1. Derivative as a rate measure

The meaning of differential coefficient can be interpreted as rate of change of the dependent variable with respect to the independent variable, for example $\frac{dy}{dx}$ is the rate of change of *y* with respect to *x*. Similarly $\frac{dv}{dt}$ and $\frac{ds}{dt}$ etc. represent the rate of change of volume and surface area w.r.t. time.

Illustration 1:

Displacement 's' of a particle at time 't' is expressed as s = $\frac{1}{2}t^3 - 6t$, find the acceleration at the time when the velocity vanishes (i.e., velocity to zero).

Solution

$$s = \frac{1}{2}t^3 - 6t$$

Thus velocity, v = $\frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right)$

And acceleration, a = $\frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t$

Velocity vanishes, when $\frac{3t^2}{2} - 6 = 0 \ \Rightarrow t = 2$

Thus acceleration at v=0 i.e when velocity vanishes is a = 3t = 3×2 = 6 units.

Illustration 2:

On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Solution

Given $x^3 = 12y$, differentiating with respect to y

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dx}{dy} = \frac{12}{3x^2}$$

The interval in which the abscissa changes at a faster rate than the ordinate, we must have

$$\Rightarrow \left|\frac{dx}{dy}\right| > 1 \text{ or } \left|\frac{12}{3x^2}\right| > 1$$

$$\text{Or } \frac{4}{x^2} > 1 \text{ or } \frac{4}{x^2} - 1 > 0 \Rightarrow \frac{4 - x^2}{x^2} > 0$$

$\Rightarrow x \in (-2, 2) - \{0\}$.

Thus $x \in (-2, 2) - \{0\}$ is the required interval in which abscissa changes at a faster rate than the ordinate.

2. Angle of Intersection of two curves

Let $y = f(x)$ and $y = g(x)$ be two given intersecting curves.

Let (x_1, y_1) be the point of intersection $\Rightarrow y_1 = f(x_1) = g(x_1)$

Slope of the tangent drawn to the curve $y = f(x)$ at (x_1, y_1) i.e., $m_1 = \left(\frac{df}{dx}\right)_{(x_1, y_1)}$

Similarly slope of the tangent drawn to the curve y = g(x) at (x_1, y_1) i.e., $m_2 = \left(\frac{dg}{dx}\right)_{(x_1, y_1)}$

If α be the angle (acute) of intersection, then $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

NOTE:

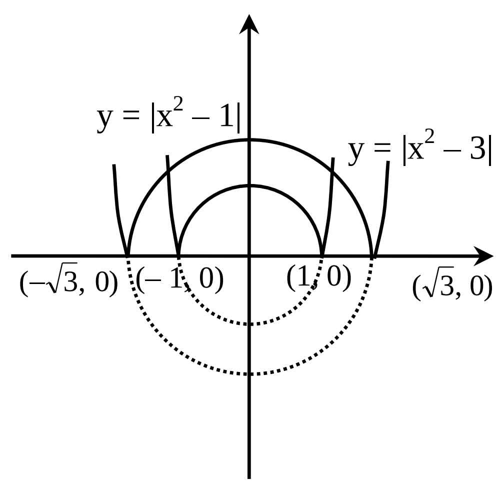
If $\alpha = 0 \Rightarrow m_1 = m_2$ means given curves will touch each other at (x_1, y_1).

If $\alpha = \pi/2 \Rightarrow m_1 m_2 = -1$ means given curves will meet at right angles.

Illustration 3:

Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Solution



For intersection point of the given curve

$|x^2 - 1| = |x^2 - 3|$

$(x^2 - 1)^2 = (x^2 - 3)^2$

$(x^2 - 1 - x^2 + 3)(x^2 - 1 + x^2 - 3) = 0$

$\Rightarrow 2x^2 = 4$

$\Rightarrow x = \mp\sqrt{2}$ and $y = |(\mp\sqrt{2})^2 - 1| = 1$

Hence, the points of intersection are $(\pm\sqrt{2}, 1)$

Since the curves are symmetrical about y-axis,

The angle of intersection at $(-\sqrt{2}, 1)$ = the angle of intersection at $(\sqrt{2}, 1)$.

