SIMPLE HARMONIC MOTION (SHM)

Periodic Motion: Motions, processes or phenomena, which repeat themselves at regular intervals, are called periodic.

Period & frequency: In a periodic motion the smallest interval of time after which the process repeats itself is called period. Usually the period is denoted by the symbol T and is measured in seconds.

The reciprocal of T gives the number of periodic motions that occur per second and is called the frequency of the periodic motion. It is represented by the symbol ν and is measured in units called hertz.

$$\nu = \frac{1}{T}$$

Oscillatory motion: If a body moves to and fro on the same path about a fixed point then its motion is called as oscillatory motion.

1. SIMPLE HARMONIC MOTION

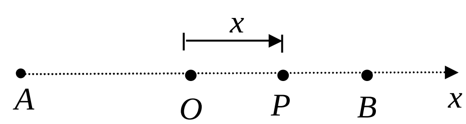
Simple harmonic motion is special type of periodic oscillatory motion in which;

(i) the particle oscillates on a straight line

(ii) the acceleration of the particle is always directed towards a fixed point on the line.

(iii) the magnitude of acceleration is proportional to the displacement of the particle from the fixed point.

This fixed point is called the centre of oscillation or the mean position. Taking this point as origin '*O*' and the line of motion as the *x*-axis, we can write the equation of simple harmonic motion based on its definition as,



$$a = -\omega^2 x$$

Where ω^2 is a positive constant. P is the particle, which is at a distance 'x' from fixed point 'O'. a is the acceleration which is directed opposite to the displacement and towards centre of oscillation 'O'.

According to Newton's laws of motion in inertial frame of reference,

$$a = F / m = -\omega^2 x \; ; \; m = \text{mass of particle}$$

$\therefore \quad F = m\omega^2 x = -kx \qquad F$ = resultant force acting on the particle

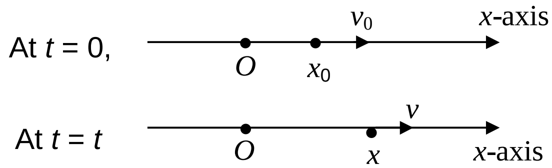
$\therefore \quad F = -kx$

i.e., the resultant force acting on the particle is proportional to displacement and directed towards mean position.

The constant $k = m\omega^2$ is called the force constant. It is to be noted that resultant force is zero at mean position so it is also the dynamic equilibrium position of the particle.

2. EQUATION OF MOTION OF A SIMPLE HARMONIC MOTION

Now we will derive the equation of motion for a particle of mass 'm' moving along x-axis under the effect of force $F = -kx$. Here k is a positive constant and x is the displacement of the particle from the assumed origin.



Suppose we start observing the motion of the particle at $t = 0$ when it is at $x = x_0$ and its velocity is $v = v_0$ as shown in the figure.

The acceleration of the particle at any instant is

$$a = F/m = \frac{-k}{m}x = -\omega^2 x \text{ ; where } \omega = \sqrt{\frac{k}{m}}$$

$$\therefore \qquad \frac{dv}{dt} = -\omega^2 x \qquad \dots (\mathbf{1})$$

$$\therefore \qquad \frac{vdv}{dx} = -\omega^2 x$$

$$\therefore \qquad \int_{v_0}^{v} vdv = \int_{x_0}^{x} \omega^2 x dx$$

where 'x' is the instantaneous position of the particle and 'v ' is the velocity at that instant

$$\therefore \qquad \left[\frac{v^2}{2}\right]_{v_0}^{v} = -\omega^2 \left[\frac{x^2}{2}\right]_{x_0}^{x}$$

$$v^2 - v_0^2 = -\omega^2 (x^2 - x_0^2)$$

$$v^2 = v_0^2 + \omega^2 x_0^2 - \omega^2 x^2$$

$$v = \omega \sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2 - x^2}$$

$\therefore$ Equation (2) can be written as

$$\therefore \qquad \frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \qquad \dots (\mathbf{2})$$

where $$A = \sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2}$$

$$\therefore \qquad \int_{x_0}^{x} \frac{dx}{\sqrt{A^2 - x^2}} = \int_{0}^{t} \omega dt$$

$$\left[\sin^{-1}\left(\frac{x}{A}\right)\right]_{x_0}^{x} = \omega t$$

$\therefore \qquad \sin^{-1}\dfrac{x}{A} - \sin^{-1}\dfrac{x_0}{A} = \omega t$

Put $\qquad \sin^{-1}\dfrac{x_0}{A} = \phi$

$\therefore \qquad \sin^{-1}\dfrac{x}{A} = (\phi + \omega t)$

$\therefore \qquad x = A\sin(\omega t + \phi) \qquad \ldots \textbf{(3)}$

and velocity ; $v = \dfrac{dx}{dt} = \omega A\cos(\omega t + \phi) \qquad \ldots \textbf{(4)}$

2.1 AMPLLITUDE

It is maximum displacement of the particle from its mean position.

Equation (3) gives the displacement of the particle. The value of 'x' is maximum when $\sin(\omega t + \phi)$ is maximum i.e., $\sin(\omega t + \phi) = \pm 1$

$\therefore \qquad x_{\max} = \pm A$

$\therefore$ 'A' which was used as constant while deriving the equation of motion is nothing but the amplitude of simple harmonic motion.

2.2 TIME PERIOD

Periodic functions $f(t)$ with period T are those functions of the variable 't' which have the property,

$$f(t + T) = f(t) \qquad \ldots \textbf{(5)}$$

Both $\sin(\omega t + \phi)$ and $\cos(\omega t + \phi)$ will repeat their values if the angle $(\omega t + \phi)$ increases by 2π or its multiple. As T is smallest time for repetition.

$$\omega(t + T) + \phi = \omega t + \phi + 2\pi$$

$\therefore \qquad \omega T = 2\pi \qquad$ or $\qquad T = \dfrac{2\pi}{\omega}$

Since $\quad \omega = \sqrt{\dfrac{k}{m}}$

$\therefore \qquad T = 2\pi\sqrt{\dfrac{m}{k}} \qquad \ldots \textbf{(6)}$

2.3 FREQUENCY AND ANGULAR FREQUENCY

Frequency is defined as the number of oscillations per second or simply as the reciprocal of time period.

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{m}{k}} \qquad \ldots (7)$$

The constant ω is called the angular frequency. The angular frequency and period in simple harmonic motion are independent of the amplitude.

2.4 PHASE

The quality $\theta = \omega t + \phi$ is called the phase. It determines the states of the particle in simple harmonic motion.

When the particle is at **mean position** $x = 0$

i.e., $A\sin(\omega t + \phi) = 0$

$\therefore$ $\omega t + \phi = n\pi; n = 0, 1, 2, 3, \ldots$

(i) consider $n = 0$; $\therefore$ $\omega t + \phi = 0$

$\therefore$ $x = 0$

and $v = \omega A \cos(\omega t + \phi) = \omega A$

i.e., the particle is crossing the mean position and is moving towards the positive direction.

(ii) consider $n = 1$

$\therefore$ $\omega t + \phi = \pi$

$\therefore$ $x = 0$

and $v = -\omega A$

i.e., again the particle is crossing the mean position but now it is moving towards the negative direction.

When the particle is at **extreme position**.

$x = x_{max}$

i.e., $A\sin(\omega t + \phi) = \pm A$

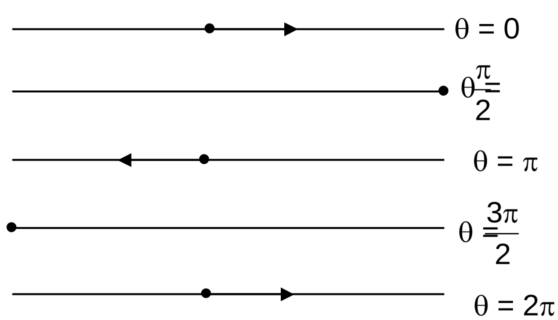
$\Rightarrow$ $(\omega t + \phi) = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

i.e., $(\omega t + \phi) = \frac{(2n+1)}{2}\pi$; $n = 0, 1, 2 \ldots$

$\therefore$ Consider $n = 1$; $\therefore$ $\omega t + \phi = \frac{3\pi}{2}$

$\therefore$ $x = -A$

and $v = 0$



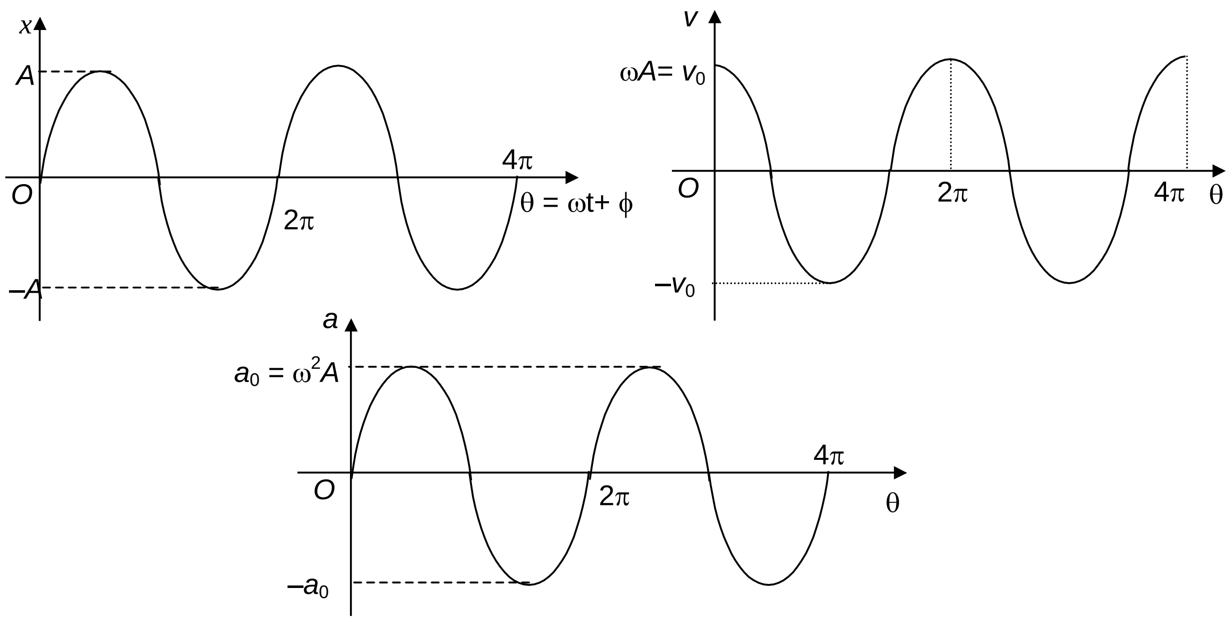
i.e., the particle is at extreme left and again its velocity is zero.

