DEFINITION

Definite integral, which is used in various field of Mathematics, Physics and Chemistry, symbolically $\int_a^b f(x)dx$ is the integration of $f(x)$ w.r.t. x with $x = a$ as lower limit and $x = b$ as upper limit.

If $\int f(x)dx = g(x)+c$, then $\int_a^b f(x)dx = \lim_{x\to b^-} g(x) - \lim_{x\to a^+} g(x)$

Generally, we write $\int_a^b f(x)dx = g(b)-g(a)$.

1. GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

Let f(x) be a function defined on a closed interval [a, b]. Then $\int_a^b f(x)dx$ represents the algebraic sum of the areas of the region bounded by the curve y = f(x), x–axis and the lines x = a, x = b. Here algebraic sum means that area which is above the x–axis will be added in this sum with + sign and area which is below the x–axis will be added in this sum with – sign. So value of the definite integral may be positive, zero or negative.

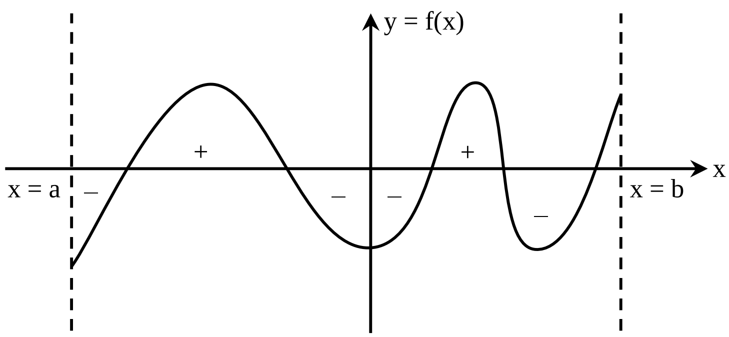
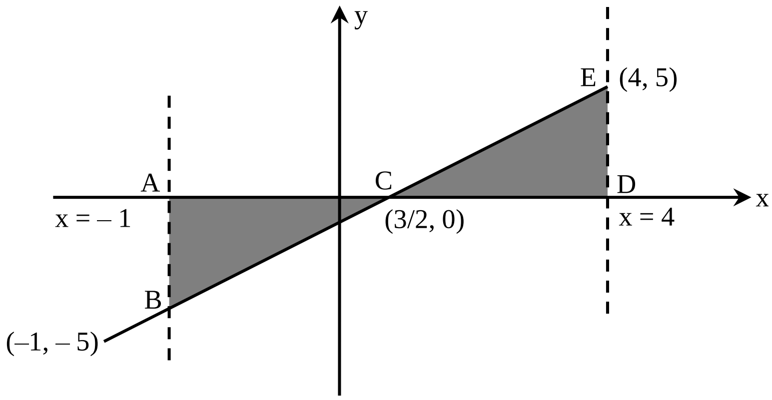


Illustration 1:

Evaluate $\int_{-1}^{4}(2x-3)\,dx$.

Solution:

$y = 2x - 3$ is a straight line, which lie below the x-axis in $\left[-1,\frac{3}{2}\right)$ and above in $\left(\frac{3}{2},4\right]$



Now area of $\Delta ABC = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

Area of $\Delta CDE = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

So $\int_{-1}^{4} (2x-3)\,dx = -\frac{25}{4} + \frac{25}{4} = 0$

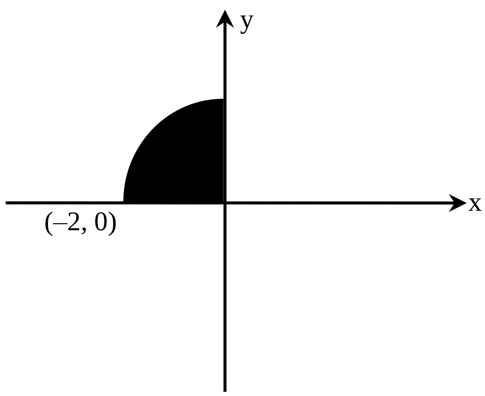
Illustration 2:

Evaluate $\int_{-2}^{0} \sqrt{4-x^2}\,dx$.

