1. Distance between Two Points

The distance between two points P(x_1, y_1) and Q(x_2, y_2) is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Illustration 1:

If M is the mid-point of the side BC of the triangle ABC, prove that $AB^2 + AC^2 = 2AM^2 + 2BM^2$.

Solution

This is a well-known theorem in pure geometry; we prove it by analytical methods. In fig. choose $\vec{BC}$ to define the positive direction of the x-axis and M to be the origin. Let the abscissa of C be a; then, since M is the mid-point of BC, the abscissa of B is –a; thus C is (a, 0) and B is (– a, 0). Let the coordinates of A be (x, y). Then

$AB^2 = [x - (-a)]^2 + [y - o]^2$

$= (x + a)^2 + y^2$ …(i)

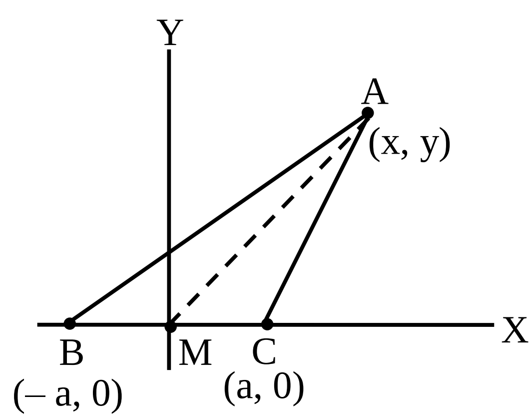
And $AC^2 = (x - a)^2 + y^2$. …(ii)

Add (i) and (ii) and simplify, then

$AB^2 + AC^2 = 2x^2 + 2a^2 + 2y^2$. …(iii)

$AM^2 = x^2 + y^2$ and $BM^2 = a^2$, so that

$AM^2 + BM^2 = x^2 + a^2 + y^2$. …(iv)



The formulae (iii) and (iv) given the desired result, namely,

$AB^2 + AC^2 = 2AM^2 + 2BM^2$.

2. Section Formula

If P(x, y) divides the line joining A(x_1, y_1) & B(x_2 , y_2) in the ratio m : n, then

(i) Internal division

$$x = \frac{mx_2 + nx_1}{m+n}; \qquad y = \frac{my_2 + ny_1}{m+n}$$

(ii) External division

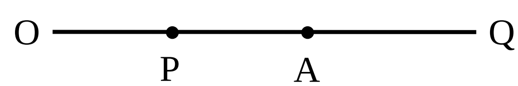
$$x = \frac{mx_2 - nx_1}{m-n}\;; \qquad y = \frac{my_2 - ny_1}{m-n}$$

The coordinates of the mid-point of the line-segment joining (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Illustration 2:

If P divides OA internally in the ratio $\lambda_1 : \lambda_2$ and Q divides OA externally in the ratio $\lambda_1 : \lambda_2$, then prove that OA is the harmonic mean of OP and OQ

Solution



$$\frac{1}{OP} = \frac{\lambda_1 + \lambda_2}{\lambda_1.OA},$$

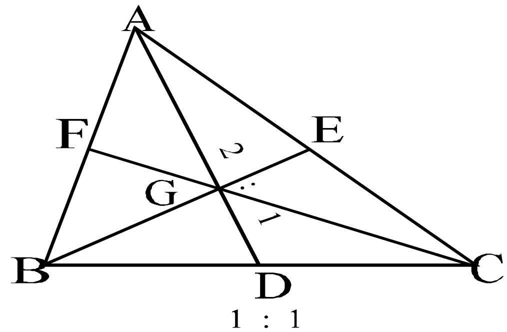
$$\frac{1}{OQ} = \frac{\lambda_1 - \lambda_2}{\lambda_1.OA}$$

$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OA}$ i.e. OP, OA and OQ are in H.P. (Harmonic Progression)

3. Centres Connected With a Triangle

(w.r.t. ΔABC, where A $\equiv (x_1, y_1)$, B $\equiv (x_2, y_2)$, C $\equiv (x_3, y_3)$, BC = a, CA = b & AB = c).

