1. Definition

A circle is the locus of a point which moves in such a way that its distance from a fixed point is constant. The fixed point is called the centre of the circle and the constant distance, the radius of the circle.

2. Equation of The Circle in Various Forms

- The simplest equation of the circle is $x^2 + y^2 = r^2$ whose centre is (0, 0) and radius r.
- The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle with centre (a, b) and radius r.
- The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is the general equation of a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Diametric form

Equation of the circle with points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$..

Parametric equation of a circle

The equation $x = a \cos\theta$, $y = a \sin\theta$ are called parametric equations of the circle $x^2 + y^2 = a^2$ and θ is called a parameter the point $(a \cos\theta, a \sin\theta)$ is also referred to as point θ. The parametric coordinates of any point on the circle $(x - h)^2 + (y - k)^2 = a^2$ are given by $(h + a \cos\theta, k + a \sin\theta)$ with $0 \leq \theta \leq 2\pi$.

The equation of a straight line joining two point α and β on the circle

$x^2 + y^2 = a^2$ is $x\cos\dfrac{\alpha+\beta}{2} + y\sin\dfrac{\alpha+\beta}{2} = a\cos\dfrac{\alpha-\beta}{2}$.

Intercepts made on axes

For the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of circle

- The intercept made on x-axis by the circle = $2\sqrt{g^2 - c}$ if $g^2 > c$.
- The intercept made on y-axis by the circle = $2\sqrt{f^2 - c}$ if $f^2 > c$.
- If $g^2 - c > 0$ $\Rightarrow$ circle cuts x-axis at two distinct points.
- If $g^2 = c$ $\Rightarrow$ circle touches the x-axis.
- If $g^2 < c$ $\Rightarrow$ circle lies completely above or below the x-axis.
- Similar in the case for intercepts on y-axis.

Illustration 1:

Find the equation of the circle whose diameter is the line joining the points (– 4, 3) and (12, – 1). Find the intercept made by it on the y-axis

Solution

The equation of the required circle is $(x + 4)(x - 12) + (y - 3)(y + 1) = 0$

On the y-axis, $x = 0$

$\Rightarrow -48 + y^2 - 2y - 3 = 0 \Rightarrow y^2 - 2y - 51 = 0$

$\Rightarrow y = 1 \pm \sqrt{52}$

Hence the intercept on the y-axis = $2\sqrt{52} = 4\sqrt{13}$

Illustration 2:

A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Solution

Let the centre of the circle be (α, β).

It lies on the line $y = x - 1$

$\Rightarrow \beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ The equation of the circle is

$$(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$$

It passes through (7, 3)

$$\Rightarrow (7 - \alpha)^2 + (3 - \alpha + 1)^2 = 9$$

$$\Rightarrow (7 - \alpha)^2 + (4 - \alpha)^2 = 9$$

$$\Rightarrow \alpha^2 - 11\alpha + 28 = 0$$

$$\Rightarrow (\alpha - 4)(\alpha - 7) = 0 \Rightarrow \alpha = 4, 7.$$

Hence the required equations are

$$x^2 + y^2 - 8x - 6y + 16 = 0$$

And $x^2 + y^2 - 14x - 12y + 76 = 0$

Equations of Tangents and Normals

If S = 0 be a curve then $S_1 = 0$ indicate the equation which is obtained by substituting $x = x_1$ and $y = y_1$ in the equation of the given curve, and T = 0 is the equation which is obtained by substituting $x^2 = xx_1$, $y^2 = yy_1$, $2xy = xy_1 + yx_1$, $2x = x + x_1$, $2y = y + y_1$ in the equation S = 0.

If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

•Equation of the tangent to $x^2 + y^2 + 2gx + 2fy + c = 0$ at $A(x_1, y_1)$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

•The equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) lying on the circle is $= \dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$. In particular, equations of the tangent and the normal to the circle $x^2 + y^2 = a^2$ at

(x_1, y_1) are $xx_1 + yy_1 = a^2$; and $\dfrac{x}{x_1} = \dfrac{y}{y_1}$ respectively.

Equation of tangent in Slope Form

The condition that the straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$ and the point of contact is $(-a^2m/c, a^2/c)$

i.e. $y = mx \pm a\sqrt{1+m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$ whatever be the value of m.

