1 NEWTON'S LAWS OF MOTION

Till the mid of 17th century most of the philosophers thought that some influence was needed to keep a body moving. They thought that a body was in its 'natural state' when it was at rest and some external influence was needed to continuously move a body; otherwise it would naturally stop moving.

Confusions about these issues were solved in 1687 when Newton presented his three laws of motion. According to him influence is needed not for all kind of motion it is needed for accelerated motion only. Before going in details about these three laws, let us summerise these three laws first.

Law 1: Everybody will remain at rest or continue to move with uniform velocity unless an external force is applied to it.

Law 2: When an external force is applied to a body of constant mass the force produces an acceleration, which is directly proportional to the force and inversely proportional to the mass of the body.

Law 3: When a body *A* exerts a force on another body *B*, *B* exerts an equal and opposite force on *A*.

1.1 FRAME OF REFERENCE

Before going in details about Newton's law, let us first define frame of reference. Suppose you are standing on your school bus with one of your friend who is properly seated in his seat. There is another friend of yours standing on bus stop waves his hand to stop the bus. The driver applies brakes and your friend in bus observes you to move forward but your friend outside the bus observes bus and you to stop together. So your two friends one in the bus and other outside the bus observe you. The person in bus finds you initially at rest and then starts moving, while a friend outside the bus observes nothing unusual. Each observer such as your friend in bus or your friend outside bus defines a reference frame. A reference system requires a co-ordinate system (made of origin and co-ordinate axes) and a set of clocks, which enable an observer to measure positions, velocities and accelerations in his or her particular reference frame. Observers in different frame may measure different displacements, velocities and accelerations.

Newton's laws are applicable for a special kind of frame of reference. In the example given earlier, the friend outside the bus is in a frame which observes you moving with bus and then comes to rest. But the friend inside the bus finds you to come in motion without any cause. So we can say that your motion can't be analysed using Newton's law with respect to your friend in bus. The first law of Newton is called "law of inertia" and the frame in which this law is applicable is called as inertial frame. In the said example your friend outside the bus defines an inertial frame.

Any reference frame which is not accelerated (either at rest or moving with uniform velocity) is called an inertial frame. Newton's first law is applicable only in an inertial frame. We generally apply Newton's first law with respect to earth by assuming it an inertial frame. In actual practice earth experiences an accelerations of 4.4×10^{-3} m/s^2 towards the sun due to its circular motion around sun. In addition earth rotates about its own axis once every 24 hours, a point on the equator experiences an addition acceleration of 3.37×10^{-2} m/s^2 towards the center of earth. However these accelerations are small compared with g and can often be neglected. In most situations we shall assume that a set of nearby points on earth's surface constitutes an inertial frame. At a later stage we will study about accelerated frame also.

1.2 NEWTON'S FIRST LAW OF MOTION

If a body is observed from an inertial frame which is at rest or moving with uniform velocity then it will remain at rest or continue to move with uniform velocity until an external force is applied on it.

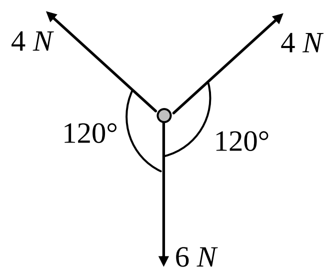
The property due to which a body remains at rest or continue its motion with uniform velocity is called as Inertia.

Force is a push or pull that disturbs or tends to disturb inertia of rest or inertia of uniform motion with uniform velocity of a body.

Hence first law of motion defines inertia, force and inertial frame of reference.

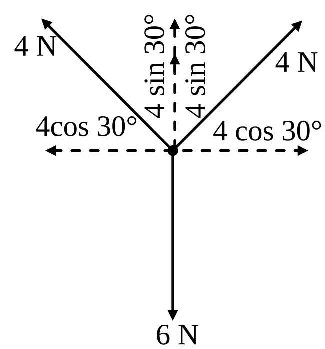
Illustration 1

The diagram shows the forces that are acting on a particle. Does the particle have any acceleration?



Solution:

To check whether the particle will have any acceleration or not, let us see net force is zero or not. Resolving the forces in horizontal and vertical directions.



Net force in horizontal direction

$$= 4 \cos 30° - 4 \cos 30° = 0$$

Net force in vertically downward direction

$$= 6 - 4 \sin 30° - 4 \sin 30° = 2 \text{ N}$$

As net force is not zero, so the particle will have acceleration.

1.3 Newton's Second Law of Motion

Newton's first law gives definition of force and inertia. Newton's second law of motion defines magnitude of force. Before stating Newton's Second's Law, Let us know about Mass.

If we attempt to change the state of rest or motion with uniform velocity, the object resists this change. Inertia is solely a property of an individual object; it is a measure of response of an object to an external force. If we take two blocks identical in shape and size; one of wood and the other of steel, the same force causes more acceleration in the wooden block. Therefore we say steel block has more inertia than the wooden block.

Mass is measure of inertia of a body. It is an internal property of a body and is independent of the body's surrounding and of the method used to measure it. Its SI unit is kg.

Mass should not be confused with weight. Mass and weight are difference quantities. We will see later, the weight of a body is equal to magnitude of force exerted by the earth on the bodies and varies with location. For example a body, which weighs 60 N on earth weights 10 N on moon. But its mass is 6 kg on earth as well as on moon.

If we push a block of ice on a smooth surface by applying a horizontal force F, the block will move with some acceleration. If we double the force the acceleration doubles, likewise if we make the force $3F$ the acceleration triples. From such observations we conclude that the acceleration of an object is directly proportional to the resultant force acting on it.

Also if we push a block of ice on a smooth surface by applying a force F, the block moves with an acceleration of a. If we double the mass, the same force causes an acceleration of $a/2$. If we triple the mass of block, the acceleration will be $a/3$.

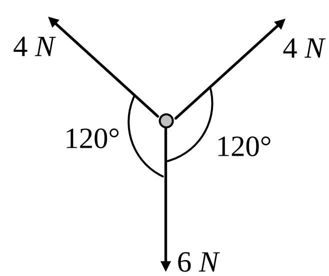
These observations are summarized, as follows: '*the acceleration of an object is directly proportional to the net force acting on it and is inversely proportional to its mass'. Thus we can relate mass, force and acceleration through following mathematical relation,*

$$\sum \vec{F}_{ext} = M\vec{a}\,. \qquad \ldots \textbf{(1)}$$

It is important to note here that it is a vector relation that is *acceleration is in the direction of net force*.

Illustration 2

Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a 5.0 kg mass. If F_1 = 20.0 N and F_2 = 15.0 N, find the acceleration.



Solution:

Acceleration will be in the direction of net force and will have the magnitude given by

$$\sum \vec{F} = M\vec{a}$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

$$\therefore \quad |\vec{F}| = \sqrt{20^2 + 15^2} = 25 \text{ N}$$

$$\therefore \quad |\vec{a}| = \frac{|\vec{F}|}{5.0} = 5 \text{ ms}^2$$

If the resultants force is at angle α with $\vec{F}_1$.

$$\tan\alpha = \frac{15}{20} \Rightarrow \alpha = 37°$$

Therefore, acceleration is **5 ms^{-2} at an angle 37°** with the direction of $\vec{F}_1$.

1.4 NEWTON'S THIRD LAW OF MOTION

We state this law as, "To every action there is equal and opposite reaction".

But what is meaning of action and reaction and which force is action and which force is reaction?

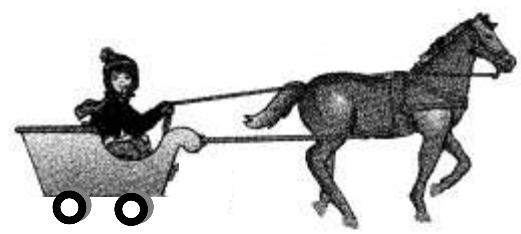
Every force that acts on a body is due to the other bodies in environment. Suppose that a body A experiences a force $\vec{F}_{AB}$ due to other body B. Also body B will experience a force $\vec{F}_{BA}$ due to A. According to Newton third law two forces are equal in magnitude and opposite in direction. Mathematically we write it as

$$\vec{F}_{AB} = -\vec{F}_{BA} \qquad \text{.....(2)}$$

Here we can take either $\vec{F}_{AB}$ or $\vec{F}_{BA}$ as action force and the other will be the reaction force. Another important thing is these two forces always acts on different bodies.

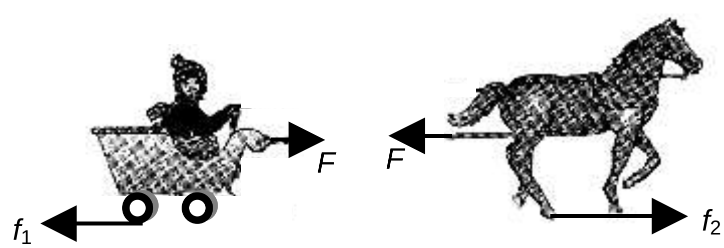
Illustration 3

A horse pulls a cart with a horizontal force, causing it to accelerate as shown in figure. Newton's third law says that the cart exerts an equal and opposite force on the horse. In view of this, how can the cart accelerate?



Solution:

The motion of any object is determined by the external forces that acts on it. If resultant of external force is non-zero, the object moves in the direction of resultant force. In this situation, the horizontal forces exerted on the cart are forward force exerted by the horse (F) and the backward contact force (f_1) due to roughness of surface. When forward force exerted on the cart exceeds the backward force, the resultant force on it is in the forward direction. This resultant force causes the cart to accelerate to the right. The horizontal force that acts on the horse are the forward contact force (f_2) due to roughness of surface and the backward force of the cart (F). The resultant of these two forces causes the horse to accelerate.



2. MOTION OF CONNECTED BODIES

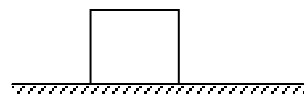
Before knowing how to apply Newton's laws of motion to solve questions based on motion of connected bodies and to know about the stepwise procedure to solve the same, let us know about the commonly used forces in such situations.

Also in a particular question some assumptions will be given and one should know how to use them while analysing the problem. Such assumptions definitely simplify the analysis at the cost of some physical reality. But in later stages we add some new techniques that permit us to be more realistic in our analysis.

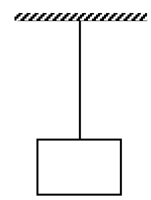
2.1 COMMONLY USED FORCES

(i) Weight of a body: It is the force with which Earth attracts a body towards its center. If M is mass of body and g is acceleration due to gravity, weight of the body is Mg. We take its direction vertically downward.

(ii) Normal Force: Let us consider a book resting on the table. It is acted upon by its weights in vertically downward direction and is at rest. It means there is another force acting on the block in opposite direction, which balances its weight. This force is provided by the table and we call it as normal force. Hence, if two bodies are in contact a contact force arises, if the surface is smooth the direction of force is normal to the plane of contact. We call this force as Normal force. We take its direction towards the body under consideration.

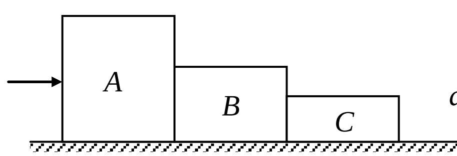


(iii) Tension in string: Let a block is hanging from a string. Weight of the block is acting in vertically downward but it is not moving, hence its weight is balanced by a force due to string. This force is called 'tension in string'. Tension is a force in a stretched string. Its direction is taken along the string and away from the body under consideration.



2.2 ASSUMPTIONS AND THEIR BENEFITS

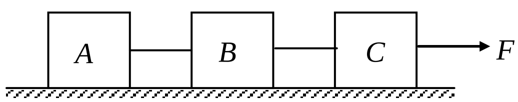
(i) If the **bodies are rigid and moving together** then their accelerations, velocities and displacements will be same. As in the figure acceleration of blocks A, B and C will be same



$a_A = a_B = a_C$

(ii) If the **surface is smooth** the contact force will only be the normal force.

(iii) If the **string is inextensible** accelerations, velocities and displacements of two blocks moving together will be same as in figure



$a_A = a_B = a_C$

(iv) If **string is massless,** tension throughout the string will be same.

(v) If the **pulley is massless and frictionless** then tension on the two sides of pulley will be same.

In later stage we will discuss about flexible string, massive and rough pulley also.

2.3 STEPS TO BE FOLLOWED TO SOLVE QUESTIONS BASED ON MOTION OF CONNECTED BODIES

In such questions you will be given a system of bodies under the action of forces and you will need to find out accelerations of different bodies and unknown forces on bodies. The following steps are needed by you to apply while solving such questions.

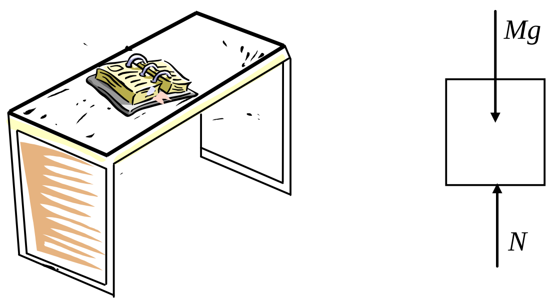
Step 1: Identify the unknown accelerations and unknown forces involved in the question.

Step 2: Draw free body diagram of different bodies in the given system.

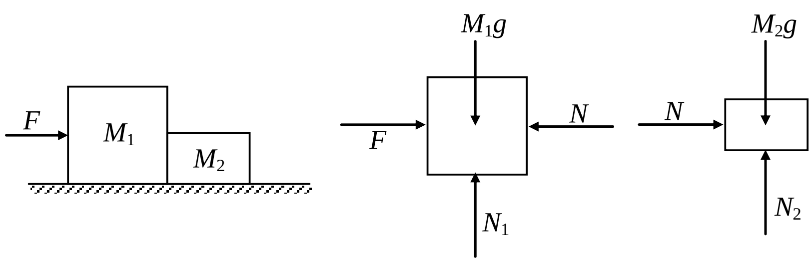
Free body diagram (FBD). It is a diagram that shows forces acting on the body making it free from other bodies applying forces on the body under consideration. Hence free body diagram will include the forces like weight of the body, normal force, tension in string and the applied force. The important thing while drawing **FBD** is the shape of the body should be taken under consideration and force should be shown in a particular way. For example weight should be applied from center of gravity of body, normal force(s) should be applied on the respective surface(s), tension should be applied on the side(s) of string(s).

Examples

(i) Free body diagram of a block resting on table

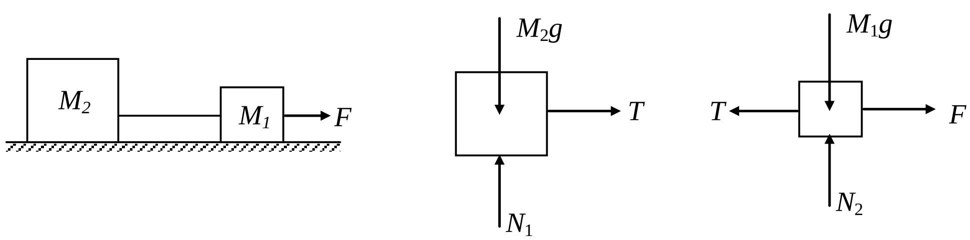


(ii) Free body diagram of bodies in contact and moving together on smooth surface.



Note that, normal force is taken normal to the surface of contact and towards the body under consideration

(iii) Free body diagram of bodies connected with strings and moving under the action of external force, on a smooth surface.



Note that, tension is acting along the string and away from the body under consideration.

Step 3: Identify the direction of acceleration and resolve the forces along this direction and perpendicular to it.

