \cos\alpha)\cos^2\beta = M\,(M+m)\,v^2\cos^2\beta + (M+m)^2\,v^2\sin^2\beta$

$$= (M+m)v^2\,[M\cos^2\beta + (M+m)\sin^2\beta]$$

$\Rightarrow \quad v^2 = \frac{2m^2g\ell}{M+m}\left[\frac{(\cos\beta-\cos\alpha)\cos^2\beta}{M+m\sin^2\beta}\right]$

$\therefore \quad v = \sqrt{\frac{2m^2g\ell}{M+m}\left[\frac{(\cos\beta-\cos\alpha)\cos^2\beta}{M+m\sin^2\beta}\right]}$

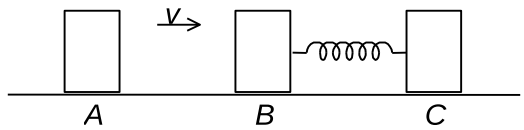
(b) In this particular case when $\beta = 0$,

$$v=\sqrt{\frac{2m^2g\ell}{M+m}\frac{(1-\cos\alpha)}{M}}=\sqrt{\frac{2m^2g\ell}{M+m}\frac{2\sin^2\frac{\alpha}{2}}{M}}$$

$$v = 2m\sin\frac{\alpha}{2}\sqrt{\frac{g\ell}{(M+m)M}}\,.$$

Example 9.

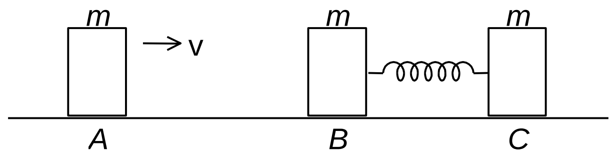
Two blocks B and C of mass m each connected by a spring of natural length ℓ and spring constant k rest on an absolutely smooth horizontal surface as shown in Figure. A third block A of same mass collides elastically block B velocity v. Calculate the velocities of blocks, when the spring is compressed as much as possible and also the maximum compression.



Solution:

Let A be the moving block and B and C the stationary blocks.

Since A and B are of equal mass, A is stopped dead and B takes off with its velocity. Now B and C move under their mutual action and reaction and so their momentum is conserved.



Let v_1 and v_2 be their instantaneous velocities when the compression of spring is x.

By the principle of conservation of momentum,

$mv = m(v_1 + v_2)$

$v_1 + v_2 = v$ (a constant)

By the principle of conservation of energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}kx^2$$

$$\Rightarrow \quad v^2 = v_1^2 + v_2^2 + \frac{k}{m}\cdot x^2$$

$$\Rightarrow \quad v^2 = (v_1 + v_2)^2 - 2v_1v_2 + \frac{k}{m}\cdot x^2$$

$$\Rightarrow \quad v^2 = v^2 - 2v_1v_2 + \frac{k}{m}x^2$$

$$\Rightarrow \quad v_1v_2 = \frac{k}{2m}\cdot x^2$$

Obviously compression (x) is maximum, when v_1v_2 is maximum under the condition that their sum $(v_1 + v_2)$ is constant.

We have, $(v_1 + v_2)^2 = (v_1 - v_2)^2 + 4v_1v_2$

$$\Rightarrow \quad v^2 = (v_1 - v_2)^2 + 4v_1v_2$$

$$\Rightarrow \quad 4v_1v_2 = v^2 - (v_1 - v_2)^2$$

Obviously v_1v_2 is maximum when $(v_1 - v_2)^2$ is minimum. But it is a real positive quantity. Its minimum value is zero.

$$(v_1v_2)_{max} = \frac{v^2}{4} \text{, when } v_1 = v_2$$

$$x_{max}^2 = \frac{2m}{k}(v_1v_2)_{max} = \frac{2m}{k}\cdot\frac{v^2}{4}$$

$$x_{max} = \sqrt{\frac{m}{2k}}\cdot v$$

Example 10.

A body A moving with velocity 10 m/s make a head on collision with a stationary body B of same mass. As a result of collision the kinetic energy of system decreases by one percent. Find the magnitude and direction of the velocity of particle A after collision.

Solution:

Let m be the mass of A and m the mass of B.

Let v_1 be the velocity of A and v_2 the velocity of B after collision.

