# 1. DEFINITION

If a function is one to one and onto from A to B, then function $g$ which associates each element $y \in$ B to one and only one element $x \in$ A, such that $y = f(x)$, then $g$ is called the inverse function of $f$ denoted by $x = g(y)$.

Usually, we denote $g = f^{-1}$ {Read as $f$ inverse}

$$x = f^{-1}(y).$$

If cos θ = x, then may be any angle whose cosine is x, and we write $\theta = \cos^{-1} x$. It means that θ is an angle whose cosine is $x$.

Thus, $\sin^{-1}\frac{1}{2}$ is an angle, whose sin is $\frac{1}{2}$, i.e. $\theta = \sin^{-1}\frac{1}{2} = n\pi + (-1)^n \frac{\pi}{6}$

where, $\frac{\pi}{6}$ is the least positive value of θ.

| Function | Domain | Range(Principal Values) |
|---|---|---|
| $\sin^{-1}x$ | $[-1, 1]$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ |
| $\cos^{-1}x$ | $[-1, 1]$ | $[0, \pi]$ |
| $\tan^{-1}x$ | **R** | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |
| $\cot^{-1}x$ | **R** | $(0, \pi)$ |
| $\sec^{-1}x$ | **R – (–1, 1)** | $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ |
| $\text{cosec}^{-1}x$ | **R – (–1, 1)** | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ |

**Note:** $\sin^{-1}x$ is not to be interpreted as $\dfrac{1}{\sin x}$.

Arc is also used for inverse e.g. $\sin^{-1}x = \text{arc}\,(\sin x)$

**Remark:**

**1.** The inverse trigonometric functions are also written arc sin x, arc cos x etc.

**2.** 1st quadrant is common to the range of all the inverse functions.

**3.** 3rd quadrant is not used in inverse function.

**4.** 4th quadrant is used in the clockwise direction i.e. $-\dfrac{\pi}{2} \le y \le 0$.

**5.** No inverse function is periodic. Basically these functions are one to one functions.

**Illustration 1:**

Find the principal values of

**(i)** $\text{cosec}^{-1}(-1)$

**(ii)** $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

**Solution:**

**(i)** Let θ be the principal value of $\text{cosec}^{-1}(-1)$.

$\theta \in (-\pi/2, \pi/2) - \{0\}$ and $\text{cosec}^{-1}(-1) = \theta$

$\therefore\ \theta = -\frac{\pi}{2}$ because $-\dfrac{\pi}{2} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$ and because $\left(-\dfrac{\pi}{2}\right) = -1$

$\therefore$ Principal value of $\text{cosec}^{-1}(-1) = -\dfrac{\pi}{2}$.

**(ii)** Let θ be the principal value of $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore$ $\theta \in (0, \pi)$ and $\cot\theta = -\frac{1}{\sqrt{3}}$

$\therefore$ $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$, because $\frac{2\pi}{3} \in (0, \pi)$ and $\cot\left(\pi - \frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\frac{1}{\sqrt{3}}$.

$\therefore$ Principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{3}$

## 2. Some Properties of Inverse Trigonometric Functions

### Property I:

- **(a)** $\sin^{-1}(\sin x) = x$; for all $x \in [-\pi/2, \pi/2]$
- **(b)** $\cos^{-1}(\cos x) = x$; for all $x \in [0, \pi]$
- **(c)** $\tan^{-1}(\tan x) = x$; for all $x \in (-\pi/2, \pi/2)$
- **(d)** $\cot^{-1}(\cot x) = x$; for all $x \in (0, \pi)$
- **(e)** $\sec^{-1}(\sec x) = x$; for all $x \in [0, \pi]$, $x \neq \pi/2$
- **(f)** $\text{cosec}^{-1}(\text{cosec}\, x) = x$; for all $x \in [\pi/2, \pi/2]$, $x \neq 0$

### Illustration 2:

Evaluate: $\tan^{-1}\{\tan(-6)\}$

### Solution

We know that, $\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$.

Here, $\theta = -6$ radians which does not lie between $-\pi/2$ and $\pi/2$

We find that

$2\pi - 6$ lie between $-\pi/2$ and $\pi/2$

Such that, $\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

$$\tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6).$$

## Property II:

**(a)** $\sin(\sin^{-1}x) = x$; for all $x \in [-1, 1]$

**(b)** $\cos(\cos^{-1}x) = x$; for all $x \in [-1, 1]$

**(c)** $\tan(\tan^{-1}x) = x$; for all $x \in R$

**(d)** $\cot(\cot^{-1}x) = x$; for all $x \in R$

**(e)** $\sec(\sec^{-1}x) = x$; for all $x \in (-\infty, -1]\cup[1, \infty)$

**(f)** $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$; for all $x \in (-\infty, -1]\cup[1, \infty)$

### Illustration 3:

Evaluate: $\cos\left\{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}$.