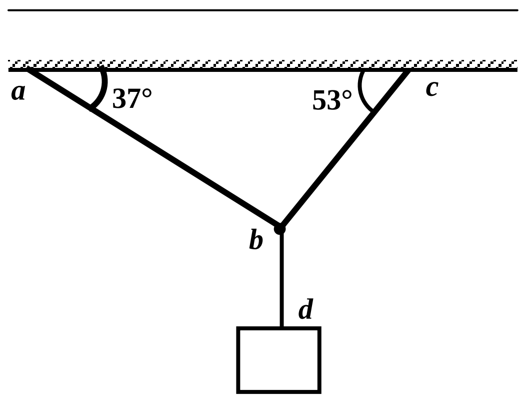
Step 4: Find net force in the direction of acceleration and apply $F = Ma$ to write equation of motion in that direction. In the direction of equilibrium take net force zero.

Step 5: If needed write relation between accelerations of bodies given in the situation

Step 6: Solve the written equations in steps 4 and 5 to find unknown accelerations and forces.

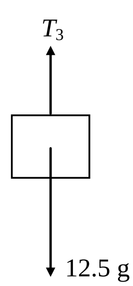
Illustration 4

A body suspended with the help of strings. A body of mass 12.5 kg is suspended with the help of strings as shown in figure. Find tension in three strings. Strings are light [$g = 10\ \text{ms}^{-2}$]

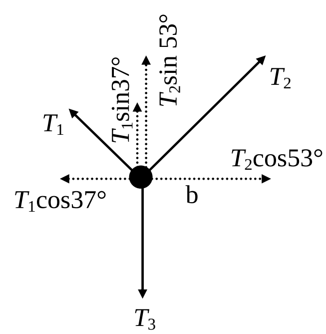


Solution:

Let the tensions in strings *ab*, *bc* and *bd* are respectively T_1, T_2, and T_3. As the body is hanging in equilibrium, we can use the condition that net force on block is zero. This will give the value of T_3. To know the values of T_1 and T_2 we need to draw *FBD* of knot *b* also.



(FBD of hanging body)



(FBD of knot b)

For equilibrium of hanging body. $T_3 = 12.5\ \text{g} = \mathbf{125\ N}$ …(i)

For equilibrium of knot,

$$T_2 \cos 53° - T_1 \sin 37° = 0 \quad \text{… (ii)}$$

and, $$T_3 - T_1 \sin 37° - T_2 \sin 53° = 0 \quad \text{… (iii)}$$

From (i), (ii) & (iii)

$T_2 = \mathbf{99.9\ N}$

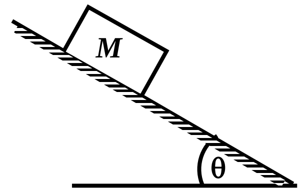
$T_3 = \mathbf{75.1\ N}$

Illustration 5

Motion of a block on a frictionless incline

A block of mass M is placed on a frictionless, inclined plane of angle θ, as shown in the figure.

Determine the acceleration of the block after it is released. What is force exerted by the incline on the block?



Solution:

When the block is released, it will move down the incline. Let its acceleration be a. As the surface is frictionless, so the contact force will be normal to the plane. Let it be N.

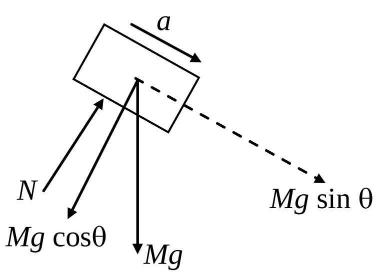
Here, for the block we can apply equation for motion along the plane and equation for equilibrium perpendicular to the plane.

i.e., $Mg \sin\theta = Ma$

⇒ $a = \mathbf{g \sin\theta}$

Also, $Mg \cos\theta - N = 0$

⇒ $N = \mathbf{Mg \cos\theta}$

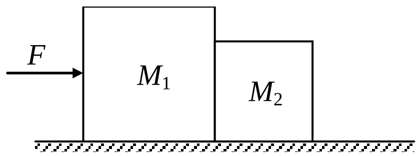


(FBD of Block)

Illustration 6

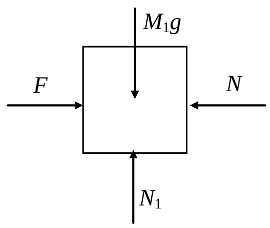
One block pushes other

Two blocks of masses M_1 and M_2 are placed in contact with each other on a frictionless horizontal surface as shown in figure. A constant force F is applied on M_1 as shown. Find magnitude of acceleration of the system. Also calculate the contact force between the blocks.

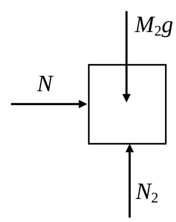


Solution:

Here accelerations of both blocks will be same as they are rigid and in contact. As the surfaces are frictionless, contact force on any surface will be normal force only. Let the acceleration of each block is a and contact forces are N_1, N_2 and N as shown in free body diagrams of blocks.



(FBD of M_1)



(FBD of M_2)

Applying, Newton's Second Law for M_1

$$F - N = M_1 a \qquad \ldots \text{(i)}$$

$$M_1 g - N_1 = 0 \qquad \ldots \text{(ii)}$$

Applying, Newton's second law for M_2

$N = M_2a$ …(iii)

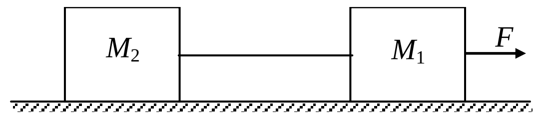
$M_2g - N_2 = 0$ … (iv)

Solving (1) and (3) $a = \dfrac{F}{M_1 + M_2}$ and $N = \dfrac{M_2F}{M_1 + M_2}$

Illustration 7

Bodies connected with strings

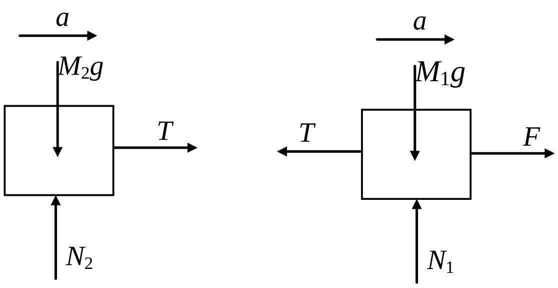
A light, inextensible string as shown in figure connects two blocks of mass M_1 and M_2. A force F as shown acts upon M_1. Find acceleration of the system and tension in string.



Solution:

Here as the string is inextensible, acceleration of two blocks will be same. Also, string is mass less so tension throughout the string will be same. Contact force will be normal force only.

Let acceleration of each block is a, tension in string is T and contact force between M_1 and surface is N_1 and contact force between M_2 and surface is N_2.



(FBD of M_2) (FBD of M_1)

Applying Newton's second law for the blocks;

For M_1 , $F - T = M_1 a$ … (i)

$M_1g - N_1 = 0$ … (ii)

For M_2 $T = M_2a$ … (iii)

$M_2g - N_2 = 0$ … (iv)

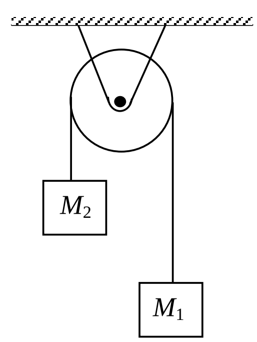
Solving (i) and (iii)

$$a = \frac{F}{M_1 + M_2} \text{ and } T = \frac{M_2F}{M_1 + M_2}$$

Illustration 8

Bodies connected with a string and the string passes over a pulley. (Atwood's Machine)

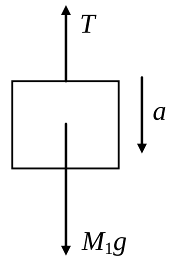
Two blocks of unequal masses M_1 and M_2 are suspended vertically over a frictionless pulley of negligible mass as shown in figure. Find accelerations of each block and tension in the string.



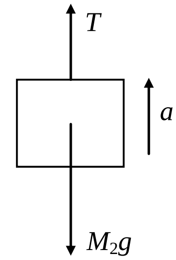
Solution:

As the string is inextensible, the magnitude of acceleration of two blocks will be same. Pulley in question is mass less and frictionless so tension in strings on two sides of pulley will be same.

Let acceleration of M_1 be 'a' (downward) then acceleration of M_2 will be 'a' (upward). Let the tension in string be T.



(FBD of block M_1)



(FBD of block M_2)

Applying Newton's second law for the blocks,

For M_1, $M_1g - T = M_1a$ ... (i)

For M_2, $T - M_2g = M_2a$ ... (ii)

Solving equation (i) & (ii), $a = \dfrac{(M_1 - M_2)g}{M_1 + M_2}$ and

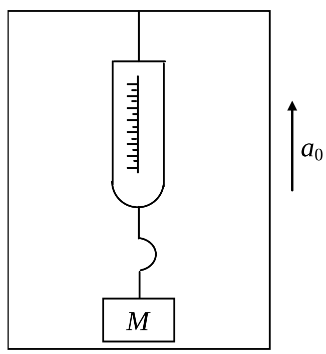
$$T = \frac{2M_1M_2}{M_1 + M_2}g$$

Illustration 9

Weighing a body in an Elevator

A block of mass M is suspended with the help of a spring balance. The spring balance is attached to the ceiling of an elevator moving with upward acceleration a_0 as shown in figure.

What is reading of spring balance?



Solution:

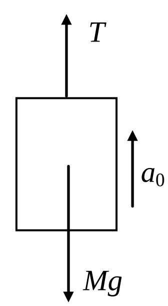
A person outside the elevator will observe the block moving with the elevator upward with an acceleration a_0. Also spring balance will give the reading according to tension in spring. So calculating reading of spring balance means to find tension in the spring of spring balance.

Let tension in spring is T.

Applying Newton's Second law for the block,

$T - Mg = Ma_0$

$\Rightarrow$ $T = \boldsymbol{M(g + a_0)}$

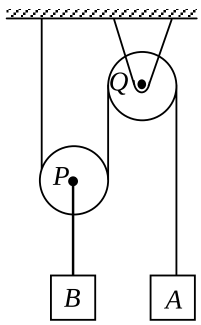


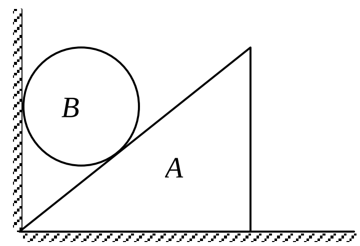
(FBD of block)

This will be the reading of spring balance. Note that the reading given by spring balance is different from the weight of block.

2.4 RELATED MOTION

Till now we had seen the case when accelerations of the different parts of a system are same. There are situations in which the accelerations of different parts of the system may not be same. We get such situations in case of moveable pulleys or bodies in contact where each body is free to move.





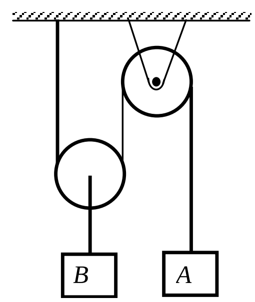
For example in the figure given above (Left), pulley *P* is movable which leads to different accelerations of block *B* and *A*.

In the figure given above (Right), triangular wedge *A* and sphere *B* will not have same acceleration.

In such cases, a relationship between accelerations can be found by considering physical properties of system. We call such relations as constrained relation.

Illustration 10

Find the relation between accelerations of blocks *A* and *B*

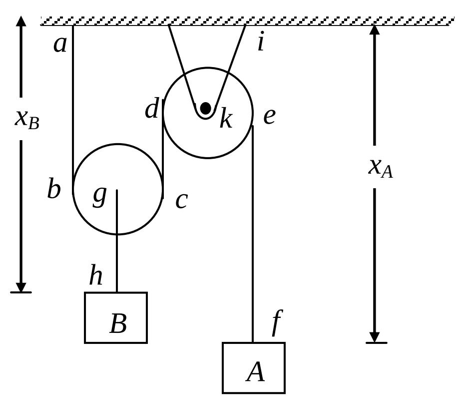


Solution:

The physical property that we can use is the inextensibility of string.

i.e., $ab + \widehat{bc} + cd + \widehat{de} + ef = \text{constant}$ … (i)

Let at any moment *A* and *B* are at distances x_A and x_B from the support as shown in figure.



Let us take $gh = \ell_1$ and $i\,k = \ell_2$ and express the length in equation (i) in terms of x_A, x_B, ℓ_1 and ℓ_2

we get,

$$x_B - \ell_1 + \widehat{bc} + (x_B - \ell_1 - \ell_2) + \widehat{de} + (x_A - \ell_2) = \text{constant}$$

Here ℓ_1, ℓ_2, $\widehat{bc}$ and $\widehat{de}$ are constant

$\therefore \quad 2x_B + x_A = \text{constant} \qquad \ldots \text{(ii)}$

let in time Δt, x_B charge to $x_B + \Delta x_B$ and x_A changes to $x_A - \Delta x_A$

[B is assumed to move downward]

then, $2(x_B + \Delta x_B) + (x_A - \Delta x_A) = \text{constant} \qquad \ldots \text{(iii)}$

From (ii) and (iii), $2\Delta x_B - \Delta x_A = 0$

Also, $\left(\frac{2\Delta x_B}{\Delta t}\right) - \left(\frac{\Delta x_A}{\Delta t}\right) = 0$

$\Rightarrow \quad 2V_B - V_A = 0$

Also, $\quad 2\Delta V_B - \Delta V_A = 0$

$\Rightarrow \quad \frac{2\Delta V_B}{\Delta t} - \frac{\Delta V_A}{\Delta t} = 0$

$\Rightarrow \quad \boldsymbol{2a_B = a_A}$

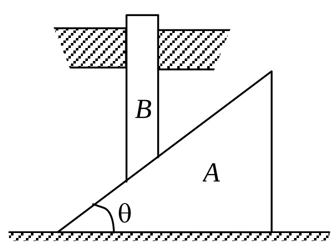
Hence magnitude of acceleration of A is two times magnitude of acceleration of B.

Here we get the relation between the acceleration by using the inextensibility of string but after some practice such relation can easily be written by observation.

Let us think B moves by a distance x during an interval of time, this will cause movement of pulley g by x. an extra length of $2x$ of string will come to the left of pulley k. This must be coming from right side of pulleys. Hence displacement of A will be $2x$. On the basis of this discussion we can say if the acceleration of block B is a, then the acceleration of A will be $2a$.

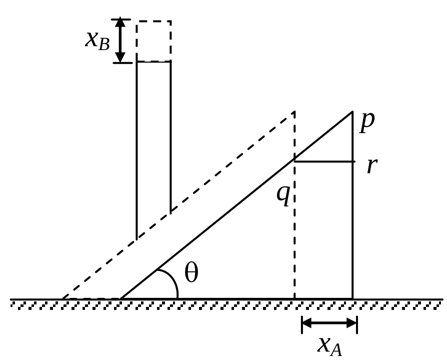
Illustration 11

In the arrangement shown A is a wedge and B is a rod. The rod is constrained to move vertical. Find relation between accelerations of A and B.



Solution:

Here the physical property that we can use is the rigidity of body. Let x_A and x_B are displacement of wedge and rod as shown.



In Δpqr, $qr = x_A$ and $pr = x_B$

$\therefore \quad \tan\theta = \frac{x_B}{x_A}$

$\Rightarrow \quad x_B = x_A \tan\theta \qquad \text{and} \qquad V_B = V_A \tan\theta$

$\Rightarrow \quad \boldsymbol{a_B = a_A \tan\theta}$

3. ACCELERATED FRAME OF REFERENCE

When Newton's laws of motion were introduced earlier, we emphasized that the laws are valid only when observations are made in an inertial frame of reference (frame at rest or moving with uniform velocity). Now we analyze how an observer in accelerated frame of reference (Non-inertial frame) would attempt to apply Newton's second law.

