1. WORK

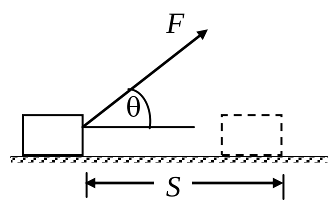
Almost all the terms we have used so far, velocity, acceleration, force etc. have same meaning in Physics as they have in our everyday life. We, however, encounter a term 'work' whose meaning in Physics is distinctly different from its everyday meaning. e.g. consider a person holding a 100 kg weight on his head moves through a distance of 100 m. In everyday language we will say he is doing work. But according to definition of work in Physics you, will the work done by the man is zero.

1.1 WORK DONE BY A CONSTANT FORCE

Consider an object undergoes a displacement S along a straight line while acted on a force F that makes an angle θ with S as shown in figure given below.

The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement.

i.e., $W = FS\cos\theta$ … (**1**)



Work done is a scalar quantity and its S.I. unit is N.m or joule (J) We can also write work done as a scalar product of force and displacement.

$$W = \vec{F}.\vec{S}. \quad \ldots (\mathbf{2})$$

From this definition, we conclude the following points:

(i) force does no work if point of application of force does not move ($S = 0$).

(ii) work done by a force is zero if displacement is perpendicular to the force ($\theta = 90°$).

(iii) if angle between force and displacement is acute ($\theta < 90°$), we say that work done by the force is positive.

(iv) if angle between force and displacement is obtuse ($\theta > 90°$), we say that work done by the force is negative.

Illustration 1

A person slowly lifts a block of mass m through a vertical height h, and then walks horizontally a distance d while holding the block. Determine work done by the person.

Solution:

The man slowly lifts the block, therefore he must be applying a force equal to the weight of the block, mg, the work done during vertical displacement is mgh, since the force is in the direction of displacement. The work done by the person during the horizontal displacement of the block is zero. Since the applied force during this process is perpendicular to displacement. Therefore total work done by the man is ***mgh***.

Illustration 2

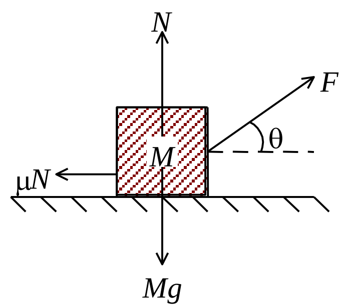
A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with horizontal. Coefficient of friction between block and surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block.

Solution:

The forces acting on the block are shown in Figure. As the block moves with uniform velocity the forces add up to zero.

$\therefore\ F\cos\theta = \mu N$ …(i)

$F\sin\theta + N = Mg$ …(ii)



Eliminating N from equations (i) and (ii),

$F\cos\theta = \mu(Mg - F\sin\theta)$

$$F = \frac{\mu Mg}{\cos\theta + \mu\sin\theta}$$

Work done by this force during a displacement d

$$W = F \cdot d\cos\theta = \frac{\mu Mgd\cos\theta}{\cos\theta + \mu\sin\theta}$$

Illustration 3

A particle moving in the xy plane undergoes a displacement $\vec{s} = (2.0\hat{i} + 3.0\hat{j})$ m while a constant force $F = (5.0\hat{i} + 2.0\hat{j})$ N acts on the particle.

(a) Calculate the magnitude of the displacement and that of the force.

(b) Calculate the work done by the force.

Solution:

(a) $s = \sqrt{x^2 + y^2} = \sqrt{(2.0)^2 + (3.0)^2} = \sqrt{\mathbf{3}}$ **m**

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2} = \mathbf{5.4\ N}$$

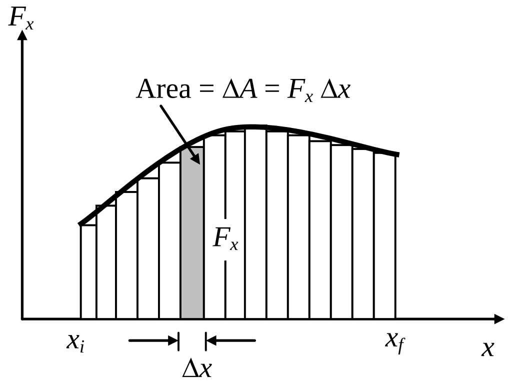
(b) Work done by force, $W = \vec{F} \cdot \vec{s}$

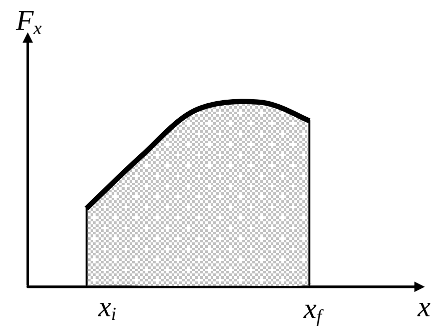
$= (5.0\hat{i} + 2.0\hat{j}) \cdot (2.0\hat{i}) + 3.0\hat{j})$ N.m

$= 10 + 0 + 0 + 6 = 16$ N.m $=$ **16J**

1.2 WORK DONE BY A VARYING FORCE

Consider a particle being displaced along the x-axis under the action of a varying force, as shown in the figure. The particle is displaced in the direction of increasing x from $x = x_i$ to $x = x_f$. In such a situation, we cannot use $W = (F \cos \theta)s$ to calculate the work done by the force because this relationship applies only when F is constant in magnitude and direction. However, if we imagine that the particle undergoes a very small displacement Δx, shown in the figure given below (left), then the x component of the force, F_x, is approximately constant over this interval, and we can express the work done by the force for this small displacement as $W_1 = F_x \, \Delta x$





This is just the area of the shaded rectangle in the figure given above (left). If we imagine that the F_x versus x curve is divided into a large number of such intervals, then the total work done for the displacement from x_i to x_f is approximately equal to the sum of a large number of such terms.

$$W \cong \sum_{x_i}^{x_f} F_x \Delta x$$

If the displacements are allowed to approach zero, then the number of terms in the sum increases without limit but the value of the sum approaches a definite value equal to the area under the curve bounded by F_x and the x axis.

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x dx$$

This definite integral is numerically equal to the area under the F_x versus x curve between x_i and x_f. Therefore, we can express the work done by F_x for the displacement of the object from x_i to x_f as

$$W = \int_{x_i}^{x_f} F_x dx \qquad \ldots \textbf{(3)}$$

This equation reduces to equation (1) when $F_x = F\cos \theta$ is constant.

If more than one force acts on a particle, the total work done is just the work done by the resultant force. For systems that do not act as particles, work must be found for each force separately. If we express the resultant force in the x-direction as $\sum F_x$, then the net work done as the particle moves from x_i to x_f is

$$W_{net} = \int_{x_i}^{x_f} (\sum F_x)\, dx \qquad \ldots \textbf{(4)}$$

Illustration 4

A force $F = (4.0\,x\,\hat{i} + 3.0y\hat{j})$ N acts on a particle which moves in the x-direction from the origin to $x = 5.0$ m. Find the work done on the object by the force.

