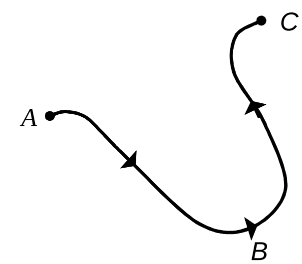
1. KINEMATIC VARIABLES

Kinematics is study of geometry of motion without considering its cause. It deals with position, distance travelled by a body and its displacement, speed and velocity and acceleration of a body. Before going in details about different kinds of motion separately, let us know about these kinematics variable first.

1.1 DISTANCE AND DISPLAEMENT

When a particle is moving its successive position in general may lie on a curve, say, *ABC* shown in figure. The curve is then called the path of the particle.

The total length of the path followed by the body is called the distance traversed by the body. Its unit in S.I. system is 'metre'. It is a scalar quantity. Sometimes we may not be interested in the actual path of the particle but only in its final position *C* relative to the initial position *A*. *The directional distance between initial and final positions of the particle AC in the figure is called displacement*. It is a vector quantity.



1.2 SPEED AND VELOCITY

Speed is the rate at which a moving body describes its paths. The path may be a curve or a straight line and its shape need not be considered to decide the speed. If a particle traverses a distance Δs during a time Δt,

Average speed, $$v = \frac{\Delta s}{\Delta t} \quad \ldots(\mathbf{1})$$

If the interval of time Δt is infinitesimally small approaching zero, this ratio is called of instantaneous speed or sometimes referred as speed of particle.

i.e., instantaneous speed $$v = \underset{\Delta t \to 0}{Lt} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \quad \ldots(\mathbf{2})$$

Speed is a scalar quantity and in S.I. system it is measured in meter/second (m/s).

Velocity is defined as the rate of change of position.

If during a time internal Δt, a particle changes its position by $\Delta \vec{r}$.

Average velocity, $$\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t} \quad \ldots(\mathbf{3})$$

Also, instantaneous velocity, $$\vec{v} = \underset{\Delta t \to 0}{Lt} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad \ldots(\mathbf{4})$$

Velocity is a vector quantity having direction same as that of displacement and is measured in meter per second (ms^{-1}).

1.3 ACCELERATION

The rate of change of velocity is called as acceleration. The change in velocity can be either in the magnitude of the velocity or in the direction of velocity or in both of them simultaneously.

If $\Delta\vec{v}$ is change in velocity which takes place in time interval Δt, then during this interval

Average acceleration, $\vec{\bar{a}} = \dfrac{\Delta\vec{v}}{\Delta t}$ …(**5**)

Also, instantaneous acceleration, $\vec{a} = \underset{\Delta t \to 0}{Lt} \dfrac{\Delta\vec{v}}{\Delta t} = \dfrac{d\vec{v}}{dt}$ …(**6**)

Acceleration is a vector quantity and is measured in meter/second^2 (ms^{-2}).

Illustration 1

A particle moves along a semicircular path of radius R in time t with constant speed. For the particle calculate

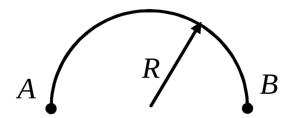
(i) distance travelled

(ii) displacement

(iii) average speed

(iv) average velocity

(v) average acceleration



Solution:

(i) Distance = length of path of particle = $\overset{\frown}{AB}$ = πR

(ii) Displacement = minimum distance between initial and final point

$= AB = 2R$

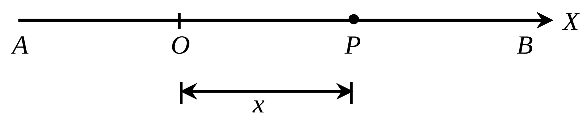
(iii) Average speed, $v = \dfrac{\text{distance}}{\text{time}} = \dfrac{\pi R}{t}$

(iv) Average velocity = $\dfrac{2R}{t}$

(v) Average acceleration = $\dfrac{\text{Change in velocity}}{\text{time taken}} = \dfrac{2v}{t} = \dfrac{2\pi R}{t^2}$

2. MOTION IN ONE DIMENSION

As discussed earlier for the analysis of motion, we need a reference frame made of origin and a set of co-ordinate axis depending on the type of motion. If a motion is taking place on a straight line, we need one axis for the analysis of motion. So we call this motion as motion in one dimension. Similarly if motion is taking place in a plane, we need two co-ordinate axes and we call motion as motion in two dimension. Three co-ordinate axes are required if motion is random motion in space and we call such motion as motion in three dimension. Here we are going to discuss motion on a straight line i.e., motion in one dimension.



Consider a particle moving on a straight line *AB*. For the analysis of motion we take origin, *O* at any point on the line and *x*-axis along the line. Generally we take origin at the point from where particle starts its motion and rightward direction as positive *x*-direction. At any moment if particle is at P then its position is given by OP = *x*.

Velocity is defined as, $v = \dfrac{dx}{dt}$

Acceleration is defined as, $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2} = \dfrac{v\,dv}{dx}$

2.1 MOTION IN A STRAIGHT LINE WITH UNIFORM VELOCITY

If motion takes place with a uniform velocity *v* on a straight line, then

displacement in time *t*, $s = v.t$ …(**7**)

acceleration of particle is zero.

2.2 MOTION IN A STRIGHT LINE WITH UNIFORM ACCELERATION

Equation of Motion

Let a particle move in a straight line with initial velocity *u* (velocity at time *t* = 0) and with uniform acceleration *a*. Let its velocity be *v* at the end of the interval of time *t* (final velocity at time t). Let *S* be its displacement at the instant *t*.

Now, acceleration $a = \dfrac{v-u}{t}$

or, $v = u + at$ … (**8**)

If '*u*' and '*a*' are in the same direction, '*a*' is positive and hence the velocity increases with time. If '*a*' is opposite to the direction of '*u*', '*a*' is negative and the velocity decreases with time.

Displacement during the time interval *t* = average velocity *x t*

$$S = \frac{u+v}{2} \times t \qquad \text{... (9)}$$

Eliminating v from equations (8) and (9), we get

$$S = \frac{u+u+at}{2} \times t$$

or $$S = ut + \frac{1}{2}at^2 \qquad \dots (\mathbf{10})$$

Another equation is obtained by eliminating t from equations (8) and (9)

$$v = u + at \quad \text{i.e.} \quad a = \frac{v-u}{t} \quad \text{and} \quad S = \frac{v+u}{2} \times t$$

$$aS = \frac{v-u}{t} \times \frac{v+u}{2} \times t = \frac{v^2 - u^2}{2}$$

or, $$v^2 - u^2 = 2aS$$

$$v^2 = u^2 + 2aS \qquad \dots (\mathbf{11})$$

***Distance traversed by the particle in the n^{th} second of its motion*:**

The velocity at the beginning of the n^{th} second = $u + a\,(n - 1)$

The velocity at the end of the n^{th} second = $u + an$

Average velocity during the n^{th} second = $\dfrac{u + a\,(n-1) + u + an}{2} = u + \dfrac{1}{2}a\,(2n-1)$

Distance traversed during this one second

$$S_n = \text{average velocity} \times \text{time} = \left[u + \frac{1}{2}a\,(2n-1)\right] \times 1$$

i.e., $$S_n = u + \frac{1}{2}a\,(2n-1) \qquad \dots (\mathbf{12})$$

The five equations (eqn. 8 to eqn. 12) derived above are very important and are to be memorized. They are very useful in solving problems in straight-line motion.

