1. Definition

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point is in a constant ratio (less than one) to its distance from a fixed line. The fixed point is called the focus and fixed line is called the directrix and the constant ratio is called the eccentricity of the ellipse.

2. Equation of an Ellipse

2.1 Standard Equation of Ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2 = a^2(1 - e^2)$

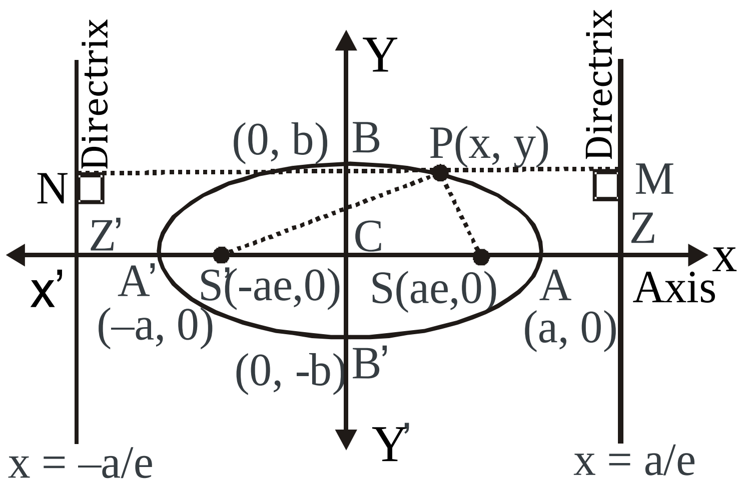
Where e is the eccentricity of the Ellipse

2.1.1 Another definition of Ellipse

An ellipse is locus of a point which moves in a plane so that the sum of its distances from fixed points is constant.

2.1.2 Various terms related with an Ellipse

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a > b$



(i) Vertices of an ellipse: The points at which the ellipse cuts the x-axis (a, 0) & (–a, 0) are called the vertices of the ellipse.

(ii) Major & Minor axis: The line segment AA′ is called the major axis and BB′ is called the minor axis. The major and minor axis taken together is called the principal axes and their lengths will be given by 2a and 2b respectively.

(iii) Centre: The point which bisects each chord of the ellipse is called the centre (0, 0).

(iv) Directrix: ZM and Z′M′ are two directrices and their equations are $x = \frac{-a}{e}$ and $x = \frac{a}{e}$.

(v) Focus: S (–ae, 0) and S′ (ae, 0) are two foci of the ellipse.

(vi) Latus Rectum: The chord which passes through either of the focus and perpendicular to the major axis is called a latus rectum.

Length of Latus Rectum:

Length of Latus rectum is given by $\frac{2b^2}{a}$

(vii) Relation between constant *a*, *b* and *e*

$$b^2 = a^2(1 - e^2) \Rightarrow e^2 = \frac{a^2 - b^2}{a^2} \quad \therefore e = \frac{\sqrt{a^2 - b^2}}{a}$$

Result:

(i) Another form of standard equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $a < b$

In this case major axis is BB′ = 2b which is along y-axis and minor axis is AA′ = 2a which is along x-axis. Foci are S(0, be) and S′(0, –be) and directrices are $y = b/e$ and $y = -b/e$.

(ii) Focal distances: The focal distance of the point (x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $a + ex$ and $a - ex$.

2.2 General equation of the ellipse

The general equation of an ellipse, whose focus is (h, k), the directrix is the line $ax + by + c = 0$ and the eccentricity is e is given by

