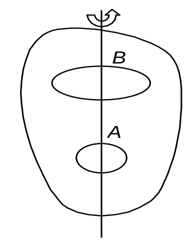
1. MOTION OF RIGID BODIES

A rigid body is a body whose deformation is negligible when subjected to external forces. In a rigid body the distance between any two points remains constant. A rigid body can undergo various types of motion. It may translate, rotate or may translate and rotate at the same time.

When a rigid body translates each particles of rigid body undergo same displacement, velocity and acceleration. To apply equation of translation, all of its mass can be assumed to be concentrated at its center of mass and we can use $\vec{F}_{ext} = M\vec{a}_{CM}$. So our study takes the form as we study in case of particle dynamics.

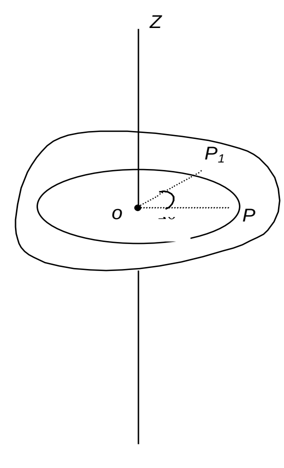
If a body rotates or when it translates and rotates simultaneously motion is different and we shall study it in details.

1.1 ROTATION (ROTATIONAL KINEMATICS)



In this motion, the particles forming the rigid body move in parallel have along circles centred on the axis of rotation. As shown in figure, different position of rigid body is undergoing circular motion in parallel planes.

Consider a rigid body is undergoing rotational motion about *zz'*. A point *P* of the rigid body is undergoing circular motion of radius *OP* = *r* . If during a time interval Δt. the body rotates through an angle $\Delta\theta$, the point *P* will subtend an angle $\Delta\theta$ at the center of motion of its circular path. $\Delta\theta$ is angular displacement of rigid body as well as of point *P*.



Average angular speed during time interval Δt is defined as,

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} \qquad \ldots(\mathbf{1})$$

Instantaneous angular speed is defined as

$$\omega = \underset{\Delta t \to 0}{Lt} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \ldots(\mathbf{2})$$

If the body rotates through equal angle in equal interval of time it is said as rotating uniformly. But if its angular speed changes with time it is said to be accelerated and its angular acceleration is defined as

Average angular acceleration $\quad \overline{\alpha} = \frac{\Delta\omega}{\Delta t} \qquad \ldots(\mathbf{3})$

Instantaneous angular acceleration $\alpha = \frac{d\omega}{dt} \qquad \ldots(\mathbf{4})$

In case of uniform rotation, angular displacement $\Delta\theta = \omega t$

In case of uniformly accelerated rotation following kinematic relations we use,

$$\left.\begin{aligned}\omega &= \omega_0 + \alpha t \\ \Delta\theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \text{ and} \\ \omega^2 &= \omega_0^2 + 2\alpha\Delta\theta\end{aligned}\right] \qquad \ldots(5)$$

Linear Velocity and Linear acceleration of a particle of rigid body

Considering again point P of rigid body. As it is moving in a circular path of radius $OP = r$,

Linear velocity = ωr along the tangent to its circular path

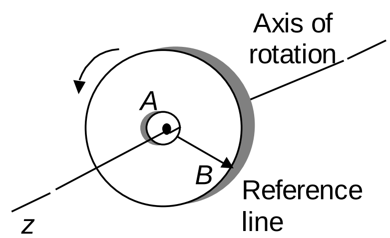
Linear acceleration will have in general two component one in radial direction having magnitude $\omega^2 r$ and one along the tangent having magnitude αr, Therefore in general point P of the rigid body will have net acceleration 'a' given by

$$a = \sqrt{(\omega^2 r)^2 + \alpha^2 r^2} \qquad \ldots(6)$$

If resultant acceleration makes an angle β with OP then $\tan\beta = \dfrac{\alpha r}{\omega^2 r}$

Illustration 1

Starting from rest at time $t = 0$, grindstone has a constant angular acceleration of 3.2 rad/s^2. At $t = 0$ the reference line AB is horizontal. Find (a) the angular displacement of the line AB (and hence of the grindstone and (b) the angular speed of the grindstone, 2.7 s later.



Solution:

(a) At $t = 0$, we have $\omega_0 = 0$, and $\alpha = 3.2$ rad/s^2. Therefore, after 2.7 s,

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 0 + (0)(2.7\text{ s}) + \frac{1}{2}(3.2\text{ rad/s}^2)(2.7\ s)^2$$

$$= 11.7\text{ rad} = 1.9\text{ rev.}$$

(b) Angular speed after 2.7 s

$$\omega_z = \omega_0 + \alpha t = 0 + (3.2\text{ rad/s}^2)(2.7\ s)$$

$$= 8.6\text{ rad/s} = 1.4\text{ rev/s.}$$

Illustration 2

If the radius of the grindstone of previous illustration is 0.24 m, calculate (a) the linear or tangential speed of a point of the rim, (b) the tangential acceleration of a point on the rim, and (c) the radial acceleration of a point on the rim, at the end of 2.7 s. (d) Repeat for a point halfway in from the rim-that is, at $r = 0.12$ m

Solution:

We have $\alpha = 3.2$ rad/s^2, $\omega = 8.6$ rad/s after 2.7 s, and $r = 0.24$ m. Then,

(a) $v_T = \omega r = (8.6 \text{ rad/s})(0.2 \text{ m}) = \mathbf{2.1\ m/s}$,

(b) $a_T = \alpha r = (3.2 \text{ rad/s}^2)(0.24 \text{ m}) = \mathbf{0.77\ m/s^2}$,

(c) $a_R = \omega^2 r = (8.6 \text{ rad/s})^2 (0.24 \text{ m}) = \mathbf{18\ m/s^2}$.

(d) The angular variables are the same for this point at $r = 0.12$ m as for a point on the rim. That is, once again $\alpha = 3.2$ rad/s^2 and $\omega = 8.6$ rad/s. With $r = 0.12$ m, we obtain for this point

$v_T = \mathbf{1.0\ m/s,}$

$a_T = \mathbf{0.38\ m/s^2,}$

$a_R = \mathbf{8.9\ m/s^2.}$

1.2 MOMENT OF A FORCE – TORQUE

To translate a body, we need to apply a force on a body, i.e., cause of translation is force and it is related with linear acceleration of the body as $\vec{F} = M\vec{a}$.

But for rotation, not only the force but its line of action and point of application is also important. Turning effect of force depends on

(i) magnitude of force

(ii) direction of force

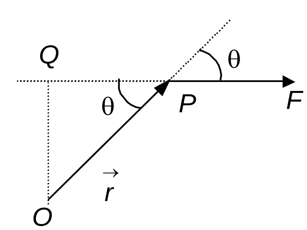
(iii) the distance of force from the axis of rotation

Taking consideration of all these we define torque of a force which gives measure of a turning effect of a force.

Consider a force $\vec{F}$ acting on a body at point P, then turning effect of this force, torque, about point O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \ldots(\mathbf{7})$$

Where $\vec{r}$ is a vector joining O to any point on the line of action of force.



$|\vec{\tau}| = |\vec{r}| \cdot |\vec{F}| \sin\theta$ where θ is angle between $\vec{r}$ and $\vec{F}$ as shown

$$\Rightarrow \qquad |\vec{\tau}| = |\vec{F}|\left(|\vec{r}|\sin\theta\right)$$

$$= |\vec{F}|OQ \quad \text{(refer figure)}$$

Hence, torque about point can also be calculated by multiplying force with the perpendicular distance from the point on the line of action of force. Direction of torque can be obtained by the definition of cross product.

To calculate torque of a force about an axis, we consider a point O on the axis and then we define $\vec{\tau} = \vec{r} \times \vec{F}$ about point O. The component of $\vec{\tau}$ vector along the axis gives the torque about the axis. If a force is parallel to the axis or intersects the axis, its torque about the axis becomes zero. If a force is perpendicular to the axis, we calculate torque as product of magnitude of force and distance of line of action of force from the axis.

1.3 MOMENT OF INERTIA

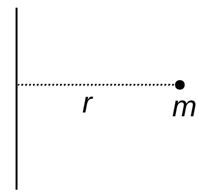
One of the most fundamental characteristics possessed by an object is its intrinsic reluctance to accept a change in its state of motion i.e., its inertia.

A body needs a force to start its translation motion and its translational inertia is better known as mass. Also force is directly proportional to mass of body and linear acceleration of body.

On the other hand the state of motion of a body can undergo change in rotation if a torque is applied. The resulting angular acceleration depends partly on the magnitude of the applied torque, however the same torque applied to different bodies produce different angular acceleration, indicating that each body has an individual amount of rotational inertia which controls the degree of change in motion. The measure of a body's rotational inertia is called **moment of inertia** and it is represented by I. The moment of inertia of a body is a function of the mass of the body, the distribution of that mass and the position of the axis of rotation.

