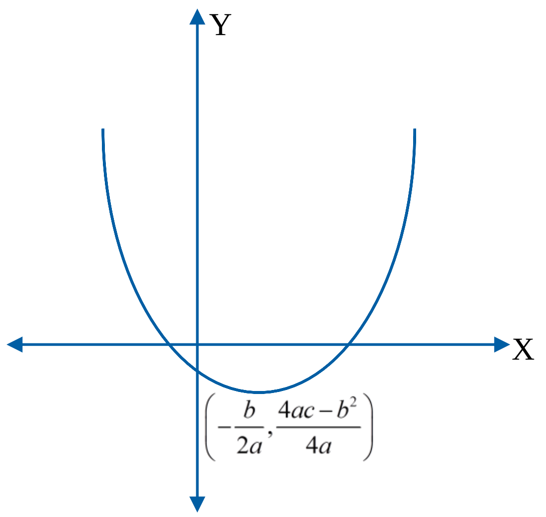
A QUADRATIC EXPRESSION

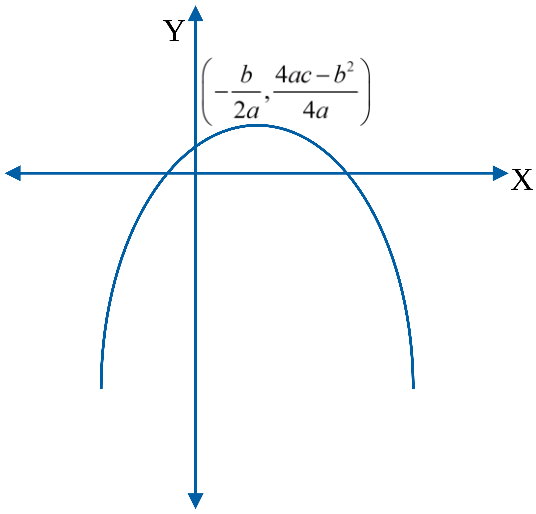
$f(x) = ax^2 + bx + c$, where $a, b, c \in R$ & $a \neq 0$, is called a quadratic expression or a quadratic polynomial function of x. It's necessary to understand characteristics of a quadratic expression for learning quadratic equations with a complete and conceptual understanding.

GRAPH OF A QUADRATIC EXPRESSION $\{y = ax^2 + bx + c\}$

The equation $y = ax^2 + bx + c$ traces a parabola on $x - y$ coordinate system. We can rewrite this equation as $\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y - \frac{4ac - b^2}{4a}\right)$. Hence the parabola will have its vertex at $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$. This provides a number of possibilities as according a is positive or negative as well as according to the sign of the quantities $-\frac{b}{2a}$ & $\frac{4ac - b^2}{4a}$. Following figures will give you a clear idea about the tracing of Quadratic Expressions.

$y = ax^2 + bx + c,\ a > 0$





$y = ax^2 + bx + c,\ a < 0$

Range Of A Quadratic Expression

As it can be undesrstood from the above graphs that when ‘a’ is positive, y i.e. $ax^2 + bx + c$ can’t be less than $\frac{4ac-b^2}{4a}$ and similarly when ‘a’ is negative, y i.e. $ax^2 + bx + c$ can’t exceed $\frac{4ac-b^2}{4a}$.

$$\text{Hence } \left\{ \begin{array}{c} ax^2+bx+c \geq \dfrac{4ac-b^2}{4a} \;\; if \;\; a>0 \\ \& \\ ax^2+bx+c \leq \dfrac{4ac-b^2}{4a} \;\; if \;\; a<0 \end{array} \right\}$$

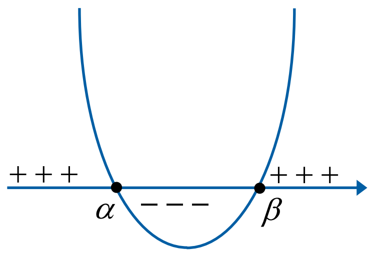
SIGN OF A QUADRATIC EXPRESSION

As we have seen in the graphs above the y coordinate of the vertex of the poarabola traced by the equation $y = ax^2 + bx + c$, is $\dfrac{4ac-b^2}{4a}$ and that the vertex of a parabola is the topmost or bottommost point on the parabola depending on the sign of 'a'.

