1. ELECTRIC CHARGE

Charge is the property associated with matter due to which it produces and experiences electrical and magnetic effects.

All bodies consist of atoms, which contain equal amount of positive and negative charges in the form of protons and electrons respectively. The number of electrons being equal to the number of protons as an atom is electrically neutral. If the electrons are removed from a body, it gets positively charged. If the electrons are transferred to a body, it gets negatively charged.

"Similar charges (charges of the same sign) repel one another; and dissimilar charges (charges of opposite sign) attract one another."

1.1 WAYS OF CHARGING A BODY

(i) Charging by friction

When two bodies are rubbed together, a transfer of electrons takes place from one body to another. The body from which electrons have been transferred is left with an excess of positive charge, so it gets positively charged. The body which receives the electrons becomes negatively charged.

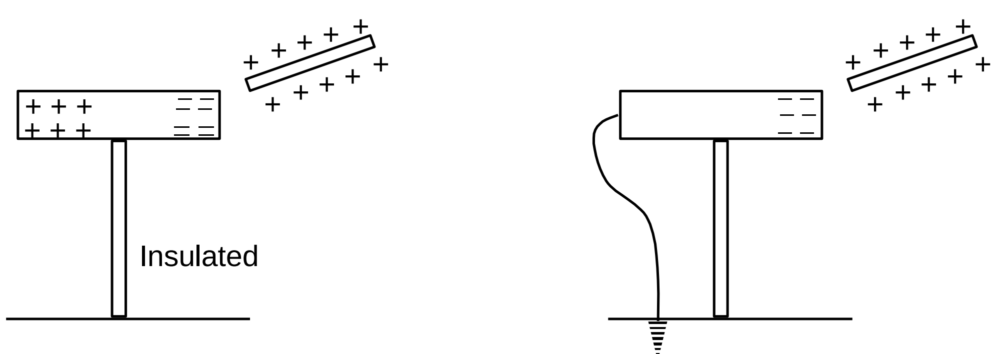
"The positive and negative charges produced by rubbing are always equal in magnitude."

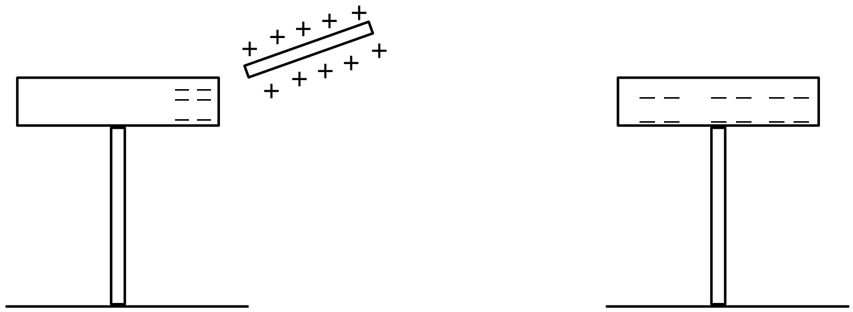
When a glass rod is rubbed with silk, it loses its electrons and gets a positive charge, while the piece of silk acquires equal negative charges.

An ebonite rod acquires a negative charge, if it is rubbed with wool (or fur). The piece of wool (or fur) acquires an equal positive charge.

(ii) Charging by electrostatic induction

If a positively charged rod is brought near an insulated conductor, the negative charges (electrons) in the conductor will be attracted towards the rod. As a result, there will be an excess of negative charge at the end of the conductor near the rod and the excess of positive charge at the far end. This is known as 'electrostatic induction'. The charges thus induced are found to be equal and opposite to each other. Now if we touch the far end with a conductor connected to the earth, the positive charges here will be cancelled by negative charges coming from the earth through the conducting wire. Now, if we remove the wire first and then the rod, the induced negative charges which were held at the outer end will spread over the entire conductor. It means that the conductor has become negatively charged by induction. In the same way one can induce a positive charge on a conductor by bringing a negative charged rod near it.





Important points regarding electrostatic induction

(a) Inducing body neither gains nor loses charges.

(b) The nature of induced charge is always opposite to that of inducing charge.

(c) Induced charge can be lesser or equal to inducing charge but it is never greater than the inducing charge.

(d) Induction takes place only in bodies (either conducting or nor conducting) and not in particles.

(iii) Charging by conduction

Let us consider two conductors, one charged and the other uncharged. We bring the conductors in contact with each other. The charge (whether negative or positive) under its own repulsion will spread over both the conductors. Thus the conductors will be charged with the same sign. This is called 'charging by conduction (through contact)'.

1.2 UNIT OF CHARGE

In SI units as current is assumed to be fundamental quantity and $I = \left(\frac{q}{t}\right)$, charge is a derived physical quantity with dimensions [AT] and unit (ampere × second) called 'coulomb (C)'.

The coulomb is related to CGS units of charge through the basic relation

$$1 \text{ coulomb} = 3 \times 10^9 \text{ esu of charge} = \frac{1}{10} \text{ emu of charge}$$

1.3 PROPERTIES OF CHARGE

(i) Charge is always associated with mass

The charge can not exist without mass though mass can exist without charge.

(ii) Charge is quantized

When a physical quantity can have only discrete values rather than any value, the quantity is said to be quantised.

Several experiments have established that the smallest charge that can exist in nature is the charge of an electron. If the charge of an election (= 1.6×10^{-19}C) is taken as the elementary unit, i.e., quanta of charge, and is denoted by e, the charge on any body will be some integral multiple of e, i.e.,

$$q = \pm ne \; ; n = 1, 2, 3, \ldots\ldots\ldots \qquad \ldots \textbf{(1)}$$

charge on a body can never be $\left(\frac{2e}{3}\right)$, (17.2) e or (10^{-5}) e etc.

(iii) Charge is conserved

A large number of experiments show that in an isolated system, total charge does not change with time, though individual charges may change, i.e., charge can neither be created nor be destroyed. This is known as the principle of conservation of charge.

(iv) Charge is invariant

This means that charge is independent of frame of reference, i.e., charge on a body does not change whatever be its speed.

1.4 CONDUCTORS AND INSULATORS

The conductors are materials, which allow electricity (electric charge) to pass through them due to the presence of free elections. e.g., metals are good conductors.

The insulators are materials, which do not allow electric charge to pass through them as there is no free electrons in them. e.g. wood, plastics, glass etc.

Illustration 1.

How many electrons must be removed from a piece of metal so as to leave it with a positive charge of 10^{-7} coulomb?

Solution:

From 'Quantization of charge', we know $Q = ne$

$$\therefore \quad n = \frac{Q}{e} = \frac{10^{-7}C}{1.6\times10^{-19}C} = \mathbf{6.3 \times 10^{11}}$$

Illustration 2.

A copper penny has a mass of 3.1 g. Being electrically neutral, it contains equal amounts of positive and negative charges. What is the magnitude of these charges? A copper atom has a positive nuclear charge of 4.6×10^{-18} C. Atomic weight of copper is 64g/mole and Avogadro's number is 6×10^{23} atoms/mole.

Solution:

1 mole i.e., 64 g of copper has 6×10^{23} atoms. Therefore, the number of atoms in copper penny of 3.1g is

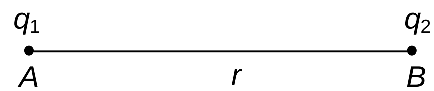
$$\frac{6\times10^{23}}{64}\times3.1 = 2.9 \times 10^{22}$$

One atom of copper has each positive and negative charge of 4.6×10^{-18} C. So each charge on the penny is

$$(4.6 \times 10^{-18}) \times (2.9 \times 10^{22}) = \mathbf{1.3 \times 10^5\ C.}$$

2. COULOMB'S LAW

"Two stationary point charges repel or attract each-other with a force which is directly proportional to the product of the magnitudes of their charges and inversely proportional to the square of the distance between them."



Let 'r' be the distance between two point charges q_1 and q_2.

According to Coulomb's law, we have $F \propto \frac{|q_1||q_2|}{r^2}$

where F is the magnitude of the mutual force that acts on each of the two charges q_1 and q_2.

or, $F = \frac{K|q_1||q_2|}{r^2}$, where K is a constant of proportionality

The value of K depends upon the medium in which two point charges are placed.

In the SI system. $K = \frac{1}{4\pi\varepsilon_0}$ for vacuum (or air)

The constant ε_0 (= 8.85×10^{-12} $C^2/N\text{-}m^2$) is called "permittivity" of the free space. Thus

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} \approx 9 \times 10^9 \frac{|q_1||q_2|}{r^2} \qquad \ldots \textbf{(2)}$$

2.1 PERMITTIVITY OF A MEDIUIM

If the medium between the two point charges q_1 and q_2 is not a vacuum (or air). Then the electrostatic force between the two charges becomes

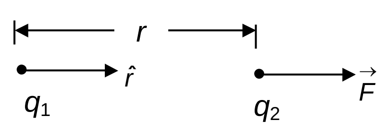
$$F = \frac{1}{4\pi\varepsilon}\frac{|q_1||q_2|}{r^2} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{|q_1||q_2|}{r^2} \qquad \ldots \textbf{(3)}$$

where $\varepsilon = \varepsilon_0\,\varepsilon_r$ is called the 'absolute permittivity' or 'permittivity' of the medium and ε_r is a dimensionless constant called 'relative permittivity' of the medium which is a constant for a given medium. ε_r is also sometimes called "dielectric constant' or 'specific inductive capacity' of the medium.

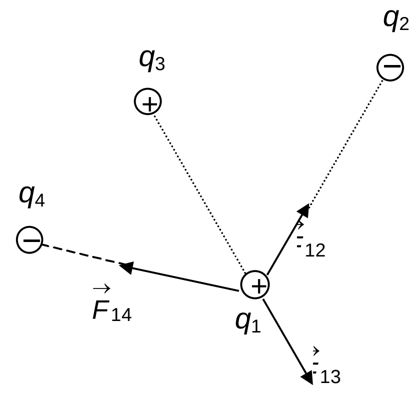
2.2 COLOUMB'S LAW IN VECTOR FORM

The vector form of Coulomb's law is $\vec{F} = \frac{Kq_1q_2}{r^2}\,\hat{r}$ $\qquad \ldots$ **(4)**

The unit vector $\hat{r}$ has its origin at the 'source of the force". For example, to find the force on q_2, the origin of $\hat{r}$ is at q_1. The signs of the charges must be explicitly included in equation (4). If F is the magnitude of the force, then $\vec{F} = +F\hat{r}$ means a repulsion, whereas $\vec{F} = -F\hat{r}$ means an attraction.



3. PRINCIPLE OF SUPERPOSITION



According to the principle of superposition, the force acting on one charge due to another is independent of the presence of charges. So, we can calculate the force separately for each pair of charges and then take their vector sum or find the net force on any charge.

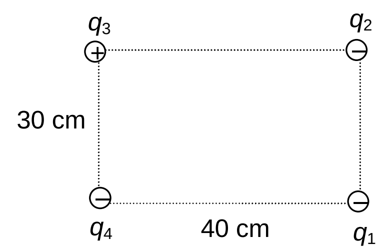
The figure shows a charge q_1 interacting with other charges. Thus, to find the force on q_1, we first calculate the forces exerted by each of the other charges, one at a time. The net force $\vec{F}_1$ on q_1 is simply the vector sum

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \qquad \ldots \textbf{(5)}$$

where $\vec{F}_{12}$ is the force on the charge q_1 due to the charge q_2 and so on.

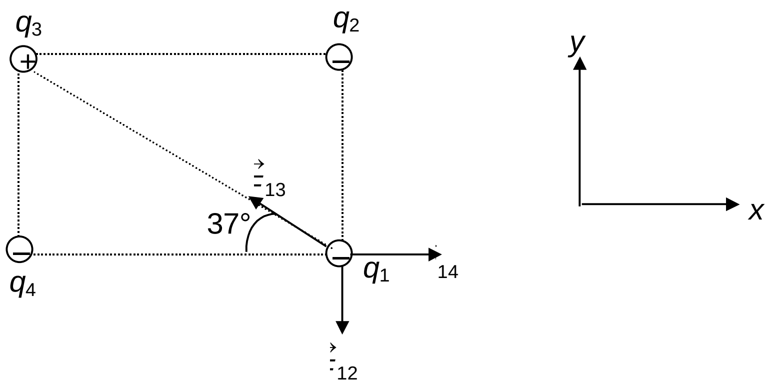
Illustration 3.

Find the net force on charge q_1 due to the three other charges as shown in the figure. Take q_1 = 50 μC; q_2 = –8 μC; q_3 = 15 μC, and q_4 = – 16 μC



Solution:

The directions of the forces on q_1 and the co-ordinate axes are shown below.



Now, $F_{12} = 9 \times 10^9 \dfrac{|q_1||q_2|}{r^2}$

$$= \frac{(9\times10^9 N - m^2/c^2 \times (5\times10^{-6} C)\times(8\times10^{-6} C)}{(3\times10^{-1} m)^2} = 4 \text{ N}$$

Similarly, F_{13} = 2.7 N and F_{14} = 4.5 N

The component of the net force are

$$(F_1)_x = 0 - F_{13} \cos 37° + F_{14} = 2.3 \text{ N}$$

$(F_1)y = -F_{12} + F_{13}\sin 37° + 0 = -2.4$ N

The net force on q_1 is $\vec{F}_1 = \mathbf{2.3\ \hat{i}\ - 2.4\ \hat{j}\ N}$

Illustration 4.

The electron and the proton in a hydrogen atom are 0.53×10^{-11} m apart. Compare the electrostatic and the gravitational forces between them.

Solution:

The magnitude of the electrostatic force is

$$F_E = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

$$= \frac{(9\times10^9\,\mathrm{N-m^2/C^2})\times(1.6\times10^{-19}\,\mathrm{C})^2}{(5.3\times10^{-11}\mathrm{m})^2}$$

$$= 8.2 \times 10^{-8}\ \mathrm{N}$$

The magnitude of the gravitational force is

$$F_G = G\frac{m_e m_p}{r^2}$$

$$= \frac{(6.67\times10^{-11}\mathrm{N-m^2/kg^2})(9.11\times10^{-31}\,\mathrm{kg})(1.67\times10^{-27}\,\mathrm{kg})}{(5.3\times10^{-11}\mathrm{m})^2}$$

$$= 3.6 \times 10^{-47}\ \mathrm{N}$$

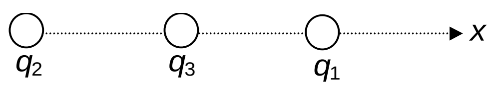
The ratio of the forces

$$\frac{F_G}{F_E} = \mathbf{4.4 \times 10^{-40}}$$

Note: The $\frac{F_G}{F_E}$ is extremely small. So when we deal with the electrical interaction between elementary particles, gravity may safely be ignored.

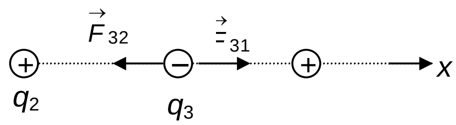
Illustration 5.

Three charges lie along the x-axis as shown in the figure. The positive charge q_1 = 15.0 μC is at x = 2.0 m, and the positive charge q_2 = 6.00 μC is at the origin. Where must a negative charge q_3 be placed on the x-axis such that the resultant force on it is zero?



Solution:

Since q_3 is negative and both q_1 and q_2 are positive, the forces $\vec{F}_{31}$ and $\vec{F}_{32}$ are both attractive. Let x be the co-ordinate of q_3



We have

$$F_{31} = \frac{K|q_3|\ |q_1|}{(2-x)^2}\ ;\ F_{32} = \frac{K|q_3|\ |q_2|}{x^2}$$

Since the net force on the change q_3 is zero,

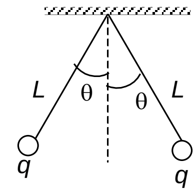
we have, $\dfrac{K|q_3|\ |q_2|}{x^2} = \dfrac{K|q_3|\ |q_1|}{(2-x)^2}$

or, $(4 - 4x + x^2)\ (6 \times 10^{-6}\text{C}) = x^2\ (15 \times 10^{-6}\ \text{C})$

Solving this quadratic equation for x, we get x = **0.775 m.**

Illustration 5.

Two identical small charged sphere, each having a mass of 3.0×10^{-2} kg, hang in equilibrium as shown below. If the length of each string is 0.15 m and the angle $\theta = 5°$, find the magnitude of the charge on each sphere.

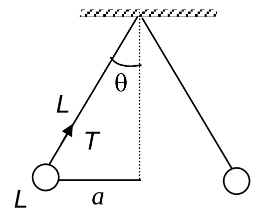


Solution:

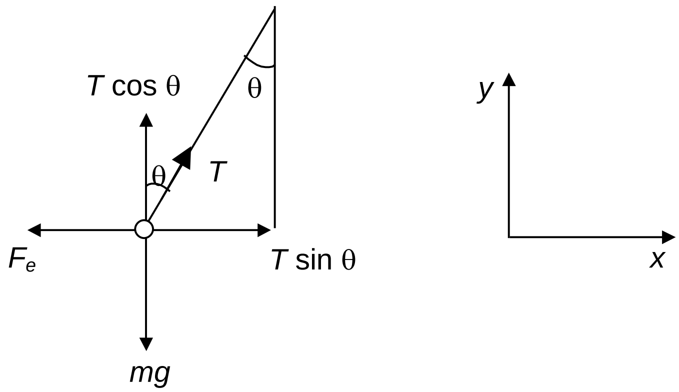
From the right angled triangle, we have $\sin\theta = \dfrac{a}{L}$

or, $a = L \sin\theta$ = (15 m) sin 5° = 0.013 m

Hence, the separation of the spheres is $2a$ = 0.026 m



F.B.D. of one of the spheres:-



Since the sphere is in equilibrium, the resultants of the forces in the horizontal and vertical directions must separately add up to zero. thus

$T \sin\theta - F_e = 0$

$\Rightarrow \quad T \sin\theta = F_e \qquad \ldots$ (i)

and $\quad T \cos\theta - mg = 0$

$\Rightarrow \quad T \cos\theta = mg \qquad \ldots$ (ii)

Dividing equation (i) by equation (ii) , we get

$$\tan\theta = \frac{F_e}{mg} \quad \text{or,} \quad F_e = mg\tan\theta$$

$$= (3 \times 10^{-2}\text{ kg}) \times (9.8\text{ m/s}^2)(\tan 5°)$$

$$= 2.6 \times 10^{-2}\text{ N}$$

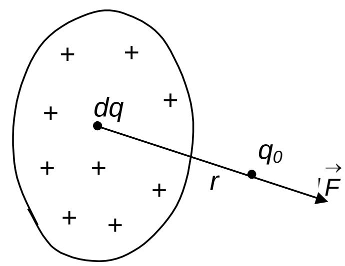
Let q be charge on each sphere.

According to Coulomb's law

$$F_e = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{|q||q|}{r^2}$$

or, $2.6 \times 10^{-2} = \dfrac{(9\times10^9\text{ N}-\text{m}^2/\text{C}^2)q^2}{(.026\text{m})^2}$ or, $|q| = \mathbf{4.4 \times 10^{-8}\ C}$

4. FORCE DUE TO CONTINUOUS CHARGE DISTRIBUTIONS



To find the force exerted by a continuous charge distribution on a point charge, we divide the charge into infinitesimal charge elements. Each infinitesimal charge element is then considered as a point charge.

The magnitude of the force dF exerted by the charge dq on the charge q_0 is given by

$$dF = \frac{1}{4\pi\varepsilon_0}\frac{|dq||q_0|}{r^2},$$

Where r is the distance between dq and q_0. The total force is then found by adding all the infinitesimal force elements, which involves the integral

$$\vec{F} = \int d\vec{F}$$

Taking $d\vec{F} = dF_x\,\hat{i} + dF_y\,\hat{j} + dF_z\hat{K}$, we have

$$\left.\begin{aligned} F_x &= \int dF_x \\ F_y &= \int dF_y \\ F_z &= \int dF_y \end{aligned}\right] \qquad \dots (\mathbf{6})$$

Because of the vector nature of the integration, the mathematical procedure must be carried out with care. The symmetry of charge distribution will usually result in a simplified calculation.

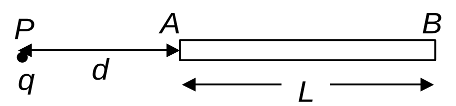
Each type of charge distribution is described (in the table given below) by an appropriate Greek letter parameter: λ, σ or ρ.

How we choose the charge element dq depends upon the particular type of the charge distribution.

Charge distribution	Relevant parameter	SI units	Charge element dq
Along a line	λ, charge per unit length	C/m	$dq = \lambda\, dx$
On a surface	σ, charge per unit area	C/m^2	$dq = \sigma\, dA$
Throughout a volume	ρ, charge per unit volume	C/m^3	$dq = \rho dV$

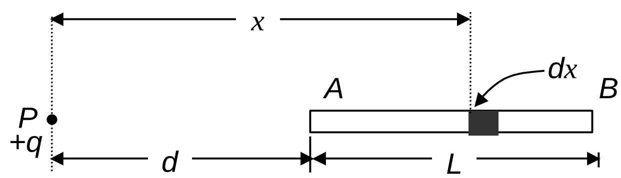
Illustration 7.