Centroid:



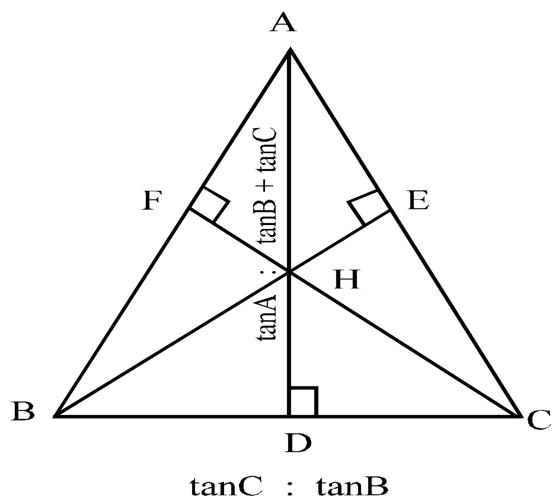
The point of concurrency of the medians of a triangle is called the centroid of the triangle. The centroid of a triangle divides each median in the ratio 2 : 1. The coordinates of centroid are given by

$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Orthocentre:

The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle. The co-ordinates of the orthocentre are given by $H \equiv \left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$

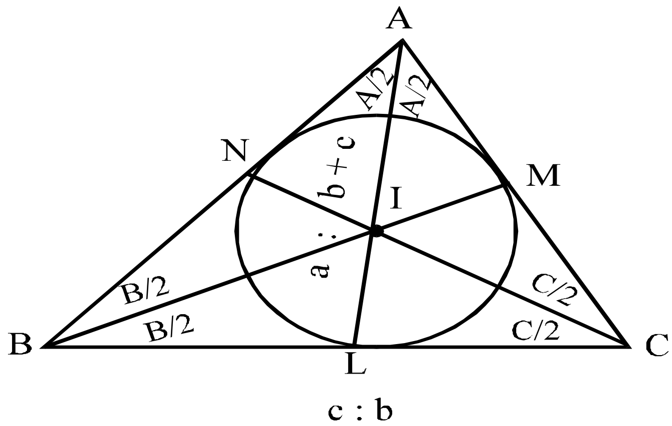
The triangle formed by joining the feet of altitudes in a Δ is called the orthic triangle. Here ΔDEF is the orthic triangle ofΔABC.



Incentre:

The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre are given by

$$\mathbf{I} \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$



Excentre:

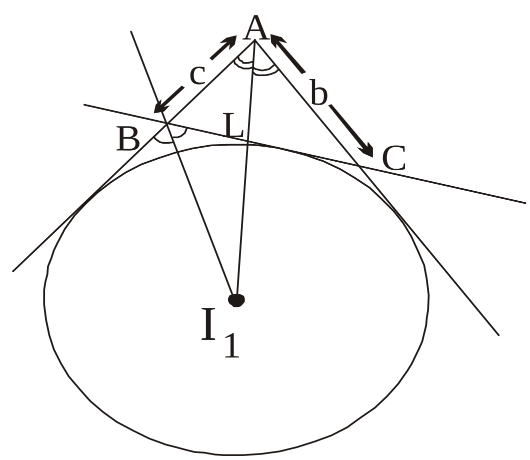
Co-ordinate of excentre opposite to ∠A is given by

$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

and similarly for excentres (I_2 & I_3) opposite to ∠B and ∠C are given by

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{-a+b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$

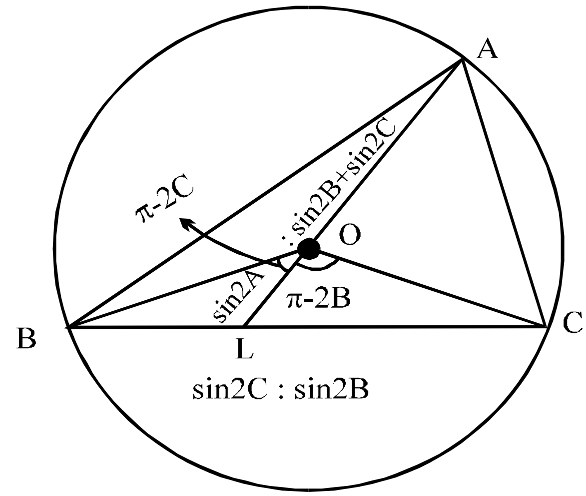
$$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$



$$\frac{BL}{LC} = \frac{c}{b}, \text{also} \frac{AI_1}{I_1L} = -\frac{b+c}{a}$$

Circumcentre:

The point of concurrency of the perpendicular bisectors of the sides of a triangle is called circumcentre of the triangle. The coordinates of the circumcentre are given by



$$O \equiv \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

Remarks:

1. Circumcentre O, Centroid G and Orthocentre H of a ΔABC are collinear. G Divides OH in the ratio 1 : 2, i.e. OG : GH = 1 : 2

2. In an isosceles triangle centroid, orthocenter, incentre and circumcentre lie on the same line and in an equilateral triangle all these four points coincide.