Consider a particle of mass m situated at a distance r from the axis as shown in the figure. Its moment of inertia I is defined as

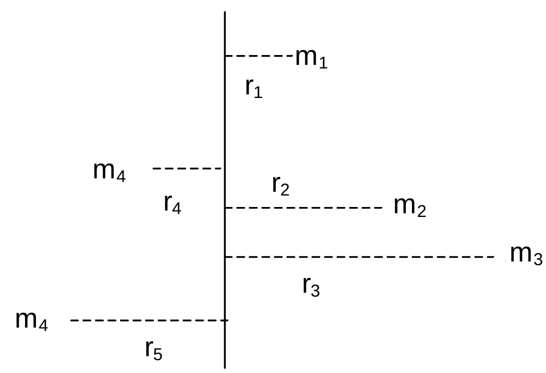
$$I = mr^2 \qquad \ldots(\mathbf{8})$$



If a system of particles is made of number of particles of masses m_1 , m_2, m_3, ... m_n at distances r_1, r_2, r_3 ,... r_n from the axis of rotation, its moment of inertia is defined as

$$\mathrm{I} = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_nr_n^2$$

$$= \sum_{i=1}^{i=n} m_i r_i^2 \qquad \ldots(\mathbf{9})$$



1.4 MOMENT OF INERTIA OF CONTINUOUS BODY

For calculating moment of inertia of a continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice depends on symmetry of body. Consider an element to of the body at a distance r from the axis of rotation. The moment of inertia of this element about the axis we define as $(dm)\, r^2$ and the discrete sum over particles becomes integral over the body:

$$\mathrm{I} = \int (dm) r^2 \qquad \ldots(\mathbf{10})$$

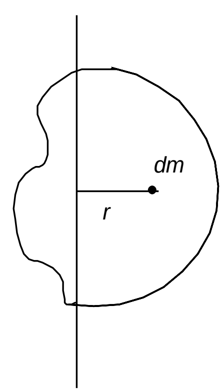


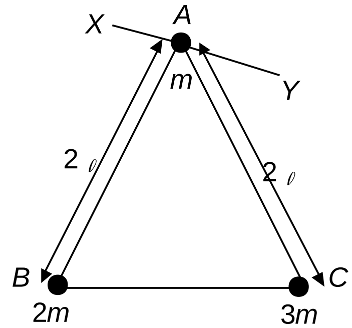
Illustration 3

Three light rods, each of length 2ℓ, are joined together to form a triangle. Three particles A, B, C of mass m, $2m$, $3m$ are fixed to the vertices of the triangle. Find the moment of inertia of the resulting body about

(a) an axis through A perpendicular to the plane ABC,

(b) an axis passing through A and the midpoint of BC.

Solution:



(a) B is distant $2\ \ell$ from the axis XY

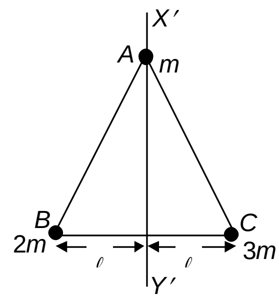
So the moment of inertia of B (I_B) about XY is $2m\,(2\ell)^2$

Similarly I_C about XY is $3m\,(2\ell)^2$

and I_A about XY is $m(0)^2$

Therefore the amount of inertia of the body about XY is

$2m\,(2\ell)^2 + 3m\,(2\ell)^2 + m(0)^2 = 20\,m\ell^2$



(b) I_A about $X'Y' = m(0)^2$

I_B about $X'Y' = 2m(\ell)^2$

I_C about $X'Y' = 3m\,(\ell)^2$

Therefore the moment of inertia of the body about $X'Y'$ is

$m(0)2 + 2m(\ell)^2 + 3m(\ell)^2 = 5\,m\ell^2$

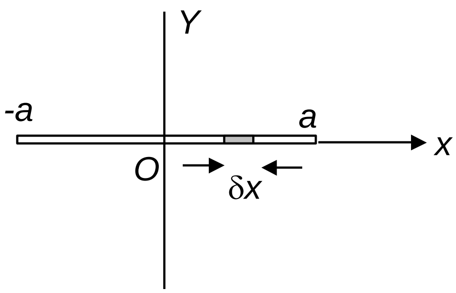
Illustration 4

A rod is of mass M and length 2a. Find moment of inertia about an axis

(a) through the centre of the rod and perpendicular to the rod,

(b) parallel to the rod and distant d from it.

Solution:



Let the rod be divided into elements of length dx, each element being approximately a particle.

(a) For a typical element, mass = $\dfrac{M}{2a}dx$

moment of inertia about $YY' = \left(\dfrac{M}{2a}dx\right)x^2$

Therefore $I_{YY'}$, the moment of inertia of the rod about YY' is given by

$$I_{XY} = \frac{M}{2a}\int_{-a}^{a} x^2 dx = \frac{1}{3}Ma^2$$

(c) In this case every element of the rod is the same distance, d, from the axis XY. The moment of inertia of an element about $XY = \left(\frac{M}{2a}dx\right)\left(d^2\right)$. Therefore the moment of inertia of the rod about $XY = \int\left(\frac{M}{2a}\right)dx\left(d^2\right) = \boldsymbol{Md^2}$

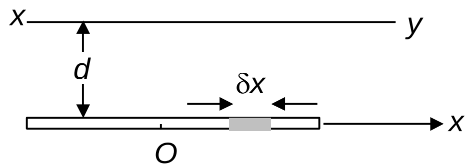
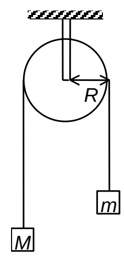


Illustration 5

The pulley shown in Figure has a moment of inertia I about its axis and its radius is R. Find the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.

Solution:

Suppose the tension in the left string is T_1 and that in the right string is T_2. Suppose the block of mass M goes down with an acceleration a and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim.



The angular acceleration of the wheel $\alpha = \frac{a}{R}$. The equations of motion for the mass M and mass m and the pulley are as follows;

$$Mg - T_1 = Ma \qquad \text{.... (i)}$$

$$T_2 - mg = ma \qquad \text{.... (ii)}$$

$$T_1R - T_2R = I\,\alpha = \frac{Ia}{R} \qquad \text{... (iii)}$$

Substituting for T_1 and T_2 from equations (i) and (ii) in equation (iii)

$$[M(g-a) - m(g+a)]R = \frac{Ia}{R}$$

Solving, we get

$$a = \frac{(M-m)gR^2}{I+(M+m)R^2}$$

1.5 RADIUS OF GYRATION

The moment of inertia of any rigid body about a specified axis can be expressed in the form MK^2 where M is the mass of the body and K is a length. This is the same as the moment of inertia of a particle of mass M distant K from the axis, and K is called the radius of gyration of the body about that axis.

i.e., $I = MK^2$ ...(**11**)

Consider, for example, a uniform rod of mass M and length 2ℓ rotating about an axis through it centre and perpendicular to the rod. If I is the moment of inertia of the rod about this axis then

$I = \frac{Ml^2}{3} = M\left(\frac{\ell}{\sqrt{3}}\right)^2$. So radius of Gyration of the rod about axis through its centre of and perpendicular to the rod = $\frac{\ell}{\sqrt{3}}$

Many times we are tempted to replace a rotating rigid body by a particle of equal mass at the centre of gravity, but the above example shows that this does not give the correct result for the moment of inertia.

1.6 TABLE OF MOMENT OF INERTIA

S. No.	Body	Dimension	Axis	Moment of Inertia
1.	Circular ring	radius r	Through its centre and perpendicular to its plane	Mr^2
2.	Circular disc	radius r	Through its centre and perpendicular to its plane	$\frac{Mr^2}{2}$
3.	Solid right circular cylinder	radius r and length ℓ	About the generating axis	$\frac{Mr^2}{2}$
4.	Solid cylinder	Radius r and length ℓ	Through its centre and perpendicular to its length	$M\left[\frac{r^2}{4}+\frac{\ell^2}{12}\right]$
5.	Uniform solid sphere	radius R	About a diameter	$\frac{2}{5}MR^2$
6.	Hollow sphere	Radius R	About a diameter	$\frac{2}{3}MR^2$
7.	Thin uniform rod	Length 2ℓ	Through its centre and perpendicular to its length	$\frac{M\ell^2}{3}$
8.	Thin rectangular sheet (lamina or block)	sides a and b	Through its centre and perpendicular to its plane	$M\left[\frac{a^2}{12}+\frac{b^2}{12}\right]$

1.7 CHANGE OF AXIS

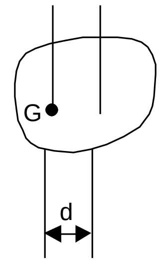
Up to this point we have usually calculated the moment of inertia of a body about an axis which passes through its centre of mass. If the moment of inertia about a different axis is required, we do not always have to go back to first principles. In some cases the following theorem provides an easy way to find the required moment of inertia.

(i) The parallel Axis Theorem

If the moment of inertia of a uniform body of mass M about an axis through G, its centre of mass, is I_G, and I_A is the moment of inertia about a parallel axis through a point A, then

$$I_A = I_G + Md^2 \qquad \ldots(\mathbf{12})$$

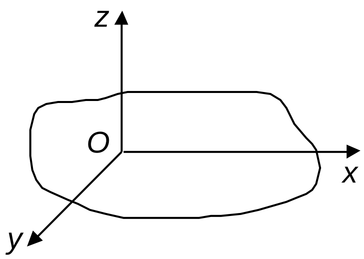
Where d is the distance between the parallel axes.



