# Introduction

If $x^2 + 1 = 0$ , then , $x = \pm\sqrt{-1}$

$\sqrt{-1}$ is represented as i. This is taken as unit of imaginaries.

If , $x^2 + x + 1 = 0$, then $x = \frac{-1 \pm \sqrt{1-4}}{2} or = \frac{-1 \pm \sqrt{3}i}{2}$

Here roots of this equation are of the form x + iy, where x and y are real numbers. Roots of this form are called complex roots.

Any number of the form x + iy (where x and y are real numbers) is called a complex number.

A complex number x + iy is also defined as an ordered pair of real numbers x and y and may be written as (x, y). If z = x + iy, then x is called the real part of the complex number and y is called the imaginary part of the complex number z. 'x' is denoted as Re(z) and 'y' is denoted as Im(z).

$i^2 = -1, i^3 = -i, i^4 = 1$ and $i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i, n \in I$

## Algebraic Operations with Complex Numbers:

**(i) Addition:** $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i\,(y_1 + y_2)$

**(ii) Subtraction:** $(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + I\,(y_1 - y_2)$

**(iii) Multiplication:** $(x_1 + iy_1)\,(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$

**(iv) Division:** $\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

**(v) Equality:** $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ & $y_1 = y_2$ .

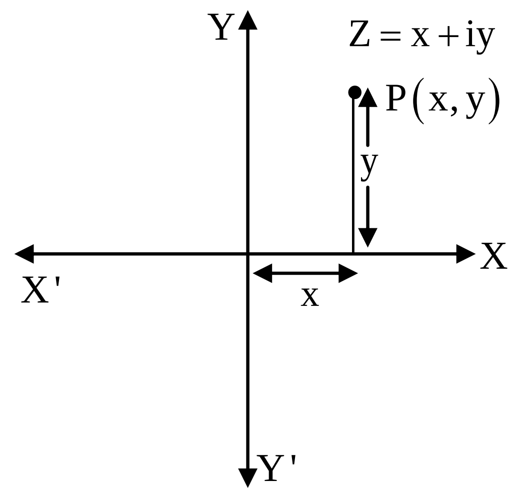
**(vi)** The complex number do not possess the property of order

i.e., $(x_1 + iy_1) <$ or $> (x_2 + iy_2)$ is not defined.

# Argand Plane and Geometrical Representation of Complex Number

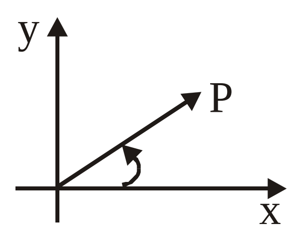
**(a)** Let O be the origin and OX and OY be the x-axis and y-axis respectively. Corresponding to each complex number x + iy there will be one and only one point P(x, y) in the xy – plane.

Thus each complex number x + iy can be represented by a point P(x, y) of the xy-plane and conversely corresponding to each point P(x, y) of the xy-plane there will be a unique complex number x + iy. The xy-plane is called the **Argand Plane, Complex Plane or Gaussian Plane,** x-axis is called the **real axis** and y-axis is called the **imaginary axis.**




**(b)** Each complex number z can be represented by a vector $\overrightarrow{OP}$, where P is the point representing the complex number z.

Thus z = $\overrightarrow{OP}$




**Note:** Any other vector $\overrightarrow{AB}$ which has the same magnitude, direction and sense as that of $\overrightarrow{OP}$ but has a different initial point, also represents the complex number z.

Complex numbers can be considered as vectors in case of sum, difference and modulus of complex numbers.

# Conjugate of a Complex Number

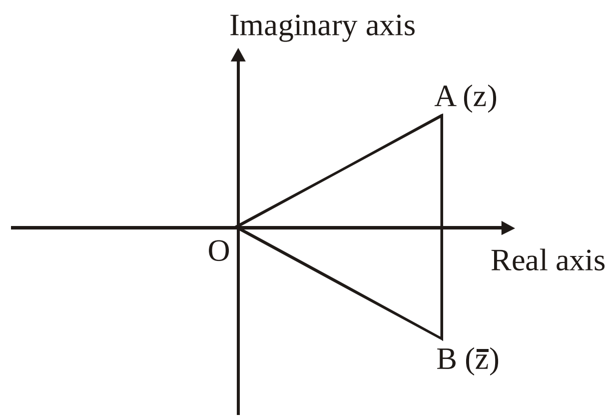
The complex numbers z = x + iy and $\bar{z} = x - iy$ where x and y are real numbers, $i = \sqrt{-1}$ and y≠0 are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i). It is represented by $\bar{z}$.

**Note that,** sum = (x + iy) + (x – iy) = 2x,

Which is real and product = (x + iy) (x – iy) = $x^2 + y^2$ which is real.

## Properties of Conjugate:

(i) $\bar{z}$ is the mirror image of z along real axis.




(ii) $z = \bar{z}$ is purely real

(iii) $z = -\bar{z}$ is purely imaginary

(iv) Re (z) = Re ($\bar{z}$)

(v) Im (z) = – Im $(\bar{z})$

(vi) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(viii) $\overline{z_1 z_2} = \left(\overline{z_1}\right)\left(\overline{z_2}\right)$

(ix) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}} (Z_2 \neq 0)$

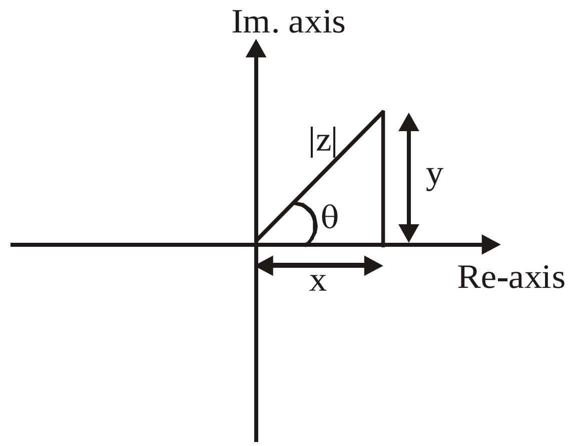
(x) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,\mathrm{Re}(z_1\bar{z}_2)$

(xi) $\overline{z^n} = \left(\bar{z}\right)^n$

(xii) If z = f($z_0$), then $\bar{z} = f\left(\bar{z}_0\right)$

# Modulus of a Complex Number

Distance of a complex number z from origin is called the modulus of the complex number z and it is denoted by |z| . Therefore if z = x + iy, then $|z| = \sqrt{x^2+y^2}$.