At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}$, $m_2 = -2x = -2\sqrt{2}$

$$\therefore \tan\theta = \left|\frac{4\sqrt{2}}{1 - 8}\right| = \frac{4\sqrt{2}}{7} \Rightarrow \theta = \tan^{-1}\frac{4\sqrt{2}}{7}$$

3. Equations of tangent and normal

Cartesian Equations: The angle ψ which the tangent at any point (x, y) on the curve $y = f(x)$ makes with x-axis, is given by

$$\tan \psi = \frac{dy}{dx} = f'(x)$$

Thus, the equation of the tangent at the point (x, y) on the curve $y = f(x)$ is

$$Y - y = f'(x)(X - x)$$

Where (X, Y) is an arbitrary point on the tangent.

The equation of the normal at (x, y) to the curve $y = f(x)$ is

$$Y - y = -\left(\frac{1}{f'(x)}\right)(X - x) \Leftrightarrow (X - x) + f'(x)\,(Y - y) = 0,\ f'(x) \neq 0$$

Illustration 4:

Find the equation of normal to the curve $x + y = x^y$, where it cuts the x-axis.

Solution

Given curve is $x + y = x^y$ .... (i)

At x-axis y = 0,

$\therefore$ eq1 $\Rightarrow x + 0 = x^0 \Rightarrow x = 1$

Now to differentiate $x + y = x^y$, take log on both sides

$\Rightarrow \ln(x + y) = y \ln x$

$$\therefore \frac{1}{x+y}\left\{1 + \frac{dy}{dx}\right\} = y\frac{1}{x} + (\ln x)\frac{dy}{dx}$$

Putting x = 1, y = 0, we get

$$\left\{1+\frac{dy}{dx}\right\}=0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)}=-1$$

$\therefore$ Slope of normal = – 1

Equation of normal is, $\frac{y-0}{x-1}=-(-1) \Rightarrow y = x - 1.$

4. Lengths of the tangent, normal, sub-tangent and sub-normal at any point of a curve

Let the tangent and the normal at any point (x, y) of the curve $y = f(x)$ meets the x-axis at T and G respectively. Draw the ordinate PM.

Then the lengths TM, MG are called the sub-tangent and sub-normal respectively.

The lengths PT, PG are sometimes referred to as the lengths of the tangent and the normal respectively.

Clearly $\angle MPG = \psi$

Also $\tan\psi = \frac{dy}{dx}$

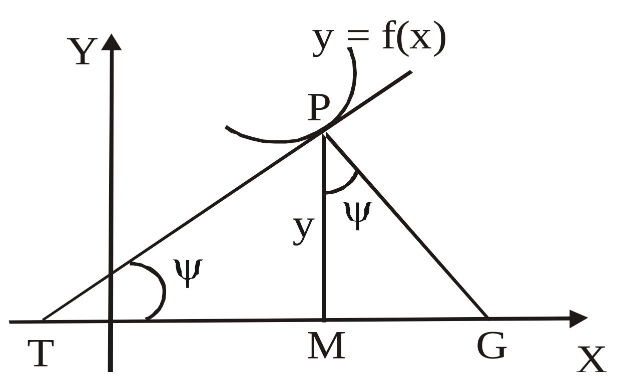
From the figure, we have

(i) Length of Tangent

$$= TP = MP\,|\mathrm{cosec}\,\psi|$$

$$= |y|\sqrt{1+\cot^2\psi}$$

$$= |y|\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}$$



(ii) Length of Sub-tangent

$$= TM = MP\ |\cot \psi| = |\ y \frac{dx}{dy}\ |$$

(iii) Length of Normal

$$= GP = MP\ |\sec \psi| = |\ y\ | \sqrt{1+\tan^2 \psi} = |\ y\ | \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$

(iv) Length of Sub-normal

$$= MG = MP\ |\mathbf{tan}\ \psi| = |\ y \frac{dy}{dx}\ |$$

Illustration 5:

Find the equation of family of curves for which the length of normal at any point P is equal to the distance of ‘P’ from origin.