A point charge q is situated at a distance 'd' from one end of a thin non-conducting rod of length L having a charge Q (uniformly distributed along its length) as shown. Find the magnitude of the electric force between the two.



Solution:

Consider an element of rod of length 'dx' at a distance x from the point charge q. Treating the element as a point charge, the force between q and the charge element will be



$$dF = \frac{1}{4\pi\varepsilon_0}\frac{qdQ}{x^2}$$

But $\quad dQ = \left(\frac{Q}{L}\right) dx$

So, $\quad dF = \frac{1}{4\pi\varepsilon_0}\frac{qQdx}{Lx^2}$

$\therefore \quad F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{L}\int_d^{(d+L)}\frac{dx}{x^2}$

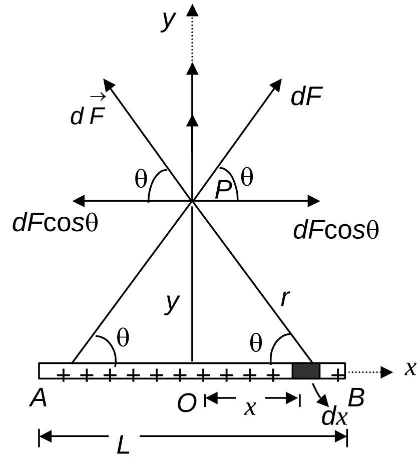
$$= \frac{1}{4\pi\varepsilon_0}\left[-\frac{1}{x}\right]_d^{d+L}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qQ}{L}\left[\frac{1}{d}-\frac{1}{d+L}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qQ}{d(d+L)}$$

Illustration 8.

Consider an element of rod of length 'dx 'at a distance x from the centre. Treating the element as a point charge, the force between q and the charge element will be



$$dF = \frac{1}{4\pi\varepsilon_0}\frac{q_0\ dq}{r^2}$$

The direction of $d\vec{F}$ is shown in the figure. But $dq = \lambda dx$

So, $$dF = \frac{1}{4\pi\varepsilon_0}\frac{q_0\lambda dx}{r^2}$$

Solution:

For every charge element dq located at position $+x$, there is another charge element dq located at $-x$. When we add the forces due to the charge elements at $+x$ and $-x$, we find the x components have equal magnitudes but point in opposite directions. So their sum is zero, i.e., $F_x = 0$

Now, $dF_y = dF \sin\theta$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_0\lambda dx}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}}$$

$\left(\because r^2 = x^2 + y^2;\ \sin\theta = \frac{y}{\sqrt{x^2+y^2}}\right)$

or, $$dF_y = \frac{1}{4\pi\varepsilon_0}\frac{q_0\lambda y dx}{(x^2+y^2)^{3/2}}$$

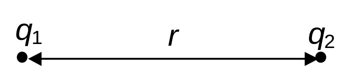
$$\therefore \qquad F_y = \int dF_y = \frac{1}{4\pi\varepsilon_0} q_0\lambda\, y \int_{-\frac{L}{2}}^{+\frac{L}{2}} \frac{dx}{(x^2+y^2)^{3/2}}$$

Evaluating the integral, we obtain

$$F_y = \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{y\sqrt{y^2+\frac{L^2}{4}}}$$

5. ELECTRIC FIELD

An electric field is defined as a region in which there should be a force on a charge brought into that region. Whenever a charge is being placed in an electric field, it experiences a force.



Electric fields that we will study are usually produced by different types of charged bodies – point charges, charged plates. Charged sphere etc. We can also define the electric field of a charged body as its region of influence within which it will exerts force on other charges.

If two point charges are placed as shown, we describe the forces on them in two ways

(i) The charge q_2 is in the electric field of charge q_1. Thus the electric field of charge q_1 exerts force on q_2.

(ii) The charge q_1 is in the electric field of charge q_2. Hence the electric field of charge q_2 exerts a force on q_1.

Electric field $\xrightarrow{\text{exerts force on}}$ charges inside it.

Electric field $\xrightarrow{\text{is created by}}$ charged bodies.

5.1 ELECTRIC INTENSITY OR ELECTRIC FIELD STRENGTH ($\vec{E}$)

The electric field intensity at a point in an electric field is the force experienced by a unit positive charge placed at that point, it is being assumed that the unit charge does not affect the field.

Thus, if a positive test charge q_0 experiences a force $\vec{F}$ at a point in an electric field, then the electric field intensity $\vec{E}$ at that point is given by

$$\vec{E} = \frac{\vec{F}}{q_0} \qquad \dots (7)$$

Important points regarding electric Intensity

(i) It is a vector quantity. The direction of the electric field intensity at a point inside the electric field is the direction in which the electric field exerts force on a (unit) positive charge.

(ii) Dimensions of the electric field intensity

$$E = \frac{F}{q_0} = \frac{[MLT^{-2}]}{[AT]} = [MLT^{-3}A^{-1}]$$

In S.I. systems, the unit of $\vec{E}$ is N/C or V/m as

$$\frac{\mathrm{N}}{\mathrm{C}} = \frac{\mathrm{N}\times\mathrm{m}}{\mathrm{C}\times\mathrm{m}} = \frac{\mathrm{J}}{\mathrm{C}\times\mathrm{m}} = \frac{\mathrm{V}}{\mathrm{m}}$$

5.2 FORCED EXERTED BY A FIELD ON A CHARGE INSIDE IT

By definition as $\vec{E} = \dfrac{\vec{F}}{q_0}$, i.e., $\vec{F} = q_0\,\vec{E}$

If q_0 is a +ve charge, force $\vec{F}$ on it is in the direction of $\vec{E}$.

If q_0 is a –ve charge, $\vec{F}$ on it is opposite to the direction of $\vec{E}$

Illustration 9.

An electron ($q = -e$) is placed near a charged body experiences a force in the positive y direction of magnitude 3.60×10^{-8} N.

(a) What is the electric field at that location?

(b) What would be the force exerted by the same charged body on an alpha particle ($q = +2e$) placed at the location initially occupied by the electron?

Solution:

Using equation (7), we have

$$E_y = \frac{|F_y|}{|q_0|} = \frac{3.60 \times 10^{-8}\,\text{N}}{1.60 \times 10^{-19}\,\text{C}} = \mathbf{2.25 \times 10^{11}\ N/C}.$$

The electric field is in the negative y direction.

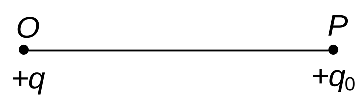
(b) The force on the alpha particle is given by

$$F_y = q_0 E_y = 2(+\,1.60 \times 10^{-19}\ \text{C})\,(2.25 \times 10^{11}\ \text{N/C}) = \mathbf{7.20 \times 10^{-8}\ N}$$

The force is in the negative y direction, the same direction as the electric field.

5.3 ELECTRIC FIELD INTENSITY DUE TO A POINT CHARGE

Let a positive test charge q_0 be placed at a distance r from a point charge q. The magnitude of force acting on q_0 is given by Coulomb's law,



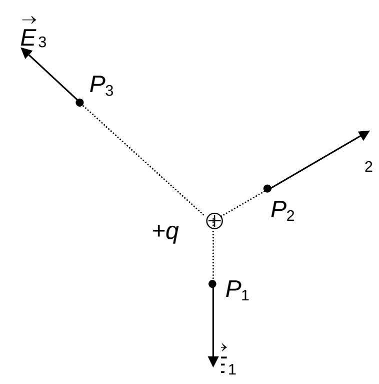
$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r^2}$$

The magnitude of the electric field at the site of the charge is

$$E = \frac{F}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \ldots\ (\mathbf{8})$$

The direction of $\vec{E}$ is the same as the direction of $\vec{F}$, along a radial line from q, pointing outward if q is positive and negative if q is negative.

The figure given below shows the direction of the electric field $\vec{E}$ at various points near a positive point charge.



5.4 ELECTRIC FIELD INTENSITY DUE TO A GROUP OF POINT CHARGES

Since the principle of linear superposition is valid for Coulomb's law, it is also valid for the electric field. To calculate the electric field strength at a point due to a group of N point charges. We first find the individual field strengths $\vec{E}_1$ due to Q_1, $\vec{E}_2$ due to Q_2, and so on

The resultant field strength is the vector sum of individual field strengths.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots$$

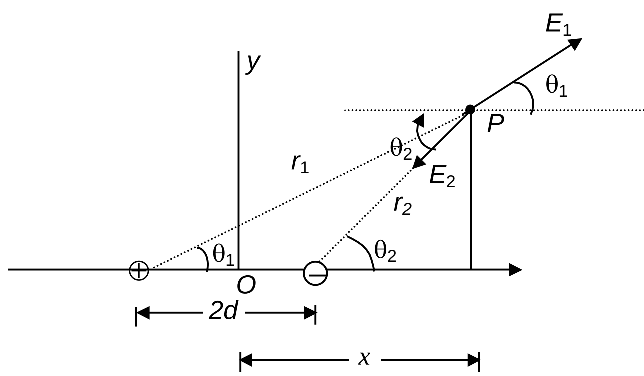
$$= \sum \vec{E}_n \ (n = 1, 2, 3, \ldots N)$$

Illustration 10.

A point charge Q_1 = 20 μC is at (–d, 0) while Q_2 = –10 μC is at (+ d, 0). Find the resultant field strength at a point with co-ordinates (x, y). Take d = 1.0 m and $x = y$ = 2m

Solution:

We have



$$r_1 = \sqrt{(x+d)^2 + y^2}$$

$$= \sqrt{13} = 3.6 \text{ m}$$

$$r_2 = \sqrt{(x-d)^2 + y^2} = \sqrt{5} = 2.2 \text{ m}$$

The magnitudes of the fields are

$$E_1 = \frac{1}{4\pi\varepsilon_0} \frac{|Q_1|}{r_1^2}$$

$$= \frac{(9.0\times10^9 \text{N.m}^2/\text{C}^2 (2\times10^{-5}\text{C})}{13\text{m}^2}$$

$$= 1.4 \times 10^4 \text{ N/C}$$

$$E_2 = \frac{1}{4\pi\varepsilon_0} \frac{|Q_2|}{r_2^2} = \frac{(9\times10^9 \text{N.m}^2/\text{C}^2 (10^{-5}\text{C})}{5\text{m}^2}$$

$$= 1.8 \times 10^4 \text{ N/C}$$

The components of the resultant field strength are

$$E_x = E_{1x} + E_{2x} = E_1 \cos\theta_1 - E_2 \cos\theta_2$$

$$E_y = E_{1y} + E_{2y} = E_1 \sin\theta_1 - E_2 \sin\theta_2$$

From the above figure, we see that $\sin\theta_1 = \frac{y}{r_1}$; $\sin\theta_2 = \frac{y}{r_2}$; $\cos\theta = \left(\frac{x+d}{n}\right)$;

and $\cos\theta_2 = \frac{x-d}{r_2}$

Therefore,

$$E_x = (1.4 \times 10^4 \text{ N/C}) \frac{3}{3.6} - (1.8 \times 10^4 \text{ N/C}) \frac{1.0}{2.2}$$

$$= 3.5 \times 10^3 \text{ N/C}$$

$$E_y = (1.4 \times 10^4 \text{ N/C}) \frac{2}{3.6} - (1.8 \times 10^4 \text{ N/C}) \frac{2}{2.2}$$

$$= -8.6 \times 10^3 \text{ N/C}$$

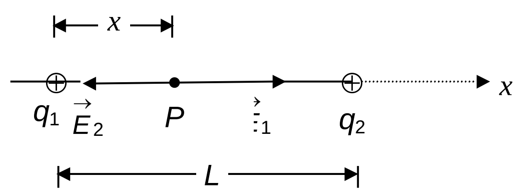
Hence, $\vec{E}$ = $\mathbf{3.5 \times 10^3}\ \hat{i}$ **-** $\mathbf{8.6 \times 10^3}\ \hat{j}$ **N/C**

Illustration 11.

A point charge q_1 of + 1.5 μC is placed at a origin of a co-ordinate system, and the second charge q_2 of + 2.3 μC is at a position $x = L$, where L = 13 cm. At what point P along the x-axis is the electric field zero?

Solution:

The point must lie between the charges because only in this region the forces exerted by q_1 and q_2 on a test charge oppose each-other. If $\vec{E}_1$ is the electric field due to q_1 and $\vec{E}_2$ is that due to q_2, the magnitudes of these vectors must be equal, or



$$E_1 = E_2$$

We then have

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{(L-x)^2}$$, where x is the co-ordinate of the point P.

Solving for x, we have

$$x = \frac{L}{1 \pm \sqrt{q_2/q_1}}$$

Substituting numerical values for L, q_1 and q_2,

We obtain

$$x = 5.8 \text{ cm and } x = -54.6 \text{ cm}$$

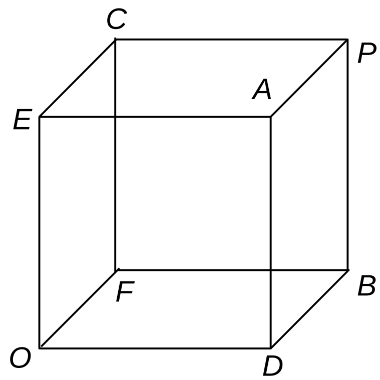
But the negative value of x is unacceptable

Hence, x = **5.8 cm**

Illustration 12.

A cube of edge '*a*' carries a point charge *q* at each corner. Calculate the resultant force on any one of the charges.

Solution:



Let us take one corner of the cube as origin O (0, 0, 0) and the opposite corner *P* as (*a*, *a*, *a*). We will calculate the electric field at *P* due to the other seven charges at corners.

Expressing the field of a point charge in vector form

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

Electric field strength at *P* due to charges at *A*, *B* and *C*

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}[\vec{AP}+\vec{BP}+\vec{CP}] = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}(\hat{i}+\hat{j}+\hat{k})$$

Electric field strength at *P* due to charges at *D*, *E* and *F*

We have,

$$DP = EP = FP = (a\sqrt{2})$$

$$\vec{E}_2 = \frac{q}{4\pi\varepsilon_0(a\sqrt{2})^3}\left[\vec{DP}+\vec{EP}+\vec{FP}\right]$$

$$= \frac{q}{4\pi\varepsilon_0(2\sqrt{2}a^3)}\left[(a\hat{j}+a\hat{k}) + (a\hat{i}+a\hat{j}) + (a\hat{i}+a\hat{k})\right]$$

$$= \frac{q}{(4\pi\varepsilon_0)(\sqrt{2}a^2)}[\hat{i}+\hat{j}+\hat{k}]$$

Electric field strength at *P* due to *O*

We have

$$OP = a\sqrt{3}$$

$$\vec{E}_3 = \frac{q}{4\pi\varepsilon_0\,(a\sqrt{3})^3}\vec{OP}$$

$$= \frac{q}{4\pi\varepsilon_0(3\sqrt{3}a^3)}[a\hat{i}+a\hat{j}+a\hat{k}]$$

$$= \frac{q}{4\pi\varepsilon_{-}(3\sqrt{3}a^2)}[\hat{i}+\hat{j}+\hat{k}]$$

Hence, the resultant field at P

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$= \frac{q[\hat{i}+\hat{j}+\hat{k})}{4\pi\varepsilon_0\ a^2}\ [1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}] \text{ outward}$$

along OP

Force on the charge at P is $F = qE$

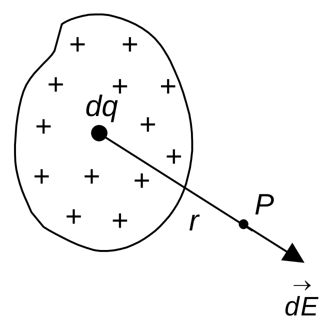
$$\Rightarrow \qquad F = \frac{q^2\sqrt{3}}{4\pi\varepsilon_0 \boldsymbol{a}^2}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{3\sqrt{3}}\right]$$

Outwards along diagonal OP

6. ELECTRIC FIELD OF CONTINUOUS CHARGE DISTRIBUTIONS

To find the field of a continuous charged distribution, we divide the charge into infinitesimal charge elements. Each infinitesimal charge element is then considered as a point charge and its field is given by

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$



At a point distant r from the element, the net field is the summation of fields of all the elements.

$$\vec{E} = \int d\vec{E}$$

Taking $d\vec{E} = dE_x\ \hat{i} + dE_y\hat{j} + dE_z\hat{k}$, we have $E_x = \int dE_x$, $E_y = \int dE_y$ and $E_z = \square dE_z$

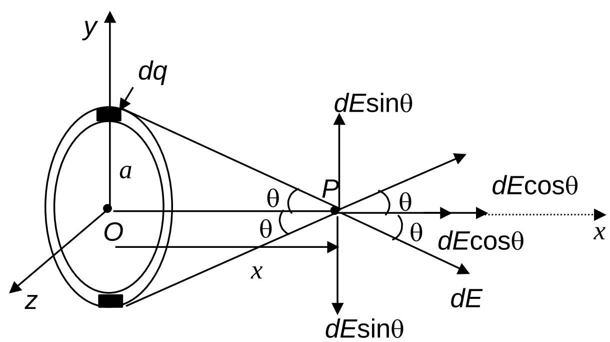
Because of the vector nature of the integration, the mathematical procedure must be carried out with care. Fortunately, in the cases we consider, the symmetry of the charge distribution usually results in a simplified calculation.

Each type of charge distribution is described (in the table given below) by an appropriate Greek-letter parameter: λ, σ or ρ. How we choose the charge element '*dq*' depends upon the particular type of charge distribution.

Charge Distribution	Relevant Parameter	SI Units	Charge Element dq
Along a line	λ, Charge per unit length	C/m	$dq = \lambda\, dx$
On a surface	σ, Charge per unit area	C/m^2	$dq = \sigma\, dA$
Throughout a volume	ρ, Charge per unit volume	C/m^3	$dq = \rho\, dV$

6.1 ELECTRIC FIELD DUE TO A UNIFORMLY CHARGE RING AT A POINT ON THE AXIS OF THE RING

Let us consider a charge Q distributed uniformly on a thin, circular, non-conducting ring of radius a. We have to find electric field E at a point P on the axis of the ring, at a distance x from the centre.



From symmetry, we observe that every element dq can be paired with a similar element on the opposite side of the ring. Every component $dE\sin\theta$ perpendicular to the x-axis is thus cancelled by a component $dE\sin\theta$ in the opposite direction. In a summation process, all the perpendicular components add to zero. Thus we only add the dE_x components.

Now, $$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(d^2 + x^2)}$$

Hence, the resultant electric field at P is given by

$$E = \int dE_x$$

$$= \int dE\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{(a^2 + x^2)}\left(\frac{x}{\sqrt{a^2 + x^2}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{x}{(a^2 + x^2)^{3/2}}\int dq$$

As we integrate around the ring, all the terms remain constant and $\int dq = Q$

So, the total field is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{x}Q}{(a^2 + \boldsymbol{x}^2)^{3/2}}$$

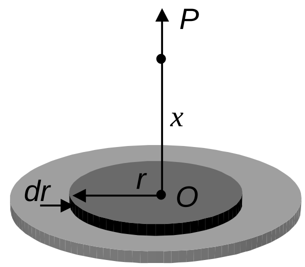
or $$E = \frac{1}{4\pi\varepsilon_0}\frac{Q\boldsymbol{x}}{(a^2 + \boldsymbol{x}^2)^{3/2}} \qquad \ldots \textbf{(9A)}$$

6.2 ELECTRIC FIELD DUE TO A UNIFORMLY CHARGE DISC AT A POINT ON THE AXIS OF THE DISC

Let us consider a flat, circular, non-conducting thin disc of radius R having a uniform surface charge density σ c/m^2. We have to find the electric field intensity at an axial point at a distance x from the disc.

Let O be the centre of a uniformly charged disc of radius R and surface charge density σ. Let P be an axial point, distant x from O, at which electric field intensity is required.

From the circular symmetry of the disc, we imagine the disc to be made up of a large number of concentric circular rings and consider one such ring of radius r and an infinitesimally small width dr



The area of the elemental ring = circumference x width = ($2\pi\, rdr$)

The charge dq on the elemental ring = $(2\pi rdr)\,\sigma$

Therefore, the electric field intensity at P due to the elementary ring is given by

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{(2\pi rdr)\,\sigma x}{(r^2+x^2)^{3/2}} = \frac{\sigma x}{2\varepsilon_0}\frac{rdr}{(r^2+x^2)^{3/2}},$$

and is directed along the x-axis. Hence, the electric intensity E due to the whole disc is given by

$$E = \frac{\sigma x}{2\varepsilon_0}\int_O^R \frac{rdr}{(r^2+x^2)^{3/2}}$$

$$= \frac{\sigma x}{2\varepsilon_0}\left[-\frac{1}{(r^2+x^2)^{1/2}}\right]_0^R = \frac{\sigma x}{2\varepsilon_0}\left[-\frac{1}{(R^2+x^2)^{1/2}}+\frac{1}{x}\right]$$

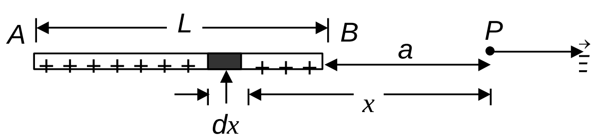
$$= \frac{\sigma}{\varepsilon_0}\left[1-\frac{x}{(R^2+x^2)^{1/2}}\right] \qquad \ldots \textbf{(9B)}$$

Illustration 13.