Calculus method of deriving equations of motion:

The acceleration of a body is defined as

$$a = \frac{dv}{dt} \quad \text{i.e.,} \quad dv = a\,dt$$

Integrating, we get, $v = at + A$

Where A is constant of integration. By the initial condition when $t = 0$, $v = u$ (initial velocity), we get $A = u$

$\therefore$ $$v = u + at$$

we know that the instantaneous velocity $v = \dfrac{dS}{dt}$.

Displacement of body for duration dt from time t to $t + dt$ is given by

$$dS = v\,dt = (u + at)\,dt$$

On integration we get,

$$S = ut + \frac{1}{2}at^2 + B \text{, where } B \text{ is integration constant.}$$

At, $t = 0$, $S = 0$ this equation yields $B = 0$.

Therefore, $S = ut + \frac{1}{2}at^2$

Acceleration = $\frac{dv}{dt} = \frac{dv}{dS}.\frac{dS}{dt} = v.\frac{dv}{dS}$

$\therefore$ $a = v\frac{dv}{dS}$

$a.dS = v.dv$

Integrating we get,

$$aS = \frac{v^2}{2} + C \text{, where } C \text{ is integration constant.}$$

Applying initial condition, when $S = 0$, $v = u$ we get

$$0 = \frac{u^2}{2} + C \quad \text{or} \quad C = \frac{u^2}{2}$$

$\therefore$ $aS = \frac{v^2}{2} - \frac{u^2}{2}$

$\therefore$ $v^2 - u^2 = 2aS$

$\Rightarrow$ $v^2 = u^2 + 2aS$

If S_1 and S_2 are the distances traversed during n seconds and $(n - 1)$ seconds

$$S_1 = u\,n + \frac{1}{2}an^2$$

$$S_2 = u(n - 1) + \frac{1}{2}a(n - 1)^2$$

$\therefore$ Displacement in n^{th} second

$$S_n = S_1 - S_2 = u_n + \frac{1}{2}an^2 - u(n - 1) - \frac{1}{2}a(n - 1)^2$$

$$S_n = u + \frac{1}{2}a(2n - 1)$$

Illustration 2

A certain automobile manufacturer claims that its super-delux sport's *car* will accelerate from rest to a speed of 42.0 ms^{-1} in 8.0 s. Under the important assumption that the acceleration is constant.

(a) determine the acceleration of car in ms^{-2} .

(b) find the distance the car travels in 8.0 s.

(c) find the distance the car travels in 8th second.

Solution

(a) We are given that u = 0 and velocity after 8 s is 42 m/s, so we can use $v = u + at$ to find acceleration

$$a = \frac{v-u}{t} = \frac{42.0-u}{8.0} = 5.25 \text{ ms}^{-2}$$

(b) distance travelled in 8.0 s,

We can use, $S = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 5.25 \times 8^2 = 168 \text{ m}$

(c) distance travelled in 8th second,

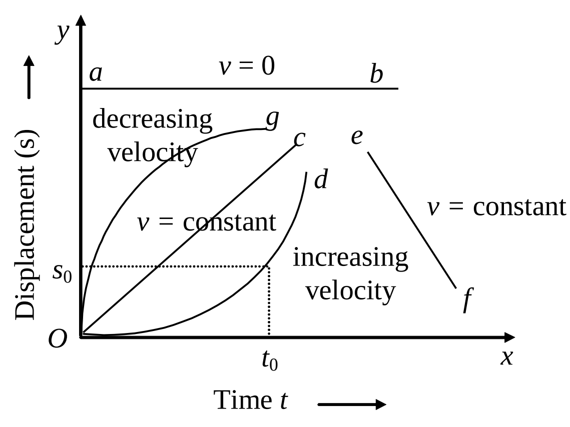
We have, $S_n = u + (2n - 1)\frac{a}{2}$

$$= (2 \times 8 - 1) \times \frac{5.25}{2} = 39.375 \text{ m}$$

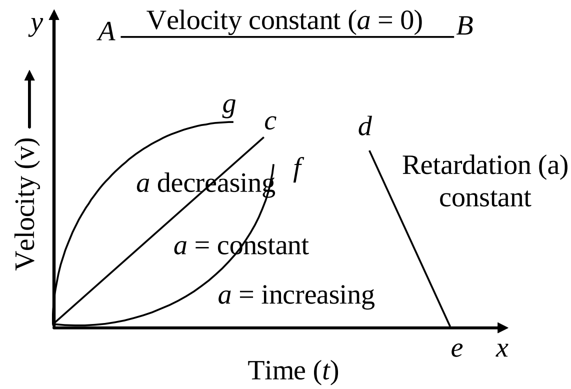
2.3 GRAPHICAL REPRESENTATION OF MOTION

(i) Displacement-time graph: If displacement of a body is plotted on *Y*-axis and time on *X*-axis, the curve obtained is called displacement-time graph. The instantaneous velocity at any given instant can be obtained from the graph by finding the slope of the tangent at the point corresponding to the time.

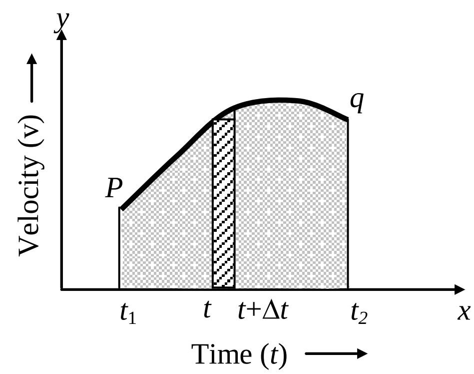
If the graph obtained is parallel to time axis, the velocity is zero (ab in figure). If the graph is an oblique straight line, the velocity is constant (*oc* and *ef* in figure). If the graph obtained is a curve (*od* in figure) whose slope increases with time, the velocity goes on increasing i.e., the motion is accelerated. If the graph obtained is a curve of type *og* in figure whose slope decreases with time, the velocity goes on decelerated.



Velocity-time graph: Similarly a graph can be drawn between velocity and time of a moving body. The curve obtained will be similar to the one shown in figure. If the graph is a straight line parallel to time axis, the velocity is constant and acceleration is zero (*AB* in figure). If the graph is an oblique straight line, the motion is uniformly accelerated (positive slope) or uniformly decelerated (negative slope). The velocity-time curve may be used to determine displacement, velocity and acceleration i.e., it is used to specify the entire motion.



To obtain the velocity at any time t we draw a perpendicular from given instant on the curve and noting the corresponding point and dropping a perpendicular from this point on the velocity axis. The slope of the tangent at the point corresponding to the particular time on the curve gives instantaneous acceleration. The area enclosed by velocity-time graph and time axis for a time interval gives the displacement during this time interval. In figure the shaded area pqt_2t_1 represents the net displacement during the time interval between t_1 and t_2.



Acceleration-time graph: It is a graph plotted between time and acceleration. If the graph is a line parallel to time axis, the acceleration is constant. If it is a straight line with positive slope, the acceleration is uniformly increasing. The co-relation of the graph explained above follows directly from the differential expressions $v=\dfrac{dS}{dt}$ and $a=\dfrac{dv}{dt}$.

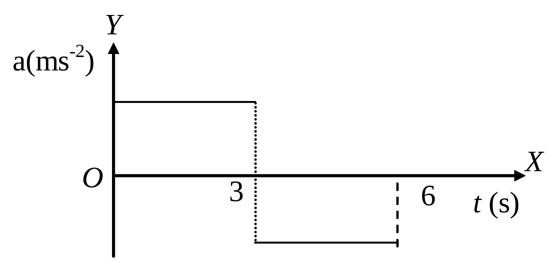
The area enclosed between acceleration-time graph and time axis gives change in velocity during this time interval.

