LIMIT OF A FUNCTION

Let $y = f(x)$ be a function of x. If at $x = a, f(x)$ takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to 'a', then the unique number so obtained is called the limit of f(x) at x = a and we write it as $\lim_{x \to a} f(x)$.

CLASSICAL DEFINITION OF LIMIT

$\lim_{x \to a} f(x) = L$ is defined to mean that, we can make the values of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.

In other words, for each number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ with the property that, whenever x is a number with $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

(1) **Meaning of '$x \to a$':** Let x be a variable and a be the constant. If x assumes values nearer and nearer to 'a' then we say 'x tends to a' and we write '$x \to a$'. It should be noted that as $x \to a$, we have $x \neq a$. By 'x tends to a' we mean that

(i) x assumes values nearer and nearer to 'a' and

(ii) We are not specifying any manner in which x should approach to 'a'. x may approach to a from left (values smaller than a) or right (values greater than a).

(2) **Left hand and right hand limit:** Consider the values of the functions at the points which are very near to a on the left of a. If these values tend to a definite unique number as x tends to a, then the unique number so obtained is called **left-hand limit** of $f(x)$ at x = a and symbolically we write it as

$$f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Similarly, we can define **right-hand limit** of $f(x)$ at $x = a$ which is expressed as

$$f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h).$$

(3) **Method for finding L.H.L. and R.H.L.**

(i) For finding right hand limit (R.H.L.) of the function, we write x + h in place of x, while for left hand limit (L.H.L.) we write x – h in place of x.

(ii) Then we replace x by 'a' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

(4) **Existence of limit:** $\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist i.e. L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ i.e. L.H.L. = R.H.L.

INDETERMINATE FORMS

The following forms are called indeterminate forms:

(i) $\frac{0}{0}$ form : $\lim_{x \to a} \frac{f(x)}{g(x)}, \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

(ii) $\frac{\infty}{\infty}$ form : $\lim_{x \to a} \frac{f(x)}{g(x)}, \lim_{x \to a} f(x) = \pm\infty, \lim_{x \to a} g(x) = \pm\infty$

(iii) $\infty-\infty$ form : $\lim\limits_{x\to a}\left(f(x)-g(x)\right)$, $\lim\limits_{x\to a}f(x)=\infty, \lim\limits_{x\to a}g(x)=\infty$

(iv) 0^0 form : $\lim\limits_{x\to a}\left(f(x)\right)^{g(x)}$, $\lim\limits_{x\to a}f(x)=0, \lim\limits_{x\to a}g(x)=0, \left(f(x), g(x)>0\right)$

(v) ∞^0 form : $\lim\limits_{x\to a}\left(f(x)\right)^{g(x)}$, $\lim\limits_{x\to a}f(x)=\infty, \lim\limits_{x\to a}g(x)=0$

(vi) 1^∞ form : $\lim\limits_{x\to a}\left(f(x)\right)^{g(x)}$, $\lim\limits_{x\to a}f(x)=1, \lim\limits_{x\to a}g(x)=\infty$

AN IMPORTANT THEOREM

If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x)\le g(x)\le h(x)\ \forall x\in$ neighborhood of $x=a$ and $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}h(x)=L\,(\text{say})$, then $\lim\limits_{x\to a}g(x)=L$. This theorem is normally applied when the $\lim\limits_{x\to a}g(x)$ can't be obtained by using conventional methods by carefully identifying the functions $f(x)$ and $h(x)$.