(ii) The perpendicular Axes Theorem

If a plane body has moments of inertia I_{OX} and I_{OY} about two perpendicular axes, OX and OY, in the plane of the body then its moment of inertia about an axis OZ, perpendicular to the plane, is $I_{OX} + I_{OY}$.

i.e., $$I_{OZ} = I_{OX} + I_{OY} \qquad \ldots(\mathbf{13})$$



Note three axes under consideration must be mutually perpendicular and concurrent, although none of them needs to pass through the centre of mass of the body.

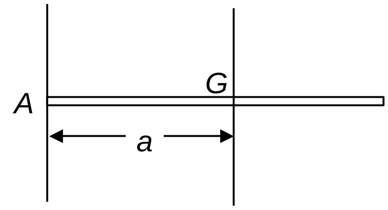
This theorem cannot be applied three-dimensional bodies.

Illustration 6

Use the parallel axis theorem to find the moment of inertia of a uniform rod of mass M and length $2a$, about a perpendicular axis through one end.

Solution:

The moment of inertia, I_G, about an axis through G and perpendicular to the rod is $\frac{1}{3}Ma^2$



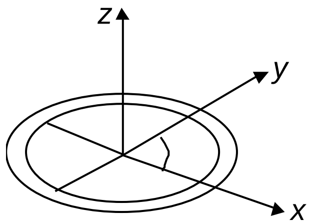
The axis through the end A is a parallel axis, therefore

$$I_A = I_G + Ma^2 = \frac{1}{3}Ma^2 + Ma^2$$

$$= \frac{4}{3}Ma^2$$

Illustration 7

Find the moment of inertia, about a diameter, of a uniform ring of mass M and radius a.



Solution:

We know that the moment of inertia, I_{OZ}, of the ring about OZ is Ma^2. We also know that, from symmetry, the moment of inertia about any one diameter is the same as that about any other diameter,

i.e., $I_{OX} = I_{OY}$

Using the perpendicular axes theorem gives

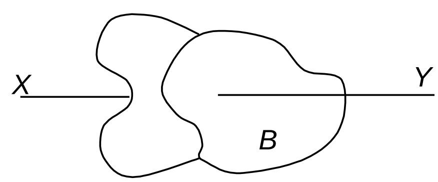
$$I_{OZ} = I_{OX} + I_{OY}$$

$$\Rightarrow \quad Ma^2 = 2I_{OX} = 2I_{OY}$$

The moment of inertia of the ring about any diameter $\frac{1}{2}Ma^2$

Moment of Inertia of Compound Bodies

Consider two bodies A and B , rigidly joined together. The moment of inertia of this compound body, about an axis XY, is required.



If I_A is the moment of inertia of body A about XY.

I_B is the moment of inertia of body B about XY

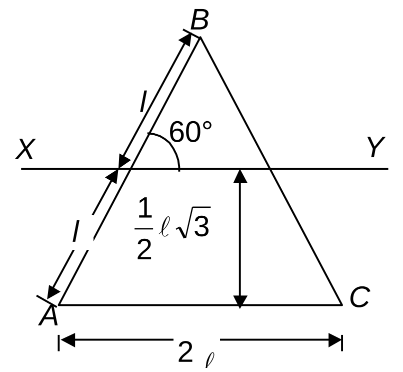
Then, Moment of Inertia of compound body

$$I = I_A + I_B$$

Illustration 8

Three uniform rods, each of length $2l$ and mass M are rigidly joined at their ends to form a triangular framework. Find the moment of inertia of the framework about an axis passing through the midpoints of two of its sides.

Solution:



The rod AB is rotating about an axis through its midpoint and inclined to AB at 60°, therefore

For rod AB, $I_{XY} = \frac{1}{3}M\ell^2 \sin^2 60° = \frac{1}{4}M\ell^2$

Similarly for rod BC $I_{XY} = \frac{1}{4}\ M\ell^2$

The rod AC is rotating about an axis parallel to AC and distant $\frac{1}{2}\ell\sqrt{3}$ from AB, therefore

For rod AC, $I_{XY} = M(\frac{1}{2}\ell\ \sqrt{3})^2 = \frac{3}{4}M\ell^2$

Hence for the whole framework

$$I_{XY} = \frac{1}{4}M\ell^2 + \frac{1}{4}\ M\ell^2 + \frac{3}{4}\ M\ell^2 = \frac{5}{4}\ M\ell^2$$

1.8 BASIC EQUATION OF ROTATION

The relation $\tau = I\alpha$ is the fundamental equation of rotation. It is the exact counterpart of the equation $F = ma$ in linear motion. An unbalanced torque is necessary to give a body an angular acceleration just as an unbalanced force is required to give a body a linear acceleration. The quantities τ, I and α play similar roles in angular motion as F, m and a play in translatory motion. Therefore for rotational motion we use $\tau = I\alpha$.

Illustration 9

A wheel having moment of inertia 2 kg m^2 about its axis rotates at 50 r.p.m. about this axis. Find the torque that can stop the wheel in one minute.

Solution:

Initial angular velocity = 50 r.p.m. $= \frac{50\times 2\pi}{60} = \frac{5\pi}{3}$ radian/s

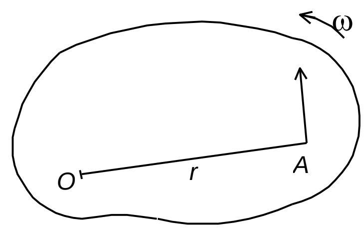
$$\textit{Using, } \omega = \omega_0 + \alpha t \quad \text{we get } \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - \frac{5\pi}{3}}{60} = -\frac{\pi}{36}\ \text{radian/sec}^2$$

The torque that can produce this deceleration = I α

$$= 2\,\text{kg.m}^2 \times \frac{\pi}{36}\text{rad/sec}^2 = \frac{\pi}{18}\text{Nm.}$$

1.9 KINETIC ENERGY OF A BODY ROTATING ABOUT A FIXED AXIS

Consider a body rotating with angular velocity ω about a fixed axis. Figure shows a section of the body taken at right angles to the axis. In the Figure, *O* represents the axis. The body may be considered to be made up of a large number of particles. Let one of the particles of mass *m* be at *A*. The angular velocity of the particle about *O* is ω. If the distance of *A* from *O* is *r*, the linear velocity is $r\omega$. The moment of inertia of the body about the axis of rotation.



The kinetic energy of the particle = $\frac{1}{2}mr^2\omega^2$. The kinetic energy of the whole body is the sum of the kinetic energies of all the particles in it and that is $\sum\frac{1}{2}mr^2\omega^2$. In this summation, ω is the same for all particles. Therefore the total kinetic energy = $\frac{1}{2}\omega^2\sum mr^2$. The quantity $\sum mr^2$ we have already seen to be the moment of inertia about the specified axis. Therefore the kinetic energy of the entire body is $\frac{1}{2}I\omega^2$, where I is

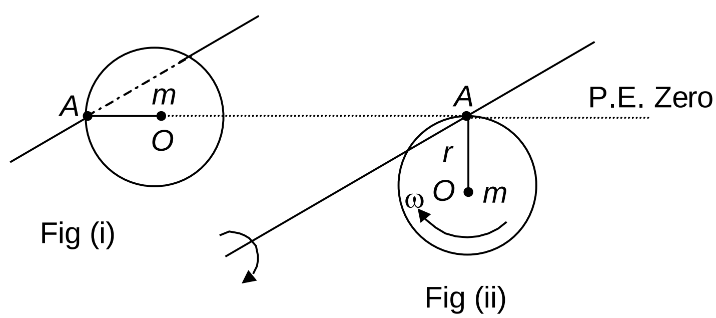
Therefore kinetic energy = $\frac{1}{2}I\omega^2$ …(**15**)

Illustration 10

A uniform circular disc of mass *m*, radius *r* and centre *O* is free to turn in its own plane about a smooth horizontal axis passing through a point *A* on the rim of the disc. The disc is released from rest in the position in which *OA* is horizontal and the disc is vertical. Find the angular velocity of the disc when *OA* first becomes vertical.

Solution:

The moment of inertia of the disc about the axis through *A* perpendicular to the disc is given by



$$I=\frac{1}{2}mr^2+mr^2 \text{ (Parallel axis theorem)}$$

i.e., $$I=\frac{3}{2}mr^2$$

Initially (Figure (i))

K.E. = 0

P.E. = 0

When AO is vertical (Figure (ii) and the angular velocity is ω

$$\text{K.E.} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{3}{2}mr^2\right)\omega^2$$

$$\text{P.E.} = -mgr$$

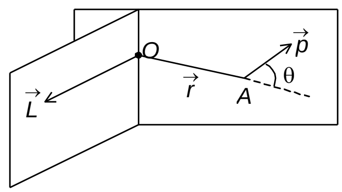
Using the principle of conservation of mechanical energy we have

$$0 + 0 = \frac{3}{4}mr^2\omega^2 - mgr$$

Hence $\omega^2 = 4g/3r$

i.e., $\omega = 2\sqrt{g/3r}$

1.10 ANGULAR MOMENTUM OF A PARTICLE



In translatory motion, the linear momentum of a single particle is expressed as $p = mv$. In rotational motion, the analogue of linear momentum is angular momentum. Consider the case of a particle A (see Figure) having linear momentum p. The angular momentum L of the particle A with respect to a fixed point O as origin is defined as

$$\vec{L} = \vec{r} \times \vec{p} \qquad \ldots \textbf{(16)}$$

where $\vec{r}$ is the vector distance of particle from origin O.