Equation of tangents in parametric form

Since parametric co-ordinates of circle $x^2 + y^2 = a^2$ is $(a \cos \theta, a \sin \theta)$, then equation of tangent at $(a \cos \theta, a \sin \theta)$ is

$x.\, a \cos \theta + y.\, a \sin \theta = a^2$ or $x \cos \theta + y \sin \theta = a$

Length Of A Tangent And Power Of A Point

The length of a tangent from an external point (x_1, y_1) to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$. Square of length of the tangent from the point P is also called the Power of a Point w.r.t. a circle.

Chord of Contact

If two tangents PT_1 & PT_2 are drawn from an outside point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact $T_1 T_2$ is: $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Equation of Pair of Tangent

The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ is $T^2 = SS_1$.

Director Circle

The locus of the point of intersection of perpendicular tangents is called director circle. If $(x - \alpha)^2 + (y - \beta)^2 = r^2$ is the equation of a circle then its director circle is $(x - \alpha)^2 + (y - \beta)^2 = 2r^2$

Equation Of The Chord With A Given Middle Point

The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid-point $M(x_1, y_1)$ is $y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)$.This on simplification can be put in the from

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

The position of a point with respect to a circle

- The point $P(x_1, y_1)$ lies outside, on, or inside a circle

  $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, accordingly as

  $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > = \text{ or } < 0$.
- The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.

Length of perpendicular (p) from the centre on the line then

- $p > r \Leftrightarrow$ the line does not meet the circle
- $p = r \Leftrightarrow$ the line touches the circle
- $0 < p < r \Leftrightarrow$ the line is a secant of the circle
- $p = 0 \Leftrightarrow$ the line is a diameter of the circle

Some important notes

- Chord of contact exists only if the point 'P' is outside the circle
- Length of chord of contact $T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$, where R is radius of the circle & L is the length of the tangent from (x_1, y_1) on the circle $S = 0$.
- Area of the triangle formed by the pair of tangents and its chord of contact $= \frac{RL^3}{R^2 + L^2}$

  Where R is radius of the circle & L is the length of the tangent from (x_1, y_1) on the circle $S = 0$.
- Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\frac{2LR}{L^2 - R^2}\right)$ where R = radius; L = length of tangent.
- Equation of the circle circumscribing the triangle PT_1T_2 is:

  $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$

Illustration 3:

Find the co-ordinates of the point from which tangents are drawn to the circle $x^2 + y^2 - 6x - 4y + 3 = 0$. Such that the mid-point of its chord of contact is (1, 1).

Solution

Let the required point be P (x_1, y_1).

The equation of the chord of contact of P with respect to the given circle is $xx_1 + yy_1 - 3(x + x_1) - 2(y + y_1) + 3 = 0$

The equation of the chord with mid-point (1, 1) is

$x + y - 3(x + 1) - 2(y + 1) + 3 = 1 + 1 - 6 - 4 + 3$

$\Rightarrow 2x + y = 3$

Equating the ratios of the coefficients of x, y and the constant terms and solving for x, y we get, $x_1 = -1$, $y_1 = 0$.

Illustration 4:

Find the pair of tangents from A (5, 10) to the circle $x^2 + y^2 + 4x - 2y - 8 = 0$ and the corresponding points of contact.

Solution

The circle is: $(x + 2)^2 + (y - 1)^2 = 13$; the centre C is $(-2, 1)$ and the radius is $\sqrt{13}$.

Let $y = mx + b$ be a tangent; then, since it passes through A,

$$b = 10 - 5m \qquad \ldots (1)$$

If p is the length of the perpendicular from centre C $(-2, 1)$ to the tangent, then $p = r = \sqrt{13}$. But, the equation of the tangent being written as $mx - y + b = 0$, p is given by

$$p = \pm \frac{-2m - 1 + b}{\sqrt{(1 + m^2)}} \qquad \ldots (2)$$

Or, by means of (1) $p = \pm \dfrac{9 - 7m}{\sqrt{(1 + m^2)}}$.

