FUNCTIONS

A function is defined as a relation between an independent variable (say 'x') and a dependent one (say 'y') such that the relation expresses a unique value for y for each value of x. Functions play an important role in Physics to mathematically define the relationship between different physical quantities.

For example, the temperature at which water boils depends on the elevation above sea level (the boiling point drops as one ascends).The displacement of an object depends on the amount of time it has spent in motion. The force required to further stretch or compress a given spring at any instant depends on the elongation or compression in the spring at that moment. Similarly, the volume of a sphere depends on the diameter of the sphere. In each case, the value of one variable quantity depends on the value of another variable quantity.

Mathematically, a function is represented as: $y = f(x)$

The set of all possible input values for x (independent variable) is called the **domain** of the function. The set of all output values of y (dependent variable) is the **range** of the function.

Examples of common functions

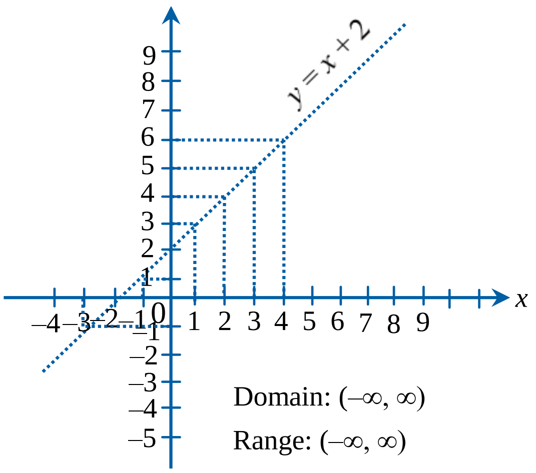
Let's take an example in which dependent variable varies with independent variable as,

$y = f(x) = x + 2$. This is an example of a Linear Function

Now, to represent the function geometrically, we can first define a set of ordered pairs of values of (x, y)

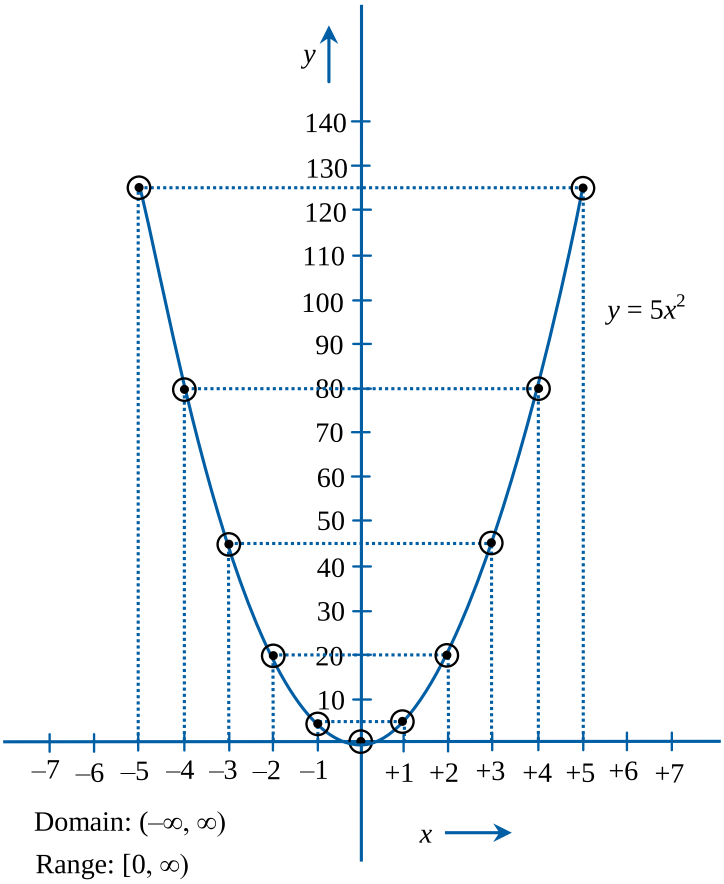
x	0	1	2	3	4	–1	–2	–3
y	2	3	4	5	6	1	0	–1

Now fit them into a "best fit" curve or graph on the X-Y Cartesian plane for a geometrical representation of the function.



Next let's take the simple quadratic function $y = 5x^2$

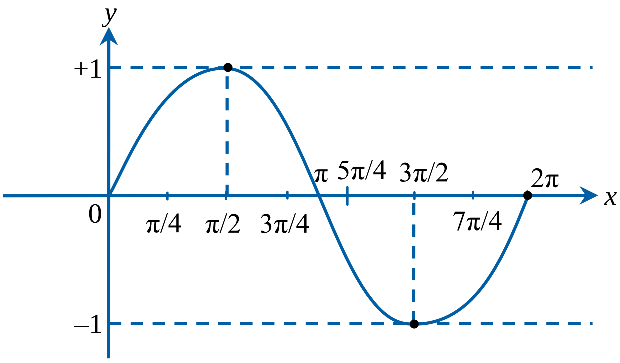
x	0	±1	±2	±3	±4	±5
y	0	5	20	45	80	125



Such a shape of graph is called a "parabola" in co-ordinate geometry.

Next, consider a trigonometrically defined function, $y = \sin x$

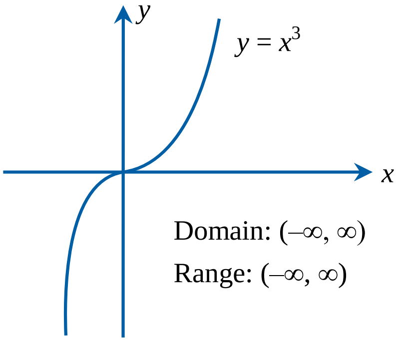
x	0	30° or $\pi/6$	45° or $\pi/4$	60° or $\pi/3$	90° or $\pi/2$	120° or $2\pi/3$	135° or $3\pi/4$
y	0	1/2	$1\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1\sqrt{2}$
x	150° or $5\pi/6$	180° or π	210°	225°	240°	270°	300°
y	1/2	0	–1/2	$-1\sqrt{2}$	$-\sqrt{3}/2$	–1	$\sqrt{3}/2$
x	315°	330°	360°				
y	$-1/\sqrt{2}$	–1/2	0				



Domain: (–∞, ∞)

Range: [–1, 1]

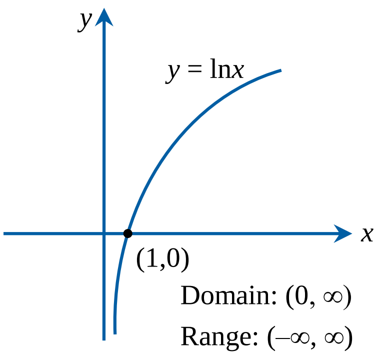
Next: $y = x^3$



Domain: (–∞, ∞)

Range: (–∞, ∞)

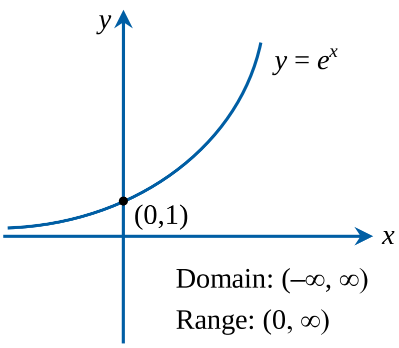
Next consider the Logarithmic function $y = \ln(x)$



Domain: (0, ∞)

Range: (–∞, ∞)

And the Exponential function



Domain: (–∞, ∞)

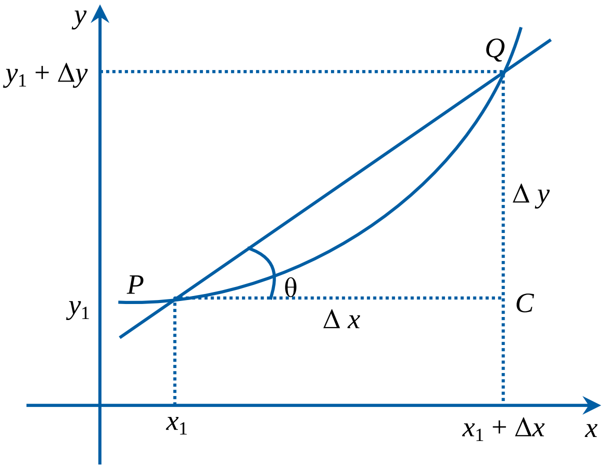
Range: (0, ∞)

RATE OF CHANGE

A very common calculation required in several areas of Physics is the "rate of change" of one parameter with respect to another. For example one might need to calculate the rate of change of Power consumed with Voltage applied, or rate of change of Velocity with time (acceleration) etc.

Consider an arbitrary function

y = f(x), the graph of which is as below.



If two points on the graph are $p(x_1, y_1)$ and $Q(x_1 + \Delta x, y_1 + \Delta y)$, then the average rate of change of y with respect to x over the interval Δx

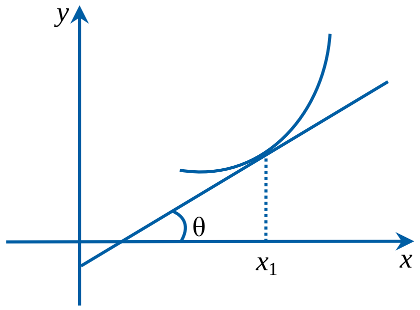
$$= \frac{(y_1 + \Delta y) - y_1}{(x_1 + \Delta x) - x_1} = \frac{f(x_1 + \Delta x) - f(x_1)}{(x_1 + \Delta x) - x_1}$$

$$= \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

This also represents the slope of the "secant line" PQ.

$$\frac{\Delta y}{\Delta x} = \frac{BC}{AC} = \tan\theta$$

However, the "instantaneous" rate of change may vary from point to point between P and Q. Thus, to know the rate of change of y at a particular value of x, say at P, we have to take very small value of Δx. As we make the value of Δx smaller and smaller, the slope of the line PQ approaches the slope of the "tangent" at P.