Solution:

Here the work done is only due to x component of force because displacement is along x-axis.

i.e., $$W = \int_{x_1}^{x_2} F_x dx$$

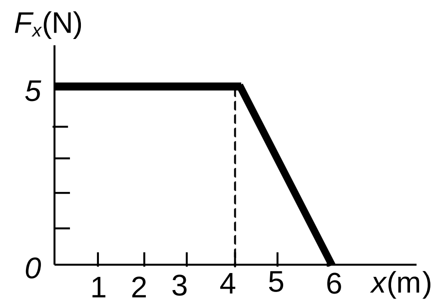
$$= \int_0^5 4x\,dx = \left[2x^2\right]_0^5 = \mathbf{50\ J}$$

Illustration 5

Force acting on a particle varies with x as shown in figure. Calculate the work done by the force as the particle moves from $x = 0$ to x = 6.0 m.

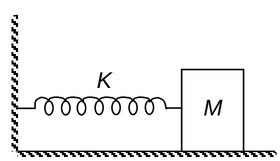
Solution:

The work done by the force is equal to the area under the curve from $x = 0$ to $x = 6.0$ m. This area is equal to the area of the rectangular section from $x = 0$ to $x = 4.0$ m plus the area of the triangular section from $x = 4.0$ m to $x = 6.0$ m. The area of the rectangle is (4.0) (5.0) N.m = 20J, and the area of the triangle is $\frac{1}{2}$(2.0 (5.0) N.m = 5.0 J. Therefore, the total work done is **25J.**



1.3 WORK DONE BY A SPRING

A common physical system for which the force varies with position is a spring-block system as shown in the figure. If the spring is stretched or compressed by the small distance from its unstreached configuration, the spring will exert a force on the block given by



$F = -kx$, where x is compression or elongation in spring. k is a constant called spring constant whose value depends inversely on un-stretched length and the nature of material of spring.

Important: The negative sign indicates that the direction of the spring force is opposite to x, the displacement of the free end.

Consider a spring block system as shown in the figure 3 and let us calculate work done by spring when the block is displaced by x_0.

At any moment if elongation in spring is x, then force on block by the spring is kx towards left. Therefore, work done by the spring when block further displaces by dx

$dW = -kxdx$ [Negative sign shows displacement is opposite to displacement]

$\therefore$ Total work done by the spring, $W = -\int_0^{x_0} kx\,dx = -\frac{1}{2}kx_0^2$

Similarly, work done by the spring when it is given a compression x_0 is $-\frac{1}{2}kx_0^2$. We can also say that work done by an agent in giving an elongation or compression of x_0 is $\frac{1}{2}kx_0^2$.

2. POWER

From a practical viewpoint, it is interesting to know not only the work done on an object but also the rate at which the work is being done. *The time rate of doing work is called power*.

If an external force is applied to an object (which we assume as a particle), and if the work done by this force is ΔW in the time interval Δt, then the average power during this interval is defined as

$$\overline{P} = \frac{\Delta W}{\Delta t} \qquad \ldots (\mathbf{5})$$

The work done on the object contributes to increasing the energy of the object. A more general definition of power is the time rate of energy transfer. The instantaneous power is the limiting value of the average power as Δt approaches zero.

i.e.,
$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad \ldots (\mathbf{6})$$

where we have represented the infinitesimal value of the work done by dW (even though it is not a change and therefore not a differential). We find from equation (2) that $dW = \vec{F}.d\vec{s}$. Therefore, the instantaneous power can be written as

$$P = \frac{dW}{dt} = \vec{F}.\frac{d\vec{s}}{dt} = \vec{F}.\vec{v} \qquad \ldots (\mathbf{7})$$

where we have used the fact that $\vec{v} = \frac{d\vec{s}}{dt}$.

The SI unit of power is 'joule per second (J/s), also called watt (W)' (after James Watt);

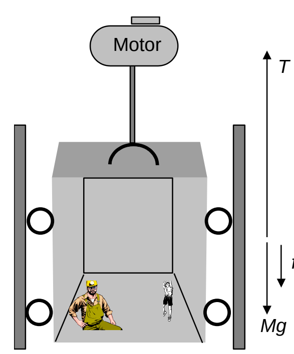
$$1\text{W} = 1\text{J/s} = 1\ \text{kg.m}^2/\text{s}^3.$$

Illustration 6

An elevator has a mass of 1000 kg and carries a maximum load of 800 kg. A constant frictional force of 4000 N retards its upward motion, as shown in the figure.

(a) What must be the minimum power delivered by the motor to lift the elevator at a constant speed of 3.00 m/s?

(b) What power must the motor deliver at any instant if it is designed to provide an upward acceleration of 1.00 m/s^2?



Solution:

(a) The motor must supply the force T that pulls the elevator upward. From Newton's second law and from the fact that $a = 0$ since v is constant, we get

$$T - f - Mg = 0,$$

Where M is the total mass (elevator plus load), equal to 1800 kg.

$$\Rightarrow \quad T = f + Mg$$

$$= 4.00 \times 10^3 \text{N} + (1.80 \times 10^3 \text{ kg})(9.80 \text{ m/s}^2) = \mathbf{2.16 \times 10^4 \text{ N}}$$

Using equation (7) and the fact that T is in the same direction as v, we have gives

$$P = Tv$$

$$= (2.16 \times 10^4 \text{ N})(3.00 \text{ m/s}) = 6.49 \times 10^4 \text{ W}$$

$$= \mathbf{64.9 \text{ kW}} = 87.0 \text{ hp}$$

(b) Application of Newton's second law to the elevator gives

$$T - f - Mg = Ma$$

$$\Rightarrow \quad T = M(a + g) + f$$

$$= (1.80 \times 10^3 \text{ kg})(1.00 + 9.80) \text{ m/s}^2 + 4.00 \times 10^3 \text{ N}$$

$$= 2.34 \times 10^4 \text{ N}$$

Therefore, using equation (7) we get the required power

$$P = Tv = \mathbf{(2.34 \times 10^4 v) \text{ W}}$$

where v is the instantaneous speed of the elevator in metres per second. Hence, the power required increases with increasing speed.

3. ENERGY

A body is said to possess energy if it has the capacity to do work. When a body possessing energy does some work, part of its energy is used up. Conversely if some work is done upon an object, the object will be given some energy. Energy and work are mutually convertible.

There are various forms of energy. Heat, electricity, light, sound and chemical energy are all familiar forms. In studying mechanics, we are however concerned chiefly with mechanical energy. This type of energy is a property of movement or position.

3.1 KINETIC ENERGY

Kinetic energy (K.E.), is the capacity of a body to do work by virtue of its motion.

If a body of mass m has a velocity v its kinetic energy is equivalent to the work, which an external force would have to do to bring the body from rest up to its velocity v.

The numerical value of the kinetic energy can be calculated from the formula

$$\text{K.E.} = \frac{1}{2} mv^2 \qquad \ldots \textbf{(8)}$$

This formula can be derived as follows.

Consider a constant force F which, acting on a mass m initially at rest, gives the mass a velocity v. If, in reaching this velocity, the particle has been moving with an acceleration a and has been given a displacement s, then

F = ma (Newton's Law)

v^2 = 2as (Motion of a particle moving with uniform acceleration)

Fs = Work done by the constant force

Combining these relationships, we have

$$\text{Work done} = ma\left(\frac{v^2}{2a}\right) = \frac{1}{2}mv^2$$

But the K.E. of the body is equivalent to the work done in giving the body is velocity.

Hence, $\quad \text{K.E.} = \frac{1}{2}mv^2$

Since both m and v^2 are always positive, K.E. is always positive and does not depend upon the direction of motion of the body.

3.2 POTENTIAL ENERGY

Potential energy is energy due to position. If a body is in a position such that if it were released it would begin to move, it has potential energy.

