1. Definition

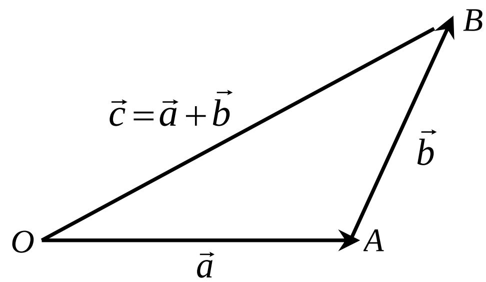
A scalar is a quantity, which has only magnitude but does not have a direction. For example time, mass, temperature, distance and specific gravity etc. are scalars.

A Vector is a quantity which has magnitude, direction and follows the law of parallelogram (addition of two vectors). For example displacement, force, acceleration are vectors.

2. Addition of Two Vectors

Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{AB}=\vec{b}$ and $\overrightarrow{OB}=\vec{c}$.

Here $\vec{c}$ is sum (or resultant) of vectors $\vec{a}$ and $\vec{b}$. It is to be noticed that the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$ and the line joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ represents vector $\vec{a}+\vec{b}$ in magnitude and direction.



2.1 Properties

(i) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$, (Vector addition is commutative)

(ii) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$, (Vector addition is associative)

(iii) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$, equality holds when $\vec{a}$ and $\vec{b}$ are like vectors

(iv) $|\vec{a}+\vec{b}|\le||\vec{a}|+|\vec{b}||$, equality holds when $\vec{a}$ and $\vec{b}$ are unlike vectors

(v) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$

(vi) $\vec{a}+(-\vec{a})=0=(-\vec{a})+\vec{a}$

3. Type of Vectors

(i) Equal Vectors

Two vectors are said to be equal if and only if they have equal magnitudes and same direction.

(ii) Zero Vector (null vector)

A vector, whose initial and terminal points are same, is called the null vector.

(iii) Like and Unlike Vectors

Two vectors are said to be

(a) Like, when they have same direction.

(b) Unlike, when they are in opposite directions. $\vec{a}$ and $-\vec{a}$ are two unlike vectors as their directions are opposite, $\vec{a}$ and $3\vec{a}$ are like vectors.

(iv) Unit Vector

A unit vector is a vector whose magnitude is unity. We write, unit vector in the direction of $\vec{a}$ as $\hat{a}$. Therefore $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

(v) Parallel vectors

Two or more vectors are said to be parallel, if they have the same support or parallel support. Parallel vectors may have equal or unequal magnitudes and direction may be same or opposite.

(vi) Position Vector

If P is any point in the space then the vector $\overrightarrow{OP}$ is called position vector of point P, where O is the origin of reference. Thus for any points A and B in the space, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

(vii) Co-initial vectors

Vectors having same initial point are called co-initial vectors.

4. Linear Combinations

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}$ ….. then the vector $r = x\vec{a} + y\vec{b} + z\vec{c} +$ … is called a linear combination of $\vec{a}, \vec{b}, \vec{c}$ ….. for any x, y, z ….. $\in$ R.

We have the following results:

(i) If $\vec{a}, \vec{b}$ are non-zero, non-collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'; y = y'$

(ii) If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are n non-zero vectors, & $k_1, k_2, \ldots, k_n$ are n scalars & if the linear combination $k_1x_1 + k_2x_2 + \ldots + k_nx_n = 0$

$\Rightarrow k_1 = k_2 = \ldots = k_n = 0$, then we say that vector $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are **Linearly Independent Vectors.**

(iii) If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are not linearly independent vectors then they are said to be **Linearly Dependent Vectors.**

Illustration 1:

Show that the vectors $5\vec{a} + 6\vec{b} + 7\vec{c}, 7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$ are coplanar (where $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors).

Solution

Let $\vec{A} = 5\vec{a} + 6\vec{b} + 7\vec{c}, \vec{B} = 7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$

$\vec{A}, \vec{B}$ and $\vec{c}$ are coplanar $\Rightarrow x\vec{A} + y\vec{B} + z\vec{C} = 0$ must have a real solution for x, y, z other than (0, 0, 0).

