Introduction

In order to locate the position of a point in a plane, we require two numbers denoting its perpendicular distances from two mutually perpendicular, intersecting, lines in the plane. In actual life, we have, in addition, to deal with points in space. As an example consider the position of an aeroplane as it flies from one station to another at different times during its flight.

For a point in space, we not only require its perpendicular distance from two perpendicular lines, in the horizontal plane, but also the height of the point from the horizontal plane. We require, therefore, not two but three numbers representing the perpendicular distances of the point from the three mutually perpendicular planes. Three numbers representing the three distances are called the coordinates of the point with reference to the three coordinate planes. Thus, a point in space has three coordinates.

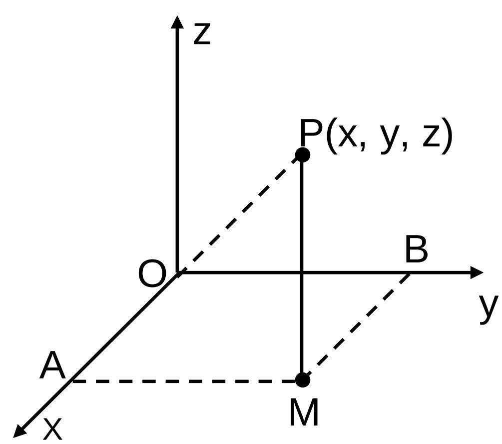
Coordinate Axes and Coordinate Planes

Consider a three dimensional coordinate system O – xyz. The axes are taken in such a way that if we place a screw at the origin and rotate it from the positive direction of the x–axis to the positive y–axis direction, the screw moves in the positive direction of the z–axis.

The plane passing through Ox, Oy axes defines the xy–plane or xOy–plane or the plane z = 0. Similarly, we define the xOz (y = 0), and the yOz (x = 0) planes.

Since the axes are mutually perpendicular, the three coordinate planes are also mutually perpendicular.

Let P (x, y, z) be any point in space. Draw perpendicular PM from P to the xy–plane. By definition PM = z.



From M, draw MA and MB perpendiculars on Ox and Oy respectively. Then, we have

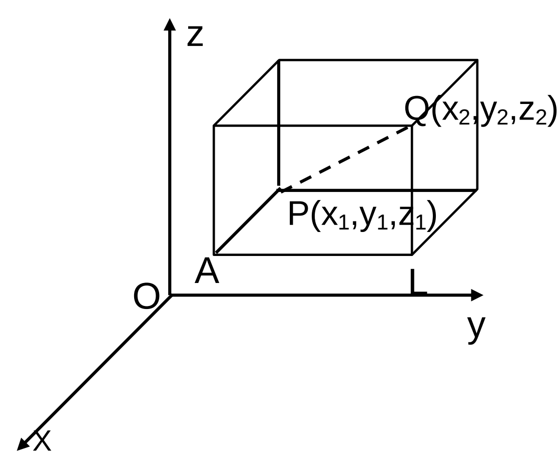
$$OA = MB = x,\ OB = AM = y,\ PM = z.$$

We find that any point in space can be represented by the triplet of numbers (x, y, z). These numbers are positive or negative according to the position of P in space. e.g., if MA cuts the x–axis on the negative side, then the x–coordinate of P is negative. To each point P, in space, would correspond an ordered triplet (x, y, z), and every ordered triplet can be represented by a point in space. There is, thus, one to one correspondence between the points in space and ordered triplets of real numbers.

- The coordinates of the origin O are (0, 0, 0).
- The coordinates of any point on the x–axis will be (x, 0, 0) and so on.
- The coordinates of any point in the yz–plane will be (0, y, z) and so on.

Distance between two points

Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points in space. Draw planes parallel to the coordinate planes through the points P and Q to form a rectangular parallelepiped with PQ as one diagonal.



Angle PAL is a right angle, so that

$$PA^2 + AL^2 = PL^2.$$

Also angle PLQ is a right angle

$$\Rightarrow QL^2 + PL^2 = PQ^2.$$

Hence $PQ^2 = QL^2 + PL^2$

$$= QL^2 + PA^2 + AL^2$$

$$= PA^2 + AL^2 + QL^2 \qquad \dots (1)$$

Now, $PA = x_2 - x_1$, $AL = y_2 - y_1$ and $QL = z_2 - z_1$.

Hence $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$$\Rightarrow PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

If one of the points is taken as the origin O (0, 0, 0) and the other as P (x, y, z), then

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

Illustration 1:

Show that the points P (– 2, 3, 5), Q (1, 2, 3) and R (7, 0, – 1) are collinear.

Solution:

The points are collinear if PQ + QR = PR.

