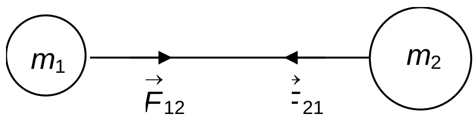
1. NEWTON'S LAW OF GRAVITATION

Newton's law of gravitation states that every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

Therefore from Newton's law of gravitation

$$\vec{F} = \frac{Gm_1 m_2}{r^2}\,\hat{r} \qquad \ldots \textbf{(1)}$$

where G is called the gravitational constant and $\hat{r}$ is the unit vector along the line joining the two mass particles.



$\vec{F}_{12} + \vec{F}_{21} = 0$

The gravitational force between two particles form an action reaction pair.

Illustration 1

A mass M is split into two parts m and $(M - m)$, which are then separated by a certain distance. What ratio $\left(\frac{m}{M}\right)$ maximizes the gravitational force between the parts?

Solution:

If *'r'* is the distance between m and $(M - m)$, the gravitational force will be

$$F = G\frac{(M-m)m}{r^2} = \frac{G}{r^2}\left[Mm - m^2\right]$$

for F to be maximum, $\frac{dF}{dm} = 0$,

i.e., $\frac{d}{dm}\left[\frac{G}{r^2}(Mm - m^2)\right] = 0$

or, $M - 2m = 0 \qquad [\because \frac{G}{r^2} \neq 0]$

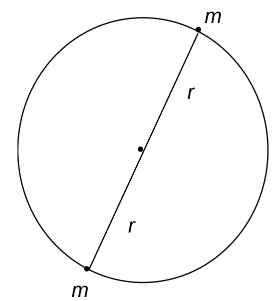
or, $\frac{m}{M} = \frac{1}{2}$, i.e., the force will be maximum when two parts are equal.

Illustration 2

Two particles of equal mass *m* are moving in a circle of radius *'r'* under the action of their mutual gravitational attraction. Find the speed of each particle.

Solution:

The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius.



Considering the circular motion of one particle, we have,

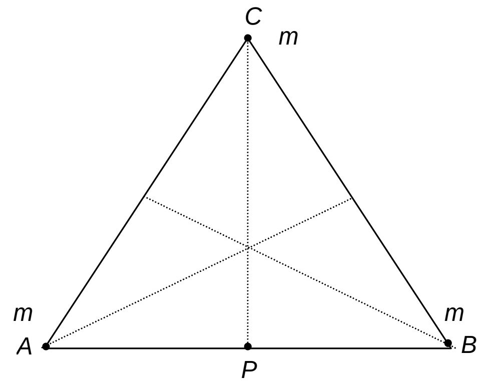
$$\frac{mv^2}{r} = \frac{Gm.m}{(2r)^2} \quad \therefore v = \sqrt{\frac{Gm}{4r}}$$

Illustration 3

Three equal particles each of mass *'m'* are placed at the three corners of an equilateral triangle of side *'a'*. Find the force exerted by this system on another particle of mass *m* placed at (a) the midpoint of a side (b) at the center of the triangle.

Solution:

As gravitational force is a two body interaction, the principle of superposition is valid, i.e., resultant force on the particle of mass *m* at *P* is



$$\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C$$

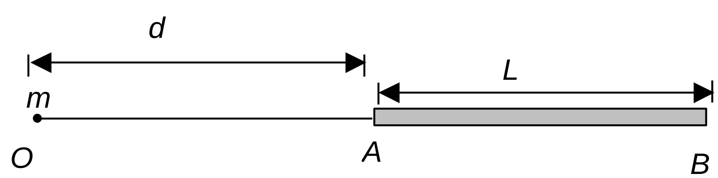
(a) As shown in the above figure, when *P* is at the midpoint of a side, $\vec{F}_A$ and $\vec{F}_B$ will be equal in magnitude but opposite in direction. So they will cancel each–other. So the point mass *m* at *P* will experience a force due to C only, i.e.,

$$F = F_C = \frac{Gmm}{(CP)^2} = \frac{Gm^2}{(a\sin 60^0)^2} = \frac{4Gm^2}{3a^2} \text{ along } PC$$

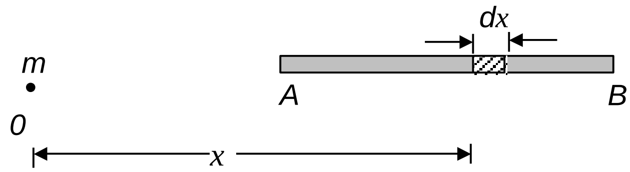
(b) From symmetry, the net force on the particle at the center of triangle = **0**

Illustration 4

Find the gravitational force of attraction on the point mass '*m*' placed at *O* by a thin rod of mass *M* and length *L* as shown in figure.



Solution:



First we need to find the force due to an element of length dx. The mass of the element is $dm = \left(\frac{M}{L}\right)dx$. so,

$$dF = G\frac{Mm}{L}\frac{dx}{x^2}$$

$\therefore$ The net gravitational force is

$$F = \frac{GMm}{L}\int_d^{d+L}\frac{dx}{x^2} = \frac{GMm}{L}\left[\frac{1}{d} - \frac{1}{L+d}\right]$$

$$= \frac{\boldsymbol{Mm}}{\boldsymbol{d(L+d)}}$$

Notice that when $d >> L$, we find $F = \frac{GMm}{d^2}$, the result for two point masses.

2. GRAVITATIONAL FIELD

(a) Gravitational field intensity due to a point mass

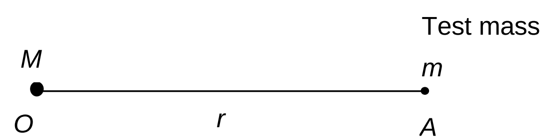
Consider a point mass M at O and let us calculate gravitational intensity at A due to this point mass.

Suppose a test mass is placed at A.

By Newton's law of gravitation, force on test mass

$$F = \frac{GMm}{r^2} \text{ along } \overrightarrow{AO}$$

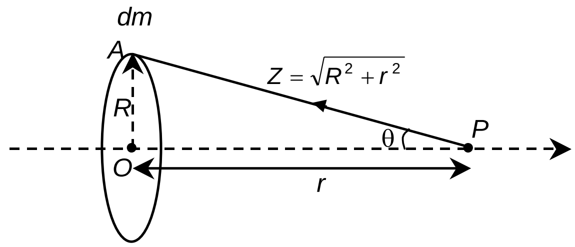
$$E = \frac{F}{m} = -\frac{GM}{r^2}\hat{e}_r \qquad \dots (2)$$



(b) Gravitational field intensity due to a uniform circular ring at a point on its axis.

Figure shows a ring of mass M and radius R. Let P is the point at a distance r from the centre of the ring. By symmetry the field must be towards the centre that is along $\vec{PO}$.

Let us assume that a particle of mass dm on the ring say, at point A. Now the distance AP is $\sqrt{R^2 + r^2}$.