## Properties of Modulus

(i) $|z| = 0 \quad z = 0$

(ii) $\mathrm{Re}(z) \le |z| \;\&\; \mathrm{Im}(z) \le |z|$

(iii) $z\overline{z} = |z|^2$

(iv) $|z_1 z_2| = |z_1|\,|z_2|$

(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

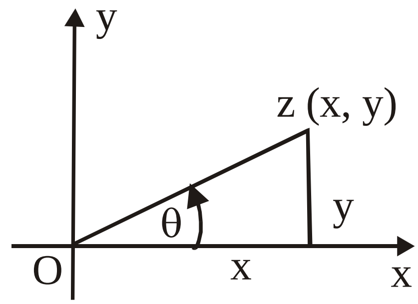
(vi) $|z_1 + z_2|^2 = (z_1 +_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + z_2\overline{z_1} = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2)$

(vii) $z^{-1} = \dfrac{\overline{z}}{|z|^2}$

(viii) $|z|^n = |z^n|$, $n \in N$

# Argument of a Complex Number

We have $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ ... (1)




and $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$ ... (2)

Value of θ, $-\pi < \theta \le \pi$ satisfying equations (1) and (2) simultaneously, is called the principal argument of z. It is also known as amplitude

**Method of calculating principal argument:**

First calculate $\alpha = \tan^{-1}\left|\dfrac{y}{x}\right|$.

Now α, π-α, π+α or 2π-α becomes the principal argument of z according as point P (z = x + iy) lies in $I^{st}$, $II^{nd}$, $III^{rd}$ or $IV^{th}$ quadrant respectively.

**Note:** *Whenever we have to calculate the argument of a complex number, it is obvious that we have to calculate the principal argument.*

## Properties of Arguments

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

In general $\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n)$,

(ii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

(iii) $\arg(-z) = \arg(z) \pm \pi$

(iv) $\arg(iy) = \dfrac{\pi}{2}$ if $y > 0$

$= -\dfrac{\pi}{2}$ if $y < 0$

(v) $\arg(z-\bar{z}) = \pm\dfrac{\pi}{2}$

(vi) $\arg(\bar{z}) = -\arg(z) = \arg\left(\dfrac{1}{z}\right)$

(vii) arg (z) = 0 or $\pi \Leftrightarrow$ z is purely real.

(viii) $\arg(z) = \pm\dfrac{\pi}{2} \Leftrightarrow z$ is purely imaginary.

**Note:** Here arg (z) means general argument of z.

## Illustration 1:

If z and w are two non-zero complex numbers such that | z | = | w | and amp z + amp w = π, then prove that $z = -\bar{w}$

## Solution:

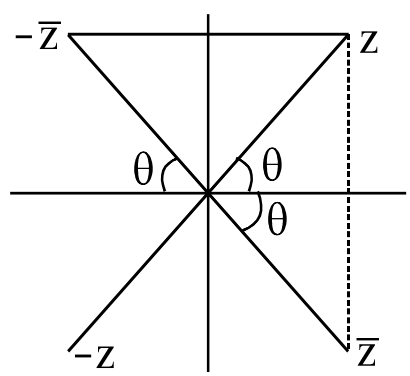
If amp z = θ ....(i)

amp $\bar{z} = -\theta$

amp $-\bar{z} = \pi - \theta$ ....(ii)

Hence amp z + amp $(-\bar{z}) = \pi$

but $w = -\bar{z} \Rightarrow z = -\bar{w}$




## Illustration 2:

If $z = \dfrac{(1+i)(1+2i)(1+3i)}{(1-i)(2-i)(3-i)}$ then find the principal value of arg z = ?

## Solution:

amp z = $(\tan^{-1}1+\tan^{-1}2+\tan^{-1}3) - [\tan^{-1}(-1)+\tan^{-1}(-1/2)+\tan^{-1}(-1/3)] + 2k\pi$

$$= \pi + \frac{\pi}{2} = \frac{3\pi}{2}, \text{ Hence amp Z} = -\frac{\pi}{2}$$

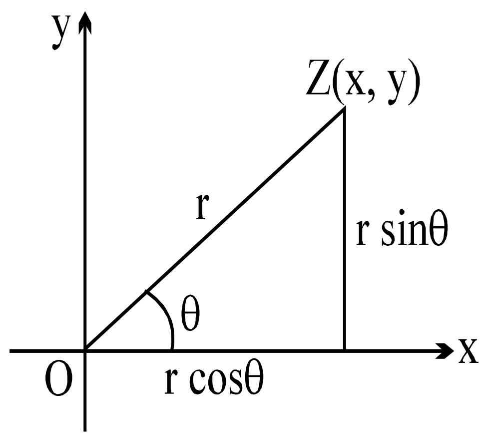
# Representation of a Complex in Different Form

## (i) Cartesian form / Algebric form:

$z = x + iy$ ; Here $|z| = \sqrt{x^2 + y^2}$ ; $\bar{z} = x - iy$ $\theta = \tan^{-1}\frac{y}{x}$

Generally this form is used in locus problems or while solving equations.

## (ii) Trigonometric form / Polar form:




$z = x + iy = r(\cos\theta + i\sin\theta) = r\,\text{CiS}\,\theta$

where $|z| = r$ ; amp $z = \theta$

Note that $(\text{CiS}\,\alpha)(\text{CiS}\,\beta) = \text{CiS}(\alpha + \beta)$

$(\text{CiS}\,\alpha)(\text{CiS}\,(-\beta)) = \text{CiS}(\alpha - \beta)$

$\frac{1}{(\text{CiS})\alpha} = (\text{CiS}\,\alpha)^{-1} = \text{CiS}(-\alpha)$

## (iii) Exponential form:

Since $e^{ix} = \cos x + i\sin x$,

Hence $z = re^{i\theta}$ is the exponential representation.

Note that

(a) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ are known as Eulers identities.

(b) $\cos ix = \frac{e^{x} + e^{-x}}{2}$ = cos hx is always positive real $\forall\, x \in R$ and is $\geq 1$.

and $\sin ix = \frac{e^{x} - e^{-x}}{2}\, i$ = i sin hx is always purely imaginary.

## Illustration 3:

If z = 1 + $\cos\frac{6\pi}{5}$ + $i\,\sin\frac{6\pi}{5}$ find $|z|$ and amp z.

## Solution:

$$z = 2\cos^2\frac{3\pi}{5} + 2i\sin\frac{3\pi}{5}\cos\frac{3\pi}{5}$$

$$= 2\cos\frac{3\pi}{5}\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right]$$

$$= -2\cos\frac{2\pi}{5}\left[-\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right]$$

$$= 2\cos\frac{2\pi}{5}\left[\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right]$$

Hence $|z| = 2\cos\frac{2\pi}{5}$; amp z = $-\frac{2\pi}{5}$

## Illustration 4:

Evaluate: (a) $i^{135}$ (b) $\left(-\sqrt{-i}\right)^{4n+3}, n \in N$ (c) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$

## Solution:

**a**. 135 leaves remainder as 3 when it is divided by 4

$$\therefore\ i^{135} = i^3 = -i$$

**b.** $\left(-\sqrt{-1}\right)^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3 = \left\{(-i)^4\right\}^n (-i)^3 = 1\times(-i)^3 = i$

**c.** $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$

**Illustration 5:**

Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ for all $n \in N$.

**Solution:**

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n\left[1 + i + i^2 + i^3\right]$$

$$= i^n\left[1 + i - 1 - i\right] = i^n(0) = 0$$

**Illustration 6:**

If $(a+b) - i(3a+2b) = 5 + 2i,$ then find a and b.