Hence sign scheme of $f(x) = ax^2 + bx + c$ depends purely on 'a' & 'D' where $D = b^2 - 4ac$.

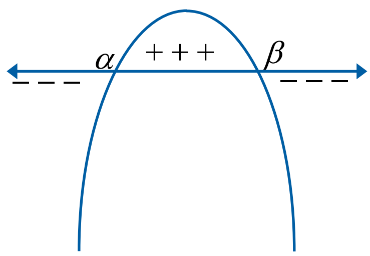
We can understand this with the help of following possible situations –

Case I. When a > 0 & D > 0



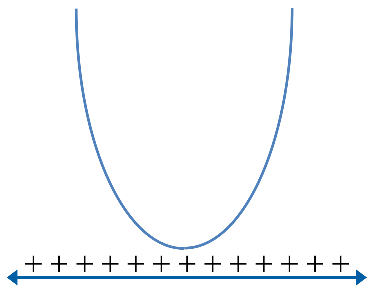
Obviously in this situation $ax^2 + bx + c = 0$ will have real and distinct roots which means the graph of $y = ax^2 + bx + c$ will be a parabola opening upwards and will meet x – axis twice as can be seen in the adjoining figure.

Case II. When a < 0 & D > 0



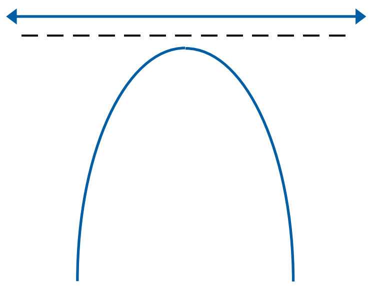
Obviously in this situation $ax^2 + bx + c = 0$ will have real and distinct roots which means the graph of $y = ax^2 + bx + c$ will be a parabola opening downwards and will meet x – axis twice as can be seen in the adjoining figure.

Case III. When a > 0 & D < 0



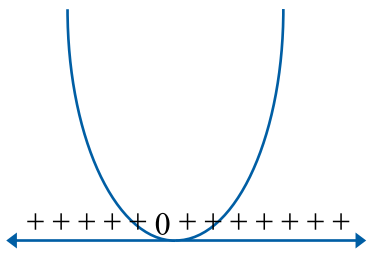
In this situation as $\dfrac{4ac-b^2}{4a}$ will be positive hence entire graph will lie above x – axis and hence $ax^2 + bx + c$ will be positive for all real values of x.

Case IV. When a < 0 & D < 0

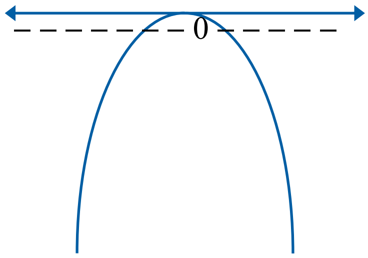


In this situation as $\dfrac{4ac-b^2}{4a}$ will be negative hence entire graph will lie below x – axis and hence $ax^2 + bx + c$ will be negative for all real values of x.

Case V. When a > 0 & D ≤ 0



In this situation as $\dfrac{4ac-b^2}{4a}$ will be zero or positive hence entire graph will lie above x – axis with vertex may be on x – axis and hence ax^2+bx+c will be non negative for all real values of x.



Case VI. **When a < 0 & D ≤ 0**

In this situation as $\dfrac{4ac-b^2}{4a}$ will be zero or negative hence entire graph will lie below x – axis with vertex may be on x – axis and hence ax^2+bx+c will be non positive for all real values of x.

Note that …

A quadratic expression has the same sign as the sign of 'a' for all real values of x if discriminant is negative.