This slope of the tangent at P thus gives the rate of change of y w.r.t. x at P. It is represented as $|dy/dx|_{x=x_1}$ .It is defined as the derivative of y w.r.t. x at the point $x = x_1$. The derivative of function $y = f(x)$ is also represented in short as $f'(x)$.

Mathematically,

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

Note:

If the value of the function y increases with an increase in x at a point, the slope of the tangent i.e. (dy/dx) is positive there and if the function decreases with an increase in x at a point, the slope of the tangent is negative at that point.

At the points on the curve $y = f(x)$, where

$dy/dx > 0$: Slope of the tangent is positive.

$dy/dx < 0$: Slope of the tangent is negative.

$dy/dx = 0$: Slope of the tangent is zero. i.e. tangent is parallel to the X-axis.

Example:

A stone falls from the top of a 150-ft cliff.

(a) What is its average speed during the first 3 sec of fall?

Take $g = 10$ m/s^2

Use the equation for vertical displacement:

$$y=ut+\frac{1}{2}gt^2$$

$u = 0$

Average speed = Average rate of change of position w.r.t time

(b) What is the speed of the stone at t = 1 sec?

Instantaneous speed = Rate of change of position w.r.t time at given moment $=\left.\frac{dy}{dt}\right|_{t=t_1}$

Solution:

(a) $y=\frac{1}{2}gt^2=5t^2$

Distance travelled in first 3 seconds $=\frac{1}{2}\times 10\times 3^2=45m$

Average speed during first 3 sec $=\frac{\Delta y}{\Delta t}=\frac{45}{3}=15$

(b) We let $y = f(t) = 5t^2$

The average speed of the stone over the interval between $t = 1$ and $t = 1+\Delta t$ seconds

$$\frac{f(1+\Delta t)-f(1)}{\Delta t}=\frac{5(1+\Delta t)^2-5(1)^2}{\Delta t}$$

$$=\frac{5\Delta t(2+\Delta t)}{\Delta t}=5(2+\Delta t)$$

The stone's speed at the instant $t = 1$ sec

$$=\lim_{\Delta t\to 0} 5(2+\Delta t)$$

Or v=5 m/s

Example:

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$.

Solution:

Let's take the two nearby points $P(2, 4)$ and $Q(2+\Delta x,4+\Delta y)$ on the parabola.

Slope of the secant line

$$PQ=\frac{\Delta y}{\Delta x}=\frac{(2+\Delta x)^2-2^2}{\Delta x}$$

$$=\frac{4+4\Delta x+(\Delta x)^2-4}{\Delta x}$$

$$=4+\Delta x$$

As Q approaches P along the curve, Δx approaches zero and secant slope approaches 4.

$$\lim_{\Delta x\to 0}(4+\Delta x)=4$$

So the slope of the tangent at point $P(2, 4)$ on the given parabola = 4.

DERIVATIVES OF SOME STANDARD FUNCTIONS:

Derivative of a constant:

If C is constant, then $\frac{dC}{dx}=0$

Proof:

We apply the definition of derivative to $f(x) = C$, C = Constant

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{C-C}{\Delta x}=\lim_{\Delta x\to 0} 0=0$$

Derivative of function $f(x) = x^n$; n is a positive integer.

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

Proof:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[(x+\Delta x)^{n-1} - (x+\Delta x)^{n-2} \cdot x + \ldots + (x+\Delta x) \cdot x^{n-2} + x^{n-1}]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2} \cdot x + \ldots + (x+\Delta x) \cdot x^{n-2} + x^{n-1}]$$

$$nx^{n-1}$$

Note:

Derivative of above two functions illustrated have been proved by the definition of the derivative (rate of change at particular point).

However, you should remember the result at this stage. Below are some more standard functions whose derivative (result) should be remembered as to use them as the tool to solve the problem wherever required.

Here n is any real number including negative numbers

$$\frac{d(x^n)}{dx} = n\,x^{n-1}$$

trigonometric function: sin function

$$\frac{d(\sin x)}{dx} = \cos x$$

cosine function

$$\frac{d(\cos x)}{dx} = -\sin x$$

tan function

$$\frac{d(\tan x)}{dx} = \sec^2 x$$

cot function

$$\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

cosec function

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cdot \cot x$$

sec function

$$\frac{d(\sec x)}{dx} = \sec x \cdot \tan x$$

Logarithmic function

$$\frac{d(\ln x)}{dx} = \frac{1}{x}$$

Exponential function

$$\frac{d(e^x)}{dx} = e^x$$

Example:

What is the derivative of function $y = x^3$?

Solution:

$$\frac{dy}{dx} = 3x^2$$

Example:

What is the derivative of function $y = \frac{1}{x^5}$

Solution:

$$y = \frac{1}{x^5} = x^{-5}$$

$$\therefore \frac{dy}{dx} = (-5)x^{-5-1}$$

$$= -5x^{-6}$$

RULES OF DIFFERENTIATION

CONSTANT MULTIPLE RULE

Let $f(x)$ & $g(x)$ be two functions such that $f(x) = cg(x)$; where c is a constant.

Then,

$$\frac{d}{dx} f(x) = \frac{d}{dx}[c\,g(x)] = c\frac{d}{dx} g(x)$$

This rule says that when a function is multiplied by a constant, its derivative is also multiplied by the same constant.

SUM RULE

The derivative of the sum of two functions is the sum of their derivatives.

If $f(x) = g(x) + h(x)$, then

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x) + h(x)]$$

$$\frac{d}{dx} g(x) + \frac{d}{dx} h(x)$$

$$\Rightarrow f'(x) = g'(x) + h'(x)$$

This rule can be extended to any number of functions, i.e if $f(x) = g_1(x) + g_2(x) + ... + g_n(x)$

$f'(x) = g_1'(x) + g_2'(x) + ... + g_n'(x)$

Example: $y = x^2 + x + 1$

$$\frac{dy}{dx} = 2x + 1 + 0 = 2x + 1$$

Example: $y = 2x^3$

$$\frac{dy}{dx} = 2(3x^2) = 6x^2$$

Example: $y = 2x^2 - x + 1$

$$\frac{dy}{dx} = 4x - 1 + 0$$

Example: $y = \ell_n x + x^2$

$$\frac{dy}{dx} = \frac{1}{x} + 2x$$

Example: $y = \sin x - \frac{1}{x^3}$

$$\frac{dy}{dx} = \cos x + 3x^{-4}$$

Example: The Length of a rectangle at any time t is given by $(t^2 + 1)m$ and the breadth of the rectangle is given by $(\sin t)m$. Find the rate at which perimeter p changes with time at $t = \frac{\pi}{6}\text{sec}$

$p = 2(t^2 + 1 + \sin t)$

$= 2t^2 + 2 + 2 \sin t$

$$\frac{dp}{dt} = 4t + 2\cos t$$

$$\frac{dp}{dt} = \frac{4\pi}{6} + 2\frac{\sqrt{3}}{2} = \frac{2\pi}{3} + \sqrt{3} \text{ (m/s)}$$

PRODUCT RULE

$$\frac{d(f(x)\cdot g(x))}{dx} = f(x)\left[\frac{d(g(x))}{dx}\right] + g(x)\left[\frac{d(f(x))}{dx}\right]$$

$$\frac{d(f(x)\cdot g(x)\cdot hx)}{dx} = d\ f(x)g(x)\frac{d(h(x))}{dx} + f(x)h(x)\frac{d(g(x))}{dx}$$

Example: $y = x(\sin x)$

$$\frac{dy}{dx} = x\cos x + \sin x(1) = x\cos x + \sin x$$

Example: $y = t(t^2 - 1)(\sin t)$

$$\frac{dy}{dt} = (1)(t^2 - 1)(\sin t) + t(2t)\sin t + t(t^2 - 1)\cos t$$

$= (t^2 - 1)\sin t + 2t^2 \sin t + t(t^2 - 1)\cos t$

$= \sin t(3t^2 - 1) + t(t^2 - 1)\cos t$

QUOTIENT RULE

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{g(x)\frac{d(f(x))}{dx} - f(x)\frac{d(g(x))}{dx}}{(g(x))^2}$$

Example: $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\cos x - (\sin x)(1)}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Example: Angular position 'θ' of a particle is given as

$$\theta = \frac{t^3}{3} - t^2 + t + 1$$

(i) Find the time at which angular velocity of particle is zero.

(ii) Also find the value of angular acceleration at that time.

(Angular velocity $\omega = \frac{d\theta}{dt}$)

Angular acceleration, $\alpha = \frac{d\omega}{dt}$

(i) $\omega = \frac{d\theta}{dt} = \frac{3t^2}{3} - 2t + 1$

$= t^2 - 2t + 1$

$w = 0$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0$

$t - 1 = 0$

$t = 1$ sec.

(ii) $\alpha = \frac{d\omega}{dt} = 2t - 2$

$\alpha_{t=1} = 2(1) - 2$

$= 0$ m/s^2

Example: Position vector of a particle as a function of time is given as

$\vec{s} = 2t^3\hat{i} + (1-t^2)\hat{j} - t^3\hat{k}$

(i) Find velocity $\vec{v}$ at any time t.