A thin insulating rod of length L carries a uniformly distributed charge Q. Find the electric field strength at a point along its axis a distance 'a' from one end.

Solution:

Let us consider an infinitesimal element of length dx at a distance x from the point P. The charge on this element is $dq = \lambda dx$, where λ ($= \frac{Q}{L}$) is the linear charge density.



The magnitude of the electric field at P due to this element is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{(\lambda dx)}{x^2}$$

and its direction is to the right since λ is positive. The total electric field strength E is given by

$$E=\frac{1}{4\pi\varepsilon_0}\lambda\int_a^{a+L}\frac{dx}{x^2}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{x}\right]_a^{a+L}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{a+L}\right]$$

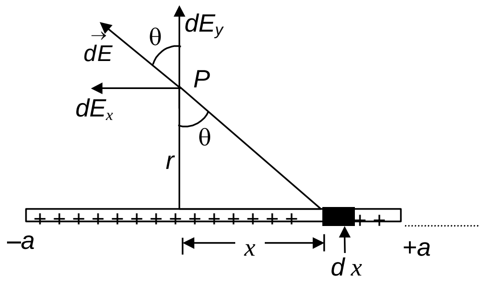
$$=\frac{Q}{(4\pi\varepsilon_0)\,a(a+L)}\quad(\because\ Q=\boldsymbol{\lambda L})$$

Illustration 14.

A uniform line charge λ (in Coulombs per meter) exists along the x-axis from $x = -a$ to $x = +a$. Find the electric field intensity E at a point P at a distance r along the perpendicular bisector.

Solution:

In all the problems, which involve distribution of charge, we choose an element of charge dq to find the element of the field dE produced at the given location. Then we sum all such dE's to find the total field E at that location.



We must note the symmetry of the situation. For each element dq located at positive x–axis, there is a similar dq (see mirror-images in origin) located at the same negative value of x. The dE_x produced by one dq is cancelled by the dE_x in the opposite direction due the other dq. Hence, all the dE_x components add to zero. So we need to sum only the dE_y components, a scalar sum since they all point in the same direction.

The element of charge is $dq = \lambda dx$

$$\therefore\qquad dE=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2+x^2}=\frac{\lambda dx}{4\pi\varepsilon_0(r^2+x^2)}$$

So, $$E=\int dE_y=\int dE\cos\theta$$

$$=\int\frac{\lambda dx}{4\pi\varepsilon_0(r^2+x^2)}\frac{r}{\sqrt{r^2+x^2}}=\frac{\lambda r}{4\pi\varepsilon_0}\int\frac{dx}{(r^2+x^2)^{3/2}}$$

The integral on the right hand side can be evaluated by substituting $x = r\tan\alpha$ and $dx = r\sec^2\alpha\, d\alpha$

$$\therefore\qquad \int\frac{dx}{(r^2+x^2)^{3/2}}=\int\frac{r\sec^2\alpha d\alpha}{r^3\sec^3\alpha}$$

$$= \int \frac{\cos\alpha}{r^2}\, d\alpha = \frac{\sin\alpha}{r^2}$$

$$\Rightarrow \qquad \int \frac{dx}{(r^2+x^2)^{3/2}} = \frac{1}{r^2}\frac{x}{\sqrt{x^2+r^2}}$$

$$\therefore \qquad E_y = \frac{\lambda r}{4\pi\varepsilon_0}\left|\frac{x}{r^2\sqrt{x^2+r^2}}\right|_{-a}^{+a} = \frac{\lambda r}{4\pi\varepsilon_0}\frac{2a}{r^2\sqrt{a^2+r^2}}$$

The net field at P is $E = E_y$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}\frac{a}{\sqrt{a^2+r^2}}$$

Note: In case of an infinite line charge, the field is everywhere perpendicular to line of charge. The field at a distance r from the line charge is calculated by taking $a \to \infty$ in the above result.

Hence, E due to infinite line charge is given by

$$E = \underset{a\to\infty}{Lt}\ \frac{\lambda}{2\pi\varepsilon_0 r}\ \frac{a}{\sqrt{a^2+r^2}}$$

$$\Rightarrow \qquad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

7. MOTION OF A CHARGED PARTICLE IN AN UNIFORM ELECTRIC FIELD

A particle of mass m and charge q in an uniform electric field $\vec{E}$ experiences a force

$$\vec{F} = q\vec{E}$$

From Newton's second law of motion,

$$\vec{F} = m\vec{a}$$

Hence, the acceleration of the charged particle in the uniform electric field is

$$\vec{a} = \frac{q\vec{E}}{m}$$

Since the field is uniform, the acceleration is constant in magnitude and direction. So we can use the equation of kinematics for constant acceleration. Now, there are two possibilities.

(a) If the particle is initially at rest

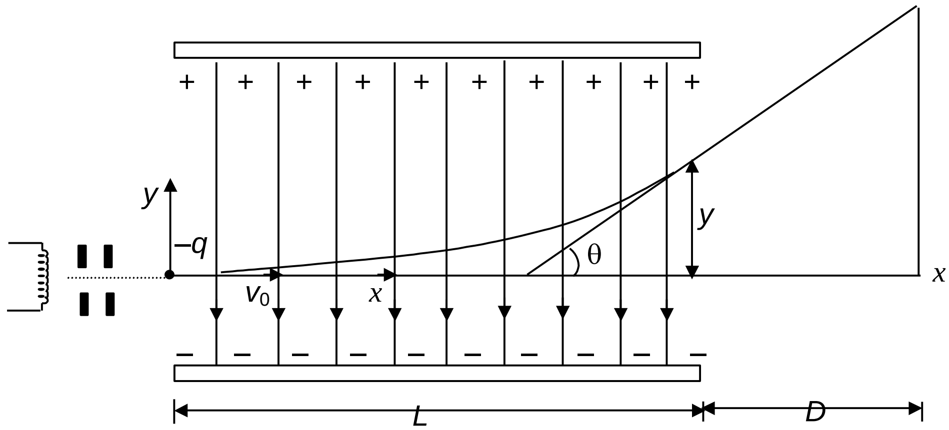
From equation $v = u + at$, we get

$$v = at = \frac{qE}{m}t \left[\because u = 0;\ a = \frac{qE}{m}\right]$$

From equation $S = ut + \frac{1}{2}at^2$, we have

$$S = \frac{1}{2}at^2 = \frac{qE}{2m}t^2$$

(b) (ii) If the particle is projected perpendicular to the field with an initial velocity v_0.



For motion along x-axis, we have $v_x = v_0$ = constant ($\because u = v_0$ and $a = 0$)

$\therefore \quad x = v_0 t \quad \ldots$ (i)

for motion along y-axis, we have

$$y = \frac{1}{2}\left[\frac{qE}{m}\right]t^2 \quad \ldots \text{(ii)}$$

$$\left[\because u = 0;\ a = \frac{qE}{m}\right]$$

Substituting the value of t from equation (i) in equation (ii),

we get

$$y = \frac{qE}{2m}\left[\frac{x}{v_0}\right]^2$$

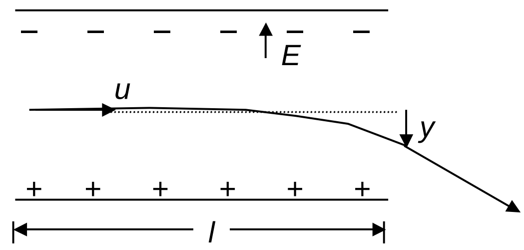
$$= \frac{qE}{2mv_0^2}x^2$$

Which is the equation of the parabola.

Illustration 15.

The electric field between the plates of a cathode ray oscillograph is 1.2×10^4 N/C. What deflection would on electron experience if it enters at right angles to the field with kinetic energy of 2 keV, the length of the plate being 1.5 cm?

Solution:



Let u = speed of the electron when it enters

We have

$$\frac{1}{2}mu^2 = 2000\,e$$

$$\Rightarrow \qquad u = \sqrt{\frac{4000e}{m}}$$

Let 't' be the time taken to cross the field. The components of acceleration are

$$a_x = 0;\ a_y = \frac{eE}{m} \text{ (taking downward direction as } +y)$$

For motion along x-axis, we have $l = ut$ … (i)

For motion along y-axis, we have $y=\frac{1}{2}\left(\frac{eE}{m}\right)t^2$ … (ii)

Substituting the value of t from equation (i) in equation (ii), we get

$$y=\frac{1}{2}\frac{eE}{m}\frac{l^2}{u^2} \Rightarrow y=\frac{1}{2}\frac{eE}{m}\frac{l^2m}{4000e}$$

$$y=\frac{El^2}{8000}=\frac{1.2\times10^4\times(1.5\times10^{-2})^2}{8000}$$

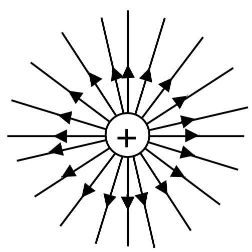
Hence, deflection = y = **3.375×10^{-4} m**

8. ELECTRIC LINES OF FORCE

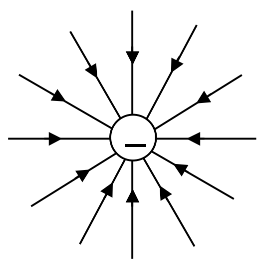
The idea of electric lines of force or the electric field lines introduced by Michel Faraday is a way to visualize electrostatic field geometrically.

The properties of electric lines of force are the following:

(i) The electric lines of force are continuous curves in an electric field starting from a positively charged body and ending on a negatively charged body.



Electric lines of force due to positive charge

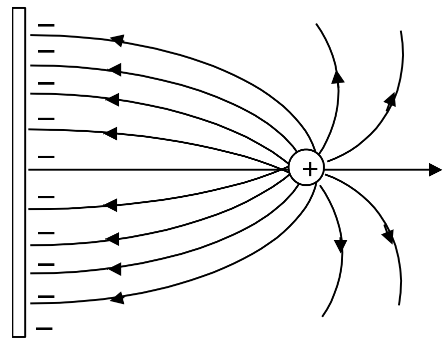


Electric lines of force due to negative charge

(ii) The tangent to the curve at any point gives the direction of the electric field intensity at that point.

(iii) Electric lines of force never intersect since if they cross at a point, electric field intensity at that point will have two directions, which is not possible.

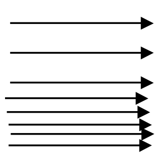
(iv) Electric lines of force do not pass but leave or end on a charged conductor normally. Suppose the lines of force are not perpendicular to the conductor surface. In this situation, the component of electric field parallel to the surface would cause the electrons to move and hence conductor will not remain equipotential which is absurd as in electrostatics conductor is an equipotential surface.



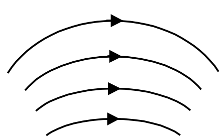
Fixed point charge near infinite metal plate

(v) The number of electric lines of force that originate from or terminate on a charge is proportional to the magnitude of the charge.

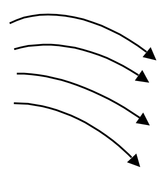
(vi) As number of lines of force per unit area normal to the area at point represents magnitude of intensity, crowded lines represent strong field while distant lines weak field. Further, if the lines of force are equidistant straight lines, the filed is uniform.



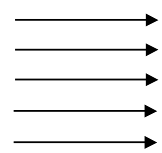
Magnitude is not constant



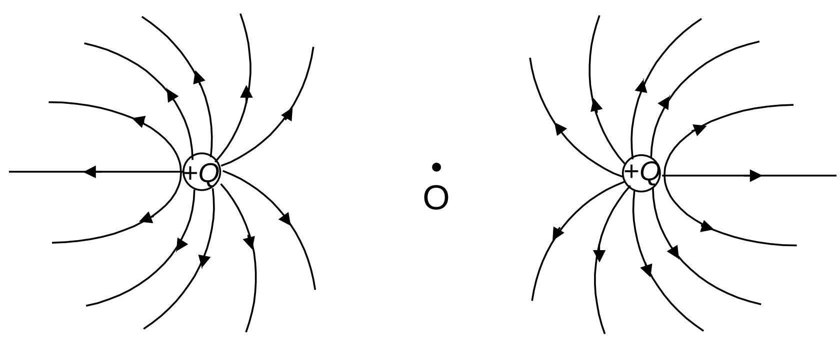
Direction is not constant



Both magnitude and direction not constant



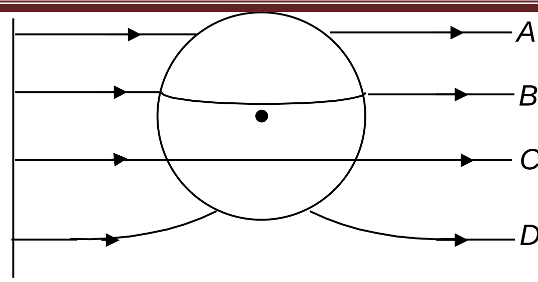
Both magnitude and direction constant.



Electric lines of force due to two equal positive charges (field is zero at 0). *O* is a null point.

Illustration 16.

A solid metallic sphere is placed in a uniform electric field. Which of the lines *A*, *B*, *C* and *D* shows the correct representation of lines of force and why?



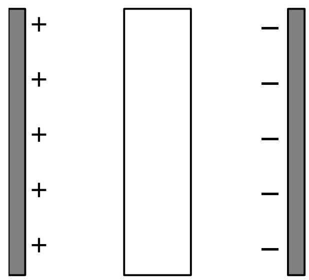
Solution:

(D)

The line (A) is wrong as lines of force start or end normally on the surface of a conductor and here it is not so. Line (B) and (C) are wrong as lines of force does not exist inside a conductor and here it is not so. Also lines of force are not normal to the surface of the conductor. Line (D) represents the correct situation, as here line of force does not exist inside the conductor and start and end normally on its surface.

Illustration 17.

A metallic slab is introduced between the two charged parallel plates as shown below. Sketch the electric lines of force between the plates.



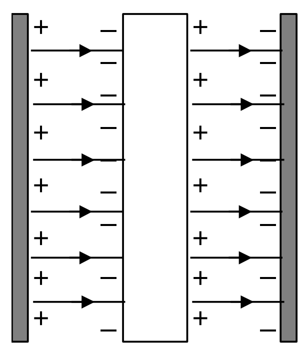
Solution:

Keeping in mind that

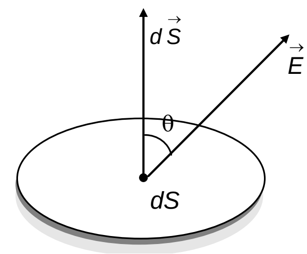
(i) Electric lines of force start from positive charge and end on negative charge.

(ii) Electric lines of force start and end normally on the surface of a conductor.

(iii) Electric lines of force do not exist inside a conductor, the lines of force are shown in the adjacent figure.



9. FLUX OF AN ELECTRIC FIELD OR ELECTRIC FULX (ϕ_E)



Let us consider a plane surface of area *S* placed in an electric field $\vec{E}$.

Electric flux through an elementary area $d\vec{S}$ is defined as the scalar product of $d\vec{S}$ and $\vec{E}$ i.e.

$d\phi_E = \vec{E}\,.\,d\vec{S}$, where $d\vec{S}$ is the area vector, whose magnitude is the area *ds* of the element and whose direction is along the outward normal to the elementary area.

Hence, the electric flux through the entire surface is given by

$$\phi_E = \int \vec{E}.d\vec{S} \qquad \ldots \textbf{(10)}$$

or, $$\phi_E = EdS\cos\theta$$

If the electric field is uniform, then $\phi_E = \int Eds\cos\theta = E\cos\theta \int ds$

When the electric flux through a closed surface is required, we use a small circular sign on the integration symbol ;

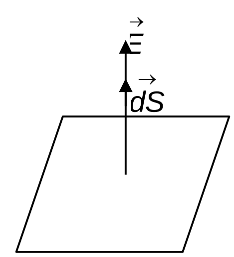
$$\phi_E = \oint \vec{E}.d\vec{S} \qquad \ldots \textbf{(11)}$$

Important points regarding electric flux:

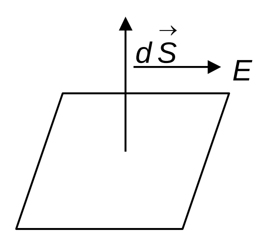
(i) The number of lines of force passing normally to the given area gives the measure of flux of electric field over the given area.

(ii) It is a real scalar physical quantity with units (volt × m).

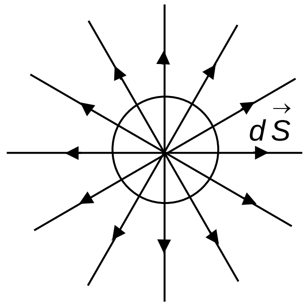
(iii) It will be maximum when cos θ = max. = 1, i.e., θ = 0°, i.e., electric field is normal to the surface with $(d\phi_E)_{max} = EdS$



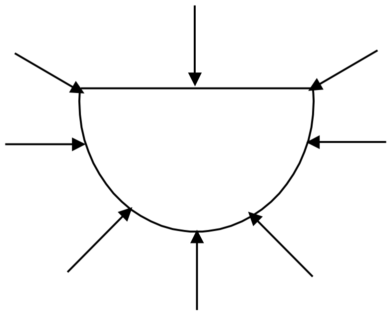
(iv) It will be minimum when |cos θ| = min = 0, i.e., θ = 90°, i.e., field is parallel to the area with $(d\phi_E)_{min} = 0$



(v) For a closed surface, ϕ_E is positive if the lines of force point outward everywhere ($\vec{E}$ will be outward everywhere, $\theta < 90°$ and $\vec{E}\,.\,d\vec{S}$ will be positive) and negative if they point inward ($\vec{E}$ is inward everywhere, $\theta > 90°$ and $\vec{E}.d\vec{S}$ will be negative)



Positive flux

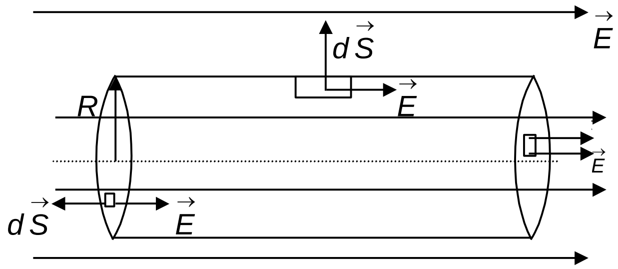


Negative flux

Illustration 18.

A closed cylinder of radius R is immersed in a uniform electric field $\vec{E}$ with its axis (a) parallel to the field, (b) perpendicular to the field. What is the electric flux through the closed cylinder in each case?

Solution:



Let us consider a closed cylinder of radius R, with its axis parallel to a uniform electric field $\vec{E}$. The electric flux through the entire surface is the sum of the electric flux through the left plane face, through the right plane face and through the curved surface.

That is, $\phi_E = \oint \vec{E}\,.\,d\vec{S}$

$$= \int \vec{E}.d\vec{S} + \int \vec{E}.d\vec{S} + \int \vec{E}.d\vec{S}$$

Left plane face Right plane face curved surface

The angle between $\vec{E}$ and $d\vec{S}$ is 180° for all patches on the left face, 0° for all patches on the right face and 90° for all patches on the curved surface. Therefore,

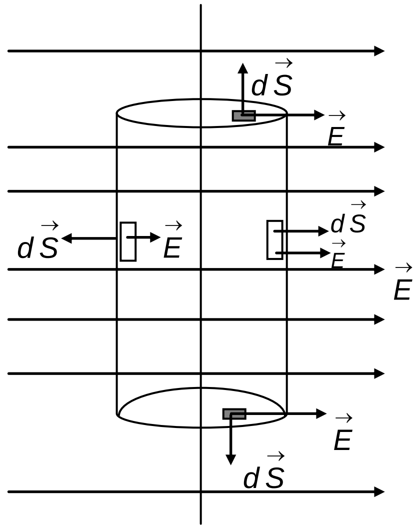
$$\phi_E = \int EdS \cos 180° + \int EdS \cos 0° + \int EdS \cos 90°$$

$$= -E\int dS + E\int dS + 0$$

$$= -E(\pi R^2) + E(\pi R^2) + 0 = 0$$

Thus, the flux through the entire cylinder is zero

(b)



Let us now consider the case when the cylinder is placed with its axis perpendicular to the field. The angle between $\vec{E}$ and $d\vec{S}$ will be 90° for all patches on the plane surfaces so that their contribution towards the flux will be zero. However the angle between $\vec{E}$ and $d\vec{S}$ will be different for different patches on the curved surfaces. But for a particular patch, if the angle is θ, then there will be an opposite patch the angle for which is (180° – θ). Thus the patches on the curved surface will mutually cancel and the net flux would be zero. Hence the flux through the entire cylinder in this case will also be zero.

