SEQUENCES and SERIES

1. Definition of a sequence

A set of elements arranged in a definite order and formed according to some definite rule relating the elements to their position or / and the preceding & succeeding elements is called a sequence.

The different elements in a sequence are called terms of the sequence and are generally denoted with respect to their positions as t_1, t_2, t_3, . . .

The n^{th} term is called the general term of the sequence and it is generally denoted by t_n

Finite Sequence

A sequence is said to be finite sequence if the number of terms in the sequence is finite. A finite sequence always has a last term. e.g. 1, 3, 9, 27, . . . , 729.

Infinite sequence

A sequence is said to be an infinite sequence if the number of terms in the sequence is infinite. An infinite sequence has no last term. e.g. $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots.$

Series

An expression consisting of the terms of a sequence, alternating with the symbol '+', is called a series. If $\{t_n\} = \{t_1, t_2, t_3, \ldots\}$ is a sequence, then S = $t_1 + t_2 + t_3 + \ldots$ is called the corresponding series.

2. Arithmetic Progression (A.P.)

A sequence $\{t_n\}$ of numbers is said to be an A.P. if $t_n - t_{n-1}$ = a constant for all $n \in N$, $n \geq 2$.
This constant is known as the common difference (c.d.) of the A.P. and is denoted by 'd'.
If three elements *a, b, c* are in A.P., then $b - a = c - b \Rightarrow 2b = a + c$.
Sequences of natural number, odd and even integers etc. are some well-known A.P.s
In general a sequence governed by a rule which linearly relates the elements with their positions is an A.P. i.e. $t_n = pn + q$, where p and q are constants.

e.g. $t_n = 3n + 2$ Generates for n = 1, 2, 3, . . . an A.P. as 5, 8, 11, . . .

$t_n = 4n - 3$ Generates for n = 1, 2, 3, . . . an A.P. as 1, 5, 9, . . .

3. General representation of an A.P.

Let us consider an A.P. with first term = a, common difference = d and the last term = l

- The general form of an A.P. is a, a + d, a + 2d, a + 3d, . . .
- Formula for n^{th} term of an A.P. is "a + (n – 1)d"
- Formula for r^{th} term from the end of an A.P. containing n terms is

"l – (r –1)d" or "a + (n – r)d"

> ***When sum of three or more elements of an A.P. is known generally the terms of the A.P. are assumed as "a – d, a, a + d" for three terms and "a –3d, a – d, a + d, a + 3d" for four terms. Similar idea may be used if it is required to select more terms.***

4. Important Results and Properties Regarding an A.P.

- If a fixed number is added to (or subtracted from) each term of an A.P., then the resulting sequence is also an A.P. with same c.d. as that of the original A.P.
- If each term of an A.P. is multiplied (or divided) by a fixed non-zero number k, then the resulting sequence is also an A.P.
- If corresponding terms of two or more A.P.'s are added then the resulting sequence will be an A.P.
- Common terms of two A.P.s are also in A.P. with common difference as L.C.M. of the common differences of the two A.P.s.
- If elements are selected at equal interval from an A.P. then those elements will also be in A.P. i.e. $t_n = \frac{1}{2}\left(t_{n-k} + t_{n+k}\right)$
- In an A.P. sum of terms symmetrically located from center is always equal to sum of the first and the last term i.e. $t_r + t_{n-r+1} = a + l$.

5. Sum of the first n terms of an A.P.

$S = a + (a+d) + (a+2d) + \dots + (a+(n-1)d)$

Also if we start from last term then

$S = l + (l-d) + (l-2d) + \dots + (l-(n-1)d)$

Adding the two series we get, $S = \frac{n}{2}\{a+l\}$.

Also as, $l = a + (n-1)d$, the formula becomes $S = \frac{n}{2}\{2a + (n-1)d\}$.

In general the expression for sum of n terms of an A.P. is always of the form ($pn^2 + qn$), where p & q are constants.