FUNDAMENTAL THEOREMS ON LIMIT OF A FUNCTION

The following theorems are very useful for evaluation of limits if $\lim\limits_{x\to 0}f(x)=L$ and $\lim\limits_{x\to 0}g(x)=M$ (L and M are real numbers) then

(1) $\lim\limits_{x\to a}\left(f(x)+g(x)\right)=L+M$

(2) $\lim\limits_{x\to a}\left(f(x)-g(x)\right)=L-M$

(3) $\lim\limits_{x\to a}\left(f(x)\times g(x)\right)=L\times M$

(4) $\lim\limits_{x\to a}k\times f(x)=k\times L$

(5) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{L}{M}, M\ne 0$

(6) If $\lim\limits_{x\to a}f(x)=+\infty$ or $-\infty$, then $\lim\limits_{x\to a}\dfrac{1}{f(x)}=0$

(7) $\lim\limits_{x\to a}\log\{f(x)\}=\log\{\lim\limits_{x\to a}f(x)\}$

(8) If $f(x)\le g(x)$ for all x, then $\lim\limits_{x\to a}f(x)\le\lim\limits_{x\to a}g(x)$

(9) $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)}=\left\{\lim\limits_{x\to a}f(x)\right\}^{\lim\limits_{x\to a}g(x)}$

(10) If p and q are integers, then $\lim\limits_{x\to a}\left(f(x)\right)^{p/q}=L^{p/q}$, provided $(L)^{p/q}$ is a real number.

(11) If $\lim\limits_{x\to a}f\left(g(x)\right)=f\left(\lim\limits_{x\to a}g(x)\right)=f(M)$ provided 'f' is continuous at

$g(x)=M$ e.g. $\lim\limits_{x\to a}\ln\left[f(x)\right]=\ln(L)$, only if $L>0$.

EVALUATING LIMIT OF A FUNCTION

We shall divide the problems of evaluation of limits in five categories.

(1) Algebraic limits:

Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x\to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method:**

If by direct substitution of the point in the given expression we do not get any indeterminate form or infinity, then the number obtained is the limit of the given expression (This method can not be used if f(x) is step wise defined i.e. f(x) changes its definition at x = a).

(ii) **Factorization method:**

In this method, numerator and denominator are factorized. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalization method:**

Rationalization is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalization the terms are factorized which on cancellation gives the result.

(iv) **Based on the form when x → ∞:**

In this case expression should be expressed as a function $\frac{1}{x}$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

Step I: Write down the expression in the form of rational function, i.e., $\frac{f(x)}{g(x)}$, if it is not so.

Step II: If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.

Step III: Use the result $\lim_{x\to\infty} \frac{1}{x^n} = 0$, where n > 0.

Important result:

- If m, n are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers,

then $$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + ... + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + ... + b_{n-1}x + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{if } m > n \end{cases}$$

- $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}, m, n \neq 0$

(2) Trigonometric limits:

To evaluate trigonometric limits, the following results are very important.

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{x}{\sin x}$ (Here note that $\frac{\sin x}{x}<1$)

(ii) $\lim_{x\to 0}\frac{\tan x}{x}=1=\lim_{x\to 0}\frac{x}{\tan x}$ (Here note that $\frac{\tan x}{x}>1$)

(iii) $\lim_{x\to 0}\frac{\sin^{-1} x}{x}=1=\lim_{x\to 0}\frac{x}{\sin^{-1} x}$ (Here note that $\frac{\sin^{-1} x}{x}>1$)

(iv) $\lim_{x\to 0}\frac{\tan^{-1} x}{x}=1=\lim_{x\to 0}\frac{x}{\tan^{-1} x}$ (Here note that $\frac{\tan^{-1} x}{x}<1$)

(v) $\lim_{x\to 0}\frac{\sin x^0}{x}=\frac{\pi}{180}$

(vi) $\lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\frac{\cos x}{x}=0$

(vii) $\lim_{x\to\infty}\frac{\sin(1/x)}{(1/x)}=1$

(3) Logarithmic limits:

To evaluate the logarithmic limits, we use following formulae

(i) $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$ up to ∞ terms where $-1<x\le 1$.