$$(x-h)^2+(y-k)^2=\frac{e^2(ax+by+c)^2}{a^2+b^2}$$

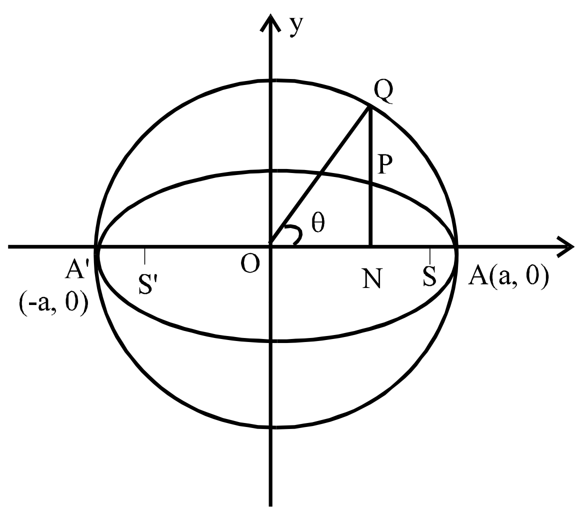
Note:

Condition for second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in x & y to represent an ellipse is given by

$h^2 - ab < 0$ & $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

2.3 Auxiliary Circle/Eccentric Angle:

A circle described on major axis as diameter is called the auxiliary circle. Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P and Q are called as the Corresponding Points on the ellipse and the auxiliary circle respectively called the Eccentric Angle of the point P on the ellipse ($0 \leq \theta \leq 2\pi$).



Note that: $\frac{\ell(PN)}{\ell(QN)}=\frac{b}{a}=\frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle."

2.4 Parametric equation of an ellipse

Clearly $x = a\cos\theta$, $y = b\sin\theta$ satisfy the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all real values of θ. Moreover any point on the ellipse can be represented as ($a\cos\theta$, $b\sin\theta$), $0 \le \theta \le 2\pi$. Hence the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $x = a\cos\theta$, $y = b\sin\theta$, where θ, is the parameter.

Equation of the chord of the ellipse whose eccentric angles are θ & ϕ,

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

Illustration 1:

Find the centre, the length of the axes, eccentricity and the foci of the ellipse.

$$12x^2 + 4y^2 + 24x - 16y + 25 = 0$$

Solution

The given equation can be written in the form

$$12(x+1)^2 + 4(y-2)^2 = 3$$

Or $$\frac{(x+1)^2}{1/4}+\frac{(y-2)^2}{3/4}=1 \qquad \ldots(1)$$

Co-ordinates of the centre of the ellipse are given by

$$x + 1 = 0 \text{ and } y - 2 = 0$$

Hence centre of the ellipse is (– 1, 2)

If a and b be the lengths of the semi minor and semi major axes, then $a^2 = 1/4$, $b^2 = 3/4$

∴ Length of major axis = $2b = \sqrt{3}$,

Length of minor axis = $2a = 1$

$\therefore\ b = \frac{\sqrt{3}}{2}, a = \frac{1}{2}$

Since, $a^2 = b^2 (1 - e^2)$

$\therefore\ 1/4 = 3/4 (1 - e^2) \Rightarrow e = \sqrt{2/3}$

$\therefore\ b \times e = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{\sqrt{2}}$

Co-ordinates of foci are given by $(0, \pm be)$

$\Rightarrow x + 1 = 0, y - 2 = \pm be$

Thus foci are $\left(-1, 2 \pm \frac{1}{\sqrt{2}}\right)$

Illustration 2:

The foci of an ellipse are $(\pm 2, 0)$ and its eccentricity is 1/2, find its equation.

Solution

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

then coordinates of foci are $(\pm ae, 0)$.

$\therefore\ ae = 2 \Rightarrow a \times \frac{1}{2} = 2 \left[\because e = \frac{1}{2}\right] \Rightarrow a = 4$

We have $b^2 = a^2 (1 - e^2)$

$\therefore\ b^2 = 16\left(1 - \frac{1}{4}\right) = 12$

Thus, the equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

3. Point and Ellipse

Let P(x_1, y_1) be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of an ellipse.

The point lies outside, on or inside the ellipse accordingly as

$S_1 = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, > 0, = 0, < 0.