Solution:

$y = \sqrt{4-x^2}$, $x \in [-2, 0]$

Represents a quarter circle in 2nd quadrant, which is above the x-axis radius of circle is 2.



so $\int_{-2}^{0} \sqrt{4-x^2}\,dx = \frac{1}{4}[\pi(2)^2] = \pi$ square unit

2. FUNDAMENTAL THEOREMS

If $f(x)$ is a continuous function on $[a, b]$, then $\frac{d}{dx}\int_a^x f(t)dt = f(x),\ x \in [a, b]$

Illustration 3:

Evaluate $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$.

Solution:

$$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$$

So $\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}}\Big|_0^1 = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + c - \sin^{-1}(0) - c$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

3. GENERAL PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt$

Illustration 4:

Evaluate $\int_4^9 \frac{dx}{\sqrt{x}(1+\sqrt{x})}dx$

Solution:

$$I = \int_4^9 \frac{dx}{\sqrt{x}(1+\sqrt{x})}dx$$

Put $1+\sqrt{x} = t$

$$\Rightarrow \frac{dx}{2\sqrt{x}} = dt \Rightarrow \frac{dx}{\sqrt{x}} = 2dt$$

Now when $x = 4$, $t = 1+\sqrt{4} = 3$

when $x = 9$, $t = 1+\sqrt{9} = 4$

So $I=\int_3^4 \frac{2dt}{t}=2[|\ell n|t||]_3^4=2(\ell n4-\ell n3)=\ell n\left(\frac{16}{9}\right)$

2. $\int_a^b f(x)dx \pm \int_a^b g(x)dx=\int_a^b [f(x)\pm g(x)]dx$

Illustration 5:

Evaluate $\int_2^3 \left(\frac{2x^2}{x^4+3x^2+1}\right)dx$.

Solution:

$$I=\int_2^3\left(\frac{2x^2}{x^4+3x^2+1}\right)dx=\int_2^3\frac{(x^2+1)}{x^4+3x^2+1}dx+\int_2^3\frac{(x^2-1)\,dx}{x^4+3x^2+1}$$

$$=\int_2^3\frac{(1+(1/x^2))\,dx}{(x-(1/x))^2+5}+\int_2^3\frac{(1-(1/x^2))\,dx}{(x+(1/x))^2+1}$$

In 1st put $x-\frac{1}{x}=t$, in 2nd put $x+\frac{1}{x}=y$

$$I=\int_{3/2}^{8/3}\frac{dt}{t^2+5}+\int_{5/2}^{10/3}\frac{dy}{y^2+1}$$

$$=\frac{1}{\sqrt{5}}\left[\tan^{-1}\left(\frac{8}{3\sqrt{5}}\right)-\tan^{-1}\left(\frac{3}{2\sqrt{5}}\right)\right]+\tan^{-1}\left(\frac{10}{3}\right)-\tan^{-1}\left(\frac{5}{2}\right)$$

$$=\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{7\sqrt{5}}{54}\right)+\tan^{-1}\left(\frac{5}{56}\right)$$

3. $\int_a^b f(x)dx=\int_a^b f(t)dt=\int_a^b f(y)dy$

Illustration 6:

Evaluate $\int_{-1}^{1} f(x)\,dx$, where $f(x)=\begin{cases}1-2x, & x\le 0\\ 1+2x, & x\ge 0\end{cases}$.

Solution:

$$\int_{-1}^{1} f(x)\,dx=\int_{-1}^{0} f(x)\,dx+\int_0^1 f(x)\,dx=\int_{-1}^{0}(1-2x)\,dx+\int_0^1(1+2x)\,dx$$

$$=[x-x^2]_{-1}^{0}+[x+x^2]_0^1=4$$

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Illustration 7:

Evaluate $\int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$.

Solution:

$$I = \int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}$

So $I = \int_{1/2}^{1/3} \frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\frac{4}{t^2}+1}} = -\int_{1/2}^{1/3} \frac{dt}{\sqrt{4+t^2}}$

$$= \ell n\left(t+\sqrt{4+t^2}\right)\Big|_{1/3}^{1/3} = \ell n\left(\frac{3}{2}\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right)$$

5. $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^{b} f(x)dx$

Illustration 8:

Evaluate $\int_{-2}^{3} |x^2-1|dx$.