**Solution:**

We have, $(a+b) - i(3a+2b) = 5 + 2i$

$\Rightarrow a + b = 5$ and $-(3a + 2b) = 2 \Rightarrow a = -12, b = 17$

**Illustration 7:**

Find the square roots of $7 - 24i$

**Solution:**

Let $\sqrt{7-24i} = x + iy.$ Then, $\sqrt{7-24i} = x + iy$

$$\Rightarrow 7 - 24i = (x+iy)^2 \quad \Rightarrow 7 - 24i = \left(x^2 - y^2\right) + 2ixy$$

$\Rightarrow x^2 - y^2 = 7$ and $2xy = -24$

On solving we get $x^2 = 16$ and $y^2 = 9 \Rightarrow x = \pm 4$ and $y = \pm 3$

Hence, $\sqrt{7-24i} = \pm(4 - 3i)$.

## Illustration 8:

Prove that $\tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right)=\frac{2ab}{a^2-b^2}\left(\text{where } a,b\in R^+\right)$

## Solution:

Let $a+ib=re^{i\theta}$

$\Rightarrow a-ib=re^{-i\theta}$

$\Rightarrow \frac{a-ib}{a+ib}=e^{-i2\theta}$ $\quad \Rightarrow \ln\left(\frac{a-ib}{a+ib}\right)=-i2\theta$

$\Rightarrow \tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right)=\tan 2\theta \quad =\frac{2b/a}{1-b^2/a^2}=\frac{2ab}{a^2-b^2}$

## Illustration 9:

If $(x+iy)^5=p+iq,$ then prove that $(y+ix)^5=q+ip.$

## Solution:

$$(x+iy)^5=(p+iq)$$

$$\Rightarrow \overline{(x+iy)^5}=\overline{p+iq} \qquad \Rightarrow (x-iy)^5=p-iq$$

$$\Rightarrow i^5(x-iy)^5=pi^5-i^6q \qquad \Rightarrow (y+ix)^5=pi+q$$

## Illustration 10:

Find the values of $\sin\theta$ if $\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)}$ is purely real or purely imaginary

## Solution:

$$z=\frac{3+2i\sin\theta}{1-2i\sin\theta}$$

Multiplying numerator and denominator by conjugate

$$z = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta} = \frac{3-4\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$$

Now,

z is purely real if sin $\theta = 0$ or $\theta = n\pi, n \in Z, z$ is purely imaginary if $3-4\sin^2\theta = 0$

$$\Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$$

**Illustration 11:**

Find the least positive integer *n* which will reduce $\left(\frac{(i-1)}{(i+1)}\right)^n$ to a real number.

**Solution:**

$$\left(\frac{i-1}{i+1}\right)^n = \left(\frac{i-1}{i+1}\times\frac{i-1}{i-1}\right)^n = \left(\frac{(i-1)^2}{i^2-1}\right)^n = \left(\frac{i^2+2i+1}{-2}\right)^n = (-i)^n$$

Hence, the required positive integer is 2.

**Illustration 12:**

Solve the equation $|z| = z+1+2i$

**Solution:**

$$|z| = z+1+2i$$

$$\Rightarrow \sqrt{x^2+y^2} = x+1+(2+y)i \Rightarrow \sqrt{x^2+y^2} = x+1 \text{ and } y = -2$$

$$\Rightarrow \sqrt{x^2+4} = x+1 \Rightarrow 2x = 3 \qquad \Rightarrow x+iy = \frac{3}{2}-2i$$

## De Moivre's Theorem

The theorem that $(\text{cis}\theta)^n = \text{cis}n\theta$ is called **De Moivre's theorem** which holds true for all whole numbers n.

**Illustration 13:**

$$\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right)$$

If prove that

**Solution:**

We have $1+\sin\theta+i\cos\theta=1+\cos\left(\frac{\pi}{2}-\phi\right)+i\sin\left(\frac{\pi}{2}-\phi\right)$

$$=2\cos^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

$$=2\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right)=2\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)e^{i(\frac{\pi}{4}-\frac{\phi}{2})}$$

Similalry $1+\sin\phi-i\cos\phi=2\cos^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$

$$=2\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)e^{-i(\frac{\pi}{4}-\frac{\phi}{2})}$$

$$\therefore\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\left(\frac{e^{i\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}}\right)^n=e^{in\left(\frac{\pi}{2}-\phi\right)}=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right)$$

## Cube Roots of Unity

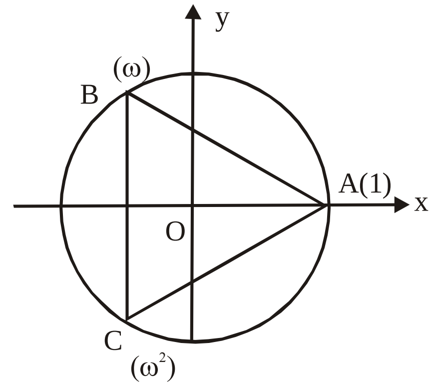
Roots of $x^3 - 1 = 0$ are called the cube roots of unity

Now $x^3 - 1 = 0$

$\Rightarrow \quad (x - 1)(x^2 + x + 1) = 0$

Therefore, $x = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

If second root be represented by, $\omega$ then third root will be $\omega^2$.




$\therefore$ Cube roots of unity are 1, $\omega$, $\omega^2$. 1 is real cube root of unity and $\omega$ and $\omega^2$ are non-real cube roots of unity.

Cube roots of unity can be taken as vertices of an equilateral triangle ABC inscribed in a circle of radius 1 and centre at origin.

## Properties of Cube Roots of Unity

(i) $1+\omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $1+ \omega^n + \omega^{2n} = 3$ (if n is multiple of 3)

(iv) $1+ \omega^n + \omega^{2n} = 0$ (if n is not a multiple of 3).

**Illustration 14:**

Find the value of the expression

$1.(2-\omega)\ (2-\omega^2) + 2.(3-\omega)\ (3-\omega^2)+\ldots + (n-1)\ (n-\omega)\ (n-\omega^2)$. where $\omega$ is an imaginary cube root of unity.

**Solution:**

We have, $(z - 1)\ (z - \omega)\ (z -\omega^2) \equiv z^3 -1$

$\therefore \quad 1(2-\omega)\ (2-\omega^2) + 2(3-\omega^2) + \ldots\ldots + (n-1)\ (n-\omega)\ (n-\omega^2)$

$$= \sum_{r=2}^{n}(r-1)(r-\omega)(r-\omega^2) = \sum_{r=2}^{n} r^3 - \sum_{r=2}^{n} 1$$

$$= \left(\sum_{r=1}^{n} r^3\right) - 1 - \left(\sum_{r=2}^{n} 1\right) = \left\{\frac{n(n+1)}{2}\right\}^2 - 1 - (n-1) = \left\{\frac{n(n+1)}{2}\right\}^2 - n$$

# The N$^{TH}$ Roots of Unity

Let x be an nth root of unity. Then $x^n = 1$, $x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ (where k is an integer)

k = 0, 1, 2, ... n – 1

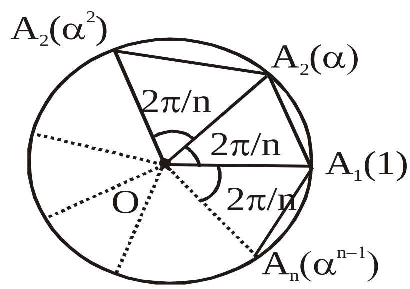
$n^{th}$ root of the $n^{th}$ degree polynomial equation $x^n - 1 = 0$ since there is no $x^{n-1}$ term the sum of the roots of unity is zero.