Roots of a Quadratic Equation

The equation $ax^2 + bx + c = 0$, could be rewritten as $\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$.

Here the quantity "$b^2 - 4ac$" is called the Discriminant of the quadratic equation and its roots are given by the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

CHARACTERISTICS OF ROOTS OF A QUADRATIC EQUATION

If $a \neq 0$ ***and a, b, c*** $\in R$***, then***

- If $D < 0$, then equation has non – real complex roots and if $(p + i\,q)$ is one root then the other must be the conjugate $(p - i\,q)$ and vice-versa.
- If $D > 0$, then equation has real and distinct roots, namely $x_1 = \frac{-b+\sqrt{D}}{2a}$, $x_2 = \frac{-b-\sqrt{D}}{2a}$
- If $D = 0$, then equation (i) has real and equal roots, $x_1 = x_2 = -\frac{b}{2a}$
- If a, b & c are rational numbers and D is a perfect square of a rational number then the roots are rational and if D is not a perfect square then the roots will be irrational and if $\left(p+\sqrt{q}\right)$ is one root then the other one must be $p-\sqrt{q}$ and vice-versa. (Here p is a rational number and q is a rational number & not a perfect square)

CHARACTERISTIC OF ROOTS UNDER PARTICULAR CONDITIONS:

- b = 0 $\Rightarrow$ roots are equal in magnitude and opposite in sign.
- a = c $\Rightarrow$ roots are reciprocal of each other.
- a & c are of opposite sign $\Rightarrow$ roots are real, distinct and are of opposite sign.
- a & c are of same sign $\Rightarrow$ roots are of opposite sign if Discriminant is greater than or equal to zero.
- a, b & c all are of same sign $\Rightarrow$ both roots are negative if Discriminant is greater than or equal to zero.
- a, – b & c are of same sign $\Rightarrow$ both roots are positive if Discriminant is greater than or equal to zero.
- a > 0, c < 0, b < 0 $\Rightarrow$ negative roots numerically greater than positive root.
- a > 0, c < 0, b < 0 $\Rightarrow$ positive root is numerically greater than negative root.
- a + b + c = 0 $\Leftrightarrow$ one of the roots is 1. Obviously in this case the other root will be c/a.

both the roots will be negative for $m \le 0$.

Relations Between Roots and Coefficients

Let α and β be the roots of $ax^2 + bx + c = 0$, then

- Sum of the roots $= -\dfrac{\text{Coeff. of } x^2}{\text{Coeff. of } x}$ i.e. $\alpha + \beta = -\dfrac{b}{a}$
- Product of the roots $= \dfrac{\text{Coeff. of } x^2}{\text{Costant term}}$ i.e. $\alpha.\beta = \dfrac{c}{a}$
- The quadratic equation whose roots are α and β is given as $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.
- Expressions like $\alpha^n \pm \beta^n$, $\dfrac{\alpha^n}{\beta^m} \pm \dfrac{\beta^n}{\alpha^m}$, $\alpha^{n-m}\beta^m \pm \alpha^m\beta^{n-m}$, *etc.* are called symmetric functions of roots, and their value may be found using simple algebraic transformation and apllying the values of sum & product of roots.

EXACTLY ONE COMMON ROOT BETWEEN TWO QUADRATIC EQUATIONS

Let us consider two quadratic equations. $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$.

then by eliminating x between the two relations we get $\dfrac{x^2}{b_1c_2 - b_2c_1} = \dfrac{x}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$

$$\Rightarrow (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$$

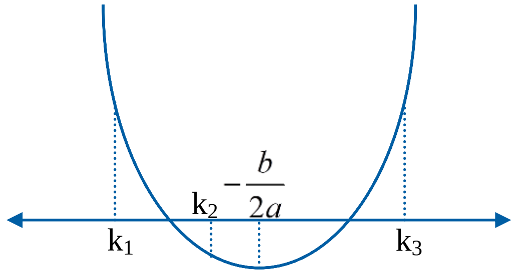
Note that . . .
above condition is applicable when the two equations have exactly one root in common. If both the roots are common then we can simply say that $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