(ii) $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$

(iv) $\lim_{x\to 0}\frac{\log_e(1-x)}{x}=-1$

(v) $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e, a>0,\neq 1$

(vi) $\lim_{x\to 0^+}\left(x\log_e x\right)=0$

(vii) $\lim_{x\to\infty}\frac{\log_e x}{x}=0$

(viii) If $\lim_{x\to a} f(x)=0$ or ∞ and $\lim_{x\to a} g(x)=0$, then $\lim_{x\to a}\{f(x)\}^{g(x)}=e^{\lim_{x\to a}\{g(x)\times\log_e f(x)\}}$

(4) Exponential limits:

To evaluate the exponential limits, we use the following results –

(i) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(ii) $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$

(iii) $\lim\limits_{x\to 0}\dfrac{e^{\lambda x}-1}{x}=\lambda \quad (\lambda \neq 0)$

(5) Limits of indeterminate form 1^{∞}:

To evaluate the exponential form 1^{∞} we use the following results.

(i) If $\lim\limits_{x\to a} f(x)=1$ and $\lim\limits_{x\to a} g(x)=\infty$, then $\lim\limits_{x\to a}\{f(x)\}^{g(x)}= e^{\lim\limits_{x\to a}(f(x)-1)g(x)}$

(ii) $\lim\limits_{x\to 0}(1+x)^{1/x}=e$

(iii) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

(iv) $\lim\limits_{x\to 0}(1+\lambda x)^{1/x}=e^{\lambda}$

(v) $\lim\limits_{x\to\infty}\left(1+\dfrac{\lambda}{x}\right)^{x}=e^{\lambda}$

(vi) $\lim\limits_{x\to\infty} a^{x}=\begin{cases}\infty, \text{if } a>1\\ 0, \text{if } a<1\end{cases}$ i.e., $a^{\infty}=\infty$, if $a>1$ and $a^{\infty}=0$ if $a<1$.

(6) L' Hospital's rule:

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$

(ii) Both are continuous at $x=a$

(iii) Both are differentiable at $x=a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x=a$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ provided that $g'(a)\neq 0$

- The above rule is also applicable if $\lim\limits_{x\to a} f(x)=\infty$ and $\lim\limits_{x\to a} g(x)=\infty$.
- If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ assumes the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\dfrac{f'(x)}{g'(x)}$ to get, $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\dfrac{f''(x)}{g''(x)}$.

  Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

(7) Using series expansion:

In finding limits, use of expansions of following functions are useful:

(i) $(1+x)^{n}=1+nx+\dfrac{n(n-1)}{2!}x^{2}+\ldots$

(ii) $a^{x}=1+x\log a+\dfrac{(x\log a)^{2}}{2!}+\ldots$

(iii) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

(iv) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots, |x| < 1$

(v) $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots,\quad \text{where } |x| < 1$

(vi) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)} = e^{1 - \frac{x}{2} + \frac{x^2}{3}} \ldots$

(vii) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$

(viii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$

(ix) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$

(x) $\sin^{-1} x = x + 1^2.\dfrac{x^3}{3!} + 3^2.1^2.\dfrac{x^5}{5!} + \ldots$

(xi) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots.$

To evaluate $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)}, \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$

Step I: Expand the series of f(x) & g(X) using above formulae.

Step II: cancel out the greatest power of x available from numerator and denominator.

(7) Limit of a sum

When number of terms of a series approach to infinity, limit of sum of the series(if it converges) can be obtained by following methods –

(i) By forming a telescoping series

If each term of the series can be represented as a difference of two terms such that positive part of each part gets cancelled by negative part of preceding or succeeding term, then limit can be obtained by first evaluating sum of n terms and then finding limit as n tends to infinity.