By the principle of conservation of momentum,

$$mv_1 + mv_2 = mv + 0$$

$$\therefore \quad v_1 + v_2 = v \text{... (i)}$$

Given, $\dfrac{K_i - K_f}{K_i} = \dfrac{1}{100}$

$$\Rightarrow \quad 1 - \frac{K_f}{K_i} = \frac{1}{100} \qquad \frac{K_f}{K_i} = 1 - \frac{1}{100} = \frac{99}{100}$$

$$\Rightarrow \quad \frac{\frac{1}{2}mv_2^2 + \frac{1}{2}mv_1^2}{\frac{1}{2}mv^2} = \frac{99}{100}$$

$$\Rightarrow \quad \frac{v_2^2 + v_1^2}{v^2} = \frac{99}{100}$$

$$\Rightarrow \quad v_2^2 + v_1^2 = \frac{99}{100}v^2$$

$$(v_1 + v_2)^2 = v^2 \qquad \text{[from (i)]}$$

$$\therefore \quad v_1^2 + v_2^2 + 2v_1v_2 = v^2$$

$$\frac{99}{100}v^2 + 2v_1v_2 = v^2$$

$\Rightarrow \quad 2v_1v_2 = \frac{v^2}{100}$ or $v_1v_2 = \frac{v^2}{200}$ $v_1 + v_2 = 10$

$$v_1v_2 = \frac{10\times10}{200} = \frac{1}{2}$$

$$v_1(10 - v_1) = \frac{1}{2}$$

$$10v_1 - v_1^2 = \frac{1}{2} \text{ or } v_1^2 - 10v_1 + \frac{1}{2} = 0$$

$$2v_1^2 - 20v_1 + 1 = 0$$

$$v_1 = \frac{20 \pm \sqrt{400-8}}{4}$$

= 5 m/s in the same direction.

Example 11.

A block of mass 37.5 kg is placed on a table of mass 12.25 kg, which can move without friction on a level floor. A particle of mass 0.25 kg moving horizontally with velocity 302 m/s strikes the block inelastically (a) Find the distance through which the block moves relative to the table before they acquire a common velocity (b) also compute the common velocity, if the coefficient of friction between block and table is 0.25.

Solution:

(a) Applying the principle of conservation of momentum to the inelastic impact, we have

$0.25 \times 302 = (0.25 + 37.5 + 12.25)v$, where v is the common velocity of the system.

$$V_{common} = \frac{0.2\times302}{50} = 1.51 \text{ m/s}$$

(b) Let u be the velocity of block immediately after impact. Then, $0.25 \times 302 = (0.25 + 37.5)$ u

$$u = \frac{0.25\times302}{37.75} = 2 \text{ m/s}$$

Let a_1 and a_2 be the retardation of the block and acceleration of the table respectively.

Then $(0.25 + 37.5)\, a_1 = 12.25\, a_2$

= kinetic frictional force

Because $F_k = \mu_k mg$

$= 0.25 \times [0.25 + 37.5]$ g

$= 0.25 \times 37.75$ g

$a_1 = 2.45$ m/s^2

$a_2 = 7.55$ m/s^2

Relative retardation of block $= a_1 + a_2 = 2.45 + 7.55 = 10$ m/s^2

$\therefore \quad v^2 = 2 \times 10 \times s$

$v = 2$ m/s

$4 = 20$ s

$s = \frac{4}{20} = \frac{1}{5} m = 0.2$ m

Example 12.

Two balls of masses m and 2m are suspended by two threads of same length ℓ from the same point on the ceiling. The ball m is pulled aside through an angle α and released from rest after a tangential velocity v_0 towards the other stationary ball is imparted to it. To what heights will the balls rise after collision, if the collision is perfectly elastic?

Solution:

The velocity acquired by m on reaching the lowest position is v (say).

Then, $\frac{1}{2}mv_0^2 + mgl\ (1-\cos\alpha) = \frac{1}{2}mv^2$

$v^2 = v_0^2 + 2gl\ (1-\cos\alpha)$

By conservation of momentum,

$mv = mv_1 + 2mv_2$

$v = v_1 + 2v_2$ or $v - v_1 = 2v_2$ … (i)

By conservation of kinetic energy,

$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}2m \cdot v_2^2$

$v^2 = v_1^2 + 2v_2^2$

$(v^2 - v_1^2) = 2v_2^2$

$(v - v_1)(v_1 + v) = 2v_2^2$ … (ii)

Using (i) in (ii), $v_1 + v = v_2$ … (iii)

Solving (i) and (iii),

$v_2 = \frac{2}{3}$ v and $v_1 = -\frac{v}{3}$.