4. Area of a Triangle

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively be the coordinates of the vertices A, B, C of a triangle ABC. Then the area of triangle ABC, is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \qquad ....(1)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad ....(2)$$

Remarks:

In case of polygon with vertices (x_1, y_1), (x_2, y_2), ... (x_n, y_n) in order, then area of polygon is given by

$$\frac{1}{2}|(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + ...+ (x_{n-1}y_n - y_{n-1}x_n)\,) + (x_ny_1 - y_nx_1)|$$

5. Locus

When a point moves in a plane under certain geometrical conditions, the point traces a path. This path of the moving point is called its locus.

Equation of locus

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

Illustration 3:

Find the locus of a variable point which is at a distance of 2 units from the y-axis.

Solution

If points on the locus are on the positive side of the y-axis, the abscissa of any point P (x, y) on the locus is given, according to the condition, by $x = 2$, whatever the value of y may be; thus, (2, 1), (2, – 10) and (2, 100) are points on the locus.

Similarly, if points on the locus are on the negative side of the y-axis, the abscissa of any point on this part of the locus is given by $x = -2$, whatever the value of the ordinate may be.

The complete equation of the locus is then $x = \pm 2$.

Illustration 4:

Find the locus of a variable point whose distance from A (4, 0) is equal to its distance from B (0, 2).

Solution

Let the coordinates of P be (x, y).

$$AP^2 = (x - 4)^2 + (y - 0)^2 = x^2 + y^2 - 8x + 16$$

And $\quad BP^2 = (x - 0)^2 + (y - 2)^2 = x^2 + y^2 - 4y + 4$

Since $\quad AP^2 = BP^2$

$\Rightarrow \quad x^2 + y^2 - 8x + 16 = x^2 + y^2 - 4y + 4$

From which $8x - 4y - 12 = 0$ or, $\quad 2x - y - 3 = 0$

Illustration 5:

Q is a variable point whose locus is 2x + 3y + 4 = 0; corresponding to a particular position of Q, P is the point of section of OQ, O being the origin, such that OP : PQ = 3 : 1.Find the locus of P?

Solution

Let Q be the point (X, Y) and P the point (x, y); the coordinates of Q satisfy the equation $2x + 3y + 4 = 0$, so that $2X + 3Y + 4 = 0$.

Apply the section-formulae for OQ, O being (0, 0); then

$$x = \frac{0 + 3X}{1 + 3}, y = \frac{0 + 3Y}{1 + 3} \text{ from which } X = \frac{4}{3}x, Y = \frac{4}{3}y.$$

Substitute these values, then the locus of P is $\frac{8}{3}x + 4y + 4 = 0$

$\Rightarrow 2x + 3y + 3 = 0.$

Illustration 6:

Find the locus of a variable point whose distance from (1, 0) is half its distance from the line x = 4.

Solution

Let S be the given point, RT the line $x = 4$ and P (x, y) any point on the locus PQ is perpendicular to RT.

The given condition is equivalent to : $PQ^2 = 4PS^2$. Now $PQ^2 = (4 - x)^2$ and $PS^2 = (x - 1)^2 + y^2$; hence $(4 - x)^2 = 4[(x - 1)^2 + y]^2$ or, on simplification, $3x^2 + 4y^2 = 12$.

6. Straight Line

Any equation of first degree of the form $ax + by + c = 0$, where a, b, c are constants always represents a straight line (at least one out of a and b is non zero)

Slope

If *a* straight line makes an angle 'θ'in anticlockwise direction with the positive direction of x-axis, $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by 'm' is $\tan\theta$. i.e. $m = \tan\theta$.

If $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \ne x_2$ are any two points, then slope of the line passing through

A and B is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Remark:

(i) If $\theta = 90°$, m does not exist and line is parallel to y - axis.

(ii) If $\theta = 0°$, $m = 0$ and the line is parallel to x-axis.

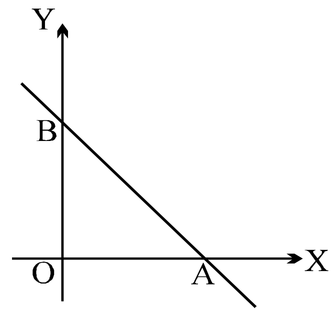
(iii) Let m_1 and m_2 be slopes of two given lines.