There are two common forms of potential energy, gravitational and elastic.

3.2 (a) Gravitational Potential Energy, it is a property of height.

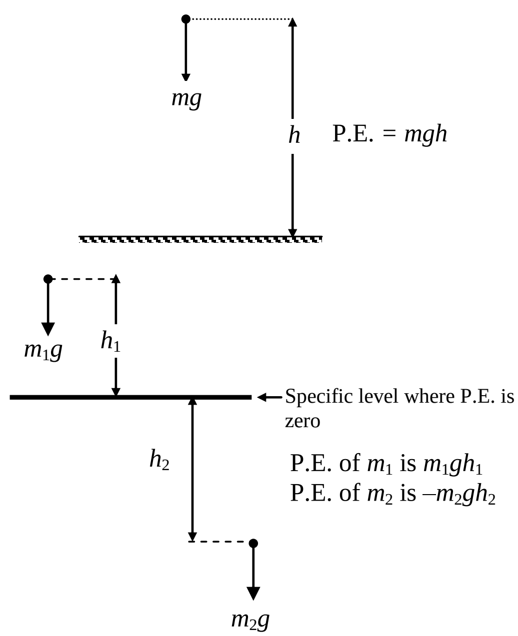
When an object is allowed to fall from one level to a lower level it gains speed due to gravitational pull, i.e., it gains kinetic energy. Therefore, in possessing height, a body has the ability to convert its height into kinetic energy, i.e., it possesses potential energy.

The magnitude of its gravitational potential energy is equivalent to the amount of work done by the weight of the body in causing the descent.

If a mass *m* is at a height *h* above a lower level the P.E. possessed by the mass is (*mg*) (*h*).

Since *h* is the height of an object above a specified level, an object below the specified level has negative potential energy.

Therefore $\text{PE} = \pm mgh$ …. (9)



The chosen level from which height is measured has no absolute position. It is important therefore to indicate clearly the zero P.E. level in any problem in which P.E. is to be calculated.

3.2 (b) Elastic Potential Energy, It is a property of stretched or compressed springs.

The end of a stretched elastic spring will begin to move if it is released. The spring therefore possesses potential energy due to its elasticity. (i.e., due to change in its configuration)

The amount of elastic potential energy stored in a spring of natural length *a* and spring constant *k* when it is extended by a length *x* is equivalent to the amount of work necessary to produce the extension.

Earlier in the lesson we saw that the work done was $\frac{1}{2}kx^2$ so

$$\text{E.P.E.} = \frac{1}{2}kx^2 \qquad \text{.... (10)}$$

E.P.E. is never negative whether due to extension or to compression.

4. WORK-ENERGY THOREM

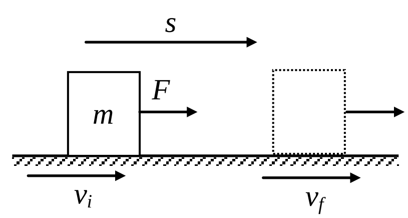
Solution using Newton's second law can be difficult if the forces in the problem are complex. An alternative approach that enables us to understand and solve such motion problems is to relate the speed of the particle to its displacement under the influence of some net force. As we shall see in this section, if the work done by the net force on a particle can be calculated for a given displacement, the change in the particle's speed will be easy to evaluate.

As shown in the figure, a particle of mass *m* moving to the right under the action of a constant net force *F*. Because the force is constant, we know from Newton's second law that the particle will move with a constant acceleration *a*. If the particle is displaced a distance *s*, the net work done by the force *F* is

$$W_{\text{net}} = Fs = (ma)s$$

We found that the following relationships are valid when a particle moves at constant acceleration

$$s = \frac{1}{2}(v_i + v_f)t\ ; \quad a = \frac{v_f - v_i}{t}$$



where v_i is the speed at $t = 0$ and v_f is the speed at time t. Substituting these expressions

$$W_{net} = m\left(\frac{v_f - v_i}{t}\right)\frac{1}{2}(v_i + v_f)t$$

$$W_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$W_{\text{net}} = K_f - K_i = \Delta K \qquad \text{... } \mathbf{(11)}$$

That is, the work done by the constant net force F_{net} in displacing a particle equals the change in kinetic energy of the particle.

Equation (11) is an important result known as the **work-energy theorem.** For convenience, it was derived under the assumption that the net force acting on the particle was constant.

Now, we shall show that the work-energy theorem is valid even when the force is varying. If the resultant force acting on a body in the x direction is $\sum F_x$, then Newton's second law states that $\sum F_x = ma$. Thus, we express the net work done as

$$W_{net} = \int_{x_i}^{x_f} \left(\sum F_x\right) dx = \int_{x_i}^{x_f} ma\, dx$$

Because the resultant force varies with x, the acceleration and speed also depend on x. We can now use the following chain rule to evaluate W_{net}.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

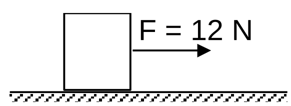
$$W_{\text{net}} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Illustration 7

A 6.0 kg block initially at rest is pulled to the right along a horizontal frictionless surface by a constant force of 12 N, as shown in the figure. Find

(a) the speed of the block after it has moved 3.0 m.

(b) the acceleration of the block and its final speed using the kinematic equation $\boldsymbol{v_f^2 = v_i^2 + 2as}$.



Solution:

The normal force balances the weight of the block, and neither of these forces does work since the displacement is horizontal. Since there is no friction, the resultant external force is the 12 N force. The work done by this force is

$$W = Fs = (12\text{ N})\,(3.0\text{ m}) = 36\text{ N.m} = 36\text{ J}$$

Using the work-energy theorem and noting that the initial kinetic energy is zero, we get

$$W = K_f - K_i = \frac{1}{2}\, mv_f^2 - 0$$

$$v_f^2 = \frac{2W}{m} = \frac{2(36\text{J})}{6.0\text{kg}} = 12\ m^2/s^2$$

$\boldsymbol{v_f = 3.5}$ **m/s**

(b) $a = 2.0\text{ m/s}^2$; $\boldsymbol{v_f = 3.5}$ **m/s.**

Note that result calculated in two ways are same.

4.1 CONSERVATIVE AND NON-CONSERVATIVE FORCE

Suppose a body is taken to a height h from the ground level, work by the done gravity on the body is equal to $-mgh$. When it is allowed to come back to the ground, the work done by it is equal to $+mgh$. So the net work performed = $-mgh + mgh = 0$. Thus, in a gravitational field if a particle describes various displacements and finally comes back to its initial position, the total work performed by the gravity taking into account all the displacements is zero. This is the case with electrostatic fields also. These forces acting in such fields are called

conservative forces. On the other hand, consider a body moving on a rough surface from A to B and then back from B to A. Work done against frictional forces only add up because in both the displacements work is done against frictional forces only. Hence frictional force cannot be considered as a conservative force. It is a non-conservative force.

4.2. CONSERVATION OF MECHANICAL ENERGY

Kinetic and Potential Energy both are forms of Mechanical Energy. The total mechanical energy of a body or system of bodies will be changed in value if

(a) an external force other than weight causes work to be done (work done by weight is potential energy and is therefore already included in the total mechanical energy),

(b) some mechanical energy is converted into another form of energy (e.g. sound, heat, light etc). Such a conversion of energy usually takes place when a sudden change in the motion of the system occurs. For instance, when two moving objects collide some mechanical energy is converted into sound energy, which is heard as a bang at the impact. Another common example is the conversion of mechanical energy into heat energy when two rough objects rub against each other.

