1. Number System

(i) Natural Numbers

The set of numbers {1, 2, 3, 4, ...} are called natural numbers, and is denoted by N.

i.e., N = {1, 2, 3, 4, ...}

(ii) Integers

The set of numbers {..., –3, –2, –1, 0, 1, 2, 3, ...} are called integers and the set is denoted by I or Z.

Where we represent;

(a) Positive integers by = {1, 2, 3, 4, ...} = Natural numbers.

(b) Negative integers by = {..., –4, –3, –2, –1}

(c) Non-negative integers (or N_0) = {0, 1, 2, 3, 4, ...} = Whole numbers

(d) Non-positive integers = {..., –3, –2, –1, 0}

(iii) Rational Numbers

A number which can be written as $\frac{a}{b}$, where a and b integers, $b \neq 0$ and H.C.F. of a and b is 1, is called a rational number and their set is denoted by Q.

i.e., $Q = \frac{a}{b}$ such that $a, b \in I$ and $b \neq 0$ and H.C.F. of a, b is 1.

Note:

(i) Every integer is a rational number as it could be written as $Q = \frac{a}{b}$ (where b = 1)

(ii) All recurring decimals are rational numbers.

e.g., $Q = 0.3333 \ldots = \frac{1}{3}$

$Q = 0.9999\ldots = 1$

(iv) Irrational Numbers

Those values which neither terminate nor could be expressed as recurring decimals are irrational numbers. (i.e., it cannot be expressed as $\frac{a}{b}$ form), their set is denoted by Q^c (i.e., complement of Q).

e.g., $\sqrt{2}$, $\pi + \sqrt{2}$, $\frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \sqrt{3}, 1+\sqrt{3}, \pm\frac{1}{\sqrt{3}}, \pi \ldots$ etc.

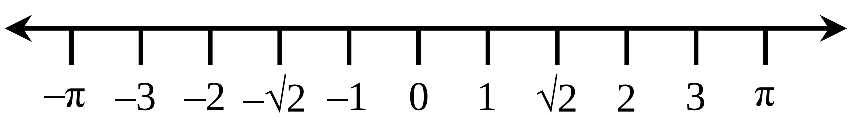
(v) Real Numbers

The set which contain both rational and irrational are called real number and is denoted by R. i.e., $R = Q \cup Q^c$

$\therefore R = \{\ldots -2, -1, 0, 1, 2, 3, \ldots, \frac{5}{6}, \frac{3}{4}, \frac{7}{9}, \frac{1}{3}, \frac{1}{7}, \frac{1}{5}, \ldots, \sqrt{2}, \sqrt{3}, \pi, \ldots\}$

Note: As from above definitions;

$N \subset I \subset Q \subset R$, it could be shown that real numbers can be expressed on number line with respect to origin as;



2. Intervals

The set of numbers between any two real numbers is called interval. The following are the types of interval.

(i) Closed Interval: [a, b] = {x : a x b}

(ii) Open Interval: (a, b) or]a, b[= {x : a < x < b}

(iii) Semi open or semi closed interval:

[a, b[or [a, b) = {x: a x < b}

]a, b] or (a, b] = {x: a < x b}

3. Inequalities

The following are some very useful points to remember:

- $a \leq b$ either $a < b$ or $a = b$
- $a < b$ and $b < c \Rightarrow a < c$
- $a < b \Rightarrow -a > -b$ i.e., inequality sign reverses if both sides are multiplied by a negative number
- $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c.\forall c \in R$
- $a < b \Rightarrow ma < mb$ if $m > 0$ and $ma > mb$ if $m < 0$
- $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$
- $\left(a + \frac{1}{a}\right) \geq 2$ for $a > 0$ and equality holds for $a = 1$
- $\left(a + \frac{1}{a}\right) \leq -2$ for $a < 0$ and equality holds for $a = -1$

4. The Absolute Value of a Real Number

The absolute value (or modulus) of a real number x (written |x|) is a non negative real number that satisfies the conditions.

$$|x| = x \text{ if } x \geq 0$$

$$|x| = -x \text{ if } x < 0$$

Example: | 2 | = 2, | –5 | = 5, | 0 | = 0

From the definition it follows that the relationship x |x| holds for any x. The properties of absolute values are

- **(i)** the inequality $|x| \leq \alpha$ means that $-\alpha \leq x \leq \alpha$; if $\alpha > 0$
- **(ii)** the inequality $|x| \geq \alpha$ means that $x \geq \alpha$ or $x \leq -\alpha$; if $\alpha > 0$
- **(iii)** $|x \pm y| \leq |x| + |y|$;
- **(iv)** $|x \pm y| \geq ||x| - |y||$;
- **(v)** $|xy| = |x|\,|y|$;
- **(vi)** $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $(y \neq 0)$.