The angular momentum is a vector quantity and its magnitude is given by

$$L = rp \sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{p}$. The direction of $\vec{L}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$.

When a force acts on a particle A, the torque about O is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

But $\vec{F} = \frac{d}{dt}(m\vec{v})$

∴ $\vec{\tau} = \vec{r} \cdot \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d}{dt}(\vec{L})$

∴ Torque is also the rate of change of angular momentum.

When $\vec{\tau} = 0$, $\frac{dL}{dt} = 0$

or L = constant.

So if total torque on a particle is zero the angular momentum of the particle is conserved. This is known as principle of conservation of angular momentum. Angular Momentum of a rotating rigid can be written as

$$L = I\omega$$

Illustration 11

A horizontal platform with a mass 100 kg rotates at 10 rpm around a vertical axis passing through its centre. A man weighing 60 kg is standing on its edge. With what velocity will the platform begin to rotate if the man moves from edge of platform to its centre? Regard the platform as a circular homogeneous disc and the man as a point mass.

Solution:

Let m_1 be the mass of the platform and m_2 be the mass of the man and r the radius of the platform.

Let I_1 be the moment of inertia of platform-man system with the man standing at the edge. Then

$$I_1 = \frac{m_1 r^2}{2} + m_2 r^2$$

Let I_2 be the moment of inertia of the system with the man at the centre of the disc.

$$I_2 = \frac{m_1 r^2}{2} + m_2 (0)^2 = \frac{m_1 r^2}{2}$$

Let ω_1 and ω_2 be the angular velocities of the platform in the two cases respectively. Using the principle of conservation of angular momentum, we get, $I_1 \omega_1 = I_2 \omega_2$

$\omega_1 = 2 \pi n_1$, where n_1 is the initial number of revolutions made by the platform per minute.

Similarly, $\omega_2 = 2 \pi n_2$, where n_2 is the final number of revolutions made by the platform per minute.

Substituting these values of ω_1 and ω_2

$$\left(\frac{m_1 r^2}{2} + m_2 r^2\right) 2\pi n_1 = \frac{m_1 r^2}{2} \cdot 2\pi n_2$$

$$\therefore \quad n_2 = n_1 \cdot \frac{m_1 r^2 + 2m_2 r^2}{m_1 r^2} = n_1 \cdot \frac{m_1 + 2m_2}{m_1} = 10 \times \frac{(100+120)}{100} = \textbf{22 rev/min}$$

1.11 TRANSLATIONAL AND ROTATIONAL QUANTITIES

S. No.	Translational motion	Rotational motion
1.	Displacement = S	Angular displacement = θ
2.	Velocity = v	Angular velocity = ω
3.	Acceleration = a	Angular acceleration = α
4.	Inertia = m	Moment of inertia = I
5.	Force = F	Torque = τ
6.	Linear momentum = mv	Angular momentum = Iω
7.	Power = Fv	Rotational power = $\tau\omega$

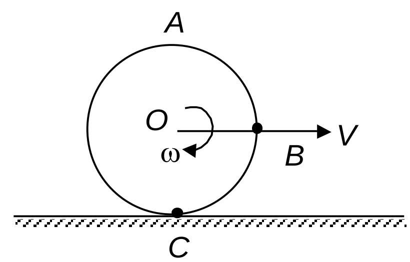
8.	Kinetic energy = $\frac{1}{2}mv^2$	Rotational K.E. = $\frac{1}{2}I\omega^2$

2. COMBINED ROTATIONAL AND TRANSNATIONAL MOTION OF A RIGID BODY: ROLLING MOTION

We already learnt about translation motion caused by a force and rotation about a fixed axis caused by a torque. Now we are going to discuss a motion in which body undergoes translation as well as rotation. Rolling is an example of such motion.

Rolling motion can be considered as combination of rotational and translational motion. For the analysis of rolling motion we deal translation separately and rotation separately and then we combine the result to analyse the overall motion.

Consider a uniform disc rolling on a horizontal surface. Velocity of its center of mass is v and its angular speed is ω as shown:

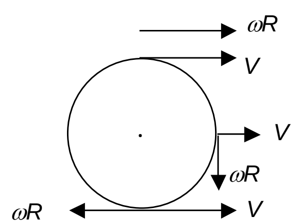


A, B and C are three points on the disc. Due to the translational motion each point A, B and C will move with center of mass in horizontal direction with velocity v. Due to pure rotational motion each point will have tangential velocity ωR , R is radius of disc. When the two motions are combined, resultant velocities of different points are given by

$$V_A = V + \omega R$$

$$V_B = \sqrt{V^2 + \omega^2 R^2}$$

$$V_C = V - \omega R$$



Similarly, if disc rolls with angular acceleration α and if its center of mass has an acceleration 'a' different points will have accelerations given by:

$$a_A = a + \alpha R$$

$$a_B = \sqrt{a^2 + \alpha^2 R^2}$$

$$a_C = a - \alpha R$$

To write equations of motion for rolling motion, we can apply $\vec{F}_{ext} = M\vec{a}_{CM}$ for translation motion and $\tau = I\alpha$ about axis passing through center of mass of body.

Rolling motion is possible in two ways – rolling without slipping and rolling with slipping. There is no relative motion at contact in case of rolling without slipping, while in case of rolling with slipping, relative motion takes place between contact points.

In the taken example, if rolling is without slipping we will have

$$V_c = 0 \Rightarrow \quad V = \omega R$$

and, $a_c = 0 \Rightarrow \quad a = \alpha R$

If rolling is with slipping, $V_c \neq$ and $a_c \neq 0$.

One more important distinction between these two kinds of rolling motion is in case of rolling with slipping the frictional force is a known force of magnitude μN, while in case of rolling without slipping frictional force will be unknown force may take any value between zero and μN

2.1 KINETIC ENERGY OF A ROLLING BODY

If a body of mass M is rolling on a plane such that velocity of its centre of mass is V and its angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 \qquad \ldots(\mathbf{17})$$

I is moment of inertia of body about axis passing through centre of mass.

In case of rolling without slipping,

$$KE = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}I\omega^2 \quad [\because V = \omega R]$$

$$= \frac{1}{2}\left[MR^2 + I\right]\omega^2$$

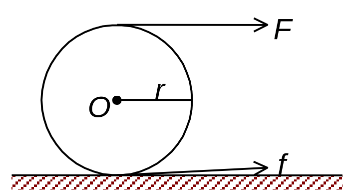
$$= \frac{1}{2}I_c\omega^2$$

I_c is moment of inertia of the body about the axis passing through point of contact.

Illustration 12

A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping find the acceleration of the centre of sphere.

Solution:



Suppose that the static friction (f) on the sphere acts towards right. Let r be the radius of sphere and a the linear acceleration of centre of sphere. The angular acceleration about the centre is $\alpha = \frac{a}{r}$ as there is no slipping. For the linear motion of centre,

$$F + f = ma \qquad \ldots \text{(i)}$$

For rotational motion about centre

$$Fr - fr = I\alpha$$

$$I = \frac{2}{5}mr^2 \quad \alpha = \frac{a}{r}$$

$$\therefore \quad r(F - f) = \frac{2}{5}mr^2 \cdot \frac{a}{r}$$

$$F - f = \frac{2}{5}ma \qquad \ldots \text{(ii)}$$

Adding (1) and (2)

$$2F = \frac{7}{5}ma \quad a = \frac{\mathbf{10F}}{\mathbf{7m}}$$

Illustration 13

A sphere of radius r and mass m is released with no initial velocity on the incline whose inclination with horizontal is 30°. It rolls without slipping. Determine

(a) the minimum value of the coefficient of friction compatible with rolling,

(b) the velocity of the centre G of the sphere after the sphere has rolled down 4 m,

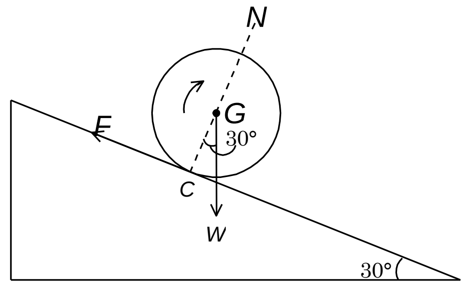
(c) the velocity of G if the sphere were to move 4 m down the same incline when there is no friction.

Solution:

The external forces W, N and F form a system equivalent to the system of effective forces represented by the vector $m\bar{a}$ and couple $I\alpha$.

Since the sphere rolls without slipping, we have $a = r\,\alpha$ where a is linear acceleration and α is angular acceleration and r the radius of sphere.

