Definition

Integration is the inverse process of differentiation. The process of finding $f(x)$, when its derivative is $f'(x)$ is given is known as integration.

1. Integrals Anti–Derivative

If $f(x)$ is a differentiable function such that $f'(x) = g(x)$, then integration of $g(x)$ w.r.t. x is $f(x) + c$. Symbolically it is written as $\int g(x)dx = f(x) + c$, here c is known as constant of integration and it can take any real value.

Function $f(x)$ (Integrand)	**Integration** $\int f(x)dx$
constant k	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c \ (n \neq -1)$
$\frac{1}{x} \ (x \neq 0)$	$\ln\lvert x \rvert + c$
$a^x \ (a > 0, a \neq 1)$	$\frac{a^x}{\ln a} + c$
e^x	$e^x + c$
sin x	$-\cos x + c$
cos x	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\text{cosec}^2 x$	$-\cot x + c$

sec x tan x	sec x + c
cosec x cot x	–cosec x + c
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x + c$
$\frac{1}{1+x^2}$	$\tan^{-1}x + c$
$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1}x + c$

2. Some Theorems

1. Two integrals of the same function can differ only by a constant.

2. (i) $\int [af(x)] + [bg(x)]dx = a\int f(x)dx + b\int g(x)dx$, where a and b are constants.

(ii) $\int f(x)dx = g(x) + c$, then $\int f(ax+b)dx = \frac{1}{a}g(ax+b) + c$, where a and b are constants and $a \neq 0$.

Illustration 1:

Evaluate: $\int \left(\sqrt{3}\sin x - \cos x\right)dx$.

Solution:

$$\int\left(\sqrt{3}\sin x-\cos x\right)dx$$

$$=\sqrt{3}\int\sin x\,dx-\int\cos dx$$

$$=-\sqrt{3}\cos x-\sin x+c$$

$$=-2\left[\cos x.\cos\frac{\pi}{6}+\sin x.\sin\frac{\pi}{6}\right]+c$$

$$=-2\cos\left(x-\frac{\pi}{6}\right)+c$$

Illustration 2:

Evaluate: $\int\sec^2(3x+5)\,dx$

Solution:

We know that $\int\sec^2 x\,dx=\tan x+c$

So $\int\sec^2(3x+5)\,dx=\frac{1}{3}\tan(3x+5)+c$

3. Integration by Substitution

It is not always possible to find the integral of a complicated function only by observation, so we need some methods of integration and integration by substitution is one of them. This method has 3 parts:

(i) Direct substitution **(ii)** Standard substitution **(iii)** Indirect substitution

3.1 Direct Substitution

If $\int f(x)dx=g(x)+c$, then in $I=\int f\left(h(x)\right)h'(x)dx$,

we put $h(x)=t\Rightarrow h'(x)dx=dt$

So $\quad I=\int f(t)dt = g(t)+c = g(h(x))+c$

Illustration 3:

Evaluate: $\int cot x\, dx$

Solution:

$$I = \int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x}$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

So $I = \int \frac{dt}{t} = \ell n\, |t| + c = \ell \text{n}\, |\sin \text{x}| + \text{c}$

Illustration 4:

Evaluate: $\int \frac{dx}{2\sqrt{x}\,(x+1)}$

Solution:

Put $x = \text{t}^2 \Rightarrow dx = 2t\, dt$

so $\quad I = \int \frac{dx}{2\sqrt{x}\,(\text{x}+1)} = \int \frac{2t\, dt}{2t\,(t^2+1)}$

$$= \int \frac{dt}{1+t^2} = tan^{-1} t + c = tan^{-1}\left(\sqrt{x}\right) + c$$

3.2 Standard Substitution

In some standard integrand or a part of it, we have standard substitution. List of standard substitution is as follows:

Integrand	Substitution
$x^2 + a^2$ or $\sqrt{x^2+a^2}$	$x = a \tan \theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a \sec \theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a \sin \theta$ or $x = a \cos \theta$
$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a \cos 2\theta$
$\left(x \pm \sqrt{x^2 \pm a^2}\right)^n$	expression inside the bracket = t
$\frac{2x}{a^2-x^2}, \frac{2x}{a^2+x^2}, \frac{a^2-x^2}{a^2+x^2}$	$x = a \tan \theta$
$2x^2 - 1$	$x = a \cos \theta$
$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}} (n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

