1. Differential Equations

An equation involving independent variable (x), dependent variable (y) and derivative of dependent variable with respect to independent variable $\left(\frac{dy}{dx}\right)$ is called a differential equation.

2. Order and Degree of a Differential Equation

The order of a differential equation is the order of the derivative of the highest order occurring in the differential equation.

The degree of a differential equation is the degree of the highest order differential coefficient appearing in it, provided it can be expressed as a polynomial equation in derivatives.

Illustration 1:

Find the order and degree of the following differential equations

(i) $(y'')^5 + 2\,(y')^6 + 3x^2 - y = 0$

(ii) $(y''')^3 + \ln\,(y'' - xy') = 0$

(iii) $y'' = \sin(y')$

(iv) $\sqrt{\frac{d^2y}{dx^2}} = 3\sqrt{\frac{dy}{dx} + 3}$

Solution

(i) Clearly, the order and degree of the given differential equation are 2 and 5 respectively.

(ii) Clearly, order of the given differential equation is 3 and degree is not defined as it cannot be written as a polynomial in derivatives.

(iii) Clearly, the order of the given differential equation is 2 and degree is not defined as it cannot be written as a polynomial equation in derivatives.

(iv) Clearly, order of the differential equation is 2 and for degree we rewrite the given differential equation as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}+3\right)^2$$

Hence the degree of the given differential equation is 3.

3. Formation of a Differential Equation Whose General Solution is given

A differential equation can be derived from its equation by the process of differentiation and other algebraic processes of elimination etc.

Note: The general solution of a differential equation of the $\mathbf{n^{th}}$ order must contain **n** and only **n** independent arbitrary constants.

Illustration 2:

By the elimination of the parameters h and k, find the differential equation of which $(x - h)^2 + (y - k)^2 = a^2$ is a solution.

Solution

Three relations are necessary to eliminate two constants. Thus, besides the given relation, we require two more and they will be obtained by differentiating the given relation twice successively. Thus we have

$$(x - h) + (y - k)\,\frac{dy}{dx} = 0 \qquad \ldots \text{(i)}$$

$$1 + (y - k)\,\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \ldots \text{(ii)}$$

From (i) and (ii), we obtain $y - k = -\dfrac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}}$ and $x - h = \dfrac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}}$.

Substituting these values in the given relation, we obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2.$$

This is the required differential equation.

Illustration 3:

Show that $y = ae^x + be^{2x} + ce^{-3x}$ is a solution of the equation $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$.

Solution

We have $y = ae^x + be^{2x} + ce^{-3x}$. … (i)

(i) $\Rightarrow y_1 = ae^x + 2be^{2x} - 3ce^{-3x}$ … (ii)

(ii) – (i) $\Rightarrow y_1 - y = be^{2x} - 4ce^{-3x}$ … (iii)

(iii) $\Rightarrow y_2 - y_1 = 2be^{2x} + 12ce^{-3x}$ … (iv)

(iv) – 2 × (iii) $\Rightarrow y_2 - y_1 - 2(y_1 - y) = 20\,ce^{-3x}$

$\Rightarrow y_2 - 3y_1 + 2y = 20ce^{-3x}$ … (v)

(v) $\Rightarrow y_3 - 3y_2 + 2y_1 = -60ce^{-3x}$ … (vi)

(vi) + 3 × (v) $\Rightarrow y_3 - 3y_2 + 2y_1 + 3(y_2 - 3y_1 + 2y) = 0$

$\Rightarrow y_3 - 7y_1 + 6y = 0$

$\Rightarrow \dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$. This is the required differential equation.

4. Solution of a Differential Equation by the Method of Variable Separation

If the coefficient of dx is only a function of x and that of dy is only a function of y in the given differential equation, then the equation can be solved using variable separation method.

Thus the general form of such an equation is

$$f(x)dx + \phi(y)dy = 0 \qquad \text{... (i)}$$

Integrating, we get $\int f(x)dx + \int \phi(y)dy = C$, Which is the solution of (i)

Illustration 4:

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

Solution

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad -\frac{\sqrt{1+x^2}}{x}dx = \frac{ydy}{\sqrt{1+y^2}}$$

Integrating we get

$$\Rightarrow \qquad -\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{ydy}{\sqrt{1+y^2}}$$

$$\Rightarrow \qquad -\left[\sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)\right] = \sqrt{1+y^2} + c$$

This is the solution to the given differential equation.