Now $x(5\vec{a} + 6\vec{b} + 7\vec{c}) + y(7\vec{a} - 8\vec{b} + 9\vec{c}) + z(3\vec{a} + 20\vec{b} + 5\vec{c}) = \vec{0}$

$\Rightarrow \quad (5x + 7y + 3z)\vec{a} + (6x - 8y + 20z)\vec{b} + (7x + 9y + 5z)\vec{c} = \vec{0}$

$5x + 7y + 3z = 0$

$6x - 8y + 20z = 0$

$7x + 9y + 5z = 0$ (As $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors)

$$\text{Now D} = \begin{vmatrix} 5 & 7 & 3 \\ 6 & -8 & 20 \\ 7 & 9 & 5 \end{vmatrix} = 0$$

So the three linear simultaneous equation in x, y and z have a non-trivial solution.

Hence $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar vectors.

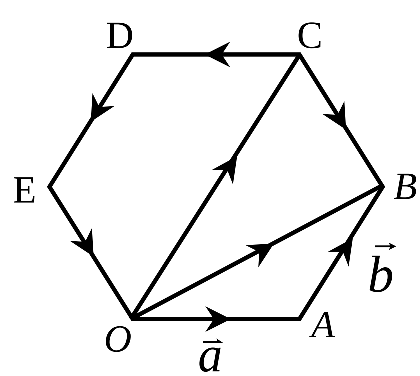
5. Multiplication of Vector by Scalars

If $\vec{a}$ is a vector and m is a scalar, then m $\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is | m | times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**.

Illustration 2:

If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order?

Solution



OABCDE is a regular hexagon. Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Join OB and OC

We have $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

Since OC is parallel to AB and double of AB.

$\therefore\ \overrightarrow{OC} = 2\overrightarrow{AB} = 2\vec{b}$

Now $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=2\vec{b}-(\vec{a}+\vec{b})=\vec{b}-\vec{a}$

$\overrightarrow{CD}=\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{DE}=-\overrightarrow{AB}=-\vec{b}$

Also $\overrightarrow{EO}=-\overrightarrow{BC}=-(\vec{b}-\vec{a})=\vec{a}-\vec{b}$

6. Collinearity and Coplanarity of Points

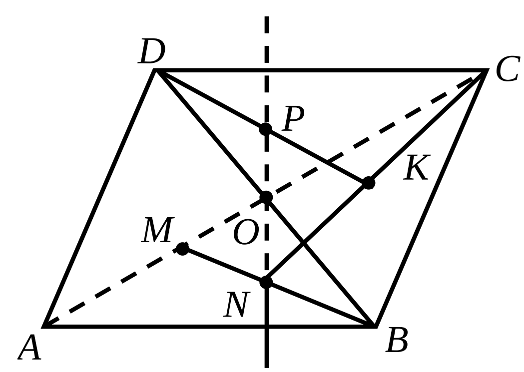
(a) The necessary and sufficient condition for three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be collinear is that there exist scalars x, y, z, not all zero, such that, $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$.

(b) The necessary and sufficient condition for four points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ to be coplanar is that there exist scalars x, y, z and u, not all zero, such that $x\vec{a}+y\vec{b}+z\vec{c}+u\vec{d}=\vec{0}$

Illustration 3:

Let 'O' be the point of intersection of diagonals of a parallelogram ABCD. The points M, N, K and P are the mid points of OA, MB, NC and KD respectively. Show that N, O and P are collinear.

Solution



Let $O\equiv(\vec{o})$, $A=(\vec{a})$, $B=(\vec{b})$

Now $M\equiv\dfrac{\vec{a}}{2}$, N $\equiv\dfrac{\frac{\vec{a}}{2}+\vec{b}}{2}=\dfrac{\vec{a}+2\vec{b}}{4}$

K $\equiv\dfrac{\frac{\vec{a}+2\vec{b}}{4}-\vec{a}}{2}=\dfrac{2\vec{b}-3\vec{a}}{8}$

$$P \equiv \frac{-\vec{b} + \frac{2\vec{b} - 3\vec{a}}{8}}{2} = \frac{-6\vec{b} - 3\vec{a}}{16}$$

$$\Rightarrow \ \overrightarrow{OP} = -\frac{3}{16}(2\vec{b} + \vec{a})$$

Also, $\overrightarrow{ON} = \frac{1}{4}(\vec{a} + 2\vec{b}) = -\frac{1}{6}\left(\overrightarrow{OP}\right)$

Hence points *N*, *O* and *P* are collinear.