Illustration 5:

$$\text{Evaluate:} \int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$$

Solution:

$$I = \int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}} = \int \frac{dx}{\left(\frac{x+3}{x-4}\right)^{15/16}(x-4)^2}$$

Put $\frac{x+3}{x-4}=t \quad \Rightarrow \left(\frac{(x-4)-(x+3)}{(x-4)^2}\right)dx=dt$

$\Rightarrow \frac{dx}{(x-4)^2}=\frac{dt}{-7}$

So $I=\frac{-1}{7}\int\frac{dt}{t^{15/16}}=\frac{-1}{7}\int t^{-15/16}dt$

$=\frac{-16}{7}t^{1/16}+c=\frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{1/16}+c$

Illustration 6:

Evaluate: $\int\frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$

Solution:

$I=\int\frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$

Put $x+\sqrt{x^2-4}=t$

$\Rightarrow\left(1+\frac{x}{\sqrt{x^2-4}}\right)dx=dt \because x+\sqrt{x^2-4}=t\Rightarrow\sqrt{x^2-4}=t-x$

$\Rightarrow x=\frac{t^2+4}{2t}\Rightarrow\sqrt{x^2-4}=\frac{t^2-4}{2t}$

so $I=\int\left(\frac{t^2-4}{2t^2}\right)\frac{1}{t^{5/3}}dt=\frac{1}{2}\int t^{-5/3}dt-2\int t^{-11/3}dt$

$$= \frac{1}{2}\frac{t^{-2/3}}{-2/3} - 2\frac{t^{-8/3}}{-8/3} + c = \frac{3}{4}t^{-8/3}[1-t^2]+c$$

Where t = $\left(x+\sqrt{x^2-4}\right)$

3.3 Indirect Substitution

If integrand $f(x)$ can be rewritten as product of two functions

$f(x) = f_1(x)\ f_2(x)$, where $f_2(x)$ is a function of integral of $f_1(x)$, then put integral of $f_1(x) = t$.

Illustration 7:

Evaluate: $\int \sqrt{\frac{x}{4-x^3}}\,dx$

Solution:

$$I = \int \sqrt{\frac{x}{4-x^3}}\,dx = \int \frac{\sqrt{x}\,dx}{\sqrt{4-x^3}}$$

Here integral of $\sqrt{x} = \frac{2}{3}x^{3/2}$ and $4 - x^3 = 4 - (x^{3/2})^2$

Put $x^{3/2} = t \Rightarrow \sqrt{x}dx = \frac{2}{3}dt$

So $I = \frac{2}{3}\int \frac{dt}{\sqrt{4-t^2}} = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right) + c$

Illustration 8:

Evaluate: $\int (\cos x - \sin x)\ (3 + 4\sin 2x)\,dx$

Solution:

$$I = \int (\cos x - \sin x)\ (3 + 4\sin 2x)\,dx$$

Here integration of cos x – sin x = sin x + cos x

and 3 + 4 sin 2x = 3 + 4((sin x + cos x)^2 – 1)

Put sin x + cos x = 1 = (cos x – sin x)dx = dt

So $I = \int (3 + 4(t^2 - 1)\,dt = \frac{t}{3}[4t^2 - 3] + c$

$$= \left(\frac{\sin x + \cos x}{3}\right)[4(\sin x + \cos x)^2 - 3] = \left(\frac{\sin x + \cos x}{3}\right)(1 + 4\sin 2x) + c$$

4. Integration by Parts

If integrand can be expressed as product of two functions, then we use the following formula. $\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x) - \int f_1'(x)\left(\int f_2(x)dx\right)dx$, where $f_1(x)$ and $f_2(x)$ are known as first and second function respectively.

Remarks:

(i) We do not put constant of integration in 1st integral; we put this only once in the end.