Hence, m is given by $(9 - 7m)^2 = 13(1 + m^2)$ or $18m^2 - 63m + 34 = 0$

or $(3m - 2)(6m - 17) = 0$

Hence, The gradients are 2/3 and 17/6; the tangents are

$$y - 10 = \frac{2}{3}(x - 5) \text{ And } y - 10 = \frac{17}{6}(x - 5)$$

Or $2x - 3y + 20 = 0$ and $17x - 6y - 25 = 0$ … (3)

The corresponding normals are

$$y - 1 = -\frac{3}{2}(x+2) \text{ and } y - 1 = -\frac{6}{17}(x+2)$$

Or $3x + 2y + 4 = 0$ and $6x + 17y - 5 = 0$

The coordinates of the point of contact of the first tangent in (3) are obtained by solving

$$2x - 3y + 20 = 0 \text{ and } 3x + 2y + 4 = 0;$$

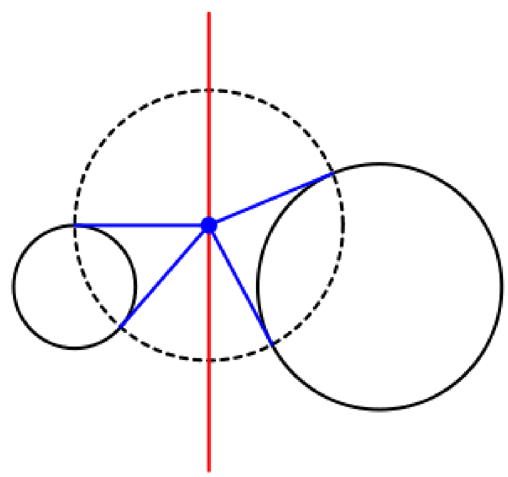
The coordinates are (–4, 4).

Similarly, the point of contact of the second tangent is (7/5, – 1/5)

3. Radical Axis

The radical axis of two circles is the locus of a point from which the tangent segments to the two circles are of equal length.

Figure: Illustration of the radical axis (red line) of two given circles (solid black). For any point **P** (blue) on the radical axis, a unique circle (dashed) can be drawn that is centered on that point and intersects both given circles at right angles, i.e., orthogonally. The point **P** has an equal **power** with respect to both given circles, because the tangents from **P** (blue lines) are radii of the orthogonal circle and thus have equal length.



Equation to the Radical Axis

In general S – S′ = 0 represents the equation of the radical Axis to the two circles i.e. $2x(g - g') + 2y(f - f') + c - c' = 0$

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ & $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$

- If $S = 0$ and $S' = 0$ intersect in real and distinct point then $S - S' = 0$ is the equation of the common chord of the two circles.
- If $S = 0$ and $S' = 0$ touch each other, then $S - S' = 0$ is the equation of the common tangent to the two circles at the point of contact.
- Radical axis is always perpendicular to the line joining the centres of the two circles.
- Radical axis will pass through the mid-point of the line joining the centres of the two circles only if the two circles have equal radii.
- Radical axis bisects a common tangent between the two circles.
- A system of circles, every two of which have the same radical axis, is called a coaxial system.
- Pairs of circles which do not have radical axis are concentric.

Illustration 5:

Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.

Solution

The given circles are $S_1 \equiv x^2 + y^2 - 6x - 4y + 9 = 0$ … (1)

And $S_2 \equiv x^2 + y^2 - 8x - 6y + 23 = 0$ … (2)

Equation of the common chord of circles (1) and (2) which is also the radical axis of the circles S_1 and S_2 is $S_1 - S_2 = 0$ or, $2x + 2y - 14 = 0$ or, $x + y - 7 = 0$ …(3)

Centre of the circle S_2 is (4, 3). Clearly, line (3) passes through the point (4, 3) and hence line (3) is the equation of the diameter of the circle (2). Hence circle (1) bisects the circumference of circle (2).

4. Family of Circles

- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ & $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are two intersecting circles, then $S + \lambda S' = 0$, $\lambda \neq -1$, is the equation of the family of circles passing through the points of intersection $S = 0$ and $S' = 0$.
- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle which is intersected by the straight line $\mu \equiv ax + by + c = 0$ at two real and distinct points, then $S + \lambda\mu = 0$ is the equation of the family of circles passing through the points of intersection of $S = 0$ and $\mu = 0$. If $\mu = 0$ touches $S = 0$ at P, then $S + \lambda\mu = 0$ is the equation of the family of circles, each touching $\mu = 0$ at P.
- The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form.

  $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

  Where, λ is a parameter.
- The equation of the family of circles which touch the line $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any value of m is

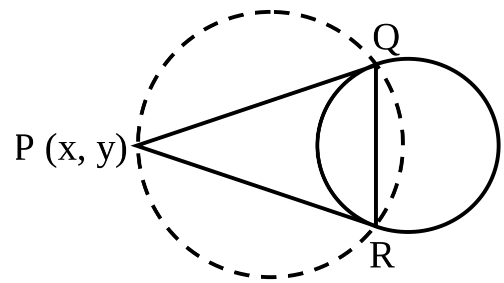
  $(x - x_1)^2 + (y - y_1)^2 + \lambda\{y - y_1 - m(x - x_1)\} = 0$.
- Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.
- Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1 L_3 + \lambda L_2L_4 = 0$ where value of λ can be found out by using condition that coefficient of x^2 = coefficient of $y^2 \neq 0$ and coefficient of $xy = 0$.