Illustration 3

At $t = 0$ a particle is at rest at origin. Its acceleration is 2ms^{-2} for the first 3s and -2 ms^{-2} for the next 3s. Plot the acceleration versus time, velocity versus time and position versus time graph.

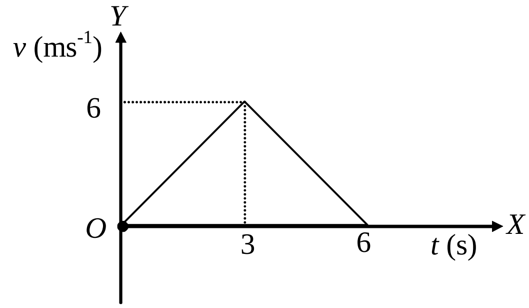
Solution

We are given that for first 3s acceleration is 2 ms^{-2} and for next 3s acceleration is -2ms^{-2}. Hence acceleration time graph is as shown in the figure.

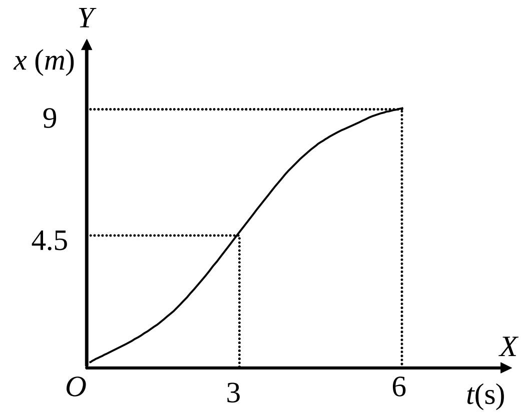


The area enclosed between a-t curve and t-axis gives change in velocity for the corresponding interval. Also at $t = 0$, $v = 0$, hence final velocity at $t = 3$s will increase to 6 ms^{-1}. In next 3s the velocity will decrease to zero. Hence the velocity time graph is as shown in figure.

Note that $v - t$ curves are taken as straight line as acceleration is constant.



Now for displacement time curve, we will use the fact that area enclosed between v–t curve and time axis gives displacement for the corresponding interval. Hence displacement in first three second is 4.5 m and in next three second is 4.5 m. Also the x-t curve will be of parabolic nature as motion is with constant acceleration. Therefore x-t curve is as shown in figure.



2.4 VERTICAL MOTION UNDER GRAVITY

When a body is thrown vertically upward or dropped from a height, it moves in a vertical straight line. If the air resistance offered by air to the motion of the body is neglected, all bodies moving freely under gravity will be acted upon by its weight only. This causes a constant vertical acceleration g having value 9.8 m/s^2, so the equation for motion in a straight line with constant acceleration can be used. In some problems it is convenient to take the downward direction as positive, in such case all the measurement in downward direction is considered as positive i.e., acceleration will be $+g$. But sometimes we may need to take upward as positive and in such case acceleration will be $-g$.

Projection of a body vertically upwards

Suppose a body is projected vertically upward from a point A with velocity u.

If we take upward direction as positive

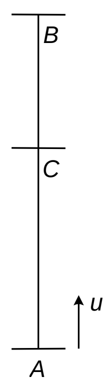
(i) At time t, its velocity $v = u - gt$

(ii) At time t, its displacement from A is given by

$$S = ut - \frac{1}{2}gt^2$$

(iii) Its velocity when it has a displacement S is given by

$$v^2 = u^2 - 2gS$$



(iv) When it reaches the maximum height from A, its velocity v = 0. This happens when $t=\frac{u}{g}$. The body is instantaneously at rest at the highest point B.

(v) The maximum height reached

$$H=\frac{u^2}{2g}$$

(vi) Total time to go up and return to the point of projection = $\frac{2u}{g}$

Since, S = 0 at the point of projection,

$$S = ut - \frac{1}{2}gt^2$$

$$0 = ut - \frac{1}{2}gt^2 \text{ or } t = \frac{2u}{g}$$

Since the time of ascent $= \frac{u}{g}$, the time of descent = $\frac{2u}{g}-\frac{u}{g}=\frac{u}{g}$

(vii) At any point C between A and B, where $AC = S$, the velocity v is given by $v=\pm\sqrt{u^2-2gS}$

The velocity of body while crossing C upwards $= +\sqrt{u^2-2gS}$

and while crossing C downwards is $-\sqrt{u^2-2gS}$. The magnitudes of the velocities are the same.

Illustration 4

A body is projected upwards with a velocity 98 m/s. Find (a) the maximum height reached, (b) the time taken to reach the maximum height, (c) its velocity at a height 196 m from the point of projection, (d) velocity with which it will cross down the point of projection and (e) the time taken to reach back the point of projection. [Take g = 9.8 m/s^2]

Solution

(a) The maximum height reached

Initial upward velocity u = 98 m/s

Acceleration $a = (-g) = -9.8$ m/s^2

Maximum height reached H is given by

$$v^2 = u^2 + 2aS$$

$$0 = 98^2 + 2(-9.8)H$$

$$H = \frac{98^2}{2\times 9.8} = \mathbf{490\ m}$$

(b) The time taken to reach the maximum height

$$t=\frac{u}{g}=\frac{98}{9.8} = \textbf{10 s}$$

(c) Velocity at a height of 196 m from the point of projection

$v^2 = u^2 + 2aS$

$v^2 = 982 + 2(-9.8)196$

$v = \pm\sqrt{5762.4} = \pm \textbf{75.91 m/s}$

+ 75.91 m/s while crossing the height upward and –75.91 m/s while crossing it downward.

(d) Velocity with which it will cross down the point of projection

$$v^2 = u^2 + 2gS$$

At the point of projection $S = 0$

$\therefore \qquad v = \pm u$

While crossing the point of projection downwards, $v = -u =$ **– 98 m/s**

The velocity has the same magnitude as the initial velocity but reversed in direction.

(e) The time taken to reach back the point of projection

$$t=\frac{2u}{g}=\frac{2\times 98}{9.8} = \textbf{20 s}$$

2.5 GENERAL MOTION IN A STRAIGHT LINE

Saying general motion means motion it is other then motion with uniform velocity or motion with uniform acceleration. In such motion we will be given the relation between two variables among position, velocity, acceleration and time and other are to be calculated.

For this we need to use, Velocity (v) = $\frac{dx}{dt}$ and Acceleration $(a)=\frac{dv}{dt}=\frac{vdv}{dx}$

and then using calculus we find the unknown variables.

Illustration 5

A particle moving in a straight line has an acceleration of $(3t - 4)$ ms^{-2} at time t seconds. The particle is initially 1 m from O, a fixed point on the line, with a velocity of 2 ms^{-1}. Find the times when the velocity is zero. Find also the displacement of the particle from O when $t = 3$.

Solution

Using $\quad a=\frac{dv}{dt}$ gives $\frac{dv}{dt}=3t-4$

$$\Rightarrow \qquad \int_0^v dv=\int_0^t (3t-4)\,dt$$

$$\Rightarrow \qquad v - 2 = \frac{3t^2}{2} - 4t$$

$$\Rightarrow \qquad v = \frac{3t^2}{2} - 4t + 2$$

The velocity will be zero when $\frac{3t^2}{2} - 4t + 2 = 0$

i.e., when $(3t - 2)(t - 2) = 0$

$$\Rightarrow \qquad t = \frac{2}{3} \text{ or } 2$$

Using $\frac{ds}{dt} = v$ we have $\frac{ds}{dt} = \frac{3t^2}{2} - 4t + 2,$

$$\Rightarrow \qquad \int_1^s ds = \int_0^3 \left(\frac{3t^2}{2} - 4t + 2 \right) dt$$

$$\Rightarrow \qquad s - 1 = \left[\frac{t^3}{2} - 2t^2 + 2t \right]_0^3 = 1.5 \qquad \Rightarrow \qquad s = 2.5$$

Therefore the particle is 2.5 *m* from *O* when t = **3 s**.