Solution:

$$\int_{-2}^{3}|x^2-1|dx = \int_{-2}^{-1}|x^2-1|dx + \int_{-1}^{1}|x^2-1|dx + \int_{1}^{3}|x^2-1|dx$$

(Here modulus function will change at the points, when $x^2 - 1 = 0$ i.e. at $x = \pm 1$)

So $I = \int_{-2}^{-1}(x^2-1)\,dx + \int_{-1}^{1}(1-x^2)\,dx + \int_{1}^{3}(x^2-1)\,dx$

$$= \frac{x^3}{3} - x \Big|_{-2}^{-1} + x + \frac{x^3}{3} \Big|_{-1}^{1} + \frac{x^3}{3} - x \Big|_{1}^{3}$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + 6 + \frac{2}{3} = \frac{28}{3}$$

6. $$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Illustration 9:

Evaluate $\int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}}$.

Solution:

$$\int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}} \qquad \dots \text{(i)}$$

$$I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{9-(-x)}}dx$$

$$I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}dx \qquad \dots \text{(ii)}$$

adding (i) and (ii), we get

$$2I = \int_1^7 \left(\frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} + \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}} \right) dx \;= \int_2^7 dx = x \Big|_2^7 = 5$$

So $I = \frac{5}{2}$

7. $$\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx$$

Illustration 10:

Evaluate $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$.

Solution:

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$$

Here $f(x)=\dfrac{1}{(1+e^{x})(1+x^{2})}$

$\Rightarrow\ f(-x)=\dfrac{1}{(1+e^{-x})\,(1+(-x)^{2})}=\dfrac{e^{x}}{(1+e^{x})(1+x^{2})}$

so $I=\displaystyle\int_{0}^{\sqrt{3}}\frac{dx}{1+x^{2}}=\tan^{-1}x\Big|_{0}^{\sqrt{3}}=\frac{\pi}{3}$

8. $$\int_{-a}^{a} f(x)dx=\begin{cases}2\displaystyle\int_{0}^{a} f(x)dx & \text{if } f(x) \text{ is an even function } \left(f(x)=f(-x)\right)\\ 0 & \text{if } f(x) \text{ is an odd function } \left(f(-x)=-f(x)\right)\end{cases}$$

Illustration 11:

Evaluate $\displaystyle\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx$

Solution:

$$I=\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx=\int_{-a}^{a}\frac{a-x}{\sqrt{a^{2}-x^{2}}}\,dx=a\int_{-a}^{a}\frac{dx}{\sqrt{a^{2}-x^{2}}}-\int_{-a}^{a}\frac{x\,dx}{\sqrt{a^{2}-x^{2}}}$$

$$=a.2\int_{0}^{a}\frac{dx}{\sqrt{a^{2}-x^{2}}}-0\quad\left(\because \frac{x}{\sqrt{a^{2}-x^{2}}}\text{ is an odd function}\right)$$

$$=2a\left[\sin^{-1}\frac{x}{a}\right]_{0}^{a}\Rightarrow 2a\,[\sin^{-1}(1)-\sin^{-1}(0)]=2a\left[\frac{\pi}{2}-0\right]=\pi a$$

Illustration 12:

Find $\displaystyle\int_{-1}^{1}x^{3}.e^{x^{4}}\,dx$.

Solution:

Let $f(x)=x^{3}e^{x^{4}}$, then $f(-x)=(-x)^{3}.\,e^{(-x)^{4}}=-x^{3}e^{x^{4}}=-f(x)$

Hence $f(x)$ is an odd function.

$\therefore\quad \displaystyle\int_{-1}^{1} f(x)\,dx=0$; or $\displaystyle\int_{-1}^{1}x^{3}e^{x^{4}}\,dx=0$

9. $$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$$

Illustration 13:

Evaluate $\int_0^{x} \frac{x\,dx}{1+\cos^2 x}$.