Illustration 1:

Determine the values of x satisfying the equality:

$$|(x^2 + 4x + 9) + (2x - 3)| = |x^2 + 4x + 9| + |2x - 3|;$$

Solution:

The equality $|a + b| = |a| + |b|$ is valid if and only if both summands have the same sign. Since $x^2 + 4x + 9 = (x + 2)^2 + 5 > 0$ at any values of x, the equality is satisfied at those values of x at which $2x - 3 \geq 0$, i.e., at $x \geq 3/2$.

5. Functions

Let A and B are two non-empty sets. A function f from set A to set B is a rule which associated each element of A to a unique element of B, denoted by $f : A \to B$

set A is called domain of function $'f'$

set B is called co-domain of function $'f'$

If element x of A corresponds to y ($\in$B) under the function f, then we say that y is the image of x and write $f(x) = y$.

(i) Any linear expression represents a function.

(ii) Range of $'f' \subseteq$ co-domain of $'f'$

(iii) $f : A \to B$ is not a function, if there is at least one element in A which does not have a f-image in B or if there is an element in A which has more than one f-images in B.

(iv) A function can also be represented as a set of ordered pairs e.g. $f = \{(1, 2), (2, 3), (3, 4), (4,4)\}$ is a function from $\{1, 2, 3, 4\}$ to $\{2, 3, 4\}$. Clearly $f = \{(1, 2), (1, –1), (2, 2), (3, 3)\}$ is not a function as $1 \to 2$ and $1 \to –1$.

Domain of the Function

Domain of the function is set of all those real numbers (x) for which $f(x)$ exists or $f(x)$ is meaningful. $f(x) \neq \infty$ or any imaginary no.)

Range of Function

Set of all the images of elements in domain is called the range.

Range = $\{f(x) : x \in \text{domain}\}$

Illustration 2:

If $f(x)=\dfrac{1+x}{1-x}$, show that $\dfrac{f(x)\,f(x^2)}{1+[f(x)]^2}=\dfrac{1}{2}$.

Solution:

$$f(x^2)=\frac{1+x^2}{1-x^2}=\frac{1+x^2}{(1+x)(1-x)}$$

$$\therefore\ \frac{f(x)\,f(x^2)}{1+[f(x)]^2}=\frac{\left(\dfrac{1+x}{1-x}\right)\dfrac{1+x^2}{(1+x)(1-x)}}{1+\dfrac{(1+x)^2}{(1-x)^2}}=\frac{\dfrac{1+x^2}{(1-x)^2}}{\dfrac{(1-x)^2+(1+x)^2}{(1-x)^2}}=\frac{1+x^2}{2+2x^2}=\frac{1}{2}$$

6. Algebraic Operation on Functions

1. Given functions f and g, their sum $f+g$, difference $f-g$, and fg are defined on dom $f\cap$ dom g as:
$(f+g)(x)=f(x)+g(x)$, $(f-g)(x)=f(x)-g(x)$ and $(f\,g)(x)=f(x)\,g(x)$.
Moreover f/g is defined on dom $f\cap\{x\in$ dom g: $g(x)\neq 0\}$ by $(f/g)(x)=f(x)/g(x)$.

2. If k is any real number and f is a function then $k\,f$ is defined on the domain of f by $(k\,f)(x)=k\,f(x)$.
We have the following formulae for domains of functions

(i) dom $(f\pm g)$ = dom $f\cap$ dom g

(ii) dom $(f\,g)$ = dom $f\cap$ dom g

(iii) dom (f/g) = dom $f\cap\{x$ □ dom g: $g(x)\neq 0\}$

(iv) dom $\sqrt{f} = \{x \in \text{dom } f\,;\, f(x) \geq 0\}$

Illustration 3:

Find the domain of following functions:

(i) $f(x) = 2^{\sin^{-1}x} + \sqrt{x+2} + \dfrac{1}{\log_{10}(x+1)}$

(ii) $f(x) = \sin^{-1}\sqrt{4-x^2}$

(iii) $f(x) = \ln(-2 + 3x - x^2)$

Solution:

(i) For f(x) to be defined $-1 \leq x \leq 1$,

$x + 2 \geq o$ i.e., $x \geq -2$,

$x + 1 > 0$ i.e., $x > -1$ and

$x + 1 \neq 1$ i.e., $x \neq 0$

so, domain of f : $(-1, 0) \cup (0, 1]$

(ii) $f(x) = \sin^{-1}\sqrt{4-x^2}$

for f (x) to be defined $0 \leq 4 - x^2 \leq 1 \Rightarrow x^2 - 4 \leq 0$ and $x^2 - 3 \geq 0$

$\Rightarrow x \in [-2, 2]$ and $x \in (-\infty, -\sqrt{3}) \cup [\sqrt{3}, \infty)$

$\Rightarrow x \in [-2, -\sqrt{3}) \cup [\sqrt{3}, 2)$

so domain (f) : $x \in [-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$

(iii) $f(x) = \ln(-2 + 3x - x^2)$

for f(x) to be defined $-2 + 3x - x^2 > 0 \Rightarrow x^2 - 3x + 2 < 0$

$\Rightarrow (x-1)(x-2) < 0 \Rightarrow x \in (1, 2)$

so domain (f) : $x \in (1, 2)$

Illustration 4:

Find the range of the following functions:

(i) $f(x) = \dfrac{1}{8-3\sin x}$

(ii) $f(x) = x^2 - 7x + 5$

(iii) $f(x) = \log_2(\log_{12}(x^2 + 4x + 4))$

Solution:

(i) $f(x) = \dfrac{1}{8-3\sin x}$. We know that $-1 \le \sin x \le 1$

$$\Rightarrow -3 \le 3\sin x \le 3 \Rightarrow 5 \le 8 - 3\sin x \le 11 \therefore \text{Range}(f) = \left[\frac{1}{11}, \frac{1}{5}\right]$$

(ii) $f(x) = x^2 - 7x + 5 \Rightarrow f(x) = \left(x - \dfrac{7}{2}\right)^2 - \dfrac{29}{4} \Rightarrow \text{Range}(f) = \left[-\dfrac{29}{4}, \infty\right)$

(iii) $f(x) = \log_2(\log_{12}(x^2 + 4x + 4))$

Since $0 < \log_{12}(x^2 + 4x + 4) < \infty \ \forall\ x \in \text{Domain}(f)$

$\Rightarrow -\infty < \log_2(\log_{12}(x^2 + 4x + 4)) < \infty$

Range (f) = $(-\infty.\infty)$

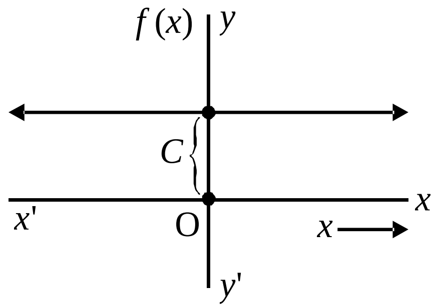
7. Some Standard Functions and Their Graphs

Constant Function

A function denoted by $f(x) = C$ (where $C \in R$) is known as constant function

Domain = R

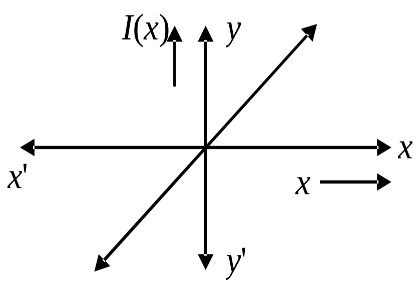
Range = C



Identity Function [I(x)]

A function which is associated to itself is known as identity function and denoted by I(x) = x

Since x can take any value so domain of this function is R, corresponding value of I(x) is also R, so range is R



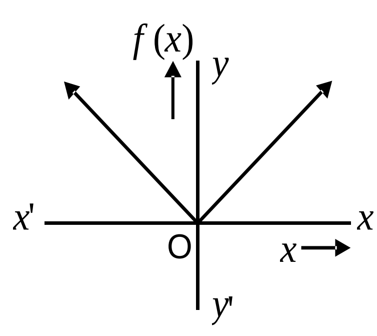
Domain = R

Range = R

Modulus Function

This is also known as absolute value function and denoted by $f(x) = |x|$

$$\text{i.e.} f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Domain of this function is set of all real numbers because $f(x)$ exists for all $x \in$ R but $|x| \geq 0$ so range is all non-negative real numbers.

Domain = R

Range = [0, ∞]

or $R^{+} \cap \{0\}$

Properties of modulus function

(a) $|x|^n = |x^n|$

(b) $|x^n| = x^n$, where n is even and $n \in z$

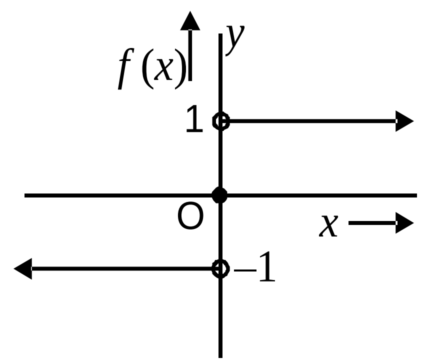
(c) $|xy| = |x||y|$

(d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, (y \neq 0)$

(e) $||x| - |y|| \leq |x + y| \leq |x| + |y|$

Signum Function

The function $f(x)$, defined as $f(x) = \begin{cases} \frac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$ is called signum function. This signum function may also defined as



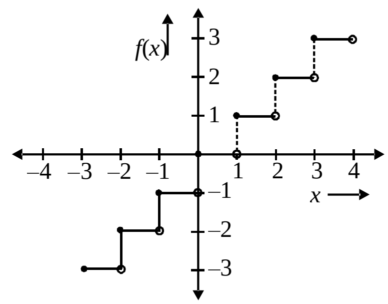
$$f(x) = \begin{cases} 1 \; ; & x > 0 \\ 0 \; ; & x = 0 \\ -1; & x < 0 \end{cases}$$

Domain = R

Range = {–1, 0, 1}

Greatest Integer Function

This function is also known as step function or floor function denoted by $f(x) = [x]$. By $[x]$ we mean greatest integer less than or equal to x. If n is an integer and x is any real number between n and n + 1



i.e. $n \le x < n + 1$, then $[x] = n$

Thus $[3.4] = 3$, $[3.99] = 3$

$[-4.99] = -5$, $[-4.001] = -5$

$[0.3] = 0$, $[-0.2] = -1$

Domain of $[x]$ is set of all real numbers because $[x]$ exist $x \in R$

But $[x]$ is always integral number so range is set of all integers Z.