3. MOTION IN TWO DEMENSIONS

Motion in a plane is called as motion in two dimensions e.g., projectile motion, circular motion etc. For the analysis of such motion our reference will be made of an origin and two co-ordinate axes x and y. Position of particle is known by knowing its co-ordinate (x, y). Velocity of particle will be resultant of velocities in x and y direction v_x and v_y. Similarly acceleration will be in the two directions. For analysis of such motion we analyse the motion along two axes independently, i.e., while dealing motion in x-direction we need not to think what is going on in y-direction and vice versa.

We have to study about projectile motion, circular motion and relative velocity under the head of motion in two dimensions. We shall discuss circular motion in next lesson.

4. PROJECTILE MOTION

A projectile is a particle, which is given an initial velocity, and then moves under the action of its weight alone. If the initial velocity is vertical, the particle moves in a straight line and such motion we had already discussed in ‘motion in one dimension as motion under gravity’. Here we are going to discuss the motion of particle which is projected obliquely near the earth’s surface. While discussing such motion we shall suppose the motion to the be within such a moderate distance from the earth’s surface, that acceleration due to gravity may be considered to remain sensibly constant. We shall also neglect the resistance of air and consider the motion to be in vacuum.

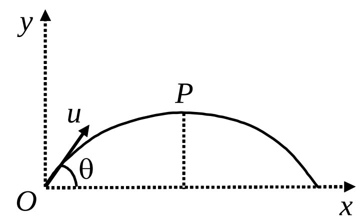
4.1 IMPORTANT TERMS USED IN PROJECTILE MOTION

When a particle is projected into air, the angle that the direction of projection makes with horizontal plane through the point of projection is called the *angle of projection*; the path, which the particle describes, is called the *trajectory*; the distance between the point of projection and the point where the path meets any plane draws through the point of projection is its *range*; the time that elapses in air is called as *time of flight* and the maximum distance above the plane during its motion is called as *maximum height* attained by the projectile.

4.2 ANALYTICAL TREATMENT OF PROJECTILE MOTION

Consider a particle projected with a velocity u of an angle θ with the horizontal from earth's surface. If the earth did not attract a particle to itself, the particle would describe a straight line; on account of attraction of earth, however, the particle describes a curved path. This curve will be proved later to be always a parabola.

Let us take origin at the point of projection and x-axis and y-axis along the surface of earth and perpendicular to it respectively as shown in figure.



By the principle of physical independence of forces, the weight of the body only has effect on the motion in vertical direction. It, therefore, has no effect on the velocity of the body in the horizontal direction, and horizontal velocity therefore remains unaltered.

Motion in x-direction:

Motion in x–direction is motion with uniform velocity.

At, $t = 0$, $x_0 = 0$ and $u_x = u\cos\theta$

Position after time t, $x = x_0 + u_{xt}$

$$\Rightarrow \quad x = (u\cos\theta)\,t \qquad \ldots \text{(i)}$$

Velocity of any time t, $v_x = u_x$

$$\Rightarrow \quad v_x = u\cos\theta \qquad \ldots \text{(ii)}$$

Motion in y-direction:

Motion in y-direction is motion with uniformly acceleration.

when, $t = 0$, $y_0 = 0$, $u_y = \text{u}\sin\theta$ and $a_y = -g$

$\therefore$ After time 't', $v_y = u_y + a_y t$

$$\Rightarrow \quad v_y = u\sin\theta - gt \qquad \ldots \text{(iii)}$$

$$y = y_0 + u_y t + \frac{1}{2}a_y t^2 \qquad \ldots \text{(iv)}$$

Also, $v_y^2 = u_y^2 + 2a_y\text{y}$

$$\Rightarrow \quad v_y^2 = u^2\sin^2\theta - 2gy \qquad \ldots \text{(v)}$$

Time of Flight (T):

Time of flight is the time during which particle moves from O to O' i.e., when $t = T$, $y = 0$

$\therefore$ From equation (iv)

$$O = u\sin\theta T - \frac{1}{2}gT^2$$

$$\Rightarrow \qquad T = \frac{2u\sin\theta}{g}, \qquad \ldots(13)$$

Range of Projectile (R) :

Range is horizontal distance traveled in time T,

i.e., $\qquad R = x$ in time T

$\therefore$ From equation (ii)

$$R = u\cos\theta . T = u\cos\theta \frac{2u\sin\theta}{g}$$

$$R = \frac{u^2\sin 2\theta}{g} \qquad \ldots(14)$$

Maximum height reached (H):

At the time particle reaches its maximum height velocity of particle becomes parallel to horizontal direction i.e., $v_y = 0$ when $y = H$

$\therefore$ From equation (v)

$$0 = u^2\sin^2\theta - 2gH$$

$$\Rightarrow \qquad H = \frac{u^2\sin^2\theta}{2g} \qquad \ldots(15)$$

Equation of trajectory:

The path traced by a particle in motion is called trajectory and it can be known by knowing the relation between x and y

From equation (i) and (iv) eliminating time t we get

$$y = x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2}\sec^2\theta \qquad \ldots(16)$$

This is trajectory of path and is equation of parabola. So we can say the path of particle is parabolic.

Velocity and direction of motion after a given time:

After time 't' $v_x = u\cos\theta$ and $v_y = u\sin\theta - gt$

Hence resultant velocity $v = \sqrt{v_x^{\,2} + v_y^{\,2}}$

$$= \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$$

If direction of motion makes an angle α with horizontal.

$$\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

$$\Rightarrow \qquad \alpha = \tan^{-1}\left(\frac{u\sin\theta - gt}{u\cos\theta}\right)$$

Velocity and direction of motion at a given height

At a height 'h', $v_x = u\cos\theta$

And $\qquad v_y = \sqrt{u^2\sin^2\theta - 2gh}$

$\therefore$ Resultant velocity $v = \sqrt{{v_x}^2 + {v_y}^2}$

$$v = \sqrt{u^2 - 2gh}$$

Note that this is the velocity that a particle would have at height h if it is projected vertically from ground with u.

Illustration 6

A particle is projected up from the ground with a velocity of 147 m/s at an angle of projection 30° with the horizontal. Find (a) the time of flight, (b) the greatest height reached, (c) the horizontal range and (d) the velocity at a height of 98 m.

Solution

(a) The time of flight $T = \frac{2u\sin\theta}{g} = \frac{2\times 147\times \sin 30^\circ}{9.8}$ = **15 s**

(b) The greatest height reached $H = \frac{u^2\sin^2\theta}{2g} = \frac{(147/2)^2}{2\times 9.8}$ = **275.6 m**

(c) The horizontal range $R = \frac{u^2\sin 2\theta}{g} = \frac{147^2\times\sin 60^\circ}{9.8}$

$= (1102.5)\sqrt{3}$ = **1909.6 m**

(d) The horizontal velocity at a height of 98 m = $u\cos\theta$

i.e., $v_x = 147\cos 30^\circ = 147\times\frac{\sqrt{3}}{2} = \frac{147\sqrt{3}}{2}$ m/s = 127.3 m/s

The vertical velocity at this height

$v_y^2 = (u\sin\theta)^2 - 2\times 9.8\times 98 = (147\sin 30^\circ)^2 - 19.6\times 98 = 3481.45$

$v_y = 59$ m/s

$v = \sqrt{v_x^2 + v_y^2}$ = **140.3 m/s** in a direction α with horizontal where α = $\mathbf{tan^{-1}}$ **(0.4635) = 24.86°**

The velocity at 98 m height has two values (i) while going up and (ii) while coming down. Both have the same magnitude except for the difference in directions.