We write $PQ = \sqrt{((1+2)^2 + (2-3)^2 + (3-5)^2} = \sqrt{9+1+4} = \sqrt{14}$,

$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$,

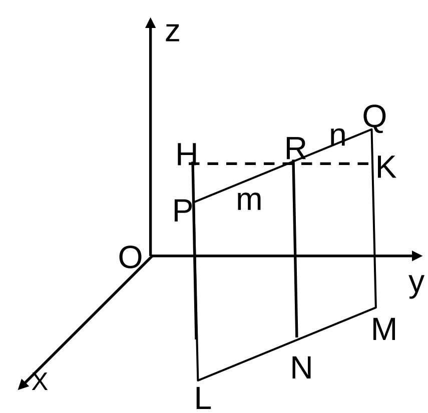
$PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2} = \sqrt{(81+9+36)} = \sqrt{26} = 3\sqrt{14}$.

We find that PR = $3\sqrt{14} = \sqrt{14} + 2\sqrt{14}$ = PQ + QR.

Hence the three given points are collinear.

Section Formula

Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points in space. Let R (x, y, z) be the point which divides the segment PQ in the ratio m : n. Draw PL, QM, RN perpendiculars to the xy–plane. These lines lie in one plane so that the point L, M and N lie on the straight line which is the intersection of this plane with the xy–plane.



Draw a line through R, parallel to the line LNM so as to meet LP in H and MQ in K. The triangles PHR and QKR are similar, so that

$$\frac{m}{n} = \frac{PR}{RQ} = \frac{PH}{QK} = \frac{LH - LP}{QM - KM} = \frac{RN - PL}{QM - RN} = \frac{z - z_1}{z_2 - z}$$

or $\quad m(z_2 - z) = n(z - z_1)$

$$\Rightarrow z = \frac{mz_2 + nz_1}{m + n}.$$

Similarly, by drawing perpendiculars to the zx–plane and the yz–plane, we get

$$y = \frac{my_2 + ny_1}{m + n}, \ x = \frac{mx_2 + nx_1}{m + n}.$$

The point R divides PQ internally or externally according as the ratio m : n is positive or negative.

- If R (x, y, z) is the middle point of PQ, then (for m = n)

  $$x = \frac{x_1 + x_2}{2}, \ y = \frac{y_1 + y_2}{2}, \ z = \frac{z_1 + z_2}{2}.$$

- If R divides PQ (externally) in the ratio m : n, then (replacing n by – n)

  $$x = \frac{mx_2 - nx_1}{m - n}, \ y = \frac{my_2 - ny_1}{m - n}, \ z = \frac{mz_1 - nz_1}{m - n}.$$

- If R (x, y, z) divides PQ in the ratio k : 1, then

  $$x = \frac{kx_2 + x_1}{k + 1}, \ y = \frac{ky_2 + y_1}{k + 1}, \ z = \frac{kz_1 + 1}{k + 1} \ (k \neq -1) \ldots (1)$$

To every value of k ($\neq -1$), there corresponds a point R on the line PQ and to every point R on PQ corresponds some value of k. Thus the general coordinates of points on a line joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are given by (1) for various value of k ($\neq -1$).

Illustration 2:

The points P (3, 2, – 4), Q (5, 4, – 6), R (9, 8, – 10) are given to be collinear. Find the ratio in which Q divides PR.

Solution

Let Q divide PR in the ratio k : 1.

The x–coordinate of Q is 5, so that

$\frac{9k+3}{k+1} = 5$ or $9k + 3 = 5k + 5 \quad \Rightarrow k = \frac{1}{2}$.

Also y–coordinate of Q is 4,

$\Rightarrow \frac{8k+2}{k+1} = 4$ which is satisfied by $k = \frac{1}{2}$.

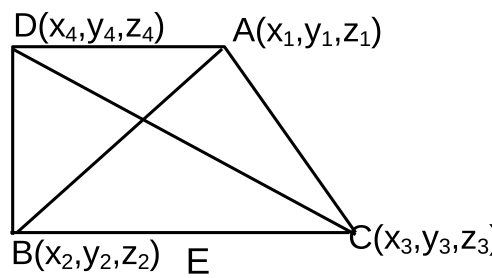
Similarly the z–coordinate is also consistent.

Hence the required ratio is $\frac{1}{2} : 1$ or 1 : 2.

Illustration 3:

If the vertices of a tetrahedron are (x_r, y_r, z_r), r = 1, 2, 3, 4, find its centroid.

Solution



Let A (x_1, y_1, z_1), B (x_2, y_2, z_2), C (x_3, y_3, z_3) and D (x_4, y_4, z_4) be the vertices of the tetrahedron.