Illustration 19.

In a region of space the electric field is given by $\vec{E} = 8\hat{i} + 4\hat{j} + 3\hat{k}$. Calculate the electric flux through a surface of area 100 units in *x*-*y* plane.

Solution:

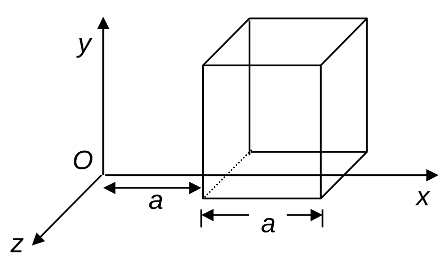
A surface of area 100 units in the *xy* plane is represented by an area vector $\vec{S} = 100\,\hat{k}$ (direction along the normal to the area). The electric flux through the surface is given by

$$\phi_E = \vec{E}.\vec{S} = (8\hat{i} + 4\hat{j} + 3\hat{k}).\,(100\hat{k})$$

$$\mathbf{= 300\ units}$$

Illustration 20.

Calculate the electric flux through a cube of side '*a*' as shown, where $E_x = bx^{1/2}$; $E_y = E_z = 0$, $a = 10$ cm and $b = 800$ N/C-m$^{1/2}$.



Solution:

The electric field throughout the region is non-uniform and its *x*-component is given by

$E_x = bx^{1/2}$, where b = 800 N/C$^{1/2}$.

For the left face perpendicular to the x-axis, we have $x = a$ = 10 cm, while for the right face $x = 2a$ = 20 cm. Hence for the left face, the x-component of the field is

$$E_x = 800 \times (10 \times 10^{-2}\ \text{m})^{1/2} = 253\ \text{N/c}$$

For the right face, we have

$$E_x' = 800 \times (20 \times 10^{-2})^{1/2} = 358\ \text{N/C}$$

The area of each face is S = 100 cm^2 = $10^{-2} m^2$

Hence, the flux through the left face

$$= -E_xS = (253)(10^{-2}) = -2.53\ \text{N-m}^2/\text{C}$$

The flux through the right face

$$= E_x'S = (358)(10^{-2}) = 3.58\ \text{N-m}^2/\text{C}$$

The net flux through the other faces is zero, because $E_y = E_z = 0$

Hence, the net flux through the cube ϕ_E = 3.58 – 2.53 = **1.05 N-m^2/C**

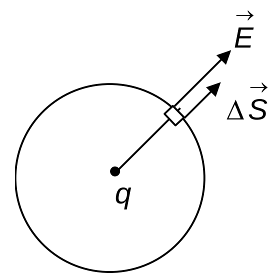
Illustration 21.

A charge q is placed at the centre of a sphere. Find the flux of the electric field through the surface of the sphere due to the enclosed charge.

Solution:

Let us take a small element ΔS on the surface of the sphere. The electric field here is radially outward and has the magnitude

$\dfrac{q}{4\pi\varepsilon_0 r^2}$, where r is the radius of the sphere.



The electric flux through this element is

$$\Delta\phi_E = \vec{E}.\Delta\vec{S} = \frac{q}{4\pi\varepsilon_0 r^2}\ \Delta S\ (\because\ \theta = 0°)$$

Hence, the electric flux through the entire sphere is given by

$$\phi_E = \Sigma\Delta\phi_E = \frac{q}{4\pi\varepsilon_0 r^2}\ \Sigma\Delta S$$

$$= \frac{q}{4\pi\varepsilon_0}(4\pi r^2) = \boldsymbol{\frac{q}{\varepsilon_0}}$$

10. GAUSS'S LAW

This law gives a relation between the electric flux through any closed hypothetical surface (called a Gaussian surface) and the charge enclosed by the surface. It states, "The electric flux (ϕ_E) through any closed surface is equal to $\frac{1}{\varepsilon_0}$ times the 'net' charge enclosed by the surface."

That is,

$$\phi_E = \oint \vec{E}.d\vec{S} = \frac{\sum q}{\varepsilon_0}, \qquad \dots (\mathbf{12})$$

where $\sum q$ denotes the algebraic sum of all the charges enclosed by the surface.

If there are several charges $+q_1$, $+q_2$, $+q_3$, $-q_4$, $-q_5$ … etc inside the Gaussian surface, then

$$\sum q = q_1 + q_2 + q_3 - q_4 - q_5 \dots$$

It is clear from equation (11) that the electric flux linked with a closed body is independent of the shape and size of the body and position of charge inside it.

Applications of Gauss's Law

Gauss's law is useful when there is symmetry in the charge distribution, as in the case of uniformly charged sphere, long cylinders, and flat sheets. In such cases, it is possible to find a simple Gaussian surface over which the surface integral given by equation (10) can be easily evaluated.

These are steps to apply the Gauss's law

(i) Use the symmetry of the charge distribution to determine the pattern of the lines

(ii) Choose a Gaussian surface for which $\vec{E}$ is either parallel to $d\vec{S}$ or perpendicular to $d\vec{S}$

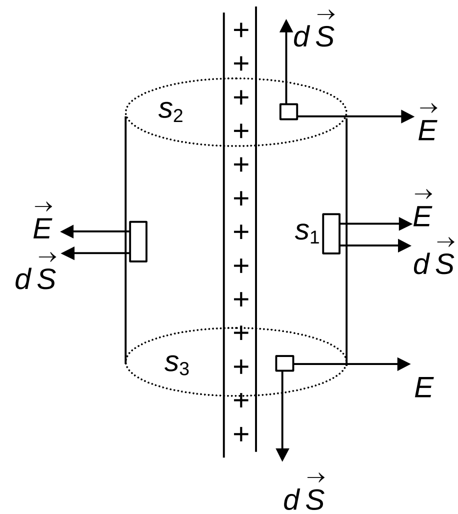
(iii) If $\vec{E}$ is parallel to $d\vec{S}$, then the magnitude of $\vec{E}$ should be constant over this part of the surface.

The integral then reduces to a sum over area elements.

10.1 FIELD DUE TO AN INFINITE LINE OF CHARGE

Consider an infinite line of charge has a linear charge density λ. Using Gauss's law, let us find the electric field at a distance 'r' from the line.

The cylindrical symmetry tells us that the field strength will be the same at all points at a fixed distance r from the line. Since the line is infinite and uniform, for every charge element on one side, there is symmetrically located element on the other side. The component along the line of the fields due to all such elements cancel in pairs. Thus, the field lines are directed radially outward, perpendicular to the line of charge.



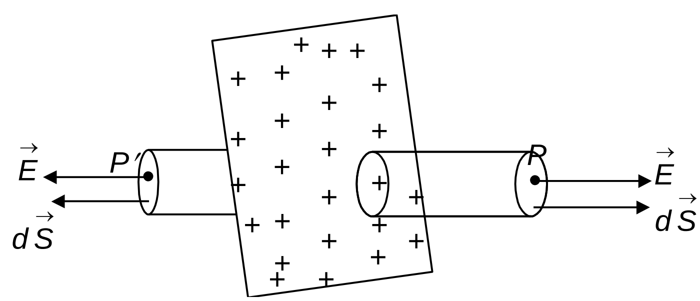
The appropriate choice of Gaussian surface is a cylinder of radius r and length L. On the flat end faces, S_2 and S_3 , $\vec{E}$ is perpendicular $d\vec{S}$, which means no flux crosses them. On the curved surface S_1, $\vec{E}$ is parallel $d\vec{S}$, so that $\vec{E}\,.\,d\vec{S} = EdS$.

The charge enclosed by the cylinder is $Q = \lambda L$.

Applying Gauss's law to the curved surface, we have

$$E \oint dS = E(2\pi r L) = \frac{\lambda L}{\varepsilon_0} \qquad \text{or,} \quad E = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad \ldots \textbf{(13A)}$$

10.2 FIELD DUE TO AN INFINITE PLANE SHEET OF CHARGE



Let us consider a thin non-conducting plane sheet of charge, infinite in extent, and having a surface charge density (charge per unit area) σ C/m^2. Let P be a point, distant r from the sheet, at which the electric intensity is required.

Let us choose a point P' symmetrical with P, on the other side of the sheet. Let us now draw a Gaussian cylinder cutting through the sheet, with its plane ends parallel to the sheet and passing through P and P'. Let A be the area of each plane end.

By symmetry, the electric intensity at all points on either side near the sheet will be perpendicular to the sheet, directed outward (if the sheet is positively charged). Thus $\vec{E}$ is perpendicular to the plane ends of the cylinder and parallel to the curved surface. Also its magnitude will be the same at P and P'. Therefore, the flux through the two plane ends is

$$\phi_E = \int \vec{E}\,.\,d\vec{S} + \int \vec{E}\,.\,d\vec{S}$$

$$= \int EdS + \int EdS$$

$$= EA + EA = 2EA$$

The flux through the curved surface of the Gaussian cylinder is zero because $\vec{E}$ and $d\vec{S}$ are at right angles everywhere on the curved surfaces.

Hence, the total flux through the Gaussian cylinder is

$$\phi_E = 2EA$$

The charge enclosed by the Gaussian surface $q = \sigma A$

Applying Gauss's law, we have

$$2EA = \frac{\sigma A}{\varepsilon_0}$$

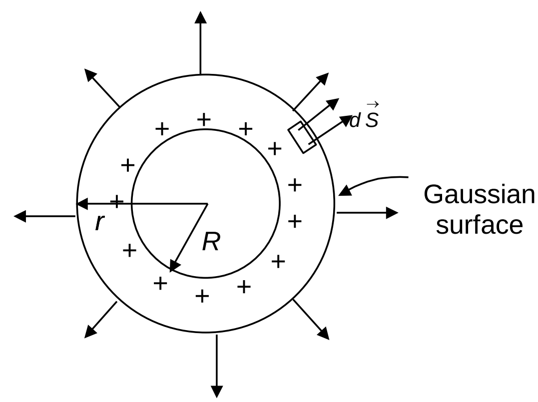
$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0} \qquad \ldots (13B)$$

10.3 ELECTRIC FIELD DUE TO A UNIFORMLY CHARGE SPHERICAL SHELL

Using Gauss's law, let us find the intensity of the electric field due to a uniformly charged spherical shell or a solid conducting sphere at

Case I: At an external point

At all points inside the charged spherical conductor or hollow spherical shell, electric field $\vec{E}$ = 0, as there is no charge inside such sphere. In an isolated charged spherical conductor any excess charge on it is distributed uniformly over its outer surface same as that of charged spherical shell or hollow sphere. Since the charge lines must point radially outward. Also, the field strength will have the same value at all points on any imaginary spherical surface concentric with the charged conducting sphere or the shell. This symmetry leads us to choose the Gaussian surface to be a sphere of radius $r > R$.



Any arbitrary element of area $d\vec{S}$ is parallel to the local $\vec{E}$, so $\vec{E}.d\vec{S} = EdS$ at all points on the surface.

According to Gauss's law

$$\oint \vec{E}.d\vec{S} = \oint EdS = E\oint dS = E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

Therefore, $$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \qquad \ldots (\mathbf{13C})$$

For points outside the charged conducting sphere or the charged spherical shell, the field is same as that of a point charge at the centre.

Case II: At an Internal Point ($r < R$)

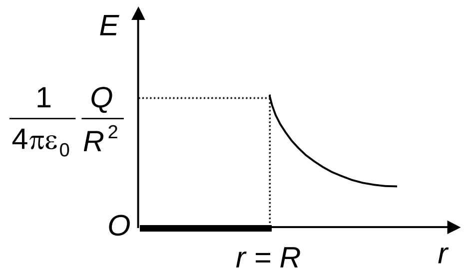
The field still has the same symmetry and so we again pick a spherical Gaussian surface, but now with radius r less than R. Since the enclosed charge is zero, from Gauss's law we have

$$E\,(4\pi r^2) = 0$$

$$\therefore \quad E = 0 \qquad \ldots \textbf{(13D)}$$

Thus, we conclude that $E = 0$ at all points inside a uniformly charged conducting sphere or the charged spherical shell.

Variation of *E* with The Distance from the centre (*r*)

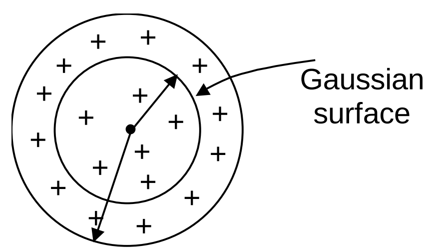


10.4 ELECTRIC FIELD DUE TO A UNIFORMALY CHARGED SPHERE

A non-conducting uniformly charged sphere of radius R has a total charge Q uniformly distributed throughout its volume. Using the Gauss's Law, Let us find the field

Case I: at an internal point ($r < R$)

Positive charge Q is uniformly distributed throughout the volume of sphere of radius R. For finding the electric field at a distance ($r < R$) from the centre, we choose a spherical Gaussian surface of radius r, concentric with the charge distribution. From symmetry the magnitude E of the electric field has the same value at every point on the Gaussian surface, and the direction of $\vec{E}$ is radial at every point on the surface.



So, applying Gauss's law

$$\oint \vec{E}.d\vec{S} = \oint EdS = E\oint dS = E\,(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$\text{Here,} \quad Q = \left(\frac{4}{3}\pi r^3\right)\rho = \left(\frac{4}{3}\pi r^3\right) \times \frac{Q}{4\pi R^3} = \frac{Qr^3}{R^3}$$

where ρ is volume density of charge.

Therefore

$$E\,(4\pi r^2) = \frac{Qr^3}{R^3\varepsilon_0} \qquad \text{or,} \qquad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3}r \qquad \ldots \textbf{(13E)}$$

The field increases linearly with distance from the centre

Case II: At an External point ($r > R$)

To find the electric field outside the charged sphere, we use a spherical Gaussian surface of radius r ($> R$). This surface encloses the entire charged sphere. So, from Gauss's law, we have

$$E\,(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

or, $$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \qquad \ldots \textbf{(13F)}$$

The field at points outside the sphere is the same as that of a point charge at the centre.

Variation of *E* with the distance from the centre (*r*)

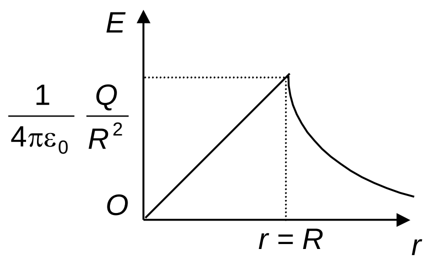
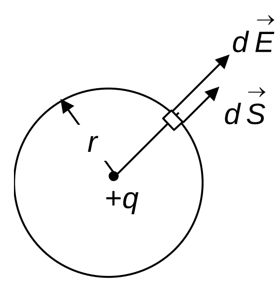


Illustration 22.

Starting with Gauss's law, calculate the electric field due to an isolated point charge q and shows that coulomb's law follows from this result.

Solution:

For this situation, we choose a spherical Gaussian surface of radius r and centered on the point charge, as shown. The electric field of a positive point charge is radially outward normal to the surface at every point. that is, $\vec{E}$ is parallel to $d\vec{S}$ at each point, and so $\vec{E}\cdot d\vec{S} = EdS$



According to Gauss's Law

$$\phi_E = \oint \vec{E}\cdot d\vec{S} = \oint EdS = \frac{q}{\varepsilon_0}$$

By symmetry, E is constant everywhere on the surface. Therefore,

$$\oint EdS = E\oint dS = E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

($\because \oint dS = 4\pi r^2$, the surface area of a sphere).

$$\therefore \qquad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

If a second point charge q_0 is placed at a point where the field is E, the electric force on this charge has a magnitude

$$F = q_0\,E = \frac{1}{4\pi\varepsilon_0}\,\frac{q\,q_0}{r^2}$$

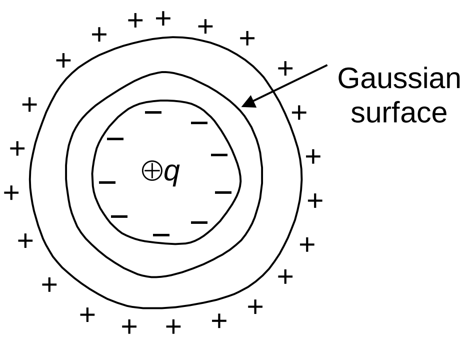
It is nothing but Coulomb's law. Hence they are equivalent.

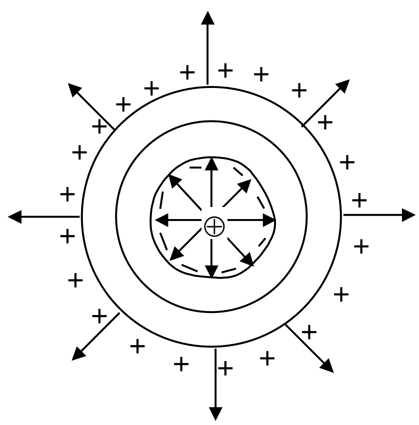
11. CONDUCTORS

A conductor contains "free" electrons, which can move freely in the material, but cannot leave it. Now, when an excess charge is given to an insulated conductor, it sets up electric field inside the conductor. The free electrons will redistribute themselves and within a fraction of a second (approx. 10^{-12} s) the internal field will vanish. Thus, in electrostatic equilibrium the value of $\vec{E}$ at all points within a conductor is zero. This idea, together with the Gauss's law can be used to prove interesting facts regarding a conductor.

11.1 CAVITY INSIDE A CONDUCTOR

Consider a charge + q_0 suspended in a cavity in a conductor. Consider a Gaussian surface just outside the cavity and inside the conductor. $\vec{E} = 0$ on this Gaussian surface as it is inside the conductor. Hence from Gauss's law





$$\oint \vec{E}\,.\,d\vec{S} = \frac{\sum q}{\varepsilon_0}$$, we have

$$\sum q = 0$$

This concludes that a charge of $-q$ must reside on the metal surface of the cavity so that the sum of this induced charge $-q$ and the original charge $+q$ within the Gaussian surface is zero. In other words, a charge q suspended inside a cavity in a conductor induces an equal and opposite charge $-q$ on the surface of the cavity. Further as the conductor is electrically neutral, a charge $+q$ is induced on the outer surface of the conductor. As field inside the conductor is zero.

The field lines coming from q cannot penetrate into the conductor, as shown in the above figure.

The same line of approach can be used to show that the field inside the cavity of a conductor is zero when no charge is kept inside it.

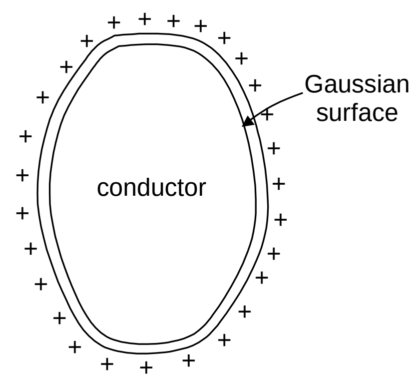
11.2 ELECTROSTATIC SHEILDING

Suppose we have a very sensitive electronic instrument that we want to protect from external fields that might cause wrong measurements. We surround the instrument with a conducting box or we keep the instrument inside the cavity of a conductor. By doing this, the charge in the conductor is so distributed that the net electric field inside the cavity becomes zero and so instrument is protected from the external fields. This is called electrostatic shielding.

Illustration 23.

Establish, on the basis of Gauss's law, the fact that when an excess charge is given to an insulated conductor. The charge resides entirely on the outer surface of the conductor

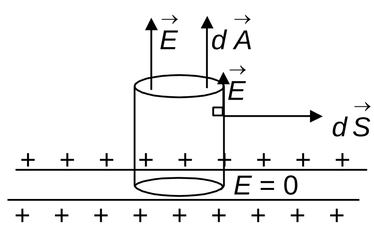
Solution:



Let us now consider the interior of a charged conducting object. Since it is a conductor, the electric field in the interior is everywhere zero. Let us consider a Gaussian surface inside the conductor as close as possible to the surface of the conductor. Since the electric intensity $\vec{E}$ is zero everywhere inside the conductor, it must be zero for every point on the Gaussian surface obviously, the flux through the surface. $\oint \vec{E}.d\vec{S}$ will be zero. Therefore, according to Gauss's law ($\oint \vec{E}.d\vec{S} = \frac{q}{\varepsilon_0}$), the net charge inside the Gaussian surface and hence inside the conductor must be zero. since there can be no charge in the interior of the conductor, any charge supplied to the conductor will reside on the surface of the conductor. In fact, the excess charge of any conductor resides on the outer surface of the conductor.

Illustration 24.