Solution of a Differential Equation of the Type $\frac{dy}{dx} = f(ax+by+c)$

Consider the differential equation $\frac{dy}{dx} = f(ax+by+c)$ … (i)

Where, $f(ax + by + z)$ is some function of '$ax + by + c$'.

Let z = $ax + by + c$.

$$\therefore \quad \frac{dz}{dx} = a + b\frac{dy}{dx} \text{ or } \frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b}$$

$$\therefore \quad \text{(i)} \Rightarrow \frac{\frac{dz}{dx} - a}{b} = f(z) \Rightarrow \frac{dz}{dx} = bf(z) + a \quad \Rightarrow \quad \frac{dz}{bf(z)+a} = dx \qquad \text{… (ii)}$$

In the differential equation (ii), the variables x and z are separated. Integrating (ii), we get

$$\int \frac{dz}{bf(z)+a} = x + C, \text{ where } z = ax + by + c$$

This represents the general solution of the differential equation (i).

Illustration 5:

Solve the differential equation $\frac{dy}{dx}$ = sin $(x + y)$ + cos $(x + y)$

Solution:

We have $\frac{dy}{dx}$ = sin $(x + y)$ + cos $(x + y)$ …(i)

Let $\quad z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

$\therefore$ (i) $\Rightarrow \dfrac{dz}{dx} - 1 = \sin z + \cos z$

$\Rightarrow \dfrac{dz}{\sin z + \cos z + 1} = dx$ (Variables are separated)

Integrating both sides, we get $\displaystyle\int \frac{dz}{\sin z + \cos z + 1} = \int 1.dx + C$

$$\Rightarrow \int \frac{dz}{\frac{2\tan z/2}{1+\tan^2 z/2} + \frac{1-\tan^2 z/2}{1+\tan^2 z/2} + 1} = x + C \Rightarrow \int \frac{\sec^2 z/2 dz}{2\tan z/2 + 2} = x + C$$

$\Rightarrow \displaystyle\int \frac{dt}{t+1} = x + C$, where $t = \tan\frac{z}{2}$. $\Rightarrow$ Log $|t + 1| = x + C$

$\Rightarrow \log\left|\tan\dfrac{x+y}{2} + 1\right| = x + C$. This is the required general solution.

Illustration 6:

Solve the differential equation. $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$. Where $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

Solution

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ … (i)

Let $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \lambda$ (say)

$\Rightarrow$ $a_1 = \lambda a_2,\ b_1 = \lambda b_2$

$\therefore$ (i) Becomes $\dfrac{dy}{dx} = \dfrac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2} \Rightarrow \dfrac{dy}{dx} = \dfrac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$ … (ii)

Let $z = a_2x + b_2y$

$\therefore$ (ii) Becomes $\dfrac{\frac{dz}{dx}-a_2}{b_2}=\dfrac{dy}{dx}=\dfrac{\lambda z+c_1}{z+c_2}$

$\Rightarrow \dfrac{dz}{dx}=\dfrac{b_2(\lambda z+c_1)}{z+c_2}+a_2 \quad \Rightarrow \dfrac{dz}{dx}=\dfrac{\lambda b_2 z+b_2c_1+a_2z+a_2c_2}{z+c_2}$

$$\Rightarrow \frac{z+c_2}{(\lambda b_2+a_2)z+b_2c_1+a_2c_2}dz=dx \qquad \text{… (iii)}$$

In the differential equation (iii), the variables x and z are separated.

Integrating (iii), we get $\displaystyle\int\frac{z+c_2}{(\lambda b_2+a_2)z+b_2c_1+a_2c_2}dz=\int 1.dx+C$

$\Rightarrow \displaystyle\int\frac{z+c_2}{(\lambda b_2+a_2)z+b_2c_1+a_2c_2}dz=x+C$, where $z = a_2x + b_2y$.

This represents the general solution of the differential equation (i)

Remark:

In differential equation (ii), $\dfrac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}$ is a function of $a_2x + b_2y$.

If we take $f(a_2x + b_2y) = \dfrac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}$, then (ii) becomes

$\dfrac{dy}{dx} = f(a_2x + b_2y)$ and we have already learnt the method of solving this type of differential equation by substituting $z = a_2x + b_2y$.