Solution

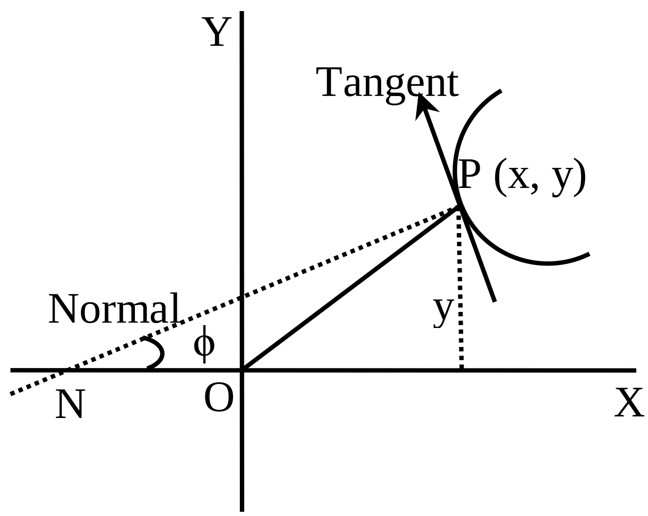
Let P(x, y) be the point on the curve.

OP = radius vector $= \sqrt{x^2 + y^2}$

PN = length of normal

Now, $\tan \phi = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)}$

$\Rightarrow$ PN $= \dfrac{y}{\sin\ \phi}$



It is given OP = PN

$$\Rightarrow \sqrt{x^2+y^2} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow x^2 + y^2 = y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] \Rightarrow x^2 = y^2\left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y}$$

$\Rightarrow$ ydy = ± x dx integrating both sides,

$y^2 = \pm x^2 + c$ is the required family of curves.

5. Rolle's Theorem

It is one of the most fundamental theorems of differential calculus and has far reaching consequences.

It states that if $y = f(x)$ be a given function and satisfies,

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) $f(a) = f(b)$

Then $f'(x) = 0$ at least once for some (a, b)

Illustration 6:

Let $f(x) = x^2 - 3x + 4$. Verify Rolle's Theorem in [1, 2].

Solution

Given function is $f(x) = x^2 - 3x + 4$

$\therefore f(1) = (1)^2 - 3(1) + 4 = 2$

$f(2) = (2)^2 - 3(2) + 4 = 2$

$f(1) = f(2) = 2$

Now, $f'(x) = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x = \frac{3}{2} \in (1, 2)$.

Hence, Rolle's Theorem is verified.

Illustration 7:

Let f(x) = (x – a) (x – b) (x – c), a < b < c, show that f(x) = 0 has two roots one belonging to (a, b) and other belonging to (b, c).

Solution

Here, f(x) being a polynomial is continuous and differentiable for all real values of x. We also have f(a) = f(b) = f(c). If we apply Rolle's Theorem to f(x) in [a, b] and [b, c] we would observe that f(x) = 0 would have at least one root in (a, b) and at least one root in (b, c).

But f(x) is a polynomial of degree two, hence f(x) = 0 cannot have more than two roots.

It implies that exactly one root of f(x) = 0 would lie in (a, b) and exactly one root of f(x) = 0 would lie in (b, c).

Remarks:

Let y = f(x) be a polynomial function of degree n. If f(x) = 0 has real roots only, then $f'(x) = 0$, $f^{n}(x) = 0$, ..., $f^{n-1}(x) = 0$ would have only real roots. It is so because if f(x) = 0 has all real roots, then between two consecutive roots of f(x) = 0, exactly one roots of $f'(x) = 0$ would lie.

Illustration 8:

Prove that if a_0, a_1, a_2, ……. a_n are real numbers such that $\frac{a_0}{n+1}+\frac{a_1}{n}+...+\frac{a_{n-1}}{2}+a_n=0$ then there exists at least one real number x between 0 and 1 such that $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$,

Solution

Consider a function f defined as

$$f(x) = \frac{a_0}{n+1}x^{n+1}+\frac{a_1}{n}x^{n}+...+\frac{a_{n-1}}{2}x^{2}+a_n x,\ x\in[0,1]$$

f being a polynomial satisfies the following conditions.

(i) f is continuous in [0, 1]

(ii) f is derivable in (0, 1)

(iii) Since f(0) = 0 and f(1) = 0 by hypothesis,

$\therefore$ f(0) = f(1)

Hence there is some $x \in (0, 1)$ such that $f'(x) = 0$

$$\Rightarrow \frac{a_0}{n+1}(n+1)x^n + \frac{a_1}{n}nx^{n-1} + \ldots + \frac{a_{n-1}}{2}.2x + a_n = 0$$

$$\Rightarrow a_0x^n + a_1x^{n-1} + \ldots. + a_{n-1}x + a_n = 0$$

6. Lagrange's Mean Value Theorem (LMVT)

If a function *f* is

(i) Continuous in a closed interval [a, b] and.