Let m rise by h_1 and 2m by h_2, then

$\frac{1}{2}mv_1^2 = mgh_1$ or $gh_1 = \frac{1}{2} \times \frac{v^2}{9} = \frac{1}{18}\left[v_0^2 + 2gl\ (1-\cos\alpha)\right]$

$h_1 = \boldsymbol{\frac{1}{18g}\left[v_0^2 + 2gl(1-\cos\alpha)\right]}$

$$\frac{1}{2}\times 2mv_2^2 = 2mgh_2$$

$$gh_2 = \frac{1}{2}\times\frac{4v^2}{9} = \frac{4}{18}\left[v_0^2 + 2gl(1-\cos\alpha)\right]$$

$$h_2 = \frac{4}{18g}\left[v_0^2 + 2g\boldsymbol{l}(1-\cos\alpha)\right].$$

Example 13.

A ball of mass m is projected with speed u into the barrel of spring gun of mass M initially at rest on a frictionless surface. The mass m sticks in the barrel at the point of maximum compression of the spring. What fraction of the initial kinetic energy of the ball is stored in the spring? Neglect the friction.

Solution:

Let v be the velocity of system after the ball of mass m sticks in the barrel. Applying law of conservation of linear momentum, we have

$$mu = (m + M)v \qquad \ldots \text{(i)}$$

The initial K.E. $\frac{1}{2}$ mu^2 of the ball is converted into elastic potential energy $\frac{1}{2}$ kx^2 of the spring and kinetic energy $\frac{1}{2}$(m + M)v^2 of the whole system. That is

$$\frac{1}{2}mu^2 = \frac{1}{2}kx^2 + \frac{1}{2}(m + M)v^2 \qquad \ldots \text{(ii)}$$

where k is the spring constant and x is its maximum compression.

Dividing equation (ii) by $\frac{1}{2}$ mu^2,

$$1 = \frac{\frac{1}{2}kx^2}{\frac{1}{2}mu^2} + \frac{\frac{1}{2}(m+M)v^2}{\frac{1}{2}mu^2} \qquad \ldots \text{(iii)}$$

$$1 = \frac{kx^2}{mu^2} + \frac{(m+M)v^2}{mu^2} \qquad \ldots \text{(iv)}$$

From equation (i), $\frac{v}{u} = \frac{m}{(M+m)}$

Substituting this value in equation (iv),

$$1 = \frac{kx^2}{mu^2} + \frac{(m+M)}{m}\cdot\frac{m^2}{(m+M)^2} = \frac{kx^2}{mu^2} + \frac{m}{m+M} \quad \frac{kx^2}{mu^2} = 1 - \frac{m}{m+M} = \frac{M}{(m+M)}$$

The energy stored in spring = $\frac{1}{2}$kx^2

Initial K.E. of the ball = $\frac{1}{2}$ mu^2.

Hence, $\frac{kx^2}{mu^2}$ represents the fraction of initial energy, which is stored in the spring.

$\therefore$ fraction = $\frac{M}{m+M}$

Example 14.

A ball is dropped on to a horizontal plate from a height h above it. If the coefficient of restitution is e, find the total distance travelled before the ball comes to rest and the total time taken.

Solution:

The impact between the ball and the plate is direct impact. If u is the velocity of the ball before impact, its velocity after is eu, where e is the coefficient of restitution.

But u = $\sqrt{2gh}$ since the ball falls from a height h.

So the velocity of the ball at the first rebound = $v_1 = eu_1 = e\sqrt{2gh}$.

The height h_1 to which it rises after first rebound

$$h_1 = \frac{v_1^2}{2g} = \frac{e^2 \cdot 2gh}{2g} = e^2h$$

The velocity u_2 at which the ball reaches the surface a second time $u_2 = v_1 = e\sqrt{2gh}$.