Once a frame of reference begins to accelerate the frame becomes non-inertial and Newton's laws do not hold good any more. To understand this in a better way, let us consider the rail-car. Suppose a body is placed on the floor of the car which we consider as smooth. The train is moving with uniform velocity and hence the position of the body with respect to the frame of reference attached to the car remains constant. Suppose brakes are applied and the train begins to decelerate. The body which was at rest on the floor, suddenly begins to slide along the floor in the forward direction even though no force of any kind acts on it. Newton's laws seem to have been violated. Conventionally we would explain this motion as due to Newton's first law and the body due to the absence of friction continues to maintain its state of uniform motion along a straight line with respect to railway track. The train has now become a non-inertial frame.

Non-inertial frames of reference are the system which are accelerated (or decelerated). Newton's laws especially first and second cannot hold good for accelerating frames of reference. Anyhow the Newton's laws of motion can be made applicable to them by applying an imaginary force on the body considered. This imaginary force is called inertial force or pseudo-force or fictitious force. The magnitude of the force is the product of mass of the body and the acceleration of the reference system. Its direction is opposite to the acceleration of the reference.

If a body of mass M is observer from a frame having acceleration $\vec{a}_{frame}$ then

$$\vec{F}_{pseudo} = -M\vec{a}_{frame} \qquad \ldots(1)$$

It should be emphasised again that no such force actually exists. But once it is introduced Newton's laws of motion will hold true in a non-inertial frame of reference.

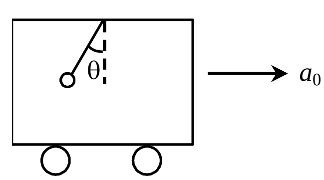
Therefore for non-inertial frame, we can write

$$\vec{F}_{ext} + \vec{F}_{pseudo} = M\vec{a}, \qquad \ldots(2)$$

where $\vec{a}$ is acceleration of body with respect to frame.

Illustration 12

A pendulum is hanging from the ceiling of a car having an acceleration a_0 with respect to the road. Find the angle made by the string with vertical at equilibrium.



Solution:

The situation is shown in figure. Suppose the mass of bob is m and the string makes an angle θ with vertical, the forces on the bob in the car frame (non-inertial frame) are indicated. The forces are

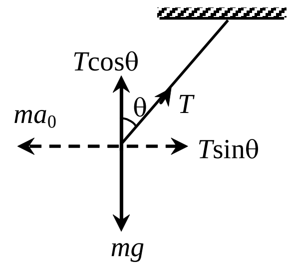
(i) tension in the string

(ii) mg vertically downwards

(iii) ma_0 in the direction opposite to the motion of car (pseudo force).

Writing the equation of equilibrium

$$T \sin \theta = ma_0 \qquad \text{and} \qquad T \cos \theta = mg$$



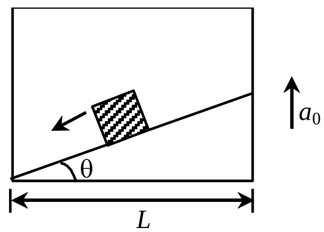
FBD of bob w.r.t car

$\therefore \quad \tan\theta = \dfrac{a_0}{g}$

$\therefore$ the string is making an angle $\boldsymbol{\tan^{-1}\left(\dfrac{a_0}{g}\right)}$ with vertical at equilibrium.

Illustration 13

A block slides down from top of a smooth inclined plane of elevation θ fixed in an elevator going up with an acceleration a_0. The base of incline has length *L*. Find the time taken by the block to reach the bottom.



Solution:

Let us solve the problem in the elevator frame. The free body force diagram is shown. The forces are

(i) *N* normal to the plane

(ii) *mg* acting vertically down

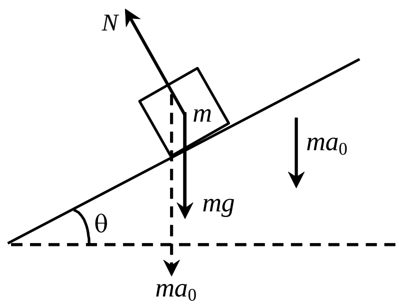
(iii) ma_0 (pseudo force).

If *a* is the acceleration of the body with respect to incline, taking components of forces parallel to the incline

$$mg \sin\theta + ma_0 \sin\theta = ma$$

$$\therefore \qquad a = (g + a_0) \sin\theta$$

This is the acceleration with respect to elevator.



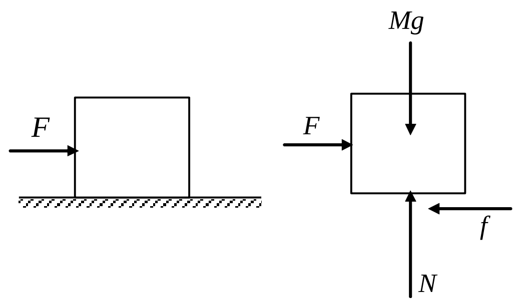
The distance traveled is $\dfrac{L}{\cos\theta}$. If *t* is the time for reaching the bottom of incline, $\dfrac{L}{\cos\theta} = 0 + \dfrac{1}{2}(g + a_0)\sin\theta . t^2$

$$\boldsymbol{t = \left[\frac{2L}{(g + a_0)\sin\theta\cos\theta}\right]^{1/2}}$$

4. FRICTION

Friction plays dual role in our life. It impedes the motion of object, causes abrasion and wear, and converts other forms of energy into heat. On the other hand, without it we could not walk, drive cars, climb roper, or use nails. Friction is a contact force that opposes the relative motion or tendency of relative motion of two bodies.

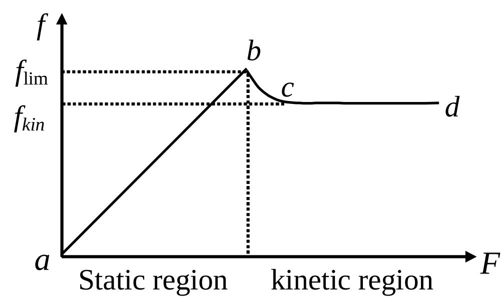
Consider a block on a horizontal table as shown in the figure. If we apply a force, acting to the right, the block remains stationary if F is not too large. The force that counteracts F and keeps the block from moving is called *frictional force*. If we keep on increasing the force, the block will remains at rest and for a particular value of applied force, the body comes to state of about to move. Now if we slightly increase the force from this value, block starts its motion with a jerk and we observe that to keep the block moving we need less effort than to start its motion.



So from this observation, we see that we have three states of block, first block does not move, second block is about to move and third block starts moving. The friction force acting in three states are called *static frictional force*, *limiting frictional force* and *kinetic frictional force* respectively. If we draw the graph between applied force and frictional force for this observation its nature is as shown in figure.

4.1 STATIC FRICTIONAL FORCE

When there is no relative motion between the contact surfaces, frictional force is called static frictional force. It is a self-adjusting force, it adjusts its value according to requirement (of no relative motion). In the taken example static frictional force is equal to applied force. Hence one can say that the portion of graph *ab* will have a slope of 45°.



4.2 LIMIING FRICTIONAL FORCE

This frictional force acts when body is about to move. This is the maximum frictional force that can exist at the contact surface. We calculate its value using laws of friction.

Laws of friction

(i) The magnitude of limiting frictional force is proportional to the normal force at the contact surface.

$$f_{\lim} \propto N \Rightarrow f_{\lim} = \mu_s N \qquad \ldots(\mathbf{3})$$

Here μ_s is a constant the value of which depends on nature of surfaces in contact and is called as 'coefficient of static friction'. Typical values of μ ranges from 0.05 to 1.5.

(ii) The magnitude of limiting frictional force is independent of area of contact between the surfaces.

4.3 KINETIC FRICTIONAL FORCE

Once relative motion starts between the surfaces in contact, the frictional force is called as kinetic frictional force. The magnitude of kinetic frictional force is also proportional to normal force.

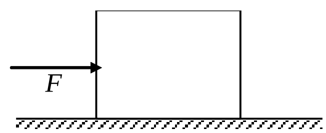
$$f_k = \mu_k N \qquad \ldots\textbf{(4)}$$

From the previous observation we can say that $\mu_k < \mu_s$

Although the coefficient of kinetic friction varies with speed, we shall neglect any variation i.e., one relative motion starts a constant frictional force starts opposing its motion.

Illustration 14

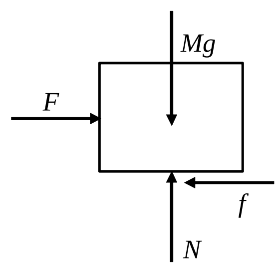
A block of mass 5 kg is resting on a rough surface as shown in the figure. It is acted upon by a force of F towards right. Find frictional force acting on block when (a) F = 5N (b) 25 N (c) 50 N



($\mu_s = 0.6$, $\mu_k = 0.5$) [$g = 10$ ms^{-2}]

Solution:

Maximum value of frictional force that the surface can offer is



$f_{\max} = f_{\lim} = \mu_s N$

$= 0.6 \times 5 \times 10$

$= 30$ newton

Therefore, if $F \le f_{max}$ body will be at rest and $f = F$

of $F > f_{max}$ body will more and $f = f_k$

(a) $F = 5\text{N} < F_{max}$

So body will not move hence static frictional force will act and,

$f_s = F =$ **5N**

(b) F = 25N < $F_{\max}$

$\therefore$ $f_s =$ **25 N**

(c) $F = 50\text{ N} > F_{max}$

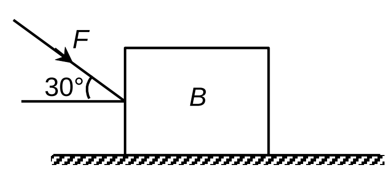
So body will move and kinetic frictional force will act, it value will be

$f_k = \mu_k N$

$= 0.5 \times 5 \times 10 =$ **25 newton**

Illustration 15

A block B slides with a constant speed on a rough horizontal floor acted upon by a force which is 1.5 times the weight of the block. The line of action F makes 30° with the ground. Find the coefficient of friction between the block and the ground.



Solution:

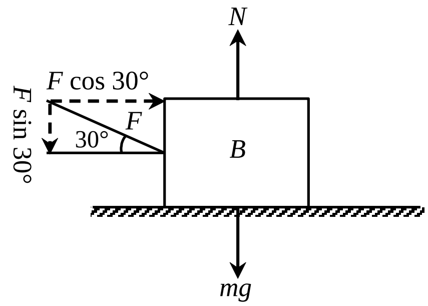
Let m be the mass of the block. The weight of the block, then is mg. It is given that

F = 1.5 mg. F can be resolved into two components $F\cos 30°$ parallel to the horizontal floor and $F\sin 30°$ perpendicular to it.

Normal reaction $N = mg + F\sin 30°$

$$= mg + \left(1.5mg \times \frac{1}{2}\right)$$

$$= mg + 0.75\ mg = 1.75\ mg$$



Hence the friction force

$$f = \mu N = \mu \times 1.75,\ mg = 1.75\ \mu\ mg$$

The body moves with constant speed. This means the force $F\cos 30°$ is just able to overcome the frictional force f

i.e., $f = \mu R = F\cos 30°$

or, $$1.75\ \mu\ mg = (1.5\ mg)\ \frac{\sqrt{3}}{2}$$

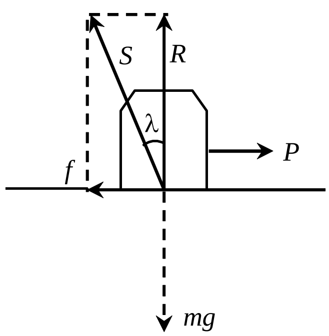
or, $$\mu = \frac{(1.5)\sqrt{3}}{2 \times 1.75} = \mathbf{0.742}$$

4.4 ANGLE OF FRICTION

The resultant of normal reaction $\vec{R}$ and the frictional for $\vec{f}$ is $\vec{S}$ which makes an angle λ with $\vec{R}$. Now,

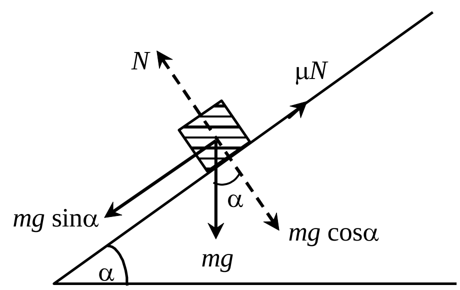
$$\tan\lambda = \frac{f}{R} = \frac{\mu R}{R} = \mu \qquad \ldots(\mathbf{5})$$

The angle λ is called the angle of friction.



4.5 ANGLE OF REPOSE

This is concerned with an inclined plane on which a body rests exerting its weight on the plane. The angle of repose of an inclined plane with respect to a body in contact with it is the angle of inclination of the plane with horizontal when the block just starts sliding down the plane under its own weight.



The limiting equilibrium of a body resting on the inclined plane is shown in figure.

The forces acting are (i) Its weight mg downward, (ii) Normal reaction, (iii) The force of limiting friction. Taking α as the angle of repose and resolving the forces along the plane and perpendicular to the plane, we get for equilibrium

$$mg \cos \alpha = N \qquad \text{...(i)}$$

$$mg \sin \alpha = f = \mu N \qquad \text{...(ii)}$$

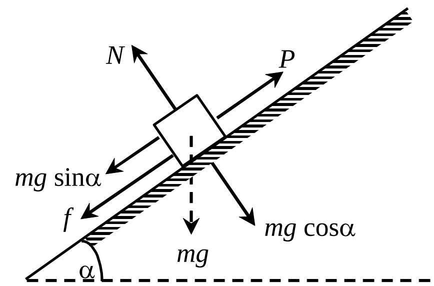
Dividing equation (ii) by (i),

$$\mu = \tan \alpha$$

$\therefore$ angle of repose $= \alpha = \tan^{-1} (\mu)$ ...(**6**)

4.6 MOTION ON A ROUGHT INCLINED PLANE

Suppose a motion up the plane takes place under the action of pull P acting parallel to the plane.



$$N = mg \cos \alpha$$

Frictional force acting down the plane,

$$F = \mu N = \mu\, mg \cos \alpha$$

Applying Newton's second law for motion up the plane.

$$P - (mg \sin \alpha + f) = ma$$

$$P - mg \sin \alpha - \mu\, mg \cos \alpha = ma$$

If $P = 0$ the block may slide downwards with an acceleration a. The frictional force would then act up the plane.

$$mg \sin \alpha - F = ma$$

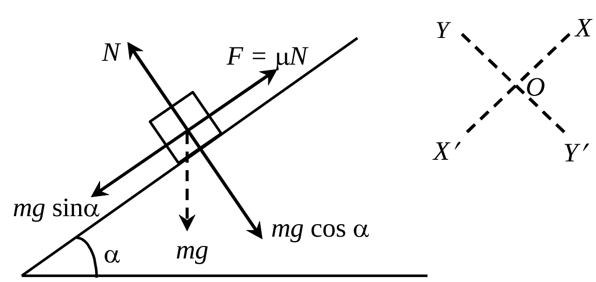
or, $$mg \sin \alpha - \mu\, mg \cos \alpha = ma$$

Illustration 16

A 20 kg box is gently placed on a rough inclined plane of inclination 30° with horizontal. The coefficient of sliding friction between the box and the plane is 0.4. Find the acceleration of the box down the incline.