Using Gauss's law, find the electric field due to an infinite conducting plate with a uniform surface charge density



Solution:

When a charge is given to a conducting plate, it distributes itself over the entire outer surface of the plate.

From the symmetry, we know that the field must be uniform and perpendicular to the plane. We now choose a cylindrical Gaussian surface. In this case, the field is zero within the conductor; therefore only one flat face has flux passing through it. If the area of the end face is *A*, from Gauss's law we have

$$E\,A = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \qquad E = \frac{\sigma}{\varepsilon_0}$$

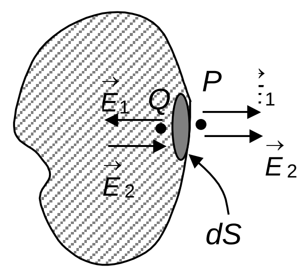
☞ Although the above equation is derived for a flat infinite conductor, it may be applied to any charged conductor without sharp points.

12. FORCE ON THE SURFACE OF A CHARGE CONDUCTOR

In a charged conductor the charge resides entirely on the surface. This shows that every element of the surface of the conductor experiences a normal outward force, which holds its charge there. This force is produced as a result of repulsion of the charge on the element by the similar charge on the rest of the surface of the conductor. Let us calculate this force.

Let dS be a small element of the surface of a charged conductor. Let σ be the surface density of charge. Let us consider a point P just outside the surface. The magnitude of the electric intensity at P is given by

$E = \frac{\sigma}{\varepsilon_0}$, and is directed along the outward drawn normal to the element.



The intensity E can be considered as made up of two parts: (i) an intensity E_1 due to the charge on the element dS, and (ii) an intensity E_2 due to the charge on the rest of the surface of the conductor. Since their directions are the same, we have

$$E_1 = E_1 + E_2 = \frac{\sigma}{\varepsilon_0} \qquad \ldots \text{(i)}$$

Let us now consider a point Q just inside the surface. The intensity at Q may again be considered as made up of two parts. The intensity due to the charge on the element dS is equal and opposite to that at P i.e., $-E_1$, since Q is very close to P but on the opposite side of the surface. The intensity due to the charge on the rest of the surface is same in magnitude and direction as at P i.e, E_2 (since Q is very close to P). But the resultant intensity at Q must be zero, since Q lies inside the conductor. Hence

$$-E_1 + E_2 = 0$$

or $$E_1 = E_2.$$

Substituting this in equation (i), we get

$$2E_2 = \frac{\sigma}{\varepsilon_0}$$

$$\therefore \qquad E_2 = \frac{\sigma}{2\varepsilon_0}$$

This gives the outward force experienced by a unit positive charge on the elements dS due to the charge on the rest of the surface. Since the charge on the element is $\sigma\, dS$, the force on dS is

$$F = E_2\,(\sigma\, dS) = \frac{\sigma^2 dS}{2\varepsilon_0}$$

Hence the force per unit area of the surface is

$$\frac{F}{dS}=\frac{\sigma^2}{2\varepsilon_0} \qquad \ldots (\mathbf{14A})$$

Whatever the sign of σ, this force acts outward along the normal to the surface.

Now, from equation (i) $E = \frac{\sigma}{\varepsilon_0}$, so that $\sigma = \varepsilon_0 E$.

Substituting this value of σ in equation (14A), the outward force per unit area of the surface

$$= \frac{(\varepsilon_0 E)^2}{2\varepsilon_0}$$

$$= \frac{\varepsilon_0 E^2}{2} \qquad \ldots (\mathbf{14B})$$

Hence the force per unit area (or electrostatic pressure) experienced by a charged conductor is $\sigma^2/2\varepsilon_0$ or $\varepsilon_0 E^2/2$ newton/meter^2 directed along the outward drawn normal to the surface.

13. ENERGY DENSITY OF AN ELECTRIC FIELD

Let us consider an electrostatic field E around a charged conductor. The charged conductor experiences a force of $\varepsilon_0 E^2/2$ Newton per meter^2 area, which is everywhere, directed along the outward drawn normal to the surface. If the conductor is placed in a medium of dielectric constant K, the normal force is $K\varepsilon_0 E^2/2$ newton.

Suppose the surface is displaced through a small distance of dl meter everywhere along its outward normal. The work done per meter^2 area of the surface

$$= \frac{K\varepsilon_0 E^2}{2}\, dl \text{ joule.}$$

The volume swept out by 1 meter^2 area is dl meter^3. Thus the work done in producing dl meter^3 of the field

$$\frac{K\varepsilon_0 E^2}{2}\, dl \text{ joule.}$$

Hence the work done is producing unit volume of the field.

$$= \frac{K\varepsilon_0 E^2}{2} \text{ joule}$$

This work is stored as energy of strain in the field. Hence the energy per meter^3, or the energy density u, of the field is

$$u = \frac{1}{2}K\varepsilon_0 E^2 \qquad \ldots (\mathbf{15})$$

14. ELECTRIC POTENTIAL

The electric potential at a point in an electric field is the external work needed to bring a unit positive charge, with constant, from infinity (point of zero potential) to the given point. Thus,

$$V = \frac{W_{ext}}{q_0} \qquad \ldots (\mathbf{16})$$

Where W_{ext} is work done in moving a charge q_0 from infinity to that point.

Important points regarding electric potential

(i) As electric field is conservative, $W_{\text{ext}} = U$.

So,

$$V = \frac{U}{q_0}$$

or, $U = q_0 V$

Thus, the electric potential at a point is numerically equal to potential energy per unit charge at that point.

(ii) It is a scalar having SI unit (J/C) called volt (V).

$$1\text{V} = \frac{1\text{J}}{1\text{C}}$$

(iii) If V_A and V_B are the electric potentials of two points A and B, the potential difference between A and B is equal to $V_B - V_A$.

Thus the potential difference between two points, A and B, is defined as

$$V_B - V_A = \frac{W_{AB}}{q_0}$$

where $W_{A\to B}$ is the work done by an external agent in moving a positive test charge q_0 from A to B .

(iv) We know that

$$\Delta V = \frac{W_{ext}}{q_0}$$

Now, $W_{\text{ext}} = \int \vec{F}_{ext} . d\vec{l}$

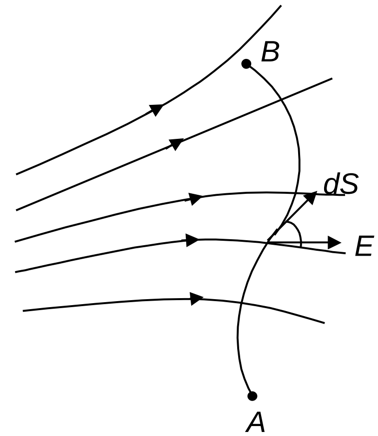
Since the external force is equal and opposite of the electrostatic force, we have

$$\vec{F}_{ext} = -q\vec{E}$$

or $W_{\text{ext}} = \int q\vec{E}. d\vec{l}$

The figure shows a curved path in a non-uniform field. The potential difference between the points A and B is given by

$$V_B - V_A = -\int_A^B \vec{E}.d\vec{l} \qquad \dots \textbf{(17)}$$



Since the electrostatic field is conservative, the value of this line integral depends only on the end points A and B and not on the path taken. So the electric potential at a point can be interpreted as the negative of the work done by the field in displacing a unit positive charge from some reference point (usually taken at infinity) to the given point.

Illustration 25.

The electric potential at point A is 20 V and at B is – 40 V. Find the work done by an external force and electrostatics force in moving an electron slowly from B to A.

Solution:

Here,

$q_0 = 1.6 \times 10^{-19}$ C; $V_A = 20\ V$;

$V_B = -40\ V$

work done by the external force = $W_{B \to A}$

$= q_0\,(V_A - V_B)$

$= (-1.6 \times 10^{-19})\,[(20 - (-40)]$

$= -9.6 \times 10^{-18}$ J

Work done by the electric force = $-(W_{B \to A})_{\text{external}}$

= $\mathbf{9.6 \times 10^{-18}}$ **J**

Illustration 26.

Find the work done by some external force in moving a charge $q = 2$ μC from infinity to a point where electric potential is 10^4V.

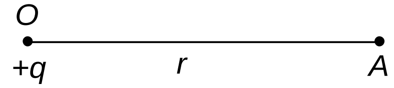
Solution:

$(E_{\infty \to A})_{\text{external}} = (2 \times 10^{-6})\,(10^4)$

= $\mathbf{2 \times 10^{-2}}$ **J**

14.1 ELECTRIC POTENTIAL AT A POINT DUE TO A POINT CHARGE

As $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$ and $V = -\int_\infty^r \vec{E}.d\vec{r}$



$$\therefore \quad V = -\int_\infty^r \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}\cdot d\vec{r} = -\frac{1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r^2}dr$$

or $$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \qquad \ldots \textbf{(18)}$$

where r is the distance of A from the point charge q.

The electric potential at A (V_A) is positive if the point charge q is positive. V_A will be negative if the point charge q is negative.

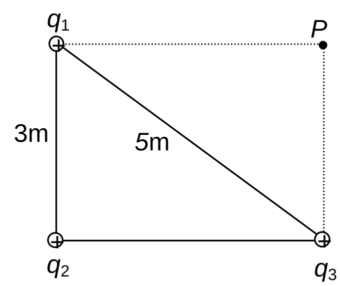
14.2 ELECTRIC POTENTIAL DUE TO A GROUP OF POINT CHARGES

The potential at any point due to a group of point charges is the algebraic sum of the potentials contributed at the same point by all the individual point charges.

$$V = V_1 + V_2 + V_3 + \ldots \qquad \ldots \textbf{(19)}$$

Illustration 27.

Three point charges $q_1 = 1\mu C$; $q_2 = 2\ \mu C$; and $q_3 = 3\ \mu C$ are fixed at a position shown. (a) What is the potential at point P at the corner of the rectangle? (b) How much work would be needed to bring a charge $q_4 = 2.5\ \mu C$ from infinity and to place it at P?



Solution:

(a) The total potential at the point P is the scalar sum

$$V_P = V_1 + V_2 + V_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3}$$

Now $$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} = \frac{(9\times10^9)(10^{-6}\,C)}{4m} = 2.25\times10^3\ V$$

Similarly, $V_2 = -3.6 \times 10^3$ V; and $V_3 = 9 \times 10^3$ V

The total potential at $P = 7.65 \times 10^3\ V$

(b) The external work is $W_{ext} = q[V_f - V_i]$

In this case, $V_i = 0$.

So, $W_{ext} = q_4 V_P = (2.5 \times 10^{-6}\ C)\ (7.65 \times 10^3\ V) = \mathbf{0.19\ J}$

14.3 ELECTRIC POTENTIAL DUE TO A CONTINUOUS CHARGE DISTRIBUTION

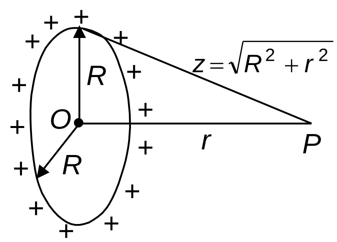
The electric potential due to a continuous charge distribution is the sum of potentials of all the infinitesimal charge elements in which the distribution may be divided.

$$V = \int dV$$

$$V = \int \frac{dq}{4\pi\varepsilon_0 r}$$

Electric Potential due to a charged ring

A charge Q is uniformly distributed over the circumference of a ring. Let us calculate the electric potential at an axial point at a distance r' from the centre of the ring.



The electric potential at P due to the charge element dq of the ring is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{Z} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(R^2 + r^2)^{1/2}}$$

Hence, the electric potential at P due to the uniformly charged ring is given by

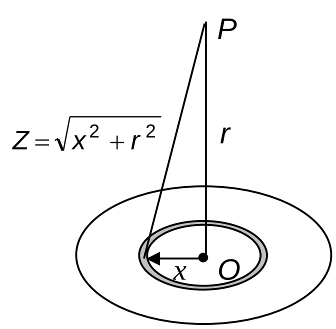
$$V = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{(R^2 + r^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\frac{1}{(R^2 + r^2)^{1/2}}\int dq$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{(R^2 + r^2)}} \qquad \text{... (19A)}$$

Electric potential due to a charged disc at a point on the axis

A non-conducting disc of radius 'R' has a uniform surface charge density σ C/m^2. Let us calculate the potential at a point on the axis of the disc at a distance 'r' from its centre

The symmetry of the disc tells us that the appropriate choice of element is a ring of radius x and thickness dx. All points on this ring are at the same distance $Z = \sqrt{x^2 + r^2}$, from the point P. The charge on the ring is $dq = \sigma dA = \sigma(2\pi x dx)$ and so the potential due to the ring is



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{Z} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi x dx)}{\sqrt{x^2 + r^2}}$$

Since potential is scalar, there are no components to worry about.

The potential due to the whole disc is given by

$$V = \frac{\sigma}{2\varepsilon_0}\int_O^R \frac{xdx}{\sqrt{x^2+r^2}} \qquad = \frac{\sigma}{2\varepsilon_0}\left[(x^2+r^2)^{1/2}\right]_0^R$$

$$= \frac{\sigma}{2\varepsilon_0}\left[(R^2+r^2)^{1/2}-r\right] \qquad \ldots \textbf{(19B)}$$

Let us see how this expression behaves at large distances, when $r >> R$.

We use binomial theorem $(1 + x)^n \approx 1 + nx$ for small x to expand the first term

$$(R^2 + r^2)^{1/2} = r\,[1 + \frac{R^2}{r^2}]^{1/2} \approx r\left[1+\frac{R^2}{2r^2}+\ldots\right]$$

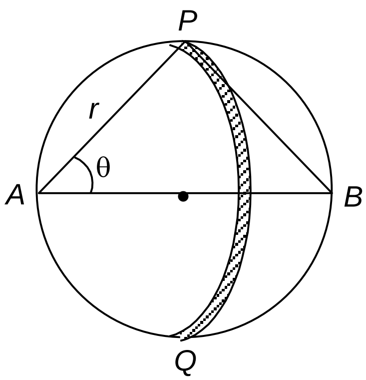
Substituting this into the expression for V, we find

$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$, where $Q = \pi r^2$ is the total charge on the disc.

Thus, we conclude that at large distances, the potential due to the disc is the same as that of a point charge Q.

Electric Potential due to a closed disc at a point on the edge

Let us calculate the potential at the edge of a thin disc of radius 'R' carrying a uniformly distributed charge with surface density σ



Let AB be a diameter and A be a point where the potential is to be calculated. From A as centre, we draw two arcs of radii r and $r + dr$ as shown. The infinitesimal region between these two arcs is an element whose area is $dA = (2r\theta)\,dr$, where 2θ is the angle subtended by this element PQ at the point A. Potential at A due to the element PQ is

$$dV = \frac{\sigma dA}{4\pi\varepsilon_0 r} = \frac{2\sigma r\theta dr}{4\pi\varepsilon_0 r} = \frac{2\sigma\theta\, dr}{4\pi\varepsilon_0}$$

From $\Delta\, APB$, we have

$$r = 2R\cos\theta \qquad \text{or,} \quad dr = -2R\sin\theta\, d\theta$$

Hence

$$dV = \frac{-4\sigma\theta R\sin\theta\, d\theta}{4\pi\varepsilon_0}$$

$$V = -\int_{\pi/2}^{0} \frac{\sigma R\theta\sin\theta d\theta}{\pi\varepsilon_0}$$

$$V = -\frac{\sigma R}{\pi\varepsilon_0}\left|-\theta\cos\theta+\sin\theta\right|_{\pi/2}^{0} \quad V = \frac{\sigma R}{\pi\varepsilon_0} \qquad \ldots \textbf{(19C)}$$

Electric Potential due to a shell

A shell of radius R has a charge Q uniformly distributed over its surface. Let us calculate the potential at a point (a) outside the shell; ($r > R$) (b) inside the shell ($r < R$).

(a) At points outside a uniform spherical distribution, the electric field is $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

since $\vec{E}$ is radial, $\vec{E}\,.\,d\vec{r} = Edr$

since $V(\infty) = 0$, we have

$$V(r) - V(\infty) = \int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2} dr \quad = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$\Rightarrow \qquad V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \ (r > R) \qquad \ldots \textbf{(19D)}$$

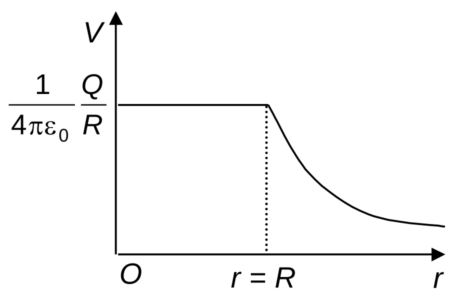
We see that the potential due to a uniformly charged shell is the same as that due to a point charge Q at the centre of the shell.

(b) At an Internal Point

At points inside the shell, $E = 0$. So, the work done in bringing a unit positive charge from a point on the surface to any point inside the shell is zero. Thus, the potential has a fixed value at all points within the spherical shell and is equal to the potential at the surface.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} \qquad \ldots \textbf{(19E)}$$

Variation of electric potential with the distance from the centre (r)

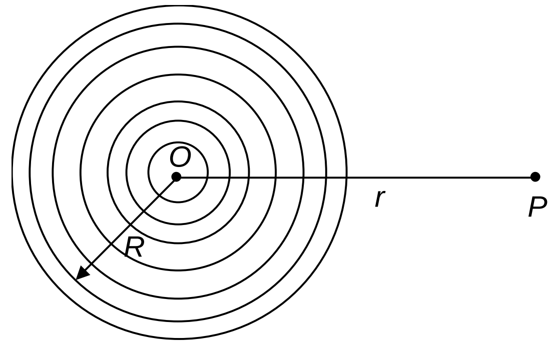


All the above results hold for a "conducting sphere' also whose charge lies entirely on the outer surface.

Electric Potential due to a non-conducting charged sphere

A charge Q is uniformly distributed throughout a non-conducting spherical volume of radius R. Let us find expressions for the potential at an (a) external point ($r > R$); (b) internal point ($r < R$); where r is the distance of the point from the centre of the sphere.

(a) At an external point



Let O be the centre of a non-conducting sphere of radius R, having a charge Q distributed uniformly over its entire volume.

Let P be a point distant r (> R) from O at which potential is required. Let ρ be the charge density

Let us divide the sphere into a large number of thin concentric shells carrying charges q_1, q_2, q_3 … etc. The potential at the point P due to the shell of charge q_1 is $\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r}$

Now, potential is a scalar quantity. Therefore the potentials V due to the whole sphere is equal to the sum of the potentials due to all the shells.

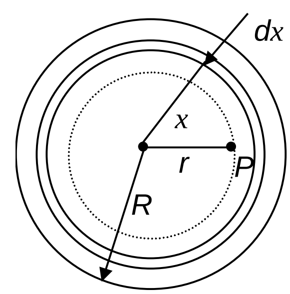
$$\therefore \qquad V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r} + \dots$$

$$= \frac{1}{4\pi\varepsilon_0 r}[q_1 + q_2 + q_3 + \dots\dots]$$

But $q_1 + q_2 + q_3 + \dots = Q$, the charge on the sphere.

$$\therefore \qquad V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad \dots \textbf{(19F)}$$

(b) Potential at an internal point



Suppose the point P lies inside the sphere at a distance r from the centre O, If we draw a concentric sphere through the point P, the point P will be external for the solid sphere of radius r, and internal for the outer spherical shell of internal radius r and external radius R.

The charge on the inner solid sphere is $\frac{4}{3}\pi r^3\rho$. Therefore, the potential V_1 at P due to this sphere is given by

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{4/3\ \pi r^3\rho}{r} = \frac{r^2\rho}{3\varepsilon_0}$$

Let us now find the potential at P due to the outer spherical shell. Let us divide this shell into a number of thin concentric shells and consider one such shell of radius x and infinitesimally small thickness dx. The volume of this shell = surface area × thickness = $4\ \pi x^2 d\ x$. The charge on this shell, $dq = 4\pi x^2\ dx\rho$. The potential at P due to this shell

$$dV_2 = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi x^2(dx)\rho}{x}$$

$$= \frac{\rho x dx}{\varepsilon_0}$$

The potential V_2 at P due to the whole shell of internal radius r and external radius R is given by

$$V_2 = \int_r^R \frac{\rho x dx}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}\left|\frac{x^2}{2}\right|_r^R$$

$$= \frac{\rho(R^2 - r^2)}{2\varepsilon_0}$$

Since the potential is a scalar quantity, the total potential V at P is given by

$$V = V_1 + V_2$$

$$= \frac{r^2\rho}{3\varepsilon_0} + \frac{\rho(R^2 - r^2)}{2\varepsilon_0}$$

$$= \frac{\rho(3R^2 - r^2)}{6\ \varepsilon_0}$$

But $$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$\therefore$ $$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2R^3}\,[3R^2 - r^2] \qquad \ldots\ (\mathbf{19G})$$

15. CALCULATION OF ELECTRIC FIELD FROM ELECTRIC POTENTIAL

In rectangular components, the electric field is

$$\vec{E} = E_x\,\hat{i} + E_y\,\hat{j} + E_z\,\hat{k};$$

and an infinitesimal displacement is $d\vec{S} = dx\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Thus,

$$dV = -\ \vec{E}.d\vec{S} \qquad \ldots\ (\mathbf{20})$$

$$= -\ [E_x\,dx + E_y\,dy + E_z\,dz]$$

for a displacement in the x-direction,

$dy = dz = 0$ and so

$dV = -E_x\,dx$. Therefore,

$$E_x = -\left(\frac{dV}{dx}\right)_{y,z\ \text{constant}}$$

A derivative in which all variables except one are held constant is called partial derivative and is written with ∂ instead of d. The electric field is, therefore,

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} \qquad \ldots \textbf{(21)}$$

Illustration 28.