Illustration 6:

Tangents PQ and PR are drawn to the circle $x^2 + y^2 = a^2$ from the point P(x_1, y_1). Prove that equation of the circumcircle of ΔPQR is

$x^2 + y^2 – xx_1 – yy_1 = 0$.

Solution



QR is the chord of contact of the tangents to the circle

$$x^2 + y^2 – a^2 = 0 \qquad \ldots (1)$$

So, its equation will be

$$xx_1 + yy_1 – a^2 = 0 \qquad \ldots (2)$$

The circumcircle of ΔPQR is a circle passing through the intersection of the circle (1) and the line (2) and the point P(x_1, y_1).

Circle through the intersection of (1) and (2) is

$$x^2 + y^2 – a^2 + \lambda(xx_1 + yy_1 – a^2) = 0 \qquad \ldots(3)$$

it will pass through (x_1, y_1) if

$$x_1^2 + y_1^2 - a^2 + \lambda(x_1^2 + y_1^2 - a^2) = 0$$

$$\lambda = -1 \text{ (since } x_1^2 + y_1^2 \neq a^2)$$

Hence equation of circle is

$$(x^2 + y^2 – a^2) – (xx_1 + yy_1 – a^2) = 0$$

Or $\quad x^2 + y^2 – xy_1 – yy_1 = 0$.

Illustration 7:

If the line $y = x + 1$ is a tangent to the circle $x^2 + y^2 + 2x + 2y + c = 0$. Then find the corresponding point of contact.

Solution

Let the pt. of contact be (α, β). This point lies on the given line $y = x + 1$

Hence we can have, $\beta = \alpha + 1$

So the point can be written as $(\alpha, \alpha + 1)$.

So, the equation of circle will be of the form

$(x - \alpha)^2 + [y - (\alpha + 1)]^2 + \lambda (x - y + 1) = 0$.

Comparing with the given circle

We get, $-2\alpha + \lambda = 2$ … (1)

And $-2(\alpha + 1) - \lambda = 2$. … (2)

Adding (1) & (2) we get, $\alpha = -\frac{3}{2}$.

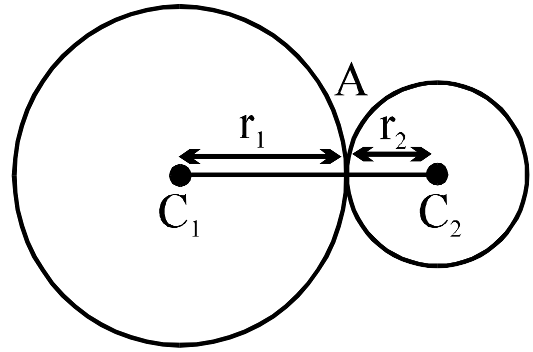
So the point is $\left(-\frac{3}{2}, -\frac{1}{2}\right)$.

5. The Condition That Two Circles Should Intersect

A necessary and sufficient condition for the two circles to intersect at two distinct points is $r_1 + r_2 > C_1C_2 > |r_1 - r_2|$, where C_1, C_2 be the centres and r_1, r_2 be the radii of the two circles.

External and Internal Contacts of Circles:

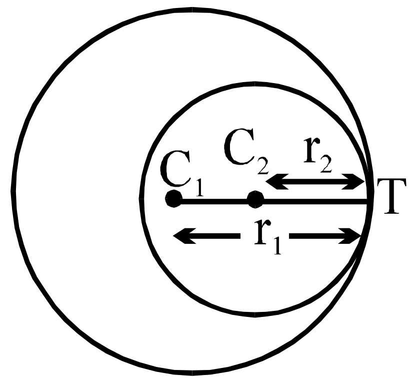
If two circles with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively, touch each other externally, $C_1C_2 = r_1 + r_2$. Coordinates of the point of contact are



$$A \equiv \left(\frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}, \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2} \right)$$

