1. Definition

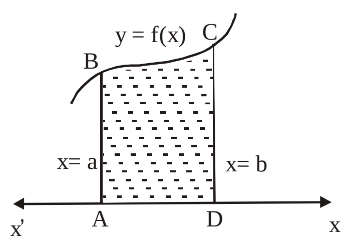
Let $f(x)$ be a continuous non-negative function in the interval $[a, b]$. The area of the region bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

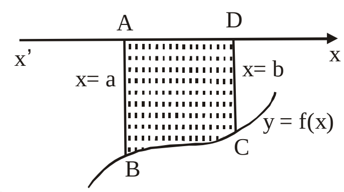
$$\int_a^b f(x)dx$$

2. Formulae for Finding the Area under by Curves

1. Area ABCDA bounded by the curve $y = f(x)$, x-axis and two ordinates $x = a$ and $x = b$ is given by

$$\int_a^b |y|\,dx = \begin{cases} \int_a^b y dx, \text{ if } y \geq 0 \text{ for } x \in [a,b] \\ -\int_a^b y dx, \text{ if } y \geq 0 \text{ for } x \in [a,b] \end{cases}$$





If however y i.e., $y = f(x)$ changes sign in interval $[a, b]$, say $y \geq 0$ in [a, c], in $[a, c]$, $y \leq 0$ in $[c, d]$ and $y \geq 0$ where $a < c < d < b$, then area bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$

$$= \int_a^b |y| dx = \int_a^c y dx - \int_c^d y dx + \int_d^b y dx$$

$= A_1 - A_2 + A_3$, where A_1, A_2, A_3 are algebraic areas.

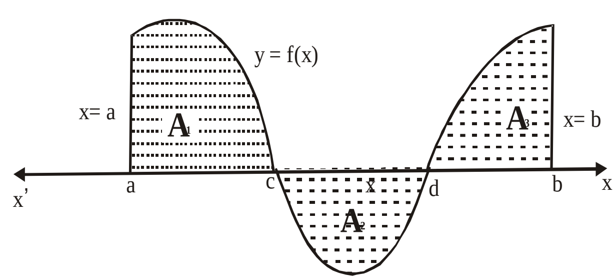
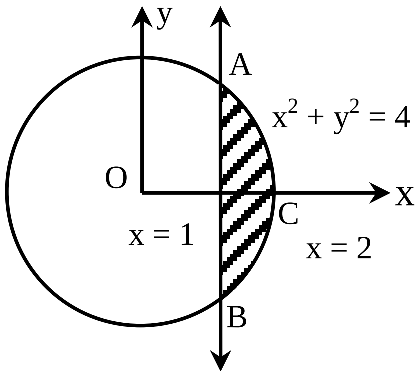


Illustration 1:

Find the area of smaller portion of the circle $x^2 + y^2 = 4$ cut off by the line $x = 1$.

Solution:

Equation of the circle is $x^2 + y^2 = 4$ and equation of the line is $x = 1$.



Required area = area ABCA

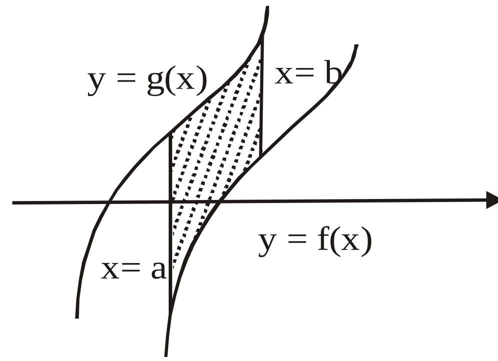
$$= 2\int_1^2 y dx = 2\int_1^2 \sqrt{4 - x^2}\, dx = 2\left[\frac{x\sqrt{2^2 - x^2}}{2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \frac{4\pi - 3\sqrt{3}}{3} \text{ sq. units}$$

2. Area ABCDA bounded by two curves $y = f(x)$, $y = g(x)$ and two ordinates $x = a$, $x = b$ is given by

$$\int_a^b | f(x) - g(x) | dx = \begin{cases} \int_a^b [f(x) - g(x)] dx, \text{ if } f(x) \geq g(x) \text{ for } a \leq x \leq b \\ -\int_a^b [f(x) - g(x)] dx, \text{ if } f(x) \leq g(x) \text{ for } a \leq x \leq b \end{cases}$$

While using this formula $f(x)$ is taken from the curve which lies above and $f(x)$ is taken from the curve which lies below.