The electric potential in a region is represented as $V = 2x + 3y - z$. Obtain expression for the electric field strength.

Solution:

We know

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

Here, $\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}[2x + 3y - z) = 2$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\ [2x + 3y - z] = 3$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}\ [2x + 3y - z] = -1$$

$\therefore \qquad \vec{E} = -\boldsymbol{(2\hat{i} + 3\hat{j} - \hat{k})}$

Illustration 29.

The electrical potential due to a point charge is given by $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$. Find

(a) the radial component of the electric field;

(b) the x-component of the electric field

Solution:

(a) The radial component is given by

$$E_r = -\frac{dV}{dr} = +\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

(b) In terms of rectangular components, the radial distance is $r = (x^2 + y^2 + z^2)^{1/2}$; therefore, the potential function

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(x^2 + y^2 + z^2)^{1/2}}$$

To find the x-component of the electric field, we treat y and z constants. Thus

$$E_x = -\frac{\partial V}{\partial x}$$

or $\qquad E_x = +\frac{1}{4\pi\varepsilon_0}\frac{Qx}{(x^2 + y^2 + z^2)^{3/2}}$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Qx}{r^3}$$

16. EQUIPOTENTIAL SURFACES

If we join the points in an electric field, which are at same potential, the surface (or curve) obtained is known as equipotential surface (curve).

Important Points Regarding Equipotential surfaces

(i) The lines of forces are always normal to equipotential surfaces

(ii) The net work done in taking a charge from A to B is zero if A and B are on same equipotential surface.

Examples

(i) In the field of a point charge, the equipotential surfaces are spheres centered on the point charge.

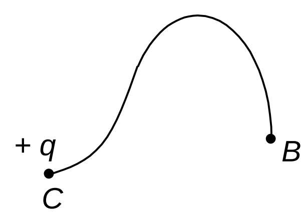
(ii) In a uniform electric field, the equipotential surfaces are planes which are perpendicular to the field lines.

(iii) In the field of an infinite line charge, the equipotential surfaces are co-axial cylinders having their axes at the line charge.

(iv) The surface of a conductor is an equipotential surface and the inside of conductor is equipotential space. Hence there is no electric field (and charge) inside the conductor's surface. The lines of forces are always normal to the surface of a conductor.

17. ELECTRIC POTENTIAL ENERGY

If a charge is moved between two points in an electric field, work is usually done against the field or by the field. In the figure, if a charge $+q$ is moved from B to C in the electric field of charge $+Q$, the work will have to be done by some outside agency in pushing the charge $+q$ against the force of field of $+Q$.



This situation is very similar to that of a mass moved in gravitational field of earth away from it. Work done against the gravitational pull of earth is stored in Gravitational potential energy and can be recovered back. Similarly in electric field, work done against an electric field is stored in the form of electric potential energy & can be recovered back. If the charge $+q$ is taken back from C to B, the electric force will try to accelerate the charge and hence to recover the potential stored in the form of kinetic energy.

As the work done against an electric field can be recovered back, electrostatic forces and fields fall under the category of conservative forces and fields. Another property of these fields is that the work done is independent of path taken from one point to the another.

17.1 POTENTIAL ENERGY OF A SYSTEM OF TWO POINT CHARGES

The potential energy possessed by a system of two-point charges q_1 and q_2 separated by a distance r is the work done required to bring them to this arrangements from infinity. This electrostatic potential energy is given by

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r} \qquad \ldots (22)$$

17.2 ELECTRIC POTENTIAL ENERGY OF A SYSTEM OF A SYSTEM OF POINT CHARGES

The electric potential energy of such a system is the work done in assembling this system starting from infinite separation between any two-point charges.

For a system of point charges q_1, q_2 …. q_n, the potential energy is

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i\, q_j}{4\pi\varepsilon_0\, r_{ij}}\ (i \neq j) \qquad \dots (23)$$

If simply means that we have to consider all the pairs that are possible.

Important points regarding Electrostatic potential energy

(i) Work done required by an external agency to move a charge q from A to B in an electric field with constant speed

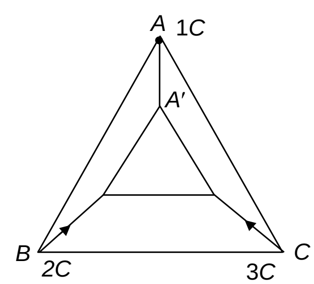
$$W_{A\to B} = q\,[V_B - V_A]$$

(ii) When a charge q is let free in an electric field, it loses potential energy and gains kinetic energy, if it goes from A to B, then loss in potential energy = gain in kinetic energy

or, $\quad q\,(V_B - V_A) = \frac{1}{2}\,mV_B^2 - \frac{1}{2}\,mV_A^2$

Illustration 30.

Three point charges 1C, 2C and 3C are placed at the corner of an equilateral triangle of side 1m. Calculate the work required to move these charges to the corners of a smaller equilateral triangle of side 0.5 m as shown



Solution:

As the potential energy of two point charges separated by a distance 'r' is given by $[U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}]$, the initial and the final potential energy of the system will be

$$U_I = \frac{1}{4\pi\varepsilon_0}\left[\frac{1\times 2}{1} + \frac{2\times 3}{1} + \frac{3\times 1}{1}\right]$$

$= 9 \times 10^9 \times 11$

$= 9.9 \times 10^{10}$ J

$$U_F = \frac{1}{4\pi\varepsilon_0}\left[\frac{1\times 2}{0.5} + \frac{2\times 3}{0.5} + \frac{3\times 1}{0.5}\right] = 9\times 10^9 \times 22$$

$= 19.8 \times 10^{10}$ J

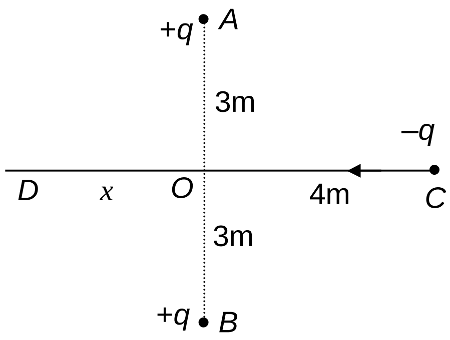
So, the work done in changing the configuration of the system

$W = U_F - U_I = (19.8 - 9.9) \times 10^{10}$

$= \mathbf{9.9 \times 10^{10}\ J}$

Illustration 31.

Two fixed equal positive charges, each of magnitude 5×10^{-5} C are located at points *A* and *B*, separated by a distance of 6 m. An equal and opposite charge moves towards them along the line *COD*, the perpendicular bisector of the line *AB*. The moving charge, when it reaches the point *C* at a distance of 4 m from *O*, has a kinetic energy of 4 joules. Calculate the distance of the farthest point *D* which the negative charge will reach before returning towards *C*.



Solution:

The kinetic energy is lost and converted to electrostatic potential energy of the system as the negative charge goes from *C* to *D* and comes to rest at *D* instantaneously.

Loss of K.E. = Gain in potential energy

$$4 = U_f - U_i$$

or, $$4 = \left[\frac{q.q}{4\pi\varepsilon_0(6)^2} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9+x^2}}\right] - \left[\frac{q.q}{4\pi\varepsilon_0(6)^2} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9+16}}\right]$$

or $$4 = \frac{2q^2}{4\pi\varepsilon_0}\left[\frac{1}{5} - \frac{1}{\sqrt{9+1}}\right]$$

or, $$4 = 2 \times (5 \times 10^{-5})^2 \times (9 \times 10^9)\left[\frac{1}{5} - \frac{1}{\sqrt{9+x^2}}\right]]$$

or, $$4 = 9 - \frac{45}{\sqrt{9+x^2}}$$

$\Rightarrow$ $x = \sqrt{72} = \mathbf{8.48\ m}$

Illustration 32.

A metal sphere of radius *R* has a charge *Q*. Find its potential energy.

Solution:

We can find the potential energy by calculating the work needed to build up the charge to the final value. Suppose at some time that the charge on the sphere is *q*. So, its potential $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$. The external work required to bring an infinitesimal charge *dq* from infinity and to deposits it on the sphere is

$$dW = Vdq = \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\right)dq$$

The total work required to give the sphere a charge Q is therefore

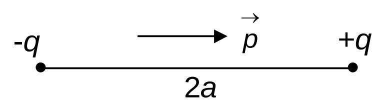
$$W = \int_O^Q \frac{1}{4\pi\varepsilon_0}\frac{q}{R}\,dq = \frac{a^2}{8\pi\varepsilon_0 R}$$

18. ELECTRIC DIPOLE

Two equal and opposite point charges placed at a short distance apart constitute an electric dipole.

18.1 ELECTRIC DIPOLE MOMENT

Electric dipole moment is a vector $\vec{p}$ directed along the axis of the dipole, from the negative to the positive charge.



The magnitude of dipole moment is

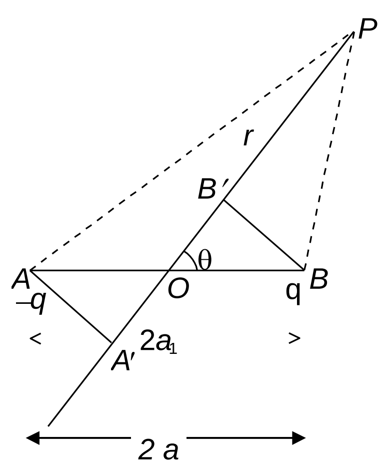
$p = (2a)\, q$

where $2a$ is the distance between the two charges. … (**24**)

18.2 EELCTRIC POTENTIAL DUE TO AN ELECTRIC DIPOLE

Suspose, the negative charge $-q$ is placed at a point A and a positive charge q is placed at a point B. The separation $AB = 2a$

The middle point of AB is O. The potential is to be evaluated at a point P where $OP = r$ and $P\hat{O}B = \theta$. Let AA' be the perpendicular from A to PO and BB' be the perpendicular from B to PO. As $2a$ is very small compared to r,



$$AP \simeq A'P = OP + OA' = r + a\cos\theta$$

Similarly, $BP \simeq B'P = OP - OB' = r - a\cos\theta$

The potential at P due to the charge $-q$ is

$$V_1 = -\frac{1}{4\pi\varepsilon_0}\frac{q}{AP} \simeq -\frac{1}{4\pi\varepsilon_0}\frac{q}{(r+a\cos\theta)}$$ and that due to the charge $+q$ is

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{BP} \simeq \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-a\cos\theta)}$$

The net potential at P due to the dipole is

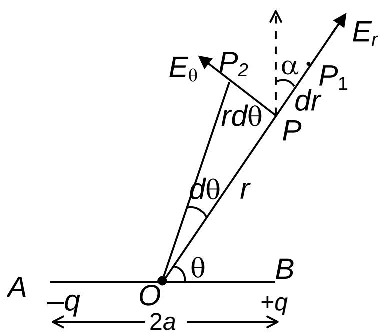
$$V = V_1 + V_2$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r-a\cos\theta}-\frac{q}{r+a\cos\theta}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{(r^2-a^2\cos^2\theta)} \simeq \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}. \qquad \ldots \textbf{(25)}$$

18.3 ELECTRIC FIELD DUE TO AN ELECTRIC DIPOLE

Consider a point P at a distance r from O making an angle θ with AB. PP_1 is a small displacement in the direction of OP and PP_2 is a small displacement perpendicular to OP. Thus PP_1 is in radial direction and PP_2 is in transverse direction. In going from P to P_1, the angle θ does not change and the distance OP changes from r to $r + dr$. Thus $PP_1 = dr$. In going from P to P_2, the angle θ changes from θ to $\theta + d\theta$ while the distance r remains almost constant so $PP_2 = r\,d\theta$.



The component of the electric field at P in the radial direction PP_1 is

$$E_r = -\frac{dV}{dr} = -\frac{\partial V}{\partial r} = -\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{r^2}\right) = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}.$$

The component of the electric field at P in the transverse direction PP_2 is

$$E_\theta = -\frac{dV}{PP_2} = -\frac{dV}{r\,d\theta} = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right) = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

The resultant electric field at P, $E = \sqrt{E_r^2 + E_\theta^2}$

$$= \frac{1}{4\pi\varepsilon_0}\sqrt{\left(\frac{2p\cos\theta}{r^3}\right)^2 + \left(\frac{p\sin\theta}{r^3}\right)^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta+1} \qquad \ldots \textbf{(26)}$$

If the resultant field makes an angle α with the radial direction OP, we have

$$\tan\alpha = \frac{E_\theta}{E_r} = \frac{\frac{p\sin\theta}{r^3}}{\frac{2p\cos\theta}{r^3}} = \frac{\tan\theta}{2}$$

or $\quad \alpha = \tan^{-1}\left(\frac{\tan\theta}{2}\right) \qquad \ldots \textbf{(27)}$

Now consider some special cases

Case I: $\theta = 0$. In this case, the point P is on the axis of the dipole

$$V = \frac{p}{4\pi\varepsilon_0 r^2} \qquad \ldots \textbf{(28A)}$$

$$E = \frac{2p}{4\pi\varepsilon_0 r^3} \qquad \ldots \textbf{(28B)}$$

Such a position of the point is called an end-on position.

Case II: $\theta = 90^\circ$. In this case, the point P is on the perpendicular bisector of the dipole

$$V = 0, \qquad \ldots \textbf{(28C)}$$

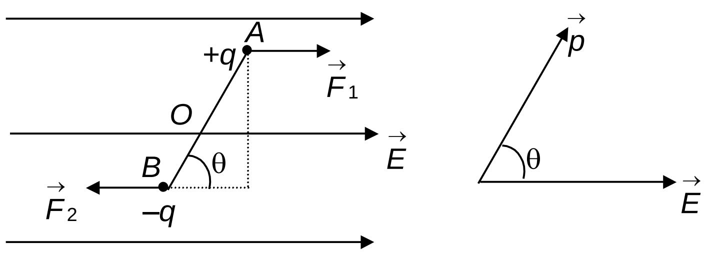
$$E = \frac{p}{4\pi\varepsilon_0 r^3} \qquad \ldots \textbf{(28D)}$$

$$\tan\alpha = \frac{\tan\theta}{2} = \infty$$

$$\alpha = 90^\circ$$

The field is anti-parallel to the dipole axis. Such a position of the point P is called a broad side on position.

18.4 DIPOLE IN AN EXTERNAL UNIFORM ELECTRIC FIELD



Suppose an electric dipole of dipole moment $|\vec{p}| = 2aq$ is placed in a uniform electric field $\vec{E}$ at an angle θ. A force $\vec{F}_1 = q\vec{E}$ will act on positive charge and $\vec{F}_2 = -q\vec{E}$ on the negative charge. Since $\vec{F}_1$ and $\vec{F}_2$ are equal in magnitude but opposite in direction, we have

$$\vec{F}_1 + \vec{F}_2 = 0$$

Thus, the net force on a dipole in a uniform electric field is zero.

The torque of $\vec{F}_1$ about O,

$$\vec{\tau}_1 = \vec{OA} \times \vec{F}_1 = q\,(\vec{OA} \times \vec{E})$$

The torque of $\vec{F}_2$ about O,

$$\vec{\tau}_2 = \vec{OB} \times \vec{F}_2 = -q\,(\vec{OB} \times \vec{E})$$

$$= (\vec{BO} \times \vec{E})$$

The net torque acting on the dipole is

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = q\ (\vec{OA} \times \vec{E}) + q\ (\vec{BO} \times \vec{E})$$

$$= q(\vec{OA} + \vec{BO}) \times \vec{E}$$

$$= q(\vec{BA} \times \vec{E})$$

or, $$\vec{\tau} = \vec{p} \times \vec{E} \qquad \ldots (\mathbf{29})$$

Thus, the magnitude of the torque is $\tau = pE \sin\theta$. The direction of torque is perpendicular to the plane of paper and inwards. Further this torque is zero at $\theta = 0°$ or $\theta = 180°$, i.e., when the dipole is parallel or antiparallel to $\vec{E}$ and maximum at $\theta = 90°$.

18.5 POTENTIAL ENERGY OF DIPOLE

When an electric dipole is placed in an electric field $\vec{E}$, a torque $\vec{\tau} = \vec{p} \times \vec{E}$ acts on it. If we rotate the dipole through a small angle $d\theta$, the work done by the torque is

$$dW = \tau\, d\theta$$

or, $$dW = -pE\sin\theta\, d\theta$$

The work is negative as the rotation $d\theta$ is opposite to the torque. The change in electric potential energy of the dipole is therefore

$$dU = -dW = pE \sin\theta\, d\theta$$

Now at angle $\theta = 90°$, the electric potential energy of the dipole may be assumed to be zero as net work done by the electric forces in bringing the dipole from infinity to this position will be zero.

$dU = pE\sin\theta\, d\theta$ from $90°$ to θ, we have

$$\int_{90°}^{\theta} dU = \int_{90°}^{\theta} pE \sin\theta\, d\theta$$

or, $$U(\theta) - U(90°) = pE\,[-\sin\theta]^{\theta}_{90°}$$

or $$U(\theta) = -\,pE\cos\theta = -\,\vec{p}.\vec{E} \qquad \ldots (\mathbf{30})$$

It the dipole is rotated from an angle θ_1 to θ_2, then

work done by external forces = $U(\theta_2) - U(\theta_1)$

or, $$W_{ext} = -pE\cos\theta_2 - (pE\cos\theta_1)$$

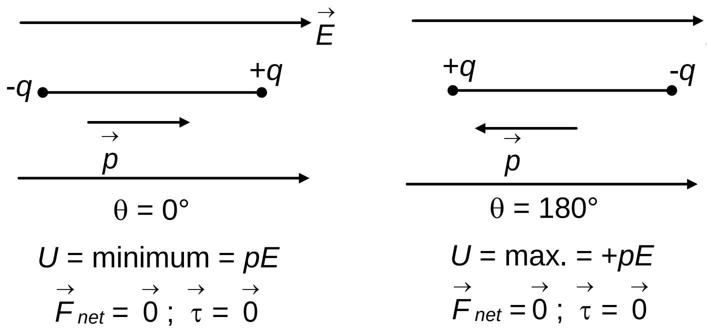
or, $$W_{ext} = pE\,(\cos\theta_1 - \cos\theta_2) \qquad \ldots (\mathbf{31})$$

Work done by electric force

$$W_{\text{electric force}} = -\,W_{ext} = pE\,[\cos\theta_2 - \cos\theta_1] \qquad \ldots (\mathbf{32})$$

18.6 EQUILIBRIUM OF DIPOLE

When an electric dipole is placed in a uniform electric field, the net force on it is zero for any position of the dipole in the electric field. But torque acting on it is zero only at $\theta = 0°$ and $180°$. Thus, we can say that at these two positions of the dipole, net force or torque on it is zero or the dipole is in equilibrium. Of this $\theta = 0°$ is the 'stable equilibrium' position of the dipole because potential energy in this position is minimum ($U = -pE\cos\theta° = -pE$) and when displaced from this position, a torque starts acting on it which is restoring in nature and which has a tendency to bring the dipole back in its equilibrium position. On the other hand, at $\theta = 180°$, the potential energy of the dipole is maximum ($U = -pE \cos 180° = +pE$) and when it is displaced from this position, the torque has a tendency to rotate it in other direction. This torque is not restoring in nature. So this equilibrium is known as 'unstable equilibrium position'.



18.7 ANGULAR SHM OF DIPOLE

When a dipole is suspended in a uniform electric field, it will align itself parallel to the field. Now if it is given a small angular displacement θ about its equilibrium, the (restoring) couple will be

$$C = -pE\sin\theta$$

or, $$C = -pE\,\theta \text{ [as } \sin\theta \approx \theta \text{, for small } \theta)$$

or, $$I\frac{d^2\theta}{dt^2} = -pE\theta$$

or, $$\frac{d^2\theta}{dt^2} = -\frac{pE}{I}\theta$$

or, $$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

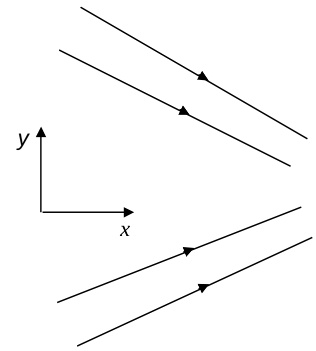
where $$\omega^2 = \frac{pE}{I}$$

This is standard equation of angular simple harmonic motion with time-period $T\left(=\frac{2\pi}{\omega}\right)$. So the dipole will execute angular SHM with time-period

$$T = 2\pi\sqrt{\frac{I}{pE}} \qquad \ldots \textbf{(33)}$$

18.8 FORCE ACTING ON A DIPOLE IN AN EXTERNAL NON-UNIFORM FIELD

When dipole lies in a non-uniform electric field the charges of the dipole experience unequal forces. Therefore, the net force on the dipole is not equal to zero. The magnitude of the force is given by the negative derivative of the potential energy with respect to distance along the axis of the dipole



$$\vec{F} = -\frac{dU}{dl} = \vec{p}.\frac{d\vec{E}}{dl}$$

Illustration 33.