7. Section Formula

Let A, B and C be three collinear points in space having position vectors $\vec{a}$, $\vec{b}$ and $\vec{r}$. Then

$$\vec{r} = \frac{m\vec{a} + m\vec{b}}{m + n}$$

8. Scalar Product of Two Vectors (Dot Product)

The scalar product, $\vec{a}.\vec{b}$ of two non-zero vectors $\vec{a}$ and $\vec{b}$ is defined as $|\vec{a}||\vec{b}|\cos\theta$, where θ is angle between the two vectors, when drawn with same initial point.

Note that $0 \le \theta \le \pi$.

Properties

(i) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (Scalar product is commutative)

(ii) $\vec{a}^2 = \vec{a}.\vec{a} = |\vec{a}|^2 = a^2$

(iii) $(m\vec{a}).\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.(m\vec{b})$ (Where m is a scalar)

(iv) $\theta = \cos^{-1}\left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$

(v) $\vec{a}.\vec{b} = 0 \Leftrightarrow$ Vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other. [$\vec{a}, \vec{b}$, are non-zero vectors].

(vi) $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vii) $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

(viii) $(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 = \vec{a}^{\,2} - \vec{b}^{\,2}$

(ix) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Then $\vec{a}.\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

(x) Maximum value of $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

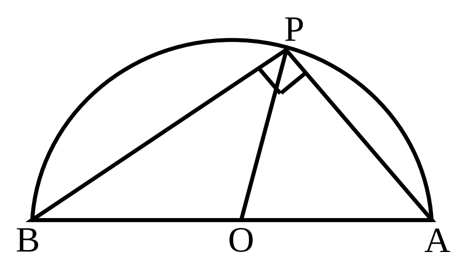
(xi) Minimum value of $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

Illustration 4:

Prove that the angle in a semi-circle is a right angle.

Solution

Let O be the centre and AB the bounding diameter of the semi-circle. Let P be any point on the circumference. With O as origin.



Let $\overrightarrow{OA} = a, \overrightarrow{OB} = -a$ and $\overrightarrow{OP} = r$

Obviously $OA = OB = OP$, each being equal to radius of the semi-circle.

$\overrightarrow{AP} = r - a$ and $\overrightarrow{BP} = r - (-a) = r + a$

$\therefore\ \overrightarrow{AP}\,.\,\overrightarrow{BP} = (r - a).(r + a) = r^2 - a^2 = OP^2 - OA^2 = 0$

$\Rightarrow$ AP and BP are perpendicular to each other, i.e., $\angle APB = 90^o$.

9. Vector (Cross) Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ are defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the both vectors $\vec{a}$ and $\vec{b}$ and $0 \le \theta \le \pi$.

Properties

(i) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(ii) $(m\vec{a})\times b = m(\vec{a}\times\vec{b}) = \vec{a}\times(m\vec{b})$ (where m is a scalar)

(iii) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow$ Vectors $\vec{a}$ and $\vec{b}$ are parallel. (Provided $\vec{a}$ and $\vec{b}$ are non-zero vectors).

(iv) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

(v) $\hat{i}\times\hat{j} = \hat{k} = -(\hat{j}\times\hat{i}),\ \hat{j}\times\hat{k} = \hat{i} = -(\hat{k}\times\hat{j}),\ \hat{k}\times\hat{i} = \hat{j} = -(\hat{i}\times\hat{k})$

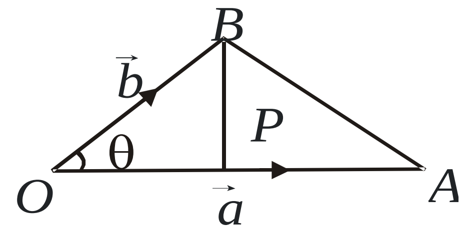
(vi) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(vii) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

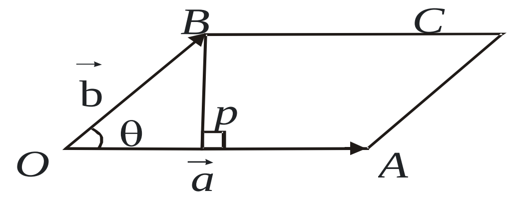
$$= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$$

(viii) $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$. (Note: we cannot find the value of θ by using this formula)



(ix) Area of triangle =

$$\frac{1}{2}ap = \frac{1}{2}ab\sin\theta = \frac{1}{2}|\vec{a}\times\vec{b}|$$



(x) Area of parallelogram =

$$ap = ab\sin\theta = |\vec{a}\times\vec{b}|$$

(xi) $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ (not commutative)

(xii) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

(xiii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(xiv) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

(xv) Lagranges Identity: for any two vectors

$$\vec{a} \text{ \& } \vec{b};(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$$

Illustration 5:

If a, b, c be three vectors such that $a + b + c = 0$, prove that $a \times b = b \times c = c \times a$ and deduce the sine rule $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Solution

Let $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ represent the vectors a, b, c respectively.

