1. WAVE MOTION

1.1 WAVES

A wave may be defined as a periodic disturbance travelling with a finite velocity through a medium. Such a wave-motion remains unchanged in type, as it progresses.

The important characteristics of a wave are:

(i) The particles of the medium traversed by a wave execute relatively small vibrations about their mean positions but the particles are not permanently displaced in the direction of propagation of the wave.

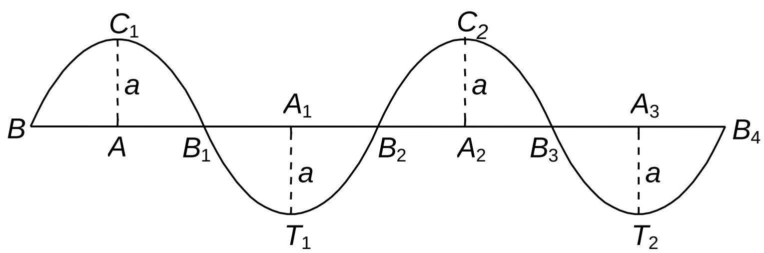
(ii) Each successive particle of the medium executes a motion quite similar to its predecessors along the line of travel of the wave but after some time.

(iii) During wave-motion only transfer of energy takes place but not a portion of the medium.

A vibration in its simplest form is called a simple harmonic motion and a particle executing such a motion may be considered as a source, which radiates waves in all directions. Usually the amplitude of vibration of the individual particles decreases as the distance of them from the source increases. But we will consider only the propagation of waves in which there is no change in amplitude as the wave progresses.

1.1 TRANSVERSE WAVES

A transverse wave is one in which particles of the medium vibrate in a direction at right angles to the line of propagation of the wave.



At any instant each particle of the medium is at its own displacement position in the process of executing a S.H.M. identical with others. The maximum displacement that every particle can have is called the amplitude of the wave.

The particles at A, A_1, A_2, ... are at their maximum displacement positions. The positions C_1, C_2, ... of the wave are called crests while the positions T_1, T_2, ... are called troughs.

Other particles of the medium are having their own individual displacements from the mean position which, sometime later would reach the maximum displacement position. Due to this it would appear as if crests and troughs travel forward in a wave-motion. As in S.H.M. the particle velocity is greatest where the displacement is least (mean position) and is least (zero) where the displacement is maximum. Successive pairs of particles such as B and B_2, A and A_2, B_1 and B_3 are in the same phase and distance between such successive pairs in the same phase is called the wavelength of the wave (written as λ).

1.2 LONGITUDINAL WAVES

If the individual particles of the medium vibrate in the direction of propagation of the wave itself the wave is called a **longitudinal wave**.

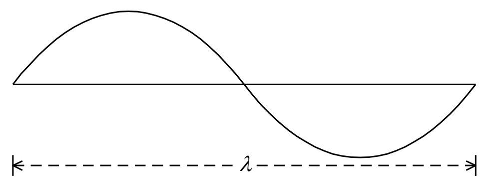
In the transverse wave-motion the displacement curve is a 'sine curve' of displacements with alternate crests and troughs. In the longitudinal wave the disturbance is passing along a succession of compressions and rarefactions. Compressions are points of maximum density and correspond to the crests of transverse wave while the rarefactions are points of minimum density corresponding to the troughs of a transverse wave. Yet the graphical representation of a longitudinal wave-motion will be quite identical with that of the transverse wave. As in transverse wave the distance between successive pairs of particles in the same phase is called wavelength (λ).

1.3 WAVE FREQUENCY

If the particles of the medium make n (also written as ν) vibrations per second, n is called the frequency of the wave. The time taken for one vibration is the wave period T and $T = \dfrac{1}{\nu}$ or $\nu = 1/T$; unit – hertz (Hz).

1.4 WAVELENGTH

We have already defined wavelength and a wavelength λ is the distance travelled by the wave in one period T; unit – metre.



1.5 WAVE VELOCITY

Wave velocity is the distance travelled by the wave in one second. Symbol v or c; unit – metre/second.

If the frequency of the wave is "n" hertz and wavelength is "λ" metres, then the wave velocity v is

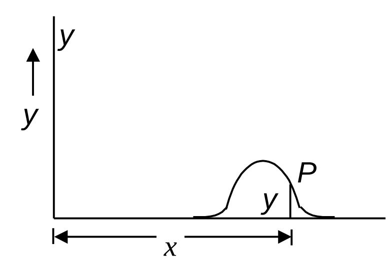
$$v = n\lambda \text{ m/s} \qquad \ldots \textbf{(1)}$$

Wave velocity = Frequency × Wavelength

2. GENERAL EQUATION OF PROGRESSIVE WAVES

Let us consider a series of particles forming a long string stretched along the x-axis as shown.

Suppose the particle at the origin O acts as the source of a disturbance, which travels as a transverse wave in the positive direction of the x-axis with a velocity v. As the wave proceeds, each successive particle is set in motion.



Let the time be measured from the instant when the particle at the origin O is passing through its equilibrium position. Then the displacement y of this particle at any instant t can be represented by

$$y = f(t)$$

where $f(t)$ is any function of time. The wave will reach a point P, distant x from O, in x/v second. Thus reach a point P, distant x from O, in x/v second. Thus the particle at P will start moving x/v second later than the particle at O. Therefore, the displacement y of the particle at P at any instant t will be the same as the displacement at the origin x/v second earlier i.e., at $(t - x/v)$. Hence the displacement at P is given by

$$y = f(t - x/v)$$

Since v is a constant, we may write it as

$$y = f\ (vt - x) \qquad \ldots \text{(i)}$$

This is the equation of a wave of any shape travelling along the positive direction of x-axis with a constant velocity v. The function f determines the exact shape of the wave.

Similarly, the equation for a wave travelling along the negative direction of x-axis would be

$$y = f\,(vt + x) \qquad \ldots \text{(ii)}$$

Thus the functions $f\,(vt \pm x)$ represent travelling waves. We can also write these functions as $f\,(x \pm vt)$.

From equation (i) and (ii) the generation equation for the wave can be written as

$$y = f_1\,(vt - x) + f_2\,(vt + x) \qquad \ldots \text{(iii)}$$

On differentiating equation (iii) twice partially with respect to 't', we get

$$\frac{\delta^2 y}{\delta t^2} = v^2\ f_1''\,(vt - x) + v^2\ \ f_2''\,(vt + x) \qquad \ldots \text{(a)}$$

where f_1'' and f_2'' are the second differentials of f_1 & f_2 respectively.

Again differentiating equation (iii) twice partially with respect to x, we get

$$\frac{\delta^2 y}{\delta x^2} = f_1''\,(vt - x) + \ f_2''\ (vt + x) \qquad \ldots \text{(b)}$$

Comparing (a) & (b)

$$\frac{\delta^2 y}{\delta t^2} = v^2\,\frac{\delta^2 y}{\delta x^2} \qquad \ldots \textbf{(2)}$$

This is general differential equation for a one dimensional wave.

2.1 EQUATION OF A SINUSOIDAL PROGRESSIVE WAVE

Particle displacement (y)

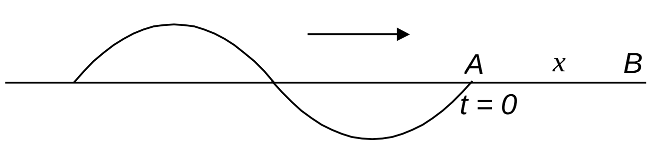
The displacement of a particle y from its mean position, taking $y = 0$ when $t = 0$ is given by the displacement-time equation for simple harmonic motion

$$y = a \sin \omega t \qquad \ldots \textbf{(3)}$$

where a is the amplitude of the oscillation and $\omega = 2\pi n$, where n is the wave frequency. Hence

$$y = a \sin 2\pi nt \qquad \ldots \textbf{(4)}$$

This equation is applicable to all individual particles affected by the wave. Suppose the wave is progressing forward with velocity v m/s. Suppose we start timing the operation when the wave reaches a certain particle A. Let us consider A as origin for the purpose of measuring distances. Then a particle B at a distance x from A will receive the wave $\frac{x}{v}$ s later than A did.



Hence its displacement will have to be considered according to

$$y = a \sin 2\pi n \left(t - \frac{x}{v}\right) \qquad \ldots (5)$$

Since $v = n\lambda$, $n = \frac{1}{T}$, the displacement equation can also take the form

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (6)$$

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \ldots (7)$$

Illustration 1.

The equation of a progressive wave is represented as

$y = 8 \sin 2\pi\,(4.7\,t - 0.01\,x)$

where x is in metres and t in seconds. Find (i) the frequency, (ii) the wavelength and (iii) the velocity of the wave.

Solution:

Comparing the given equation with the standard equation of progressive wave

$y = 8 \sin 2\pi\,(4.7\,\text{t} - 0.01\,\boldsymbol{x})$ – given equation

$y = A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$ – standard equation

(i) $\frac{t}{T} = 4.7\,t$

(or) $\frac{1}{T} = 4.7$ hertz

$\therefore$ frequency of the wave = **4.7 Hz**

(ii) $-\frac{x}{\lambda} = -\,0.01\,x$

(or) $\frac{1}{\lambda} = 0.01$

$\therefore$ wavelength $\lambda = \frac{1}{0.01}$ = **100 m**

(iii) Wave velocity = $n\lambda$ = (4.7 Hz) (100 m)

= **470 m/s**

Illustration 2.

A wave travelling in the positive x-direction has an amplitude 0.01 m, frequency 125 Hz and velocity of propagation 375 m/s. Find the displacement, velocity and acceleration of a particle in the medium situated 1.5 m from the origin at t = 5 sec.

Solution:

Let the wave be represented as

$$y = A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

Here amplitude $A = 0.01$ m

Velocity = 375 m/s

Frequency $\frac{1}{T} = n = 125$ Hz

$x = 1.5$ m

Wavelength $\lambda = \frac{v}{n} = \frac{375}{125} = 3$ m

$t = 5$ sec

Particle displacement is given by

$$y = 0.01 \sin 2\pi \left[125 \times 5 - \frac{1.5}{3}\right]$$

$$= 0.01 \sin 2\pi\,(625 - 0.5)$$

$$= 0.01 \sin 2\pi\,(624.5)$$

$$= 0.01 \sin 1249\,\pi$$

$$= 0.01 \sin \pi = 0$$

Particle velocity at that instant

$$\frac{dy}{dt} = A \cdot \frac{2\pi}{T} \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

Since $\sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$ is found to be zero

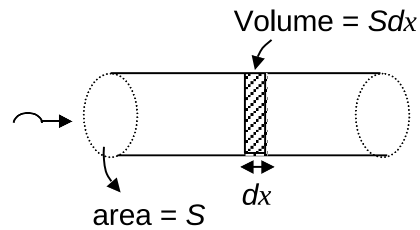
$$\cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) = 1$$

$\therefore$ particle velocity $v = 0.01 \times 2\pi \times 125 \times 1 =$ **2.5π m/s**

Particle acceleration

$$\frac{d^2y}{dt^2} = A \cdot \frac{4\pi^2}{T^2} \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) = \mathbf{0}$$

3. ENERGY OF A PLANE PROGRESSIVE WAVE



Consider a plane wave propagating with a velocity v in x-direction across an area S. An element of material medium (density = ρ kg/m^3) will have a mass $\rho(Sdx)$.

The displacement of a particle from its equilibrium position is given by the wave equation.

$$y = A \sin(\omega t - kx)$$

Total energy, $dE = \frac{1}{2} dm\, V_{max}^2$

$$dE = \frac{1}{2}(\rho S dx)(A\omega)^2$$

$$= \rho S dx\,(2\pi^2 f^2 A^2)$$

$\Rightarrow$ energy density $= \frac{dE}{(Sdx)} = 2\pi^2 f^2 A^2 \rho$ (J/m^3)

energy per unit length $= \rho S\,(2\pi^2 f^2 A^2)$ … (**8**)

$\therefore$ power transmitted $= 2\pi^2 f^2 A^2 \rho S v$ (J/s) … (**9**)

3.1 INTENSITY OF THE WAVE (I)

Intensity of the wave is defined as the power crossing per unit area

$\therefore$ $I = 2\pi^2 f^2 A^2 \rho v$ (Watt/m^2) … (**10**)

for wave propagation through a taut string,

$\rho S = \mu$, the linear density in kg/m

$\therefore$ Energy per unit length $= 2\pi^2 f^2 A^2 \mu$ … (**11**)

Illustration 3.