Some Properties of [x]

(a) $[x + k] = [x] + k$, if $k \in Z$

(b) $[-x] = -[x] - 1$

(c) $[x] + [-x] = 0$, $x \in Z$

(d) $[x] + [-x] = -1$, $x \in Z$

(e) $[x] - [-x] = 2x$, $x \in Z$

(f) $[x] - [-x] = 2[x] + 1$, $x \in Z$

(g) $x - 1 < [x] \le x$

(h) $\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots\ldots+\left[x+\frac{n-1}{n}\right] = [nx]$

(i) $[x + y] \ge [x] + [y]$

(j) $\left[\frac{(x)}{n}\right]=\left[\frac{x}{n}\right], n \in N$

(k) $[x] \ge n \Rightarrow x \ge n$, $n \in z$

(l) $[x] > n \Rightarrow x \geq n + 1, n \in z$

(m) $[x] \leq n \Rightarrow x < n + 1, n \in z$

(n) $[x] < n \Rightarrow x < n, n \in z$

Fractional Part Function

Function denoted by $f(x) = \{x\}$, known as fractional part function.

Also defined as $f(x) = x - [x]$

If $x \in Z$, then $f(x) = 0$ [i.e. $f(2) = 2 - [2] = 0$]

If $x \notin Z$, then $f(x)$ lies between 0 to 1.

i.e. $x \in Z$, $0 < f(x) < 1$ [i.e. $f(3.4) = 3.4 - [3.4] = 3.4 - 3 = 0.4$]

Fractional part function is a periodic function having period '1'.

Domain = R

Range [0, 1)

Illustration 5:

Draw the graphs of the following functions:

(i) $y = |\sin x|, x \in [0, 2\pi]$

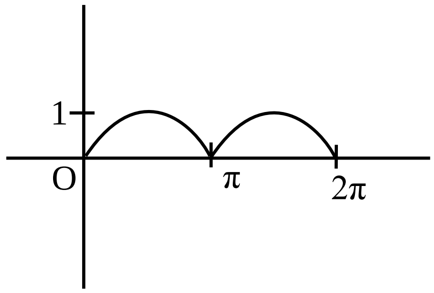
(ii) $y = \sin |x|, x \in [-2\pi, 2\pi]$

(iii) $|y| = \sin x, x \in [-2\pi, 2\pi]$

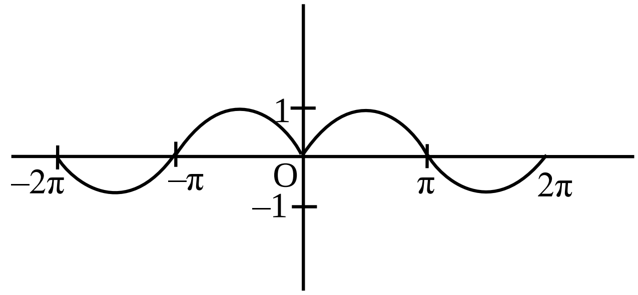
(iv) $|y| = -\sin x, x \in [-2\pi, 2\pi]$

Solution:

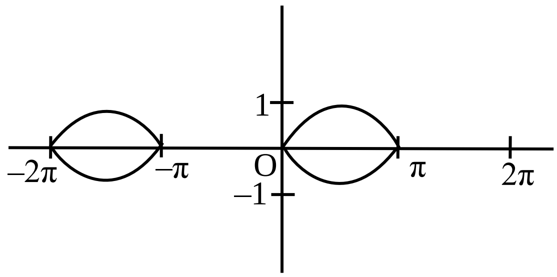
(i)



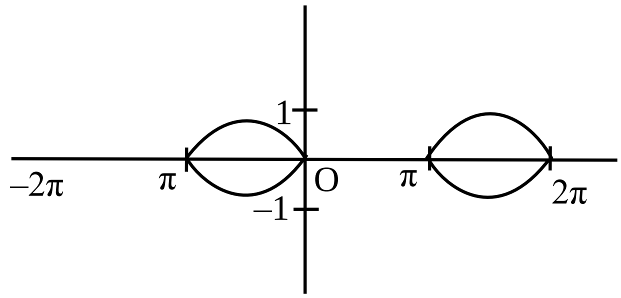
(ii)



(iii)