From above it is clear that as time increases the phase increases. An increase of 2π brings the particle to the same status in the motion. Thus, a phase $\omega t + \phi$ is equivalent to a phase $\omega t + \phi + 2\pi$.

Similarly acceleration of the particle is given by

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin(\omega t + \phi)$$

It is zero when phase (ωt + ϕ) = 0 and maximum ($\omega^2 A$) when phase $\omega t + \phi = \left(\frac{2n+1}{2}\right)\pi$. Graphically the variation of position, velocity and acceleration with the phase is shown below.



2.5 PHASE CONSTANT

The constant term ϕ in the equation (3) is called phase constant or initial phase or epoch of the particle. This constant depends on the choice of the instant $t = 0$.

Suppose we choose $t = 0$ at an instant when the particle is passing through its mean position towards right (i.e. positive direction). Then the phase $\theta = \omega t + \phi$ has to be zero. Since $t = 0$ this means $\phi = 0$. So the equation for displacement becomes;

$$x = A \sin \omega t$$

If we choose $t = 0$ when the particle is at its extreme position in the positive direction. The phase $\theta = \frac{\pi}{2}$ at this instant and hence $\phi = \frac{\pi}{2}$. Therefore equation of displacement becomes

$$x = A \cos\omega t$$

The sine form and cosine form are basically equivalent. The value of phase constant, however, depends on the form chosen, for example,

$$x = A \sin(\omega t + \phi) = A \sin(\omega t + \pi/2 + \phi')$$

$$x = A \cos(\omega t + \phi')$$

Illustration 1.

A particle executes simple harmonic motion of amplitude 4 cm and a period 3 sec. Find the velocity of the particle at (i) 2 cm from the mean position and (ii) at the mean position.

Solution:

Velocity of the particle at a distance x from the mean position is given by

$$v=\omega\sqrt{a^2-x^2}=\frac{2\pi}{T}\sqrt{a^2-x^2}$$

(i) When x = 2 cm, $v=\frac{2\pi}{3}\sqrt{4^2-2^2}$ = **7.26 cm/sec**.

(ii) At the mean position x = 0,

$$v = \omega a=\frac{2\pi}{T}.a = \frac{2\pi}{3}\times 4 = \mathbf{8.378\ cm/sec}$$

Illustration 2.

A particle executes S.H.M. of period π sec and amplitude 2 cm. Find the acceleration of it when it is (i) at the maximum displacement from the mean position and (ii) at 1 cm from the mean position.

Solution:

Acceleration f at displacement x is given by

$$f = \omega^2 x$$

(i) When x (amplitude) = 2 cm, $f=\left(\frac{2\pi}{T}\right)^2\times 2=\frac{4\pi^2}{\pi^2}\times 2=\mathbf{8 cm/sec^2}$

(ii) When x = 1 cm, f = $\left(\frac{2\pi}{T}\right)^2\times 1=\frac{4\pi^2}{\pi^2}\times 1$ = **4 cm/sec²**

Illustration 3.

A particle executes S.H.M. of time period 10 s. The displacement at any instant is given by the relation x = 10 sin ωt. Find (i) velocity of the body 2 s after it passes through the mean position and (ii) the acceleration 2 s after it passes the mean position (Amplitude is given in cm).

Solution:

(i) Velocity at any instant t is given by $v = A\omega\ \cos\omega t$

Here A = 10 cm, $\omega=\frac{2\pi}{T}=\frac{2\pi}{10}$

When t = 2 s, v = $10\times\frac{2\pi}{10}\cos\left(\frac{2\pi}{10}\times 2\right)$

= $2\pi\cos(0.4\pi)$

= **1.942 cm/s**

(ii) Acceleration at any instant t is given by

$a = -A\omega^2 \sin \omega t$

$= -10\left(\dfrac{2\pi}{10}\right)^2 \sin(0.4\pi)$

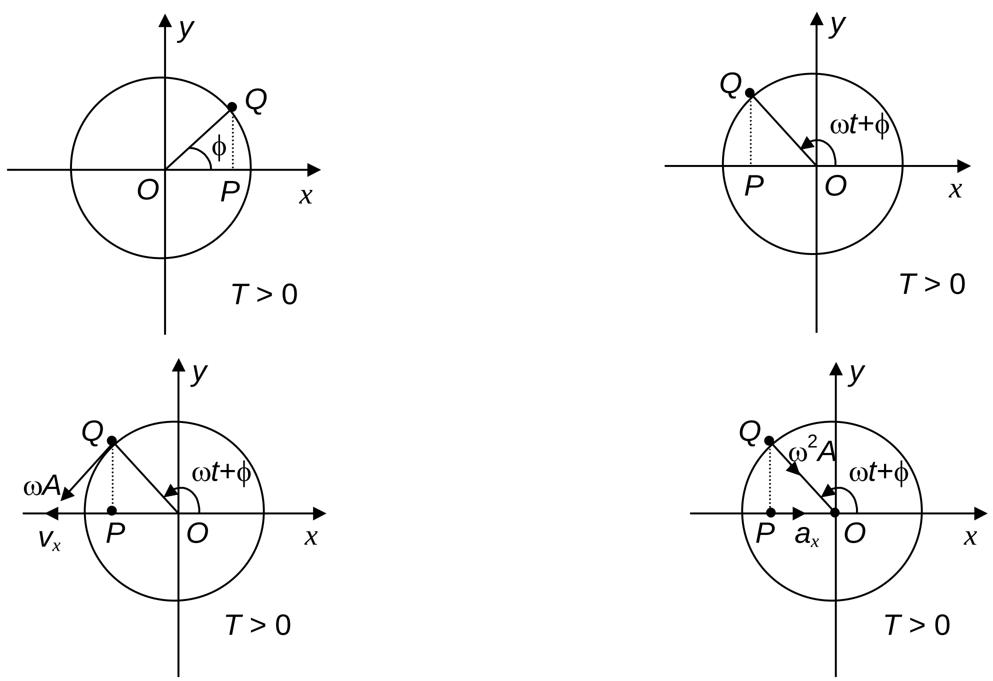
$= \mathbf{-3.755\ cm/s^2}$

acceleration is numerically equal to 3.754 cm/s^2 and is directed towards the mean position.

3. RELATION BETWEEN SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

The relation we are going to discuss is useful in describing many features of simple harmonic motion. It also gives a simple geometric meaning to the angular frequency ω and phase constant φ.

In the figure shown, *Q* is the point moving on a circle of radius *A* with constant angular speed of ω (in rad/sec). *P* is the perpendicular projection of *Q* on the horizontal diameter, along the *x*-axis. Let us take *Q* as the reference point and the circle on which it moves the reference circle. As the reference point revolves, the projected point *P* moves back and forth along the horizontal diameter.



Let the angle between the radius *OQ* and the *x*-axis at the time $t = 0$ be called φ. At any later time *t*, the angle between *OQ* and the *x*-axis is $(\omega t + \phi)$, the point *Q* moving with constant angular speed ω. The *x*-coordinate of *Q* at any time is, therefore

$$x = A\cos(\omega t + \phi)$$

i.e., *P* moves with simple harmonic motion

Thus, when a particle moves with uniform circular motion, its projection on a diameter moves with simple harmonic motion. The angular frequency ω of simple harmonic motion is the same as the angular speed of the reference point.

The velocity of Q is $v = \omega A$. the component of v along the x-axis is

$$v_x = -v \sin(\omega t + \phi)$$

$$v_x = -\omega A \sin(\omega t + \phi)$$

Which is also the velocity of P. The acceleration of Q is centripetal and has a magnitude, $a = \omega^2 A$

The component of 'a' along the x-axis is

$$a_x = -a \cos(\omega t + \phi)$$

$$a_x = -\omega^2 A \cos(\omega t + \phi)$$

which is acceleration of P.

4. ENERGY CONSIDERATIONS IN SIMPLE HARMONIC MOTION

Simple harmonic motion is defined by the equation

$$F = -kx$$

The work done by the force F during a displacement from x to $x + dx$ is

$$dW = F dx = -kx\, dx$$

The work done in a displacement from $x = 0$ to x is

$$W = \int_0^x (-kx)\, dx = -\frac{1}{2} kx^2$$

Let $U(x)$ be the potential energy of the system when the displacement is x. As the change in potential energy corresponding to a force is negative of the work done by this force,

$$U(x) - U(O) = -W = \frac{1}{2} kx^2$$

Let us choose the potential energy to be zero when the particle is at the centre of oscillation $x = 0$.