4.3 SOME IMPORTANT POINTS REGARDING PROJECTILE MOTION OVER A HORIZONTAL PLANE

(i) For a given velocity of projection, the range of horizontal plane will be maximum when angle of projection is 45°.

We have range of projectile. $R = \dfrac{u^2 \sin 2\theta}{g}$

Therefore if we keep on increasing θ range will increase and then decrease. Its value will be maximum when $\sin 2\theta$ is maximum i.e., θ = 45°

Also, maximum range $R_{\max} = \dfrac{u^2}{g}$

(ii) For a given range and given initial speed of projection, there are two possible angle of projection which are complementary angle i.e., if one is θ other will be (90°– θ).

Illustration 7

What is the least velocity with which a cricket ball can be thrown through a distance of 100 m?

Solution

Since the range is given, the least velocity of projection is that value when the angle of projection is 45°. For velocity u to be least

$$\frac{u^2 \sin 2\theta}{g} = 100 \text{ where } \theta = 45^\circ$$

or, $$\frac{u^2}{g} = 100$$

$$u^2 = 100 \times 9.8 = 980$$

$$u = \sqrt{980} = 31.3 \text{ m/s}$$

Illustration 8

Find the maximum horizontal range when the velocity of projection is 30 m/s. Find the two directions of projection to give a range of 45 m. Take g= 10 m/s^2.

Solution

(i) Maximum range $R_m = \dfrac{u^2}{g} = \dfrac{30^2}{10} = 90$ m

(ii) Now $\dfrac{u^2 \sin 2\theta}{g} = 45$

or, $\sin 2\theta = \dfrac{45\times 10}{30\times 30} = \dfrac{1}{2}$

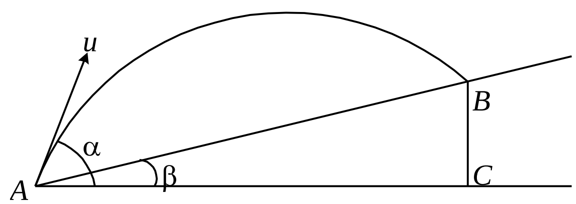
or, $2\theta = 30^\circ$ or 150° [$\because$ (180 – θ) = sin θ]

or $\theta = 15^\circ$ or 75°

Therefore for a given velocity of projection and for a given range, two directions of projection are possible.

4.4 RANGE OF PROJECTILE ON AN INCLINED PLANE THROUGH THE POINT OF PROJECTION

A particle is projected from a point *A* on an inclined plane, which is inclined at an angle β to the horizon with a velocity *u* at an elevation α. The direction of projection lies in the vertical plane through *AB*, the line of the greatest slope of the plane.



Let the particle strike the plane at *B* so that *AB* is the range on the inclined plane.

The initial velocity of projection *u* can be resolved into a component $u\cos(\alpha-\beta)$ along the plane and a component $u\sin(\alpha-\beta)$ perpendicular to the plane. The acceleration due to gravity *g* which acts vertically down can be resolved into components $g\sin\beta$ up the plane and $g\cos\beta$ perpendicular to the plane. By the principle of physical independence of forces the motion along the plane may be considered independent of the motion perpendicular to the plane. Let *T* be the time, which the particle takes to go from *A* to *B*. Then in this time the distance traversed by the projectile perpendicular to the plane is zero.

$\therefore \quad 0 = u\sin(\alpha-\beta)\,T - \dfrac{1}{2}g\cos\beta\,T^2$

$\therefore \quad T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ **…(17)**

During this time the horizontal velocity of the projectile ($u\cos\alpha$) remains constant. Hence the horizontal distance described is given by

$$AC = u\cos\alpha T = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos\beta}$$

$\therefore \quad AB = \dfrac{AC}{\cos\beta} = \dfrac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$

$\therefore$ Range on the inclined plane $= \dfrac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$ …(**18**)

Maximum range on the inclined plane

$$R = \frac{2u^2 \sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta} = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha - \beta) - \sin\beta]$$

For given values of u and β, R is maximum when

$\sin(2\alpha - \beta) = 1$ i.e., $(2\alpha - \beta) = 90^\circ$ i.e., $\alpha = (45^\circ + \beta/2)$

If R_m represents the maximum range on the inclined plane,

$$R_m = \frac{u^2}{g\cos^2\beta}(1 - \sin\beta)$$

$$R_m = \frac{u^2}{g(1+\sin\beta)}$$

For a given velocity of projection, it can be shown that there are two directions of projection which are equally inclined to the direction of maximum range.

Now $$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha - \beta) - \sin\beta]$$

For given values of u, β and R, sin (2α – β) is constant. There are two values of (2α – β) each less than 180° that can satisfy the above equation.

Let $(2\theta_1 - \beta)$ and $(2\theta_2 - \beta)$ be the two values. Then

$$2\theta_1 - \beta = 180^\circ - (2\theta_2 - \beta)$$

$$\theta_1 - \beta/2 = 90^\circ - (\theta_2 - \beta/2)$$

$$\theta_1 - (45^\circ + \beta/2) = (45^\circ + \beta/2) - \theta_2$$

Since (45° + β/2) is the angle of projection giving the maximum range, it follows that the direction giving maximum range bisects the angle between the two angles of projection that can give a particular range.

Illustration 9

A particle is projected at an angle α with horizontal from the foot of a plane whose inclination to horizontal is β. Show that it will strike the plane at right angles if, cot β = 2 tan (α – β).

Solution

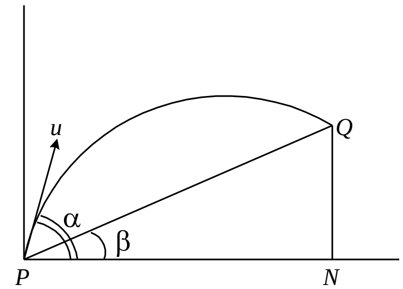
Let u be the velocity of projection so that u cos (α – β) and u sin (α – β) are the initial velocities respectively parallel and perpendicular to the inclined plane. The acceleration in these two directions are ($-g$ sin β) and ($-g$ cos β).

The initial component of velocity perpendicular to PQ is u sin (α – β) and the acceleration in this direction is ($-g$ cos β).

If T is the time the particle takes to go from P to Q then in time T the space described in a direction perpendicular to PQ is zero.

$$0 = u \sin(\alpha - \beta) \cdot T - \frac{1}{2} g \cos\beta \cdot T^2$$

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$



If the direction of motion at the instant when the particle hits the plane be perpendicular to the plane, then the velocity at that instant parallel to the plane must be zero.

∴ $u\cos(\alpha - \beta) - g\sin\beta T = 0$

$$\frac{u\cos(\alpha-\beta)}{g\sin\beta} = T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

∴ $\cot\beta = 2\tan(\alpha - \beta)$

5. RELATIVE VELOCITY

The terms 'rest' and 'motion' are only relative. For example, when we say that a train is moving with velocity 30 m.p.h, we mean is that it is the velocity with which the train moves with respect to an observer on the earth who is regarded as fixed. This is not true strictly since a person on the earth unconsciously partakes the rotatory motion of the earth round its axis and the motion of earth round the sun. In addition, he shares the motion of entire solar system through space with respect to certain fixed stars. Thus there is no absolutely fixed point on the earth about which we can measure motion. Hence a person on the earth can never realise absolute motion or absolute rest.