The $n^{th}$ roots of unity are typically represented by 1, $\alpha$, $\alpha^2$,…, $\alpha^{n-1}$

## Product of the Roots

$$1.\alpha.\alpha^2...\alpha^{n-1} = \alpha^{\frac{n(n-1)}{2}} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n\left(\frac{n-1}{2}\right)} = \cos\{\pi(n-1)\} + i\sin\{\pi(n-1)\}$$

$$\begin{cases} -1, n \text{ is even} \\ 1, n \text{ is odd} \end{cases}$$




**Note:** The points represented by n, nth roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at the origin, one vertex being on the positive real axis.

## Illustration 15:

If 1, $\alpha_1$, $\alpha_2$, ..., $\alpha_{n-1}$ are the n, nth roots of unity, prove that
$(1-\alpha_1)(1-\alpha_2).....(1-\alpha_{n-1}) = n$.

Deduce that $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}....\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$

## Solution:

1, $\alpha_1$, $\alpha_2$,.... $\alpha_{n-1}$are the roots of $x^n = 1$

$\Rightarrow \quad x^n - 1 \equiv (x-1)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})$

$\Rightarrow \quad (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1}) \equiv \frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \ldots\ldots + 1$

By taking lim $x\to1$, we get $(1-\alpha_1)(1-\alpha_2).....(1-\alpha_{n-1}) = n$

$\Rightarrow \quad |1-\alpha_1|\,|1-\alpha_2|\ldots.|1-\alpha_{n-1}| = n \Rightarrow \prod_{r=1}^{n-1}|1-\alpha_r| = n$

$$\Rightarrow \quad \prod_{r=1}^{n=1}\sqrt{\left((1-\cos\frac{2r\pi}{n}\right)^2 + \left(\sin\frac{2r\pi}{n}\right)^2} = n$$

$$\Rightarrow \quad \prod_{r=1}^{n=1}\sqrt{\left((2-\sin^2\frac{r\pi}{n}\right)^2 + \left(2\sin\frac{\pi r}{n}\cos\frac{\pi r}{n}\right)^2} = n$$

$$\Rightarrow \quad \prod_{r=1}^{n-1}2\sin\frac{r\pi}{n}\sqrt{\sin^2\frac{r\pi}{n}+\cos^2\frac{r\pi}{n}} = n$$

$$\Rightarrow \quad \prod_{r=1}^{n-1}2\sin\left(\frac{r\pi}{n}\right) = n$$

$$\Rightarrow \quad \prod_{r=1}^{n-1}\sin\left(\frac{r\pi}{n}\right) = \frac{n}{2^{n-1}}$$

## Illustration 16:

Find the value of $\sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11}-i\cos\frac{2\pi k}{11}\right)$

## Solution:

Let $S=\sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11}-i\cos\frac{2\pi k}{11}\right)=\sum_{k=1}^{10}\left(-i^2\sin\frac{2\pi k}{11}-i\cos\frac{2\pi k}{11}\right)$

$=-i\sum_{k=1}^{10}\left(\cos\frac{2\pi k}{11}+i\sin\frac{2\pi k}{11}\right)=-i\sum_{k=1}^{10}e^{i\frac{2\pi k}{11}}$

$= -i\left[\sum_{k=1}^{10}e^{i\frac{2\pi k}{11}}-1\right]$ = -i (sum of 11th roots of unity –1)

= –i(0 – 1) = i

## Illustration 17:

Find $5^{th}$ roots of unity.

## Solution:

Let $z^5=1=\text{cis}\,2k\pi$ where k is any integer

Now z = $(1)^{1/5}$ = $(\text{cis}(2k\pi))^{1/5}$ = cis(2kπ/5) where k = 0,1,2,3,4

Hence the answers are cis0, cis(2π/5) cis(4π/5) cis(6π/5) and cis(8π/5)

## Illustration 18:

If $\alpha$ be a root of equation $x^2+x+1=0$, then find the value of

$$\left(\alpha+\frac{1}{\alpha}\right)+\left(\alpha^2+\frac{1}{\alpha^2}\right)^2+\left(\alpha^3+\frac{1}{\alpha^3}\right)^2+\ldots+\left(\alpha^6+\frac{1}{\alpha^6}\right)^2$$

## Solution:

Roots of equation $x^2 + x + 1 = 0$ are ω and $\omega^2$. Hence the given expression is

$$=\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega^2+\frac{1}{\omega^2}\right)^2+\ldots+\left(\omega^6+\frac{1}{\omega^6}\right)^2=(-1)^2+(-1)^2+4+(-1)^2+(-1)^2+4=12$$

**Illustration 19:**

Show that $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{\sqrt{7}}{2}$

**Solution:**

Let $\alpha=\text{cis}\frac{2\pi}{7}$, consider $\beta=\alpha+\alpha^2+\alpha^4$, $\gamma=\alpha^3+\alpha^5+\alpha^6$

We can see that $\beta+\gamma=-1$

$$\beta\gamma=2$$

hence $\beta,\gamma$ are roots of $x^2+x+2=0$

$\Rightarrow \beta=\frac{-1+i\sqrt{7}}{2},\gamma=\frac{-1-i\sqrt{7}}{2}$

Note that $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\text{Im}(\beta)$

hence $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{\sqrt{7}}{2}$

**Illustration 20:**

Using the fact that $(a+b)^n=\sum_{r=0}^{n}{}^nC_r a^r b^{n-r}$ Where ${}^nC_r=\frac{n!}{r!(n-r)!}$

Show that $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$

**Solution:**

$(\cos\theta+i\sin\theta)^4=\cos 4\theta+i\sin 4\theta$

Hence, $\text{Re}(\cos\theta+i\sin\theta)^4=\cos 4\theta$

$\therefore \text{Re}\left({}^4C_0\cos^4\theta+i\,{}^4C_1\cos^3\theta\sin\theta-{}^4C_2\cos^2\theta\sin^2\theta-i\,{}^4C_3\cos\theta\sin^3\theta+{}^4C_4\sin^4\theta\right)=\cos 4\theta$

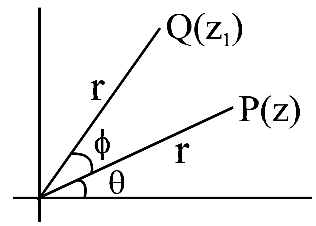
Hence $\cos 4\theta={}^4C_0\cos^4\theta-{}^4C_2\cos^2\theta\sin^2\theta+{}^4C_4\sin^4\theta$

# Vectorial Representation of a Complex

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then,

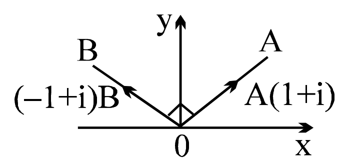
$\overrightarrow{OP} = z$ & $\left|\overrightarrow{OP}\right| = |z|$.