The equation of a progressive wave is given as $y = 0.05 \sin 2\pi \left(\frac{64t - x}{3.2}\right)$ where the amplitude and wavelength are in metres. (i) Calculate the phase velocity of the wave, (ii) also calculate the phase difference between two points at a distance 0.32 m apart, along the line of propagation, (iii) if the wave propagates through air (density = 1.3 kg/m^3) find the intensity of wave.

Solution:

(i) The progressive wave is represented by

$$y = 0.05 \sin \frac{2\pi}{3.2}(64t - x)$$

Comparing this with the standard equation

$$y = A \sin \frac{2\pi}{\lambda}(vt - x)$$

the phase velocity (wave velocity) = **64 m/s**; wavelength (λ) = **3.2 m**

(ii) The phase difference of the particles separated by a distance of λ is equal to 2π.

$\therefore$ phase difference of particles separated by a distance 0.32 m = $\frac{2\pi}{3.2} \times 0.32$

$= \frac{\pi}{5}$ **radians**.

(iii) The intensity of the sound wave is given by

$$I = \frac{1}{2}\rho v \omega^2 A^2 \;=\; \frac{1}{2}\left(1.3\frac{\text{kg}}{\text{m}^3}\right)\left(64\frac{m}{s}\right)\left(4\pi^2 n^2\right)(0.05m)^2$$

Here n is the frequency of the wave and is equal to $\frac{64}{3.2}$ = 20 Hz

$\therefore I = \frac{1}{2} \times 1.3 \times 64 \times 4\pi^2 \times 400 \times 0.0025$ W/m^2

= **1642 W/m^2**

4. WAVE SPEED

The speed of any mechanical wave, transverse or longitudinal, depends on both an inertial property of the medium and an elastic property of the medium.

4.1 TRANSVERSE WAVE IN A STRETCHED STRING

Consider a transverse pulse produced in a taut string of linear mass density μ. Consider a small segment of the pulse, of length Δl, forming an arc of a circle of radius R. A force equal in magnitude to the tension T pulls tangentially on this segment at each end.

Let us set an observer at the centre of the pulse, which moves along with the pulse towards right. For the observer any small length dl of the string as shown will appear to move backward with a velocity v.

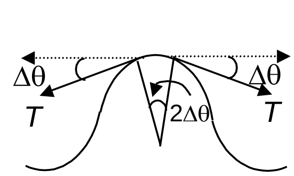
Now the small mass of the string is in a circular path of radius R moving with speed v. Therefore the required centripetal force is provided by the only force acting (neglecting gravity) is the component of tension along the radius.

The net restoring force on the element is

$$F = 2T \sin(\Delta\theta) \approx 2T(\Delta\theta) = \frac{T\Delta l}{R}$$

The mass of the segment is $\Delta m = \mu \Delta l$

The acceleration of this element towards the centre of the circle is $a = \frac{v^2}{R}$,



where v is the velocity of the pulse.

Using second law of motion

$$\frac{T\Delta l}{R} = (\mu \Delta l)\,\frac{v^2}{R}$$

or, $$v = \sqrt{\frac{T}{\mu}} \qquad \ldots \textbf{(12)}$$

5. SUPERPOSITION OF WAVES

Two or more waves can traverse the same space independent of one another. The displacement of any particle in the medium at any given time is simply the sum of displacements that the individual waves would give it. This process of the vector addition of the displacement of the particle is called superposition.

5.1 INTERFERENCE

When two waves of the same frequency, superimpose each other, there occurs redistribution of energy in the medium, which causes either a minimum intensity or maximum intensity. This phenomenon is called interference of waves.

Let the two of waves be

$$y_1 = A_1 \sin(\omega t - kx)\ ;\ y_2 = A_2 \sin(\omega t - kx + \delta)$$

According to the principle of superposition, $y = y_1 + y_2$

$$= A_1 \sin(\omega t - kx) + A_2 \sin(\omega t - kx + \delta)$$

$$= A_1 \sin(\omega t - kx)\,(A_1 + A_2 \cos\delta) + A_2 \sin\delta \cos(\omega t - kx)$$

$$y = A \sin(\omega t - kx + \phi) \qquad \ldots \textbf{(13)}$$

where, $A_1 + A_2 \cos\delta = A\cos\phi$

$$A_2 \sin\delta = A \sin\phi$$

and $$A^2 = (A_1 + A_2 \cos\delta)^2 + (A_2 \sin\delta)^2$$

$\therefore$ $$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos\delta \qquad \ldots \textbf{(14)}$$

If I_1 & I_2 are intensities of the interfering waves and δ is the phase difference, then the resultant intensity is given by,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\cos\delta \qquad \ldots \textbf{(15)}$$

Now $$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2 \text{ for } \delta = 2n\pi$$

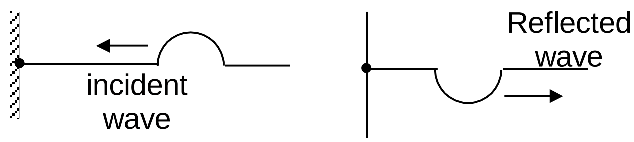
$$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2 \text{ for } \delta = (2n+1)\pi$$

5.2 REFLECTION AND TRANSMISSION OF WAVES

The nature of the reflected & transmitted wave depends on the nature of end point. There are three possibilities.

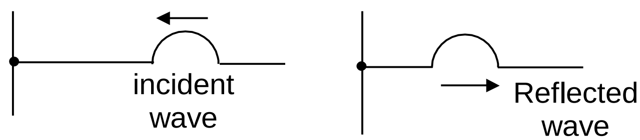
End point is fixed

Waves on reflection from a fixed end undergoes a phase change of 180°.



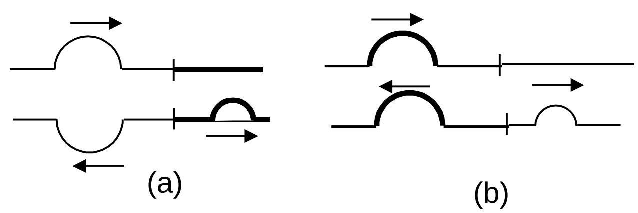
End point is free

There is no phase change in waves on reflection.



End point is neither completely fixed nor completely free

For example, consider a light string attached to a heavier string as shown in figure. If a wave pulse is produced on the light string moving towards the junction, a part of the wave is reflected and a part is transmitted on the heavier string. The reflected wave is inverted with respect to the original one (figure (a).



On the other hand, if the wave is produced on the heavier string, which moves towards the junction, a part will be reflected and a part transmitted, no inversion of wave shape will take place (as shown in figure (b).

So the rule is: if a wave enters a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.

5.3 STANDING WAVES

A standing wave is formed when two identical waves travelling in the opposite directions along the same line interfere. Consider two waves of the same frequency, speed and amplitude, which are travelling in opposite directions along a string. Two such waves may be represented by the equations.

$$y_1 = a\sin(\omega t - kx)$$

$$y_2 = a \sin(kx + \omega t)$$

Hence the resultant may be written as

$$y = y_1 + y_2 = a \sin(\omega t - kx) + \text{as in}\,(\omega t + kx)$$

$$y = 2a \sin kx \cos\omega t \qquad \ldots\ (\mathbf{16})$$

This is the equation of a standing wave.

Note:

(i) From the above equation it is seen that a particle at any particular point 'x' executes simple harmonic motion and all the particles vibrate with the same frequency.

(ii) The amplitude is not the same for different particle but varies with the location 'x' of the particle.

(iii) The points having maximum amplitudes are those for which $2a \sin kx$, has a maximum value of $2a$, these are at the positions.

$$Kx = \pi/2\ ,\ 3\ \pi/2\ ,\ 5\ \pi/2,\ \ldots$$

or $x = \lambda/4, 3\lambda/4, 5\lambda/4 \ldots.$

i.e., $x = (n + 1/2)\ \lambda/2 = (2n + 1)\ \lambda/2$; where $n = 0, 1, 2 \ldots$

There points are called antinodes.

(iv) The amplitude has minimum value of zero at positions where

$$kx = \pi, 2\pi, 3\pi, \ldots.$$

or $x = \dfrac{\lambda}{2}, \lambda, \dfrac{3\lambda}{2}, 2\lambda \ldots.$ i.e., $x = \dfrac{n\lambda}{2}$

These points are called nodes.

(v) It is clear that the separation between consecutive nodes or consecutive antinodes is $\lambda/2$.

(vi) As the particles at the nodes do not move at all, energy cannot be transmitted across them.

Illustration 4.

A progressive and a stationary wave have same frequency 300 Hz and the same wave velocity 360 m/s. Calculate

(i) the phase difference between two points on the progressive wave which are 0.4 m apart,

(ii) the equation of motion of progressive wave if its amplitude is 0.02 m,

(iii) the equation of motion of the stationary wave if its amplitude is 0.01 m and

(iv) the distance between consecutive nodes in the stationary wave.

Solution:

Wave velocity $v = 360$ m/s

Frequency $n = 300$ Hz

$$\therefore \text{ wavelength } \lambda = \frac{v}{n} = \frac{360}{300} = 1.2 \text{ m}$$

(i) The phase difference between two points at a distance one wavelength apart is 2π.

Phase difference between points 0.4 m apart is given by

$$\frac{2\pi}{\lambda} \times 0.4 = \frac{2\pi}{1.2} \times 0.4 = \boldsymbol{\frac{2\pi}{3}} \textbf{ radians.}$$

(ii) The equation of motion of a progressive wave is

$$y = A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

In the case given

$$y = \mathbf{0.02 \sin 2\pi} \left(\boldsymbol{300t - \frac{x}{1.2}}\right)$$

(iii) The equation of the stationary wave is

$$y = 2A\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi t}{T}$$

Here $2A = 2 \times 0.01 = 0.02$

$$\lambda = 1.2 \text{ m}$$

$$\frac{1}{T} = 300 \text{ Hz}$$

$$\therefore y = \mathbf{0.02 \cos \frac{2\pi x}{1.2} \sin 600\, \pi t}$$

(iv) The distance between the two consecutive nodes in the stationary wave is given by

$$\frac{\lambda}{2} = \frac{1.2}{2} = \mathbf{0.6 \text{ m}}$$

5.4 DIFFERENCES BETWEEN A TRAVELLING WAVE AND A STANDING WAVE

(1) In a travelling wave, the disturbance produced in a region propagates with a definite velocity but in a standing wave, it is confined to the region where it is produced.

(2) In a travelling wave, the motion of all the particles is similar in nature. In a standing wave, different particles move with different amplitudes.

(3) In a standing wave, the particles at nodes always remain in rest. In travelling waves, there is no particle, which always remains in rest.

(4) In a standing wave, all the particles cross their mean positions together. In a travelling wave, there is no instant when all the particles are at the mean positions together.

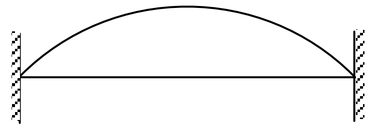
(5) In a standing wave, all the particles between two successive nodes reach their extreme positions together, thus moving in phase. In a travelling wave, the phases of nearby particles are always different.

(6) In a travelling wave, energy is transmitted from one region of space to other but in a standing wave, the energy of one region is always confined in that region.

5.5 STATIONARY WAVES IN STRINGS

A string of length L is stretched between two points. When the string is set into vibrations, a transverse progressive wave begins to travel along the string. It is reflected at the other fixed end. The incident and the reflected waves interfere to produce a stationary transverse wave in which the ends are always nodes.

Fundamental Mode



(a) In the simplest form, the string vibrates in one loop in which the ends are the nodes and the centre is the antinode. This mode of vibration is known as the fundamental mode and frequency of vibration is known as the fundamental frequency or first harmonic.

Since the distance between consecutive nodes is

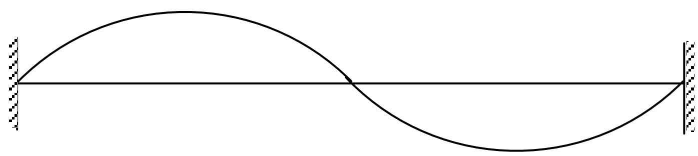
$$L = \frac{\lambda_1}{2} \quad \therefore \lambda_1 = 2L$$

If f_1 is the fundamental frequency of vibration, then the velocity of transverse waves is given as,

$$v = \lambda_1 f_1$$

or $$f_1 = \frac{v}{2L} \qquad \ldots \text{(i)}$$

First Overtone



(b) The same string under the same conditions may also vibrate in two loops, such that the centre is also the node

$$\therefore \quad L = 2\lambda_2/2 \quad \therefore \lambda_2 = L$$

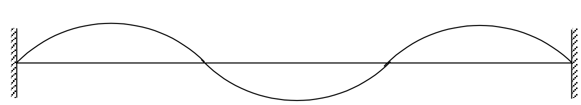
If f_2 is frequency of vibrations

$$\therefore \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{L}$$

$$\therefore \quad f_2 = \frac{v}{L} \qquad \ldots \text{(ii)}$$

The frequency f_2 is known as second harmonic or first overtone.