General term and sum of n terms of some common A.P.s are as follows:

Sequence of natural numbers $t_n = n$ & $S_n = \frac{n(n+1)}{2}$

Sequence of first n odd natural numbers $t_n = 2n-1$ & $S_n = n^2$

Sequence of first n even natural numbers $t_n = 2n$ & $S_n = n(n+1)$

6. Single Arithmetic mean (A.M.) and Arithmetic means

A.M. of the numbers $a_1, a_2, a_3, \dots, a_n$ is given by $\frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$.

If *a*, *b*, *c* are in A.P., then b is called a **single arithmetic mean** (A.M.) between a and c, i.e. $b = \frac{a+c}{2}$.

Inserting Arithmetic means

Inserting n arithmetic means between two given numbers *a* & *l* actually means identifying an A.P. consisting (n + 2) terms with *a* & *l* as respectively the first and last terms. Considering A_1, A_2, . . ., A_n to be the n arithmetic means between *a* and *l* to form an A.P. of common difference 'd' we will have

$$l = a + (n+1)d \Rightarrow d = \frac{l-a}{n+1} \quad \therefore\ A_1 = a + \frac{l-a}{n+1}, A_2 = a + \frac{2(l-a)}{n+1}, A_3 = a + \frac{n(l-a)}{n+1}, \dots$$

In general if n A.M.'s are inserted between *a* & *l*, then rth A.M. will be given by $a + \frac{r(l-a)}{n+1}$

Here take care of not getting confused between the concept of inserting A.M.'s between two given numbers and the concept of A.M. of n numbers.

Single A.M. of n A.M.'s between two numbers a & l is equal to the arithmetic mean of a & l

i.e. $\dfrac{A_1+A_2+A_3+\ldots+A_n}{n}=\dfrac{a+l}{2}$.

7. Geometric Progression (G.P.)

A sequence is said to be a G.P. if its first term is non-zero and each of the succeeding terms is equal to the preceding term multiplied by a certain non-zero number which is constant for a given sequence. The non-zero number is called the common ratio of the G.P.

If three elements a, b, c are in G.P., then $\dfrac{b}{a}=\dfrac{c}{b}\Rightarrow b^2=a\cdot c$.

In general a sequence governed by a rule which exponentially relates the elements with their positions is a G.P. i.e. $t_n=p\cdot q^n$, where p and q are constants.

e.g. $t_n=2\cdot 3^n$ generates for n = 1, 2, 3, . . . a G.P. as 6, 18, 54, . . .

$t_n=\dfrac{4^n}{3}$ generates for n = 1, 2, 3, . . . a G.P. as $\dfrac{4}{3},\dfrac{16}{3},\dfrac{64}{3},\ldots$

Note that No term of a G.P. can be zero neither can be the common ratio of a G.P. zero.

8. General representation of an A.P.

Consider A G.P. with a as the first term, last term l and r as the common ratio.

- General form of the G.P. will be $a, ar, ar^2, \ldots$
- General formula for n[th] term of the G.P. is ar^{n-1}
- General formula for n[th] term of the G.P. from the end is $\dfrac{l}{r^{n-1}}$.

When product of three or more elements of an G.P. is known generally the terms of the G.P. are assumed as "$\frac{a}{r},a,ar$" for three terms and "$\frac{a}{r^3},\frac{a}{r},ar,ar^3$" for four terms.

Similar idea may be used if it is required to select more terms.

9. Important Results and Properties Regarding A G.P.

- If each term of a G.P. be multiplied (or divided) by a fixed non-zero number, then the resulting sequence is also a G.P.
- If each term of a G.P. be raised to the same power, then the resulting sequence is also a G.P.
- If $x_1, x_2, x_3, \ldots, x_n$ is a G.P. of positive terms then $\log x_1, \log x_2, \log x_3, \ldots, \log x_n$ will be an A.P. Similarly if $x_1, x_2, x_3, \ldots, x_n$ is an A.P., then $a^{x_1}, a^{x_2}, a^{x_3}, \ldots, a^{x_n}$ (a > 0) will be a G.P.
- If a number of terms are taken from a G.P. such that

  (i) their positions are at equal interval (ii) their positions are in A.P.,
  then the selected terms will also be in G.P.
- If corresponding terms of two or more G.P.s are multiplied then the resulting terms will also be in G.P.

