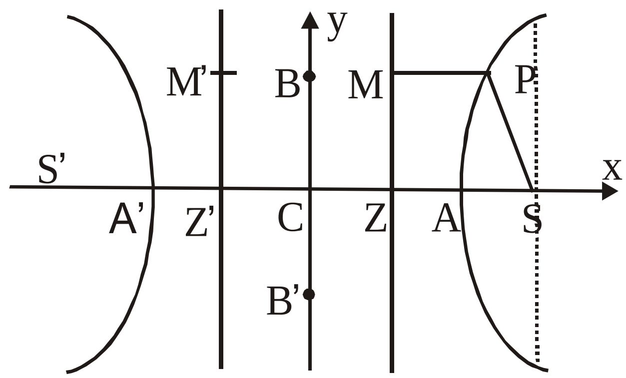
1. Definition

The hyperbola is the locus of a point which moves such that its distance from a fixed point called focus is always e times (e > 1) its distance from a fixed line called directrix.

2. Standard Form of Hyperbola

Standard equation of hyperbola S $= \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The eccentricity e of the hyperbola is given by the relation $e^2 = \left(1 + \frac{b^2}{a^2}\right)$



(i) Foci: S = (ae, 0) & ($-ae$, 0)

(ii) Equation of directories: $x = \frac{a}{e}$ & $x = -\frac{a}{e}$

(iii) Vertices: A = (a, 0) & A′ = ($-a$, 0)

(iv) Transverse Axis: The lines segment of length 2a which the foci S′ & S both called Transverse axis of the Hyperbola.

(v) Conjugate Axis: The line segment BB′(B ≡ (0, b)) and (B′ ≡ (0, $-b$)) is called the Conjugate axis of the hyperbola. The Transverse axis & the Conjugate axis of the hyperbola are together called principal axes of the hyperbola.

(vi) Length of latus rectum $= \frac{2b^2}{a}$ & $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, \frac{-b^2}{a}\right)$

Illustration 1:

Show that the equation $x^2 - 2y^2 - 2x + 8y - 1 = 0$ represents a hyperbola. Find the coordinates of the centre, length of the axes, eccentricity, latus rectum, coordinates of foci and vertices and equations of directrices of the hyperbola.

Solution

$x^2 - 2y^2 - 2x + 8y - 1 = 0 \Rightarrow (x^2 - 2x) - 2(y^2 - 4y) = 1$

$\Rightarrow (x^2 - 2x + 1) - 2(y^2 - 4y + 4) = -6$

$\Rightarrow (x - 1)^2 - 2(y-2)^2 = -6 \Rightarrow \dfrac{(x-1)^2}{\left(\sqrt{6}\right)^2} - \dfrac{(y-2)^2}{(\sqrt{3})^2} = -1$

Shifting the origin at (1, 2) without rotating the coordinate axes and denoting the new coordinates with respect to these axes by X and Y, we have

$X = (x - 1)$ and $Y = (y - 2)$ ...(i)

Using these relations, equation (i) is reduced to $\dfrac{X}{(\sqrt{6})^2} - \dfrac{Y^2}{\left(\sqrt{3}\right)^2} = -1$...(ii)

This equation is of the form $\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = -1$, where $a^2 = \left(\sqrt{6}\right)^2$ and $b^2 = \left(\sqrt{3}\right)^2$

Illustration 2:

Find the equation of the hyperbola whose foci are (8, 3) (0, 3) and eccentricity $= \dfrac{4}{3}$.

Solution

The centre of the hyperbola is the midpoint of the line joining the two foci.

So the coordinates of the centre are $\left(\frac{8+0}{2}, \frac{3+3}{2}\right)$ i.e., (4, 3).