Second Overtone



(c) The same string under the same conditions may also vibrate in three segments.

$$\therefore \quad L = \frac{3\lambda_3}{2}$$

$$\therefore \quad \lambda_3 = \frac{2}{3}L$$

If f_3 is the frequency in this mode of vibration, then,

$$f_3 = \frac{3v}{2L} \qquad \ldots \text{(iii)}$$

The frequency f_3 is known as third harmonic or second overtone.

Thus a stretched string vibrates with frequencies, which are integral multiples of the fundamental frequencies. These frequencies are known as harmonics.

The velocity of transverse wave in stretched string is given as $v = \sqrt{\frac{T}{\mu}}$. Where T = tension in the string.

μ = linear density or mass per unit length of string. If the string fixed at two ends, vibrates in its fundamental mode, then

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad \dots (\mathbf{17})$$

5.6 FREQUENCY OF THE STRETCHED STRING

In general, if the string vibrates in p loops the frequency of the string under that mode is given by

$$f = \frac{p}{2L}\sqrt{\frac{T}{\mu}} \qquad \dots (\mathbf{18})$$

Based on this relation three laws of transverse vibrations of stretched strings arise. They are law of length, law of tension and law of mass.

5.7 LAW OF LENGTH

The fundamental frequency n is inversely proportional to the length L of the stretched string.

$$f \propto \frac{1}{L} \qquad \dots (\mathbf{19})$$

(or) $nL = a$ constant (T being constant)

5.8 LAW OF TENSION

The fundamental frequency is directly proportional to the square root of the tension in the string.

$$f \propto \sqrt{T}$$

(or) $\frac{f}{\sqrt{T}} = a$ constant (L, m being constants) … (**20**)

5.9 LAW OF MASS

The fundamental frequency is inversely proportional to the square root of mass per unit length of the given string when L and T are kept constants.

$$f \propto \frac{1}{\sqrt{\mu}}$$

(or) $f\sqrt{\mu} = a$ constant (L, T being constants) … (**21**)

5.10 FREQUENCY AND DENSITY OF WIRE

In the formula $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$ for the fundamental frequency of a stretched string, if the tension T is due to a weight of M kg suspended from the free end of a wire as in a sonometer then $T = Mg$ newtons.

If the density of the material of the wire is D kg/m^3, and the radius of cross-section is r metre then the mass/unit length of the wire

$$m = \text{volume} \times \text{density}$$

$$= (\pi r^2 \times 1)(D)\text{ kg}$$

The fundamental frequency equation now becomes

$$f = \frac{1}{2L}\sqrt{\frac{Mg}{\pi r^2 D}} \qquad \ldots \mathbf{(22)}$$

Note that in this relation

M is in kg; r is in m; g is in m/s^2; D is in kg/m^3 L is in m

Illustration 5.

A wire of length 1.5 m under tension emits a fundamental note of frequency 120 Hz.

(a) What would be its fundamental frequency if the length is increased by half under the same tension?

(b) By how much the length should be shortened so that it can increase its frequency three fold?

Solution:

The variations in length take place under constant tension.

(a) Applying the relation for the stretched string

$n_1L_1 = n_2L_2$ (T being constant)

$120 \times L = n_2 \times \frac{3L}{2}$ (or) $n_1 = \frac{2 \times 120}{3} = \mathbf{80\ Hz}$

(b) Again applying the relation

$n_1L_1 = n_2L_2$

$n_1 \times L_1 = (3n_1)L_2$

$L_2 = \frac{L_1}{3} = \frac{1.5}{3} = 0.5$ m

$\therefore$ the length of the wire should be shortened by (1.5 – 0.5) m = **1 m**

Illustration 6.

Two strings A and B are made of steel and are kept under the same tension. If A has a radius twice that of B what should be the ratio of their lengths for them to have the same fundamental frequency? What should be the ratio of their lengths if the first overtone of the former should equal the third harmonic of the latter?

Solution:

	String A	String B	
Tension	T	T	(same tension)
Linear density	$\pi r^2 D$	$\pi\left(\frac{r}{2}\right)^2 D$	D = density of steel
Frequency	n	n	(same fundamental)
Lengths	L_1	L_2	

For the string A,

$$n = \frac{1}{2L_1}\sqrt{\frac{T}{\pi r^2 D}}$$

For the string B, $n = \dfrac{1}{2L_2}\sqrt{\dfrac{T}{\pi\left(\frac{r}{2}\right)^2 D}} = \dfrac{1}{2L_2}\sqrt{\dfrac{4T}{\pi r^2 D}}$

Since both have the same fundamental frequency

$$\frac{1}{2L_1}\sqrt{\frac{T}{\pi r^2 D}} = \frac{1}{2L_2}\sqrt{\frac{4T}{\pi r^2 D}}$$

$$\frac{L_1}{L_2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

If n be the fundamental frequency of string A with length L_1

$$n = \frac{1}{2L_1}\sqrt{\frac{T}{\pi r^2 D}}$$

The first overtone of A = second harmonic = $2n$

$$\therefore \qquad 2n = \frac{1}{L_1}\sqrt{\frac{T}{\pi r^2 D}}$$

Let n' be the fundamental frequency of the string B with vibrating length L_2

$$n' = \frac{1}{2L_2}\sqrt{\frac{4T}{\pi r^2 D}}$$

The third harmonic of this is 3n′

$$3n' = \frac{3}{2L_2}\sqrt{\frac{4T}{\pi r^2 D}}$$

Since $2n = 3n'$

$$\frac{1}{L_1}\sqrt{\frac{T}{\pi r^2 D}} = \frac{3}{2L_2}\sqrt{\frac{4T}{\pi r^2 D}} = \frac{3}{L_2}\sqrt{\frac{T}{\pi r^2 D}}$$

$$\frac{L_1}{L_2} = \frac{\mathbf{1}}{\mathbf{3}}$$

Illustration 7.

A thin steel wire has been stretched so that its length increases by 1%. Calculate the frequency of one metre length of the wire.

Young's modulus for steel = 20×10^{10} N/m^2 and density of steel = 7800 kg/m^3.

Solution:

Let the original length of the wire be L and ΔL be the increase in length under the stretching force T. T is the tension in the wire. Let A be the area of cross-section of the wire. Then

$$T = \frac{YA\Delta L}{L} = 20 \times 10^{10}\ \text{A}\left(\frac{\Delta L}{L}\right) = 20 \times 10^{10} \times \frac{1}{100} \times \text{A} = 2 \times 10^{9} A \text{ Newton}$$

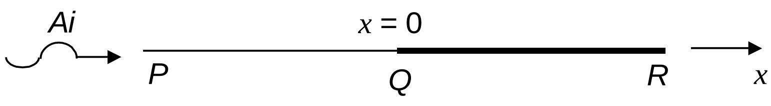
The frequency of vibration of 1 m length of the wire is given by

$$n = \frac{1}{2L}\sqrt{\frac{T}{m}} = \frac{1}{2(1)}\sqrt{\frac{2\times10^9 \times A}{(\pi r^2)D}} = \frac{1}{2}\sqrt{\frac{2\times10^9 \times A}{A\cdot D}} = \frac{1}{2}\sqrt{\frac{2\times10^9}{7800}} = \mathbf{253.2\ Hz}.$$

5.11 AMPLITUDE OF REFLECTED & TRANSMITTED WAVES

A long wire PQR is made by joining two wires PQ and QR of equal radii. PQ has a length l_1 and mass m_1, QR has length l_2 and mass m_2. The wire PQR is under a tension T. A sinusoidal wave pulse of amplitude A_i and is sent along the wire PQ from the end P. No power is dissipated during the propagation of the wave pulse.

Let us take the direction along wire as x-axis.



Let us take the direction along wire as x-axis

velocity of wave in wire $PQ = \sqrt{\frac{T}{m_1/l_1}} = \sqrt{\frac{T\,l_1}{m_1}} = v_1$ and velocity of wave in wire $PQ = \sqrt{\frac{Tl_2}{m_2}} = v_2$

Then for incident wave we can write

$$y_i = A_i \sin\omega\,[\,t - x/v_1] \qquad \text{… (i)}$$

In reflected and transmitted wave, 'ω' will not change, therefore we can write

$$y_r = A_r \sin \omega [\, t + x/v_1] \qquad \ldots \text{(ii)}$$

$$y_t = A_t \sin \omega [\, t - x/v_2] \qquad \ldots \text{(iii)}$$

Now as wave is continuous, so at the boundary ($x = 0$)

Continuity of displacement requires

$$y_i + y_r = y_t \;\text{ for } x = 0$$

Substituting from (i), (ii) and (iii) in the above,

$$A_i + A_r = A_t \qquad \ldots \text{(iv)}$$

Also, at the boundary, slope of wave will be continuous, i.e., $\frac{\delta y_i}{\delta x} + \frac{\delta y_r}{\delta x} = \frac{\delta y_t}{\delta x}$ for $x = 0$

Which gives, $A_i - A_r = \left(\frac{v_1}{v_2}\right) A_t$ ... (v)

Solving (iv) and (v) for A_r and A_t we get the required equations, i.e.,

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i \qquad \ldots \textbf{(23)}$$

and

$$A_t = \frac{2 v_2}{v_2 + v_1} A_i \qquad \ldots \textbf{(24)}$$

Illustration 8.

The displacement of medium in a sound wave is given by the equation $y_1 = A \cos (ax + bt)$ where A, a and b are positive constants. The wave is reflected by an obstacle situated at $x = 0$. The intensity of reflected wave is 0.64 times that of incident wave.

(a) Calculate the wavelength and frequency of incident wave.

(b) Write the equation of reflected wave.

(c) Express the resultant wave as a superposition of a standing wave and a travelling wave.

Solution:

The equation of the incident wave is

$$y_1 = A \cos (ax + bt) \qquad \ldots \text{(i)}$$

where A, a and b are positive constants. The incident wave is travelling along the negative x-direction as shown by solid line in the figure. The wave is reflected by an obstacle situated at $x = 0$.

Intensity of incident wave = A^2

Intensity of reflected wave = $0.64\, A^2$

$\therefore$ Amplitude of reflected wave = $0.8\, A$

The reflected wave is shown by dotted line and is travelling along the positive x-direction.

(a) The general equation of the wave is

$$y = A\cos(kx + \omega t)$$

Comparing this with equation (i), we get

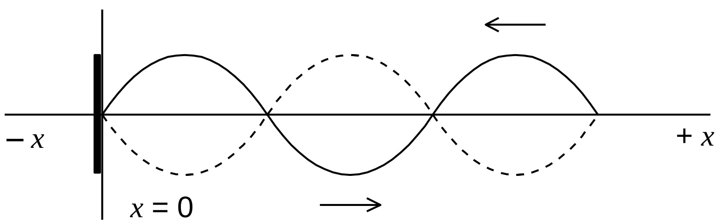
$$k = \frac{2\pi}{\lambda} = a \text{ or } \lambda = \frac{2\pi}{a}$$

$$\omega = 2\pi n = b \text{ or } n = \frac{b}{2\pi}$$

(b) The equation of the reflected wave which is travelling along positive x-direction will be represented by equation

$$\mathbf{y_2 = -0.8\,A\cos(ax - bt)}.$$

(c) The resultant wave is



$$y = A\cos(ax + bt) - 0.8\,A\cos(ax - bt)$$

$$= 0.8\,A\,[\cos(ax + bt) - \cos(ax - bt)] + 0.2A\cos(ax + bt) = y_s + y_t$$

The equation of stationary wave is

$$y_s = 0.8\,A\,[\cos(ax + bt) - \cos(ax - bt)] = \mathbf{-1.6\,A\sin ax\sin bt.}$$

and that of travelling wave is $y_t = \mathbf{0.2\,A\cos(ax + bt)}$.