Illustration 7:

Solve the differential equation $\frac{dy}{dx}=\frac{2x-y+2}{2y-4x+1}$

Solution

We have $\frac{dy}{dx}=\frac{2x-y+2}{2y-4x+1}$

$\Rightarrow \quad \frac{dy}{dx}=\frac{2x-y+2}{-4x+2y+1}$ … (i)

Here $\frac{a_1}{a_2}=\frac{2}{-4}=-\frac{1}{2}$ and $\frac{b_1}{b_2}=\frac{-1}{2}$ $\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}$

(i) $\Rightarrow \quad \frac{dy}{dx}=\frac{(2x-y)+2}{-2(2x-y)+1}$ … (ii)

Let $z = 2x - y$ $\therefore \frac{dz}{dx}=2-\frac{dy}{dx}$ or $\frac{dy}{dx}=2-\frac{dz}{dx}$

$\therefore$ (ii) $\Rightarrow \quad 2-\frac{dz}{dx}=\frac{z+2}{-2z+1} \Rightarrow \frac{2z-1}{5z}dz=dx$ (Variables are separate)

Integrating both sides, we get $\int\frac{2z-1}{5z}dz=\int dx$

$\Rightarrow \int\left(\frac{2}{5}-\frac{1}{5z}\right)dz = x + C$

$\Rightarrow \frac{2}{5}z-\frac{1}{5}\log|z| = x + C$

$\Rightarrow 2z - \log|z| = 5x + 5C$

$\Rightarrow 2(2x - y) - \log|2x - y| = 5x + C_1$ where $C_1 = 5C$

$\Rightarrow x + 2y + \log|2x - y| + C_1 = 0$

5. Homogeneous Equation

A function f(x, y) is called homogeneous function of degree n if

$$f(x, \lambda y) = \lambda^n f(x, y)$$

A differential equation of the form $\frac{dy}{dx} = f(x, y)$, where f (x, y) is a homogenous polynomial of degree zero is called a homogenous differential equation. Such equations are solved by substituting $v = y/x$ (or x/y) and then separating the variables.

Illustration 8:

Solve $x\,dy - y dx - \sqrt{x^2 - y^2}\,dx = 0$

Solution:

We have

$$x\,dy - y dx - \sqrt{x^2 - y^2}\,dx = 0 \qquad \ldots(i)$$

Which is a homogeneous equation, using the transformation

$$y = vx,\ dy = v\,dx + x\,dv$$

equation (i) reduces to

$$v\,dx + x\,dv - v dx - \sqrt{1 - v^2}\,dx = 0$$

i.e. $x\ dv - \sqrt{1 - v^2}\,dx = 0$, i.e. $\frac{dx}{x} - \frac{dv}{\sqrt{1 - v^2}} = 0$ in which the variables are separated.

Integrating, we have

$$\sin^{-1} v = \ln x + \ln C \text{ (arbitrary constant)}$$

i.e. $\sin^{-1}\left(\frac{y}{x}\right) = \ln(Cx)$, which is the required solution.

Differential Equation Reducible To Homogenous Forms

Equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}, (aB \neq Ab)$ can be reduced to a homogenous form by changing the variables x, y to X, Y by equations

x = X + h, y = Y + k, where h, k are constants to be chosen so as to make the given equation homogenous, we have

$$\frac{dy}{dx} = \frac{d(Y + k)}{dx} = \frac{dY}{dX} . \frac{dX}{dx} = \frac{dY}{dX}$$

So equation becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)}$$

Let h and k be chosen so as to satisfy the equations $ah + bk + c = 0$ and $Ah + Bk + C = 0$

This gives $h = \frac{bC - Bc}{aB - Ab}, k = \frac{Ac - aC}{aB - Ab}$, which are meaningful except when $aB \neq Ab$

Thus the reduced equation is $\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$, which can now be solved by means of the substitution $Y = VX$.