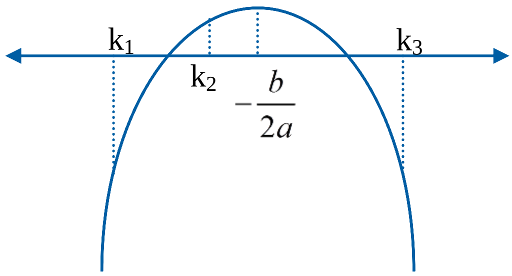
LOCATION OF ROOTS OF A QUADRATIC EQUATION:

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in$ R, be a quadratic expression and D be the Discriminant, and let α, β be the roots $\left\{\alpha = \dfrac{-b-\sqrt{D}}{2a},\ \beta = \dfrac{-b+\sqrt{D}}{2a},\ \ \alpha < \beta\right\}$.

The following figure represents graphically all possible locations of roots as compared to constants k_1, k_2 & k_3 followed by the respective conditions.

$$f(x) = ax^2 + bx + c,\ \ a > 0,\ \ D > 0$$





$$f(x) = ax^2 + bx + c,\ \ a < 0,\ \ D > 0$$

- **When a real number k_2 lies between the roots i.e. $\alpha < k_2 < \beta$**

  As it is evident from the graphs of y = f(x), the coefficient of x^2 and $f(k_2)$ are of opposite signs. Thus, the conditions are

  (i) D > 0 (ii) a f(k) < 0

- **When a real number k_3 is greater than the roots i.e. $\alpha < \beta < k_3$**

  It is evident from the graph that 'a' and f(k) are of the same sign and point representing k lies on the right hand side of the vertex of y = f(x). The conditions are

  (i) $D \geq 0$ (ii) $a.\ f(k) > 0$ (iii) $k < -b/2a$.

- **When a real number k_1 is less than the roots i.e. $k_1 < \alpha < \beta$**

  It is evident from the graph that 'a' and f(k) are of the same sign and point representing k lies on the left hand side of the vertex of y = f(x). The conditions are

  (i) $D \geq 0$ (ii) $a.\ f(k) > 0$ (iii) $k < -b/2a$

- **When both the roots lie in the interval (k_1, k_2) i.e. $k_1 < \alpha < \beta < k_2$**

  The conditions are

  (i) $D \geq 0$ (ii) $a.f(k_1) > 0$ & $a.\ f(k_2) > 0$ (iii) $k_1 < -b/2a < k_2$

- **When exactly one root lies in the interval (k_1, k_2), $k_1 < k_2$**

  then $f(k_1)$ and $f(k_2)$ are of opposite signs, hence $f(k_1)\, f(k_2) < 0$

EQUATIONS OF HIGHER DEGREE

The equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots + a_{n-1}x + a_n = 0$ when $a_0, a_1, a_2, \dots, a_n$ are constants, but $a_0 \neq 0$ is a polynomial of degree n. It has n and only n roots. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ be n roots then

$$\sum \alpha_1 = (-1)^1 \frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = (-1)^2 \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = (-1)^3 \frac{a_3}{a_0} \text{ etc.}$$

In general $\sum \alpha_1\alpha_2\alpha_3 \dots\dots \alpha_p = (-1)^p \frac{a_p}{a_0}$.

A Reciprocal Equation of the Standard form can be reduced to an equation of half its Dimensions.

Let the equation be $ax^{2m} + bx^{2m-1} + cx^{2m-2} + \dots\dots + kx^m + \dots\dots\dots cx^2 + bx + a = 0$.

Dividing by x^m and rearranging the terms, we have

$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \dots\dots + k = 0$$

Now $x^{p+1} + \frac{1}{x^{p+1}} = \left(x^p + \frac{1}{x^p}\right)\left(x + \frac{1}{x}\right) - \left(x^{p-1} + \frac{1}{x^{p-1}}\right)$

Hence writing z for $x + \frac{1}{x}$ and given to p succession the values 1, 2, 3 we obtain $x^2 + \frac{1}{x^2} = z^2 - 2$, and so on.