Again the gravitational field at P due to dm is along $\vec{PA}$ and its magnitude is

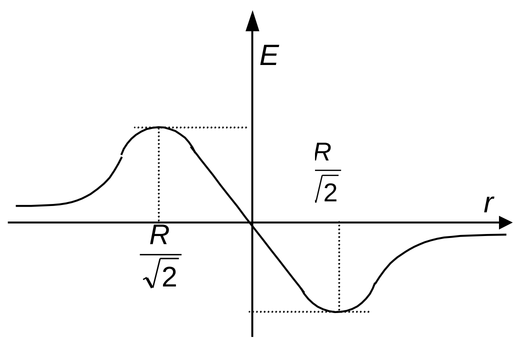
$$dE = \frac{Gdm}{Z^2}$$

$$\therefore \quad dE \cos\theta = \frac{Gdm}{Z^2}\cos\theta$$

Net gravitational field $E = \dfrac{G\cos\theta}{Z^2}\displaystyle\int dm$

$$= \frac{GM}{Z^2}\frac{r}{Z} = \frac{GMr}{(r^2+R^2)^{3/2}} \text{ along } \vec{PO} \quad \ldots (3)$$

Variation of gravitational field due to a ring as a function of its axial distance.



Important points:

(i) If $r >> R$, $r^2 + R^2 \simeq r^2$

$\therefore \; E = -\dfrac{GMr}{r^3} = -\dfrac{GM}{r^2}$ [where negative sign is because of attraction]

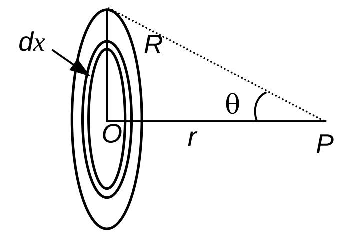
Thus, for a distant point, a ring behaves as a point mass placed at the center of the ring.

(ii) If $r << R$, $r^2 + R^2 \simeq R^2$

$$\therefore \quad E = -\frac{GMr}{R^3} \qquad \text{i.e.,} \qquad E \propto r$$

(c) Gravitational field intensity due to a uniform disc at a point on its axis

Let the mass of disc be M and its radius is R and P is the point on its axis where gravitational field is to be calculated.



Let us draw a circle of radius x and centre at O. We draw another concentric circle of radius $x + dx$. The part of disc enclosed between two circle can be treated as a uniform ring of radius x. The area of this ring is $2\pi x dx$

Therefore mass dm of the ring $= \dfrac{M}{\pi R^2} 2\pi x dx = \dfrac{2Mxdx}{R^2}$

gravitational field at P due to the ring is,

$$dE = \frac{G\left(\frac{2Mx\,dx}{R^2}\right)r}{(r^2+x^2)^{3/2}}$$

$$\int dE = \frac{2GMr}{R^2}\int_O^a \frac{xdx}{(r^2+x^2)^{3/2}}$$

$$= \frac{2GMr}{R^2}\left[-\frac{1}{\sqrt{r^2+x^2}}\right]_0^R$$

$$= \frac{2GMr}{R^2}\left[\frac{1}{r}-\frac{1}{\sqrt{r^2+R^2}}\right]$$

in terms of θ

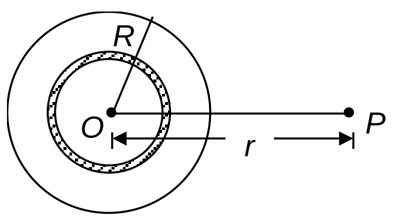
$$E = \frac{2GM}{R^2}(1-\cos\theta) \qquad \ldots (\mathbf{4})$$

(d) Gravitational field due to a uniform solid sphere

Case I:

Field at an external point

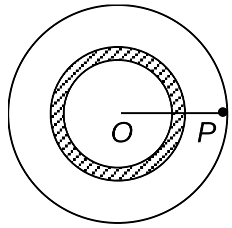
Let the mass of sphere is M and its radius is R we have to calculate the gravitational field at P.



$$\int dE = \int \frac{Gdm}{r^2} = \frac{G}{r^2}\int dm = \frac{GM}{r^2} \qquad \ldots (\mathbf{5})$$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point.

Case II



Field at an internal point

Suppose the point P is inside the solid sphere, in this case $r < R$ the sphere may be divided into thin spherical shells all centered at O. Suppose the mass of such a shell is dm. Then

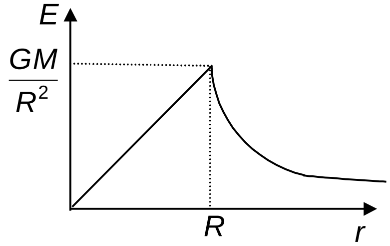
$$dE = \frac{Gdm}{r^2} \quad \text{along } PO$$

$$= \frac{G}{r^2}\int dm$$

where $$\int dm = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3}$$

$$\therefore \qquad E = \frac{GM}{R^3}r \qquad \ldots \textbf{(6)}$$

Therefore gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre $r = 0$ the field is zero. At the surface of the sphere $r = R$



$$E = \frac{GM}{R^2} \qquad \ldots \textbf{(7)}$$

☞**Gravitational field due to solid sphere is continuous but it is not differentiable function.**

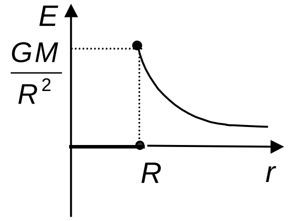
(e) Field due to uniform thin spherical shell

Case I

When point lies inside the spherical shell

$$\int dE = \frac{G}{r^2}\int m_{\text{enclosed}} = 0 \qquad \ldots (8)$$

Case II



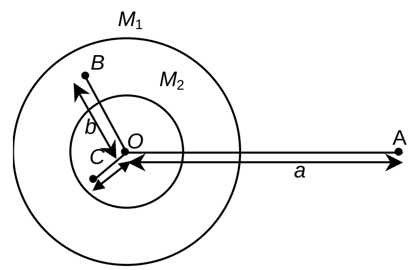
When point P lies outside the spherical shell

$$\int dE = \frac{G}{r^2}\int dm = \frac{GM}{r^2} \qquad \ldots \textbf{(9)}$$

☞ **Gravitational field due to thin spherical shell is both discontinuous and non-differentiable function.**

Illustration 5

Two concentric shells of masses M_1 and M_2 are situated as shown in figure. Find the force on a particle of mass m when the particle is located at (a) $r = a$ (b) $r = b$ (c) $r = c$. The distance r is measured from the center of the shell.



Solution

We know that attraction at an external point due to spherical shell of mass M is $\frac{GM}{r^2}$ while at an internal point is zero. So

(a) for $r = a$, the point is external for both the shell ; so

$$E_A = \frac{G(M_1 + M_2)}{a^2}$$

$$\therefore \qquad F_A = mE_A = \frac{GM\ [m_1 + M_2]}{a^2}$$

(b) For $r = b$, the point is external to the shell of mass M_2 and internal to the shell of mass M_1 ; so

$$E_B = \frac{GM_2}{b^2} + O$$

$$\therefore \qquad F_B = mE_B = \frac{GMm}{b^2}$$

(c) For $r = c$, the point is internal to both the shells, so

$$E_C = 0 + 0 = 0$$

$$\therefore \qquad F_C = m\ E_C = \mathbf{0}$$

3. GRAVITATIONAL POTENTIAL

At a point in a gravitational field, potential (V) is defined as the work done by the external agent against the gravitational field in bringing unit mass from infinity to that point.

Mathematically ,

$$V = \frac{W}{m}$$

It is a scalar having dimensions $[L^2\ T^{-2}]$ and SI unit J/kg.

$\Rightarrow$ By the definition of potential energy , $U = W$

So, $V = \frac{U}{m}$ i.e., $U = mV$

Thus, gravitational potential at a point represents potential energy of a unit point mass at that point.