Illustration 9:

Solve the differential equation $\frac{dy}{dx} = \frac{x + 2y - 5}{2x + y - 4}$

Solution

We have $\frac{dy}{dx} = \frac{x + 2y - 5}{2x + y - 4}$ …(i)

Here $\frac{a_1}{a_2}=\frac{1}{2}$ and $\frac{b_1}{b_2}=\frac{2}{1}=2$. $\quad\therefore\ \frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Let $x = X + h$ and $y = Y + k$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dY}\times\frac{dY}{dX}\times\frac{dX}{dx}=1\times\frac{dY}{dX}\times 1=\frac{dY}{dX}$$

$$\therefore \text{(i)} \Rightarrow \frac{dY}{dX}=\frac{(X+h)+2(Y+k)-5}{2(X+h)+(Y+k)-4}$$

$$\Rightarrow \qquad \frac{dY}{dX}=\frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)} \qquad \ldots\text{(ii)}$$

Let h and k be such that $h + 2k – 5 = 0$ and $2h + k – 4 = 0$.

$\therefore \qquad h = 1,\ k = 2$

$$\therefore \text{(ii)} \Rightarrow \frac{dY}{dX}=\frac{X+2y}{2X+Y} \qquad \ldots\text{(iii)}$$

This is a homogeneous differential equation

Let $Y = VX \quad \therefore\ \frac{dY}{dX}=V+X\frac{dV}{dX}$

$$\therefore \text{(iii)} \Rightarrow V+X\frac{dV}{dX}=\frac{X+2(VX)}{2X+VX}=\frac{1+2V}{2+V}$$

$$\Rightarrow \qquad X\frac{dV}{dX}=\frac{1+2V}{2+V}-V=\frac{1+2V-2V-V^2}{2+V}$$

$$\Rightarrow \qquad \frac{2+V}{1-V^2}dV=\frac{dX}{X}$$

$$\Rightarrow \qquad \int\frac{2+V}{1-V^2}dV=\int\frac{dX}{X}+\log C$$

$$\Rightarrow \quad \int \frac{2+V}{(1+V)\,(1-V)}\,dV = \log|X| + \log C$$

$$\Rightarrow \quad \int \left[\frac{1}{(1+V)(2)} + \frac{3}{2(1-V)} \right] dV = \log C\,|X|$$

$$\Rightarrow \quad \frac{1}{2}\log|1+V| + \frac{3}{2}\cdot\frac{\log|1-V|}{-1} = \log C\,|X| \quad \Rightarrow \quad \log\left|\frac{1+V}{(1-V)^3}\right| = \log C^2 X^2$$

$$\Rightarrow \quad \left|\frac{1+(Y/X)}{[1-(Y/X)]^3}\right| = C^2 X^2 \quad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = \pm C^2$$

$$\Rightarrow \quad X + Y = C_1\,(X - Y)^3 \text{ where } C_1 = \pm\, C^2$$

$$\Rightarrow \quad (x-1) + (y-2) = C_1\,[(x-1) - (y-2)]^3$$

6. Linear Differential Equation

Form: $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x alone.

Integrating factor $= e^{\int P dx}$

Multiplying the form by $e^{\int P dx}$ on both sides,

We get $e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q.e^{\int P dx}$

Or $\quad \frac{dy}{dx}.e^{\int P dx} + y.\frac{d}{dx}e^{\int P dx} = Qe^{\int P dx} \quad$ since $\frac{d}{dx}\left(e^{\int P dx}\right) = P.e^{\int P dx}$

Or $\quad \frac{d}{dx}\left(y.e^{\int P dx}\right) = Q.e^{\int P dx} \quad$ or $\quad \int \frac{d}{dx}.\left(y.e^{\int P dx}\right)dx = \int Q.e^{\int P dx} + C$

Or $\quad y.e^{\int P dx} = \int Q.e^{\int P dx} + C$

which is the required solution of the given differential equation.

Illustration 10:

Solve $2x\frac{dy}{dx}=y+6^{5/2}-2\sqrt{x}$.

Solution

The given equation can be written as

$$\frac{dy}{dx}+y\left(\frac{-1}{2x}\right)=3x^{3/2}-\frac{1}{\sqrt{x}}.$$

$$\text{I.F.}=e^{\int\frac{-1}{2x}dx}=e^{\frac{-1}{2}\ln x}=\frac{1}{\sqrt{x}}.$$

Hence, solution of the given differential equation is given by

$$y.\frac{1}{\sqrt{x}}=\int\left(3x-\frac{1}{x}\right)dx=\frac{3}{2}x^2-\ln x+C$$

$$y=\frac{3}{2}x^{5/2}-\sqrt{x}\ln x+C\sqrt{x}$$

Differential Equation Reducible To the Linear Form

$$f'(y)\frac{dy}{dx}+f(y)P(x)=Q(x) \qquad \ldots\text{(i)}$$

The transformation

$$f(y)=u \qquad \Rightarrow \qquad f'(y)\,dy=du$$

The equation (i) reduces to.

$$\frac{du}{dx}+uP(x)=Q(x)$$

which is of the linear differential equation form.