(ii) Find magnitude of velocity at t = 2 sec.

(iii) If mass of particle is 2 kg, find force acting on it.

(i) $\vec{v} = \frac{d\vec{s}}{dt} = 6t^2\hat{i} - 2t\,\hat{j} - 3t^2\hat{k}$

$\vec{v}_{at\ t=2} = 6(2)^2\hat{i} - 2(2)\,\hat{j} - 3(2)^2\hat{k} = 24\hat{i} - 4\,\hat{j} - 12\hat{k}$

(ii) $v = \sqrt{(24)^2 + (4)^2 + (12)^2}$

(iii) $\vec{a} = \frac{d\vec{v}}{dt} = 12t\ \hat{i} - 2\,\hat{j} - 6t\ \hat{k}\ \ \vec{F} = m(\vec{a}) = (24t\ \hat{i} - 4\,\hat{j} - 12t\ \hat{k})N$

Chain Rule OR Method of Substitution

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ Where $y = f(t)$ and $t = g(x)$

This is effective for functions which are "composite" in nature, i.e. $y = f\{g(x)\}$ or$g(x)$ is a function of x, say "t" $\{t = g(x)\}$ and y is a function f of $g(x)$ [or $y = f(t)$]

Example: $y = (3x + 1)^9$, find $\frac{dy}{dx}$

Substitute $t = 3x + 1$

Now, $\frac{dt}{dx} = 3$

And since $y = t^9 \Rightarrow \frac{dy}{dt} = 9t^8$

Now, $\frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt} = 3(9t^8) = 9(3x + 1)^8$

Example: $y = \sin(x^2)$ find $\frac{dy}{dx}$

$t = x^2$, $y = \sin t$

$\frac{dt}{dx} = 2x, \frac{dy}{dt} = \cos t$

$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = 2x \times \cos t$

Or $\frac{dy}{dx} = 2x\cos(x^2)$

Example: $y = [\ell n(\sec x)]$

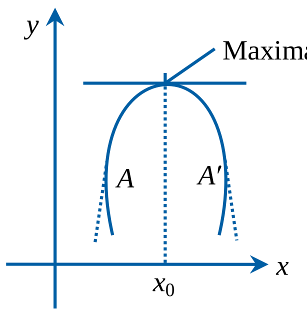
$\frac{dy}{dx} = \left(\frac{1}{\sec x}\right)\sec x \tan x = \tan x$

Example: $y = (ax^2 + b)^n$

$\frac{dy}{dx} = \left[n(ax^2 + b)^{n-1}\right] 2ax$

Maxima and Minima (An application of differentiation)

(i) Maxima



$\frac{dy}{dx} = 0$ at maxima

We can say that slope at A's left is +ve

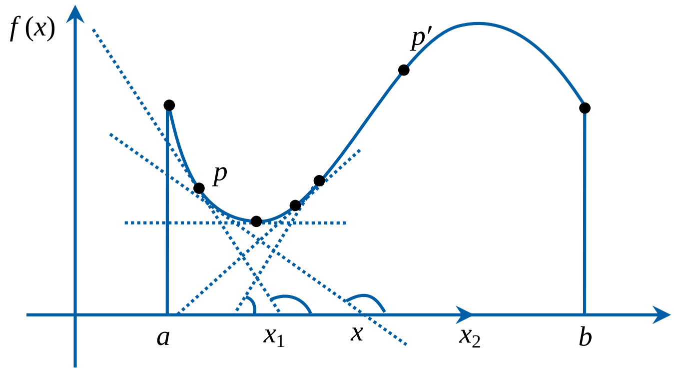
Slope at maxima is 0

Slope at A's left is –ve

In a region around $x = x_0$, the rate of change of slope is –ve

$$\frac{d\left(\frac{dy}{dx}\right)}{dx}<0 \quad \text{or} \quad \frac{d^2y}{dx^2}<0$$

(ii) Minima



$\frac{dy}{dx}=0$ at minima

At minima, the slope is increasing as it is negative before the minima and positive after it

Therefore, $\frac{d^2y}{dx^2}>0$

Example: $y = x^2 - 2x + 4$

Find maxima or minima exist for this curve

$$\frac{dy}{dx}=2x-2$$

Therefore, slope dy/dx is 0 at $2x - 2 = 0$

or $2x = 2$

or $x = 1$

$\frac{d^2y}{dx^2}=2$, which is positive, which means that at x=1 slope is zero and it is increasing, hence there is a local minima at x=1

Example: A quantity y as a function of t is given as

$$y=\frac{t^3}{3}-t^2-3t+1$$

Find the points where maxima or minima can exist. Which of these points are points of maxima/minima?

$$\frac{dy}{dx}=t^2-2t-3$$

Therefore, slope is zero at

$t^2 - 2t - 3 = 0$

$t^2 - 3t + t - 3 = 0$

$t(t-3) + 1\,(t-3) = 0$

$(t+1)\,(t-3) = 0$

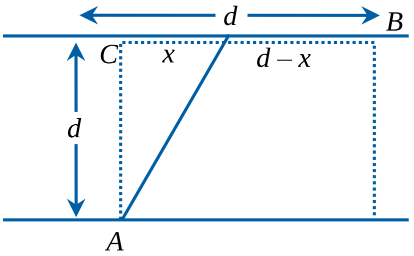
$t = -1, \quad t = 3$

$$\frac{d^2y}{dx^2} = 2t - 2$$

Therefore, at $t = -1$ second derivative is -4 (negative) : Hence Maxima exists

And at $t = 3$; second derivative is positive: Hence Minima exists

Example: A square pool with width d is shown



A person located at point A wants to reach point B. If his swimming speed is $(v/2)$ and the running speed is v. Find the shortest time in which he can reach A to B

$$t_{AC} = \frac{(d^2+x^2)^{1/2}}{\frac{v}{2}}, \quad t_{CB} = \frac{d-x}{v},$$

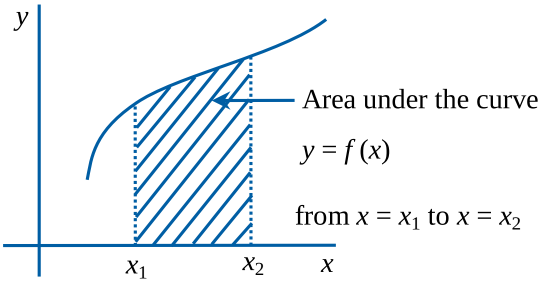
$$t = \frac{2}{v}(d^2+x^2)^{1/2} + \frac{d-x}{v}$$

$$\frac{dt}{dx} = \frac{2}{v}\left[\frac{1}{2}(d^2+x^2)^{-1/2}\right](2x) - \frac{1}{v}$$

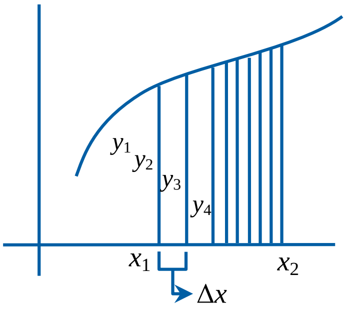
For t to be a minima $(dt/dx) = 0$ or $0 = \dfrac{2x}{v\sqrt{d^2+x^2}} - \dfrac{1}{v}$

$$2x = \sqrt{d^2+x^2} \text{ or } x = \frac{d}{\sqrt{3}}$$

Integral Calculus (Area Calculator)



The shaded area can be calculated by dividing the region into rectangular slabs of width Δx each



Area can be calculated by following expression, Area = $\sum_{i=1}^{n} y_i \, \Delta x$

This area will be an approximated one because a small portion of area is neglected with every slab. To reduce the approximation we need to use calculus *i.e.* take a limit on

$$\text{Area}= \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i \, \Delta x$$

Principal of Integral Calculus:

The variation of quantity y with x can be neglected during a very small interval of x

$$\text{Area}= \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i \, \Delta x = \int_{x_1 \leftarrow \textit{lower}\text{ limit}}^{x_2 \leftarrow \textit{upper}\text{ limit}} y \, dx$$

Or $\int_{x_1}^{x_2} f(x)\, dx$ equals the area under the graph $y = f(x)$ from x_1 to x_2

Integration as a tool in Physics

We have seen that differentiation is a rate measurer for a function *i.e.* it tells us how a function (x) changes with x.

Integration is a process whereby the “anti-derivative” of a function can be calculated, or in other words, if we know the derivative of a function, the process of integration will give us the function itself.

For applications in Physics, the velocity of a moving body is expressed as the rate of change of displacement with respect to time. Conversely, the displacement can be calculated as the integration of velocity with respect to time.

Net force of a body for example is defined as the rate of change of momentum, conversely, the change in momentum can be calculated by integrating the net force with respect to time.

Step1:

Involves developing a method to calculate the so called anti derivatives of various types of functions, and this process is called indefinite integration.

Step 2:

Then applying the limits over which the integration has to be calculated on the anti-derivative to calculate the required quantity. This process is called Definite Integration which is technically used to calculate the area enclosed by the graph of a function say $y = f(x)$ over a specified range of values of x (limits).

1. INDEFINITE INTEGRALS:

DEFINITION

A function $F(x)$ is an anti-derivative of a function $f(x)$, if $F'(x)=(x)$: for all x in the domain of $f(x)$

The set of all anti-derivatives of $f(x)$is the indefinite integral of $f(x)$ denoted by $\int f(x)dx$

The symbol $\int$ is the integral sign. The function f is the integrand of the integration and x is the variable of integration. Therefore, $\int f(x)dx = F(x)+C$

The constant C is the constant of integration or arbitrary constant.