Solution:

$$I = \int_0^{x} \frac{x\,dx}{1+\cos^2 x}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos^2(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos^2 x}$$

Addition both, we get

$$2I = \int_0^{\pi} \frac{\pi\,dx}{1+\cos^2 x} \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\cos^2 x}$$

$$= \frac{\pi}{2}\left[\int_0^{\pi} \frac{dx}{1+\cos^2 x} + \int_0^{\pi/2} \frac{dx}{1+\cos^2(\pi - x)}\right]$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{2+\tan^2 x} \qquad \text{put } \tan x = t$$

$$I = \pi \int_0^{\infty} \frac{dt}{t^2+2}$$

$$= \frac{\pi}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\Big|_0^{\infty} = \frac{\pi^2}{2\sqrt{2}}$$

10. $$\int_a^{b} f(x)dx = (b-a)\int_0^{1} [f(b-a)x + a]dx$$

Illustration 14:

Evaluate $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$

Solution:

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$$

$$=2\int_0^{\pi/2}\frac{dx}{1+2\sin^2 x} \qquad \left(\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{if } f(2a-x)=f(x)\right)$$

$$=2\int_0^{\pi/2}\frac{\sec^2 xdx}{\sec^2 x+2\tan^2 x} \quad = 2\int_0^{\pi/2}\frac{\sec^2 xdx}{1+3\tan^2 x}$$

(Note that in the beginning we cannot divide Numerator and Denominator by $\cos^2 x$, as $\cos x = 0$ at $x = \pi/2$)

$$=2\int_0^{\infty}\frac{dt}{1+3t^2}, \qquad (\tan x = t)$$

$$=2\frac{1}{\sqrt{3}}\left[\tan^{-1} t\sqrt{3}\right]_0^{\infty} = \frac{2}{\sqrt{3}}\times\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

Illustration 15:

Prove that $\int_{-5}^{-4} e^{(x+4)^2}dx = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$.

Solution:

Let $I = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$

$$=3\left[\left(\frac{2}{3}-\frac{1}{3}\right)\right]\int_0^1 e^{9\left(\left(\frac{2}{3}-\frac{1}{3}\right)x+\frac{1}{3}-\frac{2}{3}\right)^2}dx$$

$$=\int_0^1 e^{9\left(\frac{x}{3}-\frac{1}{3}\right)^2}dx = \int_0^1 e^{(x-1)^2}dx$$

Also $\int_{-5}^{-4} e^{(x+4)^2}dx = \int_0^1 e^{(x-1)^2}dx$

Alternatively: $x + 4 = 3t - 2$

4. PERIODIC PROPERTIES OF DEFINITE INTEGRAL

1. If f (x) is a periodic function with period p, then $\int_a^{a+np} f(x)dx = n\int_0^p f(x)dx,\ n \in I$

Illustration 16:

Prove that $\int_0^{n\pi+v} |\sin x|\,dx = (2n+1)-\cos v$, where n $\in$ N and $0 \le v < \pi$.

Solution:

$$I = \int_0^{n\pi+v} |\sin x|dx = \int_0^{v} |\sin x|\,dx + \int_v^{n\pi+v} |\sin x|dx = I_1 + I_2$$

$$I_1 = \int_0^{v} |\sin x|dx = \int_0^{v} \sin dx \qquad \text{(as } 0 \le v < \pi \text{ and } \sin x \ge 0, \text{ when n} \in [0, \pi])$$

$$= -\cos x \Big|_0^{v} = -\cos v + 1 = 1 - \cos v$$

$$I_2 = \int_v^{n\pi+v} |(\sin x)|dx = n\int_0^{\pi} |(\sin x)|dx = n\int_0^{\pi} \sin x\,dx = n\,[-\cos x]_0^{\pi} = 2n$$

So $I = 1 - \cos v + 2n = (2n + 1) - \cos v$

2. If f (x) is a periodic function with period p, then $\int_{mp}^{np} f(x)dx = (n-m)\int_0^{p} f(x)dx,\ n, m \in I$

Illustration 17:

Evaluate $\int_{-3/2}^{10} \{2x\}\,dx$, where {.} denotes the fractional part of x.

Solution:

$f(x) = \{2x\}$ is a periodic function with period $\frac{1}{2}$

Let $I = \int_{-3/2}^{10} \{2x\}\,dx = \int_{-3(1/2)}^{20(1/2)} \{2x\}\,dx$

$= 23\int_0^{1/2} 2x\,dx \qquad$ (as $\{2x\} = 2x - [2x]$ and when $x \in [0, 1/2)$, $[2x] = 0$)

$= 23x^2 \Big|_0^{1/2} = \frac{23}{4}$

3. If f (x) is a periodic function with period p, then $\int_{a+np}^{b+np} f(x)dx = \int_a^{b} f(x)dx,\ n \in I$

Illustration 18:

Evaluate $\int_{10\pi+\frac{\pi}{6}}^{10\pi+\frac{\pi}{3}} (\sin x + \cos x)\, dx$.