6. SOUND

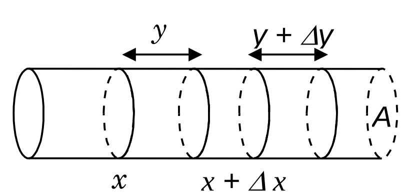
Sound is produced in a material medium by a vibrating source. Sound waves constitute alternate compression and rarefaction pulses travelling in the medium. The compression travels in the medium at a speed, which depends on the elastic and inertia properties of the medium.

The description in terms of pressure wave is more appropriate than the description in terms of the displacement wave as far as sound properties are concerned.

6.1 SOUND AS PRESSURE WAVE

A longitudinal wave in a fluid is described either in terms of the longitudinal displacements suffered by the particles of the medium or in terms of the excess pressure generated due to the compression or rarefaction.

Consider a wave going in the *x*-direction in a fluid. Suppose that at a time *t*, the particle at the undisturbed position *x* suffers a displacement *y* in the *x*-direction.



$$y = y_o \sin\omega\left(t - \frac{x}{v}\right) \quad \ldots \text{(i)}$$

A is cross-sectional area.

Increase in volume of this element at time t is

$$\Delta V = A\, dy$$

$$= A y_o \left(\frac{-\omega}{v}\right) \cos\omega\left(t - \frac{x}{v}\right) \Delta x$$

where Δy has been obtained by differentiating equation (i) with respect to *t*.

$\Rightarrow$ Volume strain is $\dfrac{\Delta V}{V} = -\dfrac{A y_o \omega \cos\omega\left(t - \frac{x}{v}\right)\Delta x}{v A \Delta x} = -\dfrac{y_o \omega}{v}\cos\omega\left(t - \dfrac{x}{v}\right) = \dfrac{\delta y}{\delta x}$

$\therefore$ volume strain $= \dfrac{\delta y}{\delta x}$ … (**25**)

The corresponding stress i.e., the excess pressure developed in the element at *x*, at time t is

$p = B\left(\dfrac{-\Delta V}{V}\right)$ where *B* is the bulk modulus of the material.

$$\Rightarrow \quad p = \frac{B y_o \omega}{v} \cos\omega\left(t - \frac{x}{v}\right) \quad \ldots \text{(ii)}$$

$$\therefore \quad P = -\frac{B\delta y}{\delta x} \quad \ldots \textbf{(26)}$$

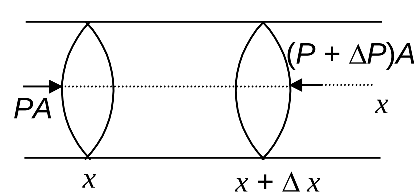
Comparing equations (1) and (2), the relation between the pressure amplitude p_0 and the displacement amplitude s_0 is

$p_o = \dfrac{B\omega}{v} y_o = B k y_o \Rightarrow y_o = \dfrac{p_o \lambda}{2\pi B}$ where *k* is a wave number. … (**27**)

As observed from equations (i) and (ii), pressure wave is 'cos θ' type, if displacement is described as 'sin θ' type.

Thus, the pressure-maxima occur where the displacement is zero and displacement-maxima occur where the pressure is at its normal level.

6.2 SPEED OF SOUND WAVE IN A MATERIAL MEDIUM



The resultant force on element of fluid (which is contained between the positions *x* and *x* + Δ *x*) is

$$\Delta F = Ap - A(p + \Delta p) = -A\Delta p$$

Acceleration, $a = \frac{\Delta F}{\rho A \Delta x}$ (using Newton's law)

Also, $a = \frac{\partial^2 y}{\partial t^2}$

$$\therefore \quad \frac{B y_o \omega^2}{\rho v^2} = \omega^2 y_o$$

$$\Rightarrow \quad v = \sqrt{\frac{B}{\rho}} \qquad \ldots \textbf{(28)}$$

6.3 POWER (W) TRANSMITTED BY WAVE:

W = $pA\frac{\partial y}{\partial t}$ (Since power = $\frac{FS}{t}$ and pressure = $\frac{F}{A}$) = $\frac{A\omega^2 y_o^2 B}{v}\cos^2\omega\left(t - \frac{x}{v}\right)$

The average of $\cos^2\theta$ over a complete cycle or over a long time is $\frac{1}{2}$.

Intensity I is the average power transmitted across unit cross-sectional area.

$$I = \frac{1}{2}\frac{\omega^2 y_o^2 B}{v}$$

$$\Rightarrow \quad I = \frac{p_o^2 v}{2B} \qquad \ldots \textbf{(29)}$$

(Since $\omega = 2\pi\nu$ and $p_0 = \frac{B\omega s_o}{v}$)

$$\Rightarrow \quad I \propto p_o^2$$

This is similar or analogous to

$$I \propto y_o^2$$

In other words, intensity is proportional to the square of the amplitude (whether displacement or pressure).

Loudness is measured in terms of sound level (in decibels i.e., *dB*) denoted by β.

$\beta = 10\log_{10}\left(\frac{I}{I_o}\right)$ where I_0 is the reference intensity equal to 10^{-12} W/m^2. … (**30**)

6.4 AUDIBLE RANGE AND ULTRASONICS

The audible range of sound waves in air in terms of frequency *n* is "20 Hz to 20 kHz".

Since velocity of sound in air at STP (Standard Temperature and Pressure) is v = 332 m/s, the wavelengths (λ) for audibility lie in the range 1.66 cm to 16.6 cm (using $v = n\lambda$).

Longitudinal mechanical waves having frequency greater than 20 kHz are called ultrasonic waves. Ultrasonic sound waves also travel with same velocity 332 m/s at S.T.P.

Hence, the wavelength of ultrasonics $\lambda < 1.66$ cm. Though human ear cannot detect these ultrasonic waves, certain creatures such as mosquito and bat show response to these. These waves can be produced by the frequency of a quartz crystal under an alternating electric field (piezo-electric effect) or by the vibrations of a ferromagnetic rod under an alternating magnetic field (magnetostriction effect). For infrasonic waves $\lambda > 16.6$ m.

6.5 LONGITUDINAL VIBRATIONS IN SOLIDS AND FLUIDS

The velocity of longitudinal waves in a solid or fluid is given by

$$v = \sqrt{\frac{E}{\rho}} \qquad \ldots (\mathbf{31})$$

where E is the elasticity of the medium.

Solids: In the case of solids the stress is tensile or unidirectional. The strain is linear and the modulus of elasticity involved is Young's modulus.

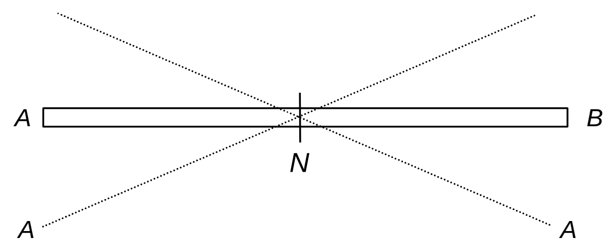
Hence $E = Y$ (Young's modulus)

$$\therefore \qquad v = \sqrt{\frac{Y}{\rho}} \qquad \ldots (\mathbf{32})$$

The velocity of longitudinal wave in a solid is given by

$v = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{Y}{\rho}}$ where Y is Young's modulus and ρ is its density.

It is possible to establish a stationary wave in a rod like AB, which is clamped in the middle and striked at one of its ends. Stationary waves are set up with the clamped point remaining as a node and the free ends as antinodes. If L is the length of the rod, λ the wavelength of the wave then we have



$$L = \frac{\lambda}{2}$$

The fundamental frequency, $n = \frac{v}{\lambda} = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{Y}{\rho}}$

Gases: In the case of gases the elasticity is bulk modulus as it brings about volume strain. Newton first assumed that the changes in gases are isothermal and the isothermal bulk modulus was taken by him as p. But later Laplace modified the formula and got the adiabatic bulk modulus which is γp, where γ is the ratio of specific heats of the gas. The velocity of sound in a gas is therefore

$$c = \sqrt{\frac{\gamma p}{\rho}}. \qquad \ldots (\mathbf{33})$$

6.6 VELOCITY OF SOUND IN AN IDEAL GASE

The motion of sound wave in air is adiabatic. In the case of an ideal gas, the relation between pressure P and volume V during an adiabatic process is given by

$$PV^{\gamma} = \text{constant}$$

Where γ is the ratio of the heat capacity at constant pressure to that at constant volume.

After differentiating, we get

$$V^{\gamma}\frac{dP}{dV}+\gamma PV^{\gamma-1}=0$$

Since $$B = -\frac{vdP}{dV} = \gamma P$$

(Laplace correction is constant to newtons' formula $v = \sqrt{\frac{P}{\rho}}$) using the gas equation $\frac{P}{\rho} = \frac{RT}{M}$ where M is the molar mass

Thus $v = \sqrt{\frac{\gamma RT}{M}}$ (T = temperature is Kelvin) … (**34**)

6.7 EFFECT OF TEMPERATURE, PRESSURE AND HUMIDITY ON VELOCITY OF SOUND IN GASES

Effect of temperature: If the specific volume of gas is v. The velocity of sound

$$= \sqrt{\frac{\gamma P}{\rho}}=\sqrt{\gamma Pv}\,.$$

If c_1 and c_2 be the velocities of sound in a gas at temperatures $t_1^{\circ}C$ and $t_2^{\circ}C$ and P_1 and P_2 the respective pressures and V_1 and V_2 the specific volumes at these temperatures.

$$c_1 = \sqrt{\gamma P_1 V_1}$$

$$c_2 = \sqrt{\gamma P_2 V_2}$$

$$\frac{c_1}{c_2} = \sqrt{\frac{P_1V_1}{P_2V_2}} = \sqrt{\frac{RT_1}{RT_2}}$$ where T and T_2 are the absolute temperatures.

$$\therefore \quad \frac{c_1}{c_2}=\sqrt{\frac{T_1}{T_2}}=\sqrt{\frac{273+t_1}{273+t_2}} \qquad \ldots (\mathbf{35})$$

If c and c_0 are the velocities at t°C and 0°C, then

$$\frac{c}{c_0}=\sqrt{\frac{273+t}{273}}$$

$$c=c_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}=c_0\left(1+\frac{t}{546}\right)$$

Calculations would show that the velocity of sound in air increases approximately at the rate of 0.6 metre/sec per degree rise of temperature.

Effect of pressure: At constant temperature the ratio $\frac{P}{\rho}$ is constant (Boyle's law). Hence the velocity of sound in gases at constant temperature is independent of pressure.

Effect of humidity: The presence of moisture in air decreases the density of air. Hence if the humidity is high the density decreases and the velocity of sound increases. Thus the velocity of sound in moist air is greater than the velocity of sound in dry air.

6.8 WAVE VELOCITY AND MOLECULAR VELOCITY IN A GAS

The velocity of propagation of sound waves through a gas is

$$v=\sqrt{\frac{\gamma P}{D}}$$

According to this relation the velocity of sound will depend on the pressure of the gas. Also according to the kinetic theory of gases the pressure of gas is

$$P=\frac{1}{3}Dc^2$$

where D is the density of gas and c is the root mean square velocity of the molecules at a particular temperature.

$$\frac{P}{D}=\frac{c^2}{3}$$

Substituting this value of $\frac{P}{D}$ in the above equation for velocity of sound in a gas

$$v=\sqrt{\frac{\gamma c^2}{3}}=c\sqrt{\frac{\gamma}{3}}$$

$$c=v\sqrt{\frac{3}{\gamma}} \qquad \ldots \textbf{(36)}$$

From the above equation if velocity of sound and γ of gas are known the r.m.s velocity can be calculated.

Illustration 9.

(a) What is the ratio of velocities of sound in hydrogen and oxygen at the same temperature?

(b) Find the temperature at which the speed of sound in oxygen is the same as that in hydrogen at 27°C.

Solution:

(a) The temperature of the two gases H_2 and O_2 are given to be the same. Now

$$v=\sqrt{\frac{\gamma RT}{M}}$$

$v_0 = \sqrt{\frac{\gamma RT}{M_O}}$ and $v_H = \sqrt{\frac{\gamma RT}{M_H}}$

γ is same for both the gases.

$\therefore \quad \frac{v_H}{v_O} = \sqrt{\frac{M_O}{M_H}} = \sqrt{\frac{32}{2}} = \sqrt{16} = 4 \qquad \therefore \quad \mathbf{v_H : v_0 = 4 : 1}$

(b) Velocity of sound in hydrogen at 27°C is given by

$$v_H = \sqrt{\frac{\gamma.R.300}{M_H}}$$

The velocity of sound in oxygen at a temperature t°C is

$$V_0 = \sqrt{\frac{\gamma.R(273+t)}{M_O}}$$

If $\quad v_H = v_0$,

$$\sqrt{\frac{\gamma R(300)}{M_H}} = \sqrt{\frac{\gamma R(273+t)}{M_O}} \quad \text{or,} \quad \frac{300}{M_H} = \frac{273+t}{M_O}$$

or $\quad 273 + t = \frac{300}{2} \times 32 = 16 \times 300$

$t = 4800 - 273 = \mathbf{4527°C}$

Illustration 10.