Example: Evaluate $\int 2xdx$

Solution: Here $\int 2xdx = x^2 + C$

x^2 is an anti-derivative of $2x$ since we know that $\frac{d}{dx}(x^2) = 2x$

C is the arbitrary constant.

The formula generates all the anti-derivatives of the function $2x$.

Anti-derivatives of some common functions

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ell n\, x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \quad \int e^x dx = e^x + C$$

Some Properties of Integration

$$\int K\, f(x)\, dx = K \int f(x)\, dx$$

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Definite Integral

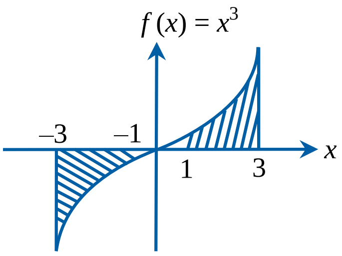
Let *f*(*x*) be a function defined on a closed interval [*a*, *b*]. Then, $\int_a^b f(x)dx$ read as the definite integration of *f*(*x*) from $x = a$ to $x = b$ represents the algebraic sum of all areas of the regions bounded by the curve $f(x)$, the *x*-axis and the straight lines $x = a$ and $x = b$.

$\int_a^b f(x)dx = F(x) + C\Big|_a^b = F(b) - F(a)$, here *F*(*x*) is the anti-derivative of the function *f*(*x*) which we have learnt how to compute using the method of indefinite integration.

Note:

1. The areas above the *x*-axis are +ve in sign
2. The areas below the *x*-axis are –ve in sign.

Example:



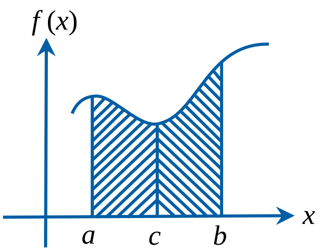
$$\int_1^3 f(x)dx = \int_1^3 x^3 dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = 20(+ve)$$

$$\int_{-3}^{-1} f(x)\,dx = \int_{-3}^{-1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-3}^{-1} = \frac{(-1)^4}{4} - \frac{(-3)^4}{4} = -20\,(-ve)$$

$$\int_{-3}^{3} f(x)dx = \int_{-3}^{3} x^4 dx = \left[\frac{x^4}{4}\right]_{-3}^{3} = \frac{(3)^4}{4} - \frac{(-3)^4}{4} = 0$$

(Since from $x = -3$ to $+\,3$, area below the curve and area above the curve are equal in magnitude but opposite in sign, hence they nullify each other)

PROPERTIES OF DEFINITE INTEGRAL



1. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^b f(t)dt$

Example: Velocity of a particle varies with time as $v = t^2$. Find the distance travelled by particle during 0 to 4 sec.

Solution: Let us take a time t at which the velocity is v

$0 < t < 4$

For the next very small time interval dt has us assume the velocity constant.

During $t + dt$ distance travelled is dx

$dx = vdt$

$$\int_0^x dx = \int_0^4 t^2 dt$$

$$x\Big|_0^x = \frac{t^3}{3}\Big|_0^4$$

$$x - 0 = \frac{4^3}{3} - \frac{0^3}{3}$$

$$\boxed{x = \frac{64}{3}}$$

Example: A particle has velocity $v = 4t^3$. Find distance travelled by particle during 2 to 3 sec. (ii). If at t= 0 the particle is located at x = 1, find position of particle as function of time

Solution: $v = 4t^3$

or $\frac{dx}{dt} = 4t^3 \Rightarrow dx = 4t^3 dt$

or $\int_a^b dx = 4\int_2^3 t^3 dt$

or distance = $(b-a) = 4\left(\frac{t^4}{4}\right)\Big|_2^3 = 65$

(ii) We will keep upper limits variable

$dx = 4t^3 dt$

or $\int_1^x dx = 4\int_0^t t^3 dt$

or $x\big|_1^x = t^4\big|_0^t$ or $x = t^4 + 1$

Example: Velocity of a particle is given as $v = 2e^t$. If particle is located at $x = 1$ at initial time. Find position of particle as function of time

Solution: $v = 2e^t$

Or $dx = 2\, e^t\, dt$

or $\int_1^x dx = 2\int_0^t e^t dt$ or $x\,|_1^2 = 2e^t\,|_0^t$

or$x = 2\,e^t - 1$

Example: Velocity of particle varies with position as $v = \sqrt{x}$.

(1) Find position as function of time, if particle starts from origin.

(2) Find velocity as function of time.

(3) Find acceleration.

Solution: $v = \sqrt{x}$ or $\dfrac{dx}{dt} = \sqrt{x}$

or $\int_0^x x^{-1/2} dx = \int_0^t dt$

or $\left.\dfrac{x^{1/2}}{1/2}\right|_0^x = t$

or $2\sqrt{x} = t$

or $\boxed{x = \dfrac{t^2}{4}}$

VECTORS AND VECTOR ALGEBRA

Measurement of Physical Quantities is an important aspect of Physics and so are addition, subtraction and other such mathematical operations involving them.

Interestingly, if we think about adding the masses of two piles of sand, or distances covered in two successive intervals of time, intuitively we just sum them up like a couple of numbers. However if we consider the addition of two forces acting simultaneously on a body lying on a smooth surface, or the addition of two successive displacements, the operation is not so straightforward and we immediately realize that information about the directions (or geometrical orientations) of the pair of forces or displacements in essential for the calculation.

Based on the above idea of whether or not the direction in which a quantity acts is essential during mathematical operations with it, all physical quantities (with a few exceptions called “tensors” which are not relevant at this stage of our study) are categorized as either scalars or vectors.

SCALARS

Physical quantities that can be completely described by information about their magnitude alone are classified as scalars. For *eg*., mass, distance travelled, volume, time, speed, current, energy, moment of inertia, temperature, power

The description of a scalar is thus done by representing it’s magnitude as a number times a standard unit, like Mass = 12 Kilograms

VECTORS

Certain physical quantities on the other hand cannot be described completely by just their magnitude as information about their direction is also required. Such quantities are classified as vectors.

For instance, we may say that a Displacement = 10 km towards North. Notice here that merely saying that Displacement = 10 km would not have been a complete description of it.

Common Vectors: Displacement, velocity, force, acceleration, momentum, weight, Torque.

The symbol used for a physical quantity which is a vector is drawn with an “arrow”, for example $\vec{v}$ for velocity or $\vec{F}$ for force. The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and represented by $|\vec{A}|$.

Note:-

A physical quantity can be classified as a vector only if -

(a) in addition to magnitude, it has a specified direction.

(b) it obeys the laws of parallelogram law of addition (vector addition).

(c) and it’s addition is commutative *i.e.* $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ then and then only it is said to be a vector. If any of the above conditions is not satisfied the physical quantity cannot be classified as a vector.

So if a physical quantity is a vector it has a direction, but the converse may or may not be true, *i.e.* if a physical quantity has a direction, it may or may not a be vector. *e.g.* time, pressure, surface tension or current etc. have directions but are not vectors.

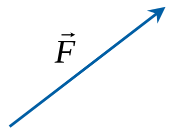
For instance, electric current has direction (since it flows from higher to lower potential), however when adding two currents, they are simply summed algebraically and not vectorially and hence it's a scalar.

REPRESENTATION OF VECTORS

Representation of a vector on the other hand requires a complete definition of it's magnitude and direction, such as

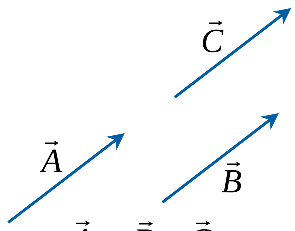
$\vec{s}$ = 30m due North, or $\vec{F}$ = 10 Newtons vertically downwards.

One of the standard approaches to vector algebra is by the application of the "geometrical method of representation", wherein a vector quantity is simply represented by an "arrow", the length of which is proportionate to it's magnitude and the direction of the arrow "head" pointing along the vector's direction



Important Points:

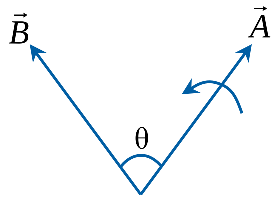
- If a vector is displaced parallel to itself it does not change



$\vec{A} = \vec{B} = \vec{C}$

Transition of a vector parallel to itself

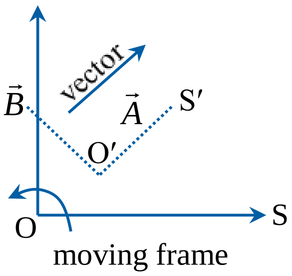
- If a vector is rotated through an angle other than a multiple of 2π (or 360^o) it changes.



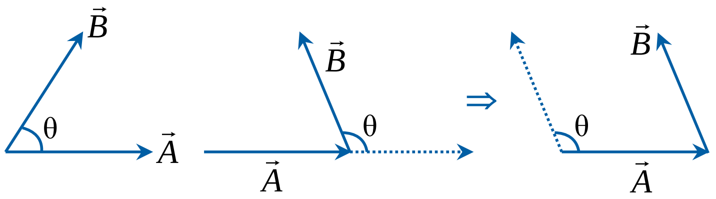
$\vec{A} \neq \vec{B}$

Rotation of a vector

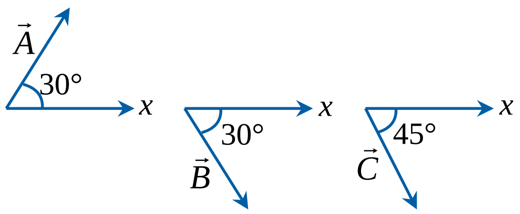
- If the frame of reference is translated or rotated the vector does not change (though its components may change).