**Note:**



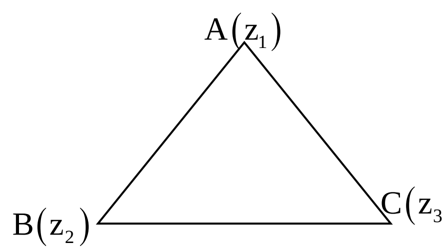

(i) If $\overrightarrow{OP} = z = re^{i\theta}$ then $\overrightarrow{OQ} = z_1 = re^{i(\theta+\phi)} = z.e^{i\theta}.e^{i\phi}$

If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are of unequal magnitude then $OQ = OP\,e^{i\phi}$




(ii) If $z = \overrightarrow{OA} = 1 + i$

then $z_1 = \overrightarrow{OB} = i\,(1 + i) = -1 + i$




(iii) Using the vectorial concept and section formula complex numbers corresponding to centroid, incentre, orthocentre and circumcentre for a triangle whose vertices are $z_1, z_2, z_3$ can be deduced.

# Centroid, Incentre, Orthocentre & Circumcentre of a triangle on a complex plane

**(i)** Centroid 'G' = $\dfrac{z_1 + z_2 + z_3}{3}$

**(ii)** Incentre 'I' = $\dfrac{a z_1 + b z_2 + c z_3}{a + b + c}$

**(iii)** Orthocentre**:** $Z_H = \dfrac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\Sigma \tan A}$

**(iv)** Circumcentre**:** $\dfrac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\Sigma \sin 2A}$

# Rotation of a Vector (Coni's Rule)

If we multiply a complex vector z with cisθ (or $e^{i\theta}$) we obtain another complex vector $z_2$ .We can conclude the following statements.

(i) $z_2 = z\,\text{cis}\theta$

(ii) $|z_2| = |z|$

(iii) $\arg z_2 = \arg z + \theta$ (since $\arg(\text{cis}\,\theta) = \theta$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$)

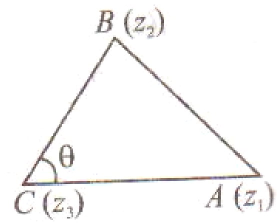
Using above three statements we can say that complex vector $z_2$ makes an angle θ with z and has same length. In other words if we rotate z by an angle θ anticlockwise about its tail without changing its length then the new vector will be z cis θ or $z_2$. This is called rotation of a complex vector or Coni's rule.

## Rule of thumb

(i) To rotate any complex vector by angle θ anticlockwise about its tail without changing its length just multiply it by cisθ.

(ii) To rotate any complex vector by angle θ clockwise about its tail without changing its length just multiply it by cis(-θ)

(iii) To rotate any complex vector by angle θ anticlockwise about its tail and also changing its length to k times the original length just multiplies it by kcisθ.




(iv) In above figure if $z_1$, $z_2$ and $z_3$ be the vertices A, B and C of ΔABC in the Argand plane as shown and $\angle ACB = \theta$, then $z_2 - z_3 = \dfrac{|z_2 - z_3|}{|z_1 - z_3|}(z_1 - z_3)e^{i\theta}$

**Illustration 21:**

Rotate A (7,6) about B(4,2) by $90^o$ anti clockwise. Find its new position $A_2$

**Solution:**

We need to find the position vector of $A_2$ or $\overrightarrow{OA_2}$ where O is the origin.

Point B will act as tail and point A will act as head. Note that $\overrightarrow{OA} = 7 + 6i$ while $\overrightarrow{OB} = 4+2i$

Hence Vector to be rotated is $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = (7+6i) – (4+2i) = 3+4i$

The new vector $\overrightarrow{BA_2} = \overrightarrow{BA}$ cis $90^o = (3+4i)(0 + i) = -4 +3i$. The position vector of $A_2$ or

$\overrightarrow{OA_2} = \overrightarrow{OB} + \overrightarrow{BA_2} = (4+2i) + (-4+3i) = 0 + 5i$ hence the new position $A_2$ is (0, 5)

## Triangle Inequality

For any two complex numbers $z_1$ and $z_2$ we know that

$$| |z_1| - |z_2| | \le |z_1 + z_2| \le |z_1| + |z_2|$$

This result is called triangle inequality.

Now let us understand its application in the questions below

**Illustration 22:**

If $|z – 3i| = 1$ then show that $2 \le |z| \le 4$.

**Solution:**

Using triangle inequality we can say that

$$| |z_1| - |z_2| | \le |z_1 + z_2| \le |z_1| + |z_2| \quad \text{.....(1)}$$

Let $z_1 = z$ while $z_2 = -3i$ .Also let $|z| = r$. Note that $|-3i| = 3$.

From (1) we can conclude that $|r – 3| \le 1 \le r + 3$

Solving both sides of above inequality we get $2 \le r \le 4$.

Try to interpret this result geometrically. *Hint: $|z – 3i| = 1$ represents a circle)*

**Illustration 23:**

For any complex number z, find the minimum value of $|z|+|z-2i|$.

**Solution:**

We have, for $z \in C$, $|2i|=|z+(2i-z)| \le |z|+|2i-z|$

$\Rightarrow 2 \le |z|+|z-2i|$ Thus, minimum value of $|z|+|z-2i|$ is 2.

Try to interpret this result geometrically.

(Hint: $|z|+|z-2i|$ represents sum of distances of a variable point z from origin and from (0,2))

**Illustration 24:**

If $|z+4| \le 3,$ then find the greatest value of $|z+1|$.

**Solution:**

$|z+1|=|z+4-3| = |(z+4)+(-3)| \le |z+4|+|-3| \le 3+3=6 \quad [\because |z+4| \le 3]$

Hence, the greatest value of $|z+1|$ is 6

Try to interpret this result geometrically.

(Hint: $|z+4| \le 3,$ represents a circular disc)

**Illustration 25:**

For complex numbers a, b and c show that | a + b + c|≤|a|+|b|+|c|

**Solution:**

We can easily say that | (a + b) + c | ≤ | a + b | +|c| ≤ |a| + |b| + |c|.

Hence proved.

**Note that** the triangle inequality will hold true for any number of complex numbers.

**Also note that the equality happens if a = b = c**

**Illustration 26:**

If all the roots of $z^3 + az^2 + bz + c = 0$ are of unit modulus, then show that $|3 - 4i + a| \le 8$

**Solution:**

Assume that the roots be $\alpha, \beta, \gamma$ such that $|\alpha| = |\beta| = |\gamma| = 1$

$\therefore \alpha + \beta + \gamma = -a$

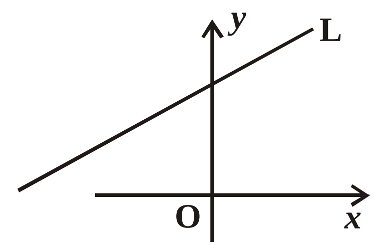
Hence $|-a| = |\alpha + \beta + \gamma| \le |\alpha| + |\beta| + |\gamma|$

$\Rightarrow \quad |a| \le 3$

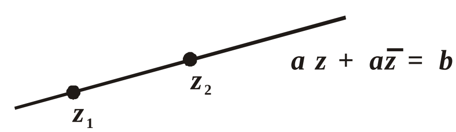
$\Rightarrow \quad |3 - 4i + a| \le |3 - 4i| + |a| \le 5 + 3 = 8$

# Locus Based On Complex Numbers

## (i) Straight line (CANONICAL FORM)




Consider a straight line L whose equation in the x-y plane is px +qy = r. Superimpose the real and imaginary axes of the Argand plane on the x & y axes, respectively.