10. Sum of the first n terms and infinite terms of a G.P.

- The sum of first n terms of a G.P. is given by $S_n = \dfrac{a(1-r^n)}{1-r} \quad where \quad r \neq 1$
- Sum to infinite terms of a G.P. is $S_\infty = \dfrac{a}{1-r}, \quad |r| < 1$

11. Single Geometric Mean (G.M.) and Geometric Means

If a, G, b are in G.P. then G is called the G.M. between a and b. In such case

$$G = \begin{cases} -\sqrt{ab} & \text{if } a, b < 0 \\ \sqrt{ab} & \text{otherwise} \end{cases}.$$

G.M. of n numbers $t_1, t_2, \ldots, t_n$ is given by $(t_1 . t_2 . t_3 \ldots t_n)^{1/n}$.

Inserting Geometric Means

Inserting n Geometric Means between two given numbers *a* & *l* actually means identifying a G.P. consisting of (n + 2) terms with *a* & *l* as respectively the first and last terms. Considering G_1, G_2, . . ., G_n to be the n Geometric Means between *a* and *l* to form a G.P. of common ratio 'r' we will have

$$l = ar^{n+2-1} \quad \Rightarrow l = ar^{n+1} \quad \Rightarrow r = \left(\frac{l}{a}\right)^{1/(n+1)}$$

$$\therefore G_1 = ar = a\left(\frac{l}{a}\right)^{\frac{1}{n+1}}, G_2 = ar^2 = a\left(\frac{l}{a}\right)^{\frac{2}{n+1}}, \ldots, G_n = ar^n = a\left(\frac{l}{a}\right)^{\frac{n}{n+1}}$$

In general if n G.M.s are inserted between a & l , then r^th^ G.M. will be given by $a\left(\frac{l}{a}\right)^{\frac{r}{n+1}}$

Here take care of not getting confused between the concept of inserting A.M.'s between two given numbers and the concept of A.M. of n numbers.

Single G.M. of n G.M.'s between two numbers a & l is equal to the Geometric Mean of a & l

i.e. $\left(G_1.G_2.G_3.\ldots G_n\right)^{\frac{1}{n}} = \left(a.l\right)^{\frac{1}{2}}$.

12. Harmonic progression (H.P.)

A sequence of non-zero real numbers t_1, t_2,..., t_n is said to be an H.P. if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \ldots, \frac{1}{t_n}$ are in A.P.

Hence if a, b, c are in H.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be in A.P. Hence $\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$.

The general form of an H.P. is $\frac{1}{\boldsymbol{a}}, \frac{1}{\boldsymbol{a}+\boldsymbol{d}}, \frac{1}{\boldsymbol{a}+2\boldsymbol{d}}, \ldots, \frac{1}{\boldsymbol{a}+(\boldsymbol{n}-1)\boldsymbol{d}}$

It must be noted here that no terms of an H.P. can be zero.

There is no formula which can represent sum of n terms of a H.P.

13. Single Harmonic mean (H.M) and Harmonic Means

If a, H, b are in H.P., then H is said to be H.M. between a and b. In such case $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P. $\quad \therefore \frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \Rightarrow H = \frac{2ab}{a+b}$.

The harmonic mean of t_1, t_2, t_3, ..., t_n is given by $\frac{1}{\boldsymbol{H}} = \frac{1}{\boldsymbol{n}}\left(\frac{1}{\boldsymbol{t}_1} + \frac{1}{\boldsymbol{t}_2} + \frac{1}{\boldsymbol{t}_3} + \ldots + \frac{1}{\boldsymbol{t}_n}\right)$.

Inserting Harmonic Means

If a, H_1, H_2, ..., H_n, b are in H.P., then H_1, H_2, H_3, ..., H_n are said to be the n H.Ms between a and b.