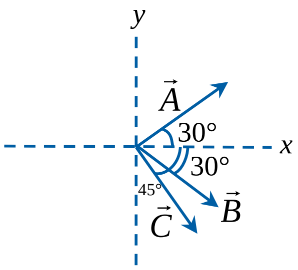
- Two vectors are called equal if their magnitudes and directions are same, and they represent values of same physical quantity.
- The Angle between two vectors is the smaller of the two angles between the vectors as observed when they are drawn from a common origin.



Example: Three vectors $\vec{A}, \vec{B}, \vec{C}$ are shown in the figure. Find angle between (i) vectors $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.



Solution: The vectors can be shifted such that all three are drawn from a common origin with the same lengths and orientations as in the above diagram.

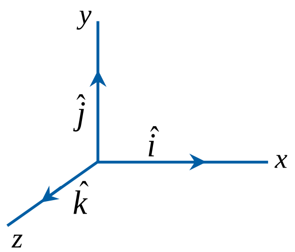


Now we can easily observe that angle between vectors $\vec{A}$ and $\vec{B}$ is 60º, between $\vec{B}$ and $\vec{C}$ is 15º and between $\vec{A}$ and $\vec{C}$ is 75º.

TYPES OF VECTORS

Unit Vectors: A vector whose magnitude is 1(unity) is known as a unit vector. Unit vectors are denoted by a special symbol $\hat{A}$. Therefore a vector $\vec{A}$ can be expressed as $\vec{A} = A\hat{A},$ where A is it's magnitude and $\hat{A}$ is a unit vector along $\vec{A}$.

Standard Unit vectors along the positive *x*, *y* and *z*-axes of a rectangular coordinate system are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$,



Null Vector: A vector whose magnitude is 0 is known as null vector. A null vector has got arbitrary direction. *eg*. Displacement $\vec{s} = \vec{0}$

Parallel or Anti Parallel Vectors: If the angle between two vectors is zero the vectors are said to be parallel *i.e*. they will have same direction.

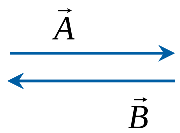
If the angle between two vectors is 180° or π then are known as anti parallel vectors.

Parallel and anti-Parallel vectors are collectively called "co-linear" vectors.

For example

$$\left.\begin{array}{ll} \vec{a} = 2m/\sec & \text{(east)} \\ \vec{b} = 5m/\sec & \text{(east)} \end{array}\right\} \text{parallel}$$

Note: If $\vec{A} = -\vec{B}$, it implies that the two vectors have identical magnitude $|\vec{A}| = |\vec{B}|$ and opposite directions



Co-planar vectors: If multiple vectors (three or more) lie on a common two-dimensional plane then they are defined co-planar.

Polar Vectors and Axial Vectors

Vectors that represent physical quantities which act along a "line of action" are termed Polar. For example force, momentum, displacement, velocity etc

Certain types of physical quantities are associated with rotation about an axis, such as angular velocity, torque, angular momentum, angular acceleration etc. Such quantities when expressed vectorially are termed "polar" vectors and the direction of the vector specifies the axis of rotation.

For example, if a fan is rotating suspended from a ceiling such that on inspection from below it, the blades appear to rotate clockwise, it's angular velocity will be assigned a direction vertically upwards (along its axis of rotation).

VECTOR OPERATIONS

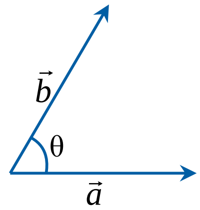
MULTIPLICATION BY A SCALAR

If $\vec{A}=m\vec{B}$ it implies that $|\vec{A}|=|m||\vec{B}|$ and their directions are parallel if m is a positive scalar, whereas their directions are opposite if m is negative.

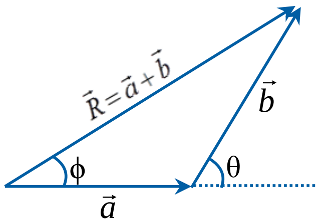
VECTOR ADDITION

Method (i): *Triangular law of vector addition*

Given two vectors (geometrically)

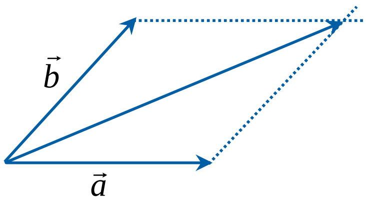


The given vectors are drawn in such a way, the tail of one coincides with the head of other. Now the vector Triangle is completed by drawing the third side. Mark out the arrow which points from initial tail to final head. This will give us the resultant vector. This method is valid for adding two vectors



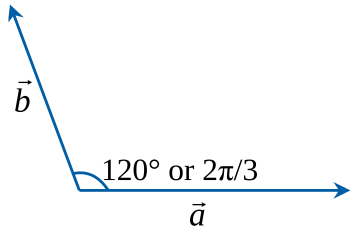
Method (ii): *Parallelogram law*

This is also used for adding two vectors.

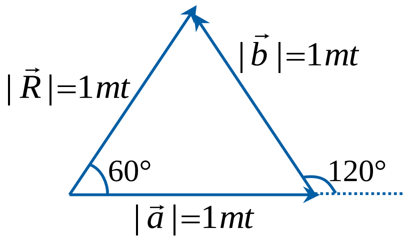


First, both vectors are drawn from a common origin, now a line from head of $\vec{b}$ parallel to $\vec{a}$ is drawn and another line from head of $\vec{a}$ parallel $\vec{b}$ to it is drawn. This will create a parallelogram. The drawn diagonal will give the resultant $\vec{R}=\vec{a}+\vec{b}$

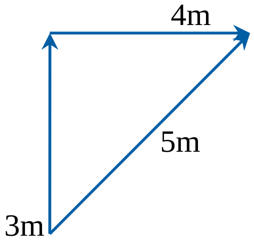
Example: Add the two vectors, given that both vectors have a magnitude of 1 and the angle between them is 120



Solution: Construction of the vector triangle for this example will result in an equilateral triangle as shown, hence the magnitude of ($\vec{a}+\vec{b}$) is also 1 and it's direction will be at 60° to $\vec{a}$.

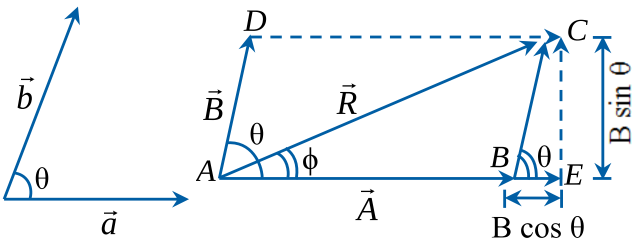


Example: A person makes a displacement of 3m towards north & then makes a displacement of 4m towards east. Find the net displacement? Plot the vectors.



Solution: Construction of the vector triangle shown above, results in a right angled triangle to be solved and the resultant of 3m due north and 4m due east works out to be 5m.

(b) Analytical method



By dropping perpendiculars from the opposite vertex of the vector parallelogram, we can observe geometrically, the magnitude of the resultant R as,

$$R^2 = B^2\sin^2\theta + (A + B\cos\theta)^2$$

$$\Rightarrow R^2 = B^2\sin^2\theta + A^2 + B^2\cos^2\theta + 2AB\cos\theta$$

$$\Rightarrow R^2 = A^2 + B^2(\sin^2\theta + \cos^2\theta) + 2AB\sin\theta$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

And, $\tan\phi = \dfrac{B\sin\theta}{A + B\cos\theta}$

Special Cases:

1. $\theta = 0°$, $\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos 0°} \Rightarrow R = \sqrt{A^2 + B^2 + 2AB} \Rightarrow R = A + B$

2. $\theta = 180°$ $\Rightarrow R = \sqrt{a^2 + b^2 + 2ab\cos 180°} \Rightarrow R = |a - b|$

3. $\theta = 90° \Rightarrow R = \sqrt{a^2 + b^2 + 2ab\cos 90°} \Rightarrow \sqrt{a^2 + b^2}$

Range of resultant

Since $R = \sqrt{a^2 + b^2 + 2ab\cos\theta}$, $-1 \le \cos\theta \le 1$

$R_{max} = \sqrt{a^2 + b^2 + 2ab} = (a + b)$ and $R_{min} = \sqrt{a^2 + b^2 - 2ab} = |a - b|$

Or $\big||\vec{a}| - |\vec{b}|\big| \le |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$, this inequality is called the 'triangle' inequality since it is defined by application of the triangle law of vector addition.

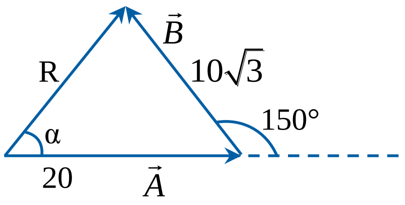
Example: Two forces of magnitudes 6N and 4N are act on the body. Which of the following can be the resultant of the two?

(1) 11 N
(2) 10 N
(3) 8 N
(4) 7 N
(5) 2 N
(6) 1 N
(7) 0

Solution: Since $2 \le R \le 10$, possible values of the resultant must lie within the range [2,10], hence Correct answers – (2), (3), (4), (5).