Rational Algebraic Equations

An equation of the form $\frac{P(x)}{Q(x)}$ where P (x) & Q (x) are algebraic polynomials and Q (x) $\neq 0$, is known as a rational equation. Consider the equation given below

- $\frac{ax^2+b_1x+c}{ax^2+b_2x+c}=\frac{Ax}{ax^2+b_3x+c}$
- $\frac{ax^2+b_1x+c}{ax^2+b_2x+c}\pm\frac{ax^2+b_3x+c}{ax^2+b_4x+c}=A$
- $\frac{Ax}{ax^2+b_1x+c}\pm\frac{Bx}{ax^2+b_2x+c}=C$

In these type of problems we divide above and below by x. and put $ax+\frac{c}{x}=t$

Rational Algebraic Inequalities

$\frac{P(x)}{Q(x)}>0,\ \frac{P(x)}{Q(x)}<0;\ \frac{P(x)}{Q(x)}\le 0,\ \frac{P(x)}{Q(x)}\ge 0$

Solving these type of inequalities, we resolve $P(x)$ and $Q(x)$ in linear factor and use wavy curve method.

Some standard forms to solve irrational equation.

- Equations of the form $\sqrt[n]{a-f(x)}+\sqrt[n]{b+f(x)}=g(x)$ suppose $u=\sqrt[n]{a-f(x)},\ v=\sqrt[n]{b+f(x)}$

  then above equation reduces to $u+v=g(x)$ and $u^n+v^n=a+b$
- Equations of the form $\sqrt[3]{f(x)}+\sqrt[3]{g(x)}=h(x)$ where $f(x)$ and $g(x)$ are functions of x, where as $h(x)$ is a function of x or constant, can be solved cubing both sides of the equation.

IRRATIONAL INEQUALITIES

We consider here some standard forms to solve irrational inequalities –

- Equations of the form $\sqrt[2n]{f(x)} < \sqrt[2n]{g(x)}, n \in N$ is equivalent to the system $\begin{cases} f(x) \geq 0 \\ g(x) > f(x) \end{cases}$

  and equations of the form $\sqrt[2n+1]{f(x)} < \sqrt[2n+1]{g(x)}, n \in N$ is equivalent to the inequation $f(x)<g(x)$

- Equations of the form $\sqrt[2n]{f(x)} < g(x), n \in N$ is equivalent to the system $\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < g^{2n}(x) \end{cases}$

  and inequalities of the form $\sqrt[2n+1]{f(x)} < g(x), n \in N$ is equivalent to the inequation $f(x)<g^{2n+1}(x)$

- Inequalities of the form $\sqrt[2n]{f(x)} > g(x), n \in N$ is equivalent to the system

  $\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases}$ and $\begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$ and inequality of the form $\sqrt[2n+1]{f(x)} > g(x), n \in N$ is equivalent to the inequality $f(x) > g^{2n+1}(x)$

EXPONENTIAL EQUATIONS

Here we consider an equation of the form $a^x = b$ where $a > 0$ then

(i) $x = \phi$ if $b < 0$ (ii) $x = Log_a b$ if $b > 0, a \neq 1$ (iii) $x = \phi$ if $a = 1$ $b \neq 1$ (iv) $x = R$ if $a = 1$ $b = 1$

Some standard forms to solve exponential equations.