The velocity v_2 after second rebound

$v_2 = ev_1 = e^2u_1$

The height h_2 it rises after rebound

$$h_2 = \frac{v_2^2}{2g} = \frac{e^4u_1^2}{2g} = e^4h$$

In general, the height h_n to which the ball rises after n^{th} rebound is given by

$h_n = e^{2n}h$

Total distance travelled

$= h + 2h_1 + 2h_2 + 2h_3 + \ldots\ldots\ldots\ldots + 2h_n + \ldots\ldots\ldots\ldots$ to ∞

$= h + 2(e^2h + e^4h + e^6h + \ldots\ldots\ldots\ldots)$

$= h + 2he^2 (1 + e^2 + e^4 + \ldots\ldots\ldots\ldots)$

$$= h + \frac{2he^2}{1-e^2} = h\left(1+\frac{2e^2}{1-e^2}\right)$$

$$= \frac{h(1+e^2)}{(1-e^2)}$$

Time taken: The time taken for the fall from a height, h = $\sqrt{\frac{2h}{g}}$

The time taken to go up to h_1 and then come down = $2\sqrt{\frac{2h_1}{g}}$

Time taken till it comes to rest is therefore given by

$$t=\sqrt{\frac{2h}{g}}+2\sqrt{\frac{2h_1}{g}}+2\sqrt{\frac{2h_2}{g}}+\ \ldots\ldots\ldots\ldots$$

$$=\sqrt{\frac{2h}{g}}+2e\sqrt{\frac{2h}{g}}+2e^2\sqrt{\frac{2h}{g}}+\ \ldots\quad \{\because\ \ h_1 = he^2,\ h_2 = he^4,\ \text{etc}\}$$

$$t=\sqrt{\frac{2h}{g}}\ \left[1+2e+2e^2+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$=\sqrt{\frac{2h}{g}}\ [1 + 2e(1 + e + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)]$$

$$=\sqrt{\frac{2h}{g}}\left[1+\frac{2e}{1-e}\right]$$

$$t=\sqrt{\frac{2h}{g}}\left[\frac{1+e}{1-e}\right]$$

Example 15.

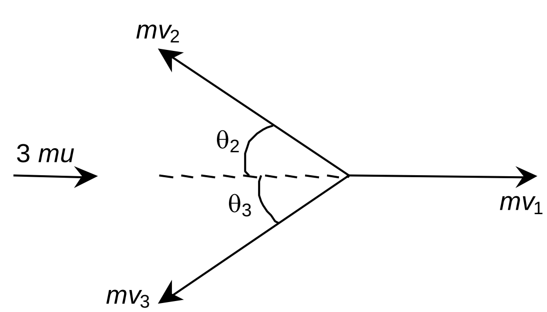
A shell flying with a velocity u = 500 m/s bursts into three identical fragments so that the kinetic energy of the system increases k times. What maximum velocity can one of the fragments obtain if k = 1.5?

Solution:

Let the mass of the shell be 3m. The mass of each fragment is m.

The particle with maximum velocity must be in the forward direction.

By law of conservation of momentum,



$3mu = mv_1 - mv_2 \cos\theta_2 - mv_3 \cos\theta_3$

$3u = v_1 - v_2 \cos\theta_2 - v_3 \cos\theta_3$

$v_1 = 3u + v_2 \cos\theta_2 + v_3 \cos\theta_3$ … (i)

Also $mv_2 \sin\theta_2 = mv_3 \sin\theta_3$ … (ii)

If v_1 is to be maximum

$\theta_2 = \theta_3 = 0$

From (2), if $\theta_2 = \theta_3$

$V_2 = v_3 = v$ (say)

Equation (i) becomes

$V_1 = 3u + 2v$

$$v = \frac{v_1 - 3u}{2} \qquad \text{… (iii)}$$

Using the principle of conservation of energy

$$\frac{1}{2}(3m)u^2 = \frac{1}{k}\left(\frac{1}{2}mv_1^2 + 2\times\frac{1}{2}mv^2\right)$$

$$3ku^2 = v_1^2 + 2v^2 \qquad \text{… (iv)}$$

Substituting for v from (iii)

$$3ku^2 = v_1^2 + \frac{1}{2}(v_1^2 + 9u^2 - 6v_1u)$$

Solving for v_1

$$v_1 = u\left[1 + \sqrt{2(k-1)}\right]$$

For u = 500 m/s and k = 1.5

$$v_1 = 500\left[1 + \sqrt{2(1.5-1)}\right] = 1000 \text{ m/s}$$