Solution:

In solving inclined plane problems, the X and Y directions along which the forces are to be considered, may be taken as shown. The components of weight of the box are



(i) $mg \sin \alpha$ acting down the plane and

(ii) $mg \cos \alpha$ acting perpendicular to the plane.

$$R = mg \cos \alpha$$

$$mg \sin \alpha - \mu N = ma$$

$$mg \sin \alpha - \mu mg \cos \alpha = ma$$

$$a = g \sin \alpha - \mu g \cos \alpha$$

$$= g (\sin \alpha - \mu \cos \alpha)$$

$$= 9.8\left(\frac{1}{2} - 0.4 \times \frac{\sqrt{3}}{2}\right)$$

$$= 4.9 \times 0.3072 = \mathbf{1.505\ m/s^2}$$

The box accelerates down the plane at 1.505 m/s^2.

Illustration 17

A force of 400 N acting horizontal pushes up a 20 kg block placed on a rough inclined plane which makes an angle of 45° with the horizontal. The acceleration experienced by the block is 0.6 m/s^2. Find the coefficient of sliding friction between the box and incline.

Solution:

The horizontally directed force 400 N and weight 20 kg of the block are resolved into two mutually perpendicular components, parallel and perpendicular to the plane as shown.

$$N = 20\text{ g} \cos 45° + 400 \sin 45° = 421.4\text{ N}$$

The frictional force experienced by the block $F = \mu N = \mu \times 421.4 = 421.4\ \mu$N.

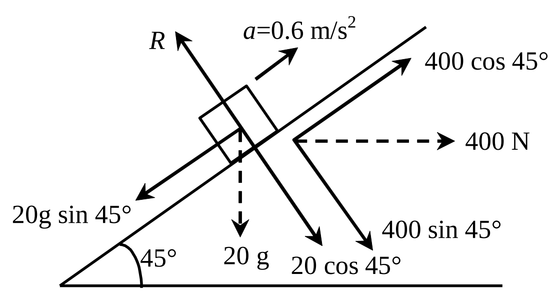
As the accelerated motion is taking place up the plane.

$$400 \cos 45° - 20g \sin 45° - f = 20a$$

$$\frac{400}{\sqrt{2}} - \frac{20 \times 9.8}{\sqrt{2}} - 421.4\mu = 20a = 20 \times 0.6 = 12$$

$$\mu = \left(\frac{400}{\sqrt{2}} = -\frac{196}{\sqrt{2}} - 12\right) \times \frac{1}{421.4}$$

$$= \frac{282.8 - 138.6 - 12}{421.4} = 0.3137$$



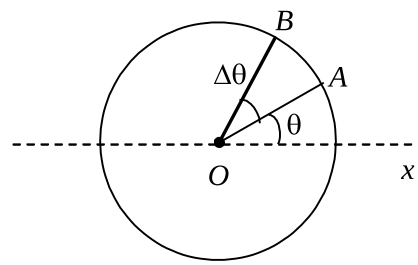
The coefficient of sliding friction between the block and the incline = 0.3137.

5. CIRCULAR MOTION

If a particle moves in such a way that its distance from a fixed point remains constant its path will be circular and its motion is called as circular motion. If we whirl a stone tied with a string at one end, its motion is circular motion. Motion of electron around the nucleus is treated as a circular motion. Motion of earth around the sun is treated as circular for some gravitational study.

5.1 ANGULAR KINEMATIC VARIABLES FOR CIRCULAR MOTION

Consider a particle moving on a circular path of center *O* and radius *R* as shown in figure. Let *OX* is a reference line (taken arbitrary) through *O*. At any moment *t* if particle is at *A*, then angle θ between *OA* and *OX* is called as its angular position. If during a time interval of Δt particle moves from position *A* to *B*, The angle subtended by the arc *AB* on the centre $\Delta\theta$ is called as angular displacement. The rate at which particle subtend angle at the center is called as angular velocity represented by ω.



∴ Average angular velocity, $\bar{\omega} = \dfrac{\Delta\theta}{\Delta t}$ and …(**7**)

Instantaneous angular velocity $\omega = \underset{\Delta t \to 0}{Lt} \dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt}$ …(**8**)

The rate of change of angular speed is called as angular acceleration represented by α

∴ Average angular acceleration, $\bar{\alpha} = \dfrac{\Delta\omega}{\Delta t}$ and …(**9**)

Instantaneous angular acceleration $\alpha = \underset{\Delta t \to 0}{Lt} \dfrac{\Delta\omega}{\Delta t} = \dfrac{d\omega}{dt}$

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{\omega d\omega}{d\theta} \qquad \ldots(\mathbf{10})$$

We have different kinds of motion in case of motion in one dimension such as motion with uniform velocity and motion with uniform acceleration. Also in those cases we have kinematic relation between the different linear variables. Similar derivation can be done in case of circular motion and such kinematic relations can be obtained for angular variables also.

For uniform circular motion; Angular displacement, $\Delta\theta = \omega t$

For circular motion with uniform angular acceleration;

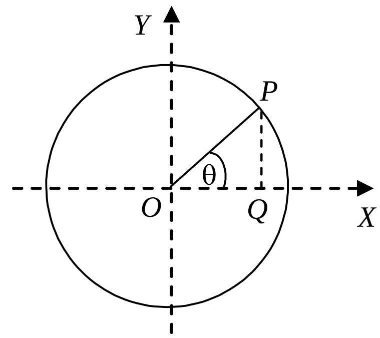
$$\left.\begin{aligned} \Delta\theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega &= \omega_0 + \alpha t \\ \omega^2 &= \omega_0^2 + 2\alpha\Delta\theta \end{aligned}\right\} \qquad \ldots(\mathbf{11})$$

5.2 LINEAR KINEMATICS FOR CIRCULAR MOTION

In the previous article we have seen different angular kinematic variables and relation between them in different types of motion. Now we will define linear velocity and linear acceleration of a particle in circular motion.

As circular motion is the motion in a plane, we take two co-ordinate axes through center for the linear kinematics analysis.

Suppose a particle P is moving on a circular path of radius R. The axes are taken as shown in figure. At any moment its angular position is θ.



Position vector of particle, $\vec{r} = \vec{OQ} + \vec{QP}$

$$= R\cos\theta\ \hat{i} + R\sin\theta\ \hat{j}$$

Also, unit vector along OP $\hat{e}_r$ can be written as,

$$\hat{e}_r = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

Unit vector along tangential direction $\hat{e}_t$ is given by

$$\hat{e}_t = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$$

Linear velocity and linear acceleration of particle in circular motion

In the previous discussion,

Position vector of particle, $\vec{r} = R\cos\theta\hat{i} + R\sin\theta\ \hat{j}$

Velocity of particle, $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$= R(-\sin\theta)\frac{d\theta}{dt}\hat{i} + R(\cos\theta)\frac{d\theta}{dt}\hat{j}$$

$$= R\omega\,[-\sin\theta\hat{i} + \cos\theta\hat{j}]\ \left[\because \frac{d\theta}{dt} = \omega\right]$$

$$= R\omega\,\hat{e}_t$$

$\therefore |\vec{v}| = R\omega$ and its direction is along the tangent to the path i.e., if a particle is moving on circular path its velocity is of magnitude equal to product of radius of circular path and its angular velocity and its direction is tangential to the circular path.

The acceleration of particle in circular motion $\vec{a} = \dfrac{d\vec{v}}{dt}$

From equation (2) $\quad \vec{a} = R\left[\omega\dfrac{d}{dt}(-\hat{i}\sin\theta + \hat{j}\cos\theta) + \dfrac{d\omega}{dt}(-\hat{i}\sin\theta + \hat{j}\cos\theta)\right]$

$$= R\omega\left[-\hat{i}\cos\theta\frac{d\theta}{dt} - \hat{j}\sin\theta\frac{d\theta}{dt}\right] + R\frac{d\omega}{dt}\left[-\hat{i}\sin\theta + \hat{j}\cos\theta\right]$$

$$= -\omega^2 R\,\hat{e}_r + \frac{Rd\omega}{dt}\hat{e}_t$$

Therefore, acceleration of particle has two parts one along the tangent having magnitude $\frac{Rd\omega}{dt}$ or $R\alpha$ or $\frac{dv}{dt}$ is called as tangential acceleration. The other component of acceleration is $\omega^2 R = \frac{v^2}{R}$ along the radial direction and is called as radial acceleration or centripetal acceleration.

Hence total acceleration of particle $a = \sqrt{{a_t}^2 + {a_r}^2} = \sqrt{(\omega^2 R)^2 + (\alpha R)^2}$ …(**12**)

If direction of acceleration makes an angle β with radius than, tan β = $\frac{a_t}{a_r}$ …(**13**)

Illustration 18

The moon orbits the earth with a period of 27.3 days at a distance of 3.84×10^8 m from the centre of earth. Find its linear speed and centripetal acceleration.

Solution:

The period of revolution of moon, T = 27.3 days = 2.36×10^6 s

Linear speed, $v = \omega R = \frac{2\pi}{T} R = \frac{2 \times 3.14 \times 3.84 \times 10^8}{2.36 \times 10^6} = 1021.83 \text{ ms}^{-1}$

Centripetal acceleration = $\frac{V^2}{R} = \frac{(1021.83)^2}{3.84 \times 10^8} = 2.72 \times 10^{-3} \text{ ms}^{-2}$

Illustration 19

A particle moves in a circle of radius 20 cm. Its linear speed at any time is given by $v = 2t$ where v is in m/s and t is in seconds. Find the radial and tangential accelerations at t = 3 seconds and hence calculate the total acceleration at this time.

Solution:

The linear speed at 3 seconds is, $v = 2 \times 3 = 6$ m/s

The radial acceleration at 3 seconds

$$= \frac{v^2}{r} = \frac{6 \times 6}{0.2} = 180 \text{ m/s}^2$$

The tangential acceleration is given by

$$\frac{dv}{dt} = 2, \text{ because } v = 2\text{t}.$$

∴ tangential acceleration is 2 m/s^2.

Total acceleration = $\sqrt{a_r^2 + a_t^2} = \sqrt{180^2 + 2^2} = \sqrt{32400 + 4}$

$$= \sqrt{\mathbf{32404}} \text{ m/s}^2$$

5.3 UNIFORM CIRCULAR MOTION AND CENTRIPETAL FORCE

If a particle moves on a circular path with a constant speed, its motion is called as a uniform circular motion. In this motion angular speed of the particle is also constant. Linear acceleration in such motion will not have any tangential component, only radial or centripetal acceleration the particle possesses. Therefore in case of uniform circular motion the particle will have acceleration towards the center only and is called as centripetal acceleration having magnitude $\frac{v^2}{R}$ or $\omega^2 R$. The magnitude of acceleration remains constant but its direction changes with time.

If a particle moving on circular path is observed from an inertial frame it has an acceleration $\omega^2 R$ or $\frac{v^2}{R}$ acting towards center. Therefore from Newton's second law of motion, there must be a force acting on the particle towards the center of magnitude $m\omega^2 R$ or $\frac{mv^2}{R}$. This required force for a particle to move on circular path is called as **centripetal force.**

$$\therefore \quad \text{centripetal force} = \frac{mv^2}{R} \qquad \text{...(14)}$$

The term 'centripetal force' merely a force towards center, it tells nothing about its nature or origin. The centripetal force may be a single force due to a rope, a string, the force of gravity, friction and so forth or it may be resultant of several forces. Centripetal force is not a new kind of force, just as 'upward force' or a 'downward force' is not a new force. Therefore while analyzing motion of particle undergoing circular motion we need not to consider centripetal force as a force, we need to consider only external forces.

Illustration 20

A ball of mass 0.5 kg is attached to the end of a cord whose length is 1.50 m. The ball is whirled in a horizontal circle. If the cord can withstand a maximum tension of 50.0 N, what is the maximum speed the ball can have before the cord breaks?

Solution:

Because the centripetal force in this case is the force T exerted by the cord on the ball, we have

$$T = m\frac{v^2}{r}$$

Solving for v, we have

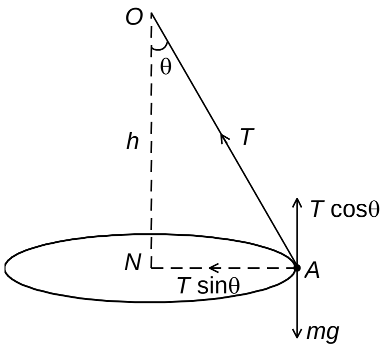
$$v = \sqrt{\frac{Tr}{m}}$$

The maximum speed that the ball can have corresponds to the maximum tension. Hence, we find

$$v_{max} = \sqrt{\frac{T_{max} r}{m}} = \sqrt{\frac{(50.0\,N)\,(1.50\,m)}{0.500\,kg}} = 12.2 \text{ m/s}$$

5.4 SOME IMPORTANT UNIFORM CIRCUALR MOTIONS

(i) Conical Pendulum: It consists of a string *OA* whose upper end *O* is fixed and a bob is tied at the free end. When the bob is drawn aside and given a horizontal push let it describe a horizontal circle with uniform angular velocity ω in such a way that the string makes an angle θ with vertical. As the string traces the surface of a cone of semi-vertical angle θ it is called conical pendulum. Let T be the tension in string, ℓ be the length and r be the radius of the horizontal circle described. The vertical component of tension balances the weight and the horizontal component supplies the centripetal force.



$$T \cos\theta = mg$$

$$T \sin\theta = mr\omega^2$$

$$\therefore \quad \tan\theta = \frac{r\omega^2}{g}$$

$$\omega = \sqrt{\frac{g\tan\theta}{r}}$$

$$r = \ell \sin\theta \text{ and } \omega = \frac{2\pi}{T}$$

T being the period i.e., time for one revolution.

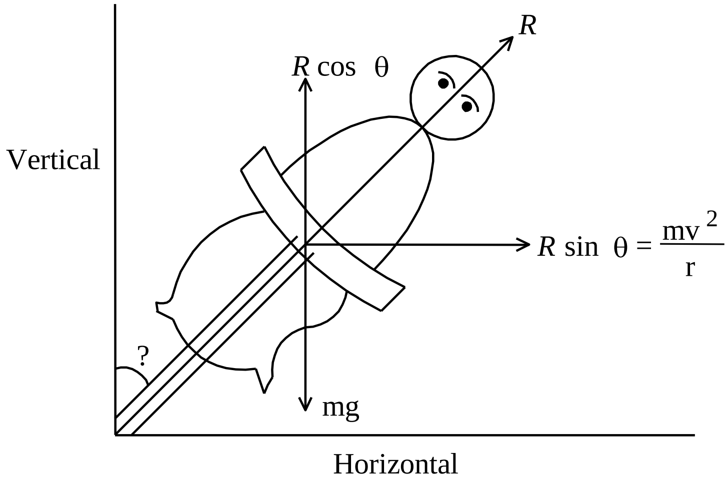
$$\therefore \quad \frac{2\pi}{T} = \sqrt{\frac{g\tan\theta}{\ell\sin\theta}}$$

$$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$$

$$= 2\pi\sqrt{h/g}\text{, where } h = \ell\cos\theta.$$

(ii) Motion of a cyclist on a circular path: Let a cyclist moving on a circular path of radius r bend away from the vertical by an angle θ.

R is the normal reaction from the ground. It can be resolved in the horizontal and vertical directions. The components are respectively equal to $R\sin\theta$ and $R\cos\theta$. The vertical component balances his weight mg. The horizontal component $R\sin\theta$ supplies the necessary force for making the circular path.



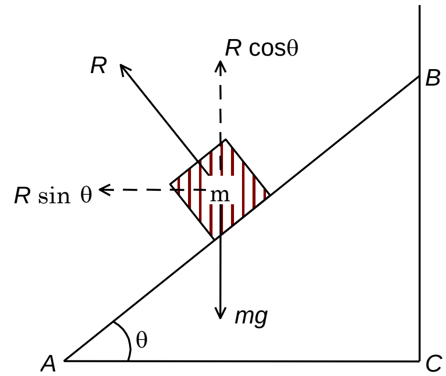
$$\therefore \quad R\sin\theta = \frac{mv^2}{r}$$

$$R\cos\theta = mg$$

$$\therefore \quad \tan\theta = v^2/rg$$

For less bending of the cyclist, v should be small and r should be great.

