TRIGONOMETRIC EQUATIONS

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TRIGONOMETRIC EQUATIONS

An equation involving one or more trigonometric ratios of unknown angle is called trigonometric equation. e.g. $\cos^2 x - 4\sin x = 1$.

It is to be noted that a trigonometric identity is satisfied for every value of the unknown angle, whereas, trigonometric equation is satisfied only for some values (finite or infinite in number) of unknown angle.

e.g. $\sin^2 x + \cos^2 x = 1$ is a trigonometric identity as it is satisfied for every value of $x \in R$.

A value of the unknown angle which satisfies the given equation is called a solution of the equation e.g. $\theta = \frac{\pi}{6}$ is a solution of $\sin\theta = \frac{1}{2}$.

GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

Now we know that $\sin x = \frac{1}{2}$ for $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots etc.$, but how can we represent all such angles generally???

Since trigonometric functions are periodic functions, solutions of trigonometric equations can be generalized with the help of the periodicity of the trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Consider sin x = a, where for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, $\sin\alpha = a$, then by the concept of values of trigonometric ratios for allied angles as well as periodic properties we do not only get $x = \alpha$ but also $x = \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha, \ldots$etc.

Now this series of angles may be generalized by observing that each of the angles is either α subtracted from an odd multiple of π or added to an even multiple of π.

$$\Rightarrow x = 2m\pi + \alpha \text{ \& } x = (2m-1)\pi - \alpha.$$

Collectively these two values give $x = n\pi + (-1)^n \alpha, n \in Z$

$$\boxed{\sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n \alpha}$$

Similarly when cos x = cos α, then $x = \pm\alpha, 2\pi \pm \alpha, 4\pi \pm \alpha, \ldots$etc.

Generalizing these values gives the formula $x = 2n\pi \pm \alpha,\ n \in Z$.

$$\boxed{\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha}$$

In the same manner $\tan x = \tan\alpha \Rightarrow x = \alpha, \pi + \alpha, 2\pi + \alpha, 3\pi + \alpha, \ldots$etc.

Generalizing these angles gives the formula $x = n\pi + \alpha,\ n \in Z$.

$$\boxed{\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha}$$

SPECIAL CASES IN GENERAL SOLUTIONS

When $\sin x = \sin\alpha$
$\Rightarrow x = n\pi + (-1)^n \alpha$, then why general solution of $\sin x = 1$ is represented as $(4n+1)\frac{\pi}{2}$ and not as $n\pi + (-1)^n \frac{\pi}{2}$???

In trigonometric equations it's a common practice to represent the general solution in most comprehensive but simplest form as well as to avoid repetition in writing solutions.

For example for some equation if we get solutions as "x = nπ & x = 2nπ", then we should give only x = nπ as the solution as the solutions given by x = 2nπ form a subset of those given by x = nπ.

Similarly if we get solution of some equation as " $x = 3n\pi, x = (3n-1)\pi$ & $x = (3n+1)\pi$", then it is advisable to write the solution as x = nπ, as all the three sets mentioned above are exhaustive subsets of x = nπ.

With this understanding, solutions of some equations like sin x = 1 or cos x = 0 are not represented using the previous article but are given as shown …..

$$\sin\theta = 1 \Rightarrow \theta = \frac{4n+1}{2}\pi$$
$$\sin\theta = -1 \Rightarrow \theta = \frac{4n-1}{2}\pi$$
$$\cos\theta = 0 \Rightarrow \frac{2n-1}{2}\pi$$
$$\cos\theta = -1 \Rightarrow \theta = (2n-1)\pi$$

Now consider the equations $\sin^2 x = \sin^2\alpha, \cos^2 x = \cos^2\alpha$ & $\tan^2 x = \tan^2\alpha$.

These equations imply $\sin x = \pm\sin\alpha, \cos x = \pm\cos\alpha$ & $\tan x = \pm\tan\alpha$ and by the concept of values of trigonometric ratios for allied angles we know that if a trigonometric ratio is equal to say k for some angle in first quadrant than in each of the other three quadrants there exists an angle for which it will be either k or –k.