(ii) Derivable in the open interval (*a*, *b*), then there exists at least one value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Geometrical Interpretation:

Let P be a point [c, f (c)] on the curve y = f(x) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

The slope of the chord AB is $\frac{f(b) - f(a)}{b - a}$ and that of the tangent P (c, f(c)) is f'(c). These being equal, it follows that there exists a point P on the curve, the tangent which is parallel to the chord AB.

Illustration 9:

If f(x) and g(x) be differentiable functions in (a, b), continuous at a and b and $g(x) \neq 0$ in [a, b], then prove that $\dfrac{g(a)f(b)-f(a)g(b)}{g(c)f'(c)-f(c)g'(c)} = \dfrac{(b-a)g(a)g(b)}{(g(c))^2}$

for at least one $c \in (a, b)$.

Solution

We have to prove (after rearranging the terms)

$$\frac{\dfrac{f(b)}{g(b)} - \dfrac{f(a)}{g(a)}}{(b-a)} = \frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}$$

Let $F(x) = \dfrac{f(x)}{g(x)}$

As f(x) and g(x) are differentiable function in (a, b), f(x) will also be differentiable in (a, b). Further f is continuous at a and b. So according to LMVT, there exists one point P [c, f (c)]

Such that, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$, which proves the required result.

Illustration 10:

If f(x) is continuous in [a, b] and differentiable in (a, b) then prove that there exists at least one $c \in (a, b)$ such that $\dfrac{f'(c)}{3c^2} = \dfrac{f(b)-f(a)}{b^3-a^3}$.

Solution

We have to prove

$(b^3 - a^3) f'(c) - (f(b) - f(a)) (3c^2) = 0$

Let us assume a function $F(x) = (b^3 - a^3) f(x) - (f(b) - f(a)) x^3$

Which will be continuous in [a, b], differentiable in (a, b) as f(x) and x^3 both are continuous.

Also F (a) = b^3 f (a) – a^3 f (b) = F (b)

So, according to Rolle's Theorem, there exists at least one such that, F′(c) = 0 which proves the required result.

7. Monotonicity

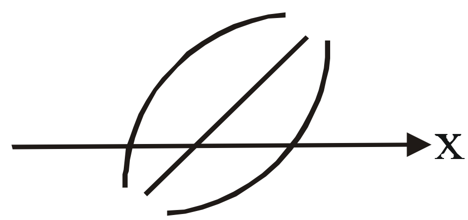
Let $y = f(x)$ be a given function with D as it's domain. Let $D_1 \subseteq D$, then

7.1 Increasing Function

$f(x)$ is said to be increasing in D_1 if for every $x_1, x_2 \in D_1$

$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

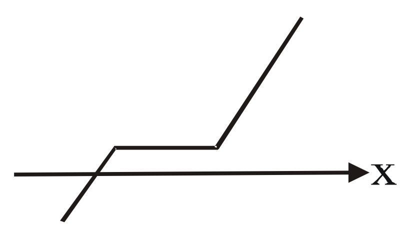
It means that there is a certain increase in the value of $f(x)$ with an increase in the value of x



Increasing function

7.2 Non-Decreasing Function

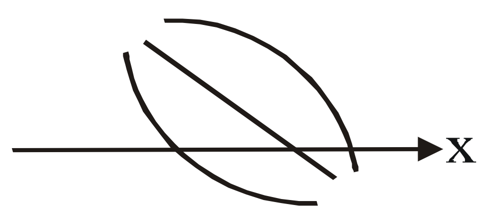
$f(x)$ is said to be non-decreasing in D_1 if for every $x_1, x_2 \in D_1$, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$. It means that the value of $f(x)$ would never decrease with an increase in the value of x



Non-Decreasing function

7.3 Decreasing Function

$f(x)$ is said to be decreasing in D_1 if for every $x_1, x_2 \in D_1$, $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ it means that there is a certain decrease in the value of $f(x)$ with an increase in the value of x.