Example: The magnitudes of two forces are 20N and $10\sqrt{3}$ respectively and the angle between them is 150°, calculate the magnitude of the resultant R and the angle made by R with respect to the 20N force.

Solution:



By application of triangle law formula

$R = \sqrt{20^2 + (10\sqrt{3})^2 + (2\times 20\times 10\sqrt{3}\times\cos 150°)} = 10,$

And $\tan\alpha = \dfrac{10\sqrt{3}\sin 150°}{20 + 10\sqrt{3}\cos 150°} = \sqrt{3} \rightarrow \alpha = 60°$

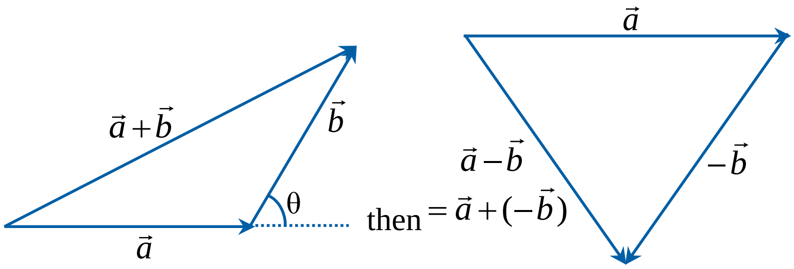
VECTOR SUBTRACTION

By triangle or parallelogram law

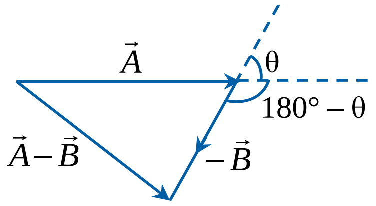
Subtraction can be treated as the addition of the negative of a vector,

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

i.e if



Then, by analytical method



$|\vec{A}-\vec{B}|=\sqrt{A^2+B^2+2AB\cos(180°-\theta)}=\sqrt{A^2+B^2-2AB\cos\theta}$, and the angle made by the vector $\vec{A}-\vec{B}$ with $\vec{A}$ is $\beta=\tan^{-1}\left[\dfrac{B\sin\theta}{A-B\cos\theta}\right]$

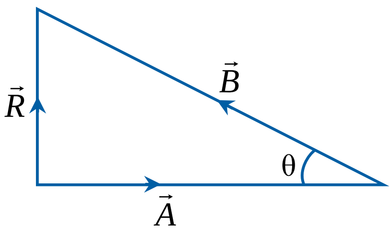
Example: Two non zero vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}-\vec{B}|=|\vec{A}+\vec{B}|$. Find angle between them?

Solution: $|\vec{A}-\vec{B}|=|\vec{A}+\vec{B}|$ $A^2+B^2+2AB\cos\theta = A^2+B^2-2AB\cos\theta$

or $4AB\cos\theta = 0$ or $\cos\theta = 0$ or $\theta = 90°$

Example: The resultant of two velocity vectors $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Magnitude of Resultant is equal to half magnitude of $\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$?

Solution: Since $\vec{R}$ is perpendicular to $\vec{A}$. Figure shows the three vectors



angle between vectors $\vec{A}$ and $\vec{B}$ is $\pi-\theta$,

Since $\sin\theta=\dfrac{R}{B}=\dfrac{1}{2}$, $\theta=30°$ and the required angle between vectors $\vec{A}$ and $\vec{B}$ is 150°.

Example: If the sum of two unit vectors is also a unit vector. Find the angle between them?

Solution: Let $\vec{A}$ and $\vec{B}$ are the given unit vectors and $\vec{C}$ is their resultant then

$$|\vec{C}| = |\vec{A}+\vec{B}|$$

$$1=\sqrt{1+1+2\cos\theta}$$

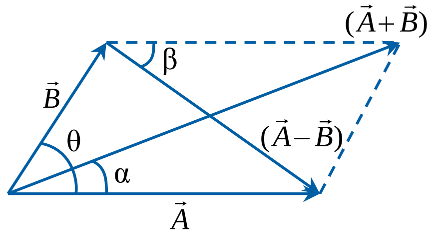
$$1=1+1+2\cos\theta$$

$$\cos\theta=-\frac{1}{2}\rightarrow\theta=120°$$

VECTOR PARALLELOGRAM AND SOME USEFUL PROPERTIES

By constructing a parallelogram with vectors $\vec{A}$ and $\vec{B}$ as adjacent side, several important properties and special cases can be understood.

Parallelogram with vectors $\vec{A}$ and $\vec{B}$



Note that on construction of the parallelogram; the diagonal from the origin of vectors $\vec{A}$ and $\vec{B}$ to the opposite vertex represents the sum while the diagonal from the 'tip' of vector $\vec{B}$ to that of $\vec{A}$ represents the difference.

Special cases:

1. If A = B, vectors $\vec{A}$ and $\vec{B}$ are of equal magnitude, then the parallelogram is infact a Rhombus, with the following important properties

Diagonals bisect perpendicularly $\rightarrow(\vec{A}+\vec{B})$ is perpendicular to $(\vec{A}-\vec{B})$

Diagonals bisect the angles between sides, or $\alpha = \theta/2$ and $\beta = 90° - \theta/2$

$|\vec{A}+\vec{B}|=2A\cos(\theta/2)$ and $|\vec{A}-\vec{B}|=2A\sin(\theta/2)$.

2. If $\theta = 90°$ or vectors $\vec{A}$ and $\vec{B}$ are perpendicular then the parallelogram is infact a Rectangle and $|\vec{A}+\vec{B}|=|\vec{A}-\vec{B}|=\sqrt{A^2+B^2}$

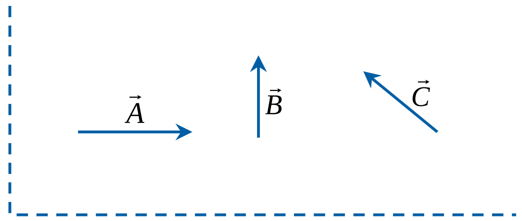
3. If $\theta < 90°$ then $|\vec{A}+\vec{B}|>|\vec{A}-\vec{B}|$ whereas

If $\theta > 90°$ then $|\vec{A}+\vec{B}|<|\vec{A}-\vec{B}|$

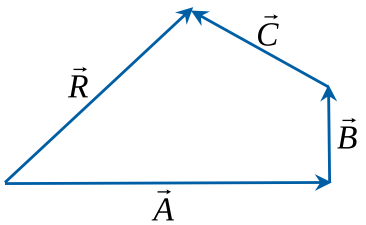
POLYGON LAW OF VECTOR ADDITION

Addition of more than two vectors can be done by an extension of the triangle law of vector addition called the polygon law, however this can be done geometrically only if all the vectors concerned are co-planar.

Example: Find the resultant sum for the three vectors shown in the diagram below



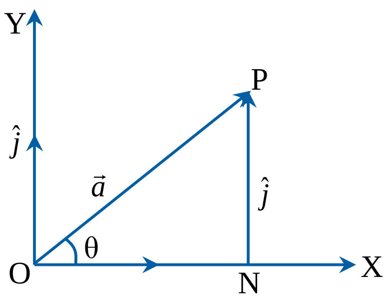
Solution: The vector arrows can be rearranged such that they are joined "head to tail" as below and the resultant will be the vector drawn from the tail of the first to the head of the last vector



COMPONENT ALGEBRA OF VECTORS

Alternate to the geometrical method of vector addition and subtraction is the 'component algebra' approach wherein a vector is expressed algebraically as the sum of mutually perpendicular components along rectangular co-ordinate axes (x, y and z axes) and operations can be implemented algebraically. The advantage of this method is that it can easily handle more than two vectors which are not co-planar.

RESOLUTION OF A VECTOR IN A TWO DIMENSIONAL PLANE



Consider the vector $\vec{a}$ drawn as the arrow OP in the diagram above. The length OP then represents it's magnitude a,

Now this vector can be expressed as the sum of two components ON and NP respectively along the X and Y axis. Geometrically ON = a cos θ and PN = a sin θ. Using the standard unit vectors $\hat{i}$ and $\hat{j}$ along the X and Y axes respectively, the vector $\vec{a}$ can be expressed in the component form as

$$\vec{a} = (a\ \cos\ \theta)\hat{i} + (a\ \sin\theta)\hat{j}$$

The coefficients of $\hat{\text{i}}$ and $\hat{\text{j}}$ are called X and Y components of the vector and are denoted by a_x and a_y respectively.

Here, $a_x = a \cos \theta$, $a_y = a \sin \theta$

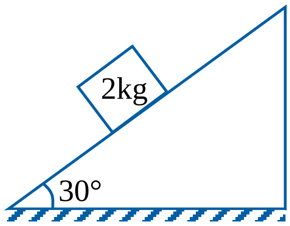
Hence, $\vec{a} = a_x\hat{i} + a_y\hat{j}$

Geometrically, these are the projections of the vector $\vec{a}$ along two coordinate axes. Conversely, if we know the components of a vector, we can find the magnitude and direction of the vector.

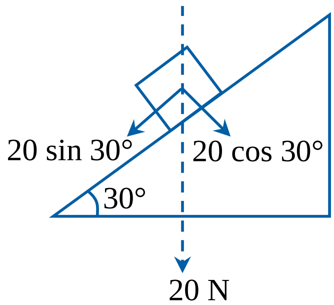
$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} \text{ and } \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right)$$

(Please note that if angle is measured anti-clockwise, it is taken as positive and if measured clockwise, it is taken as negative.)