Solution:

$f(x) = \sin x + \cos x$ is is periodic with period 2π

Let $I = \int_{10\pi+\frac{\pi}{6}}^{10\pi+\frac{\pi}{3}} (\sin x + \cos x)\, dx = \int_{\pi/6}^{\pi/3} (\sin x + \cos x)\, dx = (\sin x - \cos x)\Big|_{\pi/6}^{\pi/3}$

$$= \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right] - \left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right] = \left(\sqrt{3} - 1\right)$$

Illustration 19:

Find the value of $\int_0^{4\pi} |\sin x|\, dx$

Solution:

We know that |sin$x$| is a periodic function of π. Hence $\int_0^{4\pi} |\sin x|\, dx = 4\int_0^{4} |\sin x|\, dx = 4\int_0^{\pi} \sin x dx = 4[-\cos x]_0^{\pi} = 8$

5. DIFFERENTIATION OF DEFINITE INTEGRAL

1. If $F(x) = \int_{f_1(x)}^{f_2(x)} g(t) dt$, then $F'(x) = g\left(f_2(x)\right) f_2'(x) - g\left(f_1(x)\right) f_1'(x)$

2. If $F(x) = \int_a^b g(x,t) dt$, then $F'(x) = \int_a^b \left(\frac{\partial}{\partial x}\left(g\left(x,t\right)\right)\right) dt$, where $\frac{\partial g}{\partial x}$ represents partial derivative of *g(x, t)* w.r.to *x*.

Collectively

3. If $F(x) = \int_{f_1(x)}^{f_2(x)} g(t) dt$, then

$$F'(x) = g\left(x, f_2(x)\right) f_2'(x) - g\left(x, f_1(x)\right) f_1'(x) + \int_{f_1(x)}^{f_2(x)} \left(\frac{\partial}{\partial x}\left(g\left(x,t\right)\right)\right) dt$$

Illustration 20:

If a, b are variable real numbers such that $a + b = 4$, $a < 2$ and $f'(x) > 0\ \forall\ x \in R$, then prove that $\left(\int_0^a f(x)\,dx + \int_0^b f(x)\,dx\right)$ will increase as $(b - a)$ increases.

Solution:

Let $(b - a) = t$

$b + a = 4$

$\Rightarrow\ b = \frac{4+t}{2}, a = \frac{4-t}{2}$

Let $g(f) = \int_0^{\frac{4-t}{2}} f(x)\,dx + \int_0^{\frac{4+t}{2}} f(x)\,dx$

So, $g'(t) = f\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right) + f\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}\left[f\left(\frac{4+t}{2}\right) - f\left(\frac{4-t}{2}\right)\right]$

Now $a < 2$ and $a + b = 4$

$\Rightarrow a < b$

$\Rightarrow\ f\left(\frac{4-t}{2}\right) < f\left(\frac{4+t}{2}\right)$ (as $f'(x) > 0 \Rightarrow f(x)$ is increasing)

$\Rightarrow\ g'(t) > 0$

$\Rightarrow g(t)$ will increases as t increases

$\Rightarrow\ \int_0^a f(x)\,dx + \int_0^b f(x)\,dx$ will increases as $(b - a)$ increases

Use of differentiation to evaluate definite integrals

Illustration 21:

Evaluate $\int_0^1 \frac{x^n - 1}{\ln x}\,dx$.

Solution:

Let $f(n) = \int_0^1 \frac{x^n - 1}{\ln x}\,dx$, then

$$f'(n) = \int_0^1 \frac{x^n \ln x}{\ln x} dx = \int_0^1 x^n \, dx = \frac{1}{n+1}$$

Hence $f(n) = \ln(n+1) + c$

Now for $n = 0, f(n) = 0$

Hence $\int_0^1 \frac{x^n - 1}{\ln x} dx = \ln(n+1)$

6. APPROXIMATION IN DEFINITE INTEGRAL

1. If $f_1(x) \le f(x) \le f_2(x)\ \forall\ x \in [a, b]$, then $\int_a^b f_1(x)dx \le \int_a^b f(x)dx \le \int_a^b f_2(x)dx$

Illustration 22:

Prove that $\int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$.