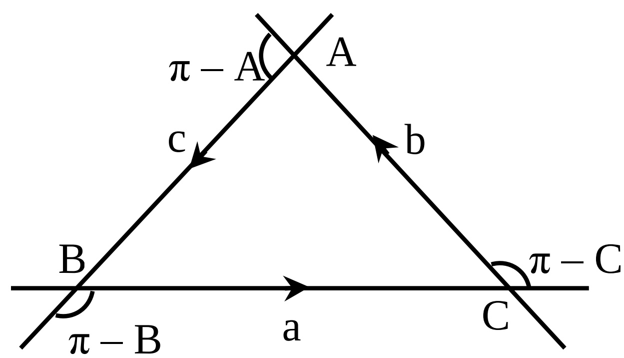
Then, we have

$a + b + c = 0,$

$\Rightarrow c = -(a + b)$

$\Rightarrow b \times c = b \times (-a - b)$

$= -b \times a = a \times b$ (since $b \times b = 0$)



Similarly, $c \times a = a \times b$

Hence, $b \times c = c \times a = a \times b$

$\Rightarrow bc \sin(\pi - A) = ca \sin(\pi - B) = ab \sin(\pi - C)$

$\Rightarrow bc \sin A = ca \sin B = ab \sin C$

Dividing by abc, we get

$$\Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

10. Scalar Triple Product (Box Product)

The scalar triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is defined as $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ and ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a} \quad \vec{b} \quad \vec{c}]$.

Properties

(i) $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = (\vec{c}\times\vec{a}).\vec{b} = -(\vec{b}\times\vec{a}).\vec{c} = -(\vec{c}\times\vec{b}).\vec{a} = -(\vec{a}\times\vec{c}).\vec{b}$

(ii) Three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(iii) The volume of the tetrahedron OABC with O as origin & the position vectors of A, B and C being $\vec{a}, \vec{b}$ & $\vec{c}$ respectively is given by $= \frac{1}{6}|[\vec{a}\ \vec{b}\ \vec{c}]|$

(iv) In a scalar triple product the position of dot & cross can be interchanged.

(v) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ *i.e.* $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(vi) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$

(vii) $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(viii) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ & $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

Illustration 6:

Prove that the formula for the volume *V* of a tetrahedron in terms of the lengths *a*, *b* and *c* of three concurrent edges and their mutual inclinations ϕ, θ and ψ is given by $V^2 = \frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\phi & \cos\psi \\ \cos\phi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}$.

Solution

Let *OABC* be the tetrahedron with *O* as origin. Let *a*, *b*, *c* be the position vectors of *A*, *B*, *C*.

Let $a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$, $c = c_1 i + c_2 j + c_3 k$.

Then

$$V = \frac{1}{6}[abc] = \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$V^2 = \frac{1}{36}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ a_1b_1 + a_2b_2 + a_3b_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ a_1c_1 + a_2c_2 + a_3c_3 & b_1c_1 + b_2c_2 + b_3c_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} |a^2| & a.b & a.c \\ a.b & |b|^2 & b.c \\ a.c & b.c & |c|^2 \end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix} a^2 & ab\cos\phi & ca\cos\psi \\ ab\cos\phi & b^2 & bc\cos\theta \\ ca\cos\psi & bc\cos\theta & c^2 \end{vmatrix}$$

$$=\frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\phi & \cos\psi \\ \cos\phi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}$$

11. Vector Triple Product

The vector triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as $\vec{a}\times(\vec{b}\times\vec{c})$.

Illustration 7:

For any vector $\vec{a}$, prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.