A gas is a mixture of two parts by volume of hydrogen and one part by volume of nitrogen. If the velocity of sound in hydrogen at 0°C is 1300 m/s find the velocity of sound in the gaseous mixture at 27°C.

Solution:

Hydrogen and nitrogen are both diatomic and hence their ratio of specific heats γ is the same. Hence we can take the γ of the mixture also to be that of diatomic gas.

Velocity of sound in hydrogen gas at 0°C

$$v_H = \sqrt{\frac{\gamma P}{D_H}} \qquad D_H = \text{Density of hydrogen.}$$

Since molecular weights of hydrogen and nitrogen are 2 and 28 respectively, nitrogen is 14 times heavier than hydrogen at the same pressure and temperature.

Considering the gaseous mixture, the density of mixture

$$D_m = \frac{2VD_H + V(14D_H)}{3V} \qquad D_{nitrogen} = 14D_H$$

$$D_m = \frac{16D_H}{3}$$

Velocity of sound in the mixture at 0°C

$$V_0 = \sqrt{\frac{\gamma P}{D_m}} \qquad D_m = \text{Density of mixture} \quad = \sqrt{\frac{\gamma P.3}{16 D_H}}$$

Now $\quad \frac{v_0}{v_H} = \sqrt{\frac{\gamma P}{D_H}.\frac{3}{16}} \Big/ \sqrt{\frac{\gamma P}{D_H}} = \sqrt{\frac{3}{16}}$

$$V_0 = \frac{1300 \times \sqrt{3}}{4} = 325\sqrt{3} \text{ m/s}$$

Velocity of sound in the mixture at a temperature 27°C is given by

$$v_t = v_0\left(1 + \frac{t}{546}\right) \text{ (approx)} = 325\sqrt{3}\left(1 + \frac{27}{546}\right) = \mathbf{591\ m/s}$$

Illustration 11.

Calculate the R.M.S. velocity of gas molecules at NTP if the velocity of sound in that gas is 300 m/s. Ratio of the specific heats of the gas is 1.5.

Solution:

R.M.S. velocity of gas molecules $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

Velocity of sound in the gas v = $\sqrt{\frac{\gamma RT}{M}}; \ \therefore \frac{v}{v_{rms}} = \sqrt{\frac{\gamma RT}{M} \times \frac{M}{3RT}} = \sqrt{\frac{\gamma}{3}}$

$$v_{\text{rms}} = v\sqrt{\frac{3}{\gamma}} = 300\sqrt{\frac{3}{1.5}} = 300\sqrt{2} = \mathbf{424.3\ m/s}$$

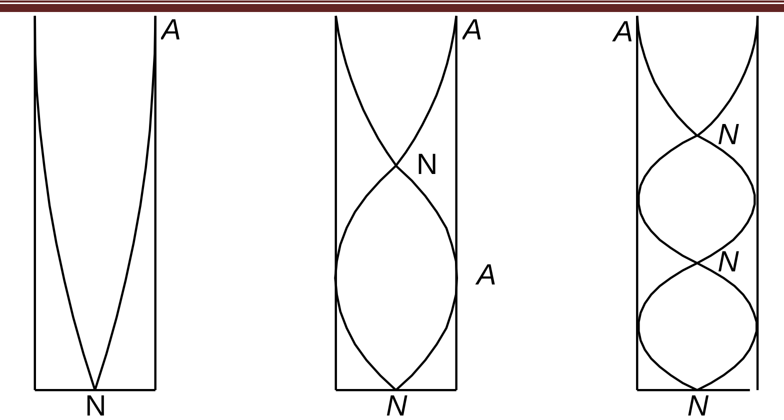
7. VIBRATIONS OF AIR COLUMNS

An air column is one, which is enclosed in a pipe. When an air column is excited the column is set into vibration and longitudinal waves are set up. The waves are reflected back at the boundaries of air columns. The incident and reflected waves produce longitudinal stationary waves.

Suppose we consider a closed tube, which is closed at one end and open at the other. At the closed end a compression is reflected as compression and a rarefaction as a rarefaction. At the open end a compression is reflected as a rarefaction while a rarefaction is reflected as a compression. Hence the closed end must be a displacement node while the open end must be a displacement antinode.

7.1 VIBRATION IN CLOSED PIPES– OVERTONES AND HARMONICS

A stationary wave pattern can be maintained in a closed tube containing a gas only for a frequency, which has one of the values making the length of the column a whole number of quarter wavelengths. It should be noted that the open end is always an antinode and the closed end a node. According to this condition there arises a number of standing waves as shown in Figure. The wave pattern, which has the lowest frequency, is called fundamental and the others are called overtones.



Fundamental First overtone Second overtone

First Mode of vibration (Fundamental)

The length of air column L is equal to $\frac{\lambda}{4}$.

$$\therefore \quad \lambda = 4L$$

$$v = n\lambda$$

$$n = \frac{v}{\lambda} = \frac{v}{4L} \qquad \ldots (\mathbf{37})$$

where n is the frequency of fundamental.

Second Mode of Vibration (First overtone)

$$L = \frac{3\lambda_1}{4}$$

$$\lambda_1 = \frac{4L}{3}$$

$$\therefore \quad \text{frequency } n_1 = \frac{v}{\lambda_1} = \frac{3v}{4L} = 3n. \qquad \ldots \text{(i)}$$

The frequency of first overtone is 3 times the value of fundamental.

Third Mode of Vibration (Second overtone)

Here $L = \frac{5\lambda_2}{4}$

$$\lambda = \frac{4L}{5}$$

$$\therefore \quad \text{frequency } n_2 = \frac{v}{\lambda_2} = \frac{5v}{4L} = 5n. \qquad \ldots \text{(ii)}$$

When an air column is excited the fundamental and a number of possible overtones are present in the vibration. Of these the loudest is the fundamental and overtones progressively become weaker in intensity. The overtones whose frequencies are integral multiples of fundamental are called harmonics. The fundamental with frequency

n itself is taken as first harmonic. The overtone with frequency $2n$ is called second harmonic and the overtone with frequency $3n$ is called third harmonic and so on.

In the case of closed type indicated above all odd harmonics are present and even harmonics are absent.

End correction: In the above discussion it is assumed that the position of antinode coincides with the open end of pipe exactly. This is not however true and it is found that antinode is a little bit displaced above the open end. If e is the end correction, then for fundamental

$$\frac{\lambda_1}{4} = (L + e)$$

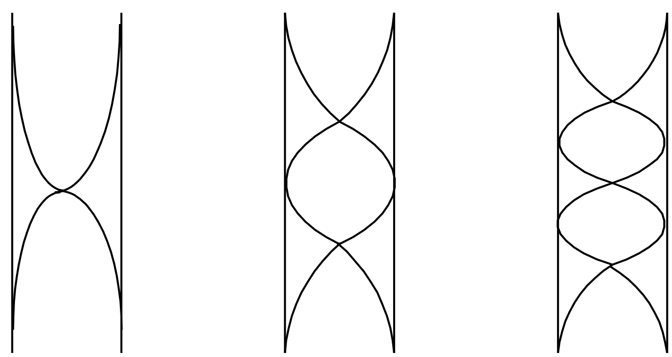
For the first overtone

$$\frac{3\lambda_2}{4} = (L + e) \text{ and so on.}$$

The end correction depends upon the diameter of the pipe. If d is the diameter the end correction $e = 0.3\ d$.

7.2 VIBRATION IN OPEN PIPES

A pipe with both ends open is called open pipe. The first three modes of vibrations, starting from fundamental in open pipes are shown in figures below.



Fundamental First overtone Second overtone

Fundamental: In the fundamental mode there is a node between antinodes at each end.

$$\therefore \qquad L = \frac{\lambda}{2} \quad \text{or} \quad \lambda = 2L$$

$$n = \frac{v}{\lambda} = \frac{v}{2L} \qquad \textbf{(38)}$$

First overtone: If λ_1 and n_1 are the wavelength and frequency of the first overtone in open pipe

$$\lambda_1 = L$$

$$n_1 = \frac{v}{\lambda_1} = \frac{v}{L} = \frac{2v}{2L} = 2n \qquad \ldots \text{(i)}$$

The frequency of first overtone is twice that of fundamental. It corresponds to second harmonic.

Second overtone: If λ_2 and n_2 be the wavelength and frequency of second overtone in the open pipe,

$$L = \frac{3\lambda_2}{2}$$

$$\lambda_2 = \frac{2L}{3}$$

$$n_2 = \frac{v}{\lambda_2} = \frac{3v}{2L} = 3n \qquad \ldots \text{(ii)}$$

This corresponds to third harmonic of the vibrating system.

In an open pipe all the harmonics, both odd and even are present.

7.3 FREE, DAMPED AND FORCED VIBRATIONS

A body capable of vibration, if excited, and set free, vibrates freely in its own natural way. The frequency of such free vibration depends on the mass, elastic property and dimensions of the body. The frequency is called free frequency or natural frequency of the body.

7.4 DAMPED VIBRATIONS

The amplitude of free vibrations of a body gradually diminishes and finally the vibrations die away after sometime. This is due to the vibratory motion being damped by forces internal and external to the body.

7.5 FORCED VIVRATIONS

If an external periodic force is applied to a body which is capable of vibration and if the frequency of the applied periodic force is not the same as the free frequency of the body, the body begins to vibrate initially with its own natural frequency but these vibrations die down quickly and the body ultimately vibrates with the frequency of the external periodic force. Such vibrations are called forced vibrations.

8. RESONANCE

Resonance is a special case of forced vibration. If the frequency of the external periodic force is the same as the natural frequency of the body, the body responds to the forced vibrations more willingly and there is a gain in the amplitude of its vibrations. This is called resonance. Resonance has vast applications in acoustics, electrical circuits and electronics.

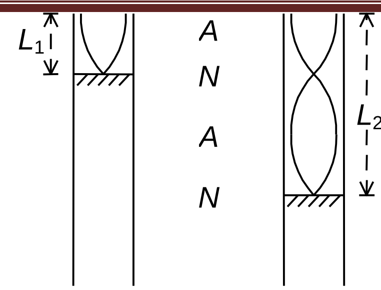
8.1 RESONANCE IN AIR COLUMNS-RESONANCE TUBES

Suppose the length of air column in a long tube can be adjusted either by dipping the tube in a reservoir of water or by allowing the water level to occupy a desired position in the tube by pressure flow, the column can be made to vibrate in resonance with an excited tuning fork kept over the mouth of the tube.

For two lengths of air column L_1 and $L_2 \sim 3L_1$, the resonance would occur and the positions correspond to the fundamental mode and the first overtone respectively. If λ be the wavelength of sound in air and v the velocity of sound in air, then

$$L_1 + e = \frac{\lambda}{4}$$

$L_2 + e = \frac{3\lambda}{4}$ where e is the end correction.



From the above equations we get

$$\frac{\lambda}{2} = L_2 - L_1 \quad \text{or,} \quad \lambda = 2(L_2 - L_1)$$

$$v = n\lambda = 2n(L_2 - L_1) \qquad \ldots \textbf{(39)}$$

where n is the frequency of vibration of the air column which is in resonance with the tuning fork of same frequency.

Illustration 12.

A rod of length 1 m clamped at its midpoint is set up with a Kundt's tube containing air. When the rod is set vibrating in its longitudinal mode of vibrations it is found that the internodal distance in the tube is 0.063 m. The temperature of air inside the tube is 10°C and the velocity of sound in air at 0°C is 331 m/s. Calculate the frequency of the note emitted by the rod and the Young's modulus of the rod. Given the density of the rod = 7600 kg/m^3.

Solution:

Velocity of sound at 0°C = 331 m/s

Velocity of sound at 10°C = 331 + (10 × 0.61) = 337.1 m/s

(Velocity of sound increases at the rate of 0.61 m/s per °C rise in temperature approximately).

Internodal distance = $\frac{\lambda_{air}}{2}$ = 0.063 m or λ_{air} = 0.126 m

Frequency of vibrations in air = $\frac{337.1}{0.126}$ = 2675 Hz

This is equal to the frequency of vibrations of the rod.