(iii) Banking of Roads: Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence on friction the roads are banked at the turn so that the outer part of the road is somewhat lifted up as compared to the inner part. The surface of the road makes an angle θ with the horizontal throughout the turn. The **Figure** shows the forces acting on a vehicle when it is moving on the banked road. *ABC* is the section of the road having a slope θ. *R* is the normal reaction and *mg* is the weight.



For vertical equilibrium, $R \cos \theta = mg$

The horizontal components $R \sin \theta$ is the required centripetal force $\frac{mv^2}{r}$

$$R \sin \theta = \frac{mv^2}{r}$$

$$\therefore \quad \tan \theta = v^2/rg$$

Above equation gives the angle of banking required which eliminates the lateral thrust in case of trains on rails or friction in case of road vehicles when rounding a curve.

(iv) Overturning and Skidding of cars

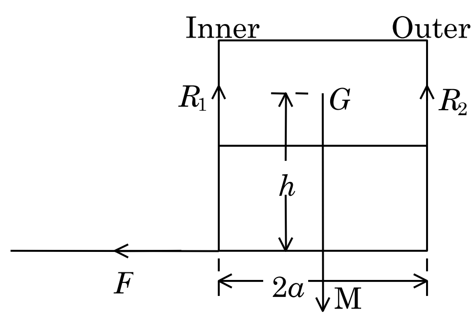
When a car takes a turn round a bend, whether the car tends to skid or topple depends on different factors. Let us consider the case of a car whose wheels are "2*a*" metre apart, and whose centre of gravity is '*h*' metres above the ground. Let the coefficient of friction between the wheels and the ground be μ.

Figure given below represents the forces on the car.

(i) The weight Mg of the car acts vertically downwards through the centre of gravity G of the car.

(ii) The normal reactions of the ground R_1 and R_2 act vertically upwards on the inner and outer wheels respectively.

(iii) The force of friction F between the wheels and the ground act towards the centre of the circle of which the road forms a part



Let the radius of the circular path be *r*, and the speed of the car be *v*.

Considering the vertical forces, since there is no vertical acceleration,

$$R_1 + R_2 = Mg \qquad \text{...(i)}$$

The horizontal force *F* provides the centripetal force for motion in a circle.

Therefore $$F = \frac{Mv^2}{r} \qquad \text{...(ii)}$$

Taking moments about G, if there is to be no resultant turning effect about the centre of gravity,

$$Fh + R_1 a = R_2 a \qquad \ldots\text{(iii)}$$

Conditions for no skidding

From equation (ii), it is seen that as the speed increases, the force required to keep the car moving in the circle also increases. However, there is a limit to the frictional force F, because,

$$F_{max} = \mu (R_1 + R_2)$$

Substituting from equation (i),

$$F_{max} = \mu Mg$$

Substituting from equation (ii),

$$\frac{Mv^2}{r} = \mu Mg$$

$\therefore$ $$v^2 = \mu rg$$

or $$v = \sqrt{\mu rg}$$

This expression gives the maximum speed v with which the car could take the circular path without skidding.

Conditions for no overturning

From equation (iii),

$$(R_2 - R_1)a = Fh$$

or $$R_2 - R_1 = \frac{Fh}{a} = \frac{Mv^2}{r} \cdot \frac{h}{a} \qquad \ldots\text{(iv)}$$

But $$R_2 + R_1 = Mg$$

Adding, $$2R_2 = Mg + \frac{Fh}{a} = Mg + \frac{Mv^2}{r} \cdot \frac{h}{a} \qquad \ldots\text{(v)}$$

$$2R_2 = M\left(g + \frac{v^2 h}{ra}\right) \qquad \ldots\text{(vi)}$$

$$R_2 = \frac{1}{2} M\left(g + \frac{v^2 h}{ra}\right)$$

Substituting for R_2 in equation (iv),

$$R_2 - R_1 = \frac{1}{2} M\left(g + \frac{v^2 h}{ra}\right) - R_1 = \frac{Mv^2 h}{ra}$$

$$R_1 = \frac{1}{2} M\left(g + \frac{v^2 h}{ra}\right) - \frac{Mv^2 h}{ra} = \frac{1}{2} \mathrm{M}\left(g + \frac{v^2 h}{ra} - \frac{2v^2 h}{ra}\right)$$

$$= \frac{1}{2} M\left(g - \frac{v^2 h}{ra}\right) \qquad \ldots\text{(vii)}$$

Equation (vi) shows that the reaction R_2 is always positive. However, equation (vii) shows that as the speed 'v' increases, the reaction R_1 decreases, and when $\frac{v^2h}{ra} = g$, R_1 becomes zero. This means that the inner wheel is no longer in contact with the ground, and the car commences to overturn outwards.

The maximum speed without overturning is given by

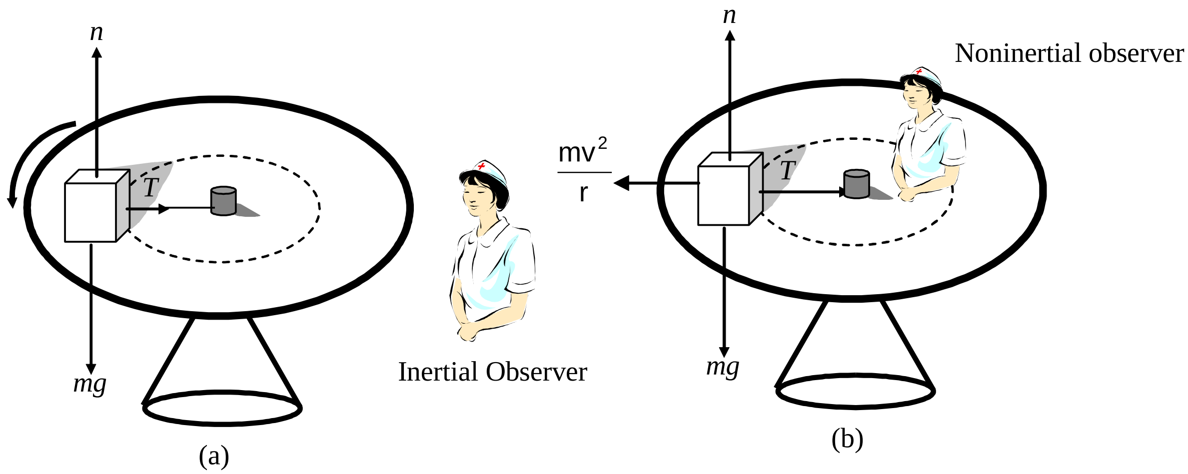
$$g = \frac{v^2h}{ra}$$

$$v = \sqrt{\frac{gra}{h}}$$

The same expression applies also to the case of a train moving on rails in a circular path of radius 'r'. Here $2a$ is the distance between the rails, and 'h' the height of the centre of gravity above the rails.

5.5 CENTRIFUGAL FORCE

An observer in a rotating system is another example of a non-inertial observer. Suppose a block of mass m lying on a horizontal, frictionless turntable is connected to a string as in figure. According to an inertial observer, if the block rotates uniformly, it undergoes an acceleration of magnitude v^2/r, where v is its tangential speed. The inertial observer concludes that this centripetal acceleration is provided by the force exerted by the string T, and writes Newton's second law $T = mv^2/r$.



According to a non-inertial observer attached to the turntable, the block is at rest. Therefore, in applying Newton's second law, this observer introduces a fictitious outward force of magnitude mv^2/r. According to the non-inertial observer, this outward force balances the force exerted by the string and therefore $T - mv^2/r = 0$.

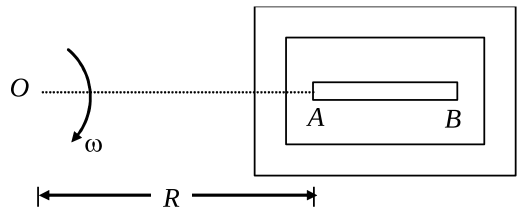
In fact, centrifugal force is a sufficient pseudo force only if we are analyzing the particles at rest in a uniformly rotating frame. If we analyze the motion of a particle that moves in the rotating frame we may have to assume other pseudo forces together with the centrifugal force. Such forces are called **Coriolis forces**. The Coriolis force is perpendicular to the velocity of the particle and also perpendicular to the axis of rotation of the frame. Once again it should be remembered that all these pseudo forces, centrifugal or Coriolis are needed only if the working frame is rotating. If we work from an inertial frame there is no need to apply any pseudo force. There should not be a misconception that centrifugal force acts on a particle because the particle describes a circle.

Therefore when we are working from a frame of reference that is rotating at a constant angular velocity ω with respect to an inertial frame. The dynamics of a particle of mass m kept at a distance r from the axis of rotation we have to assume that a force $m\omega^2 r$ acts radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

You should be careful when using fictitious forces to describe physical phenomena. Remember that fictitious forces are used only in non-inertial frames of references. When solving problems, it is often best to use an inertial frame.

Illustration 21

A table with smooth horizontal surface is fixed in a cabin that rotates with angular speed ωin a circular path of radius R. A smooth groove AB of length L ($<< R$) is made on the surface of table as shown in figure.



A small particle is kept at the point A in the groove and is released to move, find the time taken by the particle to reach the point B.

Solution:

Let us analyse the motion of particle with respect to table which is moving with cabin with an angular speed of ω. Along AB centrifugal force of magnitude $m\omega^2 R$ will act at A on the particle which can be treated as constant from A to B as $L << R$.

$\therefore$ acceleration of particle along AB with respect to cabin $a = \omega^2 R$ (constant)

Required time 't' is given by

$$S = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad L = 0 + \frac{1}{2} \times \omega^2 R t^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2L}{\omega^2 R}}$$

5.6 NON UNIFORM CIRCULAR MOTION

If the speed of the particle moving in a circle is not constant the acceleration has both radial and tangential components. The radial and tangential accelerations are

$$a_r = \omega^2 r = \frac{v^2}{r}$$

$$a_t = \frac{dv}{dt}$$

The magnitude of the resultant acceleration will be

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

If the direction of resultant acceleration makes an angle β with the radius, where then

$$\tan\beta = \frac{dv/dt}{v^2/r}$$

Now as acceleration of particle undergoing non-uniform circular motion is $a = \sqrt{(\omega^2 R)^2 + \left(R\frac{d\omega}{dt}\right)^2} = \sqrt{\left(\frac{v^2}{R}\right)^2 + \left(\frac{dv}{dt}\right)^2}$ in the direction $\tan^{-1}\left(\frac{dv/dt}{v^2/r}\right)$ with radius it need resultant force of $m\sqrt{\left(\frac{v^2}{R}\right)^2 + \left(\frac{dv}{dt}\right)^2}$ in the direction of acceleration.

Illustration 22

A car goes on a horizontal circular road of radius R, the speed increasing at a rate $\frac{dv}{dt} = a$. The friction coefficient between road and tyre is μ. Find the speed at which the car will skid.

Solution:

Here at any time t, the speed of car becomes V the net acceleration in the plane of road is $\sqrt{\left(\frac{v^2}{R}\right)^2 + a^2}$.

This acceleration is provided by frictional force. At the moment car will slide,

$$M\sqrt{\left(\frac{v^2}{R}\right)^2 + a^2} = \mu M g$$

$$\Rightarrow \qquad v = [R^2(\mu^2 g^2 - a^2)]^{1/4}$$

Motion in a vertical circular is a common example of non-uniform circular motion that we will discuss in next lesson of 'Work, Energy and Power', as it needs same idea of energy and its conservation.

Fundamental Solved Examples

Example 1.

An aeroplane, which together with its load has a mass M kg, is falling with an acceleration of a m/s^2 where a < g. Show that if a part of the load equal to $\frac{2Ma}{a+g}$ kg be thrown out, the aeroplane will begin to rise with an acceleration of a m/s^2.

Solution:

Given M is the mass of the aeroplane.

Let R be the upthrust acting on it. Since it is falling down with an acceleration a,

$$Mg - R = Ma \qquad \text{... (i)}$$

Let a mass m kg be thrown out. The remaining mass is (M – m) kg and now the plane begins to rise up with an acceleration a m/s^2.

Now $$R - (M - m)\,g = (M - m)a \qquad \text{... (ii)}$$

Adding equations (i) and (ii),

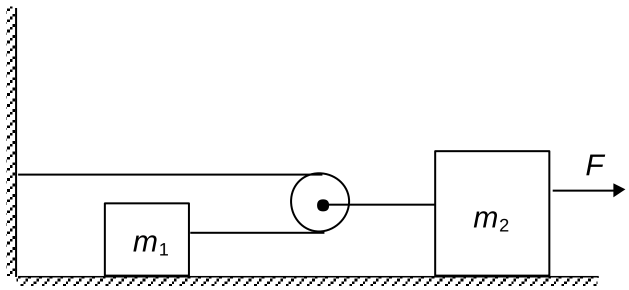
$$mg = (2M - m)a$$

or, $$m\,(g + a) = 2Ma$$

$$\Rightarrow \quad m = \frac{2Ma}{a+g}\,\text{kg}$$

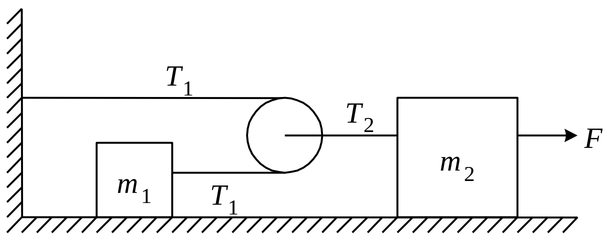
Example 2.

In the system shown below, friction and mass of the pulley are negligible. Find the acceleration of m_2 if m_1 = 300 g, m_2 = 500 g and F = 1.50 N



Solution:

When the pulley moves a distance d, m_1 will move a distance 2d. Hence m_2 will have twice as large an acceleration as m_2 has. Also because the total force on the pulley must be zero, $T_1 = (T_2/2)$.



For mass m_1, $T_1 = m_1 (2a)$ … (i)

For mass m_2, $F - T_2 = m_2 (a)$ … (ii)

Putting $T_1 = \frac{T_2}{2}$, (i) gives $T_2 = 4m_1 a$

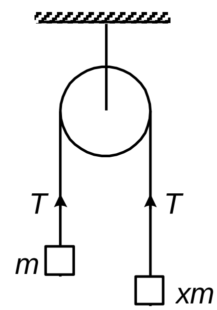
Substituting in equation (ii), $F = 4m_1a + m_2a = (4m_1 + m_2)a$

Hence $a = \frac{F}{4m_1+m_2} = \frac{1.50}{4(0.3)+0.5} = 0.88 \text{ m/s}^2$

Example 3.

A light inextensible string passing over a smooth fixed pulley attaches two masses of magnitudes m and xm. Find the two possible values of x if the acceleration of the system is g/4.

Solution:



Two cases will arise according as x < 1 or x > 1

Case 1:

When x < 1, xm < m and the mass m will fall while the mass xm will rise.