In such case $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots\ldots, \frac{1}{H_n}, \frac{1}{b}$ are in A.P. Let common difference of this A.P. be d.

$$\therefore \frac{1}{b} = \frac{1}{a} + (n+2-1)d \quad \Rightarrow \quad \frac{1}{b} - \frac{1}{a} = (n+1)d \quad \Rightarrow \quad d = \frac{a-b}{(n+1)ab}$$

$$\therefore \frac{1}{H_1}=\left[\frac{1}{a}+\frac{a-b}{(n+1)ab}\right],\ \frac{1}{H_2}=\left[\frac{1}{a}+\frac{2(a-b)}{(n+1)ab}\right],\ \ldots\ldots\ldots,\frac{1}{H_n}=\left[\frac{1}{a}+\frac{n(a-b)}{(n+1)ab}\right].$$

14. Algebraic Mean Inequalities

- If A, G & H are respectively the Arithmetic mean, Geometric mean & Harmonic mean of n positive numbers, then $A \geq G \geq H$. i.e.

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq \left(a_1.a_2.a_3.\ldots a_n\right)^{\frac{1}{n}} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_n}}.$$

- If $a_1,a_2,a_3,\ldots,a_n$ be n positive numbers and $m_1,m_2,m_3,\ldots,m_n$ are n positive rational numbers then

$$\frac{m_1.a_1+m_2.a_2+m_3.a_3+\ldots+m_n.a_n}{m_1+m_2+m_3+\ldots+m_n} \geq \left(a_1^{m_1}.a_2^{m_2}.a_3^{m_3}.\ldots a_n^{m_n}\right)^{\frac{1}{m_1+m_2+m_3+\ldots+m_n}}.$$

Here the two means are called the weighted A.M. & the weighted G.M.

- If $a_1,a_2,a_3,\ldots,a_n$ be n positive numbers and 0 < m < 1, then

$$\frac{a_1^{\ m}+a_2^{\ m}+a_3^{\ m}+\ldots+a_n^{\ m}}{n} \leq \left(\frac{a_1+a_2+a_3+\ldots+a_n}{n}\right)^m.$$

- If $a_1,a_2,a_3,\ldots,a_n$ be n positive numbers and m < 0 or m > 1, then

$$\frac{a_1^{\ m}+a_2^{\ m}+a_3^{\ m}+\ldots+a_n^{\ m}}{n} \geq \left(\frac{a_1+a_2+a_3+\ldots+a_n}{n}\right)^m.$$

15. Some more inequalities which you may find useful

Tchebychef Inequality

If $a_1,a_2,a_3,\ldots,a_n$ & $b_1,b_2,b_3,\ldots,b_n$ are real numbers such that $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$ & $b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n$, then

$$\frac{a_1b_1+a_2b_2+a_3b_3+\ldots+a_nb_n}{n} \geq \left(\frac{a_1+a_2+a_3+\ldots+a_n}{n}\right)\left(\frac{b_1+b_2+b_3+\ldots+b_n}{n}\right)$$

Weierstrass Inequality

If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers and n > 1, then

$$(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n) > 1+a_1+a_2+a_3+\ldots+a_n.$$

If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers less then unity and n > 1, then

$$(1-a_1)(1-a_2)(1-a_3)\ldots(1-a_n) > 1-a_1-a_2-a_3-\ldots-a_n.$$

Cauchy – Schwartz Inequality

If $a_1, a_2, a_3, \ldots, a_n$ & $b_1, b_2, b_3, \ldots, b_n$ are real numbers then

$$(a_1b_1+a_2b_2+a_3b_3+\ldots+a_nb_n)^2 \le (a_1^2+a_2^2+a_3^2+\ldots+a_n^2)(b_1^2+b_2^2+b_3^2+\ldots+b_n^2).$$

Equality will hold if $a_1, a_2, a_3, \ldots, a_n$ are proportional to $b_1, b_2, b_3, \ldots, b_n$.

16. Arithmetical Geometric Progression (A.G.P.)

If x_1, x_2, x_3, ..., x_n are in A.P. and y_1, y_2, y_3, ..., y_n are in G.P. then x_1y_1, x_2y_2, x_3y_3, ..., x_ny_n is called arithmetical geometric sequence.