$\therefore$ frequency of the note emitted by the rod = **2675 Hz**

$$n = \frac{1}{2l}\sqrt{\frac{q}{p}}$$

$$q = 4l^2n^2\rho = 4 \times 1^2 \times 2675^2 \times 7600 = \mathbf{2.175 \times 10^{11}\ N/m^2}$$

Illustration 13.

A tube of a certain diameter and length 48 cm is open at both ends. Its fundamental frequency is found to be 320 Hz. The velocity of sound in air is 320 m/s. Estimate the diameter of the tube.

One end of the tube is now closed. Calculate the lowest frequency of resonance of the tube.

Solution:

Let the length of the open tube be L. The end correction on both sides is e. The tube vibrates in its fundamental. Then

$$\frac{\lambda}{2} = L + 2e \text{ or } \lambda = 2(L + 2e)$$

If v be the velocity of sound in air the fundamental frequency is given by

$$n = \frac{v}{\lambda} = \frac{v}{2(L+2e)}$$

n = 320 Hz; v = 320 m/s   or, $320 = \dfrac{320}{2(L+2e)}$

or $L + 2e = 0.5$ m

$2e = 0.5 \text{ m} - 0.48 \text{ m} = 0.02$ m

$e = 0.01 \text{ m} = 1$ cm

If the diameter of the tube is d, end correction e is related to d as

$e = 0.3d$

$d = \dfrac{e}{0.3} = \dfrac{1}{0.3}$ = **3.33 cm** (approx)

Now one end of the tube is closed. The tube becomes a closed type. In its fundamental mode it has the lowest frequency. The fundamental frequency

Illustration 14.

The length of an organ pipe is 30 cm. What is the change in its length required to maintain its frequency unchanged if the temperature falls from 27°C to 7°C?

Solution:

Let L_1 and L_2 be the lengths of the pipe at t_1 = 27°C and t_2 = 7°C respectively. Now the frequency is maintained constant, irrespective of the organ pipe being a closed one or an open one, then,

$\dfrac{v}{L} = a$ constant v = velocity of sound in air

or $\dfrac{L_1}{L_2} = \dfrac{v_1}{v_2}$ v_1 = velocity at temperature $t_1^\circ C$, v_2 = velocity at temperature $t_2^\circ C$

But $\dfrac{v_1}{v_2} = \sqrt{\dfrac{T_1}{T_2}}$

$\therefore$ $\dfrac{L_1}{L_2} = \sqrt{\dfrac{273+27}{273+7}} = \sqrt{\dfrac{300}{280}} = 1.035$

Now $\dfrac{L_1 - L_2}{L_1} = \dfrac{1.035-1}{1.035} = \dfrac{0.035}{1.035} = 0.0338$

$L_1 - L_2 = 30 \times 0.0338 \approx 1$ cm

The change in length is a decrease by 1 cm

(approx). $n = \dfrac{v}{4(L+e)} = \dfrac{320}{4(0.48+0.01)} = \dfrac{80}{0.49} =$ **163.26 Hz**

9. BEATS

When two interacting waves have slightly different frequencies the resultant disturbance at any point due to the superposition periodically fluctuates causing waxing and waning in the resultant intensity. The waxing and waving in the resultant intensity of two superposed waves of slightly different frequency are known as beats.

Let the displacement produced at a point by one wave be

$$y_1 = A \sin (2\pi f_1 t - \phi_1)$$

and the displacement produced at the point produced by the other wave of equal amplitude as

$$y_2 = A \sin (2\pi f_2 t - \phi_2)$$

By the principle of superposition, the resultant displacement is

$$y = y_1 + y_2 \quad = A \sin (2\pi f_1 t - \phi_1) + A \sin (2\pi f_2 t - \phi_2)$$

$$y = 2A \sin \left\{2\pi\left(\frac{f_1 + f_2}{2}\right)t - \left(\frac{\phi_1 - \phi_2}{2}\right)\right\} \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t$$

$$y = R \sin \left\{2\pi\left(\frac{f_1 + f_2}{2}\right)t - \left(\frac{\phi_1 - \phi_2}{2}\right)\right\} \qquad \ldots (\mathbf{40})$$

where $R = 2A \cos 2\pi \left(\frac{f_1 - f_2}{2}\right)t$

The time for one beat is the time between consecutive maxima or minima.

First maxima would occur when

$$\cos 2\pi \left(\frac{f_1 - f_2}{2}\right)t = +1$$

Then $\qquad 2\pi \left(\frac{f_1 - f_2}{2}\right)t = 0$

$\therefore \qquad t = 0$

second maxima would occur when

$$\cos 2\pi \left(\frac{f_1 - f_2}{2}\right)t = -1$$

Then $\qquad 2\pi \left(\frac{f_1 - f_2}{2}\right)t = \pi$

or $\qquad t = \frac{1}{(f_1 - f_2)}$

The time for one beat = $\left\{\frac{1}{(f_1 - f_2)} - 0\right\} = \frac{1}{f_1 - f_2}$

Similarly it can also be shown time between two consecutive minima is $\dfrac{1}{(f_1 - f_2)}$

Hence frequency of beat i.e., number of beats in one second = $f_1 \sim f_2$ … (**41**)

Illustration 15.

A column of air and a tuning fork produce 4 beats per second when sounded together. The tuning fork gives the lower note. The temperature of air is 15°C. When the temperature falls to 10°C both of them produce 3 beats per second. Find the frequency of fork.

Solution:

Let λ be the wavelength and n be the frequency of fork.

At 15°C, $\dfrac{V_{15}}{\lambda} - n = 4$ or $\dfrac{V_{15}}{\lambda} = n + 4$

At 10°C, $\dfrac{V_{10}}{\lambda} - n = 3$ or $\dfrac{V_{10}}{\lambda} = n + 3$

$$\therefore \qquad \frac{V_{15}}{V_{10}} = \frac{n+4}{n+3}$$

$$\text{But} \qquad \frac{V_{15}}{V_{10}} = \sqrt{\frac{273+15}{273+10}} = \sqrt{\frac{288}{283}}$$

$$\therefore \qquad \sqrt{\frac{288}{283}} = \frac{n+4}{n+3}$$

$$\left(1+\frac{5}{283}\right)^{1/2} = \frac{n+4}{n+3}$$

$$1+\frac{5}{566} = \frac{n+4}{n+3}$$

$$\frac{5}{566} = \frac{n+4-n-3}{n+3} = \frac{1}{n+3}$$

$5n + 15 = 566$

$5n = 551$ or $n =$ **110.2 Hz**

Illustration 16.

Two tuning forks A and B when sounded together give 4 beats/sec. A is in unison with the note emitted by a length 0.96 m of a sonometer wire under a certain tension while B is in unison with 0.97 m of the same wire under the same tension. Find the frequencies of the forks.

Solution:

Let the frequency of the fork A be n. Since A is in unison with a smaller length of the sonometer wire than B which is in unison with a larger length of the wire, frequency of fork A should be larger than that of B.

$\therefore$ frequency of fork $B = (n - 4)$ Hz

Now $n \times 0.96 = (n-4)(0.97)$

$$\frac{n-4}{n} = \frac{96}{97}$$

$$1 - \frac{4}{n} = 1 - \frac{1}{97}$$

$$\frac{4}{n} = \frac{1}{97}$$

or, $n = 4 \times 97 = 388$ Hz

$\therefore$ the frequency of fork A = **388 Hz**

and that of B = **384 Hz.**

10. DOPPLER EFFECT

When a sound source and an observer are in relative motion with respect to the medium in which the waves propagate, the frequency of waves observed is different from the frequency of sound emitted by the source. This phenomenon is called Doppler effect. This is due to the wave-nature of sound propagation and is therefore applicable to light waves also. The apparent change of colour of a star can be explained by this principle.

10.1 CALCULATION OF APPARENT FREQUENCY

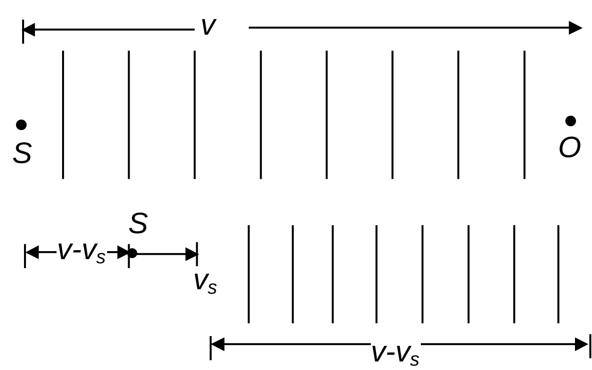
Suppose v is the velocity of sound in air, v_0 is the velocity of the observer (O), and f is the frequency of the source.

Source moves towards stationary observer

If the source were stationary the f waves sent out in one second towards the observer O would occupy a distance v, and the wavelength would be v/f .

If S moves with a velocity v_s towards O, the f waves sent out occupy a distance $(v - v_s)$ because S has moved a distance v_s towards O in 1 s. So the apparent wavelength would be

$$\lambda' = \left(\frac{v - v_s}{f}\right)$$

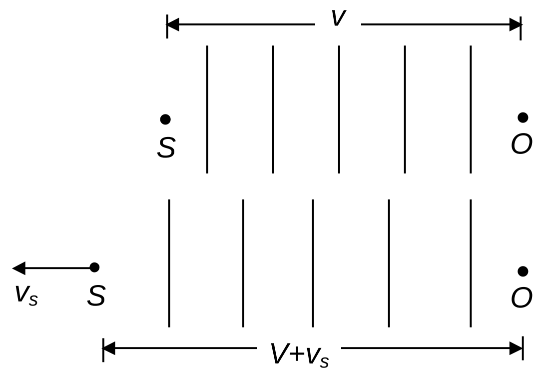


Thus, apparent frequency

$$f' = \frac{\text{velocity fo sound relative to O}}{\text{wavelength of wave reaching O}}$$

$$f' = \frac{v}{\lambda'} = f\left(\frac{v}{v - v_s}\right)$$

Source moves away from stationary observer



Now, apparent wavelength

$$\lambda' = \frac{v+v_s}{f}$$

$\therefore$ Apparent frequency

$$f' = v/\lambda'$$

or $$f' = f\left(\frac{v}{v+V_s}\right)$$

Observer, moves towards stationary source

$$f' = \frac{\text{velocity fo sound relative to O}}{\text{wavelength of wave reaching O}}$$

Here, velocity of sound relative to $O = v + v_0$

and wavelength of waves reaching $O = v/f$

$\therefore$ $$f' = \frac{v+v_0}{v/f} = f\left(\frac{v+v_0}{v}\right)$$

Observer moves away from the stationary source

$$f' = \frac{v-v_0}{v/f} = f\left(\frac{v-v_0}{v}\right)$$

Source and observer both moves towards each other

$$f' = \frac{(v+v_0)}{\left(\frac{v-v_s}{f}\right)} = f\left(\frac{v+v_0}{v-v_s}\right)$$

Both moves away from each other

$$f' = f\left[\frac{v-v_0}{v+v_s}\right]$$

Source moves towards observer but observer moves away from source

$$f' = f\left(\frac{v-v_0}{v-v_s}\right)$$

Source moves away from observer but observer moves towards source

$$f' = f\left[\frac{v+v_0}{v+v_s}\right]$$

Effect of wind on Doppler Effect in sound

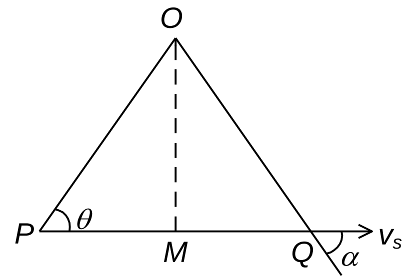
The formulae derived in the previous sections are valid only where there is no wind. If there is a wind velocity v_w, the effective velocity of sound would become $(v + v_w)$ or $(v - v_w)$ according as v_w is in the direction of v or opposite to it.

Observed frequency using Doppler's effect is given by

$$f' = \frac{[(v \pm v_W) \pm v_0)]}{[(v \pm v_W) \pm v_s]} f \qquad \dots \textbf{(42)}$$

10.2 DOPPLER EFFECT WHEN THE SOURCE IS MOVING AT AN ANGLE TO THE OBSERVER

Let O be a stationary observer and let a source of sound of frequency f be moving along the line PQ with constant speed v_s.



When the source is at P, the line PO makes angle θ with PQ, which is the direction of v_s.

The component of velocity v_s along PO is $v_s \cos\theta$ and it is towards the observer.