- Equations of the form $a^{f(x)} = 1$ $a > 0$ or $a \neq 1$ is equivalent to the equation $f(x) = 0$
- Equations of the form $f(a^x) = 0$ put $a^x = t$. then $f(t) = 0$.
- Equation of the form $ra^{f(x)} + pb^{f(x)} + qc^{f(x)} = 0$ when $r, p, q, \in R$ and $\neq 0$ and the bases satisfy the condition $b^2 = \mathrm{ac}$ is equivalent to the equation $pt^2 + qt + r = 0$, where $\mathrm{t} = (\mathrm{a}/\mathrm{b})^{\mathrm{f}(\mathrm{x})}$
- Equations of the form $pa^{f(x)} + qb^{f(x)} + c = 0$ where $p, q, c \in R, p, q, c \neq 0$ and $ab = 1$(a and b are inverse +ve numbers) is equivalent to the equation $pt^2 + ct + q = 0$ where $t = a^{f(x)}$
- Equations of the form $a^{f(x)} + b^{f(x)} = c$ where $a, b, c \in \mathrm{R}$ and a, b, c satisfies the condition $a^2 + b^2 = c$ then the solution of equation will be $f(x) = 2$ and no other solution of this equation.
- An equation of the form $\{f(x)\}^{g(x)}$ is equivalent to the equation $\{f(x)\}^{g(x)} = 10^{g(x) log\, f(x)}$ where $f(x) > 0$.

Exponential Inequalities

Exponential Inequalities of the form $a^{f(x)} > b$ $(a > 0)$ we have

(i) $x \in D_f$ if $b \leq 0$ (ii) if $b > 0$, then we have $f(x) > \log_a b$ if $a > 1$ & $f(x) < \log_a b$ if $0 < a < 1$ for $a = 1$ then $b < 1$

Some standard forms to solve exponential inequalities

- Inequalities of the form $f(a^x) \geq 0$ or $f(a^x) \leq 0$ is equivalent to the system $\begin{cases} t > 0 \quad where \quad t = a^x \\ f(t) \geq 0 \quad or \quad f(t) \leq 0 \end{cases}$
- Inequalities of the form $pa^{f(x)} + qb^{f(x)} + rc^{f(x)} \geq 0$, $pa^{f(x)} + qb^{f(x)} + rc^{f(x)} \leq 0$, Where $p, q, r \in R$ and $p, q, r \neq 0$ and bases satisfy the condition $b^2 = ac$ is equivalent to the inequalities $pt^2 + qt + r \geq 0$ or $pt^2 + qt + r \leq 0$, where $t = \left(\frac{a}{b}\right)^{f(x)}$.
- Inequalities of the form $pa^{f(x)} + qb^{f(x)} + r \geq 0$, $pa^{f(x)} + qb^{f(x)} + r \leq 0$, where $p, q, c \in R$ and $p, q, c \neq 0$ and $ab = 1$is equivalent to the inequations $pt^2 + qt + c \geq 0$, $pt^2 + qt + c \leq 0$ where $t = a^{f(x)}$

Illustration 27. Solve the inequality $(x^2 + x + 1)^x < 1$

Taking log both sides on base 10.

$x \log(x^2 + x + 1) < 0$, Which is equivalent to the collection of systems

$$\left[\begin{array}{l} \begin{cases} x > 0 \\ \text{Log}(x^2 + x + 1) < 0 \end{cases} \\ \begin{cases} x < 0 \\ \text{Log}(x^2 + x + 1) > 0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x > 0 \\ x^2 + x + 1 < 1 \end{cases} \\ \begin{cases} x < 0 \\ x^2 + x + 1 > 1 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x > 0 \\ x(x+1) < 0 \end{cases} \\ \begin{cases} x < 0 \\ x(x+1) > 0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x > 0 \\ -1 < x < 0 \end{cases} \\ \begin{cases} x < 0 \\ x > 0 \quad \text{and} \quad x < -1 \end{cases} \end{array}\right. \Rightarrow \begin{cases} x \in \phi \\ x < -1 \end{cases}$$

Consequently the interval $x \in (-\infty, -1)$is the set of all solution of the Original inequality.