### Solution

$$\cos\left\{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\} = \frac{\sqrt{3}}{2}, \text{ as } \frac{\sqrt{3}}{2} \in [-1, 1]$$

## Property III:

**(a)** $\sin^{-1}(-x) = -\sin^{-1}(x)$; for all $x \in [-1, 1]$

**(b)** $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$; for all $x \in [-1, 1]$

**(c)** $\tan^{-1}(-x) = -\tan^{-1}(x)$; for all $x \in R$

**(d)** $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$; for all $x \in R$

**(e)** $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$; for all $x \in (-\infty, -1]\cup[1, \infty)$

**(f)** $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$; for all $x \in (-\infty, -1]\cup[1, \infty)$

## Property IV:

**(a)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$; for all $x \in [-1, 1]$

**(b)** $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$; for all $x \in R$

**(c)** $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$; for all $x \in (-\infty, -1] \cup [1, \infty)$

## Property V:

**(a)** $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x$; for all $x \in (-\infty, 1] \cup [1, \infty)$

**(b)** $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$; for all $x \in (-\infty, 1] \cup [1, \infty)$

**(c)** $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x; & \text{for } x > 0 \\ -\pi + \cot^{-1}x; & \text{for } x < 0 \end{cases}$

## Property VI:

**(a)** $$\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right), \; x \in (0, 1)$$

**(b)** $$\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right),\ x \in (0,\ 1)$$

**(c)** $$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right),\ x > 0$$

## Property VII:

**(a)** $\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2}$, $\quad -1 \le x \le 1$

**(b)** $\tan(\cot^{-1}x) = \cot(\tan^{-1}x) = \dfrac{1}{x}$, $x \in \mathrm{R}$, $x \ne 0$

**(c)** $\operatorname{cosec}(\sec^{-1}x) = \sec(\operatorname{cosec}^{-1}x) = \dfrac{|x|}{\sqrt{x^2-1}}$, $|x| > 1$

## Property VIII:

**(a)** $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & x \ge 0, y \ge 0 \text{ and } xy < 1 \\ \dfrac{\pi}{2}, & x \ge 0, y \ge 0 \text{ and } xy = 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & x \ge 0, y \ge 0 \text{ and } xy > 1 \end{cases}$$

**(b)** $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \qquad x \ge 0, y \ge 0$$

## Illustration 4:

Show that $(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) = \pi$

## Solution

$$\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1}\left(\frac{2+3}{1-2\times 3}\right) \{\text{as } (2)(3) > 1\} = \pi + \tan^{-1}(-1)$$

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \tan^{-1}(1) + \pi - \tan^{-1}(1)$$

$$(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) = \pi$$

# Property IX:

**(a)** $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$

$$= \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right),\ x \geq 0, y \geq 0 \text{ and } x^2 + y^2 > 1$$

**(b)** $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, $x \geq 0$, $y \geq 0$

# Property X:

**(a)** $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, $x \geq 0$, $y \geq 0$

**(b)** $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$, $x \geq 0, y \geq 0$ and $x \leq y$

$$= \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right),\ x \geq 0, y \geq 0 \text{ and } x > y$$

## Some Other Property:

**(a)** $$2\sin^{-1}x = \begin{cases} -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -1 \leq x < -\frac{1}{\sqrt{2}} \\ \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } -1 \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & \text{if } \frac{1}{\sqrt{2}} < x \leq 1 \end{cases}$$

**(b)** $$3\sin^{-1}x = \begin{cases} -\pi - \sin^{-1}\left(3x - 4x^3\right), & \text{if } -1 \le x < -\dfrac{1}{2} \\ \sin^{-1}\left(3x - 4x^3\right), & \text{if } -1 \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}\left(3x - 4x^3\right), & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$$

**(c)** $$2\cos^{-1}x = \begin{cases} 2\pi - \cos^{-1}\left(2x^2 - 1\right), & \text{if } -1 \le x < 0 \\ \cos^{-1}\left(2x^2 - 1\right), & \text{if } -1 \le x < 0 \end{cases}$$