The apparent frequency in this case

$$f_a = \frac{v}{v - v_s \cos\theta} \cdot f \qquad \dots \textbf{(43)}$$

As the source moves along PQ, θ increases cos θ decreases and the apparent frequency continuously diminishes. At M, $\theta = 90°$ and hence

$$f_a = f$$

When the source is at Q, the component of velocity v_s is $v_s \cos\alpha$ which is directed away from the observer. Hence the apparent frequency

$$f_a = \frac{v}{v + v_s \cos\alpha} \cdot f \qquad \dots \textbf{(44)}$$

10.3 REFLECTION OF SOUND AND ECHO

Like light, sound waves can also be reflected and refracted. But the sound requires an extended reflector like a wall as its wavelength is very large compared to that of light.

The phenomenon of echo is due to reflection of sound. When a sound wave is reflected by a distant reflector like a wall or mountain cliff an observer hears two sounds, one from the source directly and the other as reflected from the reflector which is the echo.

In order to hear a clear and distinct echo a minimum distance must be maintained between the reflector and the observer. This is necessary because when a sound is heard the impression persists for a fraction of a second in the ear due to the persistence of hearing. Until this impression is removed the echo cannot be heard as a distinct sound. The persistence of hearing usually lasts for $\frac{1}{10}$ th of a second. If the time interval between the emission of the sound and the return of the reflected wave is more than $\frac{1}{10}$ th of a second the reflected sound is heard after a silent 'interval' and is called an echo.

11. LOUDNESS, PITCH AND QUALITY OF SOUND

GENERALLY

loudness of a sound depends on its intensity

pitch on the frequency and

quality of sound depends on the combination of frequencies, overtones and their relative intensities. Together these three quantities determine the characteristics of sound - musical or otherwise.

LOUDNESS

Loudness is a physiological sensation in the human ear, which is intimately connected with the intensity of the wave incident.

The ear is sensitive to sound of frequencies anywhere from 20 Hz to 20000 Hz and is most sensitive to frequencies between 2000 and 3000 Hz.

The ear can detect sounds varying from the hardly audible low intensity of 10^{-12} (W/m^2) to very high intensities of 1 W/m^2.

Because of this enormous range of intensity to which the ear responds, a logarithmic scale is used to measure loudness levels instead of a linear scale.

If we take I_0 as the standard intensity level we can define another intensity level I as a ratio $\frac{I}{I_o}$, which is dimensionless. The difference L in intensity levels of two sound waves of intensity I and I_0 is defined in units of bel as

$$L = \log\left(\frac{I}{I_o}\right) \text{ bel} = 10\log\left(\frac{I}{I_o}\right) \text{ decibels } (dB) \qquad \ldots (\mathbf{45})$$

The standard threshold of audibility I_0 is taken to be intensity of 10^{-12} W/m^2.

According to this the intensity level of an average sound which may have a loudness of 10^{-6} W/m^2 will be

$$I = 10\log\frac{10^{-6}}{10^{-12}} \text{ dB}$$

$$= 10\log 10^{6} = 60 \text{ dB}$$

It is found that the intensity of the sound level must be doubled before an observer can respond to the change in intensity and say that the sound is definitely louder.

PITCH

The pitch refers to that characteristic of sound sensation that enables one to classify a note as a high note or a low note. Pitch depends on the frequency. The higher the frequency the higher is pitch.

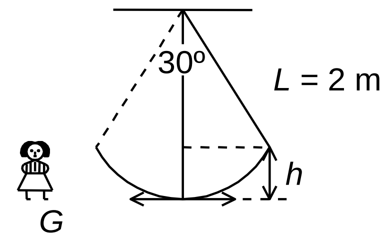
If we go up the scale by one octave we double the frequency.

QUALITY OR TIMBRE

The tone quality of any musical sound is determined by the presence of the number of overtones and their relative intensities

Illustration 17.

A boy sitting on a swing which is moving to an angle of 30° from the vertical is blowing a whistle which has a frequency of 1000 Hz. The whistle is 2.0 m from the point of support of the swing. A girl stands infront of the swing. Calculate the maximum and minimum frequencies, she will hear (Velocity of sound = 330 m/s; g = 9.8 m/s^2).



Solution:

The maximum speed of the swing is when it crosses the lowest point. If h is the height of the swing above the lowest point at its maximum displacement position, then

the potential energy = mgh

$$= mgL(1 - \cos\theta)$$

$$= 2mg(1 - \cos 30°)$$

This is equal to the K.E. at the lowest position of the swing, where v is the maximum

$$\text{K.E.} = \frac{1}{2}mv^2 = mgh = 2mg(1 - \cos 30°)$$

$$v^2 = 4 \times 9.8(1 - 0.866) = 5.25$$

$$\therefore \quad v = 2.3 \text{ m/s}$$

The maximum frequency is heard when the swing comes towards the girl.

$$\nu_{max} = \frac{C - v_o}{C - v_s} \times \nu = \frac{330 - 0}{330 - 2.3} \times 1000 = \mathbf{1007\ Hz}$$

The minimum frequency is heard when the swing goes away from the observer.

$$\nu_{min} = \frac{330}{330 + 2.3} \times 1000 = \mathbf{993\ Hz}$$

Illustration 18.

A train approaching a hill at a speed of 40 km/hr sounds a whistle of frequency 580 Hz when it is at a distance of 1 km from the hill. A wind with a speed of 40 km per hour is blowing in the direction of motion of the train. Find

(a) frequency of whistle as heard by an observer on the hill.

(b) the distance from the hill at which the echo from the hill is heard by the driver and its frequency (Velocity of sound in air is 1200 km/hour).

Solution:

(a) The apparent frequency when both source and observer are moving along the same direction is given by

$$n' = \left(\frac{c+w+v}{c+w-u}\right)n$$

According to the problem $v = 0$

$$\therefore \qquad n' = \frac{c+w}{c+w-u}\times n$$

$c = 1200$ km/hr

$w = 40$ km/hr

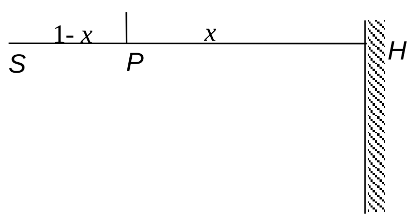
$u = 40$ km/hr

$n = 580$ Hz

$$n' = \frac{1200+40}{1200+40-40}\times 580 = \mathbf{599.33\ Hz.}$$

(b) Let echo by driver be heard when the train is at P distant x km from hill

Time taken by train to reach P



$$= \frac{\text{Distance SP}}{\text{Velocity of train}} = \frac{1-x}{40}hrs$$

The time taken by echo to reach P

$$= \frac{\text{Distance } SH}{c+w} + \frac{\text{Distance } PH}{c-w}$$

$$= \left[\frac{1}{1200+40} + \frac{x}{1200-40}\right]\text{hrs}$$

$$\therefore \qquad \frac{1-x}{40} = \frac{1}{1240} + \frac{x}{1160}$$

Solving $x = \dfrac{29}{31}$ km

If n′ is the frequency of reflected sound, $n' = 599.33$ Hz

The apparent frequency heard by the driver

$$n'' = n'\frac{(c-w)+v}{(c-w)} = 599.33\frac{(1200-40+40)}{1200-40} = \frac{1200}{1160}\times 599.33 = \mathbf{620\ Hz}$$

Illustration 19.

A metallic rod of length 1 m is rigidly clamped at its midpoint. Longitudinal stationary waves are set up in the rod in such a way that there are two nodes on either side of the midpoint. The amplitude of an antinode is 2×10^{-6} m. Write the equation of motion at a point 2 cm from the midpoint and those of the constituent waves in the rod. (Young's modulus = 2×10^{11} Nm^{-2} and density = 8000 kg m^{-3})

Solution:

The equation of standing wave can be written as

$$y = 2A \sin kx \cos \omega t$$

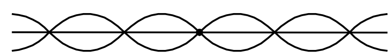
where $k = \frac{2\pi}{\lambda}$ and $\omega = \frac{2\pi v}{\lambda}$

The standing wave is obtained by adding the equation of two identical progressive waves travelling in opposite directions

$y_1 = A \sin(kx - \omega t)$; $y_2 = A \sin(kx + \omega t)$

In the present problem the length L of the rod = 1 metre.

i.e., $L = \frac{5\lambda}{2}$ or $\lambda = \frac{2}{5}$ metre.



Velocity of longitudinal wave is given by

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2\times10^{11}}{8000}} = 5\times10^3\, ms^{-1}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2/5} = 5\pi \text{ metre}^{-1}$$

$$\omega = \frac{2\pi v}{\lambda} = \frac{2\pi\times5\times10^3}{2/5} = (25\times10^3\pi)s^{-1}$$

Hence equation of standing wave is

$$y = (2 \times 10^{-6}) \sin 5\pi x \cos 25 \times 10^3 \pi t$$

Equations of component waves are

$$y_1 = (1 \times 10^{-6}) \sin (5\pi x - 25 \times 10^3 \pi t)$$

$$y_2 = (1 \times 10^{-6}) \sin (5\pi x + 25 \times 10^3 \pi t)$$

Illustration 20.

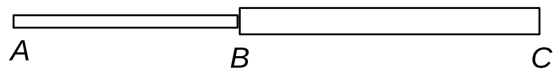
Two wires of radii r and 2r respectively are welded together end to end. The combination is used as a sonometer wire and is kept under tension T. The welded point is midway between the two bridges. What would be the ratio of the number of loops formed in the wires such that the joint is a node when stationary vibrations are set up in the wires?

Solution:

ABC is the wire where B is the midpoint of AC. AB is the half with radius r and BC is the part with radius 2r. They are of the same material and hence their volume density is the same. The welded point B will be a node for the stationary wave system formed. Let there be n_1 loops of wavelength λ_1 in AB and n_2 loops of wavelength λ_2 in BC.

Since AB = BC

$$n_1\frac{\lambda_1}{2}=n_2\frac{\lambda_2}{2} \text{ or, } \frac{\lambda_1}{\lambda_2}=\frac{n_2}{n_1} \qquad \text{... (i)}$$



Velocity of transverse wave in wire AB:

$$v_1 = \sqrt{\frac{T}{\pi r^2 D}} \text{ where T = tension}$$

D = volume density

Velocity of transverse wave in wire BC:

$$v_2 = \sqrt{\frac{T}{\pi(2r)^2 D}}$$

$$\therefore \quad \frac{v_1}{v_2} = 2 \qquad \text{... (ii)}$$

Now $\frac{v_1}{v_2}=\frac{f_1\lambda_1}{f_2\lambda_2}$, where f_1 and f_2 are frequencies

Since $f_1 = f_2$, $\frac{v_1}{v_2}=\frac{\lambda_1}{\lambda_2}=\frac{n_2}{n_1}$ = 2, using (i) and (ii).

$\therefore$ required ratio $n_1 : n_2 = 1 : 2$

Illustration 21.

Length of a sonometer wire between its fixed ends is 100 cm. Where should two bridges be placed in between the ends so as to divide the wire into 3 segments whose fundamental frequencies are in the ratio 1 : 2 : 3?

Solution:

If L_1, L_2, L_3 be the lengths of the three segments then

$L_1 + L_2 + L_3 = 100$

If n_1, n_2 and n_3 be the fundamental frequencies of the three segments then

$n_1 = n$

$n_2 = 2n$

$n_3 = 3n$,

since $n_1 : n_2 : n_3 = 1 : 2 : 3$

The tension remains the same throughout the wire. Hence

$n_1L_1 = n_1L_2 = n_3L_3$ or, $nL_1 = 2nL_2 = 3nL_3$

or, $L_1 = 2L_2 = 3L_3$

If $L_3 = x$, $L_2 = \frac{3}{2}x$, $L_1 = 3x$

$\therefore$ $L_1 + L_2 + L_3 = 3x + \frac{3}{2}x + x = 100$ cm or, $11x = 200$

$L_3 = x = \frac{200}{11} = 18.18$ cm

$L_2 = \frac{3}{2}x = \frac{54.54}{2} = 27.27$ cm

$L_1 = 3x = 54.54$ cm

The bridges should be placed in the positions of 54.54 cm from the zero end and 18.18 cm from the other end.

Illustration 22.

A solid is attached to the free end of a sonometer wire. The wire then has a frequency of 500 Hz. Then the solid is immersed in water and it is found that the wire has a frequency of 460 Hz. When the solid is immersed in a liquid the frequency of the wire is 480 Hz. Calculate the specific gravity of the solid and that of the liquid.