(a) If lines are parallel, $m_1 = m_2$ and vice versa.

(b) If lines are perpendicular, $m_1.m_2 = -1$ and vice versa.

7. Intercept of a Straight Line on the Axis

If a line AB cuts the x-axis and y-axis at A and B respectively and O be the origin then OA and OB with proper sign are called the intercepts of the line AB on x and y axes respectively.



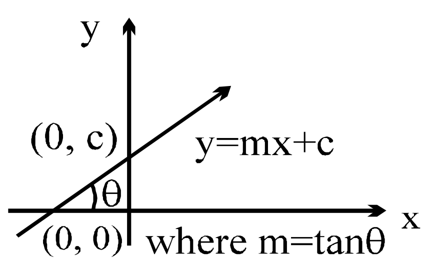
8. Standard Equations of Straight Lines

Slope-intercept form:

$y = mx + c$,

where m = slope of the line = $\tan\theta$

$c = y$ intercept

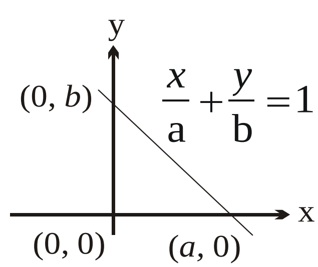


Intercept form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

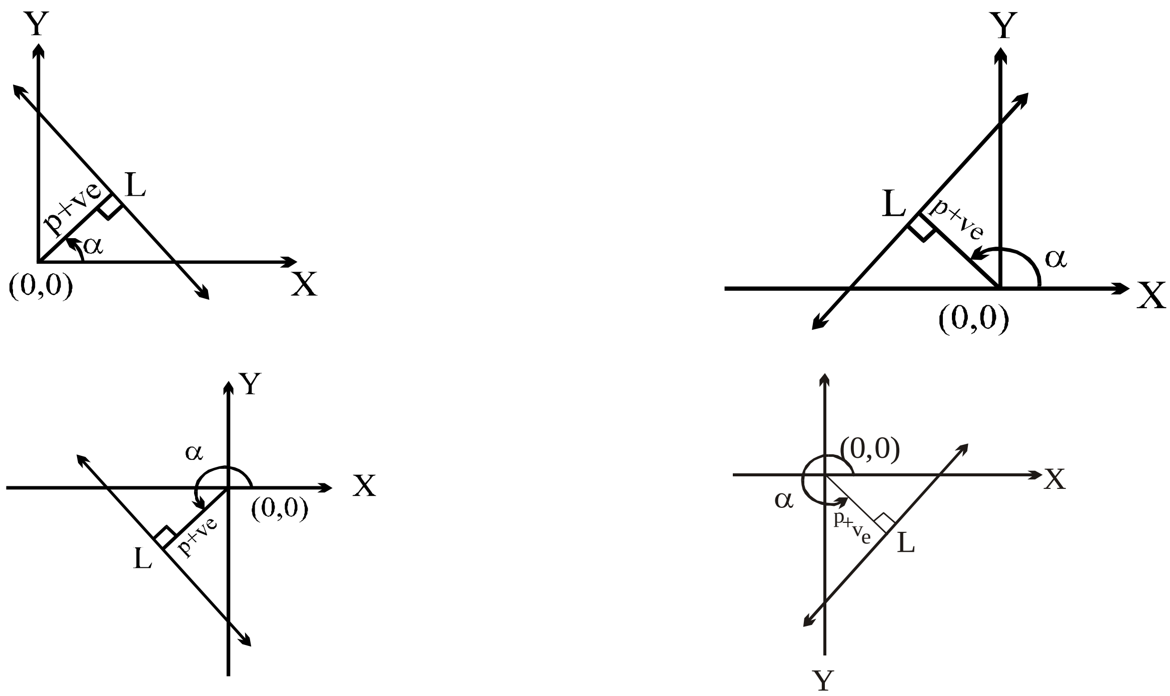
x intercept = a, length of x intercept = $|a|$

y intercept = b, length of y intercept = $|b|$



Normal form:

$x\cos\alpha + y\sin\alpha = p$, where α, is the angle which the perpendicular to the line makes with the axis of x and p is the length of the perpendicular from the origin to the line. $0 \le \alpha < 2\pi$ and p is always positive.