The equations of motion will be

for mass m, $mg - T = ma$ … (i)

for mass xm, $T - xmg = (xm) a$ … (ii)

Adding, $mg (1 - x) = (1 + x) ma$

or, $g(1 - x) = a (1 + x)$

It is given $a = \frac{g}{4}$. Putting this value,

$$(1 - x) = \left(\frac{1+x}{4}\right)$$

or, $5x = 3$ Hence $x = \mathbf{\frac{3}{5}}$

Case 2:

When x > 1, xm > m and the mass xm will fall while mass m will rise. The equations of motion will be

for mass m, $T - mg = ma$ … (iii)

for mass xm, $xmg - T = (xm)a$ … (iv)

Adding, $(x - 1) mg = (x + 1) ma$

Putting $a = \frac{g}{4}$, $4(x - 1) = (x + 1)$

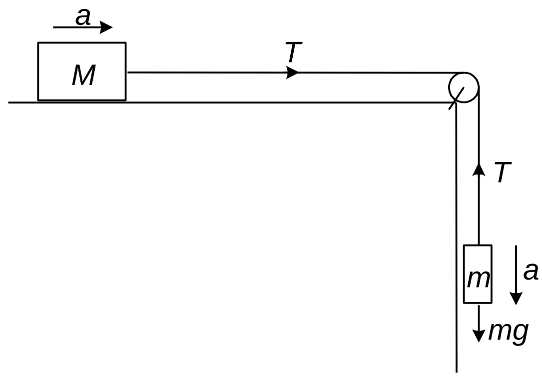
or, $3x = 5$ giving $x = \frac{5}{3}$

Thus the two possible values of x for which the acceleration of the system will be $\frac{g}{4}$ are $\mathbf{\frac{3}{5}}$ and $\mathbf{\frac{5}{3}}$.

Example 4.

A mass of 2 kg hangs freely at the end of a string, which passes over a smooth pulley fixed at the edge of a smooth table. The other end of the string is attached to a mass M on the table. If the mass on the table is doubled the tension in the string increases by one-half. Find the mass M.

Solution:



The tension in the string is given by

$$T = \frac{mM}{m+M}g \quad \ldots \text{(i)}$$

In the second case M changes to 2M and T changes to $\frac{3}{2}$ T

$$\therefore \quad \frac{3}{2}T = \frac{m(2M)}{m+(2M)} \cdot g \quad \ldots \text{(ii)}$$

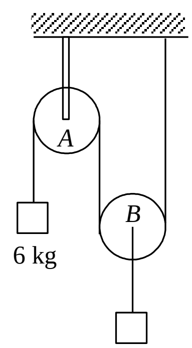
Dividing (i) by (ii), we get

$$\frac{2}{3} = \frac{m+2M}{m+M} \times \frac{1}{2}$$

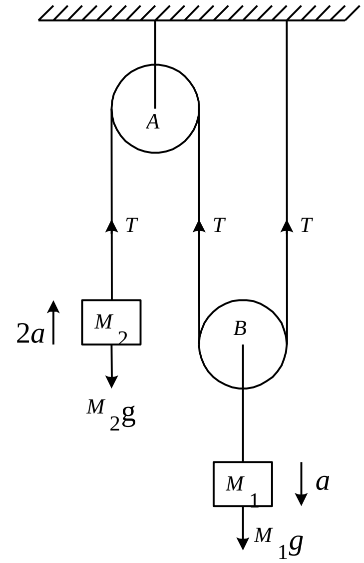
Substituting m = 2 kg, M = 1 kg

Example 5.

A mass of 15 kg and another of mass 6 kg are attached to a pulley system as shown. A is a fixed pulley while B is a movable one. Both are considered light and frictionless. Find the acceleration of 6 kg mass.



Solution:



Tension is the same throughout the string. It is clear that M_1 will descend downwards while M_2 rises up. If the acceleration of M_1 is a downwards, M_2 will have an acceleration 2a upwards.

Now, $M_1g - 2T = M_1a$

$T - M_2g = M_2 \cdot 2a$

or, $M_1g - 2M_2g = a(M_1 + 4M_2)$

$$\Rightarrow \quad a = \frac{M_1 - 2M_2}{M_1 + 4M_2} g$$

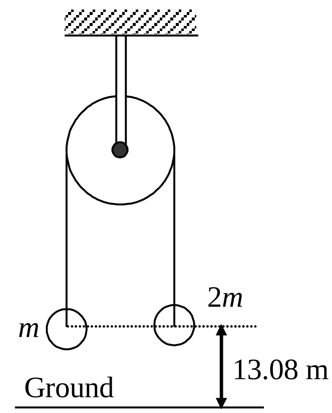
$$= \frac{15-12}{15+24} g = \frac{3}{39} g$$

$$\therefore \quad a = \frac{g}{13}$$

$\therefore$ acceleration of 6 kg mass = 2a = $\mathbf{\dfrac{2g}{13}}$

Example 6.

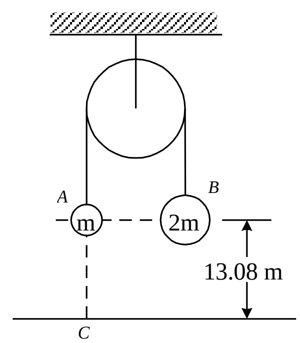
Two masses m and 2m are connected by a massless string, which passes over a pulley as shown in figure. The masses are held initially with equal lengths of the strings on either side of the pulley. Find the velocity of masses at the instant the lighter mass moves up a distance of 6.54 m. The string is suddenly cut at that instant. Calculate the time taken by each mass to reach the ground. ($g = 9.81\ m/s^2$)



Solution:

The masses A and B of m and 2m respectively are initially along the horizontal position through the line AB.

When the masses are left free, B comes down, A moves up with acceleration a.



$$\text{Now, } a = \frac{(2m - m)g}{2m + m} = \frac{g}{3}$$

The initial velocities of both of them is zero.

When the lighter mass A moves up through a height 6.54 m, its velocity v is given by

$$v = \sqrt{2 \times a \times S} = \sqrt{2 \times \frac{9.81}{3} \times 6.54} = \mathbf{6.54}\ \text{m/s}$$

Both the masses A and B have the velocity of same magnitude 6.54 m/s. At this instant the string snaps.

To calculate the time taken by mass A to reach the ground.

The position of A when the string is cut is given by 13.08 + 6.54 = 19.62 m above the ground.

Now, $S = ut + \frac{1}{2}at^2$

$$-19.62 = 6.54t + \frac{1}{2}(-9.81)t^2$$

or, $-6 = 2t - \frac{3}{2}t^2$

or, $\frac{3}{2}t^2 - 2t - 6 = 0$

or, $t = 2.78$ s

A reaches ground 2.78 s after the string snaps.

Time taken by B to reach the ground

$u = 6.54$ m/s

$a = 9.81$ m/s^2

$S = 6.54$ m

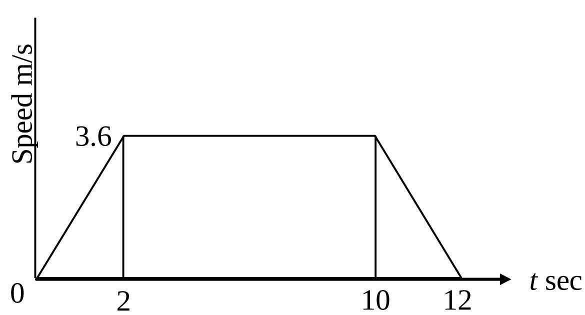
$$S = ut + \frac{1}{2} \times at^2$$

$$6.54 = 6.54\,t + \frac{1}{2} \times 9.81\,t^2$$

or $t = 0.665$ s

Example 7.

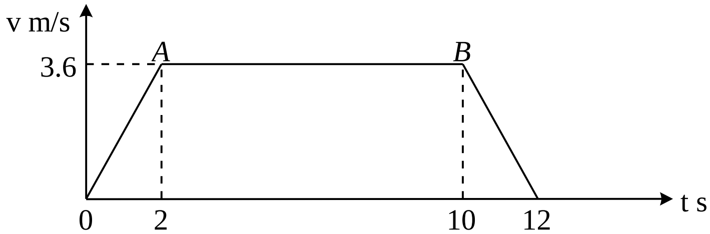
A lift is going up. The total mass of the lift and the passengers is 1500 kg. The variation in the speed of the lift is given in the graph.



What will be the tension in the rope pulling the lift at t equal to (i) 1 s (ii) 6 s and (iii) 11 s?

Solution:

The velocity-time graph of the motion of the lift is given below.



From t = 0 to t = 2 s, the lift moves up with uniform acceleration a = $\frac{3.6}{2} = 1.8 m/s^2$

If T be the tension in the rope pulling the lift with the passengers up, then

$$T - Mg = Ma$$

or $$T = M (g + a)$$

(i) At t = 1 s, T = 1500 (9.8 + 1.8) N

= 17400 N

(ii) At t = 6 s, the lift moves with uniform speed of 3.6 m/s and hence a = 0

or $$T = Mg$$

$$= 1500 \times 9.8 = 14700 \text{ N}$$

(iii) At t = 11 s, the lift is decelerating.

The deceleration = $-\frac{3.6}{2} = -1.8 m/s^2$

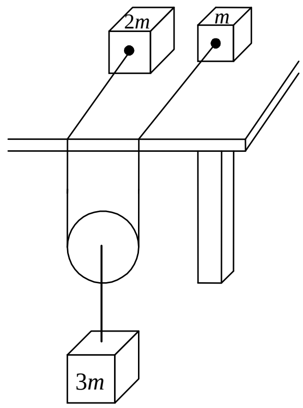
and hence the tension T in the string is

$$T = M (g + a) = 1500 (9.8 - 1.8) = 12000 \text{ N}$$

Example 8.

Two particles of masses m and 2m are placed on a smooth horizontal table. A string, which joins them, hangs over the edge supporting a light pulley, which carries a mass 3 m.

The two parts of the string on the table are parallel and perpendicular to the edge of the table. The parts of the string outside the table are vertical. Show that the acceleration of the particle of mass 3m is 9g/17.



Solution:

Let T be the tension in the string; a be the acceleration of the mass 2m; 2a be the acceleration of mass m.

$$T = m \cdot 2a$$

The mass 3m will come down with an acceleration $\frac{a+2a}{2} = \frac{3a}{2}$

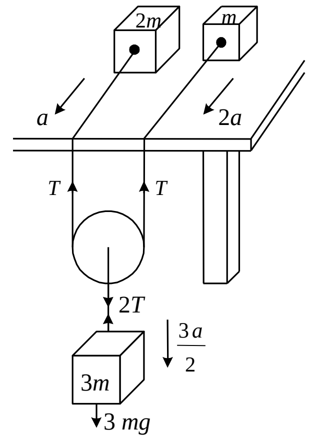
$\therefore \quad 3mg - 2T = 3m \cdot \frac{3a}{2}$

or $\quad 3mg - 4\,ma = \frac{9ma}{2}$

or $\quad \frac{17a}{2} = 3g$

or $\quad a = \frac{6}{17}g$

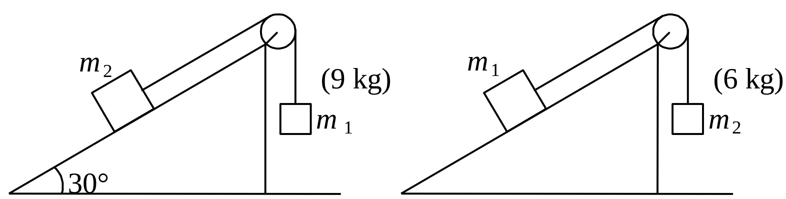
$\therefore$ the acceleration of 3m mass $= \frac{3}{2}a = \mathbf{\frac{9}{17}g}$



Example 9.

A body m_1 of mass 9 kg and another body m_2 of mass 6 kg are connected by a light inextensible string. Consider a smooth inclined plane of inclination 30° over which one of them can be placed while the other hangs vertically and freely. Show that m_1 will drag m_2 up the whole length of the plane in half the time that m_2 hanging vertically would take to draw m_1 up the plane.

Solution:



Case (i):

Let a_1 be the acceleration of the system when 9 kg mass hangs freely and T the tension in the string.

$$M_1g - T = m_1a_1$$

$$T - m_2g \sin 30° = m_2a_1$$

$$g\,(m_1 - m_2 \sin 30°) = a_1\,(m_1 + m_2) \qquad a_1 = \frac{g\left(9 - 6 \times \frac{1}{2}\right)}{15} = \frac{6g}{15} = \frac{2g}{5}$$

Case (ii):

Let a_2 be the acceleration of the system when 6 kg mass hangs freely and T′ the tension in the string.

$$m_2g - T' = m_2a_2$$

$$T' - m_1g \sin 30° = m_1a_2$$

$$g(m_2 - m_1 \sin 30°) = a_2(m_2 + m_1) \quad = g\left(6 - 9 \times \frac{1}{2}\right) = a_2(6+9) \quad a_2 = \frac{3g}{30} = \frac{g}{10}$$

If S is the length of the plane,

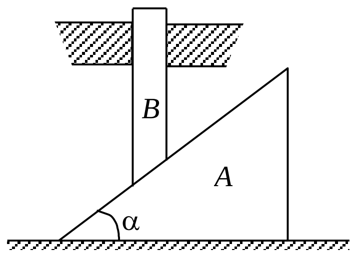
In case (i), $S = \frac{1}{2} \times a_1 t_1^2$

In case (ii), $S = \frac{1}{2} a_2 t_2^2 \qquad a_1 t_1^2 = a_2 t_2^2$

$$t_1/t_2 = \sqrt{a_2/a_1} = \sqrt{\frac{g/10}{2g/5}} = \sqrt{1/4} \Rightarrow \frac{t_1}{t_2} = \frac{\mathbf{1}}{\mathbf{2}}$$

Example 10.

Find the acceleration of rod B and wedge A in the arrangement shown in figure, if the ratio of the mass of wedge to that of rod equals n and there is no friction between any contact surfaces.

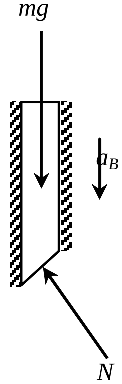


Solution:

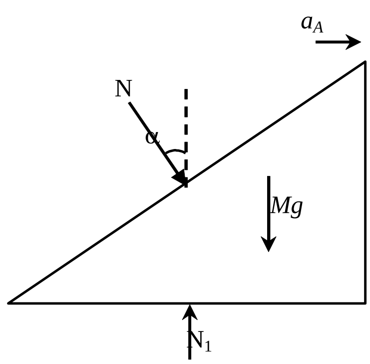
Let m be the mass of rod B and M that of wedge $\frac{M}{m} = n$

If acceleration of rod B and wedge A are a_A and a_B then

$$a_B = a_A \tan\alpha \qquad \ldots \text{(i)}$$



FBD of rod



FBD of wedge

Writing equation for motion.

for rod, $mg - N\cos\alpha = ma_B$ … (ii)

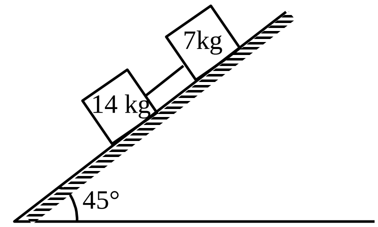
for wedge, $N\sin\alpha = Ma_A$ … (iii)

Solving equation (i), (ii) and (iii)

$$\boldsymbol{a_B = \frac{g\tan\alpha}{\tan\alpha + n\cot\alpha}} \quad , \quad \boldsymbol{a_A = \frac{g}{\tan\alpha + n\cot\alpha}}$$

Example 11.

Two masses 14 kg and 7 kg connected by a flexible inextensible string rest on an inclined plane inclined at 45° with the horizontal as shown in figure. The coefficient of friction between the plane and the 14 kg mass is 1/4 and that between the plane and the 7 kg mass is 3/8. Find the tension in the connecting string.