Illustration 3:

Find the position of the point (4, – 3) relative to ellipse $5x^2 + 7y^2 = 140$.

Solution

$5\,(4)^2 + 7\,(-3)^2 - 140 = 80 + 63 - 140 = 3 > 0$,

$\therefore$ the point (4, – 3) lies outside the ellipse $5x^2 + 7y^2 = 140$.

4. Ellipse and Line

Let the equation of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the given line be

$y = mx + c$.

Solving the line and ellipse, we get $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

i.e., $(a^2m^2 + b^2)x^2 + 2mca^2\,x + a^2(c^2 - b^2) = 0$

above equation being a quadratic in x its discriminant

$$= 4m^2c^2a^4 - 4(a^2m^2 + b^2)a^2(c^2 - b^2)$$

$$= - b^2\{c^2 - (a^2m^2 + b^2)\} = b^2\{(a^2m^2 + b^2) - c^2\}$$

Hence the line intersects the ellipse in 2 distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$ and does not intersect if $a^2m^2 + b^2 < c^2$.

$\therefore\ y = mx \pm \sqrt{(a^2m^2 + b^2)}$ touches the ellipse and condition for tangency is $c^2 = a^2m^2 + b^2$

Moreover the line $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\frac{\pm a^2 m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$.

Note: **(i)** $x \cos \alpha + y \sin \beta = p$ is a tangent if $p^2 = a^2 \cos^2\alpha + b^2 \sin^2\alpha$.

(ii) $lx + my + n = 0$ is a tangent to the ellipse if $n^2 = a^2l^2 + b^2m^2$.

4.1 Equation of the Tangent

(i) The equation of the tangent at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(ii) The equation of tangent at any point 'ϕ' is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$

(iii) Point of intersection of tangents to ellipse at points 'θ', 'ϕ' is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

4.2 Equation of the Normal

(i) The equation of the normal at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

(ii) The equation of the normal at any point 'ϕ' is $ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2$

Illustration 4:

Find the condition that the line $lx + my = n$ may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

Equation of normal to the given ellipse at $(a\cos\theta,\ b\sin\theta)$ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \qquad \ldots (1)$$

If the line $lx + my = n$ is a normal to the ellipse, then there must be a value of θ for which line (1) and line $lx + my = n$ are identical. For that value of θ, we have

$$\frac{l}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{-\left(\frac{b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)}$$

$\Rightarrow$ $$\cos\theta=\frac{an}{l(a^2-b^2)} \qquad \ldots(2)$$

And $$\sin\theta=\frac{-bn}{m(a^2-b^2)} \qquad \ldots(3)$$

Squaring and adding (3) and (4) we get

$$1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)$$

Which is the required condition

Illustration 5:

Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution

Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$,

$$\Rightarrow m = \frac{1}{2}$$

Equation of the given ellipse is $3x^2 + 4y^2 = 12$ or $\left(\frac{x^2}{4}+\frac{y^2}{3}=1\right)$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a^2 = 4$ and $b^2 = 3$

So the equation of the tangents are y = $mx \pm \sqrt{(a^2m^2+b^2)}$ = $\frac{1}{2}x \pm \sqrt{4\times\frac{1}{4}+3}$

$$\Rightarrow \quad y=\frac{1}{2}x\pm 2 \text{ Or } x - 2y \pm 4 = 0$$

5. Equation of Chord with Mid-Point (X_1, Y_1)

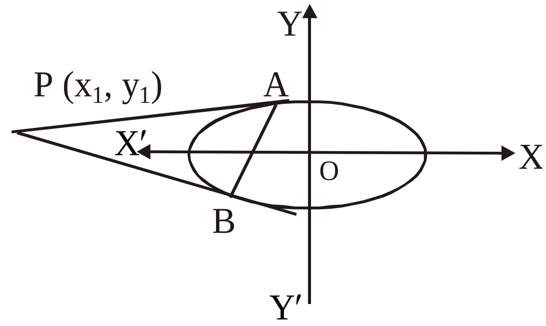
The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