LOGARITHMIC EQUATIONS & INEQUALITIES

Suppose we have an equation of the form $Log_a f(x) = b$, $(a > 0)$, $a \neq 1$ is equivalent to the equation

$f(x) = a^b \quad [f(x) > 0]$

(i) $\text{Log}_a(xy) = \text{Log}_a x + \text{Log}_a y$ (ii) $\text{Log}_a (x/y) = \text{Log}_a x - \text{Log}_a y$

(iii) $Log_a x^{2b} = 2b\, Log_a x$ (iv) $a^{bLog_a x} = x^b$ (v) $x^{Log_a y} = y^{Log_a x}$.

Some standard forms to solve logarithmic equations

- Equations of the form (a) $f(Log_a x) = 0$, $a > 0$, $a \neq 1$ and (b) $g(Log_x A) = 0$, $A > 0$

  then equation (a) is equivalent to $f(t) = 0$, where $t = Log_a x$

  and equation (b) is equivalent to $f(y) = 0$ where $y = Log_x A$

- Equations of the form $Log_a f(x) = Log_a g(x)$ $a > 0$, $a \neq 1$ is equivalent to two ways

  1st way $\begin{cases} g(x) > 0 \\ f(x) = g(x) \end{cases}$ 2nd way $\begin{cases} f(x) > 0 \\ f(x) = g(x) \end{cases}$

- $Log_{f(x)} a = \text{Log}_{g(x)} a$ $a > 0$ is equivalent to two ways 1st way $\begin{cases} g(x) > 0 \\ g(x) \neq 1 \\ f(x) = g(x) \end{cases}$ 2nd way $\begin{cases} f(x) > 0 \\ f(x) \neq 1 \\ f(x) = g(x) \end{cases}$

- Equations of the form

(a) $Log_{f(x)}\, g(x) = Log_{f(x)}\, h(x)$ is equivalent to two ways 1st way $\begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$ 2nd way $\begin{cases} h(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$

(b) $Log_{g(x)} f(x) = Log_{h(x)} f(x)$ is equivalent to two ways 1st way $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ g(x) \neq 1 \\ g(x) = h(x) \end{cases}$ 2nd way $\begin{cases} f(x) > 0 \\ h(x) > 0 \\ h(x) \neq 1 \\ g(x) = h(x) \end{cases}$

- Equations of the form $Log_{h(x)} (Log_{g(x)} f(x)) = 0$ is equivalent to the system $\begin{cases} h(x) > 0 \\ h(x) \neq 1 \\ g(x) > 0 \\ g(x) \neq 1 \\ f(x) = g(x) \end{cases}$

- Equations of the form $2m\, Log_a f(x) = Log_a g(x)$ $a > 0$, $a \neq 1$, $m \in N$ is equivalent to the system

$$\begin{cases} f(x) > 0 \\ f^{2m}(x) = g(x) \end{cases}$$

- Equations of the form (2m + 1) $Log_a f(x) = Log_a g(x)$ $a > 0$ $a \neq 1$ $m \in N$ is equivalent to $\begin{cases} g(x) > 0 \\ f^{2m+1}(x) = g(x) \end{cases}$

- Equations of the form $Log_a f(x) + Log_a g(x) = Log_a h(x)$ $a > 0$, $a \neq 1$ is equivalent to the System

  $$\begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x)g(x) = h(x) \end{cases}$$

- Equations of the form

  $Log_a f(x) - Log_a g(x) = Log_a h(x) - Log_a m(x)$, $a > 0$, $a \neq 1$ is equivalent to the equation

  $Log_a f(x) + Log_a m(x) = Log_a g(x) + Log_a h(x)$

  Which is equivalent to the system $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ m(x) > 0 \\ h(x) > 0 \\ f(x)m(x) = g(x)h(x) \end{cases}$

- $\begin{cases} Log_a f(x) > Log_a g(x) \\ a > 1 \end{cases} \Rightarrow \begin{cases} g(x) > 0 \\ a > 1 \\ f(x) > g(x) \end{cases}$

- $\begin{cases} Log_a f(x) > Log_a g(x) \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} f(x) > 0 \\ 0 < a < 1 \\ f(x) < g(x) \end{cases}$