Then $U(0) = 0$ and $U(x) = \frac{1}{2} kx^2$

$\because$ $k = m\omega^2$

$\therefore$ $U(x) = \frac{1}{2} m\omega^2 x^2$ … (**8**)

But $x = A \sin(\omega t + \phi)$

$\therefore$ $U = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi)$ … (**9**)

kinetic energy of the particle at any instant

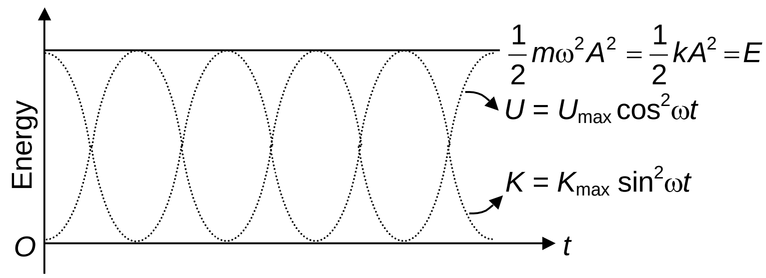
$$K = \frac{1}{2} mv^2 = \frac{1}{2} mA^2\omega^2 \cos^2(\omega t + \phi) \quad \ldots (\mathbf{10})$$

So the total mechanical energy at time 't' is

$$E = U + K$$

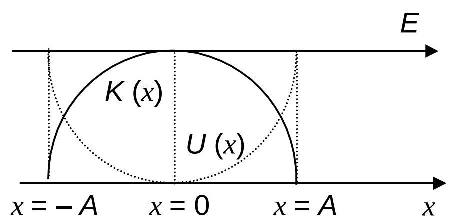
$$E = \frac{1}{2} m\omega^2 A^2 \qquad \ldots \textbf{(11)}$$

(i)



Potential, kinetic and total energy plotted as a function of time.

(ii)



Potential, kinetic and total energy plotted as a function of displacement from the equilibrium position.

Illustration 4.

If a particle of mass 0.2 kg executes S.H.M. of amplitude 2 cm and period of 6 sec find (i) the total mechanical energy at any instant (ii) kinetic and potential energies when the displacement is 1 cm.

Solution:

(i) Total mechanical energy at any instant is given by

$$E = \frac{1}{2} ma^2\omega^2$$

$$= \frac{1}{2}(0.2\text{ kg})(2\times 10^{-2} m)^2\left(\frac{2\pi}{6}\right)^2$$

$$= 0.1\times 4\times 10^{-4}\times\frac{4\pi^2}{36}$$

$$= \mathbf{4.39 \times 10^{-5}\ J}$$

(ii) K.E. at the instant when the displacement, x is given by

$$\text{K.E.} = \frac{1}{2} m\omega^2(a^2 - x^2)$$

when $x = 1$ cm,

$$\text{K.E.} = \frac{1}{2}(0.2 kg)\left(\frac{2\pi}{6}\right)^2(4\times 10^{-4} - 1\times 10^{-4})J = \mathbf{3.29 \times 10^{-5}\ J}$$

P.E. at that instant = Total energy – K.E.

$= (4.39 - 3.29) \times 10^{-5}$

$= \mathbf{1.10 \times 10^{-5}\ J}$

Illustration 5.

A body makes angular simple harmonic motion of amplitude $\frac{\pi}{10}$ rad and time period 0.05 s. If the body is at a displacement $\theta = \frac{\pi}{10}$ rad at $t = 0$, write the equation giving the angular displacement as a function of time.

Solution:

Let the required equation be

$$\theta = \theta_0 \sin(\omega t + \phi)$$

Here, $\theta_0 = \frac{\pi}{10}$ rad

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.05S} = 40\ \pi s^{-1}$$

So, $\theta = \left(\frac{\pi}{10}\text{rad}\right) \sin[(40\ \pi s^{-1})\ t + \phi]$ … (i)

At $t = 0$, $\theta = \pi/10$ rad. Putting in (i)

$\pi/10 = (\pi/10) \sin\phi$

or $\sin\phi = 1$ or $\phi = \pi/2$

Thus, $\theta = \left(\frac{\pi}{10}\textbf{rad}\right)\mathbf{\cos[40\pi s^{-1})}\boldsymbol{t}\mathbf{]}$

5. EXAMPLES OF SIMPLE HARMONIC MOTION

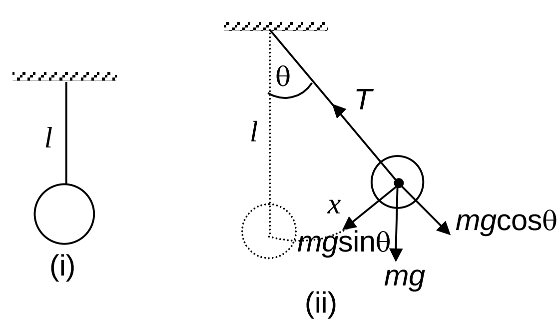
5.1 SIMPLE PENDULUM

A simple pendulum consists of a heavy particle suspended from a fixed support through a light inextensible string. The time period for simple pendulum can be found by force/ torque method and also by energy method.

(a) Force method: The mean position or the equilibrium position of the simple pendulum is when $\theta = 0$ as shown in figure (i). The length of the string is l, and mass of the bob is m.

When the bob is displaced through a distance 'x', the forces acting on it are shown in the figure (ii).

The restoring force acting on the bob to bring it to the mean position is,



$F = -\,mg \sin\theta$ (–ve sign indicates that force is directed away from the displacement towards the mean position).

For small angular displacements,

$$\text{Sin}\,\theta \approx \theta = \frac{x}{l}$$

$$\therefore \quad F = \frac{mgx}{l}$$

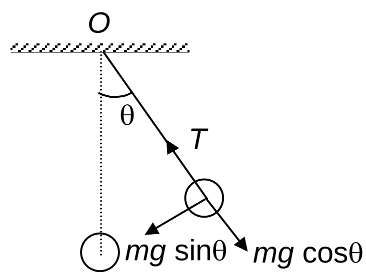
$$\therefore \quad a = -(g/l)x$$

Comparing it with equation of simple harmonic motion; $a = -\,\omega^2 x$

$$\omega^2 = g/l \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{l}}$$

$$\therefore \quad \text{Time period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \qquad \dots \textbf{(12)}$$

(b) Torque method: Now taking moment of forces acting on the bob about O,



$$\tau = -(mg\sin\theta)\, l$$

$$I\alpha = -\,mg\, l\,(\theta)\ ;\ \text{since } \sin\theta \approx \theta$$

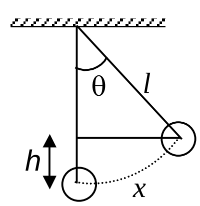
$$ml^2\alpha = \ mgl\,\theta$$

$$\therefore \quad \alpha = -\left(\frac{g}{l}\right)\theta$$

Comparing with simple harmonic motion equation; $\alpha = -\,\omega^2\theta$

$$\therefore \ \omega = \sqrt{\frac{g}{l}} \quad \text{and} \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(c) Energy method: Let the potential energy at the mean position be zero. When the bob is displaced through an angle ‘θ’, let its velocity be ‘v ’.



Then potential energy at this new position

$$= mgl\,(1 - \cos\theta) = U$$

Kinetic energy at this instant $= \frac{1}{2}\, mv^2 = K$

Total mechanical energy at this instant,

$$E = U + K = mgl\,(1 - \cos\theta) + \frac{1}{2}\, mv^2$$

We know, in simple harmonic motion, E = constant.

$\therefore \quad \frac{dE}{dt} = 0 \Rightarrow mgl\,[\sin\theta\,\frac{d\theta}{dt}] + mv\,\frac{dv}{dt} = 0$

But $\quad v = l\,\frac{d\theta}{dt}$

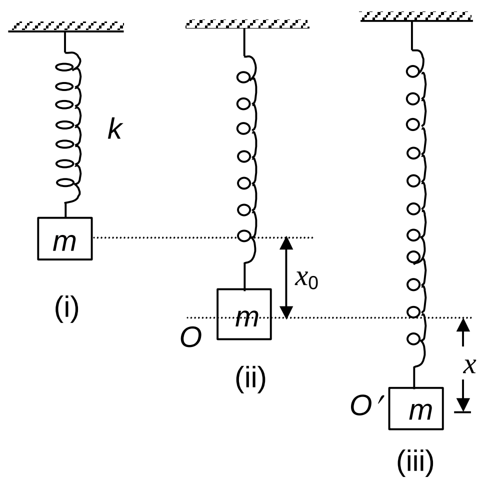
$\therefore \quad \frac{dv}{dt} = -g\sin\theta \approx -g\theta$

$a = -g\,\frac{x}{l}$

$\therefore \quad$ Time period $= T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

5.2 MASS-SPRING SYSTEM

Let a mass 'm' be attached to a massless spring of stiffness k. Because of its weight the mass will come down through a distance x_0 till it is balanced by the spring force kx_0. This position is called as mean or equilibrium position. Let the block be at rest in this position.



$\therefore \quad mg = kx_0$

or $\quad x_0 = \frac{mg}{k} \quad \ldots$ (i)

Now the block is further displaced by a distance x in the downward direction as shown in figure (iii) and is left. Forces on the block at this instant,

$F = -k(x_0 + x) + mg$

$\therefore \quad F = -kx \quad \ldots$ (ii)

This is the net restoring force acting on the block.

$\therefore \quad ma = -kx$

$\therefore \quad a = -\left(\frac{k}{m}\right)x \quad \ldots$ (iii)

Comparing this with equation of simple harmonic motion,

$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$

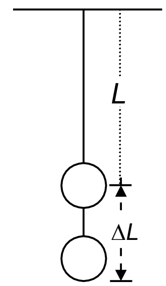
$\therefore \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \ldots$ **(13)**

Illustration 6.

A point mass m is suspended at the end of a massless wire of length L and cross-section A. If Y be the Young's modulus for the wire, obtain an expression for the frequency of oscillation for the simple harmonic motion along the vertical line.