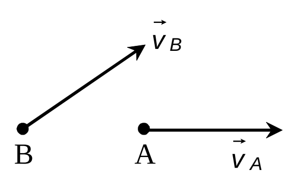
Let us consider two motor cars *A* and *B* moving in the same direction on a road with equal speed. To a person seated in *A*, if he were unconscious of his motion, the car *B* would appear to be at rest. The line joining the two cars will always remain constant in magnitude and direction. The velocity of *B* relative to *A* or the velocity of *A* relative to *B* is zero.

On the other hand if *A* is moving with 30 m.p.h and *B* with 40 m.p.h in the same direction, a person in A would observe the car *B* to be drawing away from him at the rate of 10 m.p.h. This represents the velocity of *B* relative to *A*. If, however, *B* is moving opposite to the direction of *A* with velocity 40 m.p.h., for a person in *A*, *B* appears to draw away from him at the rate of 70 m.p.h. This, therefore, represents the velocity of *B* relative to *A*.

DEFINITION OF RELATIVE VELOCITY

When the distance between two moving points *A* and *B* is altering, either in magnitude or in direction or both, each point is said to possess a velocity relative to the other. The velocity of one of the moving points, say, *A*, relative to the other point *B* is obtained by compounding with the velocity of *A*, the reversed velocity of *B*. The velocity of *A* relative to *B* is the velocity with which *A* will appear to move to *B*, if *B* is reduced to rest.

If velocity *A* is $\vec{v}_A$ and that of *B* is $\vec{v}_B$ with respect to a stationary frame, then from the definition, relative velocity of *A* with respect to *B*, $\vec{v}_{AB}$ is given by



$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B \qquad \text{...(19)}$$

If angle between $\vec{v}_A$ and $\vec{v}_B$ is θ then

$$|\vec{v}_{AB}| = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta} \qquad \ldots(\mathbf{20})$$

Also angle α made by relative velocity with v_A is given by

$$\tan\alpha = \frac{v_B \sin\theta}{v_A - v_B \cos\theta} \qquad \ldots(\mathbf{21})$$

From the above definition of relative velocity it follows that if we impress on both the moving points *A* and *B*, a velocity equal and opposite to that of *B*, then *B* would be reduced to rest and A will have two velocities (i) its own velocity and (ii) the reversed velocity of *B*. These two can be compounded into a single velocity by the parallelogram law, which will give the velocity of *A* relative to *B*.

Illustration 10

Two trains *A* and *B* have lengths 100 m and 80 m respectively. They move in opposite directions along parallel tracks at 72 km/hr and 54 km/hr respectively. What is the time taken by one train to cross the other?

Solution

Velocity of train *A*, v_A = 72 km/hr = 72 × $\frac{5}{18}$ m/s = 20 m/s

Velocity of train *B*, v_B = – 54 km/hr = – 54 × $\frac{5}{18}$ = – 15 m/s

– ve sign indicates the oppositely directed velocity.

Velocity of A relative to *B* = $v_A - v_B$ = 20 + 15 = 35 m/s.

The train *A* is now supposed to move with velocity 35 m/s while the train *B* is 'stationary'. For the train *A* to cross *B* or vice versa, the total distance to be crossed

= Length of train *B* + Length of train *A*

= 80m +100 m = 180 m

Time taken = $\frac{180\text{m}}{35\text{m/s}}$ = **5.14 s**

Illustration 11

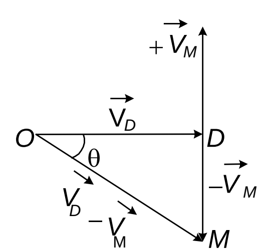
A monkey is climbing a vertical tree with a velocity of10 m/s while a dog runs towards the tree chasing the monkey with a velocity of 15 m/s. Find the velocity of the dog relative to the monkey.

Solution

Velocity of dog relative to the monkey = velocity of dog – velocity of monkey

$$= \vec{V}_D - \vec{V}_M$$

$$= \vec{V}_D + (-\vec{V}_M)$$



This velocity is directed along *OM* and its magnitude is

$$\sqrt{15^2 + 10^2} = \sqrt{225 + 100} \approx 18 \text{ m/s}.$$

This velocity makes an angle θ with the horizontal, where $\tan\theta = \frac{10}{15} = \frac{2}{3}$ or $\theta = \tan^{-1}\left(\frac{2}{3}\right)$ = **33.7°**

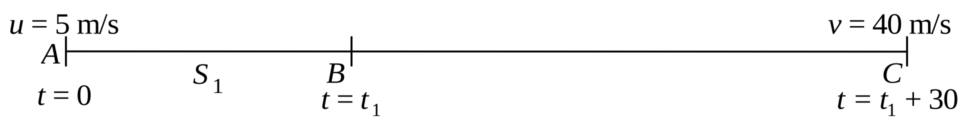
Fundamental Solved Examples

Example 1.

A particle moves along a straight path *ABC* with a uniform acceleration of 0.5 m/s^2. While it crosses *A* its velocity is found to be 5 m/s. It reaches *C* with a velocity 40 m/s, 30 seconds after it has crossed B in its path. Find the distance *AB*.

Solution:

The velocity while it crosses the point *A* is 5 m/s



Considering the displacement *AC*,

initial velocity u = 5 m/s

final velocity v = 40 m/s

acceleration a = 0.5 m/s^2

$\therefore$ time of motion $t = \frac{v-u}{a} = \frac{40-5}{0.5} = 70$ s

For the displacement *AB*,

initial velocity u = 5 m/s

acceleration a = 0.5 m/s^2

time of motion t = 70 – 30 = 40 s

$$\therefore \quad AB = S = ut + \frac{1}{2}at^2 = (5 \times 40) + \left(\frac{1}{2} \times 0.5 \times 40^2\right) = 200 + 400 = \mathbf{600\ m}$$

Example 2.

A particle moving with uniform acceleration in a straight line covers a distance of 3 m in the 8th second and 5 m in the 16th second of its motion. What is the displacement of the particle from the beginning of the 6th second to the end of 15th second?

Solution:

The distance traveled during the nth second of motion of a body is given by

$$S_n = u + \frac{1}{2}a\,(2n-1)$$

For the motion during the 8th second,

$$3 = u + \frac{1}{2}a\,(16–1) = u + \frac{15a}{2} \qquad \text{... (i)}$$

For the motion during the 16th second,

$$5 = u + \frac{1}{2}a\,(32-1) = u + \frac{31a}{2} \qquad \text{... (ii)}$$

Subtracting equations (i) from (ii)

$8a = 2$

or acceleration $a = \frac{1}{4}\,\text{ms}^2$

From equation (1), $u = 3 - \left(\frac{15}{2} \times \frac{1}{4}\right) = \frac{9}{8}\,\text{ms}^2$

Now, the velocity at the end of 5 s (velocity at the beginning of 6th second) $v_1 = u + 5a$

The velocity at the end of 15th s, $v_2 = u + 15a$

Average velocity during this interval of 10 seconds = $\frac{v_1 + v_2}{2}$

$$= \frac{(u+5a)+(u+15a)}{2} = u + 10a$$

Distance travelled during this interval

S = average velocity x time = $(u + 10\text{a}) \times t$

$$= \left(\frac{9}{8} + \frac{10}{4}\right) \times 10 = \frac{290}{8} = \mathbf{36.25\ m}$$

Example 3.

An automobile can accelerate or decelerate at a maximum value of $\frac{5}{3}$ m/s^2 and can attain a maximum speed of 90 km/hr. If it starts from rest, what is the shortest time in which can travel one kilometre, if it is to come to rest at the end of the kilometre run?