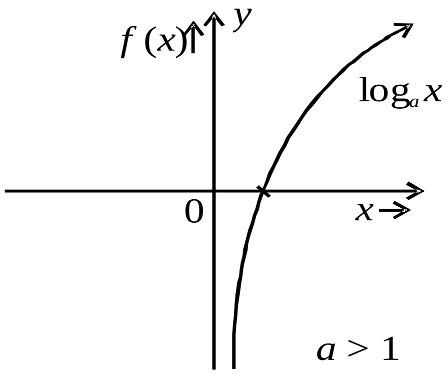
(iv)



8. Logarithmic Function

If $f : R^{+} \to R$, $f(x) = \log_{a} x$, then $f(x)$ is known as logarithmic function

Here $f(x)$ exist if $x > 0$ and $0 < a < 1$ or $a > 1$ $(a \neq 1)$



Properties of logarithmic function

(i) $\log_a m.n = \log m + \log n$

(ii) $\log_a = \log m - \log n$

(iii) $\log_a \; m^n = n\log_a m$

(iv) $\log_{a^q} b^p = \dfrac{p}{q}\log_a^{\boldsymbol{b}}$

(v) $\log_a^{\,b} = \dfrac{\log_x b}{\log_x a} = \log_x b.\log_a x$

(vi) $\log_a^b .\log_b^a = 1$

(vii) If $\log_a f(x) = y \Rightarrow \; f(x) = (a)^y$

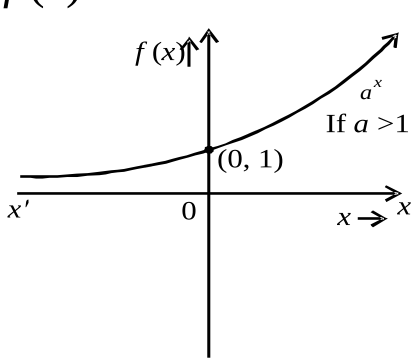
(viii) If $\log_a f(x) \geq \log_a g(x) \Rightarrow \begin{cases} f(x) \geq g(x) \text{ if } a > 1 \\ f(x) \leq g(x) \text{ if } 0 < a < 1 \end{cases}$

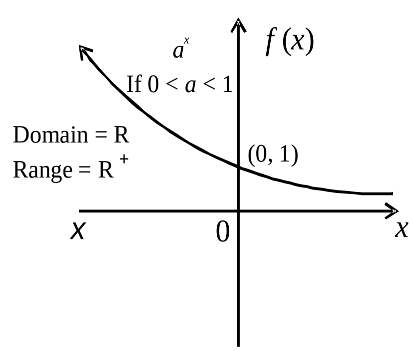
(ix) If $\log_a f(x) \geq y \quad \Rightarrow \begin{cases} f(x) \geq (a)^y \text{ if } a > 1 \\ f(x) \leq (a)^y \text{ if } 0 < a < 1 \end{cases}$

(x) If $\log_a f(x) \leq y \quad \Rightarrow \begin{cases} f(x) \leq (a)^y \text{ if } a > 1 \\ f(x) \geq (a)^y \text{ if } 0 < a < 1 \end{cases}$

Exponential Function

$f(x) = a^x$ is known as exponential function ($a > 0$)





9. Types of Function

Polynomial Function

The function $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$, where $a_0, a_1, a_2, \ldots\ldots a_n \in R$ and $n \in N$ is called a polynomial function of degree n.

Rational Function

A function defined by the quotient of two polynomial functions is called rational function. e.g. $\dfrac{x^2+1}{x^3+x+1}$ is a rational function.

Irrational Function

A function involving one or more radicals of polynomial is called an irrational function. e.g.

$$x^{\frac{3}{2}} + \sqrt{x} + x^2, \frac{x^2+2x+3}{x+\sqrt[3]{x}+5} \text{ etc.}$$

Algebraic Function

An algebraic function is one which consists of a finite number of terms involving power and roots of the variable x and simple operation, addition, subtraction, multiplication and division i.e. all rational, and irrational functions are algebraic functions.

Transcendental Function

All function which is not algebraic is called transcendental function. For example

(a) All trigometric function i.e. $\sin x$, $\cos x$ etc.

(b) All exponential function, e^x, $\log x$, a^x etc.

(c) Inverse trigonometric function $\sin^{-1} x$, $\cos^{-1} x$, etc.

- A transcendental function is not expressed in a finite number of algebraic terms.

Explicit Function

A function in which dependent variable (y) is expressed directly in terms of independent variable (say x) i.e. $y = x^3 + x^2 + 1$, $y = \dfrac{x^2 + 3x + 5}{x + 2}$, etc.

Implicit Function

A function in which we can't express dependent variable in terms of independent variable. e.g. $x^3 + y^3 + 3xy = 0$, note that we can't write y or x in terms of x, or y separately.

10. Even or Odd Function

(a) Even function

If $f(-x) = f(x)$ then $f(x)$ is said to be even function. e.g.