From the Figure, we see



$$W \sin\theta \cdot r = Ic\,\alpha$$

$$\therefore\ W \sin\theta \cdot r = \frac{7}{5}mr^2\alpha$$

But $W = mg$

$$mg \sin\theta \cdot r = \frac{7}{5}mr^2\alpha$$

But $\alpha = \frac{a}{r}$

$$mg \sin\theta \cdot r = \frac{7}{5}mr^2 \cdot \frac{a}{r}$$

$$\therefore\ a = \frac{5g\sin\theta}{7} = \frac{5\times 9.8\times \sin 30°}{7} = 3.5\text{ m/s}^2.$$

If F is the force of friction between the sphere and the inclined plane, considering the linear motion

$$mg\sin\theta - F = m\,a$$

$$mg\sin\theta - F = m\cdot\frac{5g\sin\theta}{7}$$

But $F = \mu N$

And $N = mg\cos\theta$

$$\therefore\ mg\sin\theta - \mu\, mg\cos\theta = m\cdot\frac{5g\sin\theta}{7}$$

$$\mu\, mg \cos\theta = mg\sin\theta - \frac{5}{7}mg\sin\theta$$

$$\mu\, mg \cos\theta = \frac{2}{7}mg\sin\theta$$

$$\mu = \frac{2}{7}\tan\theta = \frac{2}{7}\tan 30^\circ = \frac{2}{7}\times\frac{1}{\sqrt{3}}$$

$$\boldsymbol{\mu = 0.165}$$

(b) To calculate the velocity of the centre of the sphere after it has moved a distance 4 m.

Initial velocity = 0

$a = 3.5$ m/s^2

Distance = 4 m

Using $v^2 - u^2 = 2aS$

$$v^2 = 0 + 2 \times 3.5 \times 4 = 28$$

$$v = \mathbf{5.29\ m/s}$$

(c) To find the velocity of sliding sphere (in the absence of friction)

$u = 0$; $S = 4$ m

$$a = g \sin 30^\circ = \frac{g}{2} = 4.9\ \text{m/s}^2$$

$$v^2 = 0 + 2 \times 4.9 \times 4 = 39.2$$

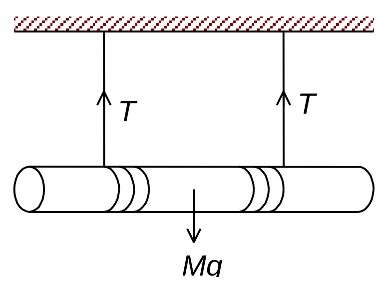
$$v = \sqrt{39.2} = \mathbf{6.26\ m/s}$$

Illustration 14

A cylinder of mass M is suspended through two strings wrapped around it as shown in Figure. Find the tension in the string and the speed of the cylinder as it falls through a distance h.

Solution:

The portion of the strings between ceiling and cylinder are at rest. Hence the points of the cylinder where the strings leave it are at rest also. The cylinder is thus rolling without slipping on the strings. Suppose the centre of cylinder falls with an acceleration a. The angular acceleration of cylinder about its axis given by



$$\alpha = \frac{a}{R} \qquad \ldots\text{(i)}$$

as the cylinder does not slip over the strings. The equation of motion for the centre of mass of cylinder is

$$Mg - 2T = Ma$$

and for the motion about the centre of mass it is

$$2T\cdot R = \left(\frac{MR^2}{2}\right)\alpha \text{ , where } I = \frac{MR^2}{2}$$

$$2\,T\,R = \frac{MR^2}{2}\cdot\frac{a}{R}$$

$$2\,\mathrm{T} = \frac{Ma}{2} \qquad \text{... (ii)}$$

From (i) and (ii) on adding

$$Mg = \frac{Ma}{2} + Ma \quad \frac{3a}{2} = g$$

$$a = \frac{2g}{3}$$

$$\therefore \qquad 2\,T = \frac{M}{2}\cdot\frac{2g}{3}$$

$$\mathbf{T = \frac{Mg}{6}}$$

As the centre of cylinder starts moving from rest, the velocity after it has fallen a height h is given by

$$v^2 = 2\left[\frac{2g}{3}\right]h \quad \text{or} \quad \mathbf{v} = \sqrt{\frac{\mathbf{4gh}}{\mathbf{3}}}$$

2.2 ANGULAR MOMENTUM OF ROLLING BODY

Angular momentum of a rolling body having angular velocity ω and velocity of center of mass v is given by

$$L = MVr + I_{CM}\,\omega$$

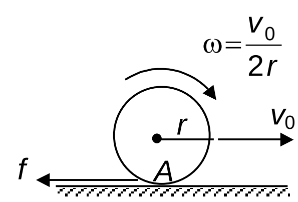
Here r is perpendicular distance of line of motion of mass from the point about which angular momentum is to be calculated.

Note that angular momentum is a vector quantity so while adding the direction of angular momentum should be given proper attention.

Illustration 15

A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and rotational velocity about the centre $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.

Solution:



Let us consider the torque about the initial point of contact *A*. The force of friction passes through this point and hence its torque is zero. The normal force and the weight balance each other. The net torque about *A* is zero. Hence the angular momentum about *A* is conserved. Initial angular momentum is,

$$L = L_{cm} + Mrv_0 = I_{cm}\,\omega + Mrv_0$$

$$= \left(\frac{2}{5}Mr^2\right)\left(\frac{v_0}{2r}\right) + Mrv_0 = \frac{6}{5}Mrv_0$$

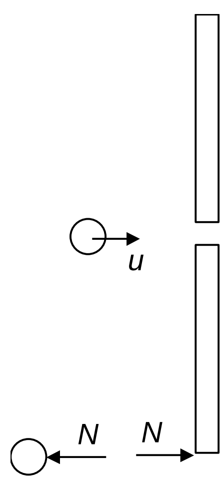
Suppose the translational velocity of the sphere, after it starts rolling, is v. The angular velocity is v/r. The angular momentum about *A* is,

$$L = L_{cm} + Mrv = \left(\frac{2}{5}Mr^2\right)\left(\frac{v}{r}\right) + mrv = \frac{7}{5}Mrv\text{. Thus, } \frac{6}{5}Mrv_0 = \frac{7}{5}Mrv \quad \text{or, } v = \frac{6}{7}v_0\text{.}$$

A body rolling without slipping on a fixed surface can also be analysed as pure rotation about the axis passing through the point of contact.

3. COLLISION OF POINT MASSES WITH RIGID BODIES (ECCENTRIC COLLISION)

In the lesson, Impulse and momentum, we had discussed about central impact in which the line of collision coincided with the line joining the center of mass of colliding bodies. Now we are going to discuss the collision in which the line of collision and line joining center of mass are different, i.e., Eccentric Collision.



Consider a uniform rod of mass *M* and length *L* resting on a frictionless surface. A small disc of mass *m* hits the rod perpendicular to its length near its end as shown in figure. The speed of disc at the time of collision is *u* , Let *e* is coefficient of restitution for the collision.

At the time of collision, forces between the rod and disc is as shown in figure. These forces on disc will cause change in velocity of disc. Let it become v_1. Force on rod will provide translational velocity v to C.M. of rod and on angular speed ω to the rod. Let us find these unknown velocities v_1, ω and v.

Taking the rod and disc as a system, $\sum \vec{F} = 0$, we can apply conservation of linear momentums to get equation:

$$Mu = mv_1 + MV \qquad \ldots \text{(i)}$$

As the forces at the time of collision are equal, opposite and collinear, of these forces torque about CM is zero so we can apply conservation of angular momentum about cm,

$$mu\frac{\ell}{2} = mv_1\frac{\ell}{2} + \frac{M\ell^2}{12}\omega \qquad \ldots \text{(ii)}$$

From the law of restitution we can write ,

$$\left(V+\omega\frac{\ell}{2}\right)-V_1=e(u-0) \qquad \text{... (iii)}$$

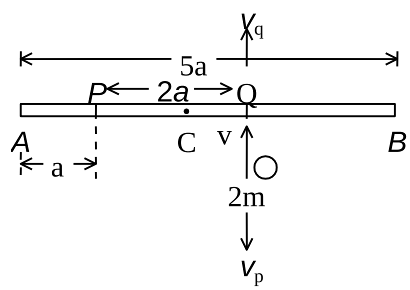
Solving these three equations we can calculate V_1 , ω and v

Here in the taken situation the rod is free to translate and rotate. If the rod were given to rotate about a fixed axis then we would not be able to apply conservation of angular momentum and in such case two unknowns can be calculated using conservation of angular momentum about the axis of rotation and law of restitution.

Illustration 16

A uniform rod *AB* of mass *m* and length 5*a* is free to rotate on a smooth horizontal table about a pivot through *P*, a point on *AB* such that *AP* = *a*. A particle of mass 2m moving on the table strikes *AB* perpendicularly at the point 2a from *P* with speed *v*, the rod being at rest. If the coefficient of restitution between them is $\frac{1}{4}$, find their speeds immediately after impact.

Solution:



Let the point of impact be *Q* so that *PQ* = 2*a*

Let *P* be the point of pivot so that *AP* = *a*

Let the velocities of *Q* and the particle after impact be v_q and v_p respectively. We can apply three principles of motion

1. Conservation of linear momentum

2. Conservation of angular momentum

3. Newton's law of restitution for collision

However the law of conservation of linear momentum will involve the unknown impulsive reaction at P. Hence we use the latter two principles only.