Let 2a and 2b be the length of transverse and conjugate axes and let e be the eccentricity. Then the equation of the hyperbola is
$\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1$ ......(i)

Now, distance between the two foci = 2ae

$\Rightarrow \quad \sqrt{(8-0)^2 + (3-3)^2} = 2ae \Rightarrow ae = 4 \Rightarrow a = 3 \left(\because e = \frac{4}{3}\right)$

Now, $b^2 = a^2 (e^2 - 1) \Rightarrow b^2 = 9\left(-1+\frac{16}{9}\right) = 7.$

Thus, the equation of the hyperbola is

$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1$ [Putting the values of a and b in (i)]

$\Rightarrow \quad 7x^2 - 9y^2 - 56x + 54y - 32 = 0$

Focal distance

The focal distance of any point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $ex_1 - a$ and $ex_1 + a$.

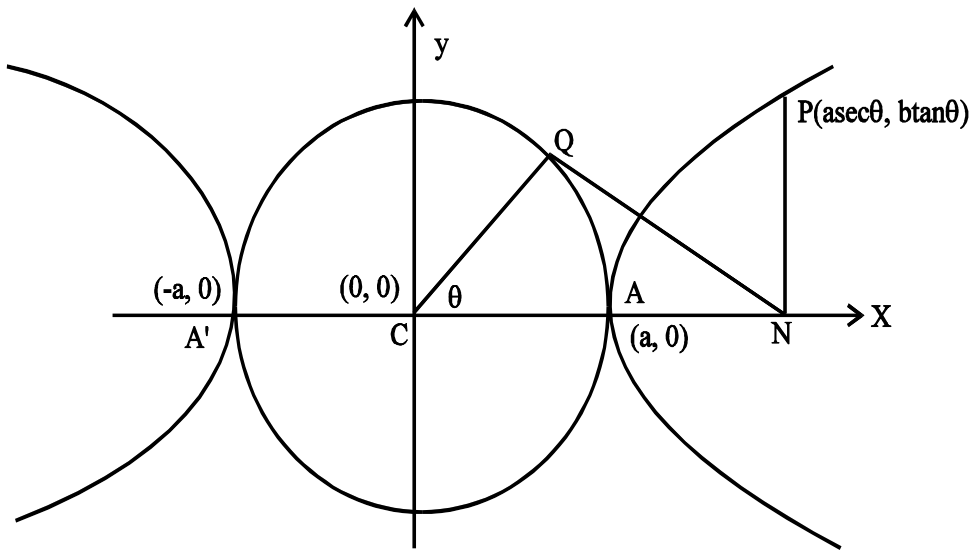
Another Definition of the Hyperbola

The difference of the focal distances of a point on the hyperbola is constant.

Auxiliary Circle

A circle drawn with centre C and AA′ as a diameter is called the Auxiliary Circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P and Q are called the "Corresponding Points" on the hyperbola and the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola where ,$0 \le \theta \le 2\pi$.



Parametric Coordinates

The equations $x = a \sec\theta$ and $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter, in other words, ($a \sec\theta$, $b \tan\theta$) is a point on the hyperbola for all values of

$\theta \ne (2n + 1)\frac{\pi}{2}, n \in I$. The point ($a \sec\theta$, $b \tan\theta$) is briefly called the point θ.

Note: Equation of a chord joining θ_1 & θ_2 is

$$\frac{x}{a}\cos\frac{\theta_1 - \theta_2}{2} - \frac{y}{b}\sin\frac{\theta_1 + \theta_2}{2} = \cos\frac{\theta_1 + \theta_2}{2}.$$

General Note:

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many proposition for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

General Form

The equation of hyperbola, whose focus is point (h, k), directrix is $lx + my + n = 0$ snd eccentricity 'e' is given by

$$(x-h)^2 + (y-k)^2 = \frac{e^2(lx+my+n)^2}{(l^2+m^2)}$$

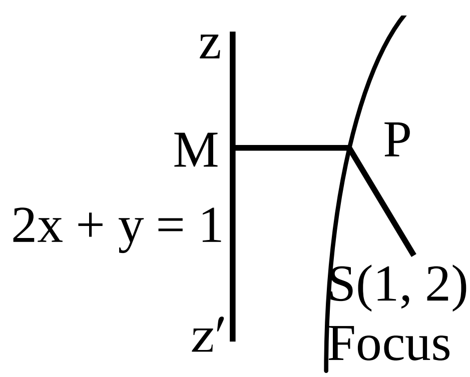
Illustration 3:

Find the equation of hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Solution

Let S(1, 2) be the focus and P (x, y) be a point on the hyperbola. Draw PM perpendicular from P on the directrix.