In general an A.G.P. may be represented as ab, (a + d)br, (a + 2d)br^2, ...[a +(n –1)d]br^{n-1}, where a, a + d, a + 2d, ..., a + (n –1)d are in A.P. and b, br, br^2, ... , br^{n-1} are in G.P.

- Generally nth term of an A.G.P. is $t_n = [a+(n-1)d]r^{n-1}$
- Sum of n terms of an A.G.P. is given by

$$S_n = \frac{a}{1-r} + \frac{dr}{(1-r)^2} - \frac{dr^n}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$
$$= \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

- If $|r| < 1$ $S_\infty = \lim_{n\to\infty} S_n = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$ $\left(\because \lim_{n\to\infty} r^n = 0\right)$

17. Series of natural numbers

- $\sum_{r=1}^{n} r = 1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$

- $$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$
- Sum of pairwise products of first n natural numbers

$$\sum_{1\le i<j\le n} i.j = \frac{1}{2}\left\{\left(\sum_{r=1}^{n} r\right)^2 - \sum_{r=1}^{n} r^2\right\}$$

In general
$1^k + 2^k + 3^k + \ldots + n^k = a_0 n + a_1 n(n-1) + a_2 n(n-1)(n-2) + \ldots + a_k n(n-1)\ldots(n-k)$
Where $a_0, a_1, a_2, \ldots, a_k$ can be found by letting n = 1, 2, 3, . . . , k – 1.

18. Method of Differences

Consider the sequence " $a_1, a_2, a_3, \ldots$ " The n[th] term of this sequence may be found as follows –

$$\begin{array}{l} S = a_1 + a_2 + a_3 + a_4 + \ldots + a_n \\ -\{S = \quad a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n\} \\ \hline 0 = a_1 + \{b_1 + b_2 + b_3 + \ldots + b_{n-1}\} - a_n \end{array}$$

$$\Rightarrow a_n = a_1 + \{b_1 + b_2 + b_3 + \ldots + b_{n-1}\}$$

Let $S = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$ . . . (1)

If the terms in bracket are in a known sequence such as A.P., G.P. , A.G.P. or powers of natural numbers, value of a_n can be found in terms of n and then $S_n = \sum_{r=1}^{n} a_r$.

19. Special Sequence Related to A.P. & H.P.

To find the sum of the series of the form . . .

- $$\frac{1}{x_1 x_2 \ldots x_r} + \frac{1}{x_2 x_3 \ldots x_{r+1}} + \ldots + \frac{1}{x_n x_{n+1} \ldots x_{n+r-1}}$$ **&**
- $$x_1 x_2 \ldots x_r + x_2 x_3 \ldots x_{r+1} + \ldots + x_n x_{n+1} \ldots x_{n+r-1}$$

where $x_1, x_2, x_3, \ldots, x_n \ldots$ are in A.P. with common difference d.

First consider the series

$$S_n = \frac{1}{x_1 x_2 \ldots x_r} + \frac{1}{x_2 x_3 \ldots x_{r+1}} + \ldots + \frac{1}{x_n x_{n+1} \ldots x_{n+r-1}}$$

Here general term of the sequence i.e. $t_n = \dfrac{1}{x_n x_{n+1} \ldots x_{n+r-1}}$

Let us define u_n as $\dfrac{1}{x_{n+1}x_{n+2} \ldots x_{n+r-2}x_{n+r-1}}$ by leaving first term in denominator of t_n .

$$\Rightarrow u_{n-1} = \frac{1}{x_n x_{n+1} \ldots x_{n+r-3}x_{n+r-2}}$$

$$\Rightarrow u_n - u_{n-1} = \frac{x_n - x_{n+r-1}}{x_n x_{n+1} \ldots x_{n+r-1}}$$

$u_n - u_{n-1} = t_n (x_n - x_{n+r-1})$

$= t_n \{[x_1 + (n-1)d] - [x_1 + (n+r-2)d]\}$

$= t_n(1-r)d$

$$\Rightarrow t_n = \frac{u_n - u_{n-1}}{d(1-r)} \quad or \quad \frac{u_{n-1} - u}{d(r-1)}$$