If neither (a) nor (b) occurs then the total mechanical energy of a system remains constant. This is the Principle of **Conservation of Mechanical Energy** and can be expressed, as,

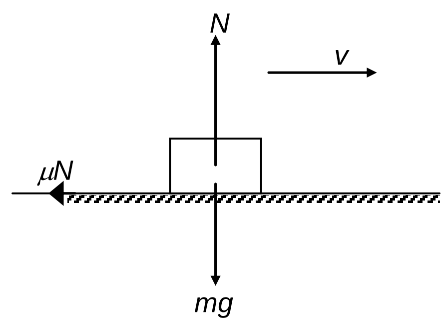
"The total mechanical energy of a system remains constant provided that no external work is done and no mechanical energy is converted into another form of energy."

When this principle is used in solving problems, a careful appraisal must be made of any external forces, which are acting. Some external forces do work and hence cause a change in the total energy of the system. Other, however, can be present without doing any work and these will not cause any change in energy.

For example, consider a mass m moving along a rough horizontal surface.

The normal reaction N is perpendicular to the direction of motion and does not do any work.

The frictional force μN, acting in the line of motion, does cause the velocity of the mass to change. The frictional force therefore does work and the total mechanical energy will change.

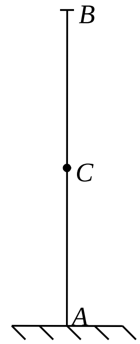


The principle of conservation of mechanical energy principle is a very useful concept to use in problem solving. It is applicable to any problem where the necessary conditions are satisfied and which is concerned with position and velocity.

Illustration 8

A body of mass 2 kg is at rest at a height of 10 m above the ground. Calculate its potential energy and kinetic energy after it has fallen through half the height. Also find the velocity at this instant.

Solution:



Total energy at B = K.E. + P.E.

$= 0 + mgh$

$= 2 \times 9.8 \times 10$

$=$ **196 J**

As it descends half the height it loses

Potential energy $= mg\,\frac{h}{2}$

$= \frac{1}{2}\,mgh$

$= 98$ J

$\therefore$ its P.E. at C = 196 – 98 **= 98J** = 98 J

The loss of potential energy = gain in kinetic energy

= 196 – 98 = **98 J**

But K.E. $= \frac{1}{2}\,mv^2$

$\therefore \quad \frac{1}{2} \times 2 \times v^2 = 98$

$\Rightarrow v^2 = 98$ or $v = \mathbf{7\sqrt{2}}$ **m/s**

Illustration 9

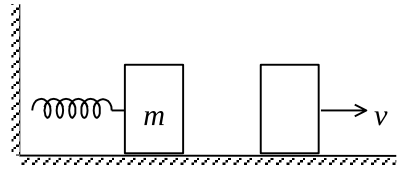
A block of mass m is pushed against a spring of spring constant k fixed at one end to a wall. The block can slide on a frictionless table as shown in Figure. The natural length of the spring is L_0 and it is compressed to half its natural length when the block is released. Find the velocity of the block as a function of its distance x from the wall.

Solution:

When the block is released the spring pushes it towards right. The velocity of the block increases till the spring acquires its natural length. Thereafter the block loses contact with the spring and moves with constant velocity. Initially the compression in the spring $= \frac{L_o}{2}$.

When the distance of block from the wall becomes x where $x < L_0$ the compression is $(L_0 - x)$. Using the principle of conservation of energy

$$\frac{1}{2}k\left(\frac{L_0}{2}\right)^2 = \frac{1}{2}k(L_o - x)^2 + \frac{1}{2}mv^2$$



Solving this, $v = \sqrt{k/m}\left[\frac{L_o^2}{4} - (L_o - x)^2\right]^{\frac{1}{2}}$

When the spring acquires its natural length $x = L_0$, we have

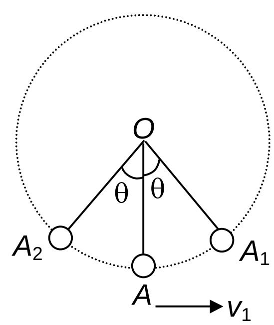
then $v = \sqrt{\frac{k}{m}\frac{L_o}{2}}$.

Thereafter the block continues with this velocity.

5. MOTION IN A VERTICLE CIRCLE

5.1 BODY SUSPENDED WITIH THE HELP OF A STRING

Imagine an arrangement like a simple pendulum where a mass m is tied to a string of length r, the other end of the string being attached to a fixed point O.

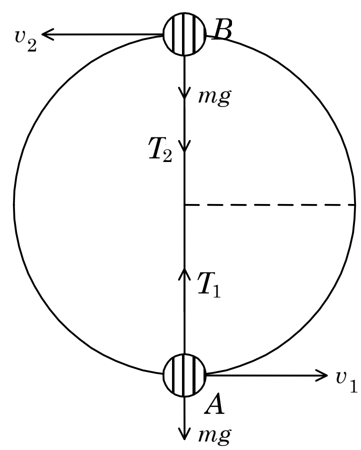


Let the body initially lie at the equilibrium position A. Let it be given a velocity v_1 horizontally. If the velocity is small then the body would make oscillations in the vertical plane like a simple pendulum.

Let us now find the velocity of projection v_1 at the lowest point A required to make the body move in a vertical circle.

The following points can be noted while considering the motion:

(i) The motion along the vertical circle is non-uniform since the velocity of body changes along the curve. Hence the centripetal acceleration v^2/r must also change, where v is the instantaneous velocity of the body at any point on the circle.



(ii) The resultant force acting on the body (in the radial direction) provides the necessary centripetal force.

(iii) The velocity decreases as the body moves up form A to B, the topmost point.

When the body is at A, the resultant force acting on it is $T_1 - mg$, where T_1 is the tension in the string.

The centripetal force $T_1 - mg = \frac{mv_1^2}{r}$

$$\therefore \qquad T_1 = mg + \frac{mv_1^2}{r} \qquad \ldots \text{(i)}$$

The tension is always positive (even if v_1 is zero). Therefore, the string will be taut when it is at A. The condition for the body to complete the vertical circle is that the string should be taut all the time i.e., the tension is greater than zero.

The region where the string is most likely to become slack is above the horizontal radius OC. Also the tension would become the least when it is at the topmost point B. Let it be T_2 at B and the velocity of the body be v_2. The resultant force on the body at B is

$$T_2 + mg = \frac{mv_2^2}{r}$$

$$T_2 = \frac{mv_2^2}{r} - mg \qquad \text{… (ii)}$$

If the string is to be taut at B, $T_2 \geq 0$.

$$\frac{mv_2^2}{r} - mg \geq 0$$

$$v_2^2 \geq rg$$

$$v_2 \geq \sqrt{rg} \qquad \text{… (iii)}$$

At A, the K.E. of the body = $\frac{1}{2}mv_1^2$

P.E. of the body = 0

Total energy at A = $\frac{1}{2}mv_1^2$

At B, the K.E. of the body = $\frac{1}{2}mv_2^2$

P.E. of the body = $mg(2r)$

Total energy at B = $2mg$r + $\frac{1}{2}mv_2^2$

Using the principle of conservation of energy, we have

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 + 2mgr$$

$$v_1^2 = v_2^2 + 4gr \qquad \text{… (iv)}$$

Combining equations (iii) and (iv)

$$v_1^2 \geq rg + 4gr$$

$$v_1 \geq \sqrt{5gr}$$

Hence, if the body has minimum velocity of $\sqrt{5gr}$ at the lowest point of vertical circle, it will complete the circle.