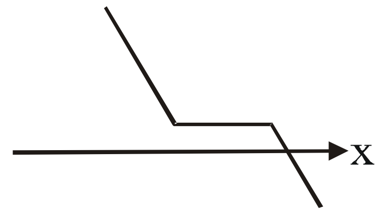


Decreasing function

7.4 Non-increasing Function

$f(x)$ is said to be non-increasing in D_1 if for every $x_1, x_2 \in D_1$, $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$.

It means that the value of $f(x)$ would never increase with an increase in the value of x



Non-Increasing function

7.5 Basic Theorems

Let $y = f(x)$ be a given function, continuous in $[a, b]$ and differentiable in (a, b). Then

- $f(x)$ is increasing in (a, b) if $f'(x) > 0 \ \forall \ x \in (a, b)$.
- $f(x)$ is non-decreasing in (a, b) if $f'(x) \geq 0 \ \forall \ x \in (a, b)$.
- $f(x)$ is decreasing in (a, b) if $f'(x) < 0 \ \forall \ x \in (a, b)$.
- $f(x)$ is non-increasing in (a, b) if $f'(x) \leq 0 \ \forall \ x \in (a, b)$.

Remarks:

(i) If $f'(x) \geq 0 \ \forall \ x \in (a, b)$ and points which make $f'(x)$ equal to zero (in between (a, b)) don't form an interval, then $f(x)$ would be increasing in (a, b).

(ii) If $f'(x) \leq 0 \ \forall \ x \in (a, b)$ and points which make $f'(x)$ equal to zero (in between (a, b)) don't form an interval, f(x) would be decreasing in (a, b).

(iii) If $f(0) = 0$ and $f'(x) \geq 0 \ \forall \ x \in R$, then $f(x) \leq 0 \ \forall \ x \in (-\infty, 0)$ and $f(x) \geq 0 \ \forall \ x \in (0, \infty)$.

(iv) If $f(0) = 0$ and $f'(x) \leq 0 \ \forall \ x \in R$ then $f(x) \geq 0 \ \forall \ x \in (-\infty, 0)$ and $f(x) \leq 0 \ \forall \ x \in (0, \infty)$.

(v) A function is said to be monotonic if it's either increasing or decreasing.

(vi) The points for which $f'(x)$ is equal to zero or doesn't exist are called critical points. Here it should also be noted that critical points are the interior points of the domain of the function.

(vii) The stationary points are the points of the domain where $f'(x) = 0$.

(viii) If $f''(x) = 0$ or does not exist at points where $f'(x)$ exists and if $f''(x)$ changes sign when passing through $x = x_0$ and $f'(x)$ doesn't change its sign then x_0 is called a point of inflection.

If $f''(x) < 0$, $x \in (a, b)$ then the curve $y = f(x)$ is convex in (a, b)

If $f''(x) > 0$, $x \in (a, b)$, then the curve $y = f(x)$ is concave in (a, b)

At the point of inflection, the curve changes its concavity.

Illustration 11:

(i) Find the critical points and the intervals of increase and decrease for

$f(x) = 3x^4 – 8x^3 – 6x^2 + 24x + 7$.

(ii) Find the intervals of monotonicity of the following functions:

(a) $f(x) = x^4 – 8x^3 + 22x^2 – 24x + 7$

(b) $f(x) = x \ln x$

Solution

(i) $f(x) = 3x^4 – 8x^3 – 6x^2 + 24x + 7$

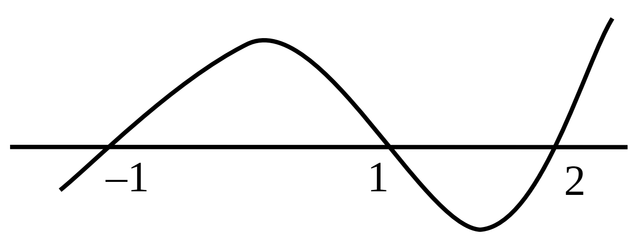
$f'(x) = 12x^3 – 24x^2 – 12x + 24 = 0$

$\Rightarrow 12(x^3 – 2x^2 – x + 2) = 0$

$\Rightarrow 12(x – 1)(x – 2)(x + 1) = 0$

Critical points are –1, 1 and 2.

sign scheme for f′ (x):



The wavy curve of the derivative is given in the figure. Hence, function increases in the interval $[–1, 1] \cup [2, \infty)$ and decreases in the interval $(–\infty, –1] \cup [1, 2]$.