Slope point form:

Equation: $y - y_1 = m(x - x_1)$, where

(a) One point on the straight line is (x_1, y_1) and

(b) The direction of the straight line i.e., the slope of the line = m

Two point form:

Equation: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, where (x_1, y_1) and (x_2, y_2) are the two given points. Here $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Illustration 7:

Reduce the line 2x – 3y + 5 = 0, in slope- intercept, intercept and normal forms.

Solution

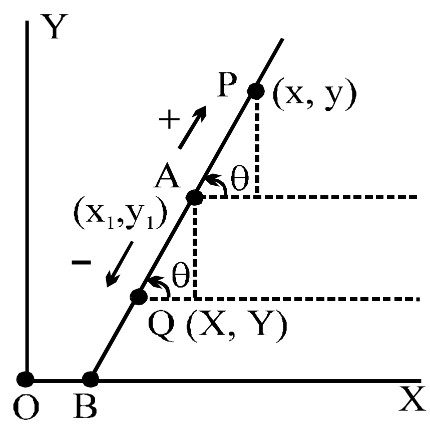
Slope-intercept form: $y = -\frac{2x}{3} + \frac{5}{3}$, tan θ = m = 2/3, c = $\frac{5}{3}$

Intercept form: $\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1,\ a = -\frac{5}{2}, b = \frac{5}{3}$

Normal form: $\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$

9. Parametric Equations of a Straight Line

For the points P (x, y) and Q (X, Y) Shown in the figure AP is regarded as a positive vector and AQ as a negative vector, as indicated by the arrows.



From the general definitions of cosθ and sinθ we have

$$\cos\theta = \frac{x - x_1}{AP},\ \sin\theta = \frac{y - y_1}{AP}$$

or $x - x_1$ = AP cosθ, $y - y_1$ = AP sinθ

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = \pm r$$

Illustration 8:

Find the distance between A (2, 3), on the line of gradient 3/4 and the point of intersection, P, of this line with $5x + 7y + 40 = 0$.

Solution

Since m = 3/4, then cosθ = 4/5 and sinq = 3/5. Any point on the line through A has, the coordinates $(2 + 4r/5, 3 + 3r/5)$. If this point is also the point of intersection, P, then these coordinates satisfy the equation of the given line:

Hence, $$5(2+\frac{4}{5}r)+7(3+\frac{3}{5}r)+40=0$$

Or, $$r(4+\frac{21}{5})+71=0 \text{ or } r=-\frac{355}{41}.$$

The distance between A and P is thus 355/41 units, the vector $\overrightarrow{AP}$ being in the negative direction of the line.

10. Collinearity of Three Given Points

Three given points A, B, C are collinear if any one of the following conditions is satisfied.

(i) Area of triangle ABC is zero.

(ii) Slope of AB = slope of BC = slope of AC.

(iii) AC = AB + BC.

(iv) Find the equation of line passing through 2 given points; if the third point satisfies the given equation of the line, then three points are collinear.

11. Reflection of a Point about a Line

The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = -2\frac{ax_1+by_1+c}{a^2+b^2}$ and the foot of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is

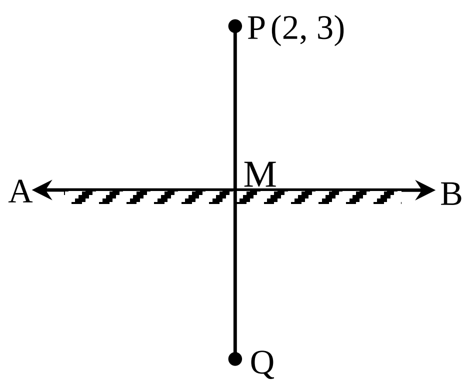
$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = -\frac{ax_1+by_1+c}{a^2+b^2}$$

Illustration 9:

Find the foot of the perpendicular drawn from the point (2, 3) to the line $3x - 4y + 5 = 0$. Also, find the image of (2, 3) in the given line.

Solution

Let AB = $3x - 4y + 5 = 0$, P ≡ (2, 3) and PM ⊥ AB. Slope of AB = $\frac{3}{4}$



$\Rightarrow$ slope of PM = $\frac{-4}{3} = \tan\theta$ (say)

$\Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{-3}{5}$

Now, $r = p = \frac{3\times 2 - 4\times 3 + 5}{\sqrt{9+16}} = \frac{6-12+5}{5} = \frac{-1}{5}$

Which is the foot of the perpendicular

$$\Rightarrow M = \left(2 - \frac{1}{5}\cos\theta, 3 - \frac{1}{5}\sin\theta\right) = \left(\frac{53}{25}, \frac{71}{25}\right)$$

Let Q be the image of P

$$\Rightarrow Q = \left(2 - \frac{2}{5}\cos\theta, 3 - \frac{2}{5}\sin\theta\right) = \left(\frac{56}{25}, \frac{67}{25}\right)$$

12. Family of Lines

(Equation of any straight line through the point of intersection of two given straight lines)

The equation of any straight line passing through the intersection of the two lines $ax + by + c = 0$, $Ax + By + C = 0$ has the general form

$ax + by + c + \lambda (Ax + By + C) = 0$

In which λ can have any real value; here, λ is parameter which can be evaluated specifically if some further condition is imposed.