(ii) Using definition of definite integration

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=n_1}^{r=n_2} f\left(\frac{r}{n}\right) = \int_a^b f(x), \text{ where } a = \lim_{n\to\infty} \frac{n_1}{n} \;\&\; b = \lim_{n\to\infty} \frac{n_2}{n}$$

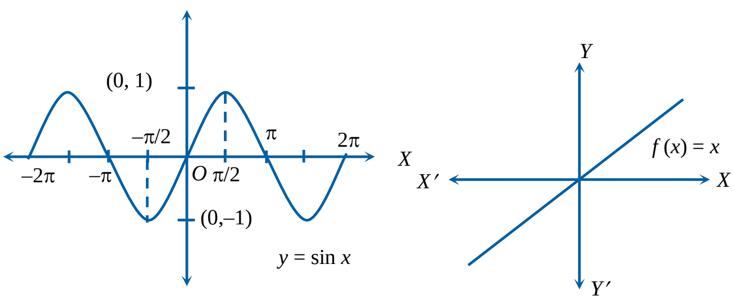
CONTINUITY

The word ‘Continuous’ means without any break or gap. If the graph of a function has no break, or gap or jump, then it is said to be continuous.

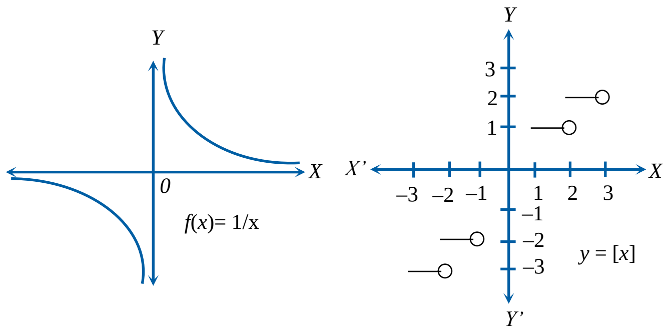
A function which is not continuous is called a discontinuous function.

While studying graphs of functions, we see that graphs of functions $\sin x$, x, $\cos x$, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous. Similarly, $\tan x$, $\cot x$, $\sec x$, $\frac{1}{x}$ etc. are also discontinuous function.

Continuous function



Discontinuous function



CONTINUITY AT A POINT

A function f(x) is said to be continuous at a point x = a in its domain iff $\lim_{x\to a} f(x) = f(a)$ i.e. a function $f(x)$ is continuous at x = a if and only if it satisfies the following three conditions:

(1) $f(a)$ exists. (‘a’ lies in the domain of f)

(2) $\lim_{x\to a} f(x)$ exist and is finite i.e. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x\to a} f(x) = f(a)$ (limit equals the value of function).

Formal definition of continuity:

The function $f(x)$ is said to be continuous at $x = a$, in its domain if for any arbitrary chosen positive number $\varepsilon > 0$, we can find a corresponding number δ depending on ε such that $|f(x) - f(a)| < \delta$ for which $0 < |x - a| < \varepsilon$.

Continuity from left & right:

Function $f(x)$ is said to be

(1) Left continuous at x = a if $\lim_{x\to a-0} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\operatorname{Lim}_{x\to a+0} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

CONTINUITY IN OPEN INTERVAL

A function $f(x)$ is said to be continuous in an open interval (a, b) iff it is continuous at every point in that interval.

This definition implies the non-breakable behavior of the function $f(x)$ in the interval (a, b).

CONTINUITY IN CLOSED INTERVAL

A function $f(x)$ is said to be continuous in a closed interval [a, b] iff,

(1) f is continuous in (a, b)

(2) f is continuous from the right at 'a' i.e. $\lim_{x\to a^+} f(x) = f(a)$

(3) f is continuous from the left at 'b' i.e. $\lim_{x\to b^-} f(x) = f(b)$.

PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$. Then

(i) $cf(x)$ is continuous at x = a, where c is any constant

(ii) $f(x) \pm g(x)$ is continuous at x = a.

(iii) $f(x) \times g(x)$ is continuous at x = a.

(iv) $\frac{f(x)}{g(x)}$ is continuous at $x = a$, provided $g(a) \neq 0$.