$\Rightarrow$ by definition of work $W = \int \vec{F}_{ext} . d\vec{r}$

$\therefore$ But, $\vec{F}_{ext} = -\vec{F}_{gravitation}$

$\therefore$ $W = -\int \vec{F}_{gravitational} . d\vec{r}$

So, $V = -\dfrac{\int \vec{F}_{gravitational} . d\vec{r}}{m} = -\int \vec{E} d\vec{r}$ $\left[\because \dfrac{\vec{F}_{gravitational}}{m} = \vec{E}\right]$

i.e., $dV = -Edr$

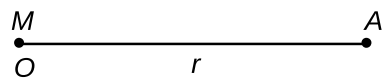
or $E = -\dfrac{dV}{dr}$ … (**10**)

So the potential can also be defined as a scalar function of position whose negative gradient. i.e., space derivative gives field intensity.

$\Rightarrow$ Negative of the slope of V *vs* r graph gives intensity.

CALCULATION OF GRAVITATIONAL POTENTIAL

(a) Gravitational potential at a point due to a point mass



We have, gravitational field due to a point mass

$$E = -\frac{GM}{r^2}$$

[The negative sign is used as gravitational force is attractive]

$\therefore$ $V = -\int Edr = -\dfrac{GM}{r} + C$

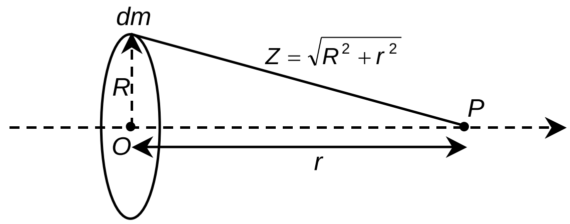
when, $r = \infty$, $V = 0$; so $C = 0$

$\therefore$ $V = -\dfrac{GM}{r}$ … (**11**)

(b) Gravitational potential at a point due to a ring

Let M be the mass and R be the radius of a thin ring.

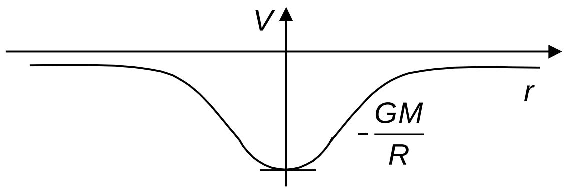
Considering a small element of the ring and treating it as a point mass, the potential at the point P is



$$dV = -\frac{G\,dm}{Z} = -\frac{G\,dm}{\sqrt{R^2 + r^2}}$$

Hence, the total potential at the point P is given by

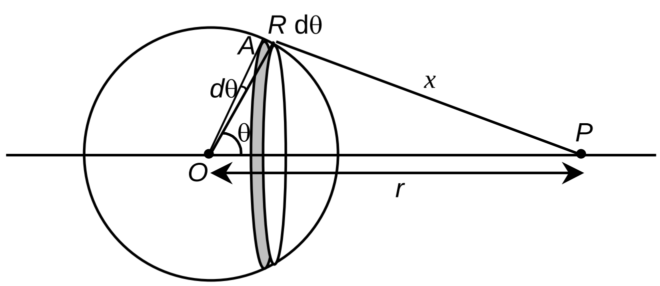
$$V = -\int \frac{G\,dm}{\sqrt{R^2 + r^2}} = -\frac{GM}{\sqrt{R^2 + r^2}} \quad \ldots \textbf{(12)}$$



at $\quad r = 0 \; \frac{dV}{dr} = 0$

$\therefore$ gravitational field is zero at the centre.

(c) Gravitational potential at a point due to a spherical shell (hollow sphere)



Consider a spherical shell of mass M and radius R. P is a point at a distance 'r' from the center O of the shell.

Consider a ring at right angles to OP. Let θ be the angular position of the ring from the line OP.

The radius of the ring = $R \sin\theta$

The width of the ring = $R\, d\theta$

The surface area of the ring = $(2\pi R \sin\theta) \,.\, R\,d\theta = 2\pi R^2 \sin\theta\, d\theta$

The mass of the ring $= (2\pi R^2 \sin\theta\, d\theta) \times \frac{M}{4\pi R^2} = \frac{M \sin\theta\, d\theta}{2}$

If 'x' is the distance of the point P from a point of the ring, then the potential at P due to the ring.

$$dV = -\frac{GM \sin\theta\, d\theta}{2x}$$

From the 'cosine-property' of the triangle *OAP*,

$$x^2 = R^2 + r^2 - 2Rr\cos\theta$$

Differentiating,

$$2x\,dx = 2\,Rr\sin\theta\,d\theta$$

$$\therefore \qquad \sin\theta\,d\theta = \frac{x\,dx}{Rr}$$

Substituting the above value of $\sin\theta\,d\theta$ in equation (i), we get

$$dV = -\frac{GM}{2x} \times \frac{x\,dx}{Rr}$$

$$= -\frac{GM}{2Rr}dx$$

Case I:

When the point *P* lies outside the shell.

$$V = -\frac{GM}{2Rr}\int_{r-R}^{R+r} dx = -\frac{GM}{2Rr}[x]_{r-R}^{r+R}$$

$$V = -\frac{GM}{2Rr}\left[(R+r)-(r-R)\right]$$

$$V = -\frac{GM}{r} \qquad \ldots (13)$$

This is the potential at *P* due to a point mass *M* at *O*.

For external point, a spherical shell behaves as a point mass supposed to be placed at it center.

Case II:

When the point *P* lies inside the spherical shell.

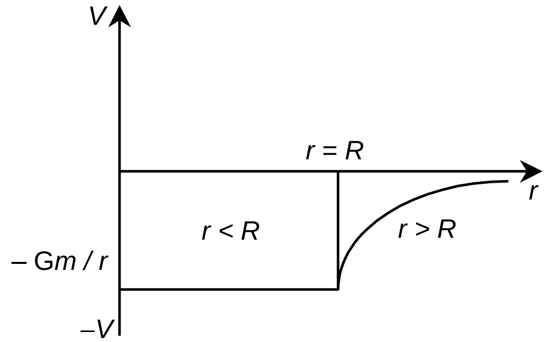
$$V = -\frac{GM}{2Rr}\int_{R-r}^{R+r} dx = -\frac{GM}{2Rr}[x]_{R-r}^{R+r}$$

or $$V = -\frac{GM}{R} \qquad \ldots (14)$$

This expression is independent of r. Thus, the potential at every point inside the spherical shell is the same and it is equal to the potential of the surface of the shell.

Thus, the gravitational field inside a spherical shell is zero everywhere.

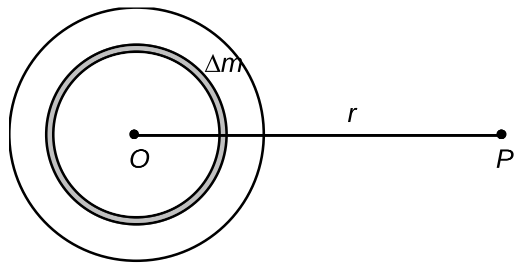
Graphical representation of the variation of *V* with *r* in case of a hollow spherical shell



(d) Gravitational potential due to a homogeneous solid sphere

Case I:

When the point *P* lies outside the sphere



Consider a homogenous solid sphere of mass *M* and radius '*R*'. *P* is a point at a distance '*r*' from the center of the sphere.

The solid sphere may be supposed to be made up of large number of thin concentric spherical shells. Consider one such shell of mass Δm.