Hence the general equation of the family of lines through the point of intersection of two given lines is $L + \lambda L' = 0$ where $L = 0$ and $L' = 0$ are the two given lines, and λ is a parameter.

Illustration 10:

Find the straight line passing through the point of intersection of $2x + 3y + 5 = 0$, $5x - 2y - 16 = 0$ and through the point $(-1, 3)$.

Solution

The equation of any line through the point of intersection of the given lines is $2x + 3y + 5 + \lambda (5x - 2y - 16) = 0$ ….(i)

But the required line passes through $(-1, 3)$, hence

$-2 + 9 + 5 + \lambda (-5 - 6 - 16) = 0.$

Hence, $\lambda = \frac{4}{9}$. Insert this value of λ in (i) and the required line is

$9(2x + 3y + 5) + 4 (5x - 2y - 16) = 0$ or, on simplification, $2x + y - 1 = 0$.

13. Concurrency of Straight Lines

The condition for three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ to be concurrent is -

(i) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(ii) There exist three constants l, m, n(not all zero the same time) such that $lL_1 + mL_2 + nL_3 = 0$, whee $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ are the three given straight lines.

(iii) The three lines are concurrent if any one of the lines passes through the point of intersection of the other two lines.

14. The Angle between Two Straight Lines

It is a convention to tell acute angle for the angle between the two lines. For this purpose

$\tan\phi = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$, where ϕ is the acute angle.

Remarks:

1. If the lines are parallel then $m_1 = m_2$

2. If the lines are perpendicular then $m_1m_2 = -1$

Illustration 11:

Find the acute angle between the lines $2x + y + 11 = 0$, $x - 6y + 7 = 0$.

Solution:

The gradients are – 2 and 1/6; the angle of slope of the first line is in the second quadrant while that of the second line is in the first quadrant; accordingly, we write: $m_2 = -2$, $m_1 = 1/6$, $\tan\phi = \frac{-2-1/6}{1+(-2)(1/6)} = \frac{13}{4}$,

And hence ϕ is an obtuse angle, If α is the acute angle between the lines then $\phi = 180^o - \alpha$, from which $\tan\phi = \tan(180^o - \alpha)$.

But $\tan(180^o - \alpha) = -\tan\alpha$. Hence $\tan\theta = -\tan\alpha$ and by (i),

$\tan\alpha = \frac{13}{4} \Rightarrow \alpha = \tan^{-1}(13/4)$

Illustration 12:

Find the equations of the two lines, each passing through (5, 6) and each making an acute angle of 45º with the line $2x - y + 1 = 0$.

Solution

For the given line, $m = 2$ corresponding to an angle of slope, θ, which is greatest than 45º (since $\tan 45^o = 1$) and less then 90º. Clearly, there will be two lines satisfying the stated requirements : one line (i) will have an angle of slope greater than θ with M denoting the corresponding gradient so that $M > m$; the other line (ii) will have an angle of slope less than θ with M' denoting the corresponding gradient so that $m > M'$.

(i) Since $\phi = 45^o$, $\tan 45^o = 1 = \frac{M-m}{1+Mn} = \frac{M-2}{1+2M}$; hence $1 + 2M = M - 2$, from which $M = -3$. The line is then $y - 6 = -3(x - 5)$ or $3x + y - 21 = 0$.

(ii) Similarly, $\tan 45^o \equiv 1 = \frac{m-M'}{1+mM'} = \frac{2-M'}{1+2M'}$ from which $M' = 1/3$.

The line is $y - 6 = \frac{1}{3}(x - 5)$ or $x - 3y + 13 = 0$.

The bisectors of the acute and the obtuse angles

If two lines are $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

will represent the equation of the bisector of the acute or obtuse angle between the lines according as $a_1a_2 + b_1b_2$ is negative or positive.