Illustration 11:

Solve $\{xy^3 (1 + \cos x) - y\}\, dx + x dy = 0$.

Solution

The given equation can be written as

$$\frac{dy}{dx} + y^3(1+\cos x) - \frac{y}{x} = 0$$

i.e. $$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{y^2 x} = -(1+\cos x)$$

using the transformation

$$\frac{-1}{y^2} = u \text{, we get } \frac{2}{y^3}dy = du$$

The above equation reduces to

$$\frac{1}{2}\frac{du}{dx} + \frac{u}{x} = -(1 + \cos x)$$

Whose I.F. = $e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

Hence, the solution of the given differential equation is given by

$$ux^2 = -2\int x^2(1+\cos x)\,dx$$

i.e. $$\frac{x^2}{2y^2} = \int x^2 dx + \int x^2 \cos x\, dx = \frac{x^3}{3} + x^2 \sin x - \int 2x \sin x\, dx$$

$$= \frac{x^3}{3} + x^2 \sin x + 2x\cos x - 2\int \cos x + C$$

$$\frac{x^2}{2y^2} = \frac{x^3}{3} + x^2 \sin x + 2x\cos x - 2\sin x + C$$

7. General Form of Variable Separation

If we can write the differential equation in the form

$f\{f_1(x, y)\}\, d\{f_1(x, y)\} + \phi\{f_2(x, y)\}\, d\{f_2(x, y)\} + \ldots =. 0,$

then each term can be easily integrated separately.

For this the following results must be memorized.

(i) $d(x + y) = dx + dy$

(ii) $d(xy) = y\,dx + x\,dy$

(iii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(v) $d(\log xy) = \frac{ydx + xdy}{xy}$

(vi) $d\left(\log\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$

(vii) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(viii) $d\left(\sqrt{x^2 + y^2}\right) = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$

Illustration 12:

Solve the differential equation $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\cos^2(x^2 + y^2)}{y^3}$

Solution

The given differential equation can be written as

$$\frac{xdx + ydy}{(ydx - xdy)/y^2} = y^2\,\frac{x}{y^3}\left(\cos^2(x^2 + y^2)\right)$$

$$\Rightarrow\ \sec^2(x^2 + y^2)\frac{1}{2}d(x^2 + y^2) = \frac{x}{y}d\left(\frac{x}{y}\right) \Rightarrow\ \frac{1}{2}\tan(x^2 + y^2) = \frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{c}{2}$$

$$\therefore\ \tan(x^2 + y^2) = \frac{x^2}{y^2} + c$$

Illustration 13:

The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country. $\left(\sqrt{2}=1.4142\right)$

Solution

Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country. Then, from equation (a) $\frac{dN}{dt}-kN=0$ which has the solution

$N = ce^{kt}$ …(i)

At $t = 0$, $N = N_0$;

hence, it follows from (i) that $N_0 = ce^{k(0)}$, or that $c = N_0$.

Thus, $N = N_0e^{kt}$ …(ii)

At $t = 2$, $N = 2N_0$.

Substituting these values into (ii), we have

$2N_0 = N_0e^{2k}$ from which $k = \frac{1}{2}\ell n2$

Substituting this value into (i) gives

$N = N_0e^{t/2 \ln 2}$ …(iii)

At $t = 3$, $N = 20{,}000$.

Substituting these values into (iii), we obtain

$20{,}000 = N_0e^{3/2 \ln 2} \Rightarrow N_0 = 20{,}000/2\sqrt{2} \approx 7071$

Illustration 14:

A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 mg. of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find **(a)** an expression for the mass of the material remaining at any time *t*, **(b)** the mass of the material after four hours, and **(c)** the time at which the material has decayed to one half of its initial mass.

Solution

(a) Let N denote the amount of material present at time t.