(ii) Order of $f_1(x)$ and $f_2(x)$ is normally decided by the rule ILATE, where I $\rightarrow$ Inverse, L $\rightarrow$ Logarithms, A $\rightarrow$ Algebraic, T $\rightarrow$ Trigonometric and E $\rightarrow$ Exponential.

Illustration 9:

Evaluate: $\int x^2 \sin x\,dx$

Solution:

$$\int x^2 \sin x\, dx$$

$$= x^2 \int \sin x\, dx - \int (2x \int \sin x\, dx)\, dx$$

$$= -x^2 \cos x + 2[x] \cos x\, dx - \int (1 \int \cos x\, dx) dx = -x^2 \cos x + 2x \sin x - 2\cos x + c$$

Illustration 10:

Evaluate: $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$

Solution:

$$I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$$

Here $\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) = \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+9}}\right)$

Put 2x + 2 = tan θ

$dx = \frac{3}{2}\sec^2\theta\, d\theta$. Also $\frac{2x+2}{\sqrt{(2x+2)^2+9}} = \frac{3\tan\theta}{3\sec\theta} = \sin\theta$

So

$$I = \frac{3}{2}\int \theta \sec^2\theta\, d\theta = \frac{3}{2}\left[\theta \int \sec^2\theta - \int (1 \int \sec^2\theta\, d\theta)\, d\theta\right] = \frac{3}{2}[\theta \tan\theta + \ell n(\cos\theta)] + c$$

$$= \frac{3}{2}\left[\frac{2x+3}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) + \ell n\left(\frac{3}{\sqrt{4x^2+8x+13}}\right)\right] + c$$

4.1 Special Use of Integration by Parts

(i) $\int f(x)dx = \int (f(x)).1dx$

Now integrate taking $f(x)$ as 1st function and 1 as 2nd function.

(ii) $\int \frac{f(x)}{g(x)^n} dx = \int \frac{f(x)}{g'(x)} . \frac{g'(x)}{g(x)^n} dx$

Now integrate taking $\frac{f(x)}{g'(x)}$ as 1st function and $\frac{g'(x)}{g(x)^n}$ as 2nd function.

(iii) If integrand is of the form $e^x f(x)$, then rewrite $f(x)$ as sum of two functions in which one is derivative of other.

$$\int e^x f(x)dx = \int e^x (g(x) + g'(x))dx = e^x g(x) + c$$

Illustration 11:

Evaluate: $\int \ell n\, x\, dx$

Solution:

$$I = \int \ell n\, x\, dx = \int (\ell n\ x.1)\, dx = \ell n\ x.x - \int \frac{1}{x}.x\, dx = x\,\ell n\, x - x + c = x(\ell n\, x - 1) + c$$

Illustration 12:

Evaluate: $\int \frac{x^2}{(x\sin x + \cos x)^2}$

Solution:

$$I = \int \frac{x^2}{(x\sin x + \cos x)^2} = \int x.\sec x \left(\frac{x\cos x}{(x\sin x + \cos x)^2} \right) dx = \frac{-x\sec x}{x\sin x + \cos x} + \tan x + c$$

Illustration 13:

Evaluate: $\int \left(\frac{x-1}{x^2+1}\right)^2 e^x \, dx$

Solution:

$$I = \left(\frac{x-1}{x^2+1}\right)^2 = \frac{x^2-2x+1}{(x^2+1)^2} = \frac{1}{(x^2+1)} + \left(\frac{-2x}{(x^2+1)}\right)$$

Here derivative of $\frac{1}{x^2+1}$ is $\frac{-2x}{(x^2+1)^2}$. So $\int e^x \left(\frac{x-1}{x^2+1}\right)^2 dx = \frac{e^x}{(x^2+1)} + c$

5. Integration by Partial Fractions

When integrand is a rational function i.e. of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are the polynomials functions of x, we use the method of partial fraction.