The particle will describe complete circle if both v_2 and T_2 do not vanish till the particle reaches the highest point.

The following points are important:

(i) If the velocity of projection at the lowest point A is less than $\sqrt{2gr}$, the particle will come to instantaneous rest at a point on the circle which lies lower than the horizontal diameter. It will then move down to reach *A* and move on to an equal height on the other side of *A*. Thus the particle executes oscillations. In this case v vanishes before T does.

We may find an expression for the tension in the string when it makes an angle θ with the vertical. At *C*, the weight of the body acts vertically downwards, and the tension in the string is towards the centre *O*.

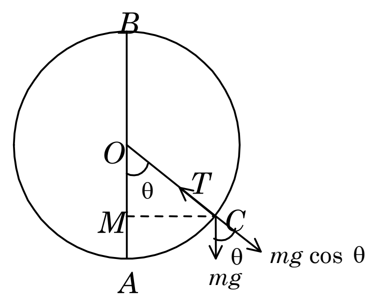
The weight mg is resolved radially and tangentially.

The radial component is mg cos θ and the tangential component is $mg \sin\theta$.

The centripetal force is $T - mg\cos\theta$

$$T - mg\cos\theta = \frac{mv^2}{r}\text{, where } v \text{ is the velocity at } C,$$

$$\text{i.e., } T = m\left(\frac{v^2}{r} + g\cos\theta\right) \qquad \ldots \text{(v)}$$



The velocity v can be expressed in terms of velocity v_1 at *A*.

The total energy at $A = \frac{1}{2}mv_1^2$

The kinetic energy at $C = \frac{1}{2}mv^2$

The potential energy at $C = mg(AM)$

$$= mg\,(AO - MO)$$

$$= mg\,(r - r\cos\theta)$$

$$= mgr\,(1 - \cos\theta)$$

The total energy at $C = \frac{1}{2}mv^2 + mgr\,(1 - \cos\theta)$

∴ From conservation of energy $\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1 - \cos\theta)$

$$v_1^2 = v^2 + 2gr\,(1 - \cos\theta)$$

$$\text{or} \quad v^2 = v_1^2 - 2gr\,(1 - \cos\theta)$$

Substituting in equation (v),

$$T = m\left[g\cos\theta + \frac{v_1^2}{r} - 2g(1 - \cos\theta)\right] = \frac{mv_1^2}{r} + 3mg\left(\cos\theta - \frac{2}{3}\right) \quad \ldots \text{(vi)}$$

This expression gives the value of the tension in the string in terms of the velocity at the lowest point and the angle θ.

Equation (v) shows that tension in the string decreases as θ increases, since the term 'g cos θ' decreases as θ increases.

When θ is 90°, cos θ = 0, and

$$T_H = \frac{mv^2}{r} \qquad \dots \text{(vii)}$$

This is obvious because, the weight is vertically downwards whereas the tension is horizontal. Hence the tension alone is the centripetal force.

(ii) If the velocity of projection is greater than $\sqrt{2gr}$ but less than $\sqrt{5gr}$, the particle rises above the horizontal diameter and the tension vanishes before reaching the highest point.

(iii) We have seen that the tension in the string at the highest point is lower than the tension at the lowest point. When the string is above the horizontal, tension may be

deduced from equation (vi), if we make θ an obtuse angle. However, in order to have a physical picture of the situation let us study this separately.

At the point D, the string OD makes an angle ϕ with the vertical. The radial component of the weight is mg cos ϕ **towards** the centre O.

$$T + mg\cos\phi = \frac{mv^2}{r}$$

$$T = m\left(\frac{v^2}{r} - g\cos\phi\right) \qquad \dots \text{(viii)}$$

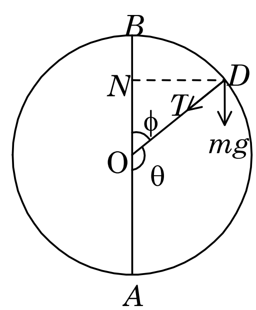
The kinetic energy at $D = \frac{1}{2}mv^2$

Potential energy at $D = mg(AN)$

$$= mg\,(AO + ON)$$

$$= mg(r + r\cos\phi)$$

$$= mgr(1 + \cos\phi)$$



From conservation of energy $\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1+\cos\theta)$

$$v^2 = v_1^2 - 2gr(1+\cos\theta)$$

Substituting in equation (viii),

$$T = m\left[\frac{v_1^2}{r} - 2g(1+\cos\phi) - g\cos\phi\right]$$

$$T = m\left[\frac{v_1^2}{r} - 3g\left(\cos\phi + \frac{2}{3}\right)\right] \qquad \ldots \text{(ix)}$$

This equation shows that the tension becomes zero, if

$$\frac{v_1^2}{r} = 3g\left(\cos\phi + \frac{2}{3}\right) \qquad \ldots \text{(x)}$$

If the tension is not to become zero,

$$v_1^2 > 3rg\left(\cos\phi + \frac{2}{3}\right)$$

Equation (x) gives the values of ϕ at which the string becomes slack.

$$\cos\phi + \frac{2}{3} = \frac{v_1^2}{3rg}$$

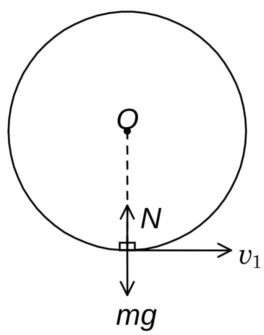
$$\cos\phi = \frac{v_1^2}{3rg} - \frac{2}{3} \qquad \ldots \text{(xi)}$$

5.2 A BODY MOVING INSIDE A HOLLOW TUBE OR SPHERE

The same discussion holds good for this case, but instead of tension in the string we have the normal reaction of the surface. If N is the normal reaction at the lowest point, then

$$N - mg = \frac{mv_1^2}{r};$$

$$N = m\left(\frac{v_1^2}{r} + g\right)$$



At the highest point of the circle,

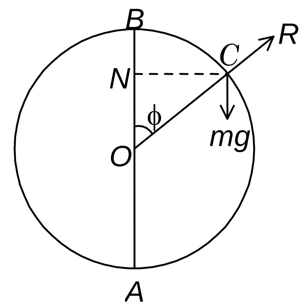
$$N + \text{mg} = \frac{mv_2^2}{r}$$

$$N = m\left(\frac{v_2^2}{r} - g\right)$$

The condition $v_1 \geq \sqrt{5rg}$ for the body to complete the circle holds for this also.

All other equations (can be) similarly obtained by replacing tension T by reaction R.

5.3 BODY MOVING ON A SPHERICAL SURFACE



The small body of mass m is placed on the top of a smooth sphere of radius r.

If the body slides down the surface, at what point does if fly off the surface?