PROPERTIES OF DISCONTINUOUS FUNCTIONS

A function 'f' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point 'a' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ or both may not exist

(ii) $\lim_{x\to a^+} f(x)$ as well as $\lim_{x\to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x\to a^+} f(x)$ as well as $\lim_{x\to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

- A function f is said to have removable discontinuity at x = a if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ but their common value is not equal to f(a). Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.
- If $\lim_{x\to a} f(x)$ does not exist, then we can not remove this discontinuity. So this become a non-removable discontinuity or essential discontinuity.
- If f is continuous at x = c and g is discontinuous at x = c, then
  (a) f + g and f – g are discontinuous
  (b) f.g may be continuous
- If f and g are discontinuous at x = c, then f + g, f – g and fg may still be continuous.

INTERMEDIATE VALUE THEOREM

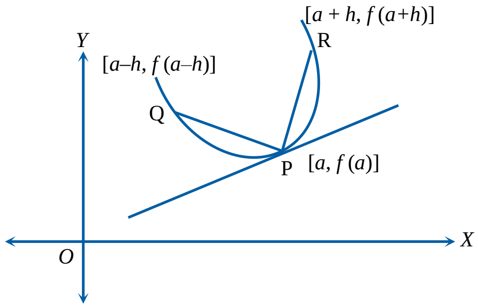
If f(x) is a continuous function for x ∈ [a, b] such that f(a) = c & f(b) = d, then for any value of f(x) lying between c & d there must exist at least one value of x lying in [a, b].

In other words, a continuous function in [a, b] must acquire all value between f(a) & f(b).

DIFFERENTIABILITY

DIFFERENTIABILITY AT A POINT

The function $f(x)$ defined on an open interval (b, c) let $P(a, f(a))$ be a point on the curve $y = f(x)$ and let $Q(a-h, f(a-h))$ and $R(a+h, f(a+h))$ be two neighboring points on the left hand side and right hand side respectively of the point P, then



slope of chord $PQ = \dfrac{f(a-h)-f(a)}{(a-h)-a} = \dfrac{f(a-h)-f(a)}{-h}$

and, slope of chord $PR = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$.

Now As $h \to 0$, point Q and R both tends to P from left hand and right hand respectively. Consequently, chords PQ and PR becomes tangent at point P.

Thus, $\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}$(slope of chord PQ)

Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P and $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0}$(slope of chord PR)

$\Rightarrow$ Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P.

Now, $f(x)$ is said to be differentiable at $x = a$ if $\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P.

In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point. i.e., "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

Right hand derivative:

Right hand derivative of $f(x)$ at $x = a$, denoted by $f'(a^+)$, is the $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

Left hand derivative:

Left hand derivative of $f(x)$ at $x = a$, denoted by $f'(a^-)$, is the $\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h}$.

Clearly, $f'(a) = \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

DIFFERENTIABILITY IN OPEN INTERVAL

A function $f(x)$ defined in an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Everywhere differentiable function:

If a function is differentiable at each $x \in R$, then it is said to be everywhere differentiable. e.g., A constant function, a polynomial function, $\sin x, \cos x$ etc. are everywhere differentiable.

Some standard results on differentiability:

(1) Every polynomial function is differentiable at each $x \in R$.

(2) The exponential function $a^x, a > 0$ is differentiable at each $x \in R$.

(3) Every constant function is differentiable at each $x \in R$.

(4) The logarithmic function is differentiable at each point in its domain.

(5) Trigonometric and inverse trigonometric functions are differentiable in their domains.

(6) The sum, difference, product and quotient of two differentiable functions is differentiable.

(7) The composition of differentiable function is a differentiable function.

(8) If a function is differentiable at a point, then it is continuous also at that point.

i.e. Differentiability $\Rightarrow$ Continuity, but the converse need not be true.

(9) If f(x) is differentiable at x = a and g(x) is not differentiable at x =a, then the product function

f(x).g(x) can still be differentiable at x = a.

(10) If f(x) and g(x) both are not differentiable at x = a then the product function f(x).g(x) can still be differentiable at x = a.

(11) If f(x) is differentiable at x = a and g(x) is not differentiable at x = a then f(x) + g(x) is also not differentiable at x = a

(12) If f(x) and g(x) both are not differentiable at x = a, then f(x) + g(x) may be a differentiable function.