If degree of $f(x)$ is less then degree of $g(x)$ and

$g(x) = (x-a_1)^{\alpha_1} \ldots\ldots.(x^2+b_1x+c_1)^{\beta_1} \ldots..,$

then we can put $\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \ldots. + \frac{A_{\alpha_1}}{(x-a)^{\alpha_1}} + \ldots.$

$$+\frac{B_1x+C_1}{(x^2+b_1x+c_1)} + \frac{B_2x+C_2}{(x^2+b_1x+c_1)^2} + \ldots.. + \frac{B_{\beta_1}x+C_{\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}} + \ldots$$

Here A_1, A_2,..., A_{α_1},..., B_1, B_2,....B_{β_1}, C_1, C_2....C_{β_1} .... are the real constants and easily calculated.

Now the function is easily integrated.

If degree of $f(x)$ is more than or equal to degree of $g(x)$, then divide $f(x)$ by $g(x)$ so that the remainder has degree less than of $g(x)$.

Illustration 14:

Evaluate: $\int \frac{dx}{(x-1)\,(x-2)\,(x-3)}$

Solution:

Put $\frac{1}{(x-1)\,(x-2)\,(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$\Rightarrow 1 = A\,(x-2)\,(x-3) + B\,(x-1)\,(x-3) + C\,(x-1)\,(x-2)$

Put $x = 1$, we get, $A = \frac{1}{2}$

$x = 2$, we get, $B = -1$

$x = 3$, we get, $C = \frac{1}{2}$

So integral $= \frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-3} = \ell n\left(\frac{\sqrt{x^2-4x+3}}{|x-2|}\right) + c$

Illustration 15:

Evaluate: $\int \frac{dx}{(x+2)(x^2+1)}$

Solution:

Let $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x^2+1)} \Rightarrow 1 = A\,(x^2+1) + (Bx + C)\,(x+2)$

Put x = – 2, we get A = $\frac{1}{5}$

Now compare the coefficients of x^2 and constant term we get 0 = A + B and 1 = A + 2C

$$\Rightarrow B = \frac{1}{5}, C = \frac{2}{5} \text{ . So } I = \frac{1}{5}\int \frac{dx}{x+2} - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{2}{5}\int \frac{dx}{x^2-1}$$

$$= \frac{1}{5}\ell n|x+2| - \frac{1}{10}\ell n(x^2+1) + \frac{2}{5}\tan^{-1}x + C$$

Illustration 16:

Evaluate: $\int \frac{x^4 dx}{(x-1)(x+1)^2}$.

Solution:

Here degree of numerator is more than the degree of denominator so first we have to divide it to reduce it to proper fraction.

$$\frac{x^4}{(x-1)(x+1)^2} = (x-1) + \frac{2x^2-1}{(x-1)(x+1)^2}$$

Put $\frac{2x^2-1}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$

$\Rightarrow 2x^2 - 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

Put $x = 1$, we get $A = \frac{1}{2}$

Put $x = -1$, we get $C = -\frac{1}{2}$

Comparing the coefficient of x^2, we get $2 = A + B \Rightarrow B = \frac{3}{2}$

So $I = \int (x-1)\,dx + \frac{1}{2}\int \frac{dx}{(x-1)} + \frac{3}{2}\int \frac{dx}{(x+1)} - \frac{1}{2}\int \frac{dx}{(x+2)^2}$

$$= \frac{x^2}{2} - x + \frac{1}{2}\ell n\,|x-1| + \frac{3}{2}\ell n|x+1| + \frac{1}{2(x+2)} + C$$

6. Algebraic Integrals

Using the technique of standard substitution and integration by parts, we can derive the following formula:

(i) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$

(iii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ (iv) $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left[x+\sqrt{x^2+a^2}\right] + c$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x+\sqrt{x^2-a^2}\right] + c$

(vi) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(vii) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left[x+\sqrt{x^2-a^2}\right] + c$

(viii) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left[x+\sqrt{x^2-a^2}\right] + c$

INTEGRAL OF THE FORM

1. $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\, dx$

Here in each case write $ax^2 + bx + c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$ put $x+\frac{b}{2a}=t$ and use the standard formulae.