**(d)** $$3\cos^{-1}x = \begin{cases} 2\pi + \cos^{-1}\left(4x^3 - 3x\right), & \text{if } -1 \le x < -\dfrac{1}{2} \\ 2\pi - \cos^{-1}\left(4x^3 - 3x\right), & \text{if } -1 \le x \le \dfrac{1}{2} \\ \cos^{-1}\left(4x^3 - 3x\right), & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$$

**(e)** $$2\tan^{-1}x = \begin{cases} -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x < -1 \\ \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x > 1 \end{cases}$$

**(f)** $$3\tan^{-1}x = \begin{cases} -\pi + \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), & \text{if } x < -\dfrac{1}{\sqrt{3}} \\ \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), & \text{if } -1 < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \end{cases}$$

## Illustration 5:

Solve the equation: $2(\sin^{-1}x)^2 - (\sin^{-1}x) - 6 = 0$

## Solution

Let, $\sin^{-1}x = y$, we get

$2y^2 - y - 6 = 0$

$2y^2 - 4y + 3y - 6 = 0$

$y = 2$ and $y = -1.5$

$\sin^{-1}x = 2$ and $\sin^{-1}x = -1.5$

Since $2 > \pi/2$ and $|-1.5| < \pi/2$, the only solution is $x = -\sin 1.5$

## Illustration 6:

Solve the equation: $\sin^{-1}6x + \sin^{-1}6\sqrt{3}x = -\pi/2$

## Solution

Let us transfer $\sin^{-1}(6\sqrt{3}\,x)$ into the right hand side of the equation and calculate the sine of the both sides of the resulting equation:

$$\sin(\sin^{-1}6x) = \sin(-\sin^{-1}6\sqrt{3}x - \pi/2)$$

$\Rightarrow 6x = -\sin(\sin^{-1}6\sqrt{3}x + \sin^{-1}1)$ {using $\sin^{-1}(-x) = -\sin^{-1}(x)$}

$\Rightarrow 6x = -\sin(\sin^{-1}\sqrt{1-108x^2})$

{using $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$}

$\Rightarrow 6x = -\sqrt{1-108x^2}$ $\quad\ldots(1)$

Squaring both sides, we get

$36x^2 = 1 - 108\,x^2$

$\Rightarrow 144\,x^2 = 1$

whose roots are $x = 1/12$ and $x = -1/12$.

Substituting $x = -1/12$ in equation (1) and inverse trigonometric equation satisfy the equation but $x = 1/12$ does not satisfy the equation (1) so only solution is $x = -1/12$.

## Illustration 7:

Solve $2\cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

## Solution

$x = \cos y$ ; where $0 \le y \le \pi$, $|x| \le 1$

$2\cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow 2\cos^{-1}(\cos y) = \sin^{-1}(2\cos y.\sqrt{1-\cos^2 y})$

$\Rightarrow 2\cos^{-1}(\cos y) = \sin^{-1}(2\cos y.\sin y)$

$\Rightarrow 2\cos^{-1}(\cos y) = \sin^{-1}(\sin 2y)$

$\Rightarrow \sin^{-1}(\sin 2y) = 2y$ for $-\pi/4 \le y \le \pi/4$

and $2\cos^{-1}(\cos y) = 2y$ for $0 \le y \le \pi$

Thus equation (i) holds only when, $y \in [0, \pi/4]$

$\Rightarrow x \in [\sqrt{2}/2, 1]$

## Illustration 8:

Solve the equation $\tan^{-1}\dfrac{x+1}{x-1}+\tan^{-1}\dfrac{x-1}{x}=\tan^{-1}(-7)$

## Solution

Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\dfrac{x+1}{x-1}\right]+\tan\left[\tan^{-1}\dfrac{x-1}{x}\right]}{1-\tan\left[\tan^{-1}\dfrac{x+1}{x-1}\right]\tan\left[\tan^{-1}\dfrac{x-1}{x}\right]}=\tan\{\tan^{-1}(-7)\}=-7$$

i.e., $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x}}{1-\dfrac{x+1}{x-1}\dfrac{x-1}{x}}=-7$

i.e., $\dfrac{2x^2-x+1}{1-x}=-7$

so that $x = 2$.