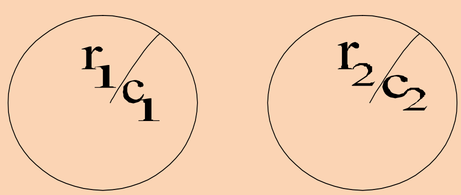
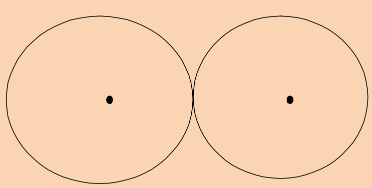
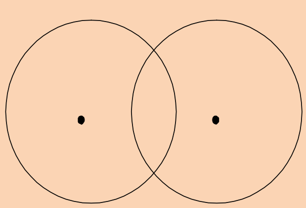
The circles touch each other internally if $C_1C_2 = r_1 - r_2$.

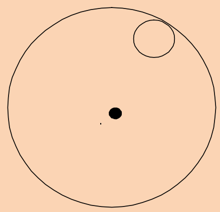
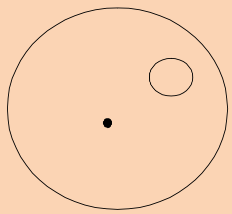
Coordinates of the point of contact are



$$T \equiv \left(\frac{r_1 x_2 - r_2 x_1}{r_1 - r_2}, \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2} \right)$$

Common Tangents to Two Circles

Circles	Number Of Tangents	Condition
	4 common tangents (2 direct and 2 transverse)	$r_1 + r_2 < C_1C_2$
	3 common tangents	$r_1 + r_2 = C_1C_2$
	2 common tangents	$\lvert r_1 + r_2 \rvert < C_1C_2 < r_1 + r_2$

	1 common tangents	$\lvert r_1 - r_2 \rvert = C_1C_2$
	0 common tangents	$C_1C_2 < r_1 + r_2$

Length of an external (or direct) common tangent & internal (or transverse) common tangent

Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles is given by:

$$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2} \ \& \ L_{\text{int}} = \sqrt{d^2 - (r_1 + r_2)^2}$$

Where, d = distance between the centres of the two circles. r_1 & r_2 are the radii of the two circles.

Note that length of internal common tangent is always less than the length of the external or direct common tangent.

- The direct common tangents to two circles meet on the line of centres and divide it externally in the ratio of the radii.
- The transverse common tangents also meet on the line of centres and divide it internally in the ratio of the radii.

Illustration 8:

Prove that the circles $x^2 + y^2 - 6x - 6y - 7 = 0$ and $x^2 + y^2 - 10x + 7 = 0$ intersect and find the coordinates of the common points.

Solution:

The first circle, S_1, is $(x - 3)^2 + (y - 3)^2 = 25$; its centre C_1 is (3, 3) and its radius is 5.

The second circle, S_2, is $(x - 5)^2 + (y - 0)^2 = 18$; i.e. $(x - 5)^2 + y^2 = 18$ its centre C_2, is (5, 0) and its radius is $3\sqrt{2}$.

Subtracting the equations of the circles we obtain, after division by 2, we get,

$$2x - 3y - 7 = 0 \ldots\ldots\ldots\ldots \text{(i)}$$

Then p_1, the perpendicular distance from C_1 (3, 3) to the line (i), is $10/\sqrt{3}$ which is less than

$r_1\ (\equiv 5)$.

Similarly p_2, the perpendicular distance from C_2 (5, 0) to (i), is $3/\sqrt{13}$ which is less than $r_2\ (\equiv 3\sqrt{2})$.

Accordingly, the circles intersect and (i) is the equation of the common chord AB.

The coordinates of A and B are obtained by solving (i) and one of the equations of the circles in this case the second equation is the simpler for this purpose. On eliminating y between the two equations concerned we obtain $9x^2 + (2x - 7)^2 - 90x + 63 = 0$

Or $13x^2 - 118x + 112 = 0$ or $(x - 8)(13x - 14) = 0$.

Thus the abscissa of A and B are 8 and 14/13; from (i) the corresponding ordinates are 3 and –21/13; A and B are the points (8, 3) and (14/13, – 21/13).

Illustration 9:

Find the equations of the common tangents to the circles

$x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

Solution

From the given equation of circles, the centres are $C_1(1, 3)$ and $C_2(-3, 1)$ respectively and the radii are $r_1 = \sqrt{1+9-9} = 1$ and $r_2 = \sqrt{9+1-1} = 3$.