Consider the point C where the mass is, at a certain instant. The forces are the normal reaction R and the weight mg. The radial component of the weight is $mg\cos\phi$ acting towards the centre. The centripetal force is

$$mg\cos\phi - R = \frac{mv^2}{r},$$

where v is the velocity of the body at O.

$$R = m\left(g\cos\phi - \frac{v^2}{r}\right) \qquad \dots \text{(i)}$$

The body files off the surface at the point where N becomes zero.

i.e., $$g\cos\phi = \frac{v^2}{r};\ \cos\phi = \frac{v^2}{rg} \qquad \dots \text{(ii)}$$

To find v, we use conservation of energy

i.e., $$\frac{1}{2}mv^2 = mg(BN)$$

$$= mg\,(OB - ON) = mgr\,(1 - \cos\phi)$$

$$v^2 = 2rg\,(1 - \cos\phi)$$

$$2(1 - \cos\phi) = \frac{v^2}{rg} \qquad \dots \text{(iii)}$$

From equations (ii) and (iii) we get

$$\cos\phi = 2 - 2\cos\phi;\ 3\cos\phi = 2$$

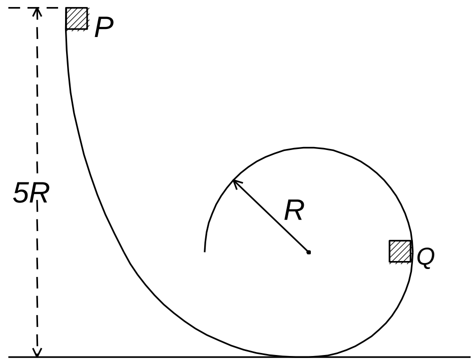
$$\cos\phi = \frac{2}{3};\ \phi = \cos^{-1}\left(\frac{2}{3}\right) \qquad \dots \text{(iv)}$$

This gives the angle at which the body goes off the surface. The height from the ground of that point = $AN = r(1 + \cos\phi)$

$$= r\left(1 + \frac{2}{3}\right) = \frac{5}{3}\,r$$

Illustration 10

A small block of mass m slides along the frictionless loop-to-loop track shown in the Figure. (a) If it starts from rest at P what is the resultant force acting on it at Q? (b) At what height above the bottom of loop should the block be released so that the force it exerts against the track at the top of the loop equals its weight?



Solution:

(a) Point Q is at a height R above the ground. Thus, the difference in height between points P and Q is $4R$. Hence, the difference in gravitational potential energy of the block between these points = $4mgR$.

Since the block starts from rest at P its kinetic energy at Q is equal to its change in potential energy. By the conservation of energy.

$$\therefore \qquad \frac{1}{2}mv^2 = 4mgR$$

$$v^2 = 8gR$$

At Q, the only forces acting on the block are its weight mg acting downward and the force N of the track on block acting in radial direction. Since the block is moving in a circular path, the normal reaction provides the centripetal force for circular motion.

$$N = \frac{mv^2}{R} = \frac{m \times 8gR}{R} = \mathbf{8\ mg}$$

The loop must exert a force on the block equal to eight times the block's weight.

(b) For the block to exert a force equal to its weight against the track at the top of the loop,

$$\frac{mv'^2}{R} = 2mg$$

or $\qquad v'^2 = 2gR$

$$\therefore \qquad mgh = \frac{1}{2}mv'^2$$

$$h = \frac{v'^2}{2g} = \frac{2gR}{2g} = R$$

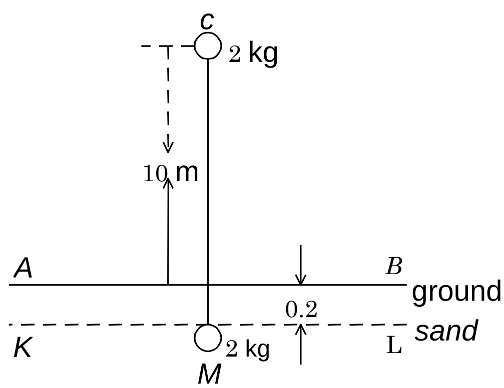
The block must be released at a **height 3R above the bottom of the loop.**

SOLVED EXAMPLES

Example 1.

A body of mass 2 kg is allowed to fall from the top of a 10 m tall building. It lands on a sandy patch of ground and buries itself 20 cm deep in the sand. Find the average resistance offered to it by sand and find the approximate time of penetration.

Solution:



AB is the ground level. The body reaches M, 0.2 m below ground level through sand. Taking M as standard (reference) position for calculating potential energy the potential energy at C = mgh = 2 × 9.8 × 10.2 = 199.92 J. On arrival at AB the total energy content of the body = 199.92 J, this is used to do work against the resistance offered by the sand.

If R is the average resistance, then

$$R \times 0.2 = 199.92$$

$$R = \frac{199.92}{0.2} = 999.6 \text{ N}$$

The velocity with which the body strikes the ground, $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 10} = 14$ m/s

The distance moved into sand = average velocity × time $= \frac{14+0}{2} \times t = 7t$

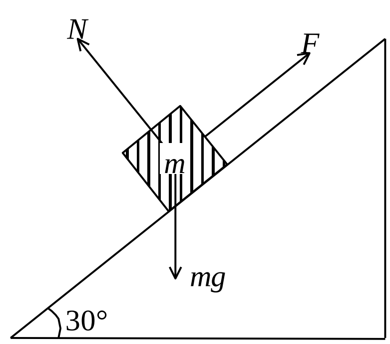
But $\qquad 7t = 0.2$

$\therefore \qquad t = \frac{0.2}{7} = 0.02857$ s

Example 2.

A block of mass 2 kg is pulled up on a smooth incline of angle 30° with horizontal. If the block moves with an acceleration of 1 m/s^2, find the power delivered by the pulling force at a time 4 seconds after motion starts. What is the average power delivered during these four seconds after the motion starts?

Solution:



The forces acting on the block are shown in Figure. Resolving forces parallel to incline,

$$F - mg \sin\theta = ma$$

$$\Rightarrow \quad F = mg \sin\theta + ma$$

$$= 2 \times 9.8 \times \sin 30° + 2 \times 1$$

$$= 11.8 \text{ N}$$

The velocity after 4 seconds = u + at

$= 0 + 1 \times 4$

$= 4$ m/s

Power delivered by force at t = 4 seconds

$=$ Force $\times$ velocity

$= 11.8 \times 4$

$= 47.2$ W

The displacement during 4 seconds is given by the formula

$V^2 = u^2 + 2aS$

$v^2 = 0 + 2 \times 1 \times S$

$\therefore \quad S = 8$ m

Work done in four seconds = Force $\times$ distance = $11.8 \times 8 = 94.4$ J

$\therefore$ average power delivered = $\dfrac{\text{workdone}}{\text{time}}$

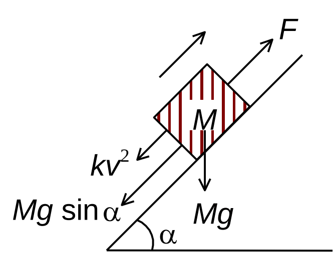
$= \dfrac{94.4}{4} = 23.6$ W

Example 3.

A motorcar of mass 1000 kg attains a speed of 64 km/hr when running down an incline of 1 in 20 with the engine shut off. It can attain a speed of 48 km/hr up the same incline when the engine is switched on. Assuming that the resistance varies as the square of the velocity, find the power developed by engine.