Then by definition. SP = ePM



$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{3}\left|\frac{2x+y-1}{\sqrt{2^2+1^2}}\right|$$

$$\Rightarrow (x-h)^2 + (y-k)^2 = \frac{e^2(lx+my+n)^2}{(l^2+m^2)}$$

$$\Rightarrow (x-1)^2 + (y-2)^2 = 3\ \frac{(2x+y-1)^2}{5}$$

$$\Rightarrow 5\{(x-1)^2 + (y-2)^2\} = 3\{2x + y - 1\}^2$$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$ (This is the required equation)

Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

e.g., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

are conjugate hyperbolas of each other.

Note: If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $\frac{1}{e_1} + \frac{1}{e_2} = 1$

2. Asymptotes

Definition: If the length of perpendicular drawn from a point on the hyperbola to a straight line tends to zero as the point moves to infinity. The straight line is called asymptotes.

Note: $\text{H} = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right), \text{C} = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1\right)$ & $\text{A} = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0\right)$

Clearly, C + H = 2A

H = hyperbola, C = Conjugate hyperbola, A = Asymptotes.

Particular Case:

When $b = a$,

The asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ is $y = \pm x$ which are at right angles.

Remarks:

(i) Equilateral hyperbola rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ from the hyperbola and the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola and the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as let $f(x, y) = 0$ represents a hyperbola. Find $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$. Then the point of intersection of $\frac{\delta f}{\delta x} = 0$ and $\frac{\delta f}{\delta y} = 0$ gives the centre of the hyperbola.

Illustration 4:

Show that the acute angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 (a^2 > b^2)$ is $2\cos^{-1}\left(\frac{1}{e}\right)$, where e is the eccentricity of the hyperbola.

Solution

Equation of the asymptotes of the given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

$\Rightarrow \quad b^2x^2 - a^2y^2 = 0$

If θ is an angle between the asymptotes, then $\tan\theta = \frac{\pm\sqrt{a^2b^2}}{b^2 - a^2} = \pm\frac{ab}{a^2 - b^2}$

As $\tan\theta = \frac{ab}{a^2 - b^2} \Rightarrow \cos(\theta/2) = \sqrt{a^2/(a^2 + b^2)} = 1/e$.

4. Rectangular or Equilateral Hyperbola

A hyperbola is called rectangular if its asymptotes are at right angles. The asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\left(\frac{b}{a}\right)x$ so they are perpendicular if $-\frac{b^2}{a^2} = -1$ i.e., $b^2 = a^2$, i.e., $a = b$.

Hence equation of a rectangular hyperbola can be written as $x^2 - y^2 = a^2$

Some important observations of rectangular hyperbola are as under:

(i) $a^2 = a^2(e^2 - 1)$ gives $e^2 = 2$ i.e., $e = \sqrt{2}$

(ii) Asymptotes are $y = \pm x$.

(iii) Rotating the axes by an angle $-\pi/4$ about the same origin, equation of the rectangular hyperbola $x^2 - y^2 = a^2$ is reduced to $xy = \frac{a^2}{2}$ or $xy = c^2, (c^2 = \frac{a^2}{2})$.

(iv) Rectangular hyperbola is also called equilateral hyperbola.

Rectangular Hyperbola referred to its asymptotes as axis of coordinates

(i) Equation of the rectangular hyperbola is $xy = c^2$ with parametric representation $x = ct, y = \frac{c}{t}, t \in \{R \sim (0)\}$.

(ii) Equation of a chord joining the points P(t_1) & Q(t_2) is
$x + t_1t_2y = c(t_1 + t_2)$

(iii) Equation of the tangent at P(x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ and at P(t) is $\frac{x}{t} + ty = 2c$.

(iv) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

(v) Equation of the normal at P(t) is $xt^3 - yt = c(t^4 - 1)$.