By the law of conservation of angular momentum, the effective impulse on the rod at *Q* is equal to the change in angular momentum of the particle and so

$$2a\,(2mv + 2mv_p) = I_p\omega \qquad \text{… (i)}$$

where I_p is the moment of inertia of AB about P.

$$I_p = \frac{1}{3}m\left(\frac{5a}{2}\right)^2 + m\left(\frac{3a}{2}\right)^2 = \frac{13ma^2}{3}$$

$$\therefore \qquad 4ma\,(v+v_p) = \frac{13ma^2}{3}\omega$$

$$12\,(v+v_p) = 13a\omega \qquad \text{… (ii)}$$

By Newton's law of restitution

$$v_p + v_q = \frac{v}{4} \qquad \text{… (iii)}$$

The angular velocity ω of the rod is such that

$$v_q = 2a\omega \qquad \text{… (iv)}$$

Substituting for v_p from (iii) in (ii)

$$12\left(v+\frac{v}{4}-v_q\right) = 13a\omega \quad 12\left(\frac{5v}{4}-2a\omega\right) = 13a\omega$$

$$15v - 24a\omega = 13a\omega \therefore \omega = \frac{15v}{37a}$$

Substituting back in (iii)

$$v_p = \frac{v}{4}-2a\frac{15v}{37a} = \frac{v}{4}-\frac{30v}{37} = -\frac{83v}{148}$$

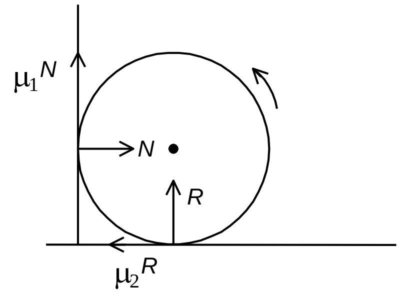
Thus the angular speed of the rod is $\mathbf{\frac{15v}{37a}}$ and the speed of the particle is $\mathbf{\frac{83v}{148}}$ after impact.

Solved Examples

Example 1.

A uniform cylinder of radius r is rotating about its axis with an angular velocity ω. It is now placed in a corner between a vertical wall and horizontal floor as shown in Figure. The coefficient of friction between wall and cylinder is μ_1 and that of cylinder and floor is μ_2. Calculate the number of rotations completed by cylinder before it comes to rest.

Solution:



The different forces acting on the cylinder are shown in Figure.

Initial kinetic energy = $\frac{1}{2}I\omega^2 = \frac{1}{4}Mr^2\omega^2$ ... (i)

where $I = \frac{Mr^2}{2}$

There is no shift in centre of gravity because of rotation.

$\therefore$ $R + \mu_1 N = Mg$ ... (ii)

$N = \mu_2 R$ ... (iii)

From (ii) and (iii)

$R + \mu_1 \mu_2 R = Mg$

$$R = \frac{Mg}{1+\mu_1\mu_2}$$

$$N = \frac{\mu_2 Mg}{1+\mu_1\mu_2}$$

Work done by frictional force = $(\mu_2 R + \mu_1 N)\, 2\pi rn$

where n is the total number of rotations made by the cylinder before it stops.

$\therefore$ $2\pi rn\,(\mu_2 R + \mu_1 N) = \frac{1}{4}Mr^2\omega^2$ (Loss of K.E)

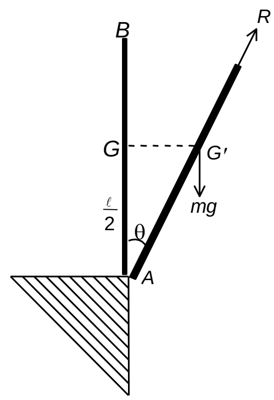
$$\therefore \quad 2\pi rn\left[\frac{\mu_2 Mg}{1+\mu_1\mu_2} + \frac{\mu_1\mu_2 Mg}{1+\mu_1\mu_2}\right] = \frac{1}{4}Mr^2\omega^2$$

$$n = \frac{1}{8\pi}\cdot\frac{Mr\omega^2(1+\mu_1\mu_2)}{Mg(\mu_2+\mu_1\mu_2)}, \quad n = \boldsymbol{\frac{\omega^2 r(1+\mu_1\mu_2)}{8\pi g\mu_2(1+\mu_1)}}$$

Example 2.

One end of a uniform rod is placed at the edge of a very rough table and the rod is released from rest in an almost vertical position. As it falls away from the table it loses contact when the force exerted by the table on the rod becomes zero. Show that this happens when the rod is inclined at an angle $\cos^{-1}\left(\frac{3}{5}\right)$ to the vertical.

Solution:



Let the length of the rod be ℓ and its mass m. The moment of inertia about one end I = $\frac{m\ell^2}{3}$.

The reaction of the table R is along the length of the rod. If the angular velocity of G about A is ω, we have

$$m\omega^2 \frac{\ell}{2} = mg\cos\theta - R$$

If it is about to fall off, R becomes zero.

$$m\frac{\omega^2\ell}{2} = mg\cos\theta$$

$$\omega^2\ell = 2g\cos\theta \qquad \text{... (i)}$$

At θ, the depth through which G has fallen = $\frac{\ell}{2}(1-\cos\theta)$

When the rod is inclined at an angle θ to the horizontal,

Loss in Potential energy = $\frac{mg\ell}{2}(1-\cos\theta)$

Gain in Kinetic energy = $\frac{1}{2}I\omega^2 = \frac{1}{2}\cdot\frac{m\ell^2}{3}\omega^2$

By the conservation of energy, Loss in P.E = Gain in K.E

$$\frac{mg\ell}{2}(1-\cos\theta) = \frac{1}{2}\frac{m\ell^2}{3}\omega^2$$

$$\omega^2\ell = 3g(1-\cos\theta) \qquad \text{... (ii)}$$

From (i) and (ii)

$2\cos\theta = 3(1-\cos\theta)$

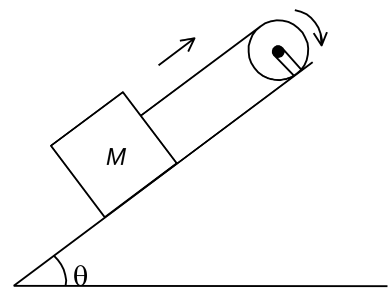
$5\cos\theta = 3$

$\cos\theta = \frac{3}{5}$ or $\theta = \cos^{-1}\frac{3}{5}$

Example 3.

A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in Figure. A string is wrapped round the wheel and its free end supports a block of mass M which can slide on the plane. Initially the wheel is rotating at speed ω in a direction such that the block slides up the plane. How far will the block move before stopping?

Solution:



Suppose the deceleration of the block is a. The linear deceleration of the rim of wheel is also a. The angular deceleration of wheel = a/r.

If the tension in the string is T, the equations of motion can be written as

Mg sin θ – T = Ma

$$T \times r = I\,\alpha = I.\frac{a}{r}.$$

Eliminating T from these equations

$$Mg\sin\theta - \frac{Ia}{r^2} = Ma$$

$$\therefore \qquad a = \frac{Mgr^2\sin\theta}{I+Mr^2}$$

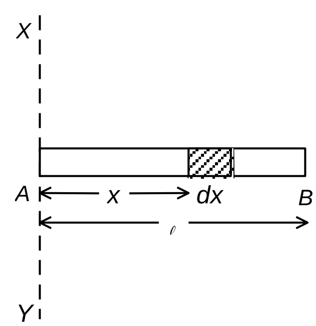
The initial velocity of block up the incline is v = ω r. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I+Mr^2)}{2Mgr^2\sin\theta} = \frac{(I+Mr^2)\omega^2}{2Mg\sin\theta}$$

Example 4.

A thin rod AB of length ℓ is such that its mass density increases uniformly from ρ at A to 4ρ at B, its total mass being M. (a) Find the moment of inertia of the rod about the axis through A perpendicular to AB. (b) If the axis through A is horizontal and AB can turn freely in a vertical plane about A, find the maximum speed it attains after being released from a position making an angle ϕ with downward vertical.

Solution:



(a) Given that the density of the thin rod AB increases uniformly from ρ at A to 4ρ at B, to find the moment of inertia of the rod AB about the axis XY through A, consider an elementary strip of the rod of length dx situated at a distance x from A.