$f(x) = \cos x$ is a even function $\quad [\because f(-x) = \cos(-x) = \cos x = f(x)]$

(b) Odd function

If $f(-x) = -f(x)$ then $f(x)$ is said to odd function. e.g. $f(x) = x^3 + \tan^3 x$ is a odd function because

$$\begin{aligned} f(-x) &= (-x)^3 + [\tan(-x)]^3 \\ &= -x^3 - \tan^3 x \\ &= -[x^3 + \tan^3 x] \\ &= -f(x) \end{aligned}$$

so $\quad f(-x) = -f(x)$

(a) Even function is symmetrical about y-axis while odd function is symmetrical about origin (i.e. in opposite quadrant)

(b) Addition and subtraction of two even function is always even function.

(c) Sum of even and odd function is neither even nor odd function.

(d) Any function 'f' can be represented as the sum of an even and an odd function.

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(n) - f(-x)]$$

Where $\frac{1}{2}[f(x) + f(-x)]$ is an even and $\frac{1}{2}[f(n) - f(-x)]$ is an odd function

(e) $f(x) = 0$ is the only function which is both odd and even..

Illustration 6:

Determine the nature of the following function for even and odd:

(i) $f(x) = \log\left(x + \sqrt{x^2+1}\right)$

(ii) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

Solution:

(i) $f(x) = \log(x + \sqrt{x^2+1}) \Rightarrow f(-x) = \log(-x + \sqrt{x^2+1})$

$$= \log\left(\frac{1}{x + \sqrt{x^2+1}}\right) = \log(x + \sqrt{x^2+1})^{-1} = \log(x + \sqrt{x^2+1}) = -f(x)$$

So, f (x) is an odd function

(ii) We have $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

$$\therefore\ f(-x) = -x\left(\frac{a^{-x}-1}{a^{-x}+1}\right) = -x\left(\frac{\frac{1}{a^x}-1}{\frac{1}{a^x}+1}\right) = -x\left(\frac{1-a^x}{1+a^x}\right) = x\left(\frac{a^x-1}{a^x+1}\right) = f(x)$$

Illustration 7:

If f is an even function defined in the interval (– 5, 5), find four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$.

Solution:

Since f is an even function, f (– x) = f (x).

$$\therefore \text{ Now } f\left(\frac{x+1}{x+2}\right) = f(x) \Rightarrow f\left(\frac{x+1}{x+2}\right) = f(-x) \Rightarrow \frac{x+1}{x+2} = x \text{ or } \frac{x+1}{x+2} = -x$$

$\therefore\ x^2 + x - 1 = 0$ or $x^2 + 3x + 1 = 0$

$$\therefore\ x = \frac{-1 \pm \sqrt{5}}{2} \text{ or } x = \frac{-3 \pm \sqrt{5}}{2}.$$

11. Periodic Function

A function 'f' defined on its domain is said to be periodic function if there exist a positive number T such that f(x + T) = f(x) $\forall$ x $\in$ D. Also both x + T and x – T should belong to D.

The least value of T, it exists is called, the period of the function.

Some Standard Functions and their Period

Function	Period
sin x	2π
cos x	2π
tan x	π
{x}	1

Illustration 8:

Find the period of f(x) = |sin x| + |cos x|

Solution:

|sin x| has period π, |cos x| has period π

Hence, according to the rule of LCM; period of f(x) must be π.

But $|\sin\left(\frac{\pi}{2}+x\right)|=|\cos x|$ and $|\cos(\frac{\pi}{2}+x)|=|\sin x|$

Since, $\frac{\pi}{2}<\pi$, period of f(x) is $\frac{\pi}{2}$

Illustration 9:

If f(x) = sin x + cos ax is a periodic function, show that a is a rational number.

Solution:

Period of sin x = 2π = $\frac{2\pi}{1}$ and period of cos ax = $\frac{2\pi}{a}$

$\therefore$ Period of sin x + cos ax = L.C.M of $\frac{2\pi}{1}$ and $\frac{2\pi}{a}$

$= \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{H.C.F. of 1 and a}} = \frac{2\pi}{\lambda}$ where λ is the H.C.F. of 1 and a.

Since λ is the H.C.F of 1 and a, $\frac{1}{\lambda}$ and $\frac{a}{\lambda}$ should be both integers.

Suppose $\frac{1}{\lambda}$ = m and $\frac{a}{\lambda}$ = n, then $\frac{\frac{a}{\lambda}}{\frac{1}{\lambda}} = \frac{n}{m}$.

Hence, a is rational number with period = 2

12. Bounded and Unbounded Function

$f(x)$ is said to be bounded above, if there exists a fixed number say M such that f(x) is never greater then M for all value of x. Similarly it's bounded below if there exists a fixed number m (say) that $f(x)$ is never less then m

i.e. M $\leq f(x) \leq$ m for all value of x.

$f(x)$ is said to be unbounded if one or both of the upper and lower (M and m) bounds of the function are infinite. e.g.

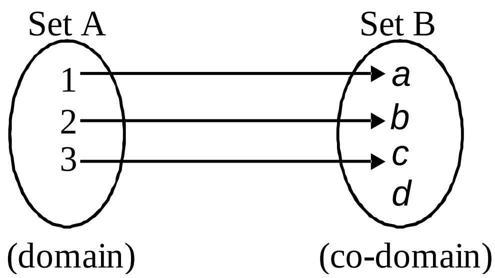
$f(x) = 3 + \sin x$ is a bounded function because maximum and minimum value of sin x are +1 and –1.