(vi) Vertex of this hyperbola is (c, c) and ($-c$, $-c$); focus is $\left(\sqrt{2}c, \sqrt{2}c\right)$ and $\left(-\sqrt{2}c, -\sqrt{2}c\right)$, the directrices are $x + y = \pm\sqrt{2}c$ and ℓL.R. $= 2\sqrt{2}c$, T.A.= C.A.

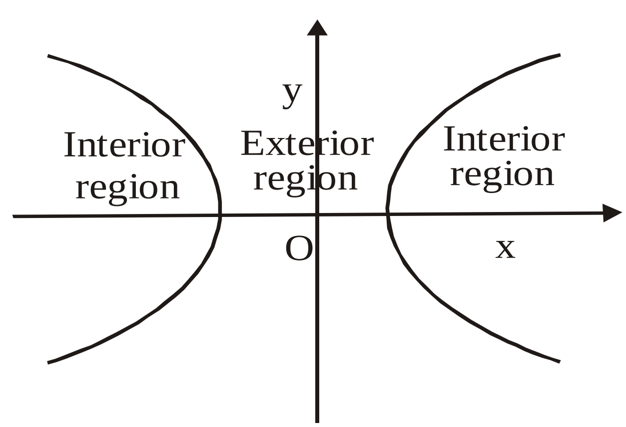
5. Position of a Point P w.r.t Hyperbola

Let S = 0 be the hyperbola and P(x_1, y_1) be the point and $S_1 \equiv S(x_1, y_1)$.
Then

$S_1 < 0 \Rightarrow$ P is in the exterior region

$S_1 > 0 \Rightarrow$ P is in the interior region

$S_1 = 0 \Rightarrow$ P lies on the hyperbola



6. Line and a Hyperbola

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as: $c^2 >, =$ or $< a^2m^2 - b^2$.

7. Tangent and Normal

1. Tangent

(i) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(ii) In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and they are $y - y_1 = m_1(x - x_1)$ and $y - y_1 = m_2(x - x_2)$, where m_1 and m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from (x_1, y_1) to the hyperbola.

(iii) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec \theta, b \tan \theta)$ is $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$.

(iv) Point of intersection of the tangents at θ_1 and θ_2 is

$$x = a\frac{\cos\frac{\theta_1 - \theta_2}{2}}{\cos\frac{\theta_1 + \theta_2}{2}},\ y = b\frac{\sin\frac{\theta_1 - \theta_2}{2}}{\cos\frac{\theta_1 + \theta_2}{2}}.$$

(v) $y = mx \pm \sqrt{a^2m^2 - b^2}$ Can be taken as the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Illustration 5:

Find the equations of the tangents to the hyperbola $3x^2 - y^2 = 3$, which are perpendicular to the line $x + 3y = 2$.

Solution

Let m be the slope of the tangent to the given hyperbola. Then,

$m \times$ (slope of the line $x + 3y = 2$) $= -1$

$\Rightarrow m\left(-\dfrac{1}{3}\right) = -1 \Rightarrow m = 3$

Now, $3x^2 - y^2 = 3 \Rightarrow \dfrac{x^2}{1} - \dfrac{y^2}{3} = 1.$

This is of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a^2 = 1$ and $b^2 = 3$.

So, the equations of the tangents are $y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow \quad y = 3x \pm \sqrt{9-3} \Rightarrow y = 3x \pm \sqrt{6}$

2. Normal

(i) The equation of the normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at point P(x_1, y_1) on the curve $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2 \equiv a^2e^2$

(ii) The equation of the normal at the point P (a sec θ, b tan θ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$

(iii) In general, four normals can be drawn to a hyperbola from any point and if α, β, γ, δ be the concentric angles of these four co-normal points, then α + β + γ + δ is an odd multiple of π.