Solution:

When the motor car is moving down the plane there is a force Mg sin α down the plane. This is opposed by the resistance, which is proportional to square of the velocity. That is



Mg sin $\alpha \propto v^2$

$$Mg \times \frac{1}{20} = kv^2, \text{ where k is a constant.}$$

$$\therefore \quad \frac{1000 \times g}{20} = k\left(64 \times \frac{5}{18}\right)^2$$

$$k = \frac{1000 \times g}{20} \times \left(\frac{18}{64 \times 5}\right)^2 \qquad \ldots (1)$$

When the engine is on, let the tractive force (force exerted by engine) be F. This is used to overcome the force due to incline and the resistance offered.

$$\therefore \quad F = k\,(48 \times 5/18)^2 + \frac{Mg}{20}$$

$$= k\,(48 \times 5/18)^2 + \frac{1000 \times g}{20}$$

Substituting the value of k from equation (1)

$$F = \frac{1000 \times g}{20} \times \left(\frac{18}{64 \times 5}\right)^2 \times \left(48 \times \frac{5}{18}\right)^2 + \frac{1000 \times g}{20}$$

$$= \frac{1000g}{20}\left[\frac{18 \times 18}{64 \times 5 \times 64 \times 5} \times \frac{48 \times 5 \times 48 \times 5}{18 \times 18} + 1\right]$$

$$= \frac{1000 \times 9.8}{20}\left[\frac{9}{16} + 1\right]$$

$$= \frac{50 \times 9.8 \times 25}{16}$$

$$= 765.6 \text{ N}$$

Power developed = Force × velocity

$$= 765.6 \times 48 \times 5/18 = 10208 \text{ W} = 10.2 \text{ kW}$$

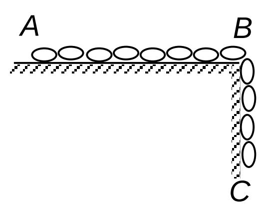
Example 4.

A uniform flexible chain of length 3 m is held on a smooth horizontal table so that 0.6 m overhangs one edge, the chain being perpendicular to the edge. If the chain be released from rest find the velocity with which it leaves the table.

Solution:

ABC is the flexible chain and BC is the free part which overhang on one side of the table. BC = 0.6 m. The C.G of the part BC lies at its middle i.e., 0.3 m below the tabletop.

Let the surface of the table be the level of zero potential energy Let M be the mass per unit length of chain. Then the mass of BC = 0.6 M



Potential energy of the chain at the start = potential energy of the overhang

$$= mgh$$

$$= 0.6Mg\,(-0.3)$$

$$= -0.18 \text{ Mg}$$

The potential energy is negative because the C.G of overhanging part lies 0.3 m below the zero potential energy level.

Kinetic energy of the chain at start = 0

$\therefore$ total energy at the start = kinetic energy + potential energy

$$= 0 + (-0.18 \text{ Mg})$$

$$= -0.18 \text{ Mg joules}$$

Let v be the velocity of chain when its full length gets detached from the table.

Its kinetic energy will be = $\frac{1}{2}$ mv^2 = $\frac{1}{2}$ (3M)v^2 = 1.5 Mv^2 joules

potential energy of the chain at that instant = Mass of chain × g × depth of C.G below table level

$$= 3M \times g \times (-1.5)$$

$$= -4.5 \text{ Mg joules}$$

$\therefore$ total energy = 1.5 Mv^2 + (– 4.5 Mg) joules

By the principle of conservation of energy

$$1.5\ Mv^2 - 4.5\ Mg = -0.18\ Mg$$

$$1.5\ v^2 = 4.5g - 0.18\ g$$

$$= 4.32\ g$$

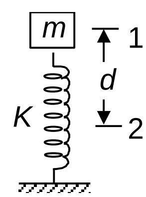
$$= 4.32 \times 9.8$$

$$v^2 = \frac{4.32 \times 9.8}{1.5}$$

v = 5.313 m/sec.

Example 5.

A block of mass m released from rest onto an ideal non-deformed spring of spring constant k from a negligible height. Neglecting the air resistance, find the compression d of the spring.



Solution:

According to work energy theorem the increment of the kinetic energy of the block is equal to the algebraic sum of the works performed by all forces acting on it.

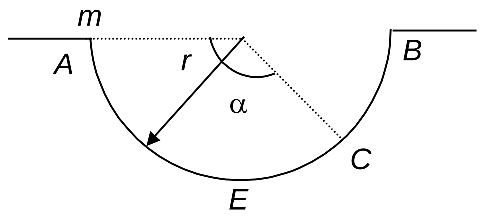
i.e., $W = \Delta T$

or, $W_{mg} + W_{sp} = 0$

or, $mg\,d + \int_0^d (-kx)\,dx = 0$ i.e., mgd $-\ \frac{1}{2}$ kd^2 = 0

Hence d = $\frac{\boldsymbol{2mg}}{\boldsymbol{k}}$

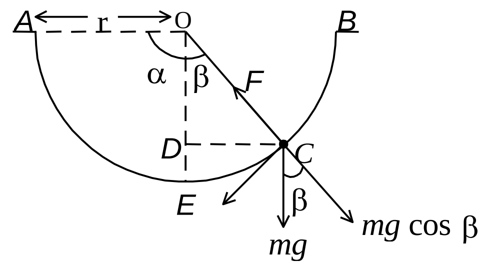
Example 6.



A small particle of mass m initially at A (see Figure) slides down a frictionless surface AEB. When the particle is at the point C, show that the angular velocity and the force exerted by the surface are

$\omega = \sqrt{2gsin\alpha in}$ and F = 3 mg sin α

Solution:



The force diagram of the mass when it is at C is shown in Figure.

At the point C the total energy = P.E. + K.E.

$$= mg(DE) + \frac{1}{2}mv^2$$

Where v is the linear velocity at C.

At the point A the total energy = mgr

By the principle of conservation of energy,

$$mgr = mg\,(DE) + \frac{1}{2}mv^2$$

But $\quad v = r\omega$, where ω is the angular velocity.

$$\therefore \quad mgr = mg(OE - OD) + \frac{1}{2}mr^2\omega^2$$

$$\Rightarrow \quad gr = g(OE - OD) + \frac{1}{2}r^2\omega^2$$

$\Rightarrow$ Now, $\quad (OE - OD) = r - r\cos\beta$

$$= r\,(1 - \cos\beta)$$

$$\therefore \quad gr = gr\,(1 - \cos\beta) + \frac{1}{2}r^2\omega^2$$

$$\omega^2 r^2 = 2gr\cos\beta$$

$$= 2gr\sin\left(\frac{\pi}{2}+\beta\right) = 2gr\sin\alpha$$

$$\omega = \sqrt{\frac{2g\sin\alpha}{r}}$$

Now, $\quad m\omega^2 r = 2mg\sin\alpha$

$\therefore$ total force at C, $F = m\omega^2 r + mg\cos\beta$

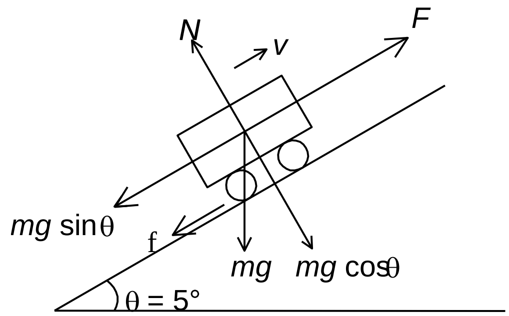
$$= 2mg\sin\alpha + mg\sin\alpha$$

$$= 3\,mg\sin\alpha$$

Example 7.

A 100000 kg engine is moving up a slope of gradient 5° at a speed of 100 m/hr. The coefficient of friction between the engine and the rails is 0.1. If the engine has an efficiency of 4% for converting heat into work, find the amount of coal, the engine has to burn up in one hour. (Burning of 1 kg of coal yields 50000 J.)

Solution:



The forces are shown in Figure.