Let z = x + iy be a point of line L. Then the equation of line L can be written as $p\left(\frac{z+\bar{z}}{2}\right) + q\left(\frac{z-\bar{z}}{2i}\right) = r$ OR $\left(\frac{p}{2} + \frac{q}{2i}\right) z + \left(\frac{p}{2} - \frac{q}{2i}\right)\bar{z} = r$, which is of the form $\bar{a}z + a\bar{z} = b$ where **a** is a complex number & **b** is a real number. This form is called canonical form of straight line. Now let $z_1$ & $z_2$ satisfy the equation

$$\therefore \bar{a}z_1 + a\bar{z}_1 = \bar{a}z_2 + a\bar{z}_2 = b$$

$$\Leftrightarrow \bar{a}(z_1 - z_2) = -a(\bar{z}_1 - \bar{z}_2)$$

$$\Leftrightarrow \frac{-a}{\bar{a}} = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

This quantity is called **complex slope** of this line and is defined as $\frac{-a}{\bar{a}}$.

**Note that** complex slope is different from the real slope we study in coordinate geometry, however they are related. A line of real slope tanα has complex slope cis2α.

## (ii) Circle

The elementary equation of a circle with centre A(a) and radius r is |z-a| = r.

However if we square both sides we can observe that it converts to its canonical form which is zz* + pz*+ p*z + q=0 where q is a real number and p is a complex number.

**Note:**

(i) If z varies so that $\text{amp}\frac{z-a}{z-a'} = \phi$, Where ϕ is a constant angle, then the point z describes an arc of a segment of a circle on $aa'$, containing an angle ϕ.

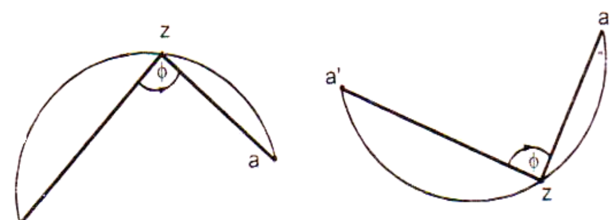
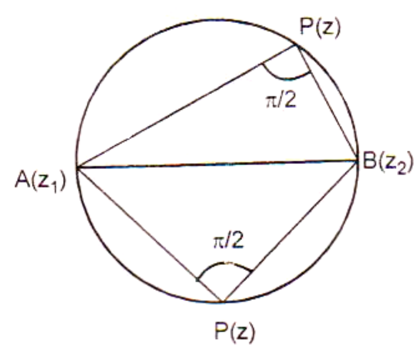



Fig. I Fig. II

The sign of ϕ determines the side of $aa'$ on which the segment lies. Thus ϕ is positive in fig.1 and negative in fig. II

(ii) If $\text{amp}\frac{z-a}{z-a'} = \phi$ if ϕ is π/2 then it represents a circle with diameter as the segment joining $A(z_1)$ and $B(z_2)$. See Fig II




(Fig.II)

(iii) If $\phi$ is $\pi$ it represents the straight line joining A($z_1$) and B($z_2$) but excluding the segment AB. See fig III


(iv) If $\phi$ is 0, then it represents the segments joining A($z_1$) and B($z_2$) see in Fig IV.


## (iii) Conic Section

### Parabola:

Locus of a point which is equidistant from a fixed line (called directrix) and a fixed point (called focus) is known as parabola. Hence equation of parabola with focus at $z_0$ and directrix as $\bar{a}z + a\bar{z} + b = 0$ is given by $|z - z_0| = \dfrac{a\bar{z} + \bar{a}z + b}{2|a|}$

### Ellipse:

Locus of a point whose sum of distances from two fixed point is a constant is called an ellipse. Thus the equation of ellipse with foci at $z_1$ and $z_2$ and length of semi major axes as 2a is $|z - z_1| + |z - z_2| = 2a$ where $2a > |z_1 - z_2|$

For $2a = |z_1 - z_2|$ it represents line segment joining A($z_1$) and B($z_2$).

### Hyperbola:

Locus of a point whose difference of distances from two fixed points is a constant is called an Hyperbola. Equation of hyperbola with foci at $z_1$ and $z_2$ and length of transverse axes as 2a is $\left||z - z_1| - |z - z_2|\right| = 2a$ where $2a < |z_1 - z_2|$. For $2a = |z_1 - z_2|$ it represents line segment joining A($z_1$) and B($z_2$) But excluding the segment AB.

### Illustration 27:

If $\left||z + 2| - |z - 2|\right| = a^2, z \in C$ representing a hyperbola for $a \in R$, then find the values of a.

**Solution:**

Here foci are at –2 and 2 at a distance 4, Hence the given equation represents a hyperbola if $a^2 < 4$ i.e $a \in (-2, 2)$.

**Illustration 28:**

Show that locus of z if $|z + 1 - i|^2 + |z - 5 - i|^2 = 36$ is a circle.

**Solution:**

$|z + 1 - i|^2 + |z - 5 - i|^2 = 36$

$\Rightarrow$ put z = x +yi to get locus of P(z) is the circle having AB as a diameter.

**Illustration 29:**

Show that locus of z if $\left|\frac{z-25}{z-1}\right| = 5$ is a circle.

**Solution:**

$\left|\frac{z-25}{z-1}\right| = 5 \Rightarrow \frac{|z-25|}{|z-1|} = 5$

$\Rightarrow$ $|z - 25|^2 = 25|z - 1|^2$ $\Rightarrow$ $(z - 25)(\bar{z} - 25) = 25(z-1)(\bar{z}-1)$

$\Rightarrow$ $|z|^2 - 25\bar{z} - 25z + 625 = 25|z|^2 - 25\bar{z} - 25z + 25$

$\Rightarrow$ $24|z|^2 = 600$ $\Rightarrow$ $|z| = 5$

Hence locus of z is the circle having centre at (0, 0) and radius 5.

**Illustration 30:**

Show that locus of z if $|z - 3| + |z + 2| = 8$ is an ellipse

**Solution:**

$|z - 3| + |z + 2| = 8$

$\Rightarrow$ PA + PB = constant = 8, where $A \equiv (3, 0)$ and $B \equiv (-2, 0)$

$\Rightarrow$ PA + PB = constant > AB

$\Rightarrow$ Locus of P(z) is an ellipse

**Illustration 31:**

Show that locus of z if |z + 5| - |z – 7| = 3 is a hyperbola

**Solution:**

$|z + 5| - |z - 7| = 3$

$\Rightarrow$ PA – PB = 3, where $A \equiv (-5, 0)$ and $B \equiv (7, 0)$

$\Rightarrow$ PA – PB = constant = 3 < AB

$\Rightarrow$ Locus of P(z) is a hyperbola.