3. Chord of Contact of Tangents Drawn from a Point Outside the Hyperbola

Chord of contact of tangents drawn from a point outside the hyperbola is

$\boxed{T = 0}$ i.e., $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$

Illustration 6:

From any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$. Then show that the area cut off by the chord of contact on the asymptotes is $4ab$

Solution:

Let P (x_1, y_1) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then, $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1.$

The chord of contact of tangent from P to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=2 \qquad \ldots \text{(i)}$$

The equations of the asymptotes are $\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$

The points of intersection of (i) with the two asymptotes are given by

$$x_1=\frac{2a}{\frac{x_1}{a}-\frac{y_1}{b}}, y_1=\frac{2b}{\frac{x_1}{a}-\frac{y_1}{b}}, x_2=\frac{2a}{\frac{x_1}{a}+\frac{y_1}{b}}, y_2=\frac{-2b}{\frac{x_1}{a}+\frac{y_1}{b}}$$

$\therefore$ Area of the triangle = $\frac{1}{2}|x_1y_2-x_2y_1|=\frac{1}{2}\left(\frac{8ab}{\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}}\right)=4ab$

8. Chord of Hyperbola with Specified Midpoint

Chord of hyperbola with specified midpoint (x_1, y_1) is $\boxed{T = S_1}$, where S_1 and T have usual meanings.

Illustration 7:

Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$, which is bisected at the point (5, 3).

Solution:

Equation of the given hyperbola can be written as $\frac{x^2}{16}-\frac{y^2}{25}=1$

Therefore, equation of the chord of this hyperbola in terms of the midpoint (5, 3) is ($T = S_1$)

$$\frac{5x}{16}-\frac{3y}{25}-1=\frac{5^2}{16}-\frac{3^2}{25}-1 \Rightarrow 125x - 48y = 481$$

Illustration 8:

Find the locus of the midpoints of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola. $9x^2 - 16y^2 = 144$

Solution

Let (h, k) be the middle point of a chord of the circle $x^2 + y^2 = 16$

Then its equation is $hx + ky - 16 = h^2 + k^2 - 16$ i.e.,

$$hx + ky = h^2 + k^2 \qquad \ldots(i)$$

Let (i) touch the hyperbola

$$9x^2 - 16y^2 = 144 \text{ i.e., } \frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \ldots(ii)$$

at the point (α, β) say, then (i) is identical with $\frac{x\alpha}{16} - \frac{y\beta}{9} = 1$ …(iii)

$$\text{Thus } \frac{\alpha}{16h} = \frac{-\beta}{9k} = \frac{1}{h^2 + k^2}$$

$$\alpha = \frac{16h}{h^2 + k^2} \text{ And } \beta = \frac{-9k}{h^2 + k^2}$$

Since (α, β) lies on the hyperbola (ii), $\frac{1}{16}\left(\frac{16h}{h^2 + k^2}\right) - \frac{1}{9}\left(\frac{9k}{h^2 + k^2}\right) = 1$

$\Rightarrow \; 16h^2 - 9k^2 = (h^2 + k^2)^2$

Hence the required locus of (h, k) is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

9. Pair of Tangents

Equation of pair of tangents from point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\boxed{SS_1 = T^2} \text{ i.e. } = \left(\frac{x^2}{a^2} - \frac{y2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

10. Director Circle

The locus of the point of intersection of two perpendicular tangents to a hyperbola is called its director circle. Its equation is $x^2 + y^2 = a^2 - b^2$.

11. Highlights

(i) Equation of the tangent at P (t) is $\frac{x}{t} + yt = 2c$ where P is the point on the curve $xy = c^2$

(ii) Equation of the normal at P(t) is $xt^3 - yt = c(t^4 - 1)$. Where P is the point on the curve $xy = c^2$.

(iii) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e., $x^2 + y^2 = a^2$ and the product of the length of these perpendiculars is b^2.

(iv) The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.

(v) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

(vi) Perpendicular from the foci on either asymptote meets it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.

(vii) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a ΔCQR of constant area equal to *ab* from the asymptotes and the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the ΔCQR in case of a rectangular hyperbola is the hyperbola itself and for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

(viii) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii. This spells reflection property of the hyperbola as an incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

(ix) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line intercepted between the point and the curve is always equal to the square on the semi conjugate axis.

(x) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ, then the eccentricity of the hyperbola is $\sec\theta$.