Let the length of the rod AB be ℓ and a, its area of cross-section. The density of the rod at distance x is given by

$$\rho_x = \rho + \frac{x(4\rho-\rho)}{\ell}$$

$$= \frac{\ell\rho + x \cdot 3\rho}{\ell}$$

Mass of the elementary strip = ρ_x a dx

$$dm = \frac{a(\ell + 3x)\rho}{\ell} \cdot dx$$

Moment of inertia of this strip about axis XY is $x^2 \cdot dm$

$$= \frac{a(\ell + 3x)\rho}{\ell} x^2 dx$$

Moment of inertia of the whole rod AB about XY will be $\int_0^\ell x^2 dm$

$$= \int_0^\ell \frac{a(\ell + 3x)\rho}{\ell} \cdot x^2 dx = \frac{a\rho}{\ell} \int_0^\ell (\ell + 3x) x^2 dx$$

$$= \frac{a\rho}{\ell} \left\{ \ell \left[\frac{x^3}{3} \right]_0^\ell + 3 \left[\frac{x^4}{4} \right]_0^\ell \right\}$$

$$= \frac{a\rho}{\ell} \left[\frac{\ell^4}{3} + \frac{3\ell^4}{4} \right]$$

$\therefore$ $\quad I = a\rho\ell^3 \cdot \frac{13}{12}$

To express this in terms of the mass of the rod we calculate M

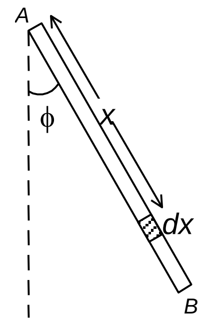
$$M = \int_0^\ell dm = \int_0^\ell \frac{a\rho}{\ell} (\ell + 3x) dx = \frac{a\rho}{\ell} \left\{ \ell [x]_0^\ell + 3 \left[\frac{x^2}{2} \right]_0^\ell \right\}$$

$$= \frac{a\rho}{\ell} \left[\ell^2 + \frac{3\ell^2}{2} \right] = \frac{5a\rho\ell^2}{2\ell} = \frac{5}{2} a\rho\ell$$

$\therefore$ $\quad a\rho\ell = \frac{2M}{5}$

Substituting this value of $a\rho\ell$ in the expression for moment of inertia

we get $I = \frac{2M}{5} \left(\frac{13}{12} \right) \ell^2 = \boldsymbol{\frac{13M\ell^2}{30}}$



(b) If the rod AB were held at an angle ϕ with downward vertical, it will attain its maximum speed when it becomes vertical.

To find maximum speed, we again consider an elementary portion in the form of a strip of length dx at a distance x from end A.

Weight of strip = $(adx\rho_x)g$

When this strip moves to the vertical position the work done by gravity is

$$dW = (adx\rho_x g)(1-\cos\phi)x$$

$$a\rho g(1-\cos\phi)\left[x\frac{(\ell+3x)}{\ell}\right]dx$$

The total work done by the whole rod on coming to the vertical position is

$$W = \int dW = a\rho g(1-\cos\phi)\int_0^{\ell}\frac{x(\ell+3x)}{\ell}dx = a\rho g(1-\cos\phi)\left[\frac{\ell^2}{2}+\frac{3}{\ell}\cdot\frac{\ell^3}{3}\right]$$

$$W = 3a\rho g(1-\cos\phi)\frac{\ell^2}{2} = 3\times\frac{2Mg}{5}(1-\cos\phi)\frac{\ell}{2} = \frac{3Mg\ell(1-\cos\phi)}{5}$$

The work done will be equal to the rotational kinetic energy of the rod.

i.e., $\frac{1}{2}I\omega^2 = W = \frac{3Mg\ell(1-\cos\phi)}{5}$

$$\omega^2 = \frac{6Mg\ell(1-\cos\phi)}{5I}$$

Substituting I = $\frac{13M\ell^2}{30}$

$$\therefore \qquad \omega^2 = \frac{6Mg\ell(1-\cos\phi)}{13M\ell^2}\times 6 = \frac{36g(1-\cos\phi)}{13\ell}$$

$$\therefore \qquad \omega \text{ (maximum angular speed)} = \mathbf{6\sqrt{\frac{g(1-\cos\varphi)}{13\ell}}}$$

Example 5.

A uniform circular disc is free to rotate in a horizontal plane about a fixed horizontal axis through its centre. It has a rough upper surface. When it is rotating with angular velocity ω a second disc of same moment of inertia is laid gently on top of it and concentrically so that initially the upper disc has no angular velocity. Prove that the frictional couple between the discs has brought their two angular velocities to the same value $\frac{\omega}{2}$. Prove also that if the frictional couple is constant and the angles through which the upper and lower discs have at that moment rotated are α and β, then $\beta = 3\alpha$.

Solution:

We have to use the principle of conservation of angular momentum. Accordingly, Total angular momentum after the two circular discs are coupled should be equal to the total angular momentum before they are coupled.

Let ω' be the angular velocity of the system of two discs after coupling. The total moment of inertia of the two discs = I + I which is equal to 2 I.

$\therefore \quad 2I\,\omega' = I\omega$

This gives $\omega' = \dfrac{\omega}{2}$

Point to note: Even though the frictional couple does work to reduce the initial angular velocity of the circular disc with the consequent loss of energy, the conservation of angular momentum still holds because angular momentum is not removed from the system.

It is given that frictional couple is constant and that at the moment the angular velocity of the system became $\dfrac{\omega}{2}$, the upper disc rotates through an angle α and the lower disc through an angle β. It is clear that both the rotations are in the same sense.

If the constant torque of couple is τ, the work done by the torque = $\tau(\beta - \alpha)$ … (i)

Also since the second disc started from rest and attained the angular velocity $\dfrac{\omega}{2}$ under the effect of torque

we have $\tau\cdot\alpha = \dfrac{1}{2}I\left(\dfrac{\omega}{2}\right)^2$

$$\tau\cdot\alpha = \frac{1}{8}I\omega^2 \qquad \ldots \text{(ii)}$$

Equating the work done given by (i) to the change in kinetic energy of the system

$$\frac{1}{2}I\omega^2 - \frac{1}{2}(2I)\left(\frac{\omega}{2}\right)^2 = \tau(\beta - \alpha)$$

$$\therefore \quad \frac{1}{4}I\omega^2 = \tau(\beta - \alpha) \qquad \ldots \text{(iii)}$$

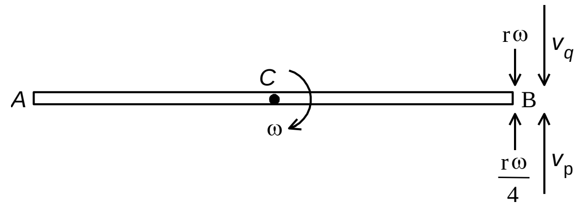
From (ii), $\dfrac{1}{4}I\omega^2 = 2\tau\alpha \;\therefore\; 2\tau\alpha = \tau(\beta - \alpha) \Rightarrow 2\alpha = \beta - \alpha$ which gives $\beta = 3\alpha$

Example 6.

A uniform rod of length 2r and mass m is rotating in a horizontal plane about a smooth fixed pivot through the centre at a steady speed of ω rad/s. A particle of mass m moving with speed $\dfrac{\omega r}{4}$ strikes the end of the rod perpendicularly. The rod and particle are moving towards each other and the coefficient of restitution is $\dfrac{1}{2}$. Find the impulsive reaction at the pivot and the new speed of rod.

Solution:

Let C be the centre of the rod which is rotating about C with an angular velocity ω. Let v_p and v_q be the velocities of the particle at end B of the rod just after impact.



Then by law of conservation of angular momentum (taken about the pivot at C) the change in angular momentum of the rod = I (ω – ω′)

where ω′ is the new angular velocity of the rod after impact.

$$\therefore \qquad r\left[\frac{m(r\omega)}{4}+mv_p\right]=I(\omega-\omega')$$

$$r\left[\frac{mr\omega}{4}+mv_p\right] = \frac{mr^2}{3}(\omega-\omega') \qquad \ldots \text{(i)}$$

$$\frac{r\omega}{4}+v_p = \frac{r}{3}(\omega-\omega') \qquad \ldots \text{(ii)}$$

By Newton's law of restitution

$$v_p-v_q=\frac{\left(r\omega+\frac{r\omega}{4}\right)}{2}=\frac{5r\omega}{8} \qquad \ldots \text{(iii)}$$

since the coefficient of restitution e = $\frac{1}{2}$

Also the velocity v_q of the rod immediately after impact is such that

$$v_q = r\,\omega' \qquad \ldots \text{(iv)}$$

Putting the value of v_q from (iv) in (iii)

$$v_p = v_q + \frac{5r\omega}{8}$$

$$= r\,\omega' + \frac{5r\omega}{8} \qquad \ldots \text{(v)}$$

Substituting the value of v_p in (ii) we get

$$\frac{r\omega}{4}+r\omega'+\frac{5r\omega}{8}=\frac{r}{3}(\omega-\omega') \quad \omega'+\frac{\omega'}{3} = \frac{\omega}{3}-\frac{\omega}{4}-\frac{5\omega}{8}$$

$$\frac{4\omega'}{3} = \left(\frac{8-6-15}{24}\right)\omega = -\frac{13\omega}{24}$$

$$\text{Hence } \omega' = -\frac{13\omega}{24}\times\frac{3}{4} = \frac{13\omega}{32} \text{ rad/s}$$

Thus the new angular speed of the rod is $\left(-\frac{13\omega}{32}\right)$ rad/s .

The negative sign shows that it reverses its direction of motion on account of impact. To find the impulsive reaction K at C, if J be the impulse at end B, at the instant of impact, then by law of conservation of linear momentum

$$J - K = mv = 0$$

where v = linear velocity of the centre of mass

= 0 since the centre of mass is fixed on the pivot

$$\therefore \quad K = J = \left[m\frac{r\omega}{4} - (-mv_p)\right] = \frac{mr\omega}{4} + mv_p$$

But from (v),

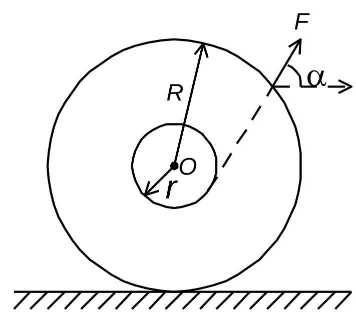
$$v_p = \frac{5r\omega}{8} + r\omega' = \frac{5r\omega}{8} - r\left(\frac{13\omega}{32}\right) = \frac{(20-13)r\omega}{32} = \frac{7r\omega}{32}$$

$$\therefore \quad K = \frac{mr\omega}{4} + \frac{7r\omega}{32} \times m = \frac{15mr\omega}{32} \text{ N-s}$$

This gives the impulsive reaction at C.