The equation of the bisector of the angle which contains a given point:

The equation of the bisector of the angle between the two lines containing the point (α,β) is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ or } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

according as $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of the same signs or of opposite signs.

Remarks:

(i) If $c_1c_2(a_1a_2 + b_1b_2) < 0$, then the origin will lie in the acute angle and if $c_1c_2(a_1a_2 + b_1b_2) > 0$, then origin will lie in the obtuse angle.

(ii) Equation of straight lines passing through $P(x_1, y_1)$ and equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines and passing through the point P.

Illustration 13:

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the -

(i) Bisector of the obtuse angle between them,

(ii) Bisector of the acute angle between them,

(iii) Bisector of the angle which contains origin and

(iv) Bisector of the angle which contains (1, 2).

Solution

Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow \frac{4x+3y-6}{5} = \pm\frac{5x+12y+9}{13}$$

$\Rightarrow 52x + 39y - 78 = \pm (25x + 60y + 45)$

$\Rightarrow 27x - 21y - 123 = 0,\ 77x + 99y - 33 = 0$

$\Rightarrow 9x - 7y - 41 = 0,\ 7x + 9y - 3 = 0.$

Let the angle between the line $4x + 3y - 6 = 0$ and

the bisector $9x - 7y - 41 = 0$ be 'θ',

$$\text{Then } \tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(i) The bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) The bisector of the acute angle is $7x - 9y - 3 = 0$

(iii) The bisector of the angle containing the origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}}=\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x + 9y - 3 = 0$$

(iv) For the point (1, 2), $4x + 3y - 6 = 4 \times 1 + 3 \times 2 - 6 > 0$

$5x + 12y + 9 = 12\times 2 + 9 > 0$

Hence equation of the bisector of the angle containing the point (1, 2) is $\frac{4x+3y-6}{5}=\frac{5x+12y+9}{13} \Rightarrow 9x - 7y - 41 = 0$

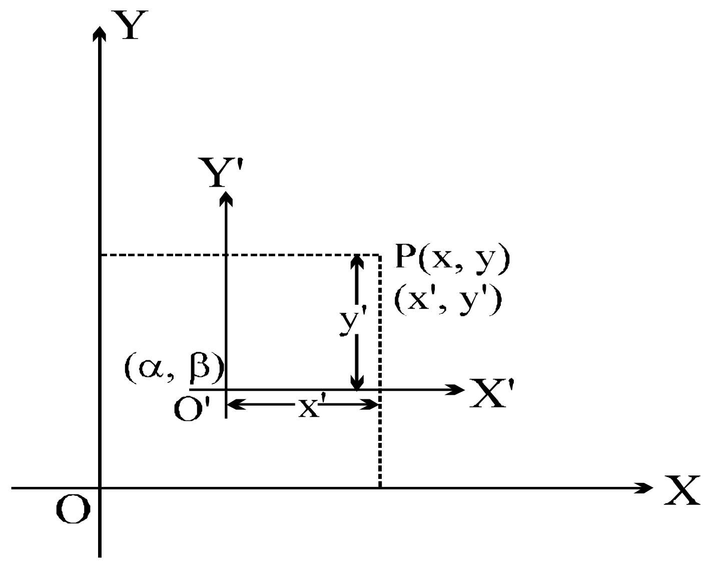
Changes of Axes (Shifting of Origin without Rotation of Axes)

Let P $\equiv (x, y)$ with respect to axes OX and OY.

Let O′ $\equiv (\alpha, \beta)$ with respect to axes OX and OY and let P $\equiv (x', y')$ with respect to axes O′X′ and O′Y′ where OX and O′X′ are parallel and OY and O′Y′ are parallel.

then $\boldsymbol{x = x' + \alpha, y = y' + \beta}$

or $\boldsymbol{x' = x - \alpha, y' = y - \beta}$



Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

Illustration 14:

Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant.