Solution:

$$0 \le x \le 1$$

$$\Rightarrow 0 \le x^3 \le x^2 \le 1$$

$$\Rightarrow -x^2 \le -x^2 \le 0$$

$$\Rightarrow 4 - x^2 - x^2 \le 4 - x^2 - x^3 \le 4 - x^2$$

$$\Rightarrow \frac{1}{\sqrt{4 - x^2}} \le \frac{1}{\sqrt{4 - x^2 - x^3}} \le \frac{1}{\sqrt{4 - 2x^2}}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4 - x^2}} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \int_0^1 \frac{dx}{\sqrt{4 - 2x^2}}$$

$$\Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \le \frac{\pi}{4\sqrt{2}}$$

2. If absolute maximum and minimum value of $f(x)$, when $x \in [a, b]$ is M and m respectively, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

Illustration 23:

Prove that $\dfrac{\pi}{\pi^3+10\pi+5} < \displaystyle\int_0^{\pi} \dfrac{dx}{x^3+10x+9\sin x+5} < \dfrac{\pi}{5}$.

Solution:

Let $f(x) = x^3 + 10x + 9\sin x + 5$

$f'(x) = 3x^2 + 10 + 9\cos x > 0 \ \forall\ x \in R$

$\Rightarrow f(x)$ is entirely increasing $\Rightarrow \dfrac{1}{f(x)}$ is decreasing in $(0, \pi)$

$\Rightarrow$ Absolute maximum of $f(x)$ in $[0, \pi]$ is $\dfrac{1}{5}$ and absolute minimum is $\dfrac{1}{\pi^3+10\pi+5}$

so $\dfrac{\pi}{\pi^3+10\pi+5} < \displaystyle\int_0^{\pi} \dfrac{dx}{x^3+10x+9\sin x+5} < \dfrac{\pi}{5}$

Illustration 24:

Estimate the integral $\displaystyle\int_1^3 \sqrt{3+x^3}\, dx$

Solution:

The function $f(x) = \sqrt{3+x^3}$ increases monotonically on the interval $[1, 3]$.

M = maximum value of $\sqrt{3+x^3} = \sqrt{3+3^3} = \sqrt{30}$

m = minimum value of $\sqrt{3+1^3} = \sqrt{4} = 2$

$b - a = 2$

$\therefore\ 2.2 \le \displaystyle\int_1^3 \sqrt{3+x^3}\, dx \le 2\sqrt{30}$ or $4 \le \displaystyle\int_1^3 \sqrt{3+x^3} \le 2\sqrt{30}$

7. DEFINITE INTEGRAL OF PIECEWISE CONTINUOUS FUNCTIONS

Suppose we have to evaluate $\displaystyle\int_a^b f(x)dx$, but either f (x) is not continuous at x = c_1, c_2, ..., cn or it is not defined at these points. In both cases we have to break the limit at c_1, c_2, ..., cn.

Illustration 25:

Evaluate $\int_1^2 [x^3 - 1]\,dx$ where [.] denotes the greatest integer function.

Solution:

$1 \le x \le 2 \quad \Rightarrow \quad 1 \le x^3 \le 8 \quad \Rightarrow \quad 0 \le x^3 - 1 \le 7$

So $\quad I = \int_1^2 [x^3 - 1]dx = \int_1^{2^{1/3}} [x^3 - 1]dx + \int_{2^{1/3}}^{3^{1/3}} [x^3 - 1]dx + ... + \int_{7^{1/3}}^{2} [x^3 - 1]dx$

Now if $x \in \left[1, 2^{\frac{1}{3}}\right)$, then $x^3 \in [1, 2)$ or $[x^3 - 1] = 0$ and so on

therefore $I = \int_1^{2^{1/3}} 0.dx + \int_{2^{1/3}}^{3^{1/3}} 1.dx + ... \int_{7^{1/3}}^{2} 6.dx$

$= [3^{1/3} - 2^{1/3}] + 2\,[4^{1/3} - 3^{1/3}] + 3\,[5^{1/3} - 4^{1/3}] + 4\,[6^{1/3} - 4^{1/3}] + 4\,[6^{1/3} - 5^{1/3}] + 6\,[2 - 7^{1/3}]$

$= 12 - [7^{1/3} + 6^{1/3} + 5^{1/3} + 4^{1/3} + 3^{1/3} + 2^{1/3}]$

Illustration 26:

Prove that $\int_a^b \frac{|x|}{x} dx = |b| - |a|$.