The potential at *P* due to the shell = $-\dfrac{G\,\Delta m}{r}$

So, the potential at *P* due to the entire sphere.

$$V = -\sum \frac{G\,\Delta m}{r} = -\frac{G}{r}\sum \Delta m$$

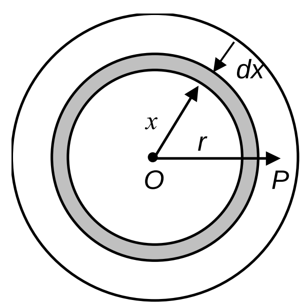
$$V = -\frac{GM}{r} \quad [\because M = \Sigma\,\Delta m] \qquad \ldots \textbf{(15)}$$

Hence for an external point, a solid sphere behaves as if the whole of its mass is concentrated at the centre.

Case II:

When the point P lies inside the sphere

Let us consider a concentric spherical surface thorough the point P. The potential at P arises out of the inner sphere and the outer thick spherical shell.



$\therefore\ V = V_1 + V_2$, where V_1 = potential due to the inner sphere and V_2 = potential due to the outer thick shell.

The mass of the inner sphere = $\dfrac{4\pi r^3}{3}\rho$, where

ρ = density of the sphere = $\dfrac{M}{\frac{4}{3}\pi R^3} = \dfrac{3M}{4\pi R^3}$

The potential at P due to this sphere

$$V_1 = -\frac{G\left[\frac{4\pi r^3}{3}\right]\rho}{r} = -\frac{4\pi G\rho}{3}r^2 \qquad \ldots \text{(i)}$$

To find V_2, consider a thin concentric shell of radius x and thickness dx.

The volume of the shell = $4\pi\, x^2\, d\,x$

The mass of the shell = $4\pi\, x^2\, d\,x\, \rho$.

The potential at P due to this shell

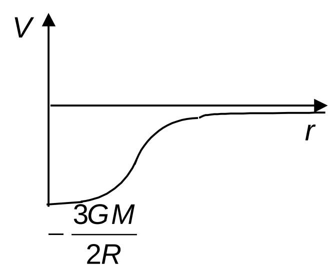
$$V_2 = -\int_r^R 4\pi G\rho x\, dx = -4\pi G\rho\left[\frac{x^2}{2}\right]_r^R = -4\pi G\rho\left[\frac{R^2}{2} - \frac{r^2}{2}\right] = -2\pi G\rho\left[R^2 - r^2\right]$$

$$\therefore \qquad V_1 + V_2 = -\frac{4\pi G\rho r^2}{3} - 2\pi G\rho\left[R^2 - r^2\right]$$

$$= -\frac{4\pi G\rho}{3}\left[r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right]$$

$$= -\frac{4\pi G\rho}{3}\left[\frac{3R^2}{2} - \frac{r^2}{2}\right]$$

$$= -\frac{4\pi G}{3}\cdot\frac{3M}{4\pi R^3}\left[\frac{3R^2 - r^2}{2}\right]$$



or, $$V = -\frac{GM}{2R^3}\left[3R^2 - r^2\right] \qquad \ldots \textbf{(16)}$$

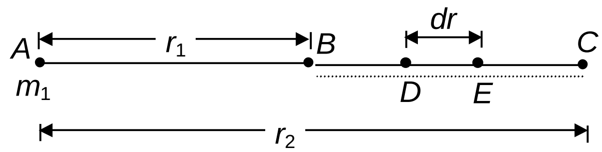
at $\quad r = 0,\ \dfrac{dV}{dr} = 0$

Hence gravitational field is 0 at the centre of a solid sphere.

4. GRAVITATIONAL POTENTIAL ENERGY

The potential energy of a system corresponding to a conservative force is defined as $\int_{U_i}^{U_f} dU = -\int \vec{F}\,.\,\vec{dr}$

i.e., The change in potential energy is equal to negative of work done



Let a particle of mass m_1 be kept fixed at point A and another particle of mass m_2 is taken from a point B to C. Initially, the distance between the particle is r_1 & finally it becomes $AC = r_2$. We have to calculate the change in gravitation in potential energy of the system of two particles.

Consider a small displacement when the distance between the particles changes from r to $r + dr$. In the figure this corresponds to the second particle going from D to E.

Force acting on second particle is; $\vec{F} = \frac{Gm_1m_2}{r_2}$ along $\vec{DA}$

$$\therefore \qquad dW = \vec{F}.\vec{dr} = -\frac{Gm_1m_2}{r^2}dr$$

$$dU = -\,dW = \int \frac{Gm_1m_2\ dr}{r^2}$$

$$\int_{r_1}^{r_2} dU = Gm_1m_2\int_{r_1}^{r_2}\frac{dr}{r^2} = Gm_1m_2\left(-\frac{1}{r}\right)_{r_1}^{r_2}$$

$$U(r_2) - U(r_1) = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \dots \text{(i)}$$

We choose the potential energy of the two particles system to be zero when the distance between them is infinity. This means $U(\infty) = 0$

Hence $U(r) = U(r) - U(\infty)$

Now in equation (i) Take $r_1 = r$ and $r_2 = \infty$

$$U(\infty) - U(r) = Gm_1m_2\left(\frac{1}{r} - \frac{1}{\infty}\right)$$

$$\therefore \qquad U(r) = -\frac{Gm_1m_2}{r_1} \qquad \dots \textbf{(17)}$$

Hence when two masses m_1 and m_2 are separated by a distance, their gravitational potential energy is

$$U(r) = \frac{-Gm_1m_2}{r}$$

Illustration 6

Three particles each of mass *m* are placed at the corners of an equilateral triangle of side *d*. Calculate

(a) the potential energy of the system,

(b) work done on the system if the side of the triangle is changed from *d* to 2*d*.

Solution:

(a) $$U = -\frac{3Gmm}{d}$$

$$= -\frac{3Gm^2}{d}$$

(b) $$U_i = -\frac{3Gm^2}{d}$$

$$U_f = -\frac{3Gm^2}{2d}$$

$\therefore$ Work done $= w = \Delta U = U_f - U_i = \dfrac{3Gm^2}{2d}$

5. VARIATION IN ACCELERATION DUE TO GRAVITY

(a) With altitude

At the surface of the earth, $g = \dfrac{GM}{R^2}$

For a height *h* above the surface of the earth,

$$g' = \frac{GM}{(R+h)^2}$$

$$\frac{g'}{g} = \frac{R^2}{(h+R)^2} = \frac{1}{\left(1+\frac{h}{R}\right)^2}$$

or, $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

So, with increase in height, *g* decreases. If $h << R$.

$$g' = g\left[1+\frac{h}{R}\right]^{-2} = g\left[1-\frac{2h}{R}\right]$$

or, $$g' = g\left[1-\frac{2h}{R}\right] \qquad \ldots (18)$$

(b) With depth

At the surface of the earth,

$$g = \frac{GM}{R^2}$$

for a point at a depth *d* below the surface,

$$g' = \frac{GM}{R^3}[R-d]$$

$\therefore$
$$\frac{g'}{g} = \left(\frac{R-d}{R}\right)$$

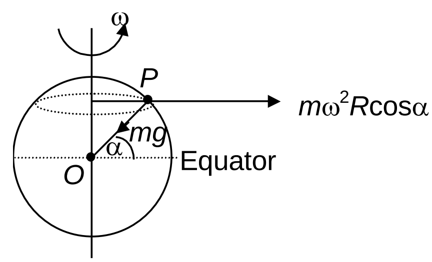
i.e.,
$$g' = g\left[1-\frac{d}{R}\right] \quad \ldots (19)$$

So with increase in depth below the surface of the earth, g decreasing and at the center of the earth it becomes zero.