Solution:

Let a force F be applied to stretch the wire by a length ΔL. If A be cross-section of the wire,



$$\text{Stress} = \frac{F}{A}$$

$$\text{Strain} = \frac{\Delta L}{L}$$

Young's modulus, $Y = \dfrac{\text{Stress}}{\text{Strain}} = \dfrac{F/A}{\Delta L/L}$ or, $F = AY\dfrac{\Delta L}{L}$

The restoring force due to elasticity $= -\dfrac{AY}{L}\Delta L$

Since the force is proportional to the displacement of the suspended mass, when made to vibrate, executes simple harmonic motions.

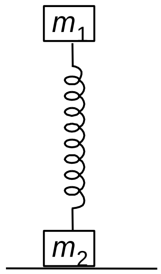
The spring factor $= \dfrac{AY}{L}$

The period of oscillation $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{m}{AY/L}} = 2\pi\sqrt{\dfrac{mL}{AY}}$

The frequency of oscillation $= \dfrac{1}{T} = \dfrac{1}{2\pi}\sqrt{\dfrac{AY}{mL}}$

Illustration 7.

A body of mass m_1 is connected to another body of mass m_2 as shown in Figure and placed on a horizontal surface. The mass m_1 performs vertical harmonic oscillations with amplitude $A = 1.6$ cm and frequency $\omega = 25$ rad/s.



Neglecting the mass of the spring find the maximum and minimum values of force that the system exerts on the surface.

(Take $m_1 = 1.0$ kg and $m_2 = 4.10$ kg)

Solution:

By compressing the spring let m_1 remain in equilibrium. Now the force with which the system presses the horizontal surface is $(m_1 + m_2)g$.

When m_1 performs vertical oscillations let it have an acceleration a at any instant, when moving down.

$\therefore$ the downward force due to oscillation $= m_1 a$

The maximum acceleration m_1 can have is $\omega^2 A$

where ω is the angular frequency and A the amplitude.

$\therefore\ F_{\max} = (m_1 + m_2)g + m_1\,\omega^2 A$

By similar reasoning,

$F_{\min} = (m_1 + m_2)g - m_1\,\omega^2 A$

$F_{\max} = (1 + 4.10) \times 9.8 + 1 \times 25^2 \times 1.6 \times 10^{-2} =$ **58.98 N**

$F_{\min} = 5.1 \times 9.8 - 625 \times 1.6 \times 10^{-2} =$ **39.98 N**

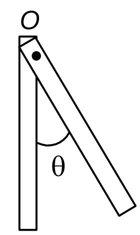
6. ANGULAR SIMPLE HARMONIC MOTION

A body free to rotate about a given axis can make angular oscillations. For example, a wooden stick nailed to a wall can oscillate about its mean position in the vertical plane.

The conditions for an angular oscillation to be angular simple harmonic motion are:

(i) When a body is displaced through an angle from the mean position ($\theta = 0$; $\tau = 0$), a resultant torque acts which is proportional to the angle displaced,

(ii) This torque is restoring in nature and it tries to bring the body towards the mean position,



If the angular displacement of the body at an instant is θ, then resultant torque on the body,

$$\tau = -k\theta \qquad \dots (\mathbf{14})$$

If the moment of inertia is I, the angular acceleration is

$$\alpha = \frac{\tau}{I} = -\frac{k}{I}\theta$$

or, $$\frac{d^2\theta}{dt^2} = -\omega^2\theta \ ; \quad \text{where } \omega = \sqrt{\frac{k}{I}} \qquad \dots (\mathbf{15})$$

Solution of equation (13) gives,

$$\theta = \theta_0 \sin(\omega t + \phi) \qquad \dots (\mathbf{16})$$

where θ_0 is the maximum angular displacement on either side.

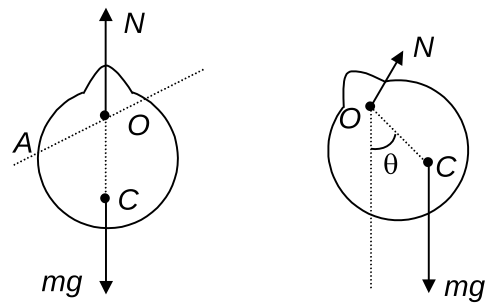
Angular velocity at time 't ' is given by

$$\Omega = \frac{d\theta}{dt} = \theta_0\omega\cos(\omega t + \phi) \qquad \dots (\mathbf{17})$$

6.1 PHYSICAL PENDULUM

Any rigid body suspended from a fixed support constitutes a physical pendulum. For example a disc suspended through a hole in it. Figure (i) shows a physical pendulum. A rigid body is suspended through a hole at *O*. When the centre of mass *C* is vertically below *O*, the body may remain at rest. This is $\theta = 0$ position.

The body is rotated through an angle θ about a horizontal axis *OA* passing through *O* and perpendicular to the plane of motion.



The torque of the forces acting on the body, about the axis *OA* is

$$\tau = mgl \sin\theta \; ; \; \{\text{where } l = OC\}$$

If moment of inertia of the body about *OA* is *I*, the angular acceleration becomes,

$$\alpha = \frac{\tau}{I} = -\frac{mgl}{I} \sin\theta$$

For small angular displacements $\sin\theta \approx \theta$

$$\therefore \qquad \alpha = -\left(\frac{mgl}{I}\right)\theta$$

Comparing with $\alpha = -\omega^2\theta$

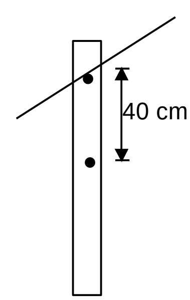
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}} \qquad \ldots \textbf{(18)}$$

Illustration 8.

A uniform meter stick is suspended through a small-hole at the 10 cm mark. Find the time period of small oscillations about the point of suspension.

Solution:

Let the mass of the stick be *M*. The moment of inertia of the stick about the axis of rotation through the point of suspension is



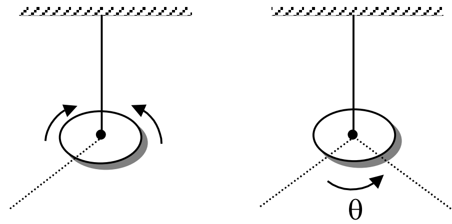
$$I = \frac{ml^2}{12} + md^2$$

Where $l = 1$ m and $d = 40$ cm

Time period $= T = 2\pi\sqrt{\frac{I}{mgd}} = \textbf{1.55 sec.}$

6.2 TORSIONAL PENDULUM

In torsional pendulum, an extended body is suspended by a light thread or a wire. The body is rotated through an angle about the wire as the axis of rotation.



The wire remains vertical during this motion but a twist 'θ' is produced in the wire. The twisted wire exerts a restoring torque on the body, which is proportional to the angle of twist,

$$\tau \; \alpha \; -\theta$$

$\tau = -k\,\theta$; k is proportionality constant and is called torsional constant of the wire.

If I be the moment of inertia of the body about the vertical axis, the angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{-k}{I}\theta = -\omega^2\theta$$

$$\therefore \quad \omega = \sqrt{\frac{k}{I}}$$

$$\therefore \quad \text{Time period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{k}} \qquad \dots (\mathbf{19})$$

Illustration 9.

The moment of inertia of the disc used in a torsional pendulum about the suspension wire is 0.2 kg – m^2. It oscillates with a period of 2s. Another disc is placed over the first one and the time period of the system becomes 2.5 s. Find the moment of inertia of the second disc about the wire.

Solution:

Let the torsional constant of the wire be k.

$$T = 2\pi\sqrt{\frac{I}{k}} = 2\pi\sqrt{\frac{0.2}{k}} = 2 \qquad \dots \text{(i)}$$

When the second disc having moment of inertia I_1 about the wire is added, the time period is,

$$2.5 = 2\pi\sqrt{\frac{0.2 + I}{k}} \qquad \dots \text{(ii)}$$

from (i) & (ii) $I_1 \approx 0.11$ kg – m^2

7. COMPOSITION OF TWO SIMPLE HARMONIC MOTION

If the particle is acted upon by two separate forces each of which can produce a simple harmonic motion, the resultant motion of the particle is a combination of two simple harmonic motions.

Let $\vec{r}_1$ denote the position of the particle at time t if the force $\vec{F}_1$ alone acts on it. Similarly, let $\vec{r}_2$ denote the position at time 't' if the force $\vec{F}_2$ alone acts on it.

According to Newton's second law of motion,

$$m\frac{d^2\vec{r}_1}{dt^2}=\vec{F}_1 \quad \text{and,} \quad \frac{md^2\vec{r}_2}{dt^2}=\vec{F}_2$$

adding them, $$m\left(\frac{d^2\vec{r}_1}{dt^2}+\frac{d^2\vec{r}_2}{dt^2}=\vec{F}_1+\vec{F}_2\right)$$

$$\therefore \qquad m\frac{d^2}{dt^2}(\vec{r}_1+\vec{r}_2)=\vec{F}_1+\vec{F}_2 \qquad \ldots \text{(i)}$$

But $\vec{F}_1+\vec{F}_2$ is the resultant force acting on the particle and so the position $\vec{r}$ of the particle when both the forces act, is given by

$$m\frac{d^2\vec{r}}{dt^2}=(\vec{F}_1+\vec{F}_2) \qquad \ldots \text{(ii)}$$

Comparing equation (i) & (ii) we can show that

$$\vec{r}=\vec{r}_1+\vec{r}_2 \quad \text{and} \quad \vec{v}=\vec{v}_1+\vec{v}_2$$

If these conditions are met at $t = 0$.

Thus the actual position of the particle is given by the vector sum of $\vec{r}_1$ & $\vec{r}_2$.