So, $2 \leq f(x) \leq 4$ for all value of x.

13. Types of Mappings or Functions

One-one Function or Injective Function:

A function is said to be one-one function if different element in a domain have different images in co-domain.



if $f(x_1) = f(x_2)$ then $x_1 = x_2$

$f(x)$ is one - one function

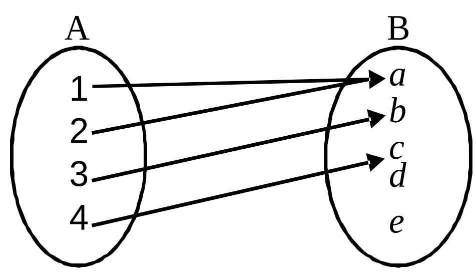
(i) Example of one-one function**:** Linear polynomial function ($ax + b$), x, e^x, log x, are always one-one functions.

(ii) If $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0 \forall x \in$ domain, then $y = f(x)$ is said to be one-one function.

Number of one-one function : If A and B are finite sets having m and n elements respectively, then number of one-one function from A to B = ${}^nP_m$, if $n \leq m = 0$, if $n < m$.

Many–one Function

A function f : A → B is said to be many one if more than one element in set A have same image in Set B.

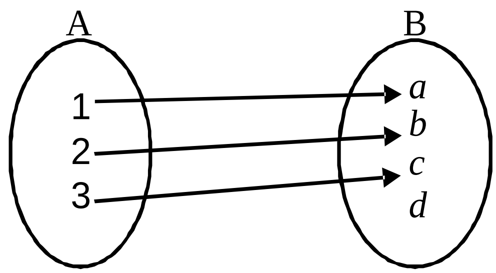


(i) All even function, modulus function, periodic functions are always many-one function.

(ii) Square function, Trigonometric function are also many–one function in their domain.

Into Function

A function $f : A \rightarrow B$ is said to be into function if there exist at least one element in set B having no any pre-image in set B.



In fig set B (co-domain) there is no pre-image, for element d, in set A, so function is into function.

Illustration 10:

Let f : $(-\infty, \infty) \rightarrow [2, \infty]$ be a function defined by $f(x) = \sqrt{x^2 - 2a + a^2}$, $a \in R$. Find a which f is onto.

Solution:

For f to be into range of the function should be $[2, \infty]$. So, $a^2 - 2a = 4$

$$\Rightarrow a = 1 \pm \sqrt{5}.$$

Illustration 11:

Show that the function $f(x) = \dfrac{x^2 - 8x + 18}{x^2 + 4x + 30}$ is not one-one.

Solution:

A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (only)

Now $f(x_1) = f(x_2) \Rightarrow \frac{x_1^2 - 8x_1 + 18}{x_1^2 + 4x_1 + 30} = \frac{x_2^2 - 8x_2 + 18}{x_2^2 + 4x_2 + 30}$

$\Rightarrow 12x_1^2 x_2 - 12x_1 x_2^2 + 12x_1^2 - 12x_2^2 - 8x_2 + 18$

$\Rightarrow (x_1 - x_2)\{12x_1 x_2 + 12(x_1 + x_2) - 312\} = 0$

$x_1 = x_2$ or $x_1 = \frac{26 - x_2}{1 + x_2}$ $\therefore$ f (x) is not one-one.

Onto Function

f : A → B, said to be onto function if every element in set B has a pre image in set A.

Range of f = co-domain of f.

Example of Onto function:

log x, linear polynomials, are always onto function.

Possible mappings are

(i) One-one and onto (bijective function)

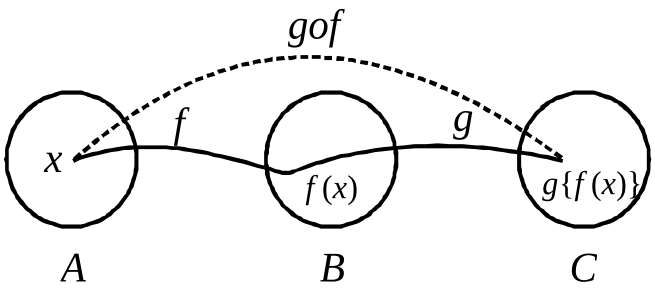
(ii) Many one and onto

(iii) One-one and into

(iv) Many one-into

14. Composition of Function

Let f : A→ B and g : B → C then the composition of g and f is denoted by gof and is defined as gof : A→ C given by $gof(x) = g\{f(x)\}$



Similarly fog is defined. Note that, gof is defined only if Range $f \subseteq$ dom g and fog is defined only if Range $\subseteq$ dom f. dom fog = {$x \in$ dom g: $g(x) \in$ dom f}.