It should be noted that value of g decreases if we move above the surface or below the surface of the earth.

(c) Due to rotation of earth

The earth is rotating about its axis from west to east. So, the earth is a non-inertial frame of reference. Everybody on its surface experiences a centrifugal force $m\omega^2 R\cos\alpha$. Where α is latitude of the place.



The net force on a particle on the surface of the earth

$$F = \sqrt{m^2g^2 + m^2\omega^4R^2\cos^2\alpha + 2(mg)(m\omega^2R\cos\alpha)[\cos(180-\alpha)]} = mg' \quad \ldots (\mathbf{20})$$

Therefore,

(i) g is maximum (= g) when cos α = minimum = 0, i.e., α = 90°, i.e., at the pole.

(ii) g is minimum (= $g - \omega^2R$) when cos α = maximum = 1, i.e., , α = 0°, i.e., at the equator.

Illustration 7

Calculate the acceleration due to gravity at the surface of Mars if its diameter is 6760 km and mass one tenth that of the earth. The diameter of earth is 12742 km and acceleration due to gravity on the earth is 9.8 m/s^2.

Solution:

We know that

$$g = \frac{GM}{R^2}$$

$$\therefore \qquad \frac{g_M}{g_E} = \left(\frac{M_M}{M_E}\right)\left(\frac{R_E}{R_M}\right)^2 = \left[\frac{1}{10}\right]\left[\frac{12742}{6760}\right]^2$$

$$\text{or,} \qquad \frac{g_M}{g_E} = 0.35$$

$\therefore \qquad g_M = 0.35 \times g_E = 9.8 \times 0.35 = \mathbf{3.48\ m/s^2}$

Illustration 8

Compute the mass and density of the moon if acceleration due to gravity on its surface is 1.62 m/s^2 and its radius is 1.74×10^6 m ($G = 6.67 \times 10^{-11}$ MKS units)

Solution:

We know that

$$g = \frac{GM}{R^2}$$

$$\therefore \qquad M = \frac{gR^2}{G} = \frac{1.62 \times (1.74 \times 10^6)^2}{6.67 \times 10^{-11}}$$

or, $\qquad M = \mathbf{7.35 \times 10^{22}\ kg}$

$$\text{Now,} \quad \rho = \frac{M}{V} = \frac{gR^2}{G\left(\frac{4}{3}\pi R^3\right)} \ \frac{3g}{4\pi GR}$$

$$\therefore \qquad \rho = \frac{3 \times 1.62}{4 \times 3.14 \times 6.67 \times 10^{-11} \times 1.74 \times 10^6} = 3.3 \times 10^3 \text{ kg/m}^3$$

6. ESCAPE SPEED

It is the minimum speed with which a body must be projected from the surface of a planet (usually the earth) so that it permanently overcomes and escapes from the gravitational field of the planet (the earth). We can also say that a body projected with escape speed will be able to go to a point which is at infinite distance from the earth.

If a body of mass m is projected with speed v from the surface of a planet of mass M and radius R, then

$$\text{K.E.} = \frac{1}{2}mv^2 \ ; \ G.P.E. = -\frac{GMm}{R}$$

Total mechanical energy (T.M.E.) of the body = $\frac{1}{2}mv^2 - \frac{GMm}{R}$

If the v' is the speed of the body at infinity, then

T.M.E. at infinity = $0 + \frac{1}{2}mv'^2 = \frac{1}{2}mv'^2$

Applying the principle of conservation of mechanical energy, we have

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv'^2$$

or, $$v^2 = \frac{2GM}{R} + v'^2$$

v will be minimum when $v' \to$ o, i.e.,

$$V_e = V_{min} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \qquad \dots (\mathbf{21})$$

$$\left[\because g = \frac{GM}{R^2}\right]$$

Important points

(i) Escape speed is independent of the mass and direction of projection of the body.

(ii) For earth as g = 9.8 m/s^2 and R = 6400 km

$$V_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 11.2 \text{ km/s}$$

Illustration 9

The masses and radii of the earth and the moon are M_1, R_1 and M_2, R_2 respectively. Their centers are at a distance d apart. What is the minimum speed with which a particle of mass m should be projected from a point midway between the two centers so as to escape to infinity?

Solution:

Potential energy of m which is midway between M_1 and M_2 is

$$U = m\,(V_1 + V_2) = m\left(-\frac{GM_1}{d/2} + \frac{-GM_2}{d/2}\right)$$

$$= -\frac{2Gm}{d}(M_1 + M_2)$$

Let v be the required speed, then

$$(\text{T.M.E.})_{\text{initial}} = -\frac{2Gm}{d}(M_1 + M_2) + \frac{1}{2}mv^2$$

As the particle reaches infinity,

$(\text{T.M.E.})_{\text{final}} = 0$

From the principle of conservation of mechanical energy, we have

$$-\frac{2Gm}{d}(M_1 + M_2) + \frac{1}{2}mv^2 = 0$$

$$v = 2\sqrt{\frac{G\,(M_1 + M_2)}{d}}$$

Illustration 10

What will be the acceleration due to gravity on the surface of the moon if its radius were $\frac{1}{4}$ th the radius of the earth and its mass $\left(\frac{1}{80}\right)th$ the mass of the earth. What will be the escape speed on the surface of the moon if it is 11.2 km/s on the surface of the earth. Given g = 9.8 m/s^2.

Solution:

As on the surface of planet,

$$g = \frac{GM}{R^2}, \text{ we have}$$

$$\frac{g_M}{g_E} = \frac{M_M}{M_E} \times \left(\frac{R_E}{R_M}\right)^2 = \frac{1}{80} \times (4)^2 = \frac{1}{5}$$

$$\therefore \quad g_M = \frac{g}{5} = \frac{9.8}{5} = \mathbf{1.96\ m/s^2}$$

Further, as escape speed = $V_e = \sqrt{\frac{2GM}{R}}$, so

$$\frac{V_M}{V_E} = \sqrt{\frac{M_M}{M_E} \times \frac{R_E}{R_M}} = \sqrt{\frac{1}{80} \times 4} = \frac{1}{\sqrt{20}}$$

$$\therefore \quad V_M = \frac{V_E}{\sqrt{20}} = \frac{11.2}{4.47} = 2.5 \text{ km/s}$$

7. MOTION OF A SATELLITE

Consider a satellite of mass *m* revolving in a circle around the earth. If the satellite is at a height *h* above the earth's surface, the radius of its orbit is $r = R + h$, where *R* is the radius of the earth. The gravitational force between *m* and *M* provides the necessary centripetal force for circular motion.

(a) Orbital velocity (V_0)

The velocity of a satellite in its orbit is called orbital velocity. Let V_0 be the orbital velocity of the satellite, then

$$\frac{GMm}{r^2} = \frac{m{v_0}^2}{r}$$

$$\Rightarrow \quad v_0 = \sqrt{\frac{GM}{r}} \qquad \ldots \textbf{(22)}$$

$$\text{or,} \quad v = \sqrt{\frac{GM}{R+h}} \qquad (\because r = R + h)$$

Important points

☞ Orbital velocity is independent of the mass of the orbiting body and is always along the tangent to the orbital.