Point I dividing $\overline{C_1C_2}$ internally in the ratio 1 : 3 is $\left(\frac{-3+3}{4} . \frac{10}{4}\right)$ or $\left(0, \frac{5}{2}\right)$

Similarly point E dividing $\overline{C_1C_2}$ externally in the ratio 1 : 3 is (3, 4). The equation of a transverse common tangent is of the form

$y - 5/2 = m(x - 0)$ or $2mx - 2y + 5 = 0$

Since this is tangent to the first circle, its distance from the centre (1, 3) is same as its radius 1.

So, $\frac{2m(1) - 2(3) + 5}{\sqrt{4m^2 + 4}} = 1$ or, $(2m - 1)^2 = 4m^2 + 4$ or, $m = -\frac{3}{4}$

Since the two circles does not touch each other externally,

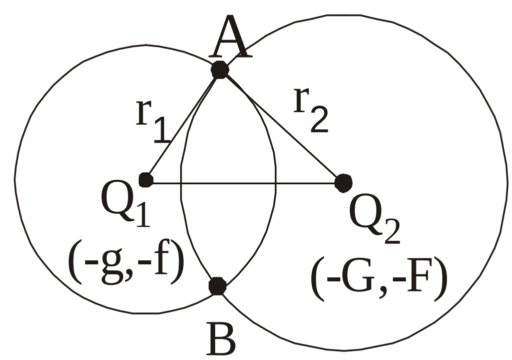
$\therefore$ Second value of $m =$ infinity, so the equations of transverse common tangent are $x = 0$ and $3x + 4 - 10 = 0$. The equation of a direct common tangent is of form $y - 4 = m(x - 3)$ or $mx - y + 4 - 3m = 0$

Since this line is tangent to the circle, we get

$\frac{m(1) - 3 + 4 - 3m}{\sqrt{m^2 + 1}} = 1$ or $(1 - 2m)^2 = m^2 + 1$ Or $(3m - 1)m = 0$ or $m = 0, \frac{4}{3}$

So the common tangents are $y - 4 = 0\ (x - 3)$ or $y = 4$ and $y - 4 = \frac{4}{3}(x - 3)$ or $4x - 3y = 0$

6. Orthogonal Circles



Two circles are said to be orthogonal if the tangents to the circles at either point of intersection are at right angles. In fig. Q_1 and Q_2 are the centres of the circles

$S_1 \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ .....(1)

$S_2 \equiv x^2 + y^2 + 2Gx + 2Fy + C = 0$ .....(2)

The circles, S_1 and S_2, intersect at A and B.

Accordingly, the condition that S_1 and S_2 should be orthogonal is that $\angle Q_1AQ_2$ should be 90°; by Pythagoras' theorem this condition is equivalent to

$Q_1Q_2^2 = Q_1A^2 + Q_2A^2 = r_1^2 + r_2^2$ .... (3)

$(g - G)^2 + (f - F)^2 = g^2 + f^2 - c + G^2 + F^2 - C$

Or, on simplification, $2(g\,G + f\,F) = c + C$. .... (4)

In numerical examples the procedure of solution should be based on the condition expressed by (3).

Note:

- The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.
- The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

Illustration 10:

If two circles cut a third circle orthogonally, prove that their common chord will pass through the centre of the third circle.

Solution

Let us take the equation of the two circles as

$$x^2 + y^2 + 2\lambda_1 x + a = 0 \quad \ldots(1)$$

$$x^2 + y^2 + 2\lambda_2 x + a = 0 \quad \ldots(2)$$

We can select axes suitable (the line of centers as x-axis and the point midway between the centre as origin) to get the above form of equation.

Let the third circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ … (3)

The circle (1) and (3) cut orthogonally

$$2\lambda_1 g = a + c \quad \ldots (4)$$

The circle (2) and (3) cut orthogonally

$$2\lambda_2 g = a + c \quad \ldots (5)$$

From (4) and (5), $2g(\lambda_1 - \lambda_2) = 0$ but $\lambda_1 \neq \lambda_2$

$$\therefore \quad g = 0$$

Hence centre of the third circle $(0, -f)$

The common chord of (1) and (2) has the equation $S_1 - S_2 = 0$

i.e. $x^2 + y^2 + 2\lambda_1 x + a - (x^2 + y^2 + 2\lambda_2 x + a) = 0$

Or $2(\lambda_1 - \lambda_2)x = 0 \;\therefore\; x = 0$

$\therefore$ $(0, -f)$ satisfies the equation $x = 0$.