Example 7.

A spool of mass m with thread wound on it, rests on a rough horizontal surface. Its moment of inertia relative to its own axis is kmR^2, where k is a numerical factor and R is the outside radius of the spool. The radius of the wound thread layer is r. The spool is pulled without sliding by the thread with a constant force F directed at an angle α to the horizontal. Find

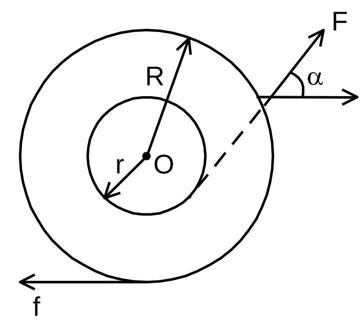


(i) the acceleration of the axis of the spool along OX,

(ii) the work done by F in t seconds from the instant the motion begins.

Solution:

Let F be the force with which the spool is pulled without sliding by the thread and f be the frictional force between the spool and the ground.



(i) Resolving the force F horizontally and vertically, the equation of motion along OX is given by

$$F\cos\alpha - f = ma \qquad \text{.... (i)}$$

Considering the couples, we get

$$fR - Fr = I\alpha = \frac{Ia}{R} \qquad \text{.... (ii)}$$

$$= kmRa$$

Therefore, eliminating f, we get

acceleration of the spool along OX,

$$a = \frac{F(\cos\alpha - \frac{r}{R})}{m(1+k)}$$

(ii) Work done by F in t seconds for the instant the motion begins is equal to the sum of the gain in translational and rotational kinetic energy.

Work done, W = $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

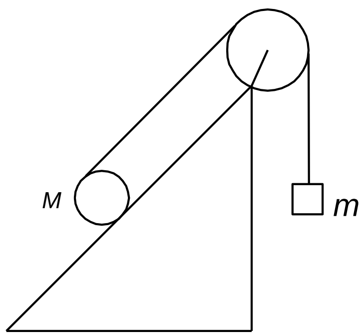
$$= \frac{1}{2}mv^2 + \frac{1}{2}kmR^2\omega^2 = \frac{mv^2}{2}(1+k)$$

$$= \frac{m(at)^2}{2}(1+k) \text{ [since v = at]}$$

$$W = \frac{F^2t^2}{2m}\frac{\left(\cos\alpha - \frac{r}{R}\right)^2}{(1+k)}$$

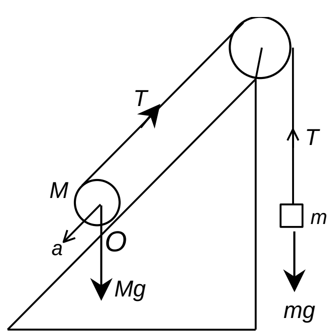
Example 8.

A uniform solid cylinder of mass M and radius 'a' rolls on a frictionless inclined plane with its axis perpendicular to the line of the greatest slope. As the cylinder rolls down, it winds up a light string, which passes over a smooth fixed light pulley and supports a freely hanging mass m, part of the string between the pulley and the cylinder being parallel to the line of the greatest slope. Prove that the tension in the string is $Mmg\frac{(3+4\sin\alpha)}{3M+8m}$ and the acceleration of the mass m is $\left[\frac{4M\sin\alpha - 8m}{3M+8m}\right]g$



Solution:

The different forces are shown in Figure.



Let T be the tension in the string and a be the acceleration of the axis of M downwards. The equation of motion for mass m is given by

$$T - mg = ma_1 \quad \text{…. (i)}$$

where a_1 is the acceleration of mass m.

As the cylinder rolls down a distance x in time t, the mass m will move a distance 2x upward.

Therefore, the acceleration of mass m is, $a_1 = 2a$

$$T - mg = 2ma \quad \text{.... (ii)}$$

Considering the couples about O, we get

$$(Mg \sin \alpha)\, r - T(2r) = I\alpha \quad \text{.... (iii)}$$

where I is the moment of inertia of the cylinder about the axis passing through the point O and α is the angular acceleration of the cylinder.

$$I = \frac{3}{2}Mr^2, \quad \alpha = \frac{a}{r}$$

Substituting the value of I in equation (iii), we have

$$(Mg \sin \alpha)\, r - 2Tr = \frac{3}{2}Mr^2 \frac{a}{r}$$

$$Mg \sin \alpha - 2T = \frac{3}{2}Ma$$

Solving, we get $T = \dfrac{(3+4\sin\alpha)Mmg}{3M+8m}$

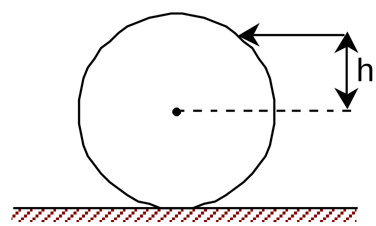
Substituting the value of T in equation (ii), we get

$$Mmg\frac{(3+4\sin\alpha)}{(3M+8m)} - mg = 2ma$$

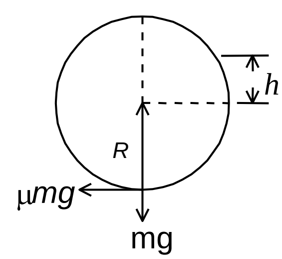
Acceleration of mass m = $2a = \left[\dfrac{\mathbf{4M\sin\alpha - 8m}}{\mathbf{8m+3M}}\right]\mathbf{g}$

Example 9.

A billiard ball initially at rest is given a sharp impulse by a cue. The cue is held horizontally at a distance h above the centre line as in Figure. The ball leaves the cue with a speed v_0 and eventually acquires a speed $\frac{9}{7}v_0$. Show that $h = \frac{4}{5}R$, where R is radius of ball.



Solution:



Let the initial angular velocity be ω_0. The angular momentum of the sphere is $I\omega_0 = \frac{2}{5}mR^2\omega_0$.

The moment of the impulse given = $mv_0 \cdot h$

$$\therefore \quad \frac{2}{5}mR^2\omega_0 = mv_0 \cdot h$$

$$\omega_O = \frac{mv_o h}{\frac{2}{5}mR^2} = \frac{5v_o h}{2R^2}$$

At time t after impact,

$$v = v_0 + \mu gt \qquad \text{... (i)}$$

$$\omega = \omega_0 - \frac{\mu mgR}{\frac{2}{5}mR^2} \cdot t = \omega_O - \frac{5\mu gt}{2R}$$

When v = ωR pure rolling begins.

$$v_O + \mu gt = R\,\omega_O - \frac{5\mu gt}{2} = \frac{5v_o h}{2R} - \frac{5}{2}\mu gt \quad \frac{7}{2}\mu gt = v_o\left(\frac{5h}{2R} - 1\right)$$

$$\frac{9}{7}v_0 = v_0 + \mu gt \qquad \text{from (i)}$$

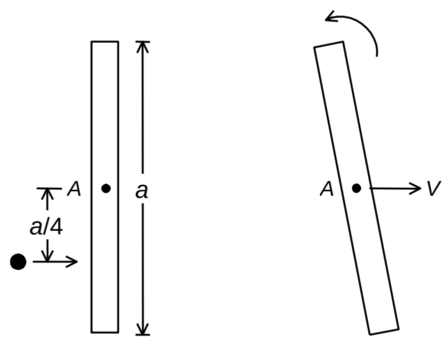
$$= v_0 + \frac{2}{7}v_o\left(\frac{5h}{2R} - 1\right) \quad \frac{5h}{2R} = 2 \Rightarrow h = \boldsymbol{\frac{4R}{5}}$$

Example 10.

A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance $\frac{a}{4}$ from the centre and stops after collision. Find the velocity of the centre of the rod and the angular velocity of the rod about its centre just after collision.

Solution:

Consider the rod and particle together as the system. As there is no external resultant force; the linear momentum of the system will remain constant. Also there is no resultant external torque on the system and so the angular momentum of the system about any line will remain constant.



Let V be the velocity of the centre of the rod and the angular velocity about centre be ω. By principle of conservation of linear momentum,

$$mv = MV \quad V = \frac{mv}{M} \qquad \text{... (i)}$$

Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about AB.

Initially the rod is at rest.

The angular momentum of the particle about AB, is L = $mv\left(\frac{a}{4}\right)$.

After collision the particle comes to rest. The angular momentum of rod about A is

$$\vec{L} = \vec{L}_{cm} + M\vec{r}_0 \times \vec{V}$$

As $\vec{r}_0 \| \vec{V}$, $\vec{r}_0 \times \vec{V} = 0$

Thus $\vec{L} = \vec{L}_{cm}$

Hence the angular momentum of rod about AB is L = Iω = $\frac{Ma^2}{12}\omega$.

Thus $\frac{mva}{4} = \frac{Ma^2}{12}\omega$ i.e., $\omega = \frac{3mv}{Ma}$