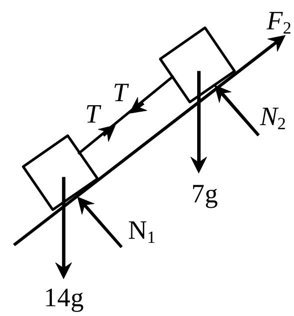


Solution:

The force diagram of the masses placed on the inclined plane is shown in Figure. Considering the motion of 14 kg mass the equation of motion can be written as

$$14g \sin 45° - f_1 - T = 14a \qquad \ldots \text{(i)}$$

where a is the acceleration down the plane.



$$N_1 = 14g \cos 45° \qquad \ldots \text{(ii)}$$

$$f_1 = \mu N_1 = \frac{1}{4} \times 14g \cos 45° \qquad \ldots \text{(iii)}$$

$$\therefore \ 14g \sin 45° - \frac{1}{4} \times 14g \cos 45° - T = 14a$$

$$\frac{14\times 9.8}{\sqrt{2}} - \frac{1}{4} \times \frac{14\times 9.8}{\sqrt{2}} - T = 14a \qquad \ldots \text{(iv)}$$

The equations of motion for 7 kg mass can be written similarly considering the motion of 7 kg mass separately.

$$T + 7g \sin 45° - f_2 = 7a \qquad \ldots \text{(v)}$$

$$N_2 = 7g \cos 45° \qquad \ldots \text{(vi)}$$

$$f_2 = \mu N_2 = \frac{3}{8} \times 7g \cos 45° \qquad \ldots \text{(vii)}$$

$$\therefore \qquad T + 7g \sin 45° - \frac{3}{8} \times 7g \cos 45° = 7a \qquad \ldots \text{(viii)}$$

$$T + \frac{7\times 9.8}{\sqrt{2}} - \frac{3}{8} \times \frac{7\times 9.8}{\sqrt{2}} = 7a$$

$$T + \frac{7\times 9.8}{\sqrt{2}} \times \frac{5}{8} = 7a$$

From (4), $\frac{14\times 9.8}{\sqrt{2}} \times \frac{3}{4} - T = 14a$

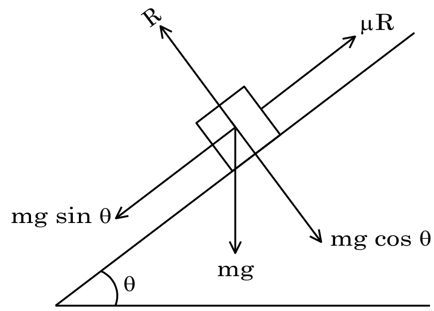
Solving the above simultaneous equations in T and a,

we get T = 4.03 N

Example 12.

A conveyor belt making an angle of 30° with the horizontal moves up with a speed of 3 m/s. A box is gently placed on it. Find how far the box will move on the belt before coming to rest, if the coefficient of friction between belt and box is 0.8.

Solution:



When the block is placed on the belt its velocity relative to the belt = – 3 m/s.

The forces acting on it are

(i) Friction up the plane = $\mu mg \cos\theta$

(ii) $mg \sin\theta$ down the plane

∴ resultant acceleration upward

$$= \frac{\mu mg\cos\theta - mg\sin\theta}{m}$$

$$= \mu g \cos\theta - g \sin\theta$$

$$= 0.8 \times 9.8 \times \frac{\sqrt{3}}{2} - 9.8 \times \frac{1}{2}$$

$$= 6.79 - 4.9 = 1.89 \text{ m/s}^2$$

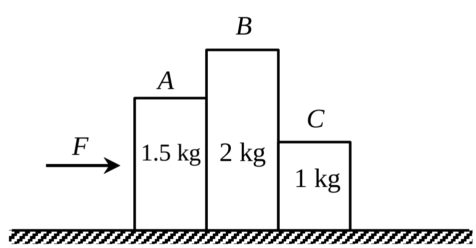
Direction up the plane is taken as positive.

Now $u = -3$ m/s; $a = 1.89$ m/s^2; $v = 0$

$$\therefore \quad S = u^2/2a = \frac{9}{2 \times 1.89} = 2.38 \text{ m}$$

Example 13.

In the system of three blocks A, B and C shown in figure, (i) how large a force F is needed to give the blocks an acceleration of 3 m/s^2, if the coefficient of friction between blocks and table is 0.27 (ii) how large a force does the block A exert on the block B?



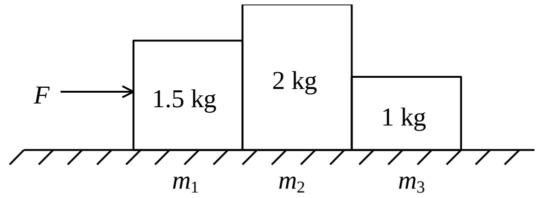
Solution:

Let a be the acceleration of the system to right. All the three frictional forces $f_1 = \mu m_1 g$, $f_2 = \mu m_2 g$ and $f_3 = \mu m_3 g$ will be directed to the left as the motion of bodies is to the right. Hence, for the whole system

$F - \mu m_1 g - \mu m_2 g - \mu m_3 g = (m_3 + m_2 + m_3)\, a$

$F = (m_1 + m_2 + m_3)\,(a + \mu g)$

$= (1.5 + 2 + 1)\,(3 + 0.2 \times 9.8) = 22.3\text{ N}$



The force exerted by the 1.5 kg block on the 2 kg block $= F - m_1 (a + \mu g)$

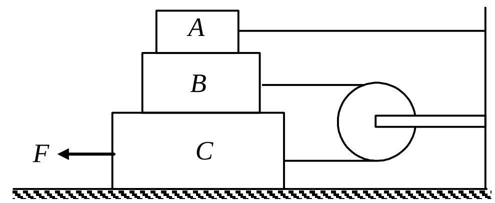
$= 22.3 - 1.5\,(3 + 0.2 \times 9.8)$

$= 22.3 - 7.44$

$= 14.86\text{ N}$

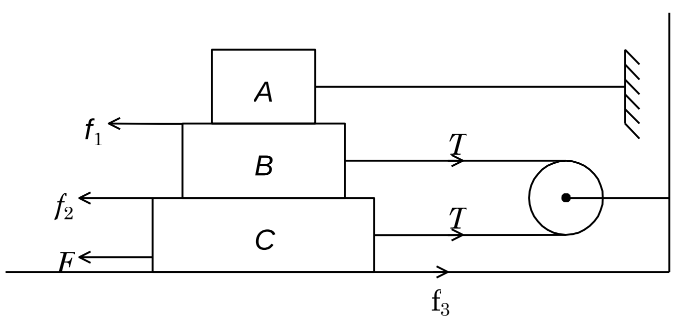
Example 14.

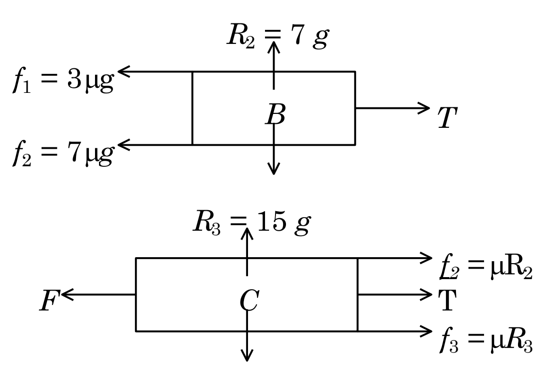
In the figure shown, blocks A, B and C weigh 3 kg, 4 kg and 8 kg respectively. The coefficient of sliding friction between any two surfaces is 0.25. A is held at rest by a massless rigid rod fixed to the wall while B and C are connected by a string passing round a frictionless pulley. Find the force needed to drag C along the horizontal surface to left at constant speed.



Assume the arrangement shown in figure is maintained all through.

Solution:





The free body diagram of B and C are separately shown in Figures.

$T = f_1 + f_2 = 3\mu g + 7\mu g = 10\mu g$

$= 10 \times 0.25 \times 9.8 = 24.5$ N

Now $\quad F = f_2 + f_3 + T$

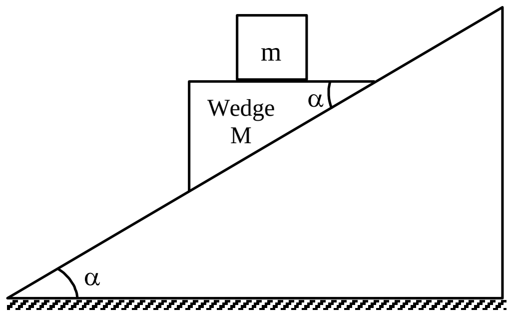
$= \mu \cdot 7g + \mu \cdot 15g + 10\mu g$

$= 0.25 \times 32 \times 9.8$

$= 78.4$ N

Example 15.

On a smooth inclined plane of angle α a smooth wedge of mass M and angle α is placed on in such a way that the upper wedge face is horizontal. On this horizontal face is placed a block of mass m. Find the resultant acceleration of the block in subsequent motion.



Solution:

Let the acceleration of the wedge M be A down the plane. With reference to this accelerated frame a pseudo force mA acts on the block m.

The force diagram for the block is shown in Figure.

The equations of motion can be written as mA cos α = ma

where a is the acceleration of the particle relative to the wedge.

$\therefore \quad A = U \cos\alpha \quad \ldots$ (i)

Considering the vertical motion of m

$R + mA \sin\alpha = mg$

$R = m(g - A\sin\alpha) \quad \ldots$ (ii)

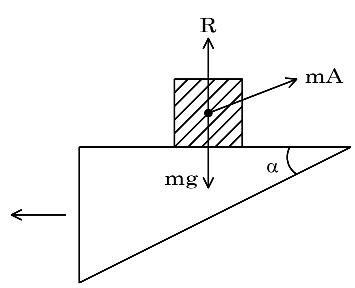
We shall now consider the free body diagram of the wedge of mass M shown in Figure.

$(R + Mg)\sin\alpha = MA \quad \ldots$ (iii)

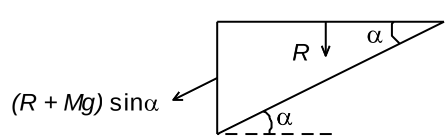
Substituting for R in equation (iii) from equation (ii)

$[m(g - A\sin\alpha) + Mg]\sin\alpha = MA$

$$\therefore \quad A = \frac{(M+m)g\sin\alpha}{M+m\sin^2\alpha} \quad \ldots \text{(iv)}$$



The acceleration of m relative to M is A cos α in the horizontal direction.



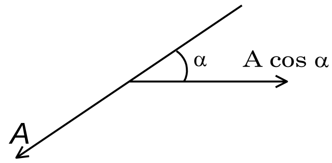
The acceleration of wedge is A down the plane. If the resultant acceleration is represented by f then by the law of parallelogram of vectors,

$f^2 = A^2 + A^2\cos^2\alpha - 2(A \cdot A\cos\alpha)\cos\alpha$

$= A^2 + A^2\cos^2\alpha - 2A^2\cos^2\alpha$

$= A^2 - A^2\cos^2\alpha = A^2\sin^2\alpha$

$f = A\sin\alpha$

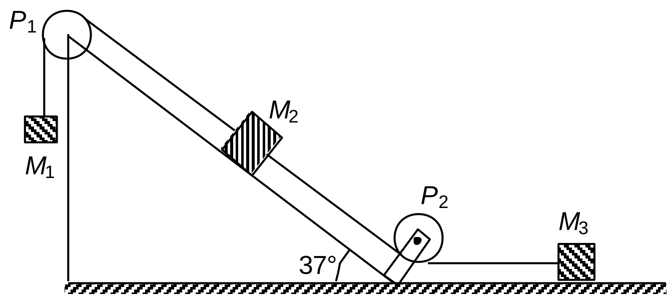


Substituting the value of A from equation (iv)

$$f = \frac{(M+m)g\sin^2\alpha}{M+m\sin^2\alpha}$$

Example 16.

Masses M_1, M_2 and M_3 are connected by light strings which pass over pulleys P_1 and P_2 as shown. The masses move such that the string between P_1 and P_2 is parallel to incline and the string between P_2 and M_3 is horizontal, $M_2 = M_3 = 4$kg. The coefficient of kinetic friction between masses and the surface is 0.25. The angle of inclination of plane is 37° to the horizontal. If the mass M_1 moves downwards with uniform velocity, find M_1 and the tension in the horizontal string. Given $g = 9.8$ m/s^2 and sin 37° = 3/5.



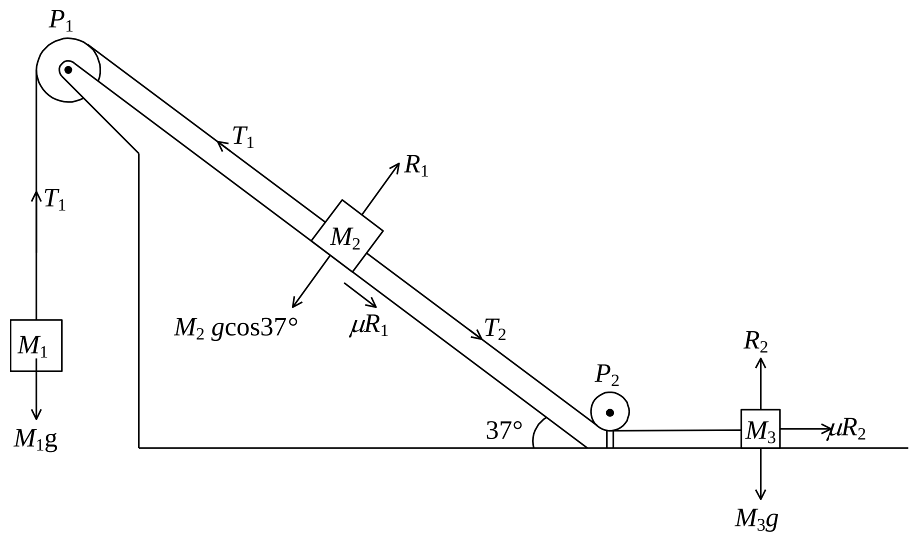
Solution:

Considering the mass M_1, since it moves down with uniform speed

$$T_1 = M_1 g \quad \ldots \text{(i)}$$

For mass M_2, $T_1 = T_2 + \mu M_2 g\cos 37° + M_2 g\sin 37°$ … (ii)

For mass M_3, $T_2 = \mu M_3 g$ … (iii)



Substituting for T_1 and T_2 from (i) and (iii) respectively in (ii),

$M_1g = \mu M_3g + \mu M_2g \cos 37° + M_2g \sin 37°$ 0

$M_1 = \mu M_3 + \mu M_2 \cos 37° + M_2 \sin 37°$

$\cos 37° = \frac{4}{5}$; $\sin 37° = \frac{3}{5}$; $\mu = \frac{1}{4}$

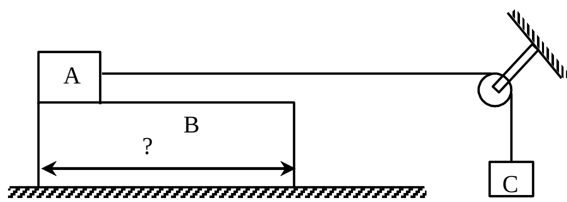
$$\therefore \quad M_1 = \frac{1}{4} \times 4 + \frac{1}{4} \times 4 \times \frac{4}{5} + 4 \times \frac{3}{5}$$

$$= 1 + \frac{4}{5} + \frac{12}{5} = 4.2 \text{ kg}$$

$$T_2 = \mu M_3 g = \frac{1}{4} \times 4 \times 9.8 = 9.8 \text{ N}$$

Example 17.

In the figure block A is one fourth the length of the block B and there is no friction between block B and the surface on which it is placed. The coefficient of sliding friction between A and B = 0.2. Block C and block A have the same mass and mass of B is four times mass of A. When the system is released, calculate the distance the block B moves when only three-fourth of block A is still on the block B.