7.1 COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS IN THE SAME DIRECTION

Let the direction be along x-axis and the simple harmonic motions produced by two forces $\vec{F}_1$ & $\vec{F}_2$ be;

$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin (\omega t + \phi)$ respectively

From above discussion, the resultant position of the particle is then,

$x = x_1 + x_2 = A_1 \sin \omega t + A_2 \sin (\omega t + \phi)$

Put $A_1 + A_2 \cos\phi = A \cos \delta$

$A_2 \sin \phi = A \sin \delta$

∴ $A = \sqrt{(A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2}$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} \qquad \dots \text{(i)}$$

and $x = A\cos\delta\sin\omega t + A\sin\delta\cos\omega t$

$$x = A\sin(\omega t + \delta) \qquad \dots \textbf{(20)}$$

and, $\tan\delta = \dfrac{A_2\sin\phi}{A_1 + A_2\cos\phi} \qquad \dots \textbf{(21)}$

The amplitude of resultant simple harmonic motion is given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} \qquad \dots \textbf{(22)}$$

It is maximum when $\phi = 0$

$$A_{max} = \sqrt{(A_1 + A_2)^2} = A_1 + A_2$$

It is minimum when $\cos\phi = -1$ i.e., $\phi = \pi$

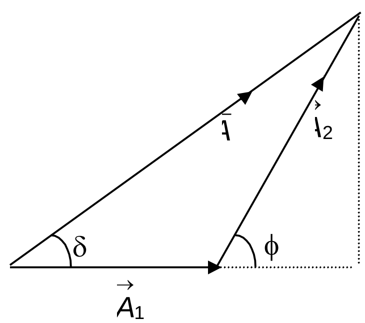
$$A_{min} = \sqrt{(A_1 - A_2)^2} = A_1 - A_2$$

7.2 VECTOR METHOD OF COMBINING TWO SIMPLE HARMONIC MOTION

$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin(\omega t + \phi)$

We draw a vector of magnitude A_1 and another vector of magnitude A_2 making an angle ϕ with first vector as shown in the figure.



The resultant $\vec{A}$ of these two vectors will represent the resultant simple harmonic motion. From vector algebra,

∴ $|\vec{A}| = A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$

& $\tan\delta = \dfrac{A_2\sin\phi}{A_1 + A_2\cos\phi}$

Illustration 10.

A particle is subjected to two simple harmonic motions $x_1 = A_1 \sin \omega t$ AND $x_2 = A_2 \sin(\omega t + \pi/3)$. Find

(a) the displacement at $t = 0$

(b) the maximum speed of the particle and

(c) the maximum acceleration of the particle

Solution:

(a) At $t = 0$, $x_1 = A_1 \sin \omega t = 0$

And $x_2 = A_2 \sin(\omega t + \pi/3) = \dfrac{A_2\sqrt{3}}{2}$

Thus resultant displacement at $t = 0$ is

$$x = x_1 + x_2 = \boldsymbol{\frac{A_2\sqrt{3}}{2}}$$

(b) $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\dfrac{\pi}{3}}$

$A = \sqrt{A_1^2 + A_2^2 + A_1A_2}$

The maximum speed is

$$V_{max} = \omega A = \boldsymbol{\omega\sqrt{A_1^2 + A_2^2 + A_1A_2}}$$

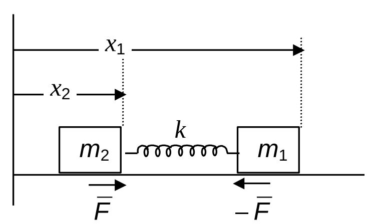
(c) The maximum acceleration is

$$a_{max} = \omega^2 A = \boldsymbol{\omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}}$$

8. TWO BODY SYSTEM

In a two body oscillations, such as shown in the figure, a spring connects two objects, each of which is free to move. When the objects are displaced and released, they both oscillate.

The relative separation $x_1 - x_2$ gives the length of the spring at any time. Suppose its unstretched length is L; then $x = (x_1 - x_2) - L$ is the change in length of the spring, and $F = kx$ is the magnitude of the force exerted on each particle by the spring as shown in the figure.



Applying newton's second law separately to the two particles, taking force component along the x-axis; we get

$$\frac{m_1 d^2x_1}{dt^2} = -kx \quad \text{and} \quad \frac{m_2 d^2x_2}{dt^2} = +kx$$

We now multiply the first of these equations by m_2 and the second by m_1, and then subtract. The result is

$$m_1m_2 \frac{d^2x_1}{dt^2} - m_1m_2\frac{d^2x_2}{dt^2} = -m_2kx - m_1kx$$

which can be written as

$$\frac{m_1m_2}{(m_1+m_2)}\frac{d^2}{dt^2}(x_1 - x_2) = -kx \qquad \ldots \text{(i)}$$

The quantity $\dfrac{m_1m_2}{m_1+m_2}$ has the dimensions of mass and is known as the reduced mass μ

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \ldots \textbf{(23)}$$

Since L is constant

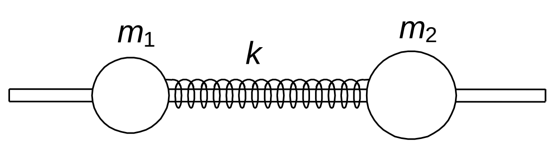
$$\frac{d}{dt}(x_1 - x_2) = \frac{d}{dt}(x + L) = \frac{dx}{dt}$$

$\therefore$ equation (i) becomes $\dfrac{d^2x}{dt^2} + \dfrac{k}{\mu}x = 0$

$$\therefore \qquad T = 2\pi\sqrt{\frac{\mu}{k}} \qquad \ldots \textbf{(24)}$$

Illustration 11.

Two balls with masses m_1 = 1 kg and m_2 = 2 kg are slipped on a thin smooth horizontal rod. The balls are interconnected by a light spring of spring constant 24 N/m. The left hand ball is imparted the initial velocity v_1 = 12 cm/s. Find (a) the oscillation frequency of the system, (b) the energy and amplitude of oscillation.



Solution:

(a) When the spring attached to two balls of masses m_1 and m_2 at the two ends gets compressed or stretched the force developed in the spring is the same. This force produces acceleration $\left(\dfrac{d^2x}{dt^2}\right)_1$ and $\left(\dfrac{d^2x}{dt^2}\right)_2$ in the two masses. If the force developed in the spring in bringing the two masses closer by compressing the spring is F then

$$F = -m_1\left(\frac{d^2x}{dt^2}\right)_1$$

$$-\left(\frac{d^2x}{dt^2}\right)_1 = \frac{F}{m_1} \qquad \ldots \text{(i)}$$

Similarly, $$-\left(\frac{d^2x}{dt^2}\right)_2 = \frac{F}{m_2} \qquad \ldots \text{(ii)}$$

Negative sign indicates that acceleration is opposite to displacement. So the relative acceleration of the system is given by

$$\left(\frac{d^2x}{dt^2}\right)_1 + \left(\frac{d^2x}{dt^2}\right)_2 = -F\left[\frac{1}{m_1} + \frac{1}{m_2}\right]$$

$$\frac{d^2x}{dt^2} = -F\,\frac{(m_1 + m_2)}{m_1 m_2}$$

$$F = -\left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d^2x}{dt^2}$$ where x is the total compression in the spring.

$$\frac{d^2x}{dt^2} = \frac{-k}{\frac{m_1 m_2}{m_1 + m_2}} \cdot x$$

so $\frac{d^2x}{dt^2}$ is proportional to x.

The system of balls performs S.H.M.

$\therefore$ $\omega_o^2 = \frac{k}{\frac{m_1 m_2}{m_1 + m_2}}$ where ω_0 is the natural frequency of oscillation.

$\omega_o^2 = \frac{k}{\mu}$ where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ called reduced mass.

$\therefore$ $\omega_o = \sqrt{\frac{k}{\mu}}$; $n = \frac{1}{2\pi}\sqrt{\frac{\mu}{k}} = \mathbf{2.65 \times 10^{-2}\ s}$

(b) The initial velocity given to the mass m_1 is v_1.

For undamped oscillation, this initial energy will remain constant.

Hence total energy of S.H.M. of two balls is given as $E = \frac{1}{2} m_1 v_1^2$

Putting the values, $E = 5$ mJ

If amplitude of oscillation is a, then

$v_1 = \omega_0 a$

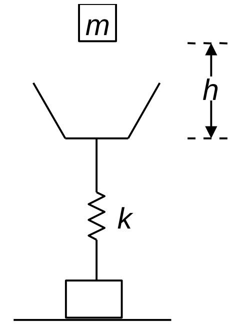
$a = \frac{v_1}{\omega_o}$

So on putting the values, we get a = **2 cm**

Illustration 12.

A body of mass m falls from a height h on to the pan of a spring balance. The masses of the pan and spring are negligible. The spring constant of the spring is k. Having stuck to the pan the body starts performing harmonic oscillations in the vertical direction. Find the amplitude and energy of oscillation.

Solution:



Suppose by falling down through a height h, the mass m compresses the spring balance by a length x.

The P.E. lost by the mass = mg (h+ x)

This is stored up as energy of the spring by compression

$$= \frac{1}{2}kx^2$$

$\therefore$ mg(h + x) = $\frac{1}{2}kx^2$ or $\frac{1}{2}kx^2$ – mgx – mgh = 0

or $$x^2 - \frac{2mgx}{k} - \frac{2mgh}{k} = 0$$

Solving this quadratic equation, we get

$$x = \frac{\frac{2mg}{k} \pm \sqrt{\left(\frac{2mg}{k}\right)^2 + \left(\frac{8mgh}{k}\right)}}{2} = \frac{mg}{k} \pm \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}}$$

In the equilibrium position, the spring will be compressed through the distance mg/k and hence the amplitude of oscillation is

$$A = \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}}$$

Energy of oscillation = $\frac{1}{2}kA^2 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2\left(1 + \frac{2kh}{mg}\right)$

$$= mgh + \frac{(mg)^2}{2k}$$

Illustration 13.