- $\begin{cases} Log_a x > 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ a > 1 \end{cases}$

- $\begin{cases} Log_a x > 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ 0 < a < 1 \end{cases}$

- $\begin{cases} Log_a x < 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ a > 1 \end{cases}$

- $\begin{cases} Log_a x < 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ 0 < a < 1 \end{cases}$

EQUATIONS INVOLVING GREATEST INTEGER AND NEAREST INTEGER

Greatest integer of x, generally represented as $[x]$, denotes the greatest integer less than or equal to x

Properties of greatest integer of x –

- $[x + n] = n + [x],\ n \in I$
- $x = [x] + \{x\}$, $\{\ \}$ denote the fractional part of x.
- $[-x]+[x]=\begin{cases} 0 & x \in I \\ -1 & x \notin I \end{cases}$
- $[x] \geq n \Rightarrow x \geq n,\ \ n \in I$ & $[x] > n \Rightarrow x \geq n+1,\ \ n \in I$
- $[x] \leq n \Rightarrow x < n+1,\ \ n \in I$ & $[x] < n \Rightarrow x < n,\ \ n \in I$
- $[x + y] \geq [x] + [y]$
- $\left[\frac{[x]}{n}\right]=\left[\frac{x}{n}\right],\ \ n \in N$
- $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\text{-----}+\left[x+\frac{n-1}{n}\right] = [nx]$

Some Important Results to Remember

Factor theorem If α is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$ the α is a root of the equation $f(x) = 0$ and the remainder obtained is $f(\alpha)$.

Every equation of an odd degree has at least one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first form is positive.

Every equation of an even degree has at least two real roots, one positive and one negative, whose last term is negative, provided that the coefficient of the first term is positive.

If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation are positive sign.

If $x = \alpha$ is root repeated m times in $f(x) = 0$, ($f(x) = 0$ is nth degree equation in x) then

$f(x) = (x - \alpha)m\, g(x)$ when $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ times in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0$, ….. $(m - (m - 1))$ times in $fm - 1\,(x) = 0$.

Lagrange's Identity If $a_1, a_2, a_3, b_1, b_2, b_3 \in R$, then:

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$$

The condition that a quadratic function $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factor if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is 0.

Law of proportions If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = ...$ then each of these ratios is also equal to:

(i) $\frac{a + c + e + ...}{b + d + f + ...}$ (ii) $\left(\frac{pa^n + qc^n + re^n + ...}{pb^n + qd^n + rf^n + ...}\right)^{1/n}$ (when $p, q, r, n \in R$) (iii) $\frac{\sqrt{ac}}{\sqrt{bd}} = \frac{\sqrt[n]{(ace....)}}{\sqrt[n]{(bdf....)}}$

Let $f(x) = 0$ be a polynomial equation and λ, μ are two real numbers. Then $f(x) = 0$ will have at least one real root or an odd number of roots between λ and μ if $f(\lambda)$ and $f(\mu)$ are of opposite signs.

But if $f(\lambda)$ and $f(\mu)$ are of same signs, then either $f(x) = 0$ has no real roots or an even number of roots between λ and μ.

(i) The number of positive roots of a polynomial equation $f(x) = 0$ (arranged in decreasing order of the degree) cannot exceed the number of changes of signs in $f(x) = 0$ as we move from left to right.

$5x^3 - 2x^2 - 3x + 7 = 0$. The number of changes of signs from left to right is 2(+ to – than to +). Then number of positive roots cannot exceed 2.

(ii) The number of negative roots of a polynomial equation $f(x) = 0$ cannot exceed the number of changes of signs in $f(-x)$. For example $8x^4 + 7x^3 - 4x^2 + 10x - 15 = 0$

Let $f(x) = 8x^4 + 7x^3 - 4x^2 + 10x - 15$, then $f(-x) = 8x^4 - 7x^3 - 4x^2 - 10x - 15$

The number of changes of signs from left to right is 1(+ to –). The number of negative roots cannot exceed 1.