☞ Close to the surface of the earth, $r = R$ as $h = 0$.

$$\therefore \qquad V_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \sqrt{10 \times 6.4 \times 10^6} \approx 8 \text{ km/s}$$

☞ Close to the surface of the planet

$$V_0 = \sqrt{\frac{GM}{R}} = \frac{V_e}{\sqrt{2}}$$

i.e., $\qquad V_e = \sqrt{2}V_0$

(b) Time period of a satellite

The time taken by a satellite to complete one revolution is called the time period (T) of the satellite

It is given by

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}}$$

or, $\qquad T = \frac{2\pi r\sqrt{r}}{\sqrt{GM}} \qquad$ or, $\qquad T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \qquad$ … (**23**)

$\Rightarrow \qquad T^2 \,\alpha\, r^3$

(c) Angular momentum of a satellite (L)

In case of satellite motion, angular momentum will be given by

$$L = mvr = mr\sqrt{\frac{GM}{r}}$$

$$L = \left(m^2GMr\right)^{1/2} \qquad \text{… (\textbf{24})}$$

☞In case of satellite motion, the net force on the satellite is centripetal force. The torque of this force about the center of the orbit is zero. Hence, angular momentum of the satellite is conserved, i.e., L = constant.

(d) Energy of a satellite

The P.E. of a satellite is

$$U = mV = -\frac{GMm}{r} \qquad \left[\because V = -\frac{GM}{r}\right]$$

The kinetic energy of the satellite is

$$K = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} \qquad \left[\because v_0 = \sqrt{\frac{GM}{r}}\right]$$

Total mechanical energy of the satellite = $-\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r}$ … (25)

Important points

☞ We have,

$$\frac{K}{E} = -1 \text{ i.e., } K = -E$$

Also, $$\frac{U}{E} = 2$$

$\Rightarrow$ $$U = 2E$$

☞ Total energy of a satellite in its orbit is negative. Negative energy means that the satellite is bound to the central body by an attractive force and energy must be supplied to remove it from the orbit to infinity.

Binding energy of the satellite

The energy required to remove the satellite from its orbit to infinity is called binding energy of the satellite, i.e.,

Binding energy $= -E = \dfrac{GMm}{2r}$

7.1. GEO-STATIONARY SATELLITE

If there is a satellite rotating in the direction of earth's rotation, i.e., from west to east, then for an observer on the earth the angular velocity of the satellite will be $(\omega_S - \omega_E)$.

However, if $\omega_S - \omega_E = 0$, satellite will appear stationary relative to the earth. Such a satellite is called 'Geo-stationary satellite' and is used for communication purposes.

The orbit of a geostationary satellite is called '**Parking orbit**'.

We know that, $T^2 = \dfrac{4\pi^2}{GM} r^3$ … (**26**)

For geostationary satellite, T = 24 hours.

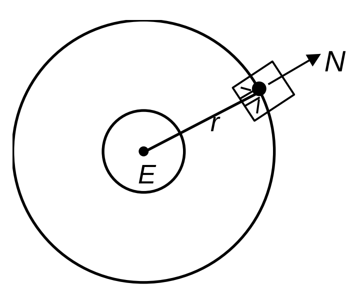
Putting this value of T in the above equation, we get

$$r \approx 42000 \text{ km}$$

or, $$h \approx 36000 \text{ km.}$$

where h is the height of the satellite from the surface of the earth.

7.2. WEIGHTLESSNESS IN A SATELLITE



The radial acceleration of the satellite is given by

$$a_r = \frac{F_r}{m} = \frac{GMm}{r^2 \times m} = \frac{GM}{r^2}$$

For a astronaut inside the satellite, we have

$$\frac{GMm_a}{r^2} - N - m_a a_r = 0$$

where m_a is mass of astronaut a_r is radial acceleration of satellite and N is normal reaction on the astronaut

or, $\dfrac{GMm_a}{r^2} - N - \dfrac{GMm_a}{r^2} = 0$

$\Rightarrow \quad N = 0$

Hence, the astronaut feels weightlessness.

Illustration 11

Calculate the orbital velocity of a satellite revolving at a height h above the earth's surface if $h = R$. Also calculate the time-period of this satellite. (g = 9. m/s^2, R = 6400 km)

Solution:

For the orbital velocity in a circular orbit, we have

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}} \quad (\because r = R + h)$$

$$\Rightarrow \quad v = \sqrt{\frac{gR^2}{2R}} = \sqrt{\frac{gR}{2}} \quad (\because GM = gR^2 \text{ and } h = R) \Rightarrow v = \sqrt{\frac{9.8 \times 6400 \times 10^3}{2}} = 5.6 \text{ km/s}$$

$$\text{Time period} = T = \frac{2\pi r}{v} = \frac{2\pi(2R)}{4\sqrt{2} \times 10^3}$$

$$\Rightarrow \quad T = \frac{4\pi \times 6400 \times 10^3}{4\sqrt{2} \times 10^3} = 3.95 \text{ hrs}$$

Illustration 12

Two planets have masses in the ratio 1 : 10 and radii in the ratio 2: 5. Compare

(a) their densities

(b) the acceleration due to gravity on their surface

(c) escape velocities from their surfaces, and

(d) the periods of revolutions of satellites near to their surfaces.

Solution:

Let M_1, M_2 by the masses and R_1, R_2 be the radii of the planets.

$$\Rightarrow \quad \frac{M_1}{M_2} = \frac{1}{10} \text{ and } \frac{R_1}{R_2} = \frac{2}{5}$$

$$\text{(a) Ratio of densities} = \frac{d_1}{d_2} \text{ or, } \frac{d_1}{d_2} - \left[\frac{M_1}{\frac{4}{5}\pi R_1^3}\right]\left[\frac{\frac{4}{3}\pi R_2^3}{M_2}\right]$$

$$\text{or,} \quad \frac{d_1}{d_2} = \frac{M_1}{M_2}\left[\frac{R_2}{R_1}\right]^3 \Rightarrow \frac{d_1}{d_2} = \left[\frac{1}{10}\right]\left[\frac{5}{2}\right]^3 = \frac{25}{16}$$

(b) Acceleration due to gravity at the surface = $g = \frac{GM}{R^2}$

$$\therefore \qquad \frac{g_1}{g_2} = \frac{M_1}{M_2}\left[\frac{R_2}{R_1}\right]^2$$

$$= \frac{1}{10}\left[\frac{5}{2}\right]^2 = \frac{5}{8}$$

(c) Escape velocity = $\sqrt{\frac{2GM}{R}}$

$$\Rightarrow \qquad \frac{v_1}{v_2} = \sqrt{\frac{M_1}{M_2}}\sqrt{\frac{R_2}{R_1}} = \sqrt{\frac{1}{10}\times\frac{5}{2}} = \mathbf{\frac{1}{2}}$$

(d) Time period of a satellite near the surface (orbit radius = R) = $\frac{2\pi}{\sqrt{GM}}R\sqrt{R}$

$$\Rightarrow \qquad \frac{T_1}{T_2} = \sqrt{\frac{M_2}{M_1}}\left[\frac{R_1}{R_2}\right]\left[\sqrt{\frac{R_1}{R_2}}\right]$$

$$= \sqrt{\frac{10}{1}}\left[\frac{2}{3}\right]\left[\sqrt{\frac{2}{5}}\right] = \frac{4}{3}$$

Illustration 13

The mean distance of mars from the sun is 1.524 times that of the earth from the sun. Find the number of years required for mars to make one revolution about the sun.