Solution:

The free body diagram for masses A, B and C are shown in Figures (a), (b) and (c).

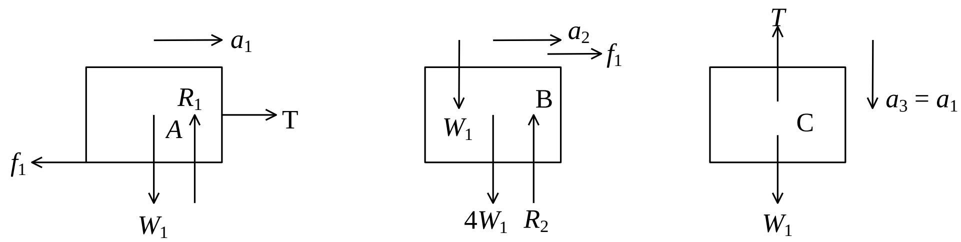


Fig. (a) **Fig. (b)** **Fig. (c)**

From Figure (b), the equations of motion for the three masses A, B and C can be written.

$T - f_1 = ma_1$

or $T - \mu W_1 = ma_1$ … (i)

$f_1 = \mu W_1 = 4ma_2$ … (ii)

$W_1 - T = ma_1$ … (iii)

Solving equations (i) and (iii), $W_1 (1 - \mu) = 2ma_1$

$W_1 = mg$

$\therefore \quad mg(1-\mu)=2ma_1$

$$a_1=\frac{g}{2}(1-\mu) \qquad \text{… (iv)}$$

From equation (ii)

$\mu W_1 = 4ma_2 \quad W_1 = mg$

$\therefore \quad \mu mg=4ma_2$

$$a_2=\frac{\mu g}{4} \qquad \text{… (v)}$$

The displacements of the blocks A and B are given by

$$x=\frac{1}{2}at^2 \; x_1=\frac{g}{4}(1-\mu)t^2$$

$$x_2=\frac{g}{8}\mu t^2$$

At the instant that three-fourth of the block A remains on block B

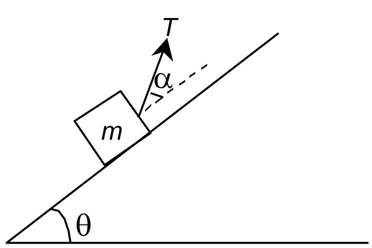
$x_2+\ell=x_1+\frac{3}{4}\frac{\ell}{4}$, where ℓ is the length of the block B.

$$\therefore \quad \frac{g}{8}\mu t^2+\ell=\frac{g}{4}(1-\mu)t^2+\frac{3\ell}{16} \quad t^2=\frac{13\ell}{2g(2-3\mu)}$$

$$x_2=\frac{g}{8}\mu\frac{13\ell}{2g(2-3\mu)}=\frac{13\mu\ell}{16(2-3\mu)}$$

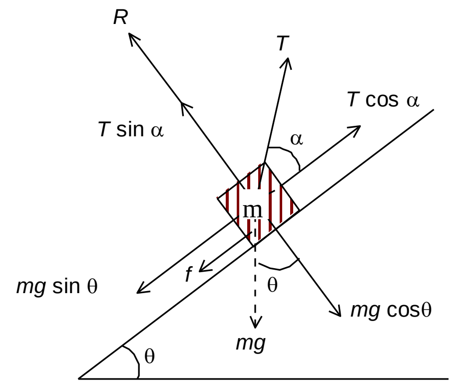
Example 18.

A block of mass m is pulled upward by means of a thread up an inclined plane forming an angle θ with the horizontal as shown in figure. The coefficient of friction is μ. Find the inclination of the thread with horizontal so that the tension in the thread is minimum. What is the value of the minimum tension?



Solution:

The different forces acting on the mass are shown in Figure. Let the mass move up the plane with an acceleration a. Writing the equation of motion



$R + T \sin\alpha = mg\cos\theta$

$R = mg\cos\theta - T\sin\alpha$ … (i)

$T\cos\alpha - mg\sin\theta - f = ma$ … (ii)

where f is the force of friction.

$f = \mu\,(mg\cos\theta - T\sin\alpha)$ … (iii)

Substituting the value of f from equation (iii) in equation (ii)

$T\cos\alpha - mg\sin\theta - \mu mg\cos\theta + \mu T\sin\alpha = ma$

$T(\cos\alpha + \mu\sin\alpha) = ma + mg\sin\theta + \mu mg\cos\theta$

$$T = \frac{ma + mg\sin\theta + \mu mg\cos\theta}{\cos\alpha + \mu\sin\alpha} \quad \text{… (iv)}$$

For T to be minimum $(\cos\alpha + \mu\sin\alpha)$ should be maximum.

$$\frac{d}{d\alpha}(\cos\alpha + \mu\sin\alpha) = 0$$

$$\frac{d^2}{d\alpha^2}(\cos\alpha + \mu\sin\alpha) = -ve$$

$$\frac{d}{d\alpha}(\cos\alpha + \mu\sin\alpha) = -\sin\alpha + \mu\cos\alpha = 0$$

$\mu = \tan\alpha$

$\alpha = \tan^{-1}(\mu)$

It can be shown that $\frac{d^2}{d\alpha^2}$ is negative.

T will have minimum value when a = 0 and $\alpha = \text{tam}^{-1}(\mu)$

From equation (iv)

$$T_{\min} = \frac{mg\sin\theta + \mu mg\cos\theta}{\cos\alpha + \mu\sin\alpha}$$

$\cos\alpha + \mu\sin\alpha = \cos\alpha + \mu\,(\mu\cos\alpha)$

$= \cos\alpha + \mu^2\cos\alpha$

$$= \cos\alpha\left(1 + \mu^2\right) = \frac{1 + \mu^2}{\sec\alpha}$$

$$= \frac{1+\mu^2}{\sqrt{1+\tan^2\alpha}} = \frac{1+\mu^2}{\sqrt{1+\mu^2}} = \sqrt{1+\mu^2}$$

$$\therefore \qquad T_{min} = \frac{mgsin\theta + \mu mgcos\theta}{\sqrt{1+\mu^2}}$$

Example 19.

A large mass M and a small mass m hang at the two ends of the string that passes through a smooth tube as shown in Figure. The mass m moves around in a circular path, which lies in the horizontal plane. The length of the string from the mass m to the top of the tube is ℓ and θ is the angle this length makes with vertical. What should be the frequency of rotation of mass m so that M remains stationary?

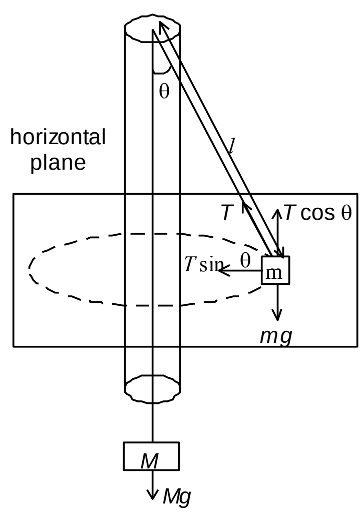
Solution:

The forces acting on mass m and M are shown in Figure. When mass M is stationary

$$T = Mg \qquad \dots(i)$$

where T is tension in string.

For the smaller mass, the vertical component of tension T cosθ balances mg and the horizontal component T sin θ supplies the necessary centripetal force.



$$T\cos\theta = mg \qquad \dots(ii)$$

$$T\sin\theta = mr\omega^2 \qquad \dots(iii)$$

ω being the angular velocity and r is the radius of horizontal circular path.

From (i) and (iii), $Mg\sin\theta = mr\omega^2$

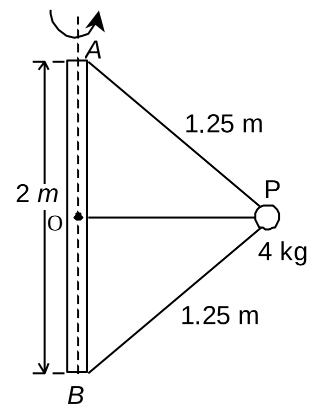
$$\omega = \sqrt{\frac{Mg\sin\theta}{mr}} = \sqrt{\frac{Mg\sin\theta}{ml\sin\theta}} = \sqrt{\frac{Mg}{ml}}$$

$$\text{Frequency of rotation} = \frac{1}{T} = \frac{1}{2\pi/\omega} = \frac{\omega}{2\pi}$$

$$\therefore \qquad \text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}$$

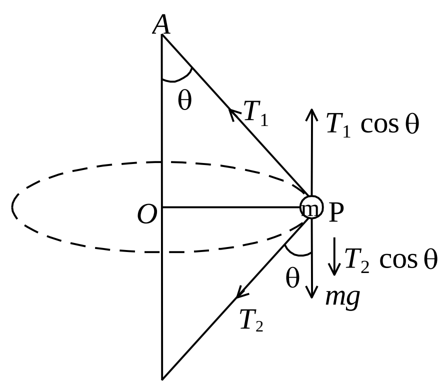
Example 20.

The 4 kg block in the Figure is attached to the vertical rod by means of two strings. When the system rotates about the axis of the rod, the two strings are extended as indicated in Figure. How many revolutions per minute must the system make in order that the tension in upper string is 60 N. What is tension in the lower string?



Solution:

The forces acting on block P of mass 4 kg are shown in the Figure. If θ is the angle made by strings with vertical, T_1 and T_2 tensions in strings for equilibrium in the vertical direction



$$T_1 \cos\theta = T_2 \cos\theta + mg$$

$$(T_1 - T_2)\cos\theta = mg$$

$$\cos\theta = \frac{1}{1.25} = \frac{4}{5}$$

$$\left[\because \cos\theta = \frac{OA}{AP} = \frac{1}{1.25}\right]$$

$$\therefore \quad T_1 - T_2 = \frac{mg}{\cos\theta} = \frac{5mg}{4} = \frac{5}{4} \times 4 \times 9.8 = 49 \text{ N}$$

Given $\quad T_1 = 60$ N

$$T_2 = T_1 - 49 = 60 \text{ N} - 49 \text{ N} = 11 \text{ N}$$

The net horizontal force ($T_1 \sin\theta + T_2 \sin\theta$) provides the necessary centripetal force $m\omega^2 r$.

$$\therefore \quad (T_1 + T_2)\sin\theta = m\omega^2 r$$

$$\Rightarrow \quad \omega^2 = \frac{(T_1 + T_2)\sin\theta}{mr}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - (4/5)^2} = \frac{3}{5}$$

$$r = OP = \sqrt{1.25^2 - 1^2} = 0.75$$

$$\therefore \quad \omega^2 = \frac{(60+11)\frac{3}{5}}{4 \times 0.75} = 14.2$$

$$\omega = \sqrt{14.2} = 3.768 \text{ rad/s}$$

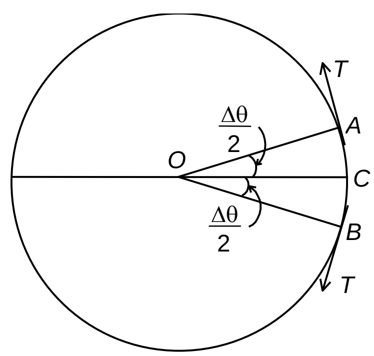
Frequency of revolution = $\dfrac{\omega}{2\pi} = \dfrac{3.768}{2 \times 3.14}$

$= 0.6$ rev/s or 36 rev/min

Example 21.

A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of ring moves with velocity v. Find the tension in the ring.

Solution:



Consider a small part ACB of the ring that subtends an angle $\Delta\theta$ at the centre as shown in Figure. Let the tension in the ring be T.

The forces on this elementary portion ACB are

(i) tension T by the part of the ring left to A

(ii) tension T by the part of the ring right to B

(iii) weight (Δm) g

(iv) normal force N by the table.

As the elementary portion ACB moves in a circle of radius R at constant speed v its acceleration towards centre is $\dfrac{(\Delta m)v^2}{R}$.

Resolving the forces along the radius CO

$$T\cos\left(90^\circ - \frac{\Delta\theta}{2}\right) + T\cos\left(90^\circ - \frac{\Delta\theta}{2}\right) = \Delta m \frac{v^2}{R} \qquad \ldots(i)$$

$$2T\sin\frac{\Delta\theta}{2} = \Delta m \frac{v^2}{R} \qquad \ldots(ii)$$

Length of the part ACB = $R\Delta\theta$. The mass per unit length of the ring is $\dfrac{m}{2\pi R}$

$\therefore$ mass of this portion ACB, $\Delta m = \dfrac{R\Delta\theta m}{2\pi R} = \dfrac{m\Delta\theta}{2\pi}$

Putting this value of Δm in (ii),

$$2T\sin\frac{\Delta\theta}{2} = \frac{m\Delta\theta}{2\pi}\frac{v^2}{R}$$

$$\therefore \qquad T = \frac{mv^2}{2\pi R}\left(\frac{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)}\right)$$

Since $\left(\frac{\frac{\Delta\theta}{2}}{\sin\left(\Delta\frac{\theta}{2}\right)}\right)$ is equal to 1,

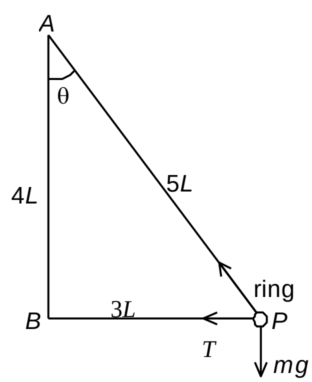
$$T = \frac{mv^2}{2\pi R}$$

Example 22.

A small smooth ring of mass m is threaded on a light inextensible string of length 8L which has its ends fixed at points in the same vertical line at a distance 4L apart. The ring describes horizontal circles at constant speed with both parts of the string taut and with the lower portion of the string horizontal. Find the speed of the ring and the tension in the string. The ring is then tied at the midpoint of the string and made to perform horizontal circles at constant speed of $3\sqrt{gL}$. Find the tension in each part of the string.

Solution:

When the string passes through the ring, the tension in the string is the same in both the parts. Also from geometry



BP = 3L and AP = 5L

$$T\cos\theta = \frac{4}{5}T = mg \qquad \ldots(i)$$

$$T + T\sin\theta = T\left(1+\frac{3}{5}\right) = \frac{8}{5}T$$

$$= \frac{mv^2}{BP} = \frac{mv^2}{3L} \qquad \ldots(ii)$$

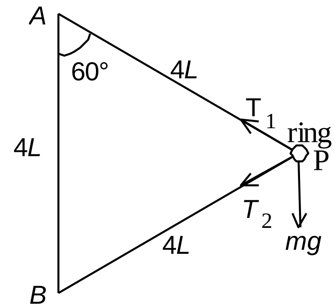
Dividing (ii) by (i),

$$\frac{v^2}{3Lg} = 2$$

$$v = \sqrt{6Lg}$$

From (i) $T = \frac{mg}{4/5} = \frac{5}{4}mg$

In the second case, ABP is an equilateral triangle.



$T_1 \cos 60^\circ = mg + T_2 \cos 60^\circ$

$$T_1 - T_2 = \frac{mg}{\cos 60^\circ} = 2mg \qquad \ldots(iii)$$

$$T_1 \sin 60^\circ + T_2 \sin 60^\circ = \frac{mv^2}{r} = \frac{9mgL}{4L \sin 60^\circ}$$

$$T_1 + T_2 = \frac{9mg}{4\sin^2 60^\circ} = 3\,mg \qquad \ldots(iv)$$

Solving equations (iii) and (iv),

$T_1 = \frac{5}{2}$ mg; $T_2 = \frac{1}{2}$ mg