Illustration 17:

Evaluate: $\int \frac{dx}{\sqrt{-x^2+4x+6}}$

Solution:

$-x^2 + 4x + 6 = -(x^2 - 4x + 4) + 10 = 10 - (x - 2)^2$

$I = \int \frac{dx}{\sqrt{10-(x-2)^2}}$ Put $x - 2 = t \Rightarrow dx = dt$

$I = \int \frac{dt}{\sqrt{10-t^2}} = \sin^{-1}\frac{t}{\sqrt{10}} + c = \sin^{-1}\left(\frac{x-2}{\sqrt{10}}\right) + c$

Illustration 18:

Evaluate: $\int \sqrt{3x^2-6x+10}\, dx$

Solution:

$3x^2 - 6x + 10 = 3(x - 1)^2 + 7$

Put $x - 1 = t$

$\Rightarrow dx = dt$

$$I = \sqrt{3}\int \sqrt{t^2 + \frac{7}{3}}\, dt = \sqrt{3}\left[\frac{t}{2}\sqrt{t^2 + \frac{7}{3}} + \frac{7}{6}\ell n\left|t + \sqrt{t^2 + \frac{7}{3}}\right|\right] + c$$

where $t = x - 1$

2. $\int \frac{(ax+b)dx}{\sqrt{cx^2+ex+f}}, \int \frac{(ax+b)dx}{cx^2+ex+f}, \int (ax+b)\sqrt{cx^2+ex+f}\, dx$

Here write $ax + b = A(2cx + e) + B$

Find A and B by comparing, the coefficients of x and constant term.

Illustration 19:

Evaluate: $\int \frac{(3x+5)\,dx}{\sqrt{x^2+4x+3}}$

Solution:

Write $3x + 5 = A\,(2x + 4) + B$

$\Rightarrow A = \frac{3}{2}, B = -1$

So $I = \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+3}} - \int \frac{dx}{\sqrt{x^2+4x+3}}$

In 1st integral put $x^2 + 4x + 3 = t$

$\Rightarrow (2x + 4)\, dx = dt$

$$I = \frac{3}{2}\int \frac{dt}{\sqrt{t}} - \int \frac{dx}{\sqrt{(x+2)^2 - 1}} = 3\sqrt{x^2+4x+3} - \ell n\left|(x+2) + \sqrt{x^2+4x+3}\right| + c$$

3. $\int \frac{(ax^2+bx+c)dx}{\sqrt{(ex^2+fx+g)}}, \int \frac{(ax^2+bx+c)dx}{(ex^2+fx+g)},$

$\int (ax2+bx+c)\sqrt{(ex^2+fx+g)}\,dx$

Here put $ax^2 + bx + c = A(ex^2 - fx + g) + B(2ex + f) + c$ and find the values of A, B and C by comparing the coefficients of x^2, x and constant term.

Illustration 20:

Evaluate: $\int \frac{(x^2+4x+7)}{\sqrt{x^2+x+1}}$

Solution:

Let $x^2 + 4x + 7 = A(x^2 + x + 1) + B(2x + 1) + C$

Comparing the coefficients of x^2, x and constant term, we get

$A = 1, A + 2B = 4, A + B + C = 7 \Rightarrow A = 1, B = \frac{3}{2}, C = \frac{9}{2}$

So $I = \int \sqrt{x^2+x+1}\,dx + \frac{3}{2}\int \frac{(2x+1)\,dx}{\sqrt{x^2+x+1}} + \frac{9}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$

Now $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$I = \left(\frac{x+\frac{1}{2}}{2}\right) + \sqrt{x^2+x+1} + \frac{3}{8}\ell n\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$

$+3\sqrt{x^2+x+1}\frac{9}{2}\ell n\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+c$

4. $\int \frac{dx}{(ax+b)\sqrt{ex^2 + fx + g}}$ for these types integral put $ax+b = \frac{1}{t}$.