Then, from equation (a) $\frac{dN}{dt} - kN = 0$

This differential equation is separable and linear, its solution is

$$N = ce^{kt} \qquad \ldots \text{(i)}$$

At $t = 0$, we are given that $N = 50$.

Therefore, from (i), $50 = ce^{k(0)}$, or $c = 50$.

Thus, $N = 50\, e^{kt}$ … (ii)

At $t = 2$, 10 percent of the original mass of 50mg. or 5mg, has decayed. Hence, at $t = 2$,

$N = 50 - 5 = 45$. Substituting these values into (ii) and solving for k, we have

$$45 = 50e^{2k} \text{ or } k = \tfrac{1}{2}\ell n \frac{45}{50}$$

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$$N = 50e^{-(1/2)\ln(0.9)\, t}$$ … (iii), where t is measured in hours.

(b) We require N at $t = 4$.

Substituting t **=** 4 into (iii) and then solving for N, we find

$$N = 50e^{-2\ln(0.9)}$$

(c) We require t when $N = 50/2 = 25$.

Substituting $N = 25$ into (iii) and solving for t, we find

$$25 = 50e^{-1/2 \ln .9t} \Rightarrow t = \ln 1/2\ (-1/2 \ln 0.9) \text{ hr.}$$

Illustration 15:

A metal bar at a temperature of 100º F is placed in a room at a constant temperature of 0ºF. If after 20 minutes the temperature of the bar is 50º F, find **(a)** the time it will take the bar to reach a temperature of 25º F and **(b)** the temperature of the bar after 10 minutes.

Solution

Use equation (a) with $T_m = 0$; the medium here is the room which is being held at a constant temperature of 0º F. Thus we have

$$\frac{dT}{dt} = kT = 0$$

Whose $\qquad T = ce^{-kt} \qquad$ …(i)

Since $T = 100$ at $t = 0$ (the temperature of the bar is initially 100º F), it follows (i) that $\quad 100 = ce^{-k(0)}$ or $100 = c$.

Substituting this value into (i), we obtain $T = 100\, e^{-kt}$ …(ii)

At $t = 20$, we are given that $T = 50$; hence, from (2),

$$50 = 100e^{-20k} \text{ from which } k = -\frac{1}{20}\ell n\frac{50}{100}$$

Substituting this value into (ii), we obtain the temperature of the bar at any time t as

$$T = 100e^{(1/20)\ln(1/2)\,t} \qquad \text{…(iii)}$$

(a) We require t when $T = 25$.

Substituting $T = 25$ into (iii), we have

$$25 = 100e^{(1/20)\ln(1/2)\,t}$$

Solving, we find that $t = 39.6$ min.

(b) We require T when $t = 10$.

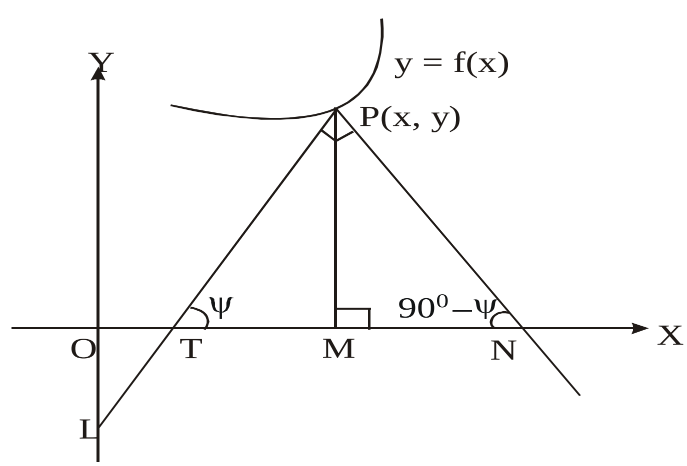
Substituting $t = 10$ into (iii) and then solving for T, we find that

$$T = 100e^{(1/20)\ln(1/2)\,10^{o}}\text{F}$$

It should be noted that since Newton's law is valid only for small temperature differences, the above calculations represent only a first approximation to the physical situation.

Geometrical Applications

Let P(x, y) be any point on the curve $y = f(x)$. Let the tangent and normal at P(x, y) to the curve meets x–axis and y–axis at T and N respectively.



Now draw perpendicular from P on x–axis.