Solution:

For planets revolving around the sun, $T^2 \alpha\, r^3$

$$\Rightarrow \qquad \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$ [T_1 : time-period of mars and T_2 : time-period of earth]

$$\Rightarrow \qquad \frac{T_1}{T_2} = \left[\frac{r_1}{r_2}\right]^{3/2}$$

$$\Rightarrow \qquad T_1 = T_2\left[\frac{r_1}{r_2}\right]^{3/2}$$ = (1 year) $[1.524]^{3/2}$ = **1.88 years**

Fundamental Solved Examples

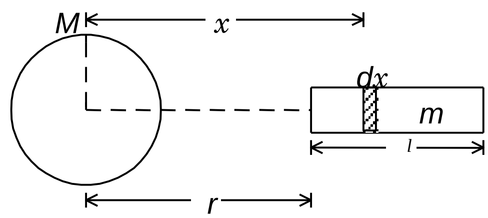
Example 1.

Find the gravitational force of attraction between a uniform sphere of mass M and a uniform rod of length l and mass m oriented as shown in the figure.

Solution:

Since the sphere is uniform its entire mass may be considered to be concentrated at its centre. The force on the elementary mass dm is

$$dF = \frac{GMdm}{x^2}$$



But dm $\frac{m}{l}dx$

$$F = \int_r^{r+\ell} \frac{GMm}{lx^2}dx = -\frac{GMm}{l}\left[\frac{1}{x}\right]_r^{r+\ell} = -\frac{GMm}{l}\left[\frac{1}{r+l} - \frac{1}{r}\right]$$

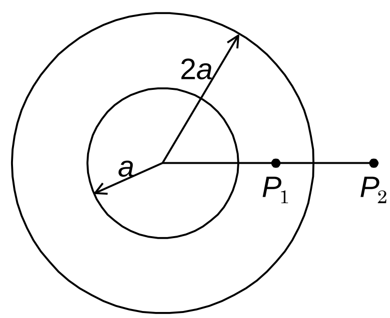
$$F = \frac{GmM}{l}\frac{l}{r(r+l)}$$

Example 2.

A uniform solid sphere of mass M and radius a is surrounded symmetrically by a thin spherical shell of equal mass and radius 2a. Find the gravitational field at a distance

(a) $\frac{3}{2}a$ from centre (b) $\frac{5}{2}a$ from centre.

Solution:



(a) The situation is indicated in the figure in the two cases.

The point P_1 is at a distance $\frac{3}{2}a$ from centre and P_2 is at a distance $\frac{5}{2}a$ from centre. As P_1 is inside the cavity of the thin spherical shell the field here due to the shell is zero. The field due to the solid sphere is

$$E = \frac{GM}{\left(\frac{3}{2}a\right)^2} = \frac{4GM}{9a^2}$$

This also represents the resultant field at P_1.

Resultant field $= \frac{4GM}{9a^2}$. The direction is towards centre.

(b) In this case P_2 is outside the sphere as well as the shell. Both may be replaced by single particles of same mass at the centre.

The field due to each of them at $P_2 = \frac{GM}{\left(\frac{5}{2}a\right)^2} = \frac{4GM}{25a^2}$

Resultant field $= \frac{4GM}{25a^2} + \frac{4GM}{25a^2} = \frac{8GM}{25a^2}$

This is also acting towards the common centre.

Example 3.

An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from earth. Determine

(a) the height of satellite above earth's surface.

(b) if the satellite is suddenly stopped, find the speed with which the satellite will hit the earth's surface after falling down.

Solution:

Escape velocity $= \sqrt{2gR}$, where g is acceleration due to gravity on surface of earth and R the radius of earth.

Orbitcal velocity $= \frac{1}{2}v_e = \frac{1}{2}\sqrt{2gR} = \sqrt{\frac{gR}{2}}$

(a) If h is the height of satellite above earth

$$\frac{mv_0^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$v_0^2 = \frac{GM}{R+h} = \frac{gR^2}{(R+h)}$$

$$\therefore \quad \left(\frac{1}{2}v_e\right)^2 = \frac{gR^2}{R+h}$$

$$\frac{gR}{2} = \frac{gR^2}{R+h} \text{ from equation (1)}$$

R + h = 2 R

h = R

(b) If the satellite is stopped in orbit, the kinetic energy is zero and its potential energy is $\frac{-GMm}{2R}$,

Total energy $= \frac{-GMm}{2R}$

When it reaches the earth let v be its velocity.

Hence the kinetic energy $=\frac{1}{2}mv^2$

Potential energy $=-\frac{GMm}{R}$

$$\therefore \frac{1}{2}mv^2-\frac{GMm}{R}=-\frac{GMm}{2R}$$

$$v^2=-2GM\left(\frac{1}{R}-\frac{1}{2R}\right)=\frac{-2gR^2}{-2R}$$

$$= gR$$

$$v=\sqrt{gR}$$

Example 4.

Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolutions are 1 hour and 8 hours respectively. The radius of orbit of S_1 is 10^4 km. When S_2 is closest to S_1, find

(i) speed of S_2 relative to S_1

(ii) the angular speed of S_2 actually observed by an astronaut in S_1.

Solution:

If r_1 and r_2 are radii of orbits of S_1 and S_2, T_1 and T_2 their respective periods, we have by Kepler's third law

$$\frac{T_1^2}{T_2^2}=\frac{r_1^3}{r_2^3}$$

$$r_2^3=r_1^3\left(\frac{T_2}{T_1}\right)^2=\left(10^4\right)^3\left(\frac{8}{1}\right)^2=\left(10^4\times4\right)^3$$

$$\therefore \quad = 4\times10^4 \text{ km}$$

(i) If the orbital speeds of satellites S_1 and S_2 are V_1 and v_2.

$$v_1=\frac{2\pi r_1}{T_1}$$

$$v_2=\frac{2\pi r_2}{T_2}$$

Speed of S_2 relative to S_1 $=\left|v_2-v_1\right|$

$$=\frac{2\pi r_2}{T_2}-\frac{2\pi r_1}{T_1}=2\pi\left(\frac{r_2}{T_2}-\frac{r_1}{T_1}\right)=2\pi\left(\frac{4\times10^4}{8}-\frac{10^4}{1}\right)$$

$$= -2\pi \times 10^4 \times \frac{1}{2} = -3.14 \times 10^4 \text{ km/hr.}$$

(ii) Angular speed of S_2 relative to S_1 $= \frac{v_2 - v_1}{r_2 - r_1} = -\frac{3.14 \times 10^4}{4 \times 10^4 - 10^4}$

$$= -1.046 \text{ rad/hour} = -\frac{1.046}{3600} \text{ rad/sec}$$

$$= -2.906 \times 10^{-4} \text{ rad/sec}$$

Example 5.

Two satellites of same mass are launched in the same orbit round the earth so as to rotate opposite to each other. They collide inelastically and stick together as wreckage. Obtain the total energy of the system before and after collision. Describe the subsequent motion of wreckage.

Solution:

Potential energy of satellite in orbit $= -\frac{G \cdot M \cdot m}{r}$

If v is the velocity in orbit, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r}$$

$\therefore$ kinetic energy $= \frac{1}{2}mv^2 = \frac{GMm}{2r}$

$\therefore$ total energy $= \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$

For the two satellites, the total energy before collision $= 2\left[-\frac{GMm}{2r}\right] = -\frac{GMm}{r}$

After collision, let v′ be the velocity of the wreckage. By law of conservation of momentum, $m\vec{v'} - m\vec{v'} = 2mv'$ {since they are rotating opposite to each other}

$\therefore \quad v' = 0$

The wreckage has no kinetic energy after collision but has potential energy

$$= \frac{-GM(2m)}{r}$$

$\therefore$ total energy after collision $= \frac{-2GMm}{r}$

After collision the centripetal force disappears and the wreckage falls down under the action of gravity.

Example 6.