Illustration 21:

Evaluate: $\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

Solution:

Put $x + 2 = \frac{1}{t} \Rightarrow dx = \frac{-dt}{t^2}$

Now $x^2 + 4x + 8 = (x + 2)^2 + 4$

So

$$I = \int \frac{-dt}{t\sqrt{\frac{1}{t^2}+4}} = -\int \frac{dt}{\sqrt{1+4t^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2+\frac{1}{4}}} = -\frac{1}{2}\ell n\left|t+\sqrt{t^2+\frac{1}{4}}\right| + c$$

$$= -\frac{1}{2}\ell n\left|\frac{1}{x+2}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + c$$

5. $\int \frac{(ax+b)dx}{(cx+e)\sqrt{ex^2 + fx + g}}$ Here put $(ax + b) = A(cx + e) + B$, find the values of A and B by comparing the coefficients of x and constant term.

Illustration 22:

Evaluate: $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+8}}$

Solution:

Let $4x + 7 = A(x + 2) + B$

$\Rightarrow A = 4, B = -1$

So $I = 4\int \frac{dx}{\sqrt{x^2+4x+8}} - \int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$$= 4\,\ell n\left(x+2+\sqrt{x^2+4x+8}\right) + \frac{1}{2}\,\ell n\left|\frac{1}{x+2} + \sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + c$$

6. $\int \frac{(ax^2+bx+c)dx}{(ex+f)\sqrt{gx^2+hx+i}}$

Here put $ax^2 + b + c$ = A($ex + f$)($2gx + h$) + B($ex + f$) + C, find the values of A, B and C by comparing the coefficients of x^2, x and constant term.

Illustration 23:

Evaluate: $\int \frac{2x^2+7x+11}{(x+2)\sqrt{x^2+4x+8}}$

Solution:

Put $2x^2 + 7x + 11 = A(x + 2)(2x + 4) + B(x + 2) + C$

Compare the coefficient of x^2, x and constant term, we get

$A = 1$, $7 = 8A + B$, $C + 2B + 8A = 11 \Rightarrow B = -1$, $C = 5$

So $I = \int \frac{2x+4}{\sqrt{x^2+4x+8}} - \int \frac{dx}{\sqrt{x^2+4x+8}} + 5\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$$=2\sqrt{x^2+4x+8}-\ell n\left|(x+2)+\sqrt{x^2+4x+8}\right|-\frac{5}{2}\ell n\left|\frac{1}{(x+2)}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right|+c$$

7. $\int \frac{xdx}{(ax^2+b)\sqrt{(cx^2+e)}}$, here put $cx^2 + e = t^2$ and integrate.

Illustration 24:

Evaluate: $\int \frac{x\,dx}{(2x^2+3)\sqrt{x^2-1}}$

Solution:

Put $x^2 - 1 = t^2$

$\Rightarrow x\,dx = t\,dt$

So $I = \int \frac{tdt}{(2t^2+5)t} = \int \frac{dt}{2t^2+5} = \frac{1}{2}\int \frac{dt}{t^2+\frac{5}{2}} = \frac{1}{10}\tan^{-1}\left(\sqrt{\frac{2}{5}}\sqrt{x^2-1}\right)+c$

8. $\int \frac{dx}{(ax^2+b)\sqrt{(cx^2+e)}}$, here 1st put $x=\frac{1}{t}$ and then expression inside the square root as y^2.

Illustration 25:

Evaluate: $\int \frac{dx}{(x^2+5)\sqrt{2x^2-3}}$

Solution:

Put $x=\frac{1}{t}$ $\quad\Rightarrow dx = -\frac{dt}{t^2}$

So $I = \int \frac{-dt}{t^2\left(\frac{1}{t^2}+5\right)\sqrt{\frac{2}{t^2}-3}} = \int \frac{-t dt}{(1+5t^2)\sqrt{2-3t^2}}$

Put $2 - 3t^2 = y^2 \Rightarrow -t\,dt = \frac{ydy}{3}$

So $I = -\frac{1}{3}\int \frac{y\,dy}{\left(\frac{13-5y^2}{3}\right)y} = \frac{1}{5}\ell n\left|\frac{y-\sqrt{13/5}}{y+\sqrt{13/5}}\right| + C$

9. $\int x^m(a+bx^n)^p dx \ \ (p \neq 0)$, here 4 cases arise

Case I: If p is a natural number, then expand $(a + bx^n)^p$ by binomial theorem and integrate.