Put n = 1, 2, 3, 4, . . . n, we get

$t_1 = \dfrac{1}{d(r-1)}(u_0 - u_1)$, $t_2 = \dfrac{1}{d(r-1)}(u_1 - u_2)$, . . . ,$t_n = \dfrac{1}{d(r-1)}(u_{n-1} - u_n)$

by adding all, we get

$$t_1 + t_2 + \ldots + t_n = \frac{1}{(r-1)d}(u_0 - u_n)$$

$$\therefore S_n = \frac{1}{(r-1)d}\left(\frac{1}{x_1x_2 \ldots x_{r-1}} - \frac{1}{x_{n+1}x_{n+2} \ldots x_{n+r-1}}\right)$$

Hence the sum of n term is

$$S_n = \frac{1}{(r-1)(x_2 - x_1)}\left(\frac{1}{x_1x_2 \ldots x_{r-1}} - \frac{1}{x_{n+1} \quad x_{n+2} \ldots x_{n+r-1}}\right)$$

case(i) when r = 2,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \ldots + \frac{1}{x_nx_{n+1}} = \left(\frac{1}{x_2 - x_1}\right)\left(\frac{1}{x_1} - \frac{1}{x_{n+1}}\right)$$

$$= \frac{x_{n+1} - x_1}{(x_2 - x_1)x_1x_{n+1}}$$

$$= \frac{nd}{dx_1x_{n+1}} = \frac{n}{x_1x_{n+1}}$$

case(ii) when r = 3, we can prove

$$\frac{1}{x_1x_2x_3} + \frac{1}{x_2x_3x_4} + \ldots + \frac{1}{x_nx_{n+1}x_{n+2}} = \frac{1}{2(x_2 - x_1)}\left(\frac{1}{x_1x_2} - \frac{1}{x_{n+1}x_{n+2}}\right)$$

Similarly consider now the series

$S_n = x_1 x_2 \ldots x_r + x_2 x_3 \ldots x_{r+1} + \ldots + x_n x_{n+1} \ldots x_{n+r-1}$

$\Rightarrow t_n = x_n x_{n+1} x_{n+2} \ldots x_{n+r-1}$ (i)

Let $u_n = x_n x_{n+1} x_{n+2} \ldots x_{n+r-1} x_{n+r}$ (ii) (Take one extra term at the end in t_n for u_n)

$\Rightarrow u_{n-1} = x_{n-1} . x_n . x_{n+1} \ldots x_{n+r-1}$

$\Rightarrow u_n - u_{n-1} = x_n x_{n+1} x_{n+2} \ldots x_{n+r-1} (x_{n+r} - x_{n-1})$

$= t_n \{[x_1 + (n + r - 1)d] - [x_1 + (n - 2)d]\}$

$= (r + 1)d\, t_n$

$\Rightarrow t_n = \frac{1}{(r+1)d}(u_n - u_{n-1})$

Put n = 1, 2, 3, . . . , n, we get

$t_1 = \frac{1}{(r+1)d}(u_1 - u_0)$, $t_2 = \frac{1}{(r+1)d}(u_2 - u_1)$, . . . , $t_n = \frac{1}{(r+1)d}(u_n - u_{n-1})$

By adding, we get

$t_1 + t_2 + t_3 + \ldots + t_n = \frac{1}{(r+1)d}(u_n - u_0)$

Hence the sum of n terms is

$S_n = \frac{1}{(r+1)(x_2 - x_1)}\left[x_n x_{n+1} \ldots x_{n+r} - x_0 x_1 x_2 \ldots x_r\right]$ $\quad$ (*Here* $x_0 = x_1 - d$)

Case(i) when r = 2,

$x_1 x_2 + x_2 x_3 + \ldots + x_n x_{n+1} = \frac{1}{3(x_2 - x_1)}(x_n x_{n+1} x_{n+2} - x_0 x_1 x_2)$

Case (ii) when r = 3,

$x_1 x_2 x_3 + x_2 x_3 x_4 + \ldots + x_n x_{n+1} x_{n+2} = \frac{1}{4(x_2 - x_1)}(x_n x_{n+1} x_{n+2} x_{n+3} - x_0 x_1 x_2 x_3)$.