Two bodies of masses M_1 and M_2 are placed at a distance d apart. What is the potential at the position where the gravitational field due to them is zero?

Solution:

Let the field be zero at a point distant x from M_1.

$$\therefore \quad \frac{GM_1}{x^2}=\frac{GM_2}{(d-x)^2}$$

$$\therefore \quad \frac{x}{d-x}=\sqrt{\frac{M_1}{M_2}}$$

$$x\sqrt{M_2}=\sqrt{M_1}\cdot d-x\sqrt{M_1}$$

$$x\left[\sqrt{M_1}+\sqrt{M_2}\right]=\sqrt{M_1}\cdot d$$

$$x=\frac{d\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}$$

$$d-x=\frac{d\sqrt{M_2}}{\sqrt{M_1}+\sqrt{M_2}}$$

Potential at this point due to both the masses will be

$$=-\frac{GM_1}{x}-\frac{GM_2}{(d-x)}=-G\left[\frac{M_1\left(\sqrt{M_1}+\sqrt{M_2}\right)}{d\sqrt{M_1}}+\frac{M_2\left(\sqrt{M_1}+\sqrt{M_2}\right)}{d\sqrt{M_2}}\right]$$

$$=-\frac{G}{d}\left(\sqrt{M_1}+\sqrt{M_2}\right)^2=-\boldsymbol{\frac{G}{d}\left(M_1+M_2+2\sqrt{M_1M_2}\right)}$$

Example 7:

The distance between earth and moon is 3.8×10^5 km and the mass of earth is 81 times the mass of moon. Deduce the position of a point on the line joining the centres of earth and moon, where the gravitational field is zero. What would be the value of gravitational field there due to earth and moon separately?

Solution:

Let x be the distance of the point of no net field from earth.

The distance of this point from moon is (r – x), where $r=3.8\times10^5$ km.

The gravitational field due to earth $=\frac{GM_e}{x^2}$ and that due to moon $=\frac{GM_m}{(r-x)^2}$. For the net field to be zero these are equal and opposite.

$$\frac{GM_e}{x^2}=\frac{GM_m}{(r-x)^2}$$

$$\frac{M_e}{M_m}=\frac{x^2}{(r-x)^2}. \text{ But } \frac{M_e}{M_m}=81$$

$$\therefore \qquad 81=\frac{x^2}{(r-x)^2}$$

$$\frac{x}{r-x}=9$$

9r – 9x = x

10x = 9 r

$$x=\frac{9}{10}r=\frac{9}{10}\times 3.8\times 10^5 = 3.42 \times 105 \text{ km}$$

The intensity of the field $=\frac{GM_e}{x^2}=\frac{R_e^2 g}{x^2}=\frac{(6.4\times 10^6)^2\times 9.8}{(3.42\times 10^8)^2}$

$= 3.43 \times 10^{-3}$ N/kg.

Example 8.

A rocket starts vertically upwards with speed v_0. Show that its speed v at a height is given by $v_0^2-v^2=\frac{2gh}{1+\frac{h}{R}}$, where R is the radius of earth. Hence deduce the maximum height reached by the rocket fired with a speed of 90% of escape velocity.

Solution:

Kinetic energy on the surface of earth $=\frac{1}{2}mv_0^2$

Potential energy on the surface of earth $=\frac{-GMm}{R}$

Total energy $=\frac{1}{2}mv_0^2-\frac{GMm}{R}$

Kinetic energy at a height $h=\frac{1}{2}mv^2$

Potential energy at this height $=\frac{-GMm}{(R+h)}$

Total energy $=\frac{1}{2}mv^2-\frac{GMm}{R+h}$

By the principle of conservation of energy,

$$\frac{1}{2}mv^2-\frac{GMm}{R+h}=\frac{1}{2}mv_0^2-\frac{GMm}{R}$$

$$\frac{1}{2}\left(v_0^2 - v^2\right) = \frac{GM}{R} - \frac{GM}{(R+h)}$$

But GM = gR^2

$$\frac{1}{2}\left(v_0^2 - v^2\right) = \frac{gR^2 h}{R(R+h)}$$

$$v_0^2 - v^2 = \frac{2gRh}{R+h} = \frac{2gh}{1+\frac{h}{R}}$$

For maximum height v = 0,

v_0 = 90% of escape velocity

$$= 0.9\sqrt{2gR}$$

$$\therefore \qquad \left(0.9\sqrt{2gR}\right)^2 - 0 = \frac{2gh_{max}}{1+\frac{h_{max}}{R}}$$

$$0.81R = \frac{h_{max}}{1+\frac{h_{max}}{R}}$$

$$0.81R + 0.81h_{max} = h_{max}$$

$$0.19h_{max} = 0.81R$$

$$h_{max} = \frac{0.81R}{0.19} = 4.26\text{ R}$$

Example 9.

An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.

(a) Determine the height of the satellite above the earth's surface.

(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits the surface of the earth. [g = 9.8 m/s^2 and R_e = 6400 km]

Solution:

(a) We know that for satellite motion.

$$v_0 = \sqrt{\frac{GM}{r}} = R\sqrt{\frac{g}{(R+h)}} \quad \left[\text{as } g = \frac{GM}{R^2} \text{ and } r = R+h\right]$$

In this problem,

$$v_0 = \frac{1}{2}v_e = \frac{1}{2}\sqrt{2gR}$$

So, $\frac{R^2 g}{R+h} = \frac{1}{2} gR$

I.e. h = R = 6400 km

(b) By conservation of ME

$$0 + \left(-\frac{GMm}{r}\right) = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right)$$

or $v^2 = 2GM\left[\frac{1}{R} - \frac{1}{2R}\right]$ [as r = R + h = R + R = 2R]

or $v = \sqrt{\frac{GM}{R}} = \sqrt{gR} = 8\,\text{km/s}$

Example 10.

A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distances from the sun are r_1 and r_2. Find the angular momentum of the planet relative to centre of sun.

Solution:

The angular momentum of planet is constant

i.e., $mv_1 r_1 = mv_2 r_2$ or $v_1 r_1 = v_2 r_2$

Total energy of planet is constant

i.e., $\frac{-GMm}{r_1} + \frac{1}{2}mv_1^2 = \frac{-GMm}{r_2} + \frac{1}{2}mv_2^2$

where M is mass of sun.

i.e., $GM\left\{\frac{1}{r_2} - \frac{1}{r_1}\right\} = \frac{v_2^2 - v_1^2}{2} = \frac{v_2^2}{2} - \frac{v_1^2}{2}$

or $GM\left\{\frac{r_1 - r_2}{r_1 r_2}\right\} = \frac{\left(\frac{v_1 r_1}{r_2}\right)^2}{2} - \frac{v_1^2}{2} = \frac{v_1^2}{2}\left\{\frac{r_1^2}{r_2^2} - 1\right\}$

i.e., $GM\left\{\frac{r_1 - r_2}{r_1 r_2}\right\} = \frac{v_1^2}{2}\frac{\left(r_1^2 - r_2^2\right)}{r_2^2}$

or $v_1^2 = \frac{2GM(r_1 - r_2)r_2^2}{(r_1^2 - r_2^2)r_1 r_2} = \frac{2GM \times r_2}{r_1(r_1 + r_2)}$

$v_1 = \sqrt{\frac{2GMr_2}{r_1(r_1 + r_2)}}$

Angular momentum of the planet $= mv_1 r_1 = \boldsymbol{m\sqrt{\frac{2GMr_1r_2}{r_1 + r_2}}}$