A dipole whose dipole moment is p lies along the x-axis $(\vec{p} = p\hat{i})$ in a non-uniform field $\vec{E} = \frac{C}{x}\hat{i}$. What is the force on the dipole?

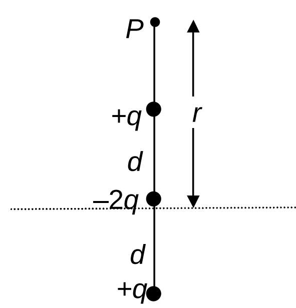
Solution:

We have $U = -\vec{p}.\vec{E} = -p\hat{i}.\frac{C}{x}\hat{i} = -\frac{pC}{x}$

Now $F = -\frac{dU}{dx} = -\frac{d}{dx}(-\frac{pC}{x}) = -\frac{pC}{x^2} \Rightarrow \vec{F} = -\frac{\mathbf{pC}}{\mathbf{x^2}}\hat{\mathbf{i}}$

Illustration 34.

An Electric quadrupole consists of two equal and opposite dipoles so arranged that their electric effects do not quite cancel each-other at distant points. In the figure given, Calculate V for the point P on the axis of this quadrupole.



Solution:

The electric potential at P is given by

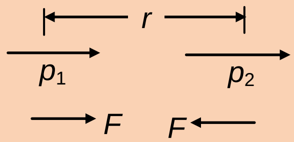
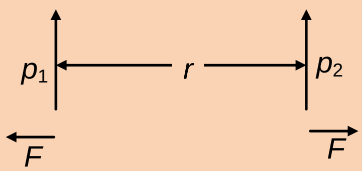
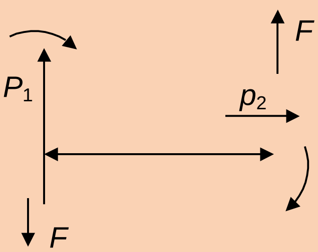
$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r-d} + \frac{-2q}{r} + \frac{q}{r+d}\right] = \frac{1}{4\pi\varepsilon_0}\frac{2qd^2}{r(r^2-d^2)} = \frac{1}{4\pi\varepsilon_0}\frac{2qd^2}{r^3\left[1-\frac{d^2}{r^2}\right]}$$

Because $d << r$, we can neglect $\frac{d^2}{r^2}$ compared to 1, in which case the potential becomes

$$V = \frac{1}{4\pi\varepsilon_0}\frac{2qd^2}{r^3}$$

18.8 INTERACTION BETWEEN DIPOLES

In this situation, one dipole is in the field of other dipole. Depending on the positions of dipoles relative to each other, force, couple and potential energy are different.

S. No.	Relative Position of Dipole	Potential energy (U)	Force (F)	Couple (C)
1.		$-\frac{1}{4\pi\varepsilon_0}\frac{2p_1p_2}{r^3}$	$-\frac{1}{4\pi\varepsilon_0}\frac{6p_1p_2}{r^4}$ (along r)	O
2.		$\frac{1}{4\pi\varepsilon_0}\frac{p_1p_2}{r^3}$	$+\frac{1}{4\pi\varepsilon_0}\frac{3p_1p_2}{r^4}$ (along r)	O
3.		O	$\pm\frac{1}{4\pi\varepsilon_0}\frac{3p_1p_2}{r^4}$ (perpendicular + or)	on $p_1 = \frac{1}{4\pi\varepsilon_0}\frac{2p_1p_2}{r^3}$ (CW) on $p_2 = \frac{1}{4\pi\varepsilon_0}\frac{p_1p_2}{r^3}$ (CW)

19. EARTHING A CONDUCTOR

Potential of earth is often taken to be zero. If a conductor is connected to the earth, the potential of the conductor becomes equal to that of the earth, i.e., zero. If the conductor was at some other potential, charges will l flow from it to the earth or from the earth to it to bring its potential to zero. The figure given below shows the symbol for earthing.

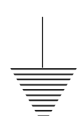
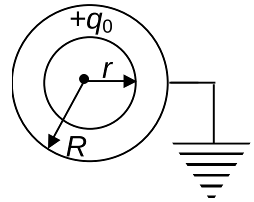


Illustration 35.

Given two concentric conducting spheres of radii r and R ($r < R$). The inner surface carries a charge q_0 and the outer sphere is earthed.



(a) find the charge on the outer sphere.

(b) find the potential of the inner surface.

Solution:

(a) Let q be a final charge on the outer sphere.

The potential of the outer sphere is given by

$$V_2 = \frac{Kq_0}{R} + \frac{Kq}{R} \qquad \{k = \frac{1}{4\pi\varepsilon_0}\}$$

Since it has been earthed, $V_2 = 0$

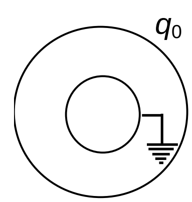
Thus, $\frac{Kq_0}{R} + \frac{Kq}{R} = 0 \quad \Rightarrow \quad q = -q_0$

(b) The potential of the inner sphere is given by

$$V_1 = \frac{kq_0}{r} - \frac{kq_0}{R} \qquad kq_0\left[\frac{1}{r} - \frac{1}{R}\right]$$

Illustration 36.

In the previous example, if the outer sphere carries a charge q_0 and the inner sphere is earthed, then find



(a) the charge on the inner sphere; and

(b) the potential of the outer sphere.

Solution:

(a) Let q be the charge on the inner sphere.

The potential of the inner sphere is given by $V_1 = \frac{kq}{r} + \frac{kq_0}{R} \qquad \{k = \frac{1}{4\pi\varepsilon_0}\}$

Since it is connected to earth, $V_1 = 0$

$$\therefore \qquad \frac{Kq}{r} + \frac{Kq_0}{R}$$

Since it is connected to earth, $V_1 = 0$

$$\therefore \qquad \frac{Kq}{r} + \frac{Kq_0}{R} = 0 \qquad \Rightarrow \qquad q = -\boldsymbol{q_0}\frac{r}{R}$$

(b) The potential of the outer sphere is given by

$$V_2 = \frac{K}{R}\left(-q_0 \frac{r}{R}\right) + \frac{Kq_0}{R} \qquad \Rightarrow V_2 = \frac{Kq_0}{R}\left[1 - \frac{r}{R}\right]$$

20. CAPACITY OF AN ISOLATED CONDUCTOR

When charge is given to an isolated body, its potential increases i.e.,

$$Q \propto V$$

or, $Q = CV$ … (**34**)

where C is a constant called capacity of the body.

if $V = 1$ then $C = Q$,

So Capacity of a body is numerically equal to the charge required to raise its potential by unity.

In SI system, the unit of capacity is (Coulomb/volt) and is called farad (F).

$$1\text{F} = \frac{1C}{1V} = \frac{3\times10^9 \text{ esu of charge}}{\left(\frac{3}{300}\right)\text{esu of potential}} = 9 \times 10^{11} \text{ esu of capacity.}$$

The capacity of a body is independent of the charge given to it and depends on its shape and size only.

21. CAPACITOR

Capacitor is an arrangement of two conductors carrying charges of equal magnitude and opposite sign and separated by an insulating medium. The following points may be carefully noted

(i) The net charge on the capacitor as a whole is zero. When we say that a capacitor has a charge Q, we mean that positively charged conductor has a charge $+Q$ and the negatively charged conductor has a charge $-Q$.

(ii) The positively charged conductor is at a higher potential than negatively charged conductor. The potential difference V between the conductors is proportional to the magnitude of charge Q and the ratio Q/V is known as capacitance C of the capacitor.

$$C = \frac{Q}{V}$$

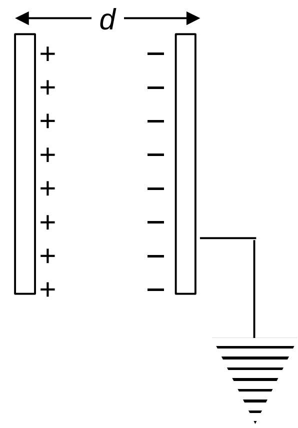
Unit of capacitance is farad (F). The capacitance is usually measured in microfarad μF.

$$1\mu F = 10^{-6} F$$

(iii) In a circuit, a capacitor is represented by the symbol:–

21.1 PARALLEL PLATE CAPACITOR

A parallel plate capacitor consists of two metal plates placed parallel to each other and separated by a distance *d* that is very small as compared to the dimensions of the plates. The area of each plate is *A*. The electric field between the plates is given by



$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

Where σ is surface charge density on either plate.

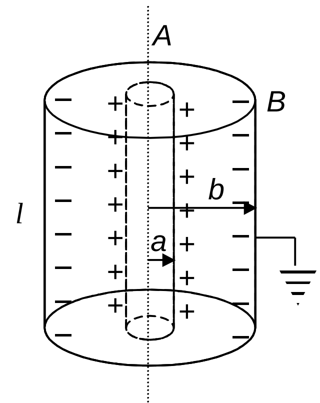
the potential difference (V) between plates is given by $V = Ed$.

or, $$V = \frac{\sigma}{\varepsilon_0} d = \frac{Q}{A\varepsilon_0} d$$

Hence, $$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d} \qquad \ldots (\mathbf{35})$$

21.2 CYLINDRICAL CAPACITOR

Cylindrical capacitor consists of two co-axial cylinders of radii *a* and *b* and length *l*. If a charge *q* is given to the inner cylinder, induced charge –*q* will reach to the inner surface of the outer cylinder. By symmetry, the electric field in region between the cylinders is radially outward.



By Gauss's theorem, the electric field at a distance *r* from the axis of the cylinders is given by

$$E = \frac{1}{2\pi\varepsilon_0 l} \frac{q}{r}$$

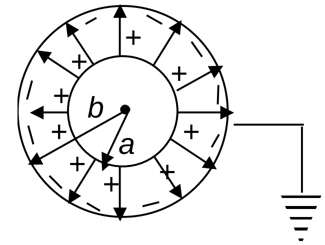
The potential difference between the cylinders is given by

$$V = -\int_b^a \vec{E}.d\vec{r} = -\frac{1}{2\pi\varepsilon_0 l} q \int_b^a \frac{dr}{r}$$

$$= \frac{-q}{2\pi\varepsilon_0 l}\left(ln\frac{a}{b}\right)$$

or, $$V = \frac{q}{V} = \frac{2\pi\varepsilon_0 l}{ln\left(\frac{b}{a}\right)} \qquad \ldots (\mathbf{36})$$

21.3 SPHERICAL CAPACITOR



A spherical capacitor consists of two concentric spheres of radii a and b as shown. The inner sphere is positively charged to potential V and outer sphere is at zero potential.

The inner surface of the outer sphere has an equal negative charge.

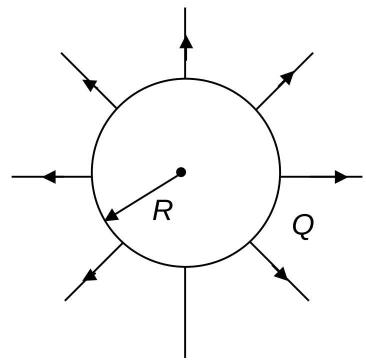
The potential difference between the spheres is

$$V = \frac{Q}{4\pi\varepsilon_0 a} - \frac{Q}{4\pi\varepsilon_0 b}$$

Hence, capacitance

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0 ab}{b-a} \qquad \dots (\mathbf{37})$$

21.4 ISOLATED SPHERE AS A CAPACITOR



A conducting sphere of radius R carrying a charge Q can be treated as a capacitor with high potential conductor as the sphere itself and the low potential conductor as a sphere of infinite radius. The potential difference between these two spheres is

$$V = \frac{Q}{4\pi\varepsilon_0 R} - 0$$

Hence, capacitance $C = \dfrac{Q}{V} = 4\pi\varepsilon_0 R$ ... (**38**)

22. ENERGY STORED IN CHARGED CAPACITOR

If dq charge is given to a capacitor at potential V

$$dW = dq\,(V)$$

or, $$E = \int_0^q \left(\frac{q}{C}\right) dq \quad [\because\ q = CV]$$

or, $$W = \frac{q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}qV$$

This work is stored as electrical potential energy i.e., a capacitor stores electrical energy

$$U = \frac{1}{2}CV^2 = \frac{q^2}{2C} = \frac{1}{2}\,qV \qquad \dots (\mathbf{39})$$

22.1 ENERGY DENSITY OF A CHARGED CAPACITOR

This energy is note localized on the charges or the plates but is distributed in the field. Since in case of a parallel plate capacitor, the electric field is only between the plates, i.e., in a volume (A × d), the energy density

$$U_E = \frac{U}{\text{volume}} = \frac{\frac{1}{2}CV^2}{A \times d} = \frac{1}{2}\left[\frac{\varepsilon_0 A}{d}\right]\frac{V^2}{Ad}$$

or, $$U_E = \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2 = \frac{1}{2}\varepsilon_0 E^2 \quad \left[\because \frac{v}{d} = E\right] \qquad \ldots (\mathbf{40})$$

22.2 FORCE BETWEEN THE PLATES OF A CAPACITOR

In a capacitor as plates carry equal and opposite charges, there is a force of attraction between the plates. To calculate this force, we use the fact that the electric field is conservative and in a conservative field $F = -\frac{dU}{dx}$.

In case of parallel plate capacitor

$$U = \frac{q^2}{2C} = \frac{1}{2}\frac{q^2 x}{\varepsilon_0 A} \quad [\text{ as } C = \frac{\varepsilon_0 A}{x}]$$

SO, $$F = -\frac{d}{dx}\left[\frac{q^2}{2\varepsilon_0 A}x\right] = \frac{-1}{2}\frac{q^2}{\varepsilon_0 A} \qquad \ldots (\mathbf{41})$$

The negative sign implies that the force is attractive.

Illustration 37.

The plates of a parallel plate capacitor are 5 mm apart and 2m^2 in area. The plates are in vacuum. A potential difference of 10,000 V is applied across a capacitor. Calculate:-

(a) the capacitance:

(b) the charge on each plate ;

(c) the electric field in space between the plates;

(d) the energy stored in the capacitor

Solution:

(a) $C = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 2}{5 \times 10^{-3}}$ = 0.00354 μF

(b) $Q = CV = (0.00354 \times 10^{-6}) \times (10{,}000) = \mathbf{3.54\ \mu C}$

The plate at higher potential has a positive charge of +3.54 μC and the plate at lower potential has a negative charge of – 3.54 μC.

(c) $E = \frac{V}{d} = \frac{10{,}000}{0.005} = 2 \times 10^6$ V/m

(d) energy = $\frac{1}{2}CV^2 = \frac{1}{2}0.00354 \times 10^{-6} \times 100 \times 10^6 = 0.177$ J

23. EFFECT OF DIELECTRIC

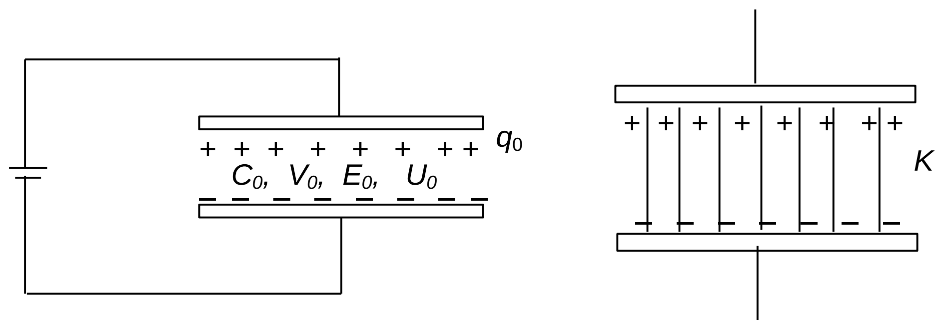
When certain non-conducting materials such as glass, paper or plastic are introduced between the plates of a capacitor, its capacity increases. These materials are called 'dielectrics' and the ratio of capacity of a capacitor when completely filled with dielectric C to that without dielectric C_0 is called 'dielectric constant K, or relative permitivity ε_r or specific inductive capacity (S. I. C) i.e.,

$$K = \frac{C}{C_0} \qquad \ldots (\mathbf{42})$$

The effect of dielectric on other physical quantities such as charges, potential difference, field and energy associated with a capacitor depends on the fact that whether the charge capacitor is isolated (i.e., charge held constant) or battery attached (i.e., potential is held constant).

23.1 INTRODUCTION OF A DIELECTRIC SLAB OF DIELECTRIC CONSTANT K BETWEEN THE PLATES

(a) When the battery is disconnected



Let q_0, C_0, V_0, E_0 and U_0 represents the charge, capacity, potential difference electric field and energy associated with charged air capacitor respectively. With the introduction of a dielectric slab of dielectric constant K between the plates and the battery disconnected.

(i) Charge remains constant, i.e., $q = q_0$, as in an isolated system charge is conserved.

(ii) Capacity increases, i.e., $C = KC_0$, as by the presence of a dielectric capacity becomes K times.

(iii) potential difference between the plates decreases, i.e., $V = \left(\frac{V_0}{K}\right)$,as

$$V = \frac{q}{C} = \frac{q_0}{KC_0} = \frac{V_0}{K} \quad [\because q = q_0 \text{ and } C = KC_0]$$

(iv) Field between the plates decreases, i.e., $E = \frac{E_0}{K}$, as

$$E = \frac{V}{d} = \frac{V_0}{Kd} = \frac{E_0}{K} \quad [\text{as } V = \frac{V_0}{K}]$$

and $$E_0 = \frac{V_0}{d}$$

(v) Energy stored in the capacitor decreases i.e.,

$$U = \left(\frac{U_0}{K}\right), \text{ as}$$

$$U = \frac{q^2}{2C} = \frac{q_0^2}{2KC_0} = \frac{V_0}{K} \text{ (as } q = q_0 \text{ and } C = KC_0]$$

(b) When the battery remains connected (potential is held constant)

(i) Potential difference remains constant, i.e., $V = V_0$, as battery is a source of constant potential difference.

(ii) Capacity increases, i.e., $C = KC_0$, as by presence of a dielectric capacity becomes K times.

(iii) Charge on capacitor increases, i.e., $q = Kq_0$, as

$$q = CV = (KC_0)V = Kq_0 \; [\because \; q_0 = C_0v]$$

(iv) Electric field remains unchanged, i.e., $E = E_0$, as

$$E = \frac{V}{d} = \frac{V_0}{d} = E_0 \text{ [as } V = V_0 \text{ and } \frac{V_0}{d} = E_0]$$

(v) Energy stored in the capacitor increases,

i.e., $U = KU_0$, as $\quad U = \frac{1}{2} CV^2 = \frac{1}{2} (KC_0) (V_0)^2 = \frac{1}{2} KU_0$

[as $C = KC_0$ and $U_0 = \frac{1}{2} C_0 V_0^2$]

Illustration 38.

A parallel plate air capacitor is made using two square plates each of side 0.2 m, spaced 1 cm apart. It is connected to a 50 V battery.

(a) What is the capacitance?

(b) What is the charge or each plate?

(c) What is the energy stored in the capacitor?

(d) What is the electric field between the plates?

(e) If the battery is disconnected and then the plates are pulled apart to a separation of 2cm. What are the answers to the above parts?

Solution:

(a) $C_0 = \frac{\varepsilon_0 A}{d} = \frac{(8.85\times10^{-12})\times0.02\times0.02}{0.01} = \mathbf{54\times10^{-5}\ \mu F}$

(b) $Q_0 = C_0V_0 = (3.54 \times 10^{-5} \times 50) = \mathbf{1.77 \times 10^{-3}\ \mu C}$

(c) $U_0 = \frac{1}{2} C_0V_0^2 = \frac{1}{2} \times (3.54 \times 10^{-11}) (50)^2 = \mathbf{4.42 \times 10^{-8}\ J}$

(d) $E_0 = \frac{V_0}{d} = \frac{50}{0.01} = \mathbf{5000\ V/m}$

(e) If the battery is disconnected the charge on the capacitor plates remains constant while the potential difference between the plates can change.

$$C = \frac{\varepsilon_0 A}{d} = \frac{C_0}{2} = \mathbf{27 \times 10^{-5}\ \mu F}$$

$$Q = Q_0 = \mathbf{1.77 \times 10^{-3}\ \mu C}$$

$$V = \frac{Q}{C} = \frac{Q_0}{C_0/2} = 2V_0$$

$$U = \frac{1}{2}CV^2 = C_0 V_0{}^2 = 8.84 \times 10^{-8}\ \text{J}$$

$$E = \frac{V}{d} = \frac{2V_0}{2d_0} = E_0 \quad = \mathbf{5000\ V/m}$$

Illustration 39.