Solution:

We can divide all the possible values of *a* and *b* in 3 cases

Case I: $\quad 0 \le a < b$

$$I = \int_a^b \frac{|x|}{x} dx = \int_a^b \frac{x}{x} dx = b - a = |b| - |a|$$

Case II: $\quad a < b \le 0$

$$I = \int_a^b \frac{|x|}{x} dx = \int_a^b \frac{-x}{x} dx = a - b = -|a| - (-|b|) = |b| - |a|$$

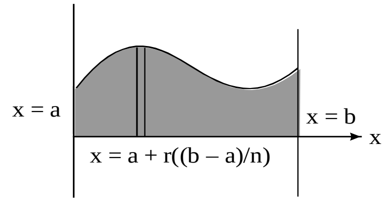
Case III: $\quad a < 0 < b$

$$I = \int_a^b \frac{|x|}{x} dx = \int_a^0 \frac{|x|}{x} dx + \int_a^b \frac{|x|}{x} dx$$

$$= \int_a^0 (-1)\,dx + \int_0^b 1 dx$$

$= a + b = -|a| + |b| = |b| - |a|$

8. DEFINITE INTEGRAL AS THE LIMIT OF A SUM



Consider $\int_a^b f(x)dx$, for simplicity, we can take $f(x) \geq 0\ \forall$ $x \in [a, b]$. Then $\int_a^b f(x)dx$ represents the area bounded by the curve $y = f(x)$ x–axis and the lines $x = a$ and $x = b$.

Now this area can be divided into n parts.

Area of the r^{th} part can be assumed a rectangle, with width equal to $\left(\frac{b-a}{n}\right)$ and height equal to $f\left[a+r\left(\frac{b-a}{n}\right)\right]$

So the area $= \sum_{r=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+r\left(\frac{b-a}{n}\right)\right)$ but this in only approximated area. To get the actual area, take rectangle with width tends to zero.

Hence, $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+r\left(\frac{b-a}{n}\right)\right)$

Specifically, $\lim_{n\to\infty}\frac{1}{n}\sum_{r=n_1}^{r=n_2} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx$, where $a = \lim_{n\to\infty}\frac{n_1}{n}$ & $a = \lim_{n\to\infty}\frac{n_2}{n}$

This is used both ways i.e. to evaluate the definite integral as a limit of sum and also used in finding the sum of infinite terms of some series.

Illustration 27:

Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+....\frac{1}{\sqrt{3n^2}}\right]$.

Solution:

$$L = \lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+....\frac{1}{\sqrt{3n^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{(1-0)}{n}\frac{1}{\sqrt{4-\left(0+r\left(\frac{1-0}{n}\right)\right)^2}}$$

Which is of the form

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\left(\frac{b-a}{n}\right)\right)$$

Here $b = 1$, $a = 0$ and $f(x) = \frac{1}{\sqrt{4-x^2}}$

So $L=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$

Illustration 28:

Evaluate $\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{1}{64n}\right]$.

Solution:

$$L = \lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{1}{64n}\right]=\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3}$$

Put $3n = m$, we get

$$L=\lim_{n\to\infty}\sum_{r=1}^{m}\frac{m^2/9}{\left(\frac{m}{3}+r\right)^3}=\lim_{n\to\infty}\sum_{r=1}^{m}\frac{3}{m}\left(\frac{1}{\left(1+\frac{3r}{m}\right)}\right)^3=\int_0^3\frac{dx}{(1+x)^3}=\frac{-1}{2(1+x)^2}\Big|_0^3=\frac{15}{32}$$

Illustration 29:

Show that $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{6n}\right)=\ln 6$

Solution:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right)=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+5n}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{5n}\left(\frac{1}{n+r}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\left(\frac{1}{1+\frac{r}{n}}\right)$$

$\because$ Lower limit of $r = 1$

$\therefore$ Lower limit of integration = $\lim_{n\to\infty}\frac{1}{n}=0$

$\because$ Upper limit of $r = 5n$.

$\therefore$ Upper limit of integration = $\lim_{n\to\infty}\frac{5n}{n}=5$

from (1) $\int_0^5 \frac{1}{1+x}\,dx = \left|\ln(1+x)\right|_0^5$ = ln6 – ln1 = ln6