The value $x = 2$ is a solution of the equation

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi+\tan^{-1}(-7)$$

## Illustration 9:

Find the sum to the $n$ term of the series

$$\operatorname{cosec}^{-1}\sqrt{10}+\operatorname{cosec}^{-1}\sqrt{50}+\operatorname{cosec}^{-1}\sqrt{170}+\ldots+\operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$$

## Solution

Let $\theta = \operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$

$\Rightarrow \operatorname{cosec}^2\theta = (n^2 + 1)(n^2 + 2n + 2)$

$= (n^2 + 1)n^2 + 2n(n^2 + 1) + 2(n^2 + 1)$

$= (n^2 + n + 1)^2 + 1$

$\Rightarrow \cot^2\theta = (n^2 + n + 1)^2$

$$\Rightarrow \tan\theta = \frac{1}{n^2+n+1} = \frac{(n+1)-n}{1+(n+1)n}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left[\frac{(n+1)-n}{1+(n+1)n}\right] = \tan^{-1}(n+1) - \tan^{-1} n$$

Thus, sum of $n$ terms of the given series

$= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + (\tan^{-1} 4 - \tan^{-1} 3) + \ldots + (\tan^{-1}(n + 1) - \tan^{-1} n)$

$\Rightarrow \tan^{-1}(n + 1) - \pi/4$

## Illustration 10:

Find the sum of the first $n$ terms of the series

$$\cot^{-1}(3) + \cot^{-1}(7) + \cot^{-1}(13) + \cot^{-1}(21) + \ldots$$

## Solution

Let $t_r$ denote the $r^{\text{th}}$ term of the series 3, 7, 13, 21, … and

$$S = 3 + 7 + 13 + 21 + \ldots + t_n$$

Also $S = 3 + 7 + 13 + \ldots + t_{n-1} + t_n$

Subtracting we get

$$0 = 3 + 4 + 6 + \ldots + 2n - t_n$$

$\Rightarrow t_n = 3 + 4 + 6 + \ldots + 2n$

$$= 3 + \frac{1}{2}(n-1)(4+2n) = n^2 + n + 1$$

Let $T_r = \cot^{-1}(r^2 + r + 1) = \tan^{-1}\left(\frac{1}{r^2 + r + 1}\right)$

$$= \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right) = \tan^{-1}(r+1) - \tan^{-1} r$$

Thus, the sum of the first $n$ terms of the given series is

$$\tan^{-1}(n+1) - \tan^{-1}(1) = \tan^{-1}\left[\frac{n+1-1}{1+1(n+1)}\right] = \tan^{-1}\left(\frac{n}{n+2}\right)$$

## Illustration 11:

Find the sum $\sum_{k=1}^{n} \tan^{-1}\left(\frac{2k}{2+k^2+k^4}\right)$

## Solution

We first try to put $\tan^{-1}[(2k)/(2+k^2+k^4)]$ in the form

$\tan^{-1}[(x-y)/(1+xy)]$

Let $x - y = 2k$ and $xy = 1 + k^2 + k^4$

$\Rightarrow x(x-2k) = 1 + k^2 + k^4 \Rightarrow x^2 - 2kx + k^2 = 1 + 2k^2 + k^4$ (since $y = x - 2k$)

$\Rightarrow (x-k)^2 = (k^2+1)^2 \Rightarrow x - k = (k^2+1)$

$\Rightarrow x = k^2 + k + 1$ and $y = k^2 - k + 1$

Therefore $\sum_{k=1}^{n} \tan^{-1}\left(\frac{2k}{2+k^2+k^4}\right) = \sum_{k=1}^{n} \tan^{-1}\left[\frac{(k^2+k+1)-(k^2-k+1)}{1+(k^2+k+1)(k^2-k+1)}\right]$

$$= \sum_{k=1}^{n} \tan^{-1}[\tan^{-1}(k^2+k+1) - \tan^{-1}(k^2-k+1)]$$

$= (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 7 - \tan^{-1} 3) + (\tan^{-1} 13 - \tan^{-1} 17) + \ldots + [\tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)]$

$= \tan^{-1}(n^2+n+1) - \tan^{-1} 1 = \tan^{-1}(n^2+n+1) - \frac{\pi}{4}$

## Illustration 12:

Find the sum of the series $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}+\tan^{-1}\frac{1}{32}+.......\infty$.

## Solution

We have

$$\tan^{-1}\left(\frac{1}{2r^2}\right)=\tan^{-1}\left(\frac{2}{4r^2}\right)$$

$$=\tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)$$

$$= \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$$

Thus,

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{n}[\tan^{-1}(2r+1)-\tan^{-1}(2r-1)]$$

$$= \tan^{-1}(2n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}(2n+1) - \pi/4$$

$$\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)=\lim_{n\to\infty}[\tan^{-1}(2n+1)-\pi/4]$$

$$= \tan^{-1}(\infty) - \pi/4 = \pi/2 - \pi/4 = \pi/4$$