Hence $x = \alpha, \pi\pm\alpha, 2\pi\pm\alpha, 3\pi\pm\alpha, ...$etc. In general form this gives $x = n\pi\pm\alpha$

$$\left.\begin{array}{l}\sin^2 x = \sin^2\alpha \\ \cos^2 x = \cos^2\alpha \\ \tan^2 x = \tan^2\alpha\end{array}\right\} \Rightarrow x = n\pi\pm\alpha$$

PROFILES OF TRIGONOMETRIC EQUATIONS (BASIC)

Equations of type f(x) = g(x), where f & g are complimentary trigonometric ratios

Such equations may be solved by writing one of f & g in terms of other using complimentary allied angle relations.

Equations which can be solved by factorization

By the use of various trigonometric identities it may sometimes be possible to factorize a trigonometric expression into simple factors of type (a f(x) + b) or f(x) = g(x), where a & b are constants and f(x) & g(x) denotes any of the six trigonometric ratios.

Equations reducible to quadratic or higher degree equations

Equations of the type $a\cos x + b\sin x = c$ could be reduced to a quadratic equation by substituting

$\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$. In general a function of $\sin x, \cos x$ & $\tan x$ can be reduced to a quadratic or higher degree polynomial in $\tan\dfrac{x}{2}$ by the above substitutions and $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$.

Also equations involving trigonometric ratios of multiple angles may be reduced to polynomial equations.

Solving equations by transforming a sum of Trigonometric functions into a product

Basic approach here is to obtain a form f(x) g(x) = 0 so that we can write f(x) = 0 & g(x) = 0 and solve further.

Solving equations by transforming a product of trigonometric functions into a sum

EXTRENIOUS ROOTS & LOSS OF ROOTS

Why sometimes we are supposed to reject some solutions though we get them in our calculations whereas sometimes we don't get all the solution by doing calculations???

Sometimes because of transformations we get some angles as solutions of transformed equation but which may not be not in the domain of original equation. Such values are called EXTRENEOUS ROOTS. In case of equations involving any

T – Ratios other than sine & cosine we should always perform a check for domain.

While using substitutions take care that sin x & cos x are defined for all real angles while tan x is not defined for odd multiples of 90^0, hence changing sine & cosine into tangents of half angles will result in change of domain which may cause a LOSS OF ROOTS.
In such a case always check that whether those angles which were in the domain of given equation but not in the transformed equation satisfy the given equation?

Also note that it is usually not advisable to increase degree of given equation as many a times it leads to equations which may not be solvable by algebraic methods or may lead to values of trigonometric ratios for which angles may not be well known.

Conclusively following points must be kept in mind while solving equations…

- While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values.
- Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of genuine solution.
- The answer should not contain such values of angles, which make any of the terms undefined.
- Domain should not be changed. If it is changed, necessary corrections must be incorporated.
- Check that the denominator is not zero at any stage while solving equations. Sometimes you may find that your answers differ from those in the answer sheet in their notations. This may be due to the different methods of solving the same problem. Whenever you come across such situation, you must check their authenticity to ensure that your answer is correct.
- While solving trigonometric equations you may get same set of solution repeated in your answer. It is necessary for you to exclude these repetitions, sometimes different solution sets consist partly of common values. In all such cases the common part must be presented only once.

PROFILES OF TRIGONOMETRIC EQUATIONS (ADVANCED)

Solving equations by introducing an Auxiliary argument

Equations of the type $a\sin x + b\cos x = c, -\sqrt{a^2+b^2} \le c \le \sqrt{a^2+b^2}$, can be solved by dividing both the sides by $\sqrt{a^2+b^2}$.

Equations which can be solved by substitution

Equations of the form $R(\sin\theta \pm \cos\theta, \sin\theta\cos\theta) = 0$, where R($x$, z) is a polynomial, can be solved by the substitution $\cos x \pm \sin x = t \,\&\, \sin x.\cos x = \mp\frac{t^2-1}{2}$.

Solving equations with the use of Boundary Values of the functions involved

An equation of the form f(x) = g(y) can be solved if min. of f(x) = max. of g(x) = k or vice versa. In this situation we can break down the equation into two equations as f(x) = k & g(y) = k.

Simultaneous trigonometric equations

Equations of type f(x) = a & g(x) = b, where f(x) & g(x) are functions of T – Ratios and a & b are numbers such that both f(x) & g(x) can acquire these values for some common angle…

Step I: Identify an angle satisfying the given pair of equations in first 4 quadrants.

Step II: now form a series of angles by recurrently adding 2π.

Step III: Notice that these angles are in A.P., hence generalize using concept of A.P.

Simultaneous equations in two variables

Step I: Solve the equations independently.

Step II: solve as algebraic simultaneous equation.