(ii) (a) we have $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7,\ x \in R$

$\Rightarrow f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3)$

From the sign scheme for (x), we can see that f(x) decreases in $(-\infty, 1)$ increases in [1, 2] decreases in [2, 3] and increases in $[3, \infty]$.

(b) We have $f(x) = x \ln x,\ x > 0$

$\Rightarrow f'(x) = 1 + \ln x < 0\ \forall\ x < e^{-1}$

$\Rightarrow$ f(x) decreases in $(0, e^{-1})$ increases in $[e^{-1}, \infty]$.

Illustration 12:

Prove the following inequalities:

(a) $\ln(1 + x) > x - \dfrac{x^2}{2}\ \forall\ x \in (0, \infty)$

(b) $\sin x < x < \tan x\ \forall\ x \in \left(0, \dfrac{\pi}{2}\right)$

Solution

(a) Consider the function

$f(x) = \ln(1 + x) - x + \dfrac{x^2}{2},\ x \in (0, \infty)$

Then $f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0\ \forall\ x \in (0, \infty)$

$\Rightarrow$ f(x) increases in $(0, \infty) \Rightarrow f(x) > f(0^+) = 0$

i.e., $\ln(1 + x) > x - \dfrac{x^2}{2}$

which is the desired result.

(b) Consider the function

$f(x) = \tan x - x,\ x \in \left(0, \frac{\pi}{2}\right)$

$f'(x) = \sec^2 x - 1 > 0\ \forall\ x \in \left(0, \frac{\pi}{2}\right)$

Thus f(x) increases in $\left(0, \frac{\pi}{2}\right) \Rightarrow f(x) > f(0) = 0$

i.e., $\tan x > x$

Now, consider the function

$g(x) = x - \sin x,\ x \in \left(0, \frac{\pi}{2}\right)$

$g'(x) = 1 - \cos x > 0\ \forall\ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ g(x) increases in $\left(0, \frac{\pi}{2}\right) \Rightarrow g(x) \geq g(0) = 0$ i.e., $\sin x < x$

8. Maxima and Minima of a Function

8.1 concept of Local maxima and minima

A function $f(x)$ is said to have local maximum at any point $x = a$ if

$f(a) > f(a - h)$ and $f(a) > f(a + h)$, where $h > 0$ (very small quantity)

A function $f(x)$ is said to have local minimum at any point $x = a$ if

$f(a) < f(a - h)$ and $f(a) < f(a + h)$, where $h > 0$ (very small quantity)

A local maximum or a local minimum is also called a local extremum.

8.2 Tests for Local Maxima/Minima

8.2.1. Test for Local Maximum/Minimum at x = a, if f(x) is Differentiable at x = a.

Test for Local Maximum/Minimum at $x = a$,

if f(x) is Differentiable at $x = a$.

If $f(x)$ is differentiable at $x = a$ and if it is a critical point of the function (i.e., $f'(a) = 0$) then we have the following three tests to decide whether $f(x)$ has a local maximum or local minimum or neither at $x = a$.

First Derivative Test:

If $f'(a) = 0$ and $f'(x) = 0$ changes it's sign while passing through the point $x = a$, then

(i) $f(x)$ would have a local maximum at $x = a$ if $f'(a - 0) > 0$ and $f'(a + 0) < 0$. It means that $f'(x)$ should change its sign from positive to negative.

(ii) $f(x)$ would have a local minimum at $x = a$ if $f'(a - 0) < 0$ and $f'(a + 0) > 0$. It means that should change its sign from negative to positive.

(iii) If $f(x)$ doesn't change its sign while passing through $x = a$, then $f(x)$ would have neither a maximum nor a minimum at $x = a$.

Second Derivative Test:

This test is basically the mathematical representation of the first derivative test. It simply says that,

(i) If $f'(a) = 0$ and $f''(a) < 0$ then $f(x)$ would have a local maximum at $x = a$.