Solution:

The solid in air, in water and in liquid have three different weights due to the buoyancy.

Let its weight in air be W_1, that in water be W_2 and that in liquid be W_3.

The frequency of the wire is given by $n = \frac{1}{2L}\sqrt{\frac{T}{m}}$

and in the present case n = a constant $\times \sqrt{T}$

or, $T = kn^2$ (k is a constant).

Now , $W_1 = k\,(500)^2$ $W_2 = k\,(460)^2$ $W_3 = k\,(480)^2$

From the principle of Archimedes we have the specific gravity of solid = $\frac{\text{Weight of solid in air}}{\text{Loss of weight in water}}$

$$= \frac{W_1}{W_1 - W_2} = \frac{k(500)^2}{k(500)^2 - k(460)^2} = \frac{500^2}{500^2 - 460^2} = \frac{250000}{38400} = 6.51$$

Specific gravity of liquid

$$= \frac{\text{Loss of weight of solid in liquid}}{\text{Loss of weight of solid in water}} = \frac{k(500)^2 - k(480)^2}{k(500)^2 - k(460)^2} = \frac{500^2 - 480^2}{500^2 - 460^2} = 0.51$$

Illustration 23.

Calculate the velocity of sound in a mixture of oxygen, nitrogen and argon at 0°C. The mixture consists of the gases oxygen, nitrogen and argon in the mass ratio 2 : 7 : 1. (Given R = 8.3 J mol^{-1} K^{-1}. Ratio of specific heats of the gases are argon 1.67, oxygen 1.4, nitrogen 1.4. The molecular weights of the respective gases are 40, 32 and 28).

Solution:

The relation for the velocity of sound in a gas v = $\sqrt{\frac{\gamma RT}{M}}$

Considering the mixture of gas while all the constituents of the mixture occupy the same volume their masses vary. Let m_0, m_N, m_A be the fractions of masses of the respective gases and M_0,M_N,M_A be their respective molecular weights. Now the velocity of sound in the mixture can be given by the relation,

$$v = \sqrt{RT}\left[\frac{\gamma_O m_O}{M_O}+\frac{\gamma_N m_N}{M_N}+\frac{\gamma_A m_A}{M_A}\right]^{1/2} = \left[8.3\times 273\left(\frac{1.4\times\frac{2}{10}}{32\times 10^{-3}}+\frac{1.4\times\frac{7}{10}}{28\times 10^{-3}}+\frac{1.67\times\frac{1}{10}}{40\times 10^{-3}}\right)\right]^{1/2}$$

$$= [8.3 \times 273 \times 1000(8.75 \times 10^{-3} + 35 \times 10^{-3} + 4.175 \times 10^{-3})]^{1/2}$$

$$= [8.3 \times 273 \times 47.925]^{1/2} \quad = 329.5 \text{ m/s}$$

Illustration 24.

If v be the velocity of sound in dry air at NTP, find the velocity of sound in moist air saturated with water vapour at NTP. Take the molecular weight of air = 28.8 and that of water vapour = 18. γ for air = 1.4; γ for water vapour = 1.33. Atmospheric pressure at NTP is P and the saturation vapour pressure of water at 0°C is p.

Solution:

The velocity of sound in dry air at NTP v = $\sqrt{\frac{\gamma P}{D}}=\sqrt{\frac{1.4P}{D}}$

where D is the density of dry air at NTP.

Let us consider the moist air at NTP. It is composed of dry air at pressure (P – p) and water vapour at pressure p.

Density of dry air at 0°C and pressure (P – p) = D_a = D · $\frac{(P-p)}{P}$

Density of water vapour at 0°C and pressure P = D_w = D × $\frac{p}{P}\times\frac{18}{28.8}$

Effective density of moist air at NTP

$$D_m = D_a + D_w = \frac{D}{P}\left[(P-p)+\frac{18}{28.8}p\right]$$

$$\therefore \qquad D_m = \frac{D}{P}[P - 0.375p] = D\left[1 - 0.375\,\frac{p}{P}\right]$$

The ratio of specific heats for moist air

$$\gamma_m = 1.4\frac{P-p}{P} + 1.33\frac{p}{P} = 1.4 - 1.4\frac{p}{P} + 1.33\frac{p}{P} = 1.4 - 0.07\frac{p}{P}$$

$$= 1.4\left(1 - \frac{0.07}{1.4}\frac{p}{P}\right)$$

$$\therefore \qquad \gamma_m = 1.4\left(1 - 0.05\frac{p}{P}\right)$$

Now the velocity of sound in moist air at NTP

$$v_m = \sqrt{\frac{\gamma_m P}{D_m}} = \sqrt{\frac{1.4\left(1-0.05\frac{p}{P}\right)P}{D\left(1-0.375\frac{p}{P}\right)}}$$

$$= v_0\sqrt{\frac{273+16}{273}} = v_0\sqrt{\frac{289}{273}}$$

$$v_m = \boldsymbol{v}\left[\left(\mathbf{1}+\mathbf{0.325}\frac{\boldsymbol{p}}{\boldsymbol{P}}\right)\right]^{\boldsymbol{1/2}}$$

This relation can be used to find the velocity of sound in moist air at any temperature.

Illustration 25.

The second overtone of an open pipe has the same frequency as the first overtone of a closed pipe 2 m long. What is the length of the open pipe?

Solution:

Let L_0 be the length of the open pipe. The fundamental frequency of the pipe is given by

$$n_o = \frac{v}{\lambda_f} = \frac{v}{2L_o}, \text{ v = velocity of sound in air}$$

The second overtone of the open pipe has a frequency $3n_o = \frac{3\cdot v}{2L_o}$ Hz

The length of the closed pipe L_C = 2 m

The frequency of the fundamental emitted by the closed pipe

$$n_c = \frac{v}{\lambda} = \frac{v}{4L_c}$$

The first overtone of the closed pipe has a frequency

$$3n_c = \frac{3v}{4L_c} = \frac{3v}{4\times 2} = \frac{3v}{8} \text{ Hz}$$

Now $3n_0 = 3n_c$ or, $\frac{3v}{4L_o} = \frac{3v}{8}$

or $2L_0 = 8$ or $L_0 = 4m$

Illustration 26.

Two identical steel wires are under tension and are in unison. When the tension in one of the wires is increased by 1 percent 4 beats per second is heard. Find the original frequency of the wires.

Solution:

By unison of the wires, we mean that they have the same frequency of vibration. Since they are identical in linear density and length they must be under the same tension to be in unison.

Let the tension in one wire be increased to $T_1 = \frac{101T}{100}$

where T is the original tension. Let n_1 be the new fundamental frequency.

Now $\frac{n_1}{n} = \sqrt{\frac{T_1}{T}} = \sqrt{\frac{101}{100}}$

But $n_1 > n$ and $n_1 - n = 4$

$\therefore$ $\frac{n+4}{n} = \sqrt{\frac{101}{100}}$ $1 + \frac{4}{n} = \left(1 + \frac{1}{100}\right)^{1/2} \approx 1 + \frac{1}{200}$

$\therefore$ $\frac{4}{n} = \frac{1}{200}$; $n = 800$ Hz.

Illustration 27.

A column of air at 51°C and a tuning fork produce 4 beats per second when sounded together. As the temperature of the air column is decreased the number of beats per second tends to decrease and when the temperature is 16°C, the two produce one beat per second. Find the frequency of the tuning fork.

Solution:

Let the velocity of sound in air at 51°C be v_1.

Let the corresponding wavelength and frequency be λ_1 and n_1 in the air column.

When the temperature decreases to 16°C, the corresponding quantities be v_2, λ_2 and n_2 respectively.

But the length of the tube remains the same and hence λ of stationary waves in both the cases will be the same. Hence

$v_1 = n_1\lambda$ and $v_2 = n_2\lambda$

or $\frac{v_1}{v_2} = \frac{n_1}{n_2}$

Since $v_t = v_0\sqrt{1 + \frac{t_1}{273}}$

$$v_1 = v_0\sqrt{\frac{273+t_1}{273}} = v_0\sqrt{\frac{273+51}{273}} = v_0\sqrt{\frac{324}{273}}$$

$$v_2 = v_0\sqrt{\frac{273+16}{273}} = v_0\sqrt{\frac{289}{273}} \quad \text{or,} \quad \frac{v_1}{v_2} = \sqrt{\frac{324}{289}} = \frac{18}{17}$$

or $$\frac{n_1}{n_2} = \frac{18}{17}$$

If n be the frequency of the tuning fork $n_1 = n \pm 4$

$n_2 = n \pm 1$

Since $n_1 > n_2$, $n_1 = n + 4$

$n_2 = n + 1$

$18(n + 1) = 17(n + 4)$

$18n + 18 = 17n + 68$

$n = 68 - 18 = 50$ Hz

It is also possible $n_2 = n - 1$

$\therefore$ $18(n - 1) = 17(n + 4)$

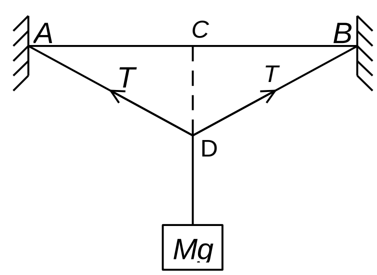
$18n - 18 = 17n + 68$

$n = 68 + 18 = 86$ Hz

$\therefore$ frequency of the fork = 50 Hz or 86 Hz

Illustration 28.

A wire of length 2 m is kept just taut horizontally between two walls. A mass hanging from its midpoint depresses it by 10 mm. The frequency of transverse vibrations of each segment of the wire is found to be the same as the fundamental frequency of air column in a closed tube of length 5 m. (Velocity of sound in air = 360 m/s). Find the Young's modulus of the material of the wire if the density of the material of the wire is 7800 kg/m^3.



Solution:

AB = 2 m; AC = CB = 1 m = 100 cm; CD = 10 mm = 1 cm

The extension produced in the segment AC of the wire due to the load Mg is given by

$$\Delta L = AD - AC = \sqrt{100^2+1^2} - 100$$

$$= 100\left(1+\frac{1}{100^2}\right)^{1/2} - 100 = 100\left(1+\frac{1}{2\times10^4}\right) - 100 = 0.005 \text{ cm.}$$

The tension T in the wire is therefore given by T = $\frac{Ya\Delta L}{L}$.

Frequency of the fundamental, n = $\frac{1}{2L}\sqrt{\frac{T}{m}}$

$$n^2 = \frac{1}{4L^2}\cdot\frac{T}{m} = \frac{1}{4L^2}\cdot\frac{Ya\Delta L}{L}\cdot\frac{1}{a\cdot D}$$ (D = density)

or, $$Y = \frac{4L^3n^2D}{\Delta L} = \frac{4(1)^3(n^2)\ (7800)}{5\times10^{-5}}$$

The frequency of the wire is the same as the fundamental of a closed tube of 5 m length.

$$n = \frac{v}{4L} = \frac{360}{4\times5} = 18 \text{ Hz}$$

∴ the Young's modulus of the material of the wire

$$Y = \frac{4\times(1)^3\times(18)^2\times(7800)}{5\times10^{-5}} = 20.22\times10^{10}\text{N/m}^2$$

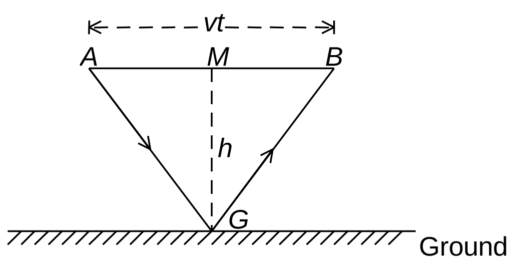
Illustration 29.

A pilot of an aeroplane travelling horizontally at 198 km/hr fires a gun and hears the echo from the ground after an interval of 3 seconds. If the velocity of sound is 330 m/s, find the height of the aeroplane from the ground.

Solution:

When the plane is in position A, the gun is fired. When it reaches position B, it hears the echo.

This echo is due to sound travelling along the path AGB. GM gives the altitude of the plane.



Let GM = h; AG = GB = x

The total distance travelled by the sound = 2x

Now, $\frac{2x}{v} = t$ $\frac{2x}{330} = 3$

∴ $$x = \frac{3\times330}{2} = 495 \text{ m}$$

Now, AB = velocity of plane × time of travel = $198\times\frac{5}{18}$ m/s × 3 s = 55 × 3 = 165 m

∴ $$AM = \frac{165}{2} = 82.5 \text{ m}$$

Now $$h = \sqrt{495^2 - 82.5^2} = 488$$