Solution

Let the origin be shifted to the point (h, k) and let P (x, y) be any point on the curve and (x_1, y_1) be the coordinates of P with respect to new axes then

$$x = x_1 + h \text{ and } y = y_1 + k$$

Hence, new equation will be

$$(y_1 + k)^2 + 4(y_1 + k) + 8(x_1 + h) - 2 = 0$$

$$y_1^2 + (2k + 4) y_1 + 8x_1 + (k^2 + 4k + 8h - 2) = 0$$

Thus new equation of the curve will be

$$y^2 + (2k + 4) y + 8x + (k^2 + 4k + 8h - 2) = 0$$

Since this equation is required to be free from the term containing y and the constant, we have

$$2k + 4 = 0 \text{ and } k^2 + 4k + 8h - 2 = 0$$

$$\therefore \quad k = -2 \text{ and } h = \frac{3}{4}$$

Hence, the point of which the origin be shifted is $\left(\frac{3}{4}, -2\right)$

Rotation of the axes (To change the direction of the axes of co-ordinates, without changing the origin, both systems of co-ordinates being rectangular):

Let OX, OY be given rectangular axes with respect to which the coordinates of a point P are (x, y). Suppose that OU, OV are the two perpendicular lines obtained by rotating OX, OY respectively through an angle α in the counter-clockwise sense. We take OU, OV as a new pair of coordinate axes, with respect to which the coordinates of P are (x', y'), then

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

or, $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$ (in matrix form)

Illustration 15:

If (x, y) and (X, Y) be the co-ordinate of the same point referred to two sets of rectangular axes with the same origin and if $ux + vy$, where u and v are independent of x and y, becomes $VX + UY$, show that $u^2 + v^2 = U^2 + V^2$.

Solution

Let the axes rotate an angle θ, and if (x, y) be the point with respect to old axes and (X, Y) be the coordinates with respect to new axes then

We get

$$\begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases}$$

Then, $ux + vy = u(X\cos\theta - Y\sin\theta) + v(X\sin\theta + Y\cos\theta)$

$$= (u\cos\theta + v\sin\theta)X + (-u\sin\theta + v\cos\theta)Y$$

But given new cure $VX + UY$

Then $VX + UV = (u\cos\theta + v\sin\theta)\,X + (-u\sin\theta + v\cos\theta)\,Y$

On comparing the coefficients of X & Y, we get

$$u\cos\theta + v\sin\theta = V \qquad \ldots (1)$$

And $$-u\sin\theta + v\cos\theta = U \qquad \ldots (2)$$

Squaring and adding (1) and (2), we get $u^2 + v^2 = U^2 + V^2$

15. Pair of Straight Lines

The general equation of degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$.

The homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin.

If lines through the origin whose joint equation is $ax^2 + 2hxy + by^2 = 0$, are $y = m_1x$ and $y = m_2x$,

Then $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$ and $y^2 + \dfrac{2h}{b}xy + \dfrac{a}{b}x^2 = 0$ are identical.

If θ is the angle between the two lines,

Then $\tan\theta = \pm\dfrac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \pm\dfrac{2\sqrt{h^2-ab}}{a+b}$

The lines are perpendicular if $a + b = 0$ and coincident if $h = ab$.

Note:

(i) **(a)** $h^2 > ab \Rightarrow$ lines are real & distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0).

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by

$$ax^2 + 2hxy + by^2 = 0, \text{ then } m_1 + m_2 = -\frac{2h}{b} \text{ \& } m_1m_2 = \frac{a}{b}$$

(iii) The equation to the straight lines bisecting the angle between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$

(iv) A homogeneous equation of degree n represents n straight lines (in general) passing through origin.

16. Joint Equation Of Pair Of Lines Joining The Origin And The Points Of Intersection Of A Line And A Curve

If the lines $lx + my + n = 0$, $(n \neq 0)$ i.e. the line not passing through origin) cuts the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at two points A and B, then the joint equation of straight lines passing through A and B and the origin is given by homogenizing the equation of the curve by the equation of the line i.e.

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$ is the equation of the lines OA and OB.

Illustration 16:

Prove that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = 2hk$ and the curve $(x - k)^2 + (y - h)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.

Solution

Making the equation of the curve homogeneous with the help of that of the line, we get

$$x^2 + y^2 - 2(kx + hy)\left(\frac{hx+ky}{2hk}\right) + (h^2 + k^2 - c^2)\left(\frac{hx+ky}{2hk}\right)^2 = 0$$

or $4h^2k^2x^2 + 4h^2k^2y^2 - 4hk^2x(hx + ky) - 4h^2ky(hx + ky) + (h^2 + k^2 - c^2)(h^2x^2 + k^2y^2 + 2hxy) = 0$

This is the equation of the pair of lines joining the origin to the points of intersection of the given line and the curve. They will be at right angles if coefficient of x^2 + coefficient of $y^2 = 0$ i.e.

$$(h^2 + k^2)(h^2 + k^2 - c^2) = 0$$

$$\Rightarrow \quad h^2 + k^2 = c^2 \text{ (since } h^2 + k^2)$$