$\therefore \quad PM = y$

If tangent at P makes angle ψ with positive direction of x–axis, then

$$\frac{dy}{dx} = \tan\psi$$

(a) Length of Sub–tangent: TM is defined as sub-tangent. In Δ PTM

$$TM = |\,y\cot\psi\,| = \left|\frac{y}{\tan\psi}\right| = \left|y\frac{dx}{dy}\right|$$

(b) Length of Sub–normal: MN is defined as sub-normal. In Δ PMN

$$MN = |y \cot(90° - \psi)| = |y \tan\psi| = \left|y\frac{dy}{dx}\right|$$

(c) Length of Tangent: PT is defined as length as length of tangent. In Δ PMT

$$PT = |y \operatorname{cosec}\psi| = \left|y\sqrt{(1+\cot^2\psi)}\right| = \left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

(d) Length of normal: PN is defined as length of normal. In Δ PMN

$$PN = |y \operatorname{cosec}(90° - \psi)| = |y \sec\psi|$$

$$= \left|y\sqrt{(1+\tan^2\psi)}\right| = \left|y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\right|$$

(e) Intercepts made by the tangent on the coordinate axes: The equation of tangent at P(x, y) is $Y - y = \frac{dy}{dx}(X - x)$ ....(i)

Putting Y = 0 in (i), we get $X = x - y\frac{dx}{dy}$

Hence the tangent of intercept OT that the tangent cuts off from the x–axis is $x - y\frac{dx}{dy}$

Putting X = 0 in (i), we get $Y = y - x\frac{dy}{dx}$

Hence the length of intercept OL that the tangent cuts off from the y–axis is $y - x\frac{dy}{dx}$

Illustration 16:

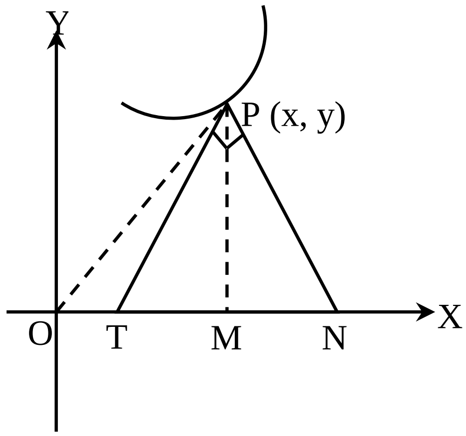
Find the curve for which the length of normal is equal to the radius vector.

Solution

Here radius vector = OP

And length of normal = PN

According to question PN = OP



$$\Rightarrow y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{(x^2+y^2)} \quad \Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 = x^2$$

$$\Rightarrow \pm y\frac{dy}{dx} = x \text{ Or } x\,dx \pm y\,dy = 0$$

Integrating, we get $x^2 \pm y^2 = c^2$

This equation represents a circle or equilateral hyperbola as +ve or –ve sign.

Illustration 17:

Find the curve for which the intercept cut off by any tangent on y–axis is proportional to the square of the ordinate of the point of tangency.

Solution

Let P(x, y) be any point on the curve

$\therefore$ Length of intercept on y-axis =   (according to question)

Length of intercept on y-axis $\propto y^2$

$$\Rightarrow y - x\frac{dy}{dx} = ky^2$$

Where k is constant of proportionality

Or $\frac{y}{x} - \frac{dy}{dx} = \frac{ky^2}{x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -\frac{ky^2}{x}$

Put $y^{-1} = v \qquad \therefore -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

Then $\frac{dv}{dx} - \frac{v}{x} = -\frac{k}{x}$ or $\frac{dv}{dx} + \frac{v}{x} = \frac{k}{x}$

Which is linear differential equation

$$\therefore \text{I.F.} = e^{\int 1/x dx} = e^{\ell n x} = x$$

$\therefore$ The solution is $v.x = \int \frac{k}{x} x\,dx + c$

$$\Rightarrow vx = kx + c \quad \Rightarrow \quad \frac{x}{y} = kx + c \qquad \left(\because v = \frac{1}{y}\right)$$

$$\Rightarrow \frac{c_1}{x} + \frac{c_2}{y} = 1 \qquad \left(\text{where} -\frac{c}{k} = c_1 \text{ and } \frac{1}{k} = c_2\right)$$