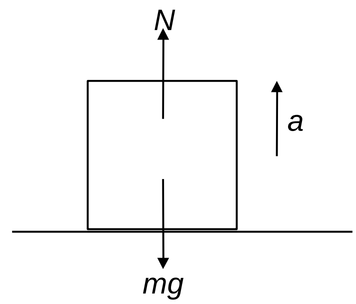
A particle rests on a horizontal plane, which is displaced up and down, with S.H.M. of frequency 50 Hz. Determine the maximum amplitude of motion for the particle to remain in contact with the plane. If the amplitude of motion is half of this value, upto what value can the frequency of vibration be increased, the particle still remaining in contact with the plane.

Solution:

Let N be the reaction of the plane at any instant and ‘a’ the acceleration of the plane.

When the plane is moving up

N – mg = ma   or   N = ma + mg



Since N ≠ 0 during the upward motion of the vibration of the plane the particle will not lose contact with the plane whatever be the value of the upward acceleration of the plane.

Now let us consider the downward motion of the particle during the vibration. The force equation becomes

$mg - N = ma$ or, $N = mg - ma = m(g - a)$

If a = g, N becomes zero and at that instant the particle loses contact with the plane. Hence the condition for the particle to be in contact with the plane is that the downward acceleration, at any instant, should be smaller than g. The maximum value of the downward acceleration is

$$a = -\omega^2 A$$

where A is the amplitude of vibration.

$$\therefore \quad \omega^2 A < g \quad \text{or} \quad A < \frac{g}{\omega^2}$$

$$\text{or} \quad A < \frac{g}{4\pi^2\nu^2}$$

$$\text{Now} \quad \frac{g}{4\pi^2\nu^2} = \frac{9.8}{4\pi^2 50^2} = 9.93 \times 10^{-5}\, m$$

Hence the maximum amplitude permissible = 9.93×10^{-5} m

If the amplitude be halved, let the maximum angular frequency be ω_1.

$$\text{Now} \quad \omega_1^2 \frac{A}{2} = g$$

Since $g = \omega^2 A$, we have

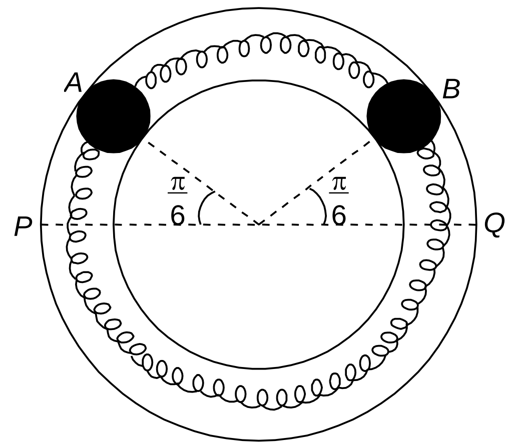
$$\omega_1^2 \frac{A}{2} = \omega^2 A$$

$$\omega_1^2 = 2\omega^2 \quad \text{or} \quad 4\pi\nu_1^2 = 2 \times 4\pi\nu^2$$

$$\text{or} \quad \nu_1^2 = \mathbf{2} \times \mathbf{50^2} \quad \text{or} \quad \nu_1 = \mathbf{50\sqrt{2}}\ \text{Hz}$$

Illustration 14.

Two identical balls A and B each of mass 0.1 kg are attached to two identical massless springs. The spring-mass system is constrained to move inside a rigid smooth pipe bent in the form of a circle as shown in the Figure. The pipe is fixed in a horizontal plane. The centres of the balls can move in a circle of radius 0.06 m. Each spring has a natural length of 0.06 π m and spring constant 0.1 N/m. Initially, both the balls are displaced by- an angle $\theta = \frac{\pi}{6}$ radian with respect to the diameter PQ of the circle as shown in the Figure and released from rest.



(a) Calculate the frequency of oscillation of ball B.

(b) Find the speed of ball A when A and B are at the two ends of the diameter PQ.

(c) What is the total energy of the system?

Solution:

(a) When the balls are displaced by an angle $\theta = \frac{\pi}{6}$ radian with respect to the diameter PQ of the circle, the restoring force acting on the ball A or B

$$= K(\Delta x) + K(\Delta x)$$

$$= 2K(\Delta x), \text{ where } \Delta x = 2r\theta.$$

Effective force constant of the spring-mass system is,

$K_{effective} = 2K$

Frequency of oscillation, $\nu = \frac{1}{2\pi}\sqrt{\frac{2K}{\mu}}$

where μ is the reduced mass of the system.

$$\mu = \frac{m \times m}{(m+m)} = \frac{m}{2}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{2K}{m/2}} = \frac{1}{3.14}\sqrt{\frac{K}{m}} = \frac{1}{3.14}\sqrt{\frac{0.1}{0.1}} = \frac{\mathbf{1}}{\mathbf{3.14}}\mathbf{s}^{-1}$$

(b) Circumference of the circle = $2\pi r$

$$= 2\pi(0.06 \text{ m})$$

Natural length of each of the two springs = π (0.06) m

P and Q are the equilibrium positions of the balls.

K.E. of the balls at the equilibrium position = Potential energy at the extreme of vibration

$$P.E. = \frac{1}{2} \times K_{\text{effective}} \times (\Delta x)^2$$

$$= \frac{1}{2}(2K)\left[\frac{2\pi(0.06)}{2\pi}\left(2 \times \frac{\pi}{6}\right)\right]^2 = 3.94 \times 10^{-4} \text{ J}$$

Kinetic energy of the balls, A and B = $KE_A + KE_B$ = P.E.

$2KE_A$ = P.E.

$$2\left(\frac{1}{2}mv^2\right) = 3.94 \times 10^{-4} \text{ joule}$$

Speed of the ball A when A and B are at the two ends of the diameter PQ, v = 0.0628 ms^{-1}.

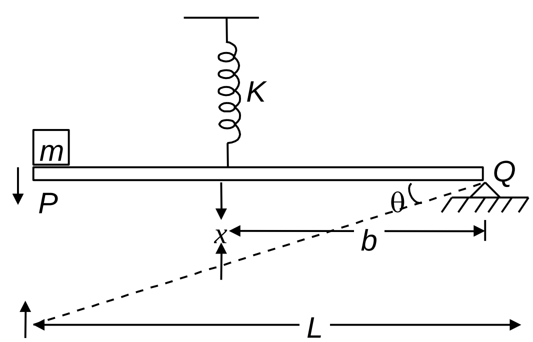
(c) Total energy of the system will be equal to the total potential energy at the extreme of vibration.

T.E = 3.94×10^{-4} Joule

Illustration 15.

The rod PQ of mass, M is attached as shown to a spring of constant K. A small block of mass, m is placed on the rod at its free end P. If end P is moved down through a small distance x ′ and released, determine the period of vibration.

Solution:



Method I: Using energy equation, moment of inertia of the system about Q is

$$I = \left[\frac{ML^2}{3} + mL^2\right]$$

where L is the length of the rod.

Elastic potential energy of the spring = $\frac{1}{2}Kx^2$

Rotational energy of the rod PQ = $\frac{1}{2}I\omega^2$

By the law of conservation of energy,

$$\frac{1}{2}I\omega^2 = \frac{1}{2}Kx^2$$

$$\frac{1}{2}\left[\frac{ML^2}{3} + mL^2\right]\frac{v^2}{L^2} = K\left(\frac{x'^2 b^2}{L^2}\right); \text{ where } x = \frac{x'b}{L}$$

Differentiating with respect to time,

$$\left(\frac{\frac{M}{3}+m}{2}\right)L^2 \times 2v\frac{dv}{dt}=\frac{1}{2}K\mathbf{b}^2 \times 2x'\frac{dx'}{dt}\text{; where } \frac{dx'}{dt} = \text{v = velocity of m}$$

Acceleration of the block, $\frac{dv}{dt}=\frac{Kb^2x'}{\left(\frac{M}{3}+m\right)L^2}$

Acceleration of the block is directly proportional to its linear displacement.

This represents an SHM with angular frequency Ω given by

$$\Omega^2 = \frac{Kb^2}{\left(\frac{M}{3}+m\right)L^2}$$

Period of vibration $\frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{\left(\frac{M}{3}+m\right)L^2}{Kb^2}}$

Method II: Using torque equation, when the system is displaced through small angle θ, tension in the string, T = Kx = K(bθ), ($\because$ x = bθ)

$\Rightarrow$ Restoring torque, $\tau = T \times b = Kb\theta = -I\ddot{\theta}$

where $I = \frac{ML^2}{3} + mL^2$

$$\Rightarrow \quad \ddot{\theta} = -\left(\frac{Kb^2}{I}\right)\theta = -\left\{\frac{Kb^2}{\left(\frac{M}{3}+m\right)L^2}\right\}\theta$$

This is an angular SHM with angular frequency, $\Omega = \sqrt{\frac{Kb^2}{\left(\frac{M}{3}+m\right)L^2}}$

$\Rightarrow$ Period of vibration = $\frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{\left(\frac{M}{3}+m\right)L^2}{Kb^2}}$

Illustration 16.

A simple pendulum of length L and mass m is suspended in a car that is travelling with a constant speed V around a circular track of radius R. If the pendulum makes small oscillations about its equilibrium position, what will be the frequency of oscillations?