Example: A mass of 2 kg lies on an inclined plane as shown in figure. Resolve its weight along and perpendicular to the plane. (Assume g = 10 m/s^2)



Solution: Weight = mg = 20 Newtons

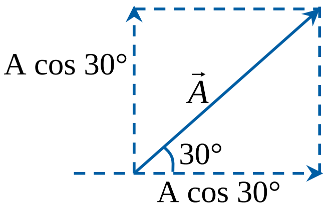


Component along the plane = 20 sin 30 = 10 N

Component perpendicular to the plane = 20 cos 30 = 10N

Example: A vector makes an angle of 30º with the horizontal. If horizontal component of the vector is 250, find magnitude of vector and its vertical component?

Solution: Let vector is



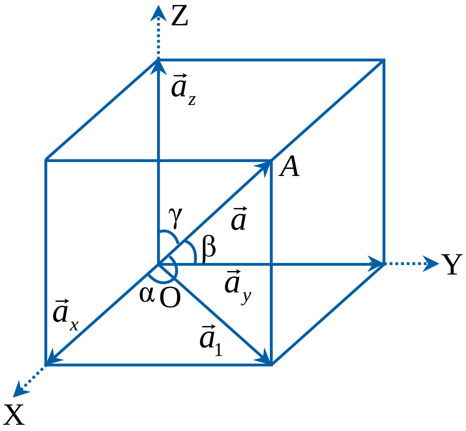
$A_x = A \cos 30° = 250$ ∵Þ $A = 500/\sqrt{3}$

$A_y = A \sin 30° = 500/\sqrt{3} \times 1/2 = 250/\sqrt{3}$

VECTORS IN THREE DIMENSIONAL COMPONENT NOTATION

In three dimensions, a vector would be expressed as the sum of three mutually perpendicular components. Let $\hat{i}, \hat{j}$ and $\hat{k}$ be unit vectors parallel to X, Y and Z axes, respectively. Then,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$



$$|\vec{a}| = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

Direction Cosines:

The angles which $\vec{a}$ make with *x*, *y* and *z* axis are α, β and γ respectively.

The direction cosines of the vector are accordingly defined as

$$\cos\gamma = \frac{a_z}{a}, \quad \cos\beta = \frac{a_y}{a} \quad \text{and } \cos\alpha = \frac{a_x}{a}$$

Example: Prove that $\cos^2\gamma + \cos^2\beta + \cos^2\alpha = 1$

Solution: L.H.S. can be simplified to $= \frac{a_z^2}{a^2} + \frac{a_y^2}{a^2} + \frac{a_x^2}{a^2} = \frac{a_z^2 + a_y^2 + a_x^2}{a^2} = \frac{a^2}{a^2} = 1$

COMPONENT ALGEBRA

To add (or subtract) two vectors, we simply add (or subtract) their components.

If $\vec{c} = \vec{a} + \vec{b}$, then $c_x = a_x + b_x$, $c_y = a_y + b_y$ and $c_z = a_z + b_z$

Example: If $\vec{a} = 3\hat{i} - 2\hat{j}$ and $\vec{b} = 2\hat{i} + 7\hat{j}$, find the magnitude and direction of

(a) $\vec{a} + \vec{b}$ (b) $\vec{a} - \vec{b}$

Solution: (a) $\vec{a} + \vec{b} = (3\hat{i} - 2\hat{j}) + (2\hat{i} + 7\hat{j}) = 5\hat{i} + 5\hat{j}$

$$\therefore \left|\vec{a} + \vec{b}\right| = \sqrt{5^2 + 5^2} = 5\sqrt{2} = 7.05$$

$$\tan\theta = \frac{5}{5} = 1 \text{ or } \theta = 45°$$

(b) $\vec{a}-\vec{b}=(3\hat{i}-2\hat{i})-(2\hat{i}-7\hat{j})=1\hat{i}-9\hat{j}$

$\therefore\ |\vec{a}-\vec{b}|=\sqrt{1^2+9^2}=\sqrt{82}$

$\tan\theta=\frac{-9}{1}=-9$ or $\theta=\tan^{-1}(-9)$

Example: $\vec{A}=\hat{i}+2\hat{j}-3\hat{k}$, when a vector $\vec{B}$ is added to $\vec{A}$ we get a unit vector along x-axis. Find the value of $\vec{B}$? Also find its magnitude

Solution: $\vec{A}+\vec{B}=\hat{i}$ or $\hat{i}+2\hat{j}-3\hat{k}+\vec{B}=\hat{i}$

So $\vec{B}=-2\hat{j}+3\hat{k}$ and $|\vec{B}|=\sqrt{2^2+3^2}=\sqrt{13}$

Example: In the above question find a unit vector along $\vec{B}$?

Solution: $\hat{B}=\frac{\vec{B}}{B}=\frac{-2\hat{j}+3\hat{k}}{\sqrt{13}}$

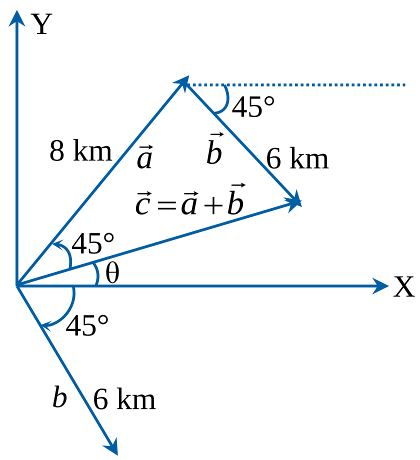
Example: Find the angle between two vectors $\vec{a}=2\hat{i}+2\hat{j}$ and $\vec{b}=5\sqrt{3}\hat{i}+5\hat{j}$

Solution: $\tan\theta_1=\frac{2}{2}=1\Rightarrow\theta_1=45°$ and $\tan\theta_2=\frac{5}{5\sqrt{3}}=\frac{1}{\sqrt{3}}\Rightarrow\theta_2=30°$

Therefore, $\theta_1-\theta_2=150°$

Example: A man walks 8 km north-east and 6 km south-east. If the origin of the coordinates is taken as the origin, x-axes is taken along east and y-axes is taken along north, (a) find the components of his total displacement? (c) What are his final distance and direction from starting point?

Solution: The successive displacements are shown on the figure below.



(a) For the sake of finding components, the tail of vector has been shifted to origin.

The components of vector $\vec{a}$ are $a_x = a\cos(45°) = 8(0.707) = 5.66$ km

and $a_y = a\sin(45°) = 8(0.707) = 5.66$ km

The component of $\vec{b}$ are $\boldsymbol{b_x = 6\cos(-45°) = 6(0.7007) = 4.24}$ **km** and $\boldsymbol{b_y}$ **= 6 sin(–45°) = –6(0.7007) = –4.24 km**

(b) The components of the resultant displacement $\vec{c} = \vec{a} + \vec{b}$ are

$c_x = a_x + b_x = 5.66 + 4.24 = 9.90$ km

and $c_y = a_y + b_y = 5.66 - 4.24 = 1.42$ km

(c) The total distance from the origin is the magnitude of $\vec{c}$

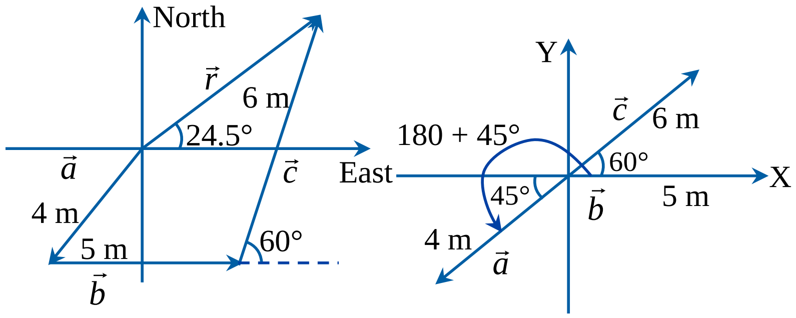
$$c = \sqrt{c_x^2 + c_y^2} = 10km$$

The direction of man's final position is given by $\tan\theta = \frac{c_y}{c_x} = \frac{1.42}{9.90} = 0.143$ or $\theta = 8.13°$

Illustration 4: A particle undergoes three displacements in a plane. The first time, it moves 4m south-west, the second time 5m east and third time 6m in a direction 60° north of east. Draw a vector diagram and determine the total displacement of the particle from the starting point.

Solution: The successive displacements are shown in the figure.

For the sake of finding components, the tails of vector $\vec{b}$ and $\vec{c}$ have been shifted to the origin in second diagram.



The components of vector $\vec{a}$ are

$$a_x = a\cos(180° + 45°) = -a\cos(45°) = -4 \times \left(\frac{1}{\sqrt{2}}\right)m \quad \text{and}$$

$$a_y = a\sin(180° + 45°) = -a\sin(45°) = -4 \times \left(\frac{1}{\sqrt{2}}\right)m$$

The components of vector $\vec{b}$ are

$b_x = b\cos(0) = 4 \times 1 = 5m$ and $b_y = b\sin(0) = 5 \times 0 = 0$

The components of $\vec{c}$ are

$$c_x = 6\cos(60°) = 6 \times \frac{1}{2} = 3m \text{ and } c_y = 6\sin(60°) = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}m$$

The components of resultant displacement $\vec{r}$ are

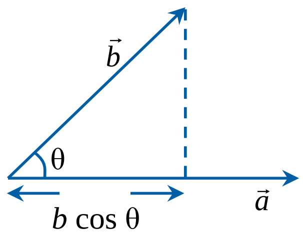
$$r_x = a_x + b_x + c_x = \left(-\frac{4}{\sqrt{2}} + 5 + 3\right) = 5.17m,\ r_y = a_y + b_y + c_y = \left(-\frac{4}{\sqrt{2}} + 0 + 3\sqrt{3}\right) = 2.37m$$

Therefore $|\vec{r}| = \sqrt{(5.17)^2 + (2.37)^2} = 5.68m$

$\tan\theta = \dfrac{2.37}{5.17} = 0.458 \quad \text{or } \theta = 24.5°$

SCALAR or DOT PRODUCT

The dot or scalar product of two vectors $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.