Whose mid-point is (x_1, y_1) is given by T= S_1,

Where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ and $S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

6. Chord of Contact

If PA and PB be the tangents through point P(x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or T = 0 at (x_1, y_1)



7. Pair of Tangents

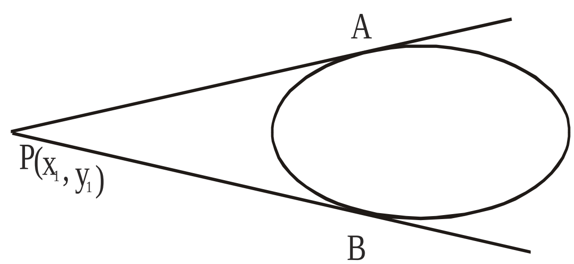
Let P (x_1, y_1) be any point outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let a pair of tangents PA, PB be drawn to it from P.

Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$,

Where $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ and

$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$



Note: The locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are perpendicular to each other is called the **director circle** and its equation is given by $x^2 + y^2 = a^2 + b^2$.

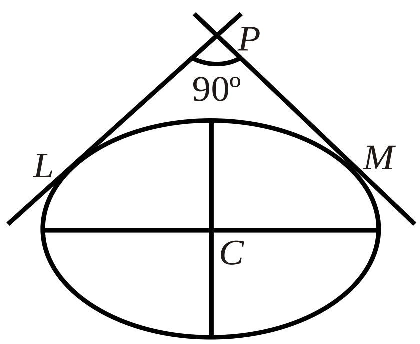
8. Director Circle

Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Illustration 6:

An ellipse slides between two perpendicular straight lines. Then the locus of its centre is

Solution



The lines PL and PM are two perpendicular tangents intersecting at P which lies on director circle

$$x^2 + y^2 = a^2 + b^2$$

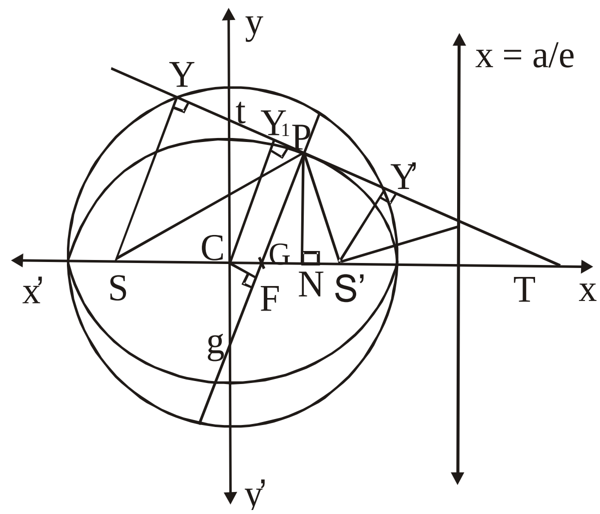
Hence $CP^2 = a^2 + b^2 =$ constant

Thus the locus of C ,the centre of ellipse is a circle of radius $\sqrt{(a^2 + b^2)}$.

9. Some Important Points to Remember

(i) The product of the lengths of the perpendicular segments from the foci on any tangent to the ellipse (at P) is b^2 and the feet of these perpendiculars lie on the auxiliary circle.

(ii) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then



(a) PF. PG = b^2

(b) PF. Pg = a^2

(c) PG. Pg = SP. S'P

(d) CG. CT = CS^2

(e) Locus of the mid-point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis].

(iii) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well-known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa.

(iv) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

(v) The circle on any focal distance as diameter touches the auxiliary circle.

(vi) If the tangent at the point P of a standard ellipse meets the axes in T and t and CY_1 is the perpendicular on it from the centre then,

(a) Tt. $PY_1 = a^2 - b^2$ **(b)** least value of Tt is $a + b$.