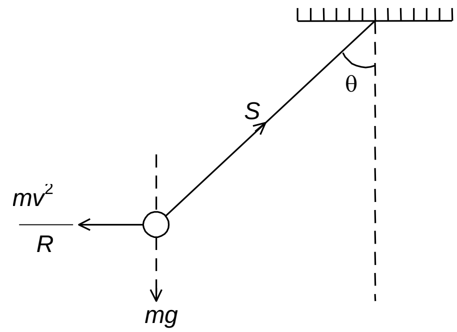
Solution:

When the car comes round a circle it is an accelerated frame of reference. A fictitious force $\frac{mV^2}{R}$ is to be introduced to the simple pendulum as a centrifugal force. If θ be the angular displacement of the pendulum in its new equilibrium position, then

S cos θ = mg

S sin θ = $\frac{mV^2}{R}$

where S is the tension in the string.



$$S=\sqrt{(mg)^2+\left(\frac{mV^2}{R}\right)^2}$$

$$= m\sqrt{g^2+\frac{V^4}{R^2}}$$

Let the pendulum be slightly displaced so that it makes an angle (θ + dθ) with the vertical and then let go.

The restoring forces = S sin dθ ≈ $Sd\theta=\frac{Sx}{L}$

where x is the linear displacement and L the length of the pendulum and x = Ldθ.

The restoring force/unit displacement k = $\frac{Sx}{L}\Big/x$

$$= \frac{S}{L}$$

The period of oscillation of the pendulum

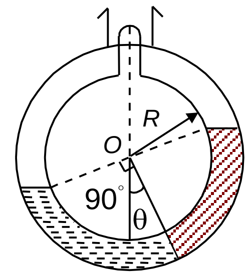
$$T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{m}{S/L}} = 2\pi\sqrt{\frac{Lm}{S}}$$

$$= 2\pi\sqrt{\frac{Lm}{m\left(g^2+\frac{V^4}{R^2}\right)^{1/2}}} = 2\pi\sqrt{\frac{L}{\left(g^2+\frac{V^4}{R^2}\right)^{1/2}}}$$

$$\text{Frequency } =\frac{1}{T} =\frac{1}{2\pi}\sqrt{\frac{\left(g^2+\frac{V^4}{R^2}\right)^{1/2}}{L}}$$

Illustration 17.

Two non-viscous, incompressible and immiscible liquids of densities ρ and $1.5\,\rho$ are poured into the two limbs of a circular tube of radius R and small cross-section kept fixed in a vertical plane as shown in the figure. Each liquid occupies one-fourth of the circumference of the tube.



(a) Find the angle θ that the radius vector to the interface makes with the downward vertical in the equilibrium position.

(b) If the whole liquid is given a small displacement from its equilibrium position, show that the resulting oscillations are simple harmonic. Find the time period of these oscillations.

Solution:

(a) Let ρ and σ be the densities of the liquids, and let OC make angle θ with the downward vertical. The pressure at the lowest point P is given by the pressure due to either the liquid left of P or the liquids to the right of P. The two pressures are equal.

Vertical height of liquid column AP = LP

$= (R - R\sin\theta)$

Vertical height of liquid column PC = NP

$= (R - R\cos\theta)$

Vertical height of liquid column BC = MN

$= MO + ON$

$= R(\sin\theta + \cos\theta)$

Thus equating the pressures due to the liquids on either side of ρ at P,

$LP.\ \rho g = MN \cdot \sigma g + NP \cdot \rho g$

or $\quad \rho[(R - R\sin\theta) - (R - R\cos\theta)] = \sigma[\sin\theta + \cos\theta]R$

or $\quad \rho[R\cos\theta - R\sin\theta] = \sigma[R\cos\theta + R\sin\theta]$

or $$\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{\sigma}{\rho}$$

which gives $$\frac{\sin\theta}{\cos\theta} = \frac{\rho - \sigma}{\rho + \sigma}$$

$$\therefore \quad \tan\theta = \frac{\rho - \sigma}{\rho + \sigma}$$

In the problem, it is given that $\sigma \to \rho$ and $\rho \to (1.5\,\rho)$

Hence $$\tan\theta = \frac{(1.5-1)\rho}{(1.5+1)\rho} = \left(\frac{0.5}{2.5}\right)$$

$$\therefore \quad \theta = \tan^{-1}(0.2) = 11°31'$$

(b) Let A = cross-sectional area of tube.

Mass of liquid column $AC = \frac{2\pi R}{4} A\,(1.5\rho)$

Mass of liquid column $CB = \frac{2\pi R}{4} A\rho$

Moment of inertia of the whole liquid about $O = I = \left(\frac{\pi R A}{2}\right)(1.5+1)\,\rho R^2$

Let y be the small displacement from equilibrium position P towards left side.

If α is the corresponding angular displacement,

$\alpha = \frac{y}{R}$ or $y = \alpha R$

Torque about $O = I\frac{d^2\alpha}{dt^2}$

$$= \frac{\pi R A}{2} \times 2.5\rho R^2 \frac{d^2\alpha}{dt^2}$$

Restoring torque due to the displaced liquid = – [Ay (1.5 ρ) g + Ay ρg] × Rcos θ

where 'R cos θ' is perpendicular distance of gravitational force from axis of rotation.

$\therefore \quad T_{restoring} = -2.5\, A\alpha\rho g R^2 \cos\theta$

$$\left(\frac{\pi R A}{2}\right) 2.5\rho R^2 \frac{d^2\alpha}{dt^2} = -2.5 A\alpha\rho g R^2 \cos\theta$$

$$\frac{d^2\alpha}{dt^2} = -2g\frac{\cos\theta\alpha}{\pi R}$$

(i.e) angular acceleration $\propto$ angular displacement

$\therefore \quad \omega = \sqrt{2g\cos\theta/\pi R}$

$\therefore \quad$ time period $= \frac{2\pi}{\omega} = \boldsymbol{2\pi\sqrt{\frac{\pi R}{2g\cos\theta}}}$

Illustration 18.

A particle of mass m is performing simple harmonic motion in a straight line with amplitude r and period T. Find the law of force. When at a distance Kr from the centre of oscillation, it collides with a stationary particle of the same mass and coalesces with it. If the law of force is the same, find the new period of oscillation and the amplitude.

Solution:

Amplitude given = r

Period = T; hence angular frequency $\omega = \frac{2\pi}{T}$

The equation of motion is therefore

$$m\ddot{x} = -\frac{4\pi^2 m}{T^2}x \qquad \ldots(i)$$

With the initial condition,

x = r when $\dot{x} = 0$, $\quad t = \frac{T}{4}$

The law of force is therefore $F_x = -\frac{4\pi^2 mx}{T^2}$ at distance x.

The solution of the equation can be obtained by integrating equation (i) with the given initial condition.

Putting $\omega^2 = \frac{4\pi^2}{T^2}, \ddot{x} = -\omega^2 x$ ... (ii)

Multiplying by $2\dot{x}$ and integrating,

$(\dot{x})^2 = -\omega^2 x^2 + C$

where the integration constant C is given by the condition.

When x = r, $\dot{x}$ = 0 ∴ C = + ω^2 r^2

Hence $(\dot{x})^2 = \omega^2(r^2 - x^2)$

or $\quad \dot{x} = \omega\sqrt{r^2 - x^2}$ … (iii)

Integrating again, $\int \frac{dx}{\sqrt{r^2 - x^2}} = \omega t + C$

where C is the integration constant.

or $\quad \sin^{-1}\left(\frac{x}{r}\right) = \omega t + C$

At $\quad$ t = 0, x = 0, and so, C = 0

∴ $\quad$ x = r sin (ωt) … (iv)

This also satisfies the condition, that x = r when $t = \frac{T}{4}$.

When the particle is at a distance Kr from the centre of oscillation

i.e., $\quad$ x = Kr, … (v)

the velocity of the particle is

$$v^2=\omega^2[r^2-(Kr)^2]$$

$$v^2=\frac{4\pi^2}{T^2}[1-K^2]r^2$$

$$v=\frac{2\pi r}{T}\sqrt{1-K^2} \qquad \ldots \text{(vi)}$$

If it collides with a stationary particle of same mass m and coalesces with it, the law of conservation of linear momentum for this collision gives,

$$m{\cdot}v = 2m{\cdot}v' \qquad \ldots \text{(vii)}$$

where v′ is the new velocity of the new system.

Hence $v'=\frac{v}{2}=\frac{1}{2}\cdot\frac{2\pi r}{T}\sqrt{1-K^2}$

$$=\frac{\pi r}{T}\sqrt{1-K^2} \qquad \ldots \text{(viii)}$$

Putting $v'=\frac{dx}{dt}$, where x is the new coordinate for the particle now of mass 2m, we have

$$\left(\frac{dx}{dt}\right)_{x=Kr}=\frac{\pi r}{T}\sqrt{1-K^2} \qquad \ldots \text{(ix)}$$

Since the law of force remains the same, we have

$$(2m)\ \ddot{x}=-\omega^2 mx$$

or $\qquad \ddot{x}=-\frac{\omega^2}{2}x \qquad \ldots$(x)

or $\qquad \ddot{x}=-\omega'^2 x$

where ω′ = the new angular velocity = $\frac{\omega}{\sqrt{2}}$

The new periodic time $T'=\frac{2\pi}{\omega'}=\frac{2\sqrt{2}\pi}{\omega}=\frac{2\sqrt{2}\pi}{2\pi/T} = \sqrt{2}T$

Hence after the collision, the combined mass 2m will oscillate with the new period $\sqrt{2}T$.