Solution:

In order that the time of motion be shortest, the car should attain the maximum velocity with the maximum acceleration after the start, maintain the maximum velocity for as along as possible and then decelerate with the maximum retardation possible, consistent with the condition that, the automobile should come to rest immediately after covering a distance of 1 km.

Let t_1 be the time of acceleration, t_2 be the time of uniform velocity and t_3 be the time of retardation.

Now, maximum velocity possible = 90 km/hr = 90 × $\frac{5}{18}$ m/s = 25 m/s

$$t_1 = \frac{v-u}{a} = \frac{25-0}{\frac{5}{3}} = 15 \text{ s}$$

Similarly, the time of retardation is also given by

$$t_3 = \frac{0-25}{-\frac{5}{3}} = 15 \text{ s}$$

During the period of acceleration, the distance covered

= average velocity *x* time

$$= \frac{25+0}{2} \times 15 = 187.5 \text{ m}$$

During the period of retardation, the distance covered is the same and hence

= 187.5 m

$\therefore$ the total distance covered under constant velocity = 1000 – 375 = 625 m

Time of motion under constant velocity, $t_2 = \frac{625}{25} = 25s$

$\therefore$ the shortest time of motion = $t_1 + t_2 + t_3$ = 15 + 25 + 15 = **55 seconds**

Example 4.

A stone is dropped into a well and the sound of the splash is heard $3\frac{1}{8}$ seconds later. If the velocity of sound in air is 352.8 m/s, find the depth of the well. g = 9.8 m/s^2.

Solution:

Let *x* metres be the depth of the well and *t* the time taken by the stone to reach the surface of water.

In this case $u = 0$, $a = 9.8$ m/s^2

Now in the relation, $S = ut + \frac{1}{2}at^2$,

we have

$$x = 0 + \frac{1}{2}(9.8)\, t^2$$

or $\quad x = 4.9t^2 \qquad$ ... (i)

Time taken by sound to travel distance x up (Motion of sound wave is not affected by gravity)

$$= \left(3\frac{1}{8} - t\right) \text{ seconds}$$

Distance traveled = Velocity of sound × Time taken

$$x = 352.8\left(\frac{25}{8} - t\right) \qquad \text{... (ii)}$$

From equations (i) and (ii),

$$4.9t^2 = 352.8\left(\frac{25}{8} - t\right)$$

or, $4.9t^2 + 352.8t - 1102.5 = 0$

or, $t^2 + 72t - 225 = 0$

Solving, $t = 3$ or -75 s.

Since the negative value of t has no meaning, $t = 3$ s.

This gives $x = 4.9t^2 = 4.9 \times 9 =$ **44.1 m**

Hence the depth of the well = **44.1 m**

Example 5.

A circus artist maintains four balls in motion making each in turn rise to a height of 5 m from his hand. With what velocity does he project them and where will the other three balls be at the instant when the fourth one is just leaving his hand? (take $g = 10$ m/s^2.)

Solution:

Obviously, to maintain proper distances, the artists must throw the balls after equal intervals of time. Let the interval of time be t, so that when the fourth ball is just leaving his hand, the first ball would have travelled for time $3t$, the second for time $2t$ and the third for time t. The second obviously would just have reached the maximum height of 5 m.

If v be the initial velocity of throw of each ball, then for the second ball we have,

$$v_2 = 0 = v - g\,(2t) \qquad \text{... (i)}$$

and $$s_2 = 5 = v\,(2t) - \frac{1}{2}g(2t)^2 \qquad \text{... (ii)}$$

These gives, $v = 2gt$

and $v \cdot 2t = 2t^2 + 5$

or $v = 20t$

and $v \cdot 2t = 20t^2 + 5$

Solving for t, we get $20t^2 = 5$ or $t = \frac{1}{2}$ second.

Therefore, $v = 20 \times \frac{1}{2} = 10$ m/s. Thus each ball is thrown up with initial velocity of **10 m/s**.

For the first ball, which would have come down for time $(3t - 2t) = t$, we have

$$S = 0 + \frac{1}{2} g \cdot t^2 = \frac{1}{2} \cdot 10 \cdot \left(\frac{1}{2}\right)^2 = \frac{5}{4} = \mathbf{1.25\ m}$$

Therefore, it will be at a height of $(5 - 1.25) =$ **3.75 m** from the hand and going downwards.

For the third ball, which will have risen up for time t,

$$S_3 = vt - \frac{1}{2}gt^2 = 10\left(\frac{1}{2}\right) - \frac{1}{2}\cdot 10\cdot\left(\frac{1}{2}\right)^2 = 5 - 1.25 = \mathbf{3.75\,m}$$

Example 6.

A stone is projected from the point on the ground in such a direction so as to hit a bird on the top of a telegraph post of height h and then attain the maximum height $2h$ above the ground. If at the instant of projection the bird were to fly away horizontally with uniform speed, find the ratio between horizontal velocities of the bird and stone if the stone still hits the bird while descending.

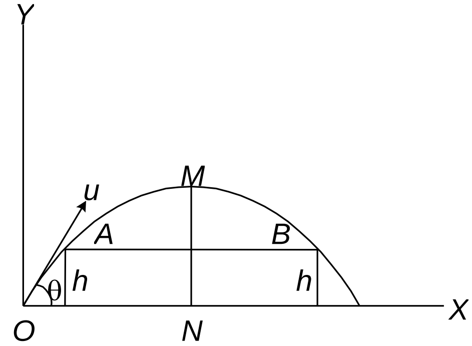
Solution:

The situation is shown in Figure. Let θ be the angle of projection and u the velocity of projection.

Maximum height $MN = 2h$

$$MN = 2h = \frac{u^2 \sin^2\theta}{2g}$$

$\therefore \quad u \sin\theta = 2\sqrt{gh} \qquad \text{... (i)}$



Let t be the time taken by stone to attain the vertical height h above the ground.

$\therefore \quad h = (u\sin\theta)t - \frac{1}{2}gt^2$

$$t^2 - \left(\frac{2u\sin\theta}{g}\right)t + \frac{2h}{g} = 0$$

$$t = \frac{u\sin\theta}{g} \pm \sqrt{\frac{u^2\sin^2\theta}{g^2} - \frac{2h}{g}}$$

Substituting the value of u sin θ from (i),

$$t = \frac{2\sqrt{gh}}{g} \pm \sqrt{\frac{4gh}{g^2} - \frac{2h}{g}} \quad = \sqrt{\frac{4h}{g}} \pm \sqrt{\frac{2h}{g}}$$

$$t_1 = \sqrt{\frac{4h}{g}} - \sqrt{\frac{2h}{g}} \qquad t_2 = \sqrt{\frac{4h}{g}} + \sqrt{\frac{2h}{g}}$$

where t_1 and t_2 are time to reach A and B respectively shown in the figure. If v is the horizontal velocity of bird, then

$$AB = vt_2.$$

AB is also equal to $u\cos\theta\,(t_2 - t_1)$, where $u\cos\theta$ is constant horizontal velocity of stone.

$$t_2 - t_1 = 2\sqrt{\frac{2h}{g}}$$

$\therefore \qquad u\cos\theta \cdot 2\sqrt{\dfrac{2h}{g}} = vt_2$

$$\frac{v}{u\cos\theta} = \frac{2\sqrt{\dfrac{2h}{g}}}{t_2} = \frac{2\sqrt{\dfrac{2h}{g}}}{\sqrt{\dfrac{2h}{g}}\cdot(\sqrt{2}+1)}$$

$$= \frac{2}{\sqrt{2}+1} = \mathbf{2(\sqrt{2}-1)}$$

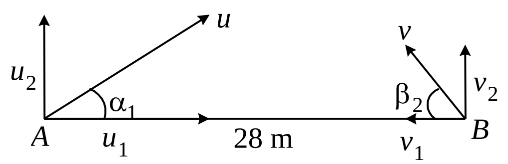
Example 7.