If $a < c < d < b$ and

$f(x) \geq g(x)$ for $a \leq x \leq c$

$f(x) \leq g(x)$ for $c \leq x \leq d$

$f(x) \geq g(x)$ for $d \leq x \leq b$

Then shaded area $= \int_a^c [f(x) - g(x)]dx + \int_d^c [g(x) - f(x)]dx + \int_d^b [f(x) - g(x)]dx$

$$= \int_a^c [f(x) - g(x)]dx - \int_c^d [f(x) - g(x)]dx + \int_b^d [f(x) - g(x)]dx$$

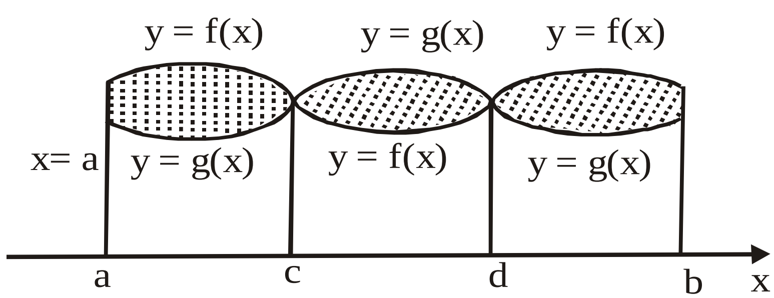


Illustration 2:

Find the area included between the line $y = x$ and the parabola $x^2 = 4y$.

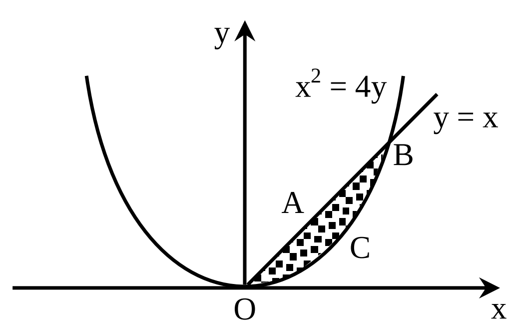
Solution:

Equation of parabola is $x^2 = 4y$ and equation of line is $y = x$

Solving we get $x^2 - 4x$

or, $x(x - 4) = 0$

$\therefore \ x = 0, 4$



$\therefore$ line $y = x$ cuts parabola at two points O and B, x co-ordinate of O is 0 and x coordinate of B is 4

Required area = area OCBAO = $\int_0^4 (y_1 - y_2)\,dx = \int_0^4 \left(x - \frac{x^2}{4}\right) dx$

$$= \left[\frac{x^2}{2} - \frac{x^3}{12}\right]_0^4 = \left[\frac{16}{2} - \frac{64}{12}\right] = \frac{8}{3} \text{sq. units.}$$

3. Area ABCDA enclosed by the curve $x = f(y)$, y-axis and two abscissae $y = c$ and $y = d$ is given by

$$\int_c^d |x|\,dy = \begin{cases} \int_c^d x\,dy, & \text{if } x \geq 0 \text{ for } c \leq y \leq d \\ -\int_c^d x\,dy, & \text{if } x \leq 0 \text{ for } c \leq y \leq d \end{cases}$$

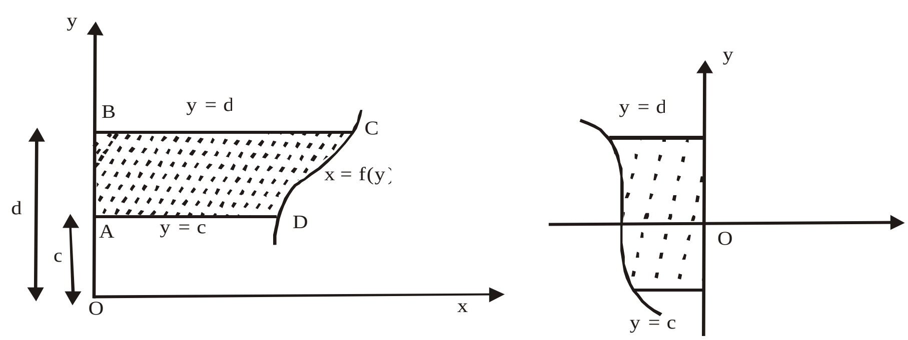


Illustration 3:

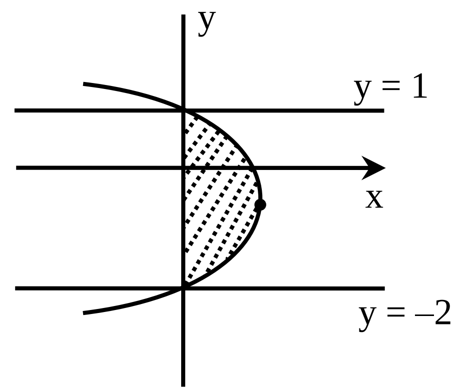
Find the area bounded by the curve $x = 2 - y - y^2$ and y-axis.