The midpoint of BC is

$$E\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right).$$

If G_1 is the centroid of ΔABC, then G_1 divides AE in the ratio 2 : 1.

Hence G_1 is $\left[\frac{2\frac{x_2+x_3}{2}+x_1}{2+1}, \frac{2\frac{y_2+y_3}{2}+y_1}{2+1}, \frac{2\frac{z_2+z_3}{2}+z_1}{2+1}\right]$

i.e., G_1 is $\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right]$.

If G (x, y, z) is the centroid of the tetrahedron, it divides DG_1 in the ratio 3 : 1.

$$\text{Hence } x = \frac{3\frac{x_1+x_2+x_3}{3}+1}{3+1} = \frac{x_1+x_2+x_3+x_4}{4}.$$

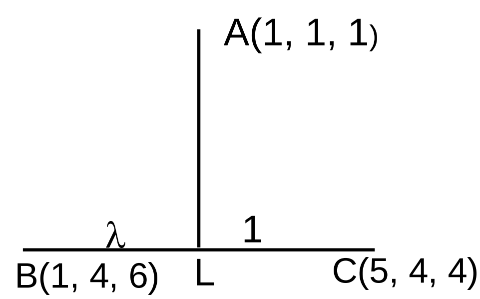
Hence the centroid is $\left[\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right]$.

Solved Problems

Illustration 4:

Find the coordinates of the foot of the perpendicular from A (1, 1, 1) on the line joining B (1, 4, 6) and C (5, 4, 4).

Solution:



Draw AL perpendicular to the line BC. Let L divide BC in the ratio $\lambda : 1$ so that the coordinates of L are $\left(\frac{5\lambda+1}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{4\lambda+6}{\lambda+1}\right)$.

In triangle, ALC,

$AC^2 = AL^2 + LC^2$,

$$\text{Hence } 16+9+9 = \left(\frac{4\lambda}{\lambda+1}\right)^2 + \left(\frac{3\lambda+3}{\lambda+1}\right)^2 + \left(\frac{3\lambda+5}{\lambda+1}\right)^2 + \left(\frac{-4}{\lambda+1}\right)^2 + (0)^2 + \left(\frac{2}{\lambda+1}\right)^2$$

$$\text{or, } 34 = \frac{34\lambda^2+48\lambda+54}{\lambda^2+2\lambda+1}$$

or, $20\lambda = 20 \Rightarrow \lambda = 1$.

Hence the foot of the perpendicular L is (3, 4, 5).

Illustration 5:

Show that the point (1, – 1, 2) is common to the lines PQ and RS where P is (6, – 7, 0), Q (16, – 19, – 4), R (0, 3, – 6), S (2, – 5, 10).

Solution:

Let the point L(1, –1, 2) divide the line in the ratio k : 1, so that L is

$\left(\frac{16k+6}{k+1}, \frac{-19k-7}{k+1}, \frac{-4k}{k+1}\right)$, $(k \neq -1)$.

Hence $\frac{16k+6}{k+1} = 1$ or $16k + 6 = k + 1$

$\Rightarrow k = -\frac{1}{3}$. The y and z coordinates also give the value of $k = -\frac{1}{3}$.

Any point on RS is $\left(\frac{2t}{t+1}, \frac{-5t+3}{t+1}, \frac{10t-6}{t+1}\right)$. If this is the same as L, then

$\frac{2t}{t+1} = 1 \Rightarrow t = 1$.

With this value of t, the y and z coordinates are –1 and 2 respectively. Hence point L is common to PQ and RS.

Illustration 6:

A (3, 2, 0), B (5, 3, 2) and C (– 9, 6, – 3) are the vertices of a triangle ABC. The bisector AD of angle A meets BC at D. Find the coordinates of D.

Solution:

Here $AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2} = \sqrt{4+1+4} = 3$,

and $AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2} = \sqrt{144+16+9} = 13$.

Since AD is the bisector of angle A, D divides BC in the ratio AB:AC i.e. 3 : 13.

$\therefore$ The coordinates of D is $\left(\frac{13\times5+3(-9)}{16}, \frac{13\times3+3(6)}{16}, \frac{13\times2+3(-3)}{16}\right)$ or $\left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$.

Illustration 7:

Find the ratio in which the line segment joining the points (2, 4, –3) and (–3, 5, 4) is divided by the xy-plane.

Solution:

Let the xy-plane divide the given line segment in the ratio k : 1.

Hence the point L $\left(\frac{-3k+2}{k+1}, \frac{5k+4}{k+1}, \frac{4k-3}{k+1}\right)$

lies in the xy–plane i.e. $\frac{4k-3}{k+1} = 0 \Rightarrow k = \frac{3}{4}$.

Hence the given line segment is divided in the ratio 3 : 4.