Geometrical Interpretation: From the diagram above, it is evident that the scalar product can be interpreted as the magnitude of vector $\vec{a}$ multiplied by the component or projection of $\vec{b}$ along the direction of $\vec{a}$ (which is b cos θ)

Properties of Scalar Product

1. Commutative property: $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

2. Distributive property: $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

3. $\vec{a}\cdot\vec{b} = 0$ if θ = π/2

$\vec{a}\cdot\vec{b} > 0$ if θ is an acute angle and $\vec{a}\cdot\vec{b} < 0$ if θ is an obtuse angle.

4. $\vec{a}\cdot\vec{a} = a^2$

5. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$, whereas, $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$

Accordingly, $\vec{a}\cdot\vec{b} = (a_x i + a_y \hat{j} + a_z \hat{k})\cdot(b_x i + a_y \hat{j} + b_z \hat{k})$

Or $\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$

6. The dot product of a vector with any unit vector is the component of that vector along that direction

(Therefore $\vec{a}\cdot\hat{i} = a_x$, $\vec{a}\cdot\hat{j} = a_y$, and $\vec{a}\cdot\hat{k} = a_z$)

Example: A(1, 2, 0), B(1, 6, 3) and P(7, 10, 0) in meters. A body is displaced from A to B during the AB displacement a force of 50N acts on the body in direction AP. Find the work done by the force during displacement AB, given that the work done is defined as = $\vec{F}\cdot\vec{s}$?

Solution: The displacement $\vec{s} = \overrightarrow{AB} = 4\hat{j} + 3\hat{k}\,m$ and the force acting on the body $\vec{F} = 10\hat{n}$ where $\hat{n}$ is a unit vector along the direction of AP, $\hat{n} = \frac{\overrightarrow{AP}}{AP} = \frac{6\hat{i} + 8\hat{j}}{\sqrt{6^2 + 8^2}} = 0.6\hat{i} + 0.8\hat{j}$, therefore $\vec{F} = 6\hat{i} + 8\hat{j}$ Newtons. Now $W = \vec{F}\cdot\vec{s} = (6\hat{i} + 8\hat{j})\cdot(4\hat{j} + 3\hat{k}) = 32\,Joules$

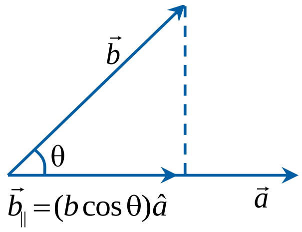
Some Applications of Scalar Products

1. To find the angle between two given vectors, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$

Example: Find the angle between the vectors $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{j} - \hat{k}$

Solution: $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{(\hat{i} + \hat{j})\cdot(\hat{j} - \hat{k})}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$, hence $\theta = 60°$

2. To find the component (or projection) of a vector along another vector



As shown, the component of $\vec{b}$ along the $\vec{a}$ is given by $\vec{b}_{\parallel}$ = (b cos θ) $\hat{a}$, where $b\cos\theta = \frac{\vec{a}\cdot\vec{b}}{a}$

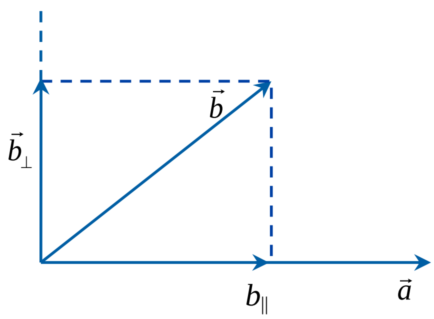
Example: Find the Projection of the Vector $\vec{a} = \hat{i} + \hat{j}$ along the vector $\vec{b} = 3\hat{i} + 4\hat{j}$.

Solution: Here, the component of $\vec{a}$ along $\vec{b}$ is $\vec{a}_{\parallel}$ = (a cos θ) $\hat{b}$, where

$a\cos\theta = \frac{\vec{a}\cdot\vec{b}}{b} = \frac{7}{5}$ and $\hat{b} = \frac{\vec{b}}{b} = \frac{3\hat{i} + 4\hat{j}}{5}$

Therefore, $\vec{a}_{\parallel} = \frac{7}{5}\left(\frac{3\hat{i} + 4\hat{j}}{5}\right) = \frac{7}{25}(3\hat{i} + 4\hat{j})$

3. To find the component or projection of a vector, perpendicular to another vector.



The vector $\vec{b}$ shown above can be expressed as the sum of two projections $\vec{b}_{\perp}$ and $\vec{b}_{\parallel}$ which are perpendicular to and parallel to the vector $\vec{a}$.

Now, since $\vec{b} = \vec{b}_\perp + \vec{b}_\parallel$, $\vec{b}_\perp$ can be calculated by subtraction of $\vec{b}_\parallel$ from $\vec{b}$.

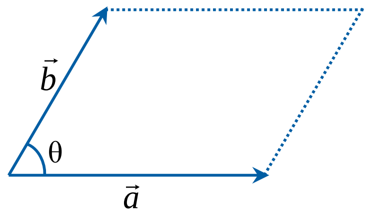
VECTOR OR CROSS PRODUCTS

The cross product of two vectors is defined by

$\vec{a} \times \vec{b} = |a||b| \sin\theta \hat{n}$, where $\hat{n}$ is a unit vector pointing perpendicular to the plane of $\vec{a}$ & $\vec{b}$.

Though, there are two directions perpendicular to any plane: 'in' and 'out'.

The ambiguity is resolved by right hand rule: Let your fingers point in the direction of first vector and curl around (via the smaller angle) toward second, then your thumb indicates the direction of $\hat{n}$



In the figure shown above, $\vec{a} \times \vec{b}$ points out of the page and $\vec{b} \times \vec{a}$ points into the page. Infact $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Geometrically, $|\vec{a} \times \vec{b}|$ is the area of the parallelogram generated by $\vec{a}$ and $\vec{b}$. If two vectors are parallel or anti-parallel, their vector product is zero.

In particular, $\vec{a} \times \vec{a} = 0$,

Because $\hat{i}, \hat{j}$ and $\hat{k}$ are mutually perpendicular unit vectors in a right handed coordinate system. (x axes to the right, y axes up and z axis out of the page, or any rotated version thereof),

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$,

$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$

$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$

$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

Therefore $\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

$= (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j} + (a_xb_y - a_yb_x)\hat{k}$

And this information can be expressed more compactly by the determinant of the matrix

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Example: Derive the formulae for magnitude of the resultant of two vectors and it's angle with the first one using scalar and vector products.

Solution: Let $\vec{c} = \vec{a} + \vec{b}$

Therefore, $\vec{c} \cdot \vec{c} = \left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right)$ or $c^2 = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = a^2 + b^2 + 2ab\cos\theta$

$c = \sqrt{a^2 + b^2 + 2ab\cos\theta}$

Now, let $\vec{c} \cdot \vec{a} = c \; a\cos \; \alpha$, where α is the angle between $\vec{c}$ and $\vec{a}$

$\left|\vec{c} \times \vec{a}\right| = c \; a\sin \; \alpha$, $\tan\alpha = \dfrac{\left|\vec{c} \times \vec{a}\right|}{\vec{c} \cdot \vec{a}}$

Where $\left|\vec{c} \times \vec{a}\right| = \left(\vec{a} + \vec{b}\right) \times \vec{a} = \left|\vec{b} \times \vec{a}\right| = ab\sin\theta$

And $\vec{c} \cdot \vec{a} = \left(\vec{a} + \vec{b}\right) \cdot \vec{a} = a^2 + ab\cos\theta$,

Hence $\tan\alpha = \dfrac{ab\sin\theta}{a^2 + ab\cos\theta} = \dfrac{b\sin\theta}{a + b\cos\theta}$

Example: Find $\vec{A} \times \vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$

Solution: $\vec{A} \times \vec{B} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix}$$

$= 10\hat{j} + 5\hat{k}$

SCALAR TRIPLE PRODUCT

Another useful vector operation is the scalar triple product of three vectors. The definition of this is $\vec{A} \cdot (\vec{B} \times \vec{C})$

Notice, that if $\vec{A}$ is perpendicular to $(\vec{B} \times \vec{C})$, it must be co-planar to the plane of $\vec{B}$ and $\vec{C}$,

Alternately if $(\vec{B} \times \vec{C}) = 0$, then $\vec{B}$ and $\vec{C}$ are co-linear which again implies that $\vec{A}$, $\vec{B}$ and $\vec{C}$ are co-planar.

Hence the condition $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ implies that the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ $\vec{A}$, are co-planar.

In the event that the three vectors are "non-coplanar" the triple product represents the volume of a "parallelepiped" formed with three edges from a vertex being the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$.

Note that $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$ and $\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{A} \cdot (\vec{C} \times \vec{B})$