Solution:

The required area $= \int_{-2}^{1} x\,dy$

$$= \int_{-2}^{1} (2 - y - y^2)\,dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1} = \frac{9}{2} \text{sq.units}$$



4. Area bounded by the two curves $x = f(y)$, $x = g(y)$ and two abscissae $y = c$ and $y = a$ is given by

area ABCDA $= \int_{c}^{d} |x_1 - x_2|\,dy$

$$= \begin{cases} \int_{c}^{d} (x_1 - x_2)\,dy, & \text{if } x_1 \geq x_2 \text{ for } c \leq y \leq d \\ -\int_{c}^{d} (x_1 - x_2)\,dy, & \text{if } x_1 \leq x_2 \text{ for } c \leq y \leq d \end{cases}$$

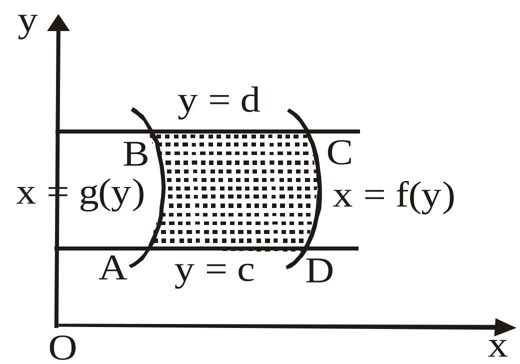


Illustration 4:

Determine the area enclosed by the two curves given by $y^2 = x + 1$ and $y^2 = -x + 1$.

Solution:

Given curves are

$y^2 = x + 1$ ….(1) and $y^2 = -x + 1$ ….(2)

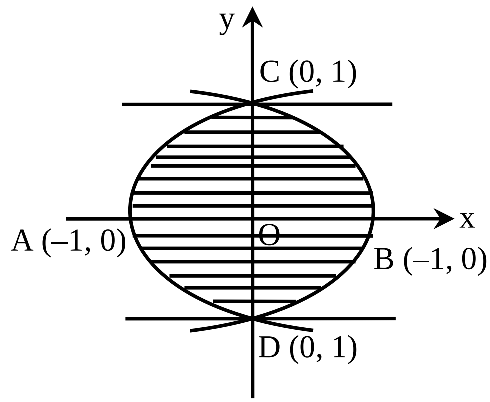
Curve (1) is the parabola having axis $y = 0$ and vertex $(-1, 0)$.

Curve (2) is the parabola having axis $y = 0$ and vertex $(1, 0)$

$(1) - (2) \Rightarrow 2x = 0 \Rightarrow x = 0$

From (1), $x = 0 \Rightarrow y = \pm 1$

Required area= $\int_{-1}^{1} (x_1 - x_2)\, dy$



$= \int_{-1}^{1} [(1 - y^2) - (y^2 - 1)]\, dy = 2\int_{-1}^{1} (1 - y^2)\, dy$

$= 2\left[y - \frac{y}{3} \right]_{-1}^{1} = 2\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] = \frac{8}{3}$ sq. units

3. Curve Sketching

For the evaluation of area of bounded regions it is very essential to know the rough sketch of the curves. The following points are very useful to draw a rough sketch of a curve.

While constructing the graph of $f(x, y) = 0$, it is expedient to follow the procedure given below:

(i) Find the set of permissible values of x (Domain).

(ii) Check if the curve is symmetrical about x–axis, y–axis, origin.

The symmetry of the curve is judged as follows:

- **(a)** If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.
- **(b)** If all the powers of x are even, the curve is symmetrical about the axis of y.
- **(c)** If powers of x and y both are even, the curve is symmetrical about the axis of x as well as y.

(d) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(e) If on interchanging the signs of x and y both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(iii) Find dy/dx and equate it to zero to find the points on the curve where you have horizontal tangents.

(iv) Find the points where the curve crosses the x-axis and also the y-axis.