Case II: If p is a negative integer and m and n are rational number, put $x = t^k$, when k is the LCM of denominators of m and n.

Case III: If $\frac{m+1}{n}$ is an integer and p is rational number, put $(a + bx^n) = t^k$, when k is the denominator of p.

Case IV: If $\frac{m+1}{n}$ is an integer, put $\frac{a+bx^n}{x^n} = t^k$, where k is the denominator of p.

Illustration 26:

Evaluate: $\int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1}$

Solution:

Here $p = -1$, is a negative integer and m and n are rational numbers.

Put $\quad x = t^3$

$\Rightarrow \quad dx\ 3t^2\ dt$

So $\quad I = \int t^{-2}(1+t^2)^{-1}\,3t^2 dt = \int \frac{3\,dt}{1+t^2} = 3\tan^{-1}(x^{1/3}) + c$

Illustration 27:

Evaluate: $\int x^{\frac{-1}{3}}\left(1+x^{\frac{1}{3}}\right)^{1/4} dx.$

Solution:

Here $m = -\frac{1}{3}, n = \frac{1}{3}, p = \frac{1}{4}$

$\frac{m+1}{n} = 2$, which is an integer

So $\quad \left(1+x^{1/3}\right) = t^4 \Rightarrow \frac{dx}{3x^{2/3}} = 4t^3 dt$

$$I = 12\int (t^4-1)t^4\,dt = -\frac{4}{15}\left(1+x^{1/3}\right)^{5/4}[4+9x^{1/3}] + c$$

Illustration 28:

Evaluate: $\int x^{-11}(1+x^4)^{-1/2}\,dx$

Solution:

Here $m = -11$, $m = 4$, $p = -\frac{1}{2}$

$\frac{m+1}{n} + p = -\frac{10}{4} - \frac{1}{2} = -3$, which is an integer.

So put $\frac{1+x^4}{x^4}=t^2 \Rightarrow 1+\frac{1}{x^4}=t^2 \Rightarrow \frac{-4}{x^5}dx=2t\,dt$

So $\quad I=\int \frac{dx}{x^{13}\left(1+\frac{1}{x^4}\right)^{1/2}}=-\frac{1}{4}\int (t^2-1)^2.\frac{1}{t}.2t\,dt$

$$=-\frac{1}{2}\int (t^2-2t^2+1)\,dt=\frac{t^5}{-10}+\frac{t^3}{3}-\frac{t}{2}+c$$

Where $t=\sqrt{1+\frac{1}{x^4}}$.

7. Trigonometric Integrals of the Form

1. $\int\left(\frac{f(\sin x,\cos x)}{g(\sin x,\cos x)}\right)dx=\int \mathrm{R}(\sin x,\cos x)dx$, where f and g both are polynomials in sin x and cos x. Here we can convert them in algebraic by putting $\tan\frac{x}{2}$ after writing $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

Some-time instead of putting the above substitution we go for below procedure.

(i) If R(–sin x, cos x) = – R(sin x, cos x), put cos x = t

(ii) If R(sin x, – cos x) = R(sin x, cos x) put tan x = t

(iii) If R(–sin x, cos x) = R(sin x, cos x) put tan x = t

Illustration 29:

Evaluate: $\int \frac{dx}{\sin x\,(2\cos^2 x-1)}$

Solution:

Here R(sin x, cos x) = $\frac{1}{\sin x\,(2\cos^2 x-1)}$

R(sin x, cos x) = $\frac{1}{-\sin x\,(2\cos^2 x-1)}$ = R – (sin x, cos x)

So we put cos = $t \Rightarrow -\sin x\, dx = dt$

$$I = \int \frac{\sin dx}{(1-\cos^2 x)\ (2\cos^2 x-1)} = \int \frac{dt}{(t^2-1)\ (2t^2-1)}$$

$$= \int \frac{dt}{t^2-1} - 2\int \frac{dt}{2t^2-1} = \frac{1}{2}\,\ell n\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{\sqrt{2}}\,\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$$

Illustration 30:

Evaluate: $\int \frac{\cos x\, dx}{\sin^2 x\,(\sin x+\cos x)}$

Solution:

Here R(sin x, cos x) = $\frac{\cos x\, dx}{\sin^2 x\,(\sin x+\cos x)}$

R(– sin x, – cos x) = R(sin x + cos x)

So put tan $x = t \Rightarrow \sec^2 x\, dx = dt$

$$I = \int \frac{\cos x \sec^2 x\, dx}{\sec^2 x \sin^2 x\,(\sin x+\cos x)} = \int \frac{dt}{t^2(1+t)}$$

Let $\dfrac{1}{t^2(1+t)} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{(1+t)}$ or $1 = At(1+t) + B(1+t) + ct^2$

Put $t = 0$, we get $B = 1$, put $t = -1$, we get $C = 1$

compare the coefficients of t^2, we get $0 = A + C \Rightarrow A = -1$

So $I = -\int \dfrac{dt}{t} + \int \dfrac{dt}{t^2} + \int \dfrac{dt}{1+t} = \ell n \left| \dfrac{1+\tan x}{\tan x} \right| - \cot x + c$

2. $\displaystyle\int \left(\frac{p \sin x + q \cos x + r}{a \sin x + b \cos x + c} \right) dx$

Here put

$p \sin x + q \cos x + r = \mathrm{A}(a \sin x + b \cos x + c) + \mathrm{B}(a \cos x - b \sin x) + \mathrm{C}$

Values of A, B and C can be obtained by comparing the coefficients of sin x, cos x and constant term. By this technique the given integral becomes sum of 3 integrals in which 1st two are very easy and in 3rd we can put $\tan \dfrac{x}{2} = t$.

Illustration 31:

Evaluate: $\displaystyle\int \frac{(5 \sin x + 6)\, dx}{\sin x + 2 \cos x + 3}$

Solution:

Let $5 \sin x + 6 = A(\sin x + 2 \cos x + 3) + B(\cos x - 2 \sin x) + C$

Equating the coefficients of sin x, cos x and constant term, we get

$$\left.\begin{aligned} A - 2B &= 5 \\ 2A + B &= 0 \\ 3A + C &= 6 \end{aligned}\right\} \Rightarrow A = 1,\ B = -2,\ C = 3$$

$$I=\int dx-2\int\frac{(\cos x-2\sin x)\,dx}{\sin x+2\cos x+3}+3\int\frac{dx}{\sin x+\cos x+3}x-2\ell n|\sin x+2\cos x+3|+3\ell_1$$

Put $\tan\frac{x}{2}=t\Rightarrow\sec^2\frac{x}{2}dx=2dt$

So

$$\ell_1=\int\frac{2dt}{t^2+2t+5}=\int\frac{2dt}{(t+1)^2+4}=\tan^{-1}\left(\frac{t+1}{2}\right)+C=\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{2}\right)+C$$

3. $\int\sin^p x\cos^q x\,dx$, Where p and q are rational number such that $\frac{p+q-2}{2}$ is a negative integer, then put $\tan x=t$ or $\cot x=t$.

Illustration 32:

Evaluate: $\int\sin^{-7/5}x\cos^{-3/5}dx$

Solution:

Here $p=-\frac{7}{5},\ q=-\frac{3}{5}$

$$\frac{p+q-2}{2}=-2$$

$$I=\int\sin^{-7/5}\cos^{-3/5}x\,dx=\int\frac{\cos^{-3/5}x}{\sin^{-3/5}x\sin^2x}dx=\int(\cot x)^{-3/5}\operatorname{cosec}^2x\,dx$$

Put $\cot x=t\Rightarrow\operatorname{cosec}^2x=-dt$. So $I=-\int t^{-3/5}dt=-\frac{5}{2}(\cot x)^{2/5}+c$

Illustration 33:

If $I_n = \int \tan^n x\,dx$, then prove that $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Solution:

Here $I_n = \int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$

$= \int \tan^{n-2} x(\sec^2 x - 1)\,dx = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= \int \tan^{n-2} x \sec^2 x\,dx - I_{n-2}$

$$I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.