**Illustration 32:**

Show that locus of z if if $z^2 - 8z - 2z\bar{z} = \bar{z}(8 - \bar{z})$ is a parabola

**Solution:**

Let z = x + iy, then $\bar{z} = x - iy$

$\Rightarrow z^2 - 8z - 2z\bar{z} = \bar{z}(8 - \bar{z}) \Rightarrow z^2 - 2z\bar{z} + \bar{z}^2 = 8(z + \bar{z})$

Now put z= x + yi to get $-4y^2 = 16x$ which is a parabola

**Illustration 33:**

If $z_1, z_2, z_3, z_4$ are the affixes of four points in the Argand plane, z is the affix of a point such that $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$, then prove that $z_1, z_2, z_3, z_4$ are concyclic.

**Solution:**

We have, $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$ Therefore, the point having affix z is equidistant from the four points having affixes $z_1, z_2, z_3, z_4$. Thus, z is the affix of the centre of a circle which means $z_1, z_2, z_3, z_4$ are concyclic.

## Illustration 34:

If $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then find the value of $|z_1 + z_2 + z_3|$.

## Solution:

$|z_1| = 1 \Rightarrow z_1\bar{z}_1 = 1, |z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4, |z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9$

Also, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + z_1\bar{z}_1z_2z_3| = 12$

$\Rightarrow |z_1z_2z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$ $\qquad \Rightarrow |z_1||z_2||z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2$ $\qquad \Rightarrow |z_1 + z_2 + z_3| = 2$

## Illustration 35:

If $|z - i\,\mathrm{Re}(z)| = |z - \mathrm{lm}(z)|$, then prove that z lies on the bisectors of the quadrants.

## Solution:

$|z - i\,\mathrm{Re}(z)| = |z - \mathrm{lm}(z)|$

$\Rightarrow |x + iy - ix| = |x + iy - y|$

$\Rightarrow x^2 + (x - y)^2 = (x - y)^2 + y^2$

$\Rightarrow x^2 = y^2$ $\qquad \Rightarrow |x| = |y| \Rightarrow$ z lies on the bisectors the quadrants.

## Illustration 36:

Let $|(\bar{z}_1 - 2\bar{z}_2)/(2 - z_1\bar{z}_2)| = 1$ and $|z_2| \neq 1$, where $z_1$ and $z_2$ are complex numbers, show that $|z_1| = 2$.

## Solution:

$\left|\frac{\bar{z}_1 - 2\bar{z}_2}{2 - z_1\bar{z}_2}\right| = 1 \Rightarrow |\bar{z}_1 - 2\bar{z}_2|^2 = |2 - z_1\bar{z}_2|^2$

Using $|a+b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}(a\bar{b})$.on both sides

$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2 \qquad \Rightarrow |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \left(|z_2|^2 - 1\right)\left(|z_1|^2 - 4\right) = 0 \qquad \Rightarrow |z_1| = 2\left(\text{as}\,|z_2| \neq 1\right)$

**Illustration 37:**

Find the greatest value of $|z_1 + z_2 + z_3|$, if $|z_1 - 1| \leq 1, |z_2 - 2| \leq 2, |z_3 - 3| \leq 3,$

Also find the minimum value.

**Solution:**

$$|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6| \leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$$

$$\leq 1 + 2 + 3 + 6 = 12$$

Hence the greatest value is 12.

The minimum value will be zero can you guess why?

(Hint: all $z_1$, $z_2$ and $z_3$ form discs with origin on its circumference)

**Illustration 38:**

Prove that for all the roots of the equation

$|\sin\theta_1|z^3 + |\sin\theta_2|z^2 + |\sin\theta_3|z + |\sin\theta_4| = 3$ , $|z|$ is greater than 2/3

**Solution:**

Since $|\sin\theta_1|z^3 + |\sin\theta_2|z^2 + |\sin\theta_3|z + |\sin\theta_4| = 3$

$$\Rightarrow |3| = \left||\sin\theta_1|z^3 + |\sin\theta_2|z^2 + |\sin\theta_3|z + |\sin\theta_4|\right|$$

$$\leq 1|z^3 + z^2 + z + 1| < |z|^3 + |z|^2 + |z| + 1$$

since $|\sin\theta_k| < 1.$

$$< 1 + |z| + |z|^2 + |z|^3 + |z|^4 + .....\infty \left(\because |z| < 1\right)$$

$$\Rightarrow \quad 3 < \frac{1}{1-|z|} \qquad \Rightarrow 3 - 3|z| < 1 \qquad \Rightarrow |z| > \frac{2}{3}$$

**Illustration 39:**

Find the greatest and the least value of $|z_1+z_2|$ if $z_1=24+7i$ and $|z_2|=6$

**Solution:**

$|z_1+z_2|\le|z_1|+|z_2|=|24+7i|+6$ = 25 + 6 =31

Also, $|z_1+z_2|=|z_1-(-z_2)|\ge||z_1|-|z_2||$ $\Rightarrow$ $|z_1+z_2|\ge|25-6|=19$

Hence the least value of $|z_1+z_2|$ is 19 and the greatest value is 25.

**Illustration 40:**

Prove that $|z_1+z_2|=|z_1|+|z_2|\Rightarrow\arg(z_1)=\arg(z_2).$

**Solution:**

$$|z_1+z_2|=|z_1|+|z_2|$$

$$\Rightarrow|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\Rightarrow|z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\bar{z}_2)=|z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\Rightarrow2\,\mathrm{Re}(z_1\bar{z}_2)=2|z_1||z_2|\Rightarrow\cos(\theta_1-\theta_2)=1\Rightarrow\arg(z_1)=\arg(z_2)$$

**Illustration 41:**

If $\arg(z_1)=170^0$ and $\arg(z_2)=70^0$, then find the principal argument of $z_1z_2$

**Solution:**

$$\arg(z_1z_2)=\arg(z_1)+\arg(z_2)=170^0+70^0=240^0$$

Thus $z_1z_2$ lies in third quadrant. Hence its principal argument is $-120^0$

## Illustration 42:

If $z_1$ and $z_2$ are conjugate to each other then find $\arg(-z_1z_2)$.

## Solution:

$z_1$ and $z_2$ are conjugate to each other i.e $z_2 = \overline{z}_1$, Therefore

$\arg(-z_1z_2) = \arg\left(-|z_1|^2\right) =$ arg(negative real number) $= \pi$

## Illustration 43:

If $0 < \alpha < \pi/2,$ then find the modulus and argument of $(1+\cos 2\alpha)+i\sin 2\alpha$

## Solution:

$z = (1+\cos 2\alpha)+i\sin 2\alpha = 2\cos^2\alpha + 2i\sin\alpha\cos\alpha$

$= 2\cos\alpha[\cos\alpha + i\sin\alpha]$ hence $|z| = 2\cos\alpha$ and $\arg(z) = \alpha$

Can you guess if $\pi/2 < \alpha < \pi,$ then what is the new modulus and argument?