(v) Find the period of the curve if it is periodic

(vi) Find the asymptote(s) of the curve, if any

(vii) Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

Illustration 5:

Construct the graph of $f(x) = \dfrac{x^2-1}{x^2+1}$ and find the area bounded by $y = f(x)$ and x-axis.

Solution:

Here, $f(x) = \dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

(i) The function $f(x)$ is well defined for all real x.

$\Rightarrow$ Domain of $f(x)$ is R.

(ii) $f(-x) = f(x)$, so it is an even function and hence graph is symmetrical about y-axis.

(iii) Obviously function is non-periodic.

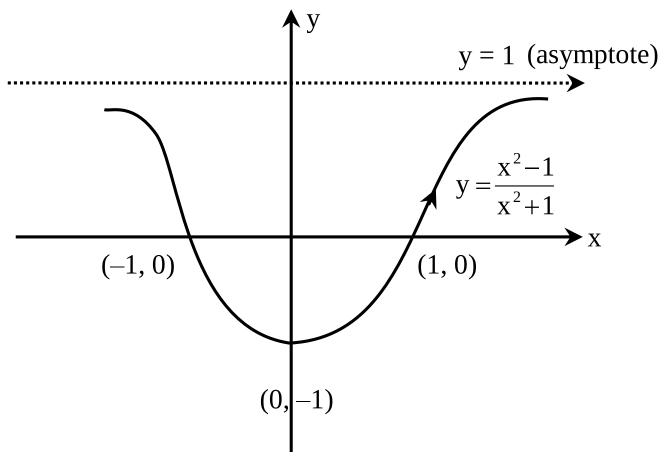
(iv) $f(x) \to 1^-$ for $x \to \infty$

(we are considering $x > 0$ only as curve is symmetrical about y-axis).

Hence $y = 1$ is an asymptote of the curves. It may be observed that $f(x) < 1$ for any $x \in R$ and consequently its graph lies below the line $y = 1$ which is the asymptote to the graph of the given function.

(v) Again $\dfrac{2}{x^2+1}$ decreases for $(0, \infty)$, thus f(x) increases for $(0, \infty)$.

(vi) The greatest value $\to 1$ for $x \to +\infty$ and the least value is -1 for $x = 0$. Thus its graph is as shown in figure.



Required area $= -\int_{-1}^{1}\dfrac{x^2-1}{x^2+1}dx = -[x]_{-1}^{1} + 2[\tan^{-1}x]_{-1}^{1} = (\pi-2)$ sq. units.

Illustration 6:

Construct the graph of $f(x) = xe^x$. Find the area bounded by $y = f(x)$ and its asymptote.

Solution:

(i) The function is well defined for all real $x \Rightarrow$ domain of $f(x)$ is R.

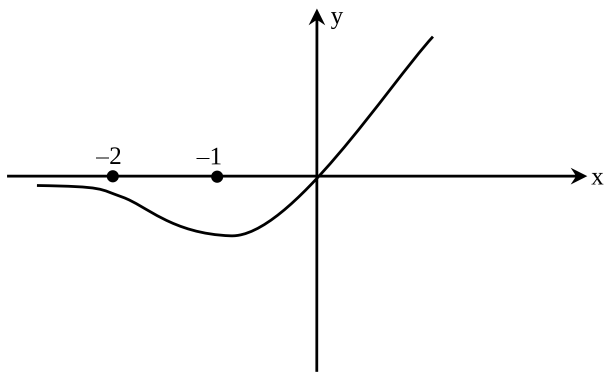
(ii) There is no symmetry in the graph.

(iii) Obviously function is non-periodic.

(iv) $f(x) \to 0^-$ as $x \to -\infty$. Hence $y = 0$ is an asymptote of the curve.

(v) $f'(x) = (x+1)e^x \Rightarrow f(x)$ increases for $x \geq -1$ and decreases for $x \leq -1$. Hence $x = -1$ is the point of absolute minima. Minimum value = $f(-1) = -\frac{1}{e}$.

(vi) $f''(x) = (x+2)e^x \Rightarrow f(x)$ is concave up for $x > -2$ and concave down for $x < -2$ and hence $x = -2$ is a point of inflexion.



The required area = $-\int_{-\infty}^{0} xe^x dx = -[xe^x]_{-\infty}^{0} + \int_{-\infty}^{0} e^x dx = 0 + [e^x]_{-\infty}^{0} = 1$ sq. unit.