Net force to more the engine up the slope.

$F = \mu N + mg \sin\theta$

$= mg(\mu \cos\theta + \sin\theta)$

If the engine has to apply an upward force equal to F,

power of engine, P = Fv

where v is the velocity equal to 100 m/hr.

Work done by engine, W = Pt = Fvt

Efficiency of engine, $\eta = \frac{\text{Output}}{\text{Input}} = \frac{Fvt}{\text{Energy used}}$

Energy used by engine $= \frac{Fvt}{\eta} = \frac{mg(\mu\cos\theta + \sin\theta)vt}{\eta}$

m = 100000 kg, $\mu = 0.1$, $\theta = 5°$, v = 100 m/hr, t = 1 hr

$\eta = \frac{4}{100} = 0.04$

Energy used by engine $= \frac{100000 \times 9.8\ (0.1\cos 5^\circ + \sin 5^\circ) \times 100}{0.04}$

$= \frac{10^5 \times 9.8(0.1 \times 0.9962 + 0.0872) \times 100}{4 \times 10^{-2}}$

$= \frac{9.8 \times 0.1868 \times 10^9}{4}$

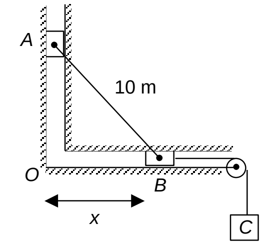
$= 4.577 \times 10^8$ J

As 1 kg coal yields 50000 J, we have the amount of coal burnt up

$= \frac{4.577 \times 10^8}{5 \times 10^4} = 9.154 \times 10^3$ kg.

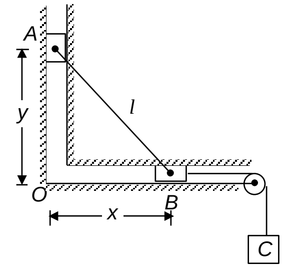
Example 8:

Two bodies A and B connected by a light rigid bar 10 m long move in two frictionless guides as shown in the Figure. If B starts from rest when it is vertically below A, find the velocity of B when x = 6 m. Assume $m_A = m_B = 200$ kg and $m_C = 100$ kg.



Solution:

At the instant, when the bar is as shown in the Figure,



$X^2 + y^2 = L^2$

$$\therefore\ 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0 \qquad \text{... (i)}$$

$$\therefore\ x\frac{dx}{dt}=-y\frac{dy}{dt} \qquad \text{... (ii)}$$

where $\frac{dx}{dt}$ = velocity of A and $\frac{dy}{dt}$ = velocity of B.

Applying the law of conservation of energy, loss of potential energy of A, if it is going down when the rod is vertical to the position shown in the Figure = $m_A g\,[10 - 8] = 2 \times 200 \times 9.8$

C moves down 6 m since B moves 6 m along x-axis

Loss of potential energy of C (= mgh) = $100 \times 9.8 \times 6$

Total loss of potential energy = $200 \times 9.8 \times 2 + 100 \times 9.8 \times 6$

$= 100 \times 9.8 \times 10 = 9800$ J.

This must be equal to kinetic energy gained

Kinetic energy gained = $\frac{1}{2}m_A(v_A)^2+\frac{1}{2}m_B(v_B)^2+\frac{1}{2}m(v_C)^2$

$$= \frac{1}{2}\times 200\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\times 200\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\times 100\left(\frac{dx}{dt}\right)^2$$

$$= 100\left(\frac{dy}{dt}\right)^2 + 150\left(\frac{dx}{dt}\right)^2$$

$$= 100\left[\frac{x}{y}\,\frac{dx}{dt}\right]^2 + 150\left(\frac{dx}{dt}\right)^2 \text{ from (ii)}$$

$$= 100\left[\frac{6}{8}\frac{dx}{dt}\right]^2 + 150\left(\frac{dx}{dt}\right)^2$$

$$= \left[100 \times \frac{9}{16} + 150\right]\left(\frac{dx}{dt}\right)^2$$

$$= \frac{3300}{16} v_B^2$$

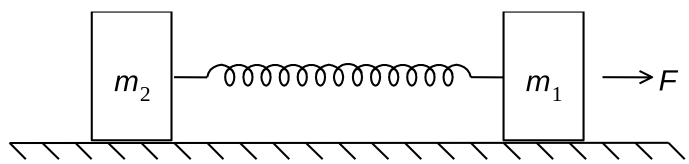
$$\therefore \frac{3300}{16} v_B^2 = 9800$$

$$\therefore v_B = \sqrt{\frac{98 \times 16}{33}} = 7 \times 4\sqrt{\frac{2}{33}} = 6.9 \text{ ms}^{-1}.$$

$\therefore$ velocity of B at the required moment = 6.9 ms $^{-1}$.

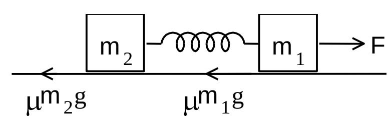
Example 9.

Two bars of masses m_1 and m_2 connected by a non-deformed light spring rest on a horizontal plane. The coefficient of friction between bars and surface is μ. What minimum constant force has to be applied in the horizontal direction to the bar of mass m_1 in order to shift the other bar?



Solution:

Let F be the force applied as shown in the Figure. If F moves through x, the work done will be F × x. This is used to work against friction μm_1 g and store energy in the spring.



Work done against friction = $\mu m_1 g x$;

Energy stored = $\frac{1}{2}kx^2$, where k is the spring constant.

$$\therefore \quad F \cdot x = \mu m_1 gx + \frac{1}{2}kx^2 \qquad \text{... (i)}$$

When the mass m_2 moves, the tension in the spring balances the force of friction at m_2.

$$\therefore \qquad kx = \mu m_2 g \qquad \text{... (ii)}$$

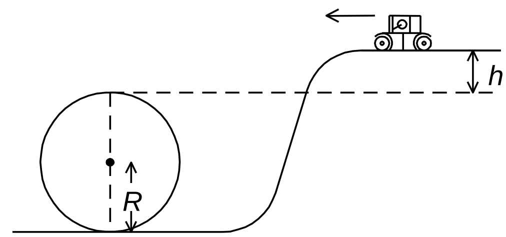
$\therefore$ combining equations (i) and (ii),

$$F \cdot x = \mu m_1 gx + \frac{1}{2}\mu m_2 g \cdot x$$

$$F = \mu g\left(m_1 + \frac{m_2}{2}\right)$$

Example 10.

The Figure shows a loop-to-loop track of radius R. A car without engine starts from a platform at a distance h above the top of the loop and goes around the loop without falling off the track. Find the minimum value of h for a successful looping. Neglect friction.



Solution:

The speed of the car at the top-most point of the loop is v. The gravitational potential energy is zero at the platform and the car starts with negligible speed. (This is assumed.)

According to the law of conservation of energy,

$0 = -mgh + \frac{1}{2}mv^2 \text{ or } v^2 = 2gh$, where m is mass of the car. … (i)

The car has radial acceleration $\left(\frac{v^2}{R}\right)$. The forces acting on the car are mg due to gravity and reaction N due to contact with the track. These forces are radial at the top of the loop.

$\therefore \quad mg + N = \frac{mv^2}{R}$

i.e., $\quad mg + N = \frac{2mgh}{R}$ … (ii)

For h to be minimum, N should be minimum or zero,

i.e., 2 mgh = Rmg

or, $\quad \boldsymbol{h_{min} = \frac{R}{2}}$