In the last example, suppose that the battery is kept connected while the plates are pulled apart. What are the answers to the parts (a), (b), (c) and (d) in that case?

Solution:

If the battery is kept connected, the potential difference across the capacitor plates always remains equal to the emf of battery and hence is constant.

$$V = V_0 = \mathbf{50\ V}$$

$$C = \frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 A}{2d} = \frac{C_0}{2} = \mathbf{27 \times 10^{-5}\ \mu F}$$

$$Q = CV = \frac{C_0 V_0}{2} = \frac{Q_0}{2} = \mathbf{8.85 \times 10^{-4}\ \mu C}$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{C_0}{2}\right)V_0{}^2 = \frac{U_0}{2} = \mathbf{2.21 \times 10^{-8}\ J}$$

$$E = \frac{V}{d} = \frac{V_0}{2d_0} = \frac{E_0}{2} = \mathbf{2500\ V/m}$$

Illustration 40.

A parallel plate capacitor has plates of area 4 m^2 separated by a distance of 0.5 mm. The capacitor is connected across a cell of emf 100 V.

(a) Find the capacitance, charge and energy stored in the capacitor.

(b) A dielectric slab of thickness 0.5 mm is inserted inside this capacitor after it has been disconnected from the cell. Find the answers to part (a) if $K = 3$.

Solution:

(a) $C_0 = \dfrac{\varepsilon_0 A}{d} = \dfrac{8.85 \times 10^{-12} \times 4}{5 \times 10^{-3}} = \mathbf{7.08 \times 10^{-2}\ \mu F}$

$Q_0 = C_0V_0 = (7.08 \times 10^{-2} \times 100)\ \mu C = \mathbf{7.08\ \mu C}$

$U_0 = \frac{1}{2} C_0V_0^2 = \mathbf{3.54 \times 10^{-6}\ J}$

(b) As the cell has been disconnected

$Q = Q_0$

$C = \frac{K\varepsilon_0 A}{d} = KC_0 = \mathbf{0.2124\ \mu F}$

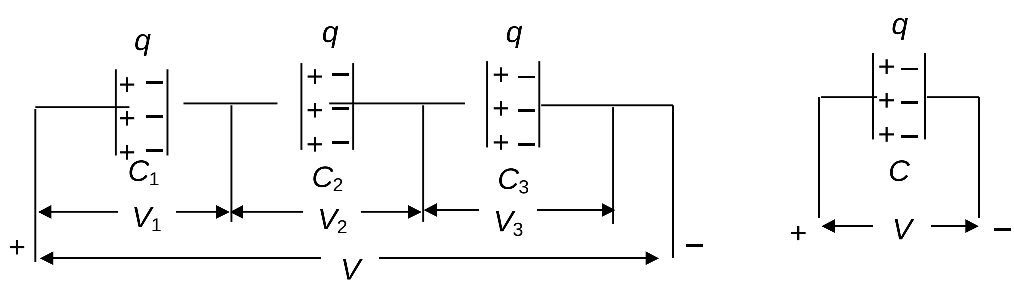
$V = \frac{Q}{C} = \frac{Q_0}{KC_0} = \frac{V_0}{K} = \mathbf{\frac{100}{3} V}$

$U = \frac{Q_0^2}{2C} = \frac{Q_0^2}{2KC_0} = \frac{U_0}{K} = \mathbf{118 \times 10^{-6}\ J}$

24. GROUPING OF CAPACITORS

Replacing a combination of capacitors by a single equivalent capacitor is called 'grouping of capacitors'. It simplifies the problem and is divided into two types

24.1 SERIES COMBINATION OF CAPACITORS



Capacitors are said to be in series if charge on each individual capacitor is same.

In this situation,

$$V = V_1 + V_2 + V_3$$

We know, $V = \left(\frac{q}{C}\right)$, so

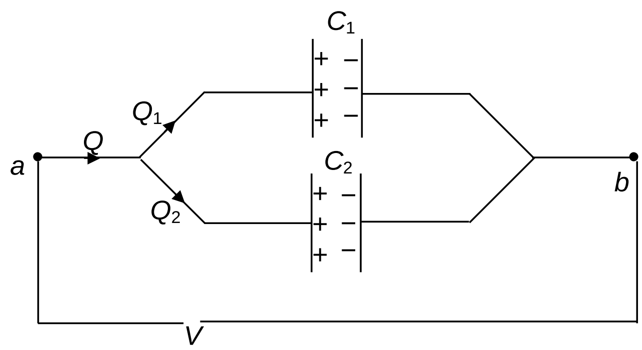
$$\frac{q}{C} = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}$$

or, $$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \qquad \ldots \textbf{(43)}$$

In case the two capacitors connected in series, we have

$$V_1 = \left(\frac{C_2}{C_1 + C_2}\right)V;\ V_2 = \left(\frac{C_1}{C_1 + C_2}\right)V$$

24.2 PARALLEL COMBINATION OF CAPACITORS



When capacitors are connected in parallel, the potential difference V across each is same and the charge on C_1, C_2 is different, i.e., Q_1 and Q_2,

The total charge Q is given as

$$Q = Q_1 + Q_2$$

$$Q = C_1V + C_2V \text{ or } \frac{Q}{V} = C_1 + C_2$$

Hence, the equivalent capacitance between a and b is

$$C = C_1 + C_2$$

The charges on capacitors is given as

$$Q_1 = \left(\frac{C_1}{C_1 + C_2}\right)Q$$

$$Q_2 = \left(\frac{C_2}{C_1 + C_2}\right)Q$$

In case of more than two capacitors.

$$C = C_1 + C_2 + C_3 + \ \ldots.. \qquad \ldots \textbf{(44)}$$

Illustration 41.

Two capacitors of capacitances C_1 = 6 μF and C_2 = 3μF are connected in series across a cell of emf 18 V. Calculate :-

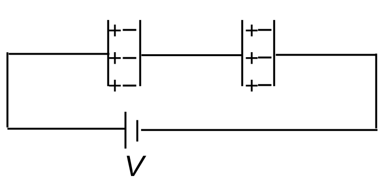
(i) The equivalent capacitance

(ii) the potential difference across each capacitor

(iii) the charge on each capacitor

Solution:

(i) $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$



$$\Rightarrow C = \frac{C_1C_2}{C_1+C_2} = \frac{6\times 3}{6+3} = \mathbf{2\ \mu F}$$

(ii) $V_1 = \left(\dfrac{C_2}{C_1+C_2}\right)V = \left(\dfrac{6}{6+3}\right)\times 18 = \mathbf{12\ V}$

(iii) $Q_1 = Q_2 = C_1V_1 = C_2V_2 = \boldsymbol{CV}$

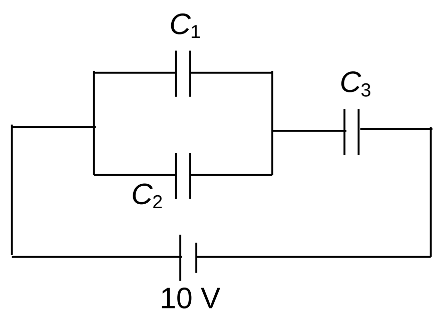
Illustration 42.

In the circuit shown above, the capacitors are C_1 = 15 μF ; C_2 = 10 μF and C_3 = 25 μF. Find

(i) the equivalent capacitance of the circuit.

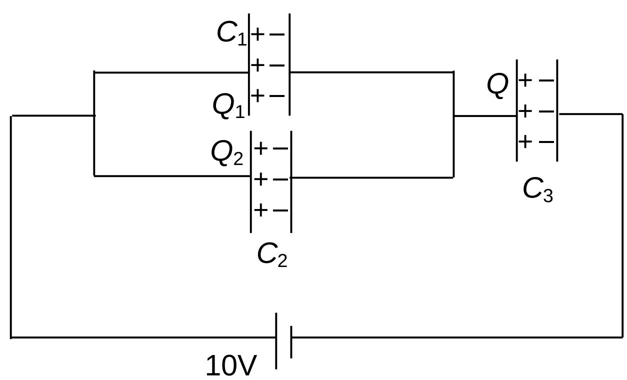
(ii) the charge on each capacitor, and

(iii) the potential difference across each capacitor.



Solution:

(i) $\dfrac{(C_1+C_2)C_3}{(C_1+C_2)+C_3} = \left(\dfrac{25\times 25}{25+25}\right)\mu F = \mathbf{12.5\ \mu F}$



(ii) Q = total charge supplied by the cell = CV = (12.5 × 10) μC = **125 μC.**

Charge on $C_1 = Q_1 = \left(\dfrac{C_1}{C_1+C_2}\right)Q = \left(\dfrac{15}{15+10}\right)\times 125 = \mathbf{75\ \mu C}$

charge on $C_2 = Q_2 = \left(\dfrac{C_2}{C_1+C_2}\right)Q = \left(\dfrac{10}{15+10}\right)\times 125 = \mathbf{50\ \mu C}$

Charge on $C_3 = Q$ = 125 μC

(iii) P.D. across $C_1 = V_1 = \dfrac{Q_1}{C_1} = \dfrac{75}{15} = \mathbf{5\ V}$

P.D. across $C_2 = V_2 = V_1 = \mathbf{5V}$

P.D. across $C_3 = V_3 = \frac{Q_3}{C_3} = \frac{125}{25}$ = **5V**

25. REDISTRIBUTION OF CHARGE

If there are two spherical conductors of radii R_1 and R_2 at potentials V_1 and V_2 respectively, far apart from each-other (so that charge on one does not affect the potential of the other). The charge on them will be

$$q_1 = C_1V_1 \text{ and } q_2 = C_2 V_2$$

The total charge of the system

$$q = q_1 + q_2$$

The total capacity of the system

$$C = C_1 + C_2$$

Now if they are connected through a wire, charge will flow from conductor at higher potential to that at lower potential till both acquire same potential.

$$V = \frac{(q_1 + q_2)}{(C_1 + C_2)} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2} = \frac{R_1V_1 + R_2V_2}{R_1 + R_2} \quad (\because C \propto R)$$

If q_1' and q_2' are the charges on two conductors after sharing, then

$$q_1' = C_1V \text{ and } q_2' = C_2V \text{, where}$$

$$q_1' + q_2' = (q_1 + q_2) = q$$

So, $$\frac{q_1'}{q_2'} = \frac{C_1}{C_2} = \frac{R_1}{R_2} \quad [\text{ as } C \propto R)$$

i.e., charge is shared in proportion to capacity.

25.1 LOSS OF ENERGY DURING REDISTRIBUTION OF CHARGE

Initial potential energy of the system is

$$U_I = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$$

Final potential energy = $U_F = \frac{1}{2}(C_1 + C_2)V^2$

Putting $V = \frac{R_1V_1 + R_2V_2}{R_1 + R_2}$ and simplifying, we get

$$U_F - U_I = -\frac{C_1C_2}{2(C_1 + C_2)}(V_1 \sim V_2)^2$$

Now as C_1, C_2 and $(V_1 \sim V_2)^2$ are always positive, there is decrease in energy of the system, i.e., in sharing energy is lost. This energy is lost mainly as heat when charge flows from one body to the other through the connecting wire and also as light and sound if sparking takes place.

Illustration 43.

Two isolated metallic solid spheres of radius R and $2R$ are charged such that both of these have same charge density σ. The spheres are located far away from each-other and connected by a thin conducting wire. Find the new charge density on the bigger sphere.

Solution:

As charge density σ on both spheres is same, total charge

$$q = q_1 + q_2 = 4\pi (R)^2\sigma + 4\pi (2R)^2\sigma = 20\pi R^2\sigma \qquad \dots \text{(i)}$$

Now in sharing, charge is shared in proportion to capacity i.e., radius, so charge on the bigger sphere

$$q_2' = \left(\frac{R_2}{R_1 + R_2}\right)q = \left(\frac{2R}{R+2R}\right)q = \frac{2q}{3}$$

So charge density on bigger sphere after sharing

$$\sigma_2' = \frac{q_2'}{4\pi(2R)^2} = \frac{\left(\frac{2}{3}\right)q}{16\pi R^2} = \frac{q}{2\pi R^2}$$

Putting the value of q from equation (i), we get

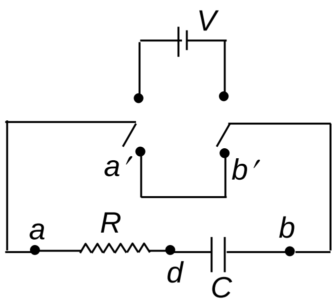
$$\sigma_2' = \frac{20\pi R^2\sigma}{24\pi R^2} = \mathbf{\frac{5\sigma}{6}},$$

26. R-C CIRCUIT

26.1 CHARGING OF A CAPACITOR

The circuit diagram for charging and discharging a capacitor is given here. A resistor R and a capacitor C are connected with a double throw switch, by which the battery V can be connected in the circuit.

Initially, the capacitor is uncharged and when the switch is thrown to include the battery in the circuit, charge flows to the capacitor through the resistance R. This is called the charging current. The current continues till the voltage V_{db} across the capacitor is equal to the voltage V of the battery. If during the charging process the instantaneous current is I at an instant, and the potential difference between a and d, and that between d and b are V_{ad} and V_{db} respectively.



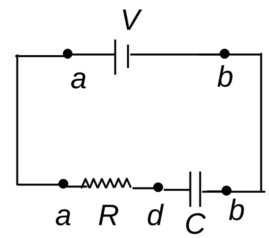
$$V_{ad} = iR \text{ and } V_{db} = \frac{q}{C} \qquad \dots \text{(i)}$$

where q is the charge on the capacitor at the instant

$$V_{ab} = V = V_{ad} + V_{db} = iR + \frac{q}{C} \qquad \dots \text{(ii)}$$

where V is the constant voltage of the battery.

$$i=\frac{V}{R}-\frac{q}{RC} \qquad \text{… (ii)}$$



Initially, as soon as the connection is made there is a current $I_0=\frac{V}{R}$, since the charge on the capacitance is zero

As the charging continues, q increases and i decreases and finally becomes zero. At that time,

$$\frac{V}{R}-\frac{q}{RC}=0$$

$q = CV = Q_0$, where Q_0 is the final charge on the capacitor.

In equation (iii), i may be written as $\frac{dq}{dt}$, so

$$\frac{dq}{dt}=\frac{V}{R}-\frac{q}{RC}$$

$$\frac{dq}{VC-q}=\frac{dt}{RC}$$

Integrating both sides, we have

$$\int_0^q \frac{dq}{VC-q} \int_0^t \frac{dt}{RC}$$

$$-\left[\ln(VC-q)\right]_0^q=\frac{1}{RC}\left[t\right]_0^t$$

or, $$-\ln\left(\frac{VC-q}{VC}\right)=\frac{t}{RC}$$

or, $$\left(1-\frac{q}{VC}\right)=e^{\frac{-t}{RC}}$$

or, $$\frac{q}{CV}=\left(1-e^{\frac{-t}{RC}}\right)$$

or, $$q = \mathrm{CV}\left(1-e^{-\frac{t}{RC}}\right)$$

or, $$q = q_0\left(1-e^{\frac{-t}{RC}}\right) \qquad \text{… } \mathbf{(45)}$$

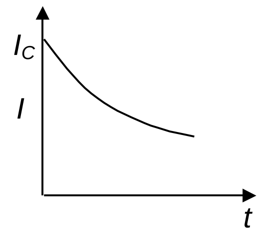
Differentiating with respect to time.

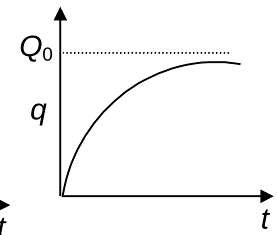
$$i=\frac{dq}{dt}=\frac{d}{dt}[VC(1-e^{-t/RC})]=\frac{V}{R}e^{-t/RC}=I_0e^{-t/RC}$$

We see that the charge and current both follow the exponential law.

When time $t=RC, q=Q_0\left(1-\frac{1}{e}\right)$ and $i=\frac{I_0}{e}$

This means that at $t = RC$, charge has increased to $1-\frac{1}{e}$ = 63% of its final value. The current has decreased to $\frac{1}{e}$ = 37% of its initial value. The time $t = RC$ is called the **time constant** of the circuit.





The half-life of the circuit t_h is the time at which the current is half its initial value and the charge on the capacitor is half its final value $t_h = RC\ln 2 = 0.693RC$

26.2 DISCHARGING OF A CAPACITOR

If in the circuit, the switch is thrown to the down position, the battery is removed from the circuit and the capacitor is resistance R.

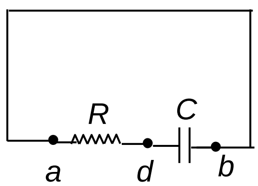
When the switch is thrown, the charge on the capacitor is Q_0 and the potential difference V_{ab} is zero.

Thus $V_{ad}+V_{db}=0$

$$V_{ad}=-V_{db}=iR$$

$$i=\frac{q}{RC}$$

when $t=0, q=Q_0$ and $I_0=\frac{Q_0}{RC}=\frac{V_0}{R}$, where V_0 is the initial potential difference across the capacitor.



When the capacitor is discharging both the charge and the current decrease, in equation $i=\frac{q}{RC}, i=-\frac{dq}{dt}$, since q is decreasing

$$\frac{dq}{dt}=-\frac{q}{RC}$$

$$\frac{dq}{q}=-\frac{dt}{RC}$$

$$\ln q=-\frac{t}{RC}+\text{Constant}$$

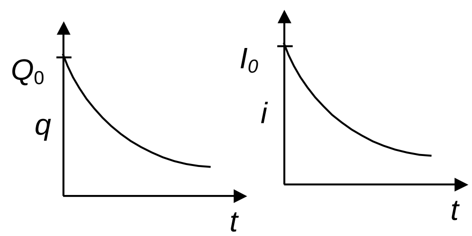
At $\quad t=0, q=Q_0$

Constant $= lnQ_0$

$$lnq = -\frac{t}{RC} + lnQ_0;\ ln\frac{q}{Q_0} = -\frac{t}{RC}$$

$$q = Q_0 e^{-t/RC} \qquad \dots (\mathbf{46})$$

Differentiating the equation $i = \frac{q}{RC}$



$$\frac{dl}{dt} = -\frac{l}{RC}$$

$$\frac{di}{i} = -\frac{dt}{RC}$$

$$lni = -\frac{t}{RC} + \text{Constant}$$

$$ln\frac{i}{l_0} = \frac{-t}{RC}$$

$$i = l_0 e^{-t/RC} \qquad \dots (\mathbf{47})$$

At time $\quad t = CR, q = \frac{Q_0}{e} \quad i = \frac{I_0}{e}$

Illustration 44.

A capacitor of a capacity 0.1 pF is first charged and then discharged through a resistance of 10 megaohms. Find the time in which the potential will fall to half its original value.

Solution:

The charge at any instant

$$q = q_0 e^{-t/RC},\ t = RC\log_e\left(\frac{q_0}{q}\right)$$

If V is the potential of the capacitor after time t and V_0 is the initial potential, then

$$\frac{q_0}{q} = \frac{CV_0}{CV} = \frac{V_0}{V}$$

$$t = RC\log_e\left(\frac{V_0}{V}\right) = 10^7 \times 0.1 \times 10^{-6} \log_e{}^2$$

$= \mathbf{0.6931\ s}$

Illustration 45.

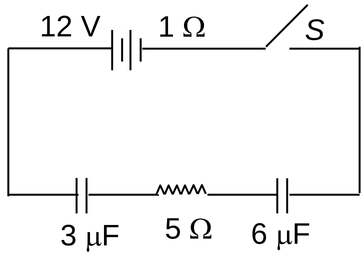
Two capacitors are charged in series by a 12 V battery.

(a) What is the time constant of the circuit?

(b) After being closed for the time CR, the switch is opened. What is the potential difference across the 6 μF capacitor?

Solution:

(a) Effective capacitance $=\frac{6\times3}{6+3}$ = $\mathbf{2\mu F}$



Total resistance $=1+5=6\Omega$

Time constant $=CR=2\times10^{-6}\times6$ **= 12μs**

(a) After 12 μs, the charge accumulated on the equivalent capacitor of 2μs would be $q=q_0(1-e^{-1})=CV\left(1-\frac{1}{6}\right)=(1-0.368)CV=0.632\times12\times2\times10^{-6}$

This is the charge on each plate of the capacitors.

Voltage on 6 μF $=\frac{0.632\times24\times10^{-6}}{6\times10^{-6}}$ = **2.53 V**