Solution

$$[\hat{i}\times(\vec{a}\times\hat{i})]+[\hat{j}\times(\vec{a}\times\hat{j})]+[\hat{k}\times(\vec{a}\times\hat{k})]$$

$$=[(\hat{i}.\hat{i})\vec{a}-(\hat{i}.\vec{a})\hat{i}][(\hat{i}.\hat{j})\vec{a}-(\hat{j}.\vec{a})\hat{j}]+[(\hat{k}.\hat{k})\vec{a}-(\hat{k}.\hat{a})\hat{k}]$$

$$=\vec{a}-(\hat{i}.\vec{a})\hat{i}+\vec{a}-(\hat{j}.\vec{a})\hat{j}+\vec{a}-(\hat{k}.\vec{a})\hat{k} \quad [\because \hat{i}.\hat{i}=\hat{j}\hat{j}=\hat{k}\hat{k}=1]$$

$$=3\vec{a}-[(\hat{i}.\vec{a})\hat{i}+(\hat{j}.\vec{a})\hat{j}+(\hat{k}.\hat{a})\hat{k}]$$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$\hat{i}.\vec{a}=\hat{i}\,(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=a_2\hat{i}^{\,2}+a_2\,(\hat{i}.\hat{j})+a_3\,(\hat{i}.\hat{k})=a_1\,(1)+a_2\,(0)+a_3\,(0)=a_1$

Similarly, $\hat{j}.\hat{a}=a_2, \hat{k}.\vec{a}=a_3$

$\therefore$ L.H.S. = $3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=3\vec{a}-\vec{a}=2\vec{a}$ = R.H.S.

Properties

If at least one of $\vec{a}$, $\vec{b}$ & $\vec{c}$ is a zero vector or $\vec{b}$ and $\vec{c}$ are collinear vectors or $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, only then $\vec{a}\times(\vec{b}\times\vec{c})=\vec{0}$.

(i) In all other cases $\vec{a}\times(\vec{b}\times\vec{c})$ will be a non-zero vector in the plane of non-collinear vectors and perpendicular to the vector $\vec{a}$.

(ii) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

12. Reciprocal System of Vectors

Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ be a system of three non-coplanar vectors. Then the system of vectors $\vec{a}', \vec{b}'$ & $\vec{c}'$ which satisfies

$\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ & $\vec{a}.\vec{b}'=a'.\vec{c}'=\vec{b}.\vec{a}'=\vec{b}.\vec{c}'=\vec{c}.\vec{a}'=\vec{c}.\vec{b}'=0$ is called the reciprocal system to the vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ and is given by

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Properties

(i) $\vec{a}.\vec{b}'=a.\vec{c}'=\vec{b}.\vec{a}'=\vec{b}.\vec{c}'=\vec{c}.\vec{a}'=\vec{c}.\vec{b}'=0$

(ii) The scalar triple product [a b c] formed from three non-coplanar vectors a, b, c is the reciprocal of the scalar triple product formed from reciprocal system.

13. Solving Of Vector Equation

Solving a vector equation means determining an unknown vector (or a number of vectors satisfying the given conditions)

Generally, to solve vector equations, we express the unknown as the linear combination of three non-coplanar vectors as

$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})$ as $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar and find x, y, z using given conditions. Sometimes we can directly solve the given conditions it would be more clearly from some examples.

Illustration 8:

Solve the vector equation for $\vec{r}$:

$\vec{r}\times\vec{b}=\vec{a}\times\vec{b}, \vec{r}.\vec{c}=0$, Provided that $\vec{c}$ is not perpendicular to $\vec{b}$.

Solution

We are given; $\vec{r}\times\vec{b}=\vec{a}\times\vec{b}$

$$\Rightarrow \quad (\vec{r}-\vec{a})\times\vec{b}=0$$

Hence $(\vec{r}-\vec{a})$ and $\vec{b}$ are parallel

$$\Rightarrow \quad \vec{r}-\vec{a}=t\,\vec{b} \qquad \ldots(\text{i})$$

And we know $\vec{r}.\vec{c}=0$,

$\therefore$ Taking dot product of (i) by $\vec{c}$ we get

$$\vec{r}.\vec{c}-\vec{a}.\vec{c}=t(\vec{b}.c) \Rightarrow 0-\vec{a}.\vec{c}=t(\vec{b}.\vec{c})$$

Or $$t=-\left(\frac{\vec{a}.\vec{c}}{\vec{b}.\vec{c}}\right) \qquad \ldots(\text{ii})$$

$\therefore$ from (i) and (ii) solution of $\vec{r}$ is;

$$\vec{r}=\vec{a}-\left(\frac{\vec{a}.\vec{c}}{\vec{b}.\vec{c}}\right)\vec{b}.$$