Also, if r′ be the new amplitude, we will have

$$\left(\frac{dx}{dt}\right)^2=\omega'^2(r'^2-x^2) \qquad \ldots \text{(xi)}$$

Putting the condition (ix) in this,

$$\left[\frac{\pi r}{T}\sqrt{1-K^2}\right]^2 = \omega'^2(r'^2 - K^2 r^2)$$

or $$\frac{\pi^2 r^2(1-K^2)}{T^2} = \frac{4\pi^2}{T'^2}(r'^2 - K^2 r^2)$$

or $$\frac{\pi^2 r^2(1-K^2)}{T^2} = \frac{4\pi^2}{2T^2}(r'^2 - K^2 r^2)$$

or $$r^2(1-K^2) = 2(r'^2 - K^2 r^2)$$

or $$r^2(1+K^2) = 2r'^2$$

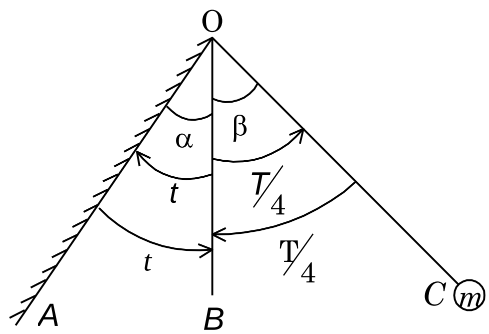
$$r'^2 = \frac{r^2}{2}(1+K^2) \qquad \text{Hence } r' = r\sqrt{\frac{1}{2}(1+K^2)}$$

Thus the new amplitude of oscillation = $\boldsymbol{r\sqrt{\frac{1}{2}(1+K^2)}}$

Illustration 19.

A ball is suspended by a thread of length ℓ at the point O on the wall, forming a small angle α with the vertical. Then the thread with the ball was deviated through small angle β ($\beta > \alpha$) and set free. Assuming the collision of the ball against the wall to be perfectly elastic, find the oscillation period of such a pendulum.

Solution:



As β is a small angle, the motion of the ball is S.H.M. After perfectly elastic collision the velocity of the ball is simply reversed. As shown in figure, the time period of one oscillation will be

$$T' = \frac{T}{4} + \frac{T}{4} + t + t = \frac{T}{2} + 2t$$

where $T = 2\pi\sqrt{\frac{l}{g}}$ or $\frac{T}{2} = \pi\sqrt{\frac{l}{g}}$

$\theta = \theta_0 \sin \omega t$

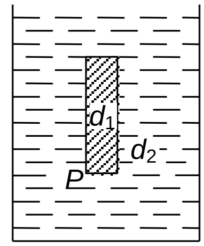
$\alpha = \beta \sin \omega t$, where t is the time taken from B to A.

$$t = \frac{1}{\omega}\sin^{-1}\left(\frac{\alpha}{\beta}\right) = \sqrt{\frac{l}{g}}\sin^{-1}\left(\frac{\alpha}{\beta}\right)$$

$$T' = \pi\sqrt{\frac{l}{g}} + 2\sqrt{\frac{l}{g}}\sin^{-1}\left(\frac{\alpha}{\beta}\right) = \boldsymbol{2\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{\alpha}{\beta}\right)\right]}$$

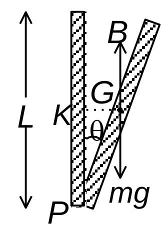
Illustration 20.

A thin rod of length L and area of cross-section S is pivoted at its lowest point P inside a stationary, homogeneous and non-viscous liquid. The rod is free to rotate in a vertical plane about a horizontal axis passing through P. The density d_1 of the material of the rod is smaller than the density d_2 of the liquid. The rod is displacement by a small angle θ from its equilibrium position and then released. Show that the motion of the rod is simple harmonic and determine its angular frequency in terms of the given parameters.



Solution:

Let the rod be displaced through an angle θ. The different forces on the rod are shown in the figure. The force acting upwards at the middle point G of the rod



= upward thrust – weight of rod = B – mg

= weight of displaced liquid – weight of rod

= $LSd_2g - LSd_1g = LSg(d_2 - d_1)$

Moment of the couple restoring it to the original position

$$= LSg(d_2 - d_1)\,KG = LSg(d_2 - d_1)\frac{L}{2}\sin\theta$$

Torque $\tau = LSg(d_2 - d_1)\dfrac{L}{2}\sin\theta = I\left(\dfrac{d^2\theta}{dt^2}\right)$

where I is moment of inertia of the rod about the axis through O.

$$\therefore \qquad I\left(\frac{d^2\theta}{dt^2}\right) = -\frac{L^2Sg}{2}(d_2-d_1)\theta \text{ [because } \theta \text{ is small } \sin\theta \simeq \theta]$$

So, $\dfrac{d^2\theta}{dt^2} \propto \theta$. Hence it executes S.H.M.

$$\therefore \qquad \frac{d^2\theta}{dt^2} = \frac{-L^2Sg}{2I}(d_2-d_1)\theta$$

But $\quad I = \dfrac{ML^2}{3}$

$$\therefore \qquad \frac{d^2\theta}{dt^2} = \frac{-L^2Sg(d_2-d_1)\theta}{\frac{2ML^2}{3}} = -\frac{3}{2}\frac{Sg(d_2-d_1)}{M}\theta = -\frac{3}{2}\frac{Sg(d_2-d_1)\theta}{LSd_1} = -\frac{3}{2}\frac{g}{L}\left(\frac{d_2-d_1}{d_1}\right)\theta$$

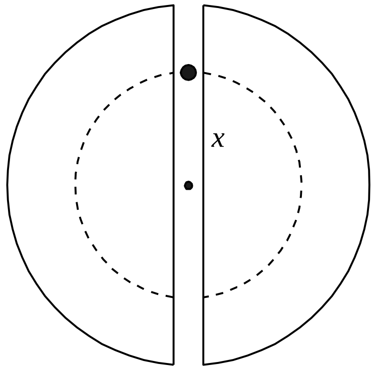
But $\quad \dfrac{d^2\theta}{dt^2} = -\omega^2\theta \qquad \omega = \sqrt{\dfrac{3g(d_2-d_1)}{2Ld_1}}$

Illustration 21.

Assume that a narrow tunnel is dug between two diametrically opposite points of earth. Treat the earth as a solid sphere of uniform density. Show that if a particle is released in this tunnel it will execute S.H.M. Find the time period.

Solution:

Consider the situation shown in the figure. Suppose at an instant t the particle in the tunnel is at a distance x from centre of earth. Let us draw a sphere of radius x with its centre at the centre of the earth. Only the part of the earth within the sphere will exert a net force of attraction on the particle.



Mass of this part $M' = \dfrac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi R^3} M = \dfrac{x^3}{R^3}.M$

$\therefore$ the force of attraction $F = \dfrac{G\left(\dfrac{x^3}{R^3}\right)Mm}{x^2} = \dfrac{GMm}{R^3}.x$

The force acts towards the centre of the earth. Thus the resultant force on the particle is opposite to the displacement from centre of earth and is proportional to it. The particle therefore executes S.H.M in the tunnel with the centre of earth as mean position.

$\therefore \quad m.\dfrac{d^2x}{dt^2} = \dfrac{GMm}{R^3}.x \quad \dfrac{d^2x}{dt^2} = \dfrac{GM}{R^3}.x$

$\therefore$ comparing this equation with the standard equation of S.H.M we see $\omega^2 = \dfrac{GM}{R^3}$

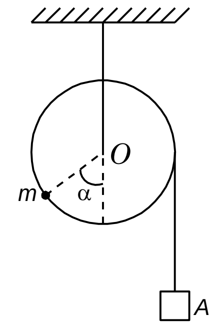
$\therefore$ period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{R^3}{GM}}$

But $\dfrac{GM}{R^2} = g$ (acceleration due to gravity)

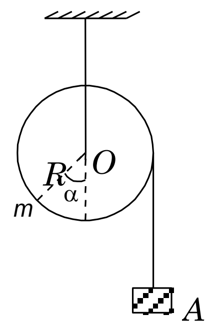
$\therefore$ period $= \boldsymbol{2\pi\sqrt{\dfrac{R}{g}}}$.

Illustration 22.

A uniform cylindrical pulley of mass M and radius R can freely rotate about the horizontal axis O. The free end of a thread tightly wound on the pulley carries a dead weight A. At a certain angle α it counter balances a point mass m fixed at the rim of the pulley. Find the frequency of small oscillations of the arrangement.



Solution:



Considering rotational equilibrium about O, we have

$m'gR = mgR\sin\alpha$, where m' is mass of A

$\therefore \quad m' = m\sin\alpha$

Consider a small angular displacement by θ in anti clockwise direction.

Then unbalanced torque τ in the clockwise direction

$= m'gR - mgR\sin(\alpha+\theta)$

Moment of inertia of the system = $\frac{1}{2}MR^2 + m'R^2 + mR^2$

$$\therefore \quad m'gR - mgR\sin(\alpha+\theta) = \left(\frac{1}{2}MR^2 + m'R^2 + mR^2\right)\frac{d^2\theta}{dt^2}$$

Putting $m' = m\sin\alpha$, we have

$$mgR\sin\alpha - mgR\sin(\alpha+\theta) = \left(\frac{1}{2}MR^2 + (m\sin\alpha)R^2 + mR^2\right)\frac{d^2\theta}{dt^2}$$

$$mgR\sin\alpha - mgR[\sin\alpha\cos\theta + \cos\alpha\sin\theta] = \frac{1}{2}R^2[M + 2m\sin\alpha + 2m]\frac{d^2\theta}{dt^2}$$

$$mgR\sin\alpha - mgR\sin\alpha - mgR\theta\cos\alpha = \frac{1}{2}R^2[M + 2m\sin\alpha + 2m]\frac{d^2\theta}{dt^2}$$

Because $\sin\theta = \theta$ and $\cos\theta = 1$ when θ is small

$$-2mg\theta\cos\alpha = [MR + 2mR(1+\sin\alpha)]\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{2mg\cos\alpha}{MR + 2mR(1+\sin\alpha)}\cdot\theta$$

$\therefore \qquad \frac{d^2\theta}{dt^2}$ is proportional to θ.

The motion is simple harmonic.

$$\omega^2 = \frac{2mg\cos\alpha}{MR + 2mR\,(1+\sin\alpha)}$$

$\therefore \qquad \text{period} = \frac{2\pi}{\omega}$

Frequency $= \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2\,mg\,\cos\alpha}{MR + 2mR\,(1+\sin\alpha)}}$