Illustration 12:

If $f(x) = x^2 + 1$, $g(x) = \frac{1}{x-1}$, then find (fog) (x) and (gof) (x).

Solution:

Give, $f(x) = x^2 + 1$ ….(i) $g(x) = \frac{1}{x-1}$ ….(2)

Now $(fog)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = f(z)$, where $z = \frac{1}{z-1}$

$= z^2 + 1 \quad [\because f(x) = x^2 + 1]$

$= \left(\frac{1}{x-1}\right)^2 + 1 = \frac{1}{(x-1)^2} + 1$

Note: Domain of fog (x) is $x \in R - \{1\}$.

$(gof)(x) = g(f(x)) = g(x^2 + 1) = g(u)$, where $u = x^2 + 1 = \frac{1}{u-1} = \frac{1}{x^2+1-1} = \frac{1}{x^2}$

Note: Domain of gof (x) is $x \in R - \{0\}$

Illustration 13:

If $f(x) = \begin{cases} 2+f(x), & f(x) \geq 0 \\ 2-f(x), & f(x) < 0 \end{cases}$, then find (fof) (x).

Solution:

$$(fof)(x) = f(f(x)\} = \begin{cases} 2+f(x), & f(x) \geq 0 \\ 2-f(x), & f(x) < 0 \end{cases}$$

$$= \begin{cases} 2+2+x, & 2+x \geq 0 \text{ and } x \geq 0 \\ 2-(2+x), & 2+x < 0 \text{ and } x \geq 0 \\ 2+2-x, & 2-x \geq 0 \text{ and } x < 0 \\ 2-(2-x), & 2-x < 0 \text{ and } x < 0 \end{cases} = \begin{cases} 4+x, & x \geq 0 \\ -x, & x \in \phi \\ 4-x, & x < 0 \\ x, & x \in \phi \end{cases}$$

Hence $(fof)(x) = \begin{cases} 4+x, & x \geq 0 \\ 4-x, & x < 0 \end{cases}$

15. Inverse Function

Two functions f and g are inverse of each other if $f\,g\{(x)\} = x$ for $x \in$ dom g and $g\{f(x)\} = x$ for $x \in$ dom f, i.e., go$f$ = $I_{dom}\,f$ and $f$og = $I_{dom}\,g$ where $I_{dom}\,f$ is identity function on dom f and $I_{dom}\,g$ is identity function on dom g. We denote g by f^{-1} or f by g^{-1}.

Existence of inverse function

A function need not have an inverse. e.g. the function $f(x) = x^2$ has no inverse (where dom f = R). To have an inverse, a function must be both one-one and onto, i.e. bijective.

Illustration 14:

Find the inverse of the function $f(x) = \ell n\ (x^2 + 3x + 1)$; $x \in [1, 3]$ and assuming it to be an onto function.

Solution:

Given $f(x) = \ell n\ (x^2 + 3x + 1)$

$$\therefore f'(x) = \frac{2x+3}{(x^2+3x+1)} > 0\ \forall\ x \in [1, 3]$$

Which is strictly increasing function. Thus f(x) in injective, given that f(x) is onto. Hence the given function f(x) is invertible.

Now $f(f^{-1}(x)) = x \Rightarrow \ell n\ ((f^{-1}(x))^2 + 3(f^{-1}(x)) + 1) = x$

$$\Rightarrow \quad (f^{-1}(x))^2 + 3(f^{-1}(x)) + 1 - e^x = 0$$

$$\therefore\ f^{-1}(x) = \frac{-3 \pm \sqrt{9 - 4.1(1-e^x)}}{2} = \frac{-3 \pm \sqrt{(5+4e^x)}}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{-3 + \sqrt{(5+4e^x)}}{2} \qquad (\text{as } f^{-1}(x) \in [1, 3])$$

$$\text{Hence } f^{-1}(x) = \frac{-3 + \sqrt{(5+4e^x)}}{2}$$

Illustration 15:

Find the inverse of the function f(x) = $\begin{cases} x, & x<1 \\ x^2, & 1\le x\le 4 \\ 8\sqrt{x}, & x>4 \end{cases}$

Solution:

Given f(x) = $\begin{cases} x, & x<1 \\ x^2, & 1\le x\le 4 \\ 8\sqrt{x}, & x>4 \end{cases}$

Let f(x) = y $\Rightarrow$ x = f^{-1} (y) ….(i)

$$\therefore\ x=\begin{cases} y, & y<1 \\ \sqrt{y}, & 1\le\sqrt{y}\le 4 \\ \dfrac{y^2}{64}, & \dfrac{y^2}{64}>4 \end{cases}$$

$$\Rightarrow f^{-1}(y)=\begin{cases} y, & y<1 \\ \sqrt{y}, & 1\le y\le 16 \\ \dfrac{y^2}{64}, & y>16 \end{cases}$$ [From (i)]

Hence $f^{-1}(x)=\begin{cases} x, & x<1 \\ \sqrt{x}, & 1\le x\le 16 \\ \dfrac{x^2}{64}, & x>16 \end{cases}$