(ii) If $f'(a) = 0$ and $f''(a) > 0$ then $f(x)$ would have a local minimum at $x = a$.

(iii) If $f'(a) = 0$ and $f''(a) = 0$ then this test fails and the existence of a local maximum or minimum at $x = a$ is decided on the basis of the n^{th} derivative test.

n^{th} Derivative Test

It is nothing but the general version of the second derivative test, It says that if, $f'(a) = f''(a) = f'''(a) = \ldots f^{n\prime}(a) = 0$ and $f^{(n+1)\prime}(a) \neq 0$ (all derivatives of the function up to order n vanishes and $(n + 1)^{th}$ order derivative does not vanish at $x = a$), then $f(x)$ would have a local maximum or local minimum at $x = a$ if n is odd natural number and that $x = a$ would be a point of local maxima if $f^{(n+1)\prime}(a) < 0$ and would be a point of local minima if $f^{(n+1)\prime}(a) > 0$. However if n is even, then f has neither a maxima nor a minima at $x = a$.

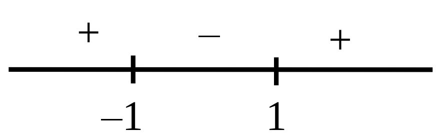
It is clear that the last two tests are basically the mathematical representation of the first derivative test. But that shouldn't diminish the importance of other tests. It is very difficult to decide whether function $f(x)$ changes it's sign or not while passing through point x = a, and the remaining tests may come handy in these kind of situations.

Illustration 13:

Let $f(x) = x + \frac{1}{x}$, $x \neq 0$. Discuss the maximum and minimum values of f(x).

Solution

Here, $f'(x) = 1 - \frac{1}{x^2} \Rightarrow f'(x) = \frac{x^2 - 1}{x^2} = \frac{(x-1)\,(x+1)}{x^2}$

Sign scheme for f′(x): 

Using number line rule,

We have maximum at x = –1 and minimum at x = 1

$\therefore$ At x = – 1 we have local maximum $\Rightarrow f_{max}(x) = -2$

And at x = 1 we have local minimum $\Rightarrow f_{min} = 2$

8.2.2. Test for Local Maximum/Minimum at x = a if f (x) is not differentiable at x = a

Case 1: When f(x) is continuous at x = a and (a – h) and (a + h) exist and are non-zero, then If $f'(a-h) > 0$ and $f'(a+h) < 0$ then $x = a$ will be a point of local maximum.

If $f'(a-h) < 0$ and $f'(a-h) > 0$ then $x = a$ will be a point of local minimum.

Case 2: When $f(x)$ is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the information about the existence of local maxima/minima from the basic definition local maxima/minima.

Case 3: If $f(x)$ is not continuous at $x = a$ and $f'(a-h)$ and/or $f'(a-h)$ are not finite, then compare the values of $f(x)$ at the neighboring points of $x = a$.

Illustration 14:

Let f(x) $=\begin{cases} x^3+x^2+10x, & x<0 \\ -3\sin x & x\geq 0 \end{cases}$. Investigate x = 0 for local maxima/minima.

Solution

Clearly f(x) is continuous at x = 0 but not differentiable at x = 0 as f(0) = f(0 – 0) = f(0 + 0) = 0

$$f'_{-}(0) = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h} = \lim_{h\to 0}\frac{-h^3+h^2-10h-0}{-h}=10$$

But $f'_{+}(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{-3\sinh}{h}=-3$

Since $f'_{-}(0)>0$ and $f'_{+}(0)<0$, x = 0 is the point of local maximum.

Illustration 15:

Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maximum and minimum of f(x) in [0, 2] and in (1, 3).

Solution

$f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$

First of all let us discuss [0, 2].

Clearly the critical point of f(x) in [0, 2] is x = 1.

$f(0) = 6,\ f(1) = 11,\ f(2) = 10$

Thus x = 0 is the point of global minimum of f(x) in [0, 2] and x = 1 is the point of global maximum.

Now let us consider (1, 3)

Clearly, x = 2 is the only critical point in (1, 3),

$f(2) = 10,\ \lim_{x \to 1+0} f(x) = 11$ and $\lim_{x \to 3-0} f(x) = 15$

Thus x = 2 is the point of global minimum in (1, 3) and the global maximum in (1, 3) does not exist.