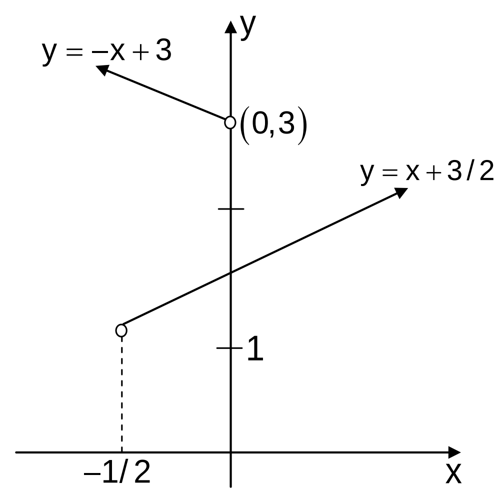
## Illustration 44:

Find the point of intersection of the curves $\arg(z-3i) = 3\pi/4$ and $\arg(2z+1-2i) = \pi/4$

## Solution:

Given loci are as follows:

$\arg(z-3i) = \frac{3\pi}{4}$ which is a ray that starts from $3i$ and makes an angle $3\pi/4$ with positive real axis as shown in the figure.




$\arg(2z+1-2i) = \frac{\pi}{4}$

$\Rightarrow \arg\left[2\left(z+\frac{1}{2}-i\right)\right] = \frac{\pi}{4}$ $\quad\Rightarrow \arg\left[z-\left(-\frac{1}{2}+i\right)\right] = \frac{\pi}{4}$

This is a ray that starts from point -1/2 + *I* and makes an angle $\pi/4$ with positive real axis as shown in the figure. From the figure it is obvious that the system of equations has no solution.

### Illustration 45:

Write $\frac{(1+7i)}{(2-i)^2}$ in polar form

### Solution:

$$z=\frac{1+7i}{4-4i+i^2}=\frac{1+7i}{3-4i}=\left(\frac{1+7i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right)=\frac{-25+25i}{25}=-1+i$$

$\because r=|z|=\sqrt{(-1)^2+(1)^2}=\sqrt{2}$ Since the point (-1, 1) respecting z lies in the second quadrant,

Therefore $\theta=\arg(z)=\pi-\alpha=\pi-\pi/4=3\pi/4.$ Hence $z=\sqrt{2}\text{cis}\frac{3\pi}{4}$

### Illustration 46:

Find the value of expression $\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)$....to $\infty$

### Solution:

$$\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)\text{...to}\,\infty$$

$$=\cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+....\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+...\right)$$

$$=\cos\left[\frac{\pi}{2}\left(\frac{1}{1-\frac{1}{2}}\right)\right]+i\sin\left[\frac{\pi}{2}\left(\frac{1}{1-\frac{1}{2}}\right)\right]=\cos\pi+i\sin\pi=-1$$

## Illustration 47:

If $z=\cos\theta+i\sin\theta$ be a root of the equation $a_0z^n+a_1z^{n-1}+a_2z^{n-2}+....+a_{n-1}z+a_n=0$, then prove that

(i) $a_0+a_1\cos\theta+a_2\cos2\theta+....+a_n\cos n\theta=0$

(ii) $a_1\sin\theta+a_2\sin2\theta+...+a_n\sin n\theta=0$

## Solution:

Dividing the given equation by $z^n$,

we get $a_0+a_1z^{-1}+a_2z^{-2}+....+a_{n-1}z^{1-n}+a_nz^{-n}=0$

Now, $z=\cos\theta+i\sin\theta=e^{i\theta}$ satisfies the above equation.

Hence, $a_0+a_1e^{-i\theta}+a_2e^{-2i\theta}+...+a_{n-1}e^{-i(n-1)\theta}+a_ne^{-in\theta}=0$

$\Rightarrow(a_0+a_1\cos\theta+a_2\cos2\theta+...+a_n\cos n\theta)-i(a_1\sin\theta+a_2\sin2\theta+...+a_n\sin n\theta)=0$

$\Rightarrow a_0+a_1\cos\theta+a_2\cos2\theta+....+a_n\cos n\theta=0$ and $a_1\sin\theta+\theta_2\sin2\theta+....+a_n\sin n\theta=0$

## Illustration 48:

If $\omega$ is a cube root of unity, then find the value of the following:

(i) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(iii) $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$

## Solution:

(i) If $\omega$ is a complex cube root of unity, then $\omega^3=1$ and $1+\omega+\omega^2=0$

$$\therefore(1+\omega-\omega^2)(1-\omega+\omega^2)=(-2\omega^2)(-2\omega)=4$$

(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=(1-\omega)^2(1-\omega^2)^2=(1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$

$$=(-3\omega)(-3\omega^2)=9\omega^3=9$$

(iii) $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=\dfrac{\omega\left(a+b\omega+c\omega^2\right)}{\left(b\omega+c\omega^2+a\right)}+\dfrac{\omega^2\left(a+b\omega+c\omega^2\right)}{\left(c\omega^2+a+b\omega\right)}$

$=\omega+\omega^2=-1$

## Illustration 49:

In $\Delta ABC$, (A($z_1$), B($z_2$) and C($z_3$)) is inscribed in the circle $|z|=5$. If H($Z_H$) be the orthocentre of triangle ABC, then find $z_H$.

## Solution:

Circumecentre of $\Delta ABC$ is clearly origin. Let G($z_G$) be its centroid. Then,

$z_G=\frac{1}{3}\left(z_1+z_2+z_3\right)$ Now we know that OG : GH = 1:2 (euler line)

$\Rightarrow z_G=\dfrac{2\times 0+1\times z_H}{3}$

$\Rightarrow z_H=3_{z_G}=z_1+z_2+z_3$

## Illustration 50:

Let $z_1$, $z_2$, $z_3$ be three complex numbers and a, b, c be real numbers not all zero, such that a + b + c = 0 and $az_1+bz_2+cz_3=0$. Show that $z_1,z_2,z_3$ are collinear.

## Solution:

$az_1+bz_2-\left(a+b\right)z_3=0$

$\Rightarrow az_1+bz_2=\left(a+b\right)z_3$

$\Rightarrow z_3=\dfrac{az_1+bz_2}{a+b}$

Hence $z_3$ divides the line segment joining $z_1$and $z_2$ in a ratio a : b hence they are collinear

## Illustration 51:

Show that $e^{2mi\cot^{-1}p}\left(\frac{(pi+1)}{(pi-1)}\right)^m = 1$

## Solution:

Let $\cot^{-1} p = \theta$. Then $\cot\theta = p$.

Now.

L.H.S = $e^{2mi\theta}\left(\frac{i\cot\theta+1}{i\cot\theta-1}\right)^m$ $= e^{2mi\theta}\left[\frac{i(\cot\theta-i)}{i(\cot\theta+1)}\right]^m$ $= e^{2mi\theta}\left(\frac{\cot\theta-i}{\cot\theta+i}\right)^m$

$= e^{2mi\theta}\left(\frac{\cos\theta-i\sin\theta}{\cos\theta+i\sin\theta}\right)^m$

$= e^{2mi\theta}\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m = e^{2mi\theta}\left(e^{-2i\theta}\right)^m = e^{2mi\theta}e^{-2mi\theta} = e^0 = 1 = \text{R.H.S}$

# Chapter Ends Here