Two particles are projected at the same instant from two points A and B on the same horizontal level where AB = 28 m, the motion taking place in a vertical plane through AB. The particle from A has an initial velocity of 39 m/s at an angle $\sin^{-1}\left(\dfrac{5}{13}\right)$ with AB and the particle from B has an initial velocity of 25 m/s at an angle $\sin^{-1}\left(\dfrac{3}{5}\right)$ with BA. Show that the particles would collide in mid-air and find when and where the impact occurs.

Solution:

AB = 28 m.

At A, a particle is projected with velocity u = 39 m/s. u_1 and u_2 are its horizontal and vertical components respectively. The angle u makes with AB is α_1.



Given that $\qquad \sin\alpha_1 = \dfrac{5}{13} \therefore \cos\alpha_1 = \dfrac{12}{13}$.

Similarly, for the particle projected from B, with velocity v = 25 m/s, v_1 and v_2 are the horizontal and vertical components respectively.

sin $\qquad \alpha_2 = \dfrac{3}{5} \;\therefore\; \cos\alpha_2 = \dfrac{4}{5}$.

Now $\qquad u_2 = u\sin\alpha_1 = 39 \times \dfrac{5}{13} = 15$ m/s.

$\qquad v_2 = v\sin\alpha_2 = 25 \times \dfrac{3}{5} = 15$ m/s.

The vertical components of the velocities are the same at the start. Subsequently at any other instant t their vertical displacement are equal and have a value

$$h = 15t - 4.9t^2$$

which means that the line joining their positions at the instant t continues to be horizontal and the particles come closer to each other.

Their relative velocity in the horizontal direction

$= 39 \cos \alpha_1 + 25 \cos \alpha_2$

$= 39 \times \frac{12}{13} + 25 \times \frac{4}{5} = 36 + 20 = \mathbf{56\ m/s}$

Time of collision $= \frac{AB}{56} = \frac{28}{56}$ = **0.5 s**, after they were projected.

Height at which the collision occurs $= ut - \frac{1}{2}at^2 = 15(0.5) - \frac{1}{2}(9.8)(0.5)^2$

= **6.275 m**

The horizontal distance of the position of collision from *A*

$= 39 \times \frac{12}{13} \times 0.5$ s = **18 m**

Example 8.

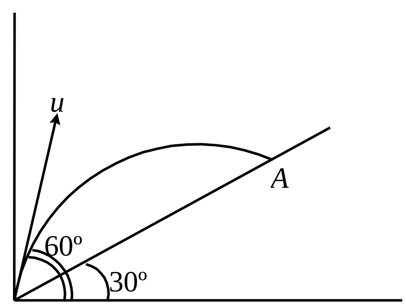
A shell is fired from a gun from the bottom of a hill along its slope. The slope of the hill is $\alpha = 30°$ and the angle of barrel to the horizontal is $\beta = 60°$. The initial velocity of shell is 21 m/s. Find the distance from the gun to the point at which the shell falls.

Solution:

We can write the equation of motion as $x = ut \cos \beta$

$$y = ut \sin \beta - \frac{gt^2}{2}$$

$$OA = \ell$$



At the moment the shell falls to the ground

$$x = \ell \cos \alpha = \ell \cos 30°$$

$$y = \ell \sin \alpha = \ell \sin 30°$$

$$\ell \cos \alpha = ut \cos \beta \qquad \text{... (i)}$$

$$\ell \sin \alpha = ut \sin \beta - \frac{gt^2}{2} \qquad \text{... (ii)}$$

$$\therefore \qquad t = \frac{\ell \cos\alpha}{u \cos\beta}$$

$$\ell \sin\alpha = \frac{\ell \cos\alpha \sin\beta}{\cos\beta} - \frac{g\ell^2 \cos^2\alpha}{2u^2 \cos^2\beta} \qquad \ell = \frac{2u^2 \sin(\beta - \alpha)\cos\beta}{g \cos^2\alpha}$$

Substituting $u = 21$ m/s, $\alpha = 30^\circ$, $\beta = 60^\circ$ and $g = 9.8$ m/s^2, we get **ℓ = 30 metres**

Example 9.

A man can swim at a velocity V_1 relative to water in a river flowing with speed V_2. Show that it will take him $\frac{V_1}{\sqrt{V_1^2 - V_2^2}}$ times as long to swim a certain distance upstream and back as to swim the same distance and back perpendicular to the direction of the stream ($V_1 > V_2$).

Solution:

Suppose the man swims a distance x up and the same distance down the stream.

Velocity of man upstream relative to the ground = $V_1 - V_2$.

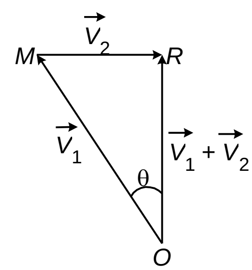
Time taken for this, $t_1 = \frac{x}{V_1 - V_2}$

Velocity of man downstream relative to the ground = $V_1 + V_2$

Time taken for this, $t_2 = \frac{x}{V_1 + V_2}$

Total time taken $t_1 + t_2 = \frac{x}{V_1 - V_2} + \frac{x}{V_1 + V_2} = \frac{2V_1 x}{V_1^2 - V_2^2}$

Next the man intends crossing the river perpendicular to the direction of the stream. If he wants to cross the river straight across he must swim in a direction OM such that the vector sum of velocity of man + velocity of river will give him a velocity relative to the ground in a direction perpendicular to the direction of the stream. In the Figure the velocity relative to the ground is $\overrightarrow{OR}$ and the magnitude of $\overrightarrow{OR} = \sqrt{V_1^2 - V_2^2}$



Now the man swims a distance x up and x down perpendicular to the river flow. Time taken for this, $t = \frac{2x}{\sqrt{V_1^2 - V_2^2}}$

Then the ratio, $\frac{t_1 + t_2}{t} = \frac{2V_1 x}{V_1^2 - V_2^2} \div \frac{2x}{\sqrt{V_1^2 - V_2^2}}$

$$= \frac{2V_1 x}{V_1^2 - V_2^2} \times \frac{\sqrt{V_1^2 - V_2^2}}{2x} = \frac{V_1}{\sqrt{V_1^2 - V_2^2}}$$

Example 10.

A man walking eastward at 6 km/hr finds that the wind seems to blow directly from north. On doubling his velocity, the wind appears to come N 30° E. Find the velocity of the wind.

Solution:

Actual velocity of the man = 6 km/hr eastward.

The direction of the relative velocity of the wind in this case is North to South.

If $\overrightarrow{OA}$ represents the velocity of man and $\overrightarrow{AB}$ represents the relative velocity of the wind, then

velocity of man + relative velocity of wind = velocity of wind $= \overrightarrow{OB}$ (say)

It is also given that when the velocity of the man is doubled (i.e., 12 km/hr) the wind seems to blow from a direction N 30°E. Representing this by vector $\overrightarrow{OC}$ = New velocity of man = 12 km/hr.

The direction of the relative velocity of the wind in this case is CB. The two directions of the relative velocity meet at B. Hence $\overrightarrow{OB}$ should give the real velocity of the wind. From the geometry of the Figure, it is clear that OBC is an equilateral triangle. Hence the magnitude of the real velocity of the wind = **12 km/hr**. Its direction is towards **30° east of south** (S 30°E).