# PERMUTATIONS and COMBINATIONS

## Historical Note

In a broad range of fields of Mathematics and other subjects, one needs to deal with problems that involve combinations made up of letters, numbers or any other objects. e.g. an investigating chemist analyzes relations involving atoms and molecules, a linguist examines the meanings of the words formed by various combinations of letters. *The field of mathematics that studies problems of how many different combinations (subject to certain restrictions), can be built out of a specific number of objects, is called* ***Combinatorial Mathematics (combinatorics)*****.**

This branch of mathematics has its origin in the $16^{th}$ century, in the gambling games. It is quite natural that the first combinatorial problems had to do mainly with gambling, such as, in how many ways can a certain sum, in throws of two or three dice, be scored, or in how many ways is it possible to get two kings in a card game. One of the first to enumerate the various combinations achieved in games of dice was the Italian mathematician *Tartagalia*. In the $17^{th}$ century, French scholars, *Pascal* and *Fermat*, made a theoretical investigation into the problems of combinatorics. Further advances in the theory of combinations were connected with the names of *Jacob Bernoulli*, *Leibnitz* and *Euler*. During recent years, combinatorial mathematics has seen extensive developments associated with greater interest in problems of **Discrete Mathematics.**

## Fundamental principle of multiplication

If an operation can be performed in m different ways and another operation can be performed in n different ways, then the two operations in succession can be performed in mn ways. The principle can also be generalized, for even more than two operations.

## Fundamental principle of addition

If an operation can be performed in m different ways and another operation can be performed in n different ways and the two operations are mutually exclusive, then either of the two operations can be performed in (m + n) ways.

## Combinations:

Each of the groups or selections which can be made by taking some or all of a number of objects is called a combination.

### Selection of Objects without Repetition

The number of selections (combinations or groups) that can be formed from n different objects taken r ($0 \le r \le n$) at a time is $\frac{n!}{r!(n-r)!} = {}^nC_r$

### Selection of objects with repetition

The number of combinations of n distinct objects, taken r at a time when each may occur once, twice, thrice,….. upto r times in any combination is ${}^nH_r = {}^{n+r-1}C_r$ .

### Properties of combinatorial operator

**1.** ${}^nC_r = {}^nC_{n-r}$

**2.** ${}^nC_r = {}^nC_s \Rightarrow r = s$ OR $r + s = n$

**3.** ${}^nC_{r-1} + {}^nC_r = {}^{(n+1)}C_r$

**4.** $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{(n-r+1)}{r}$

**5.** When n is even the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$ and when n is odd the greatest value of ${}^nC_r$ is at $r = \frac{n-1}{2}$ and $r = \frac{n+1}{2}$.

## Permutations:

The number of permutations of n objects, taken r at a time, is the total number of arrangements of r objects, selected from n objects where the order of the arrangement is important.

### Without Repetition:

(a) Arranging n objects, taken r at a time is equivalent to filling r places from n things.

| | | | | | | |
|---|---|---|---|---|---|---|
| **r–Places:** | 1 | 2 | 3 | 4 | ............ | r |
| **Number of Choices:** | n | n-1 | n-2 | n-3 | | n - (r-1) |

The number of ways of arranging = The number of ways of filling r places

$$= n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

(b) The number of arrangements of n different objects taken all at a time = ${}^{n}p_{n} = n!$

### With Repetition:

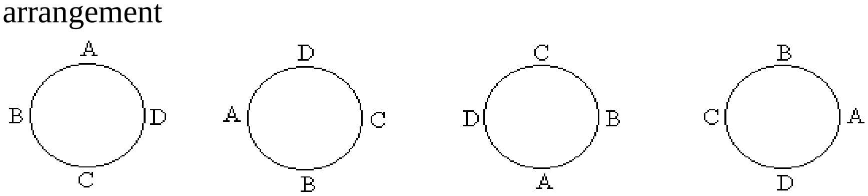
The number of permutations (arrangements) of n different objects, taken r at a time, when each object may occur once, twice, thrice…. upto r times in any arrangement is the number of ways of filling r places where each place can be filled by any one of n objects

| | | | | | | |
|---|---|---|---|---|---|---|
| **r - Places:** | 1 | 2 | 3 | 4 | ............ | r |
| **Number of Choices:** | **n** | **n** | **n** | **n** | | **n** |

The number of permutations = The number of ways of filling r places = $(n)^r$

### Circular and Linear permutations:

Suppose four persons *A*, *B*, *C*, D to be seated at a round table. Let us look at one such circular arrangement



In the above figure; *A*, *B*, *C*, D are shifted one position in any one particular direction, in the anti-clockwise direction. We can see all four arrangements are identical, since the relative position of *A*, *B*, *C*, D is the same.

But in case the four persons are to be seated in a row, the above four arrangement will be ABCD, DABC, CDAB, BCDA

Thus, it is clear that corresponding to four different linear arrangements there will be only one circular arrangement. Hence, the total number of circular arrangements in the above case = $\frac{4!}{4} = 3! = (4-1)!$ where as the number of linear permutations = 4!

In general, number of circular permutations is (n – 1)! while that of linear permutations is n!

## Important results for Combinations

1. The number of selections of r dissimilar objects out of n dissimilar objects = ${}^{n}C_{r}$
2. The number of selections of one or more dissimilar objects out of n dissimilar objects = $2^{n}-1$
3. The number of selections of r dissimilar objects out of n dissimilar objects such that p particular objects are always included = ${}^{n-p}C_{r-p}$
4. The number of selections of r dissimilar objects out of n dissimilar objects such that p particular objects are never included = ${}^{n-p}C_{r}$
5. Number of ways to select r objects out of n objects in which $n_1$ are of identical of one type, $n_2$ are of identical of one type,…, $n_k$ are of identical of one type and rest are different is given by coefficient of $x^r$ in the expansion of
$$\left(1+x+..+x^{n_1}\right)\left(1+x+...+x^{n_2}\right)...\left(1+x+...+x^{n_k}\right)\left(1+x\right)^{n-\left(n_1+n_2+...+n_k\right)}$$
6. Number of ways to select r objects out of n objects placed in a row such that no two or more adjacent objects are chosen = ${}^{n-r+1}C_{r}$
7. Number of ways to select r objects out of n objects placed in a circle such that no two or more adjacent objects are chosen = ${}^{n-r+1}C_{r} - {}^{n-r-1}C_{r-2}$

## Important results for Permutations

1. The number of permutations of n dissimilar things taken r at a time when repetition of things allowed any number of times is $n^r$.
2. The number of permutations of n things taken all at a time when p of them are alike and rest all are different is $\frac{n!}{p!}$.
3. If in $(n_1 + n_2 + ... + n_k)$ objects, $n_1$ objects are alike of one kind, $n_2$ objects are alike of second kind,... $n_k$ objects are alike of $k^{th}$ kind, then the number of permutations of all the objects taken all at a time is $\frac{(n_1+n_2+...+n_k)!}{n!n_2!...n_k!}$.
4. The number of circular permutations of n different things taken 'r' at a time is ${}^nC_r\,(n-1)!$.
5. The number of circular permutations of n different things taken all at a time = $(n – 1)!$
6. The number of circular permutations of n different things taken all at a time (If anti clockwise & clockwise arrangements are same) = $\frac{(n-1)!}{2}$
7. The number of permutations of n different things taken r at a time in which 'p' particular things do not occur is ${}^{(n-p)}C_r \times r!$.
8. The number of permutations of n different things taken r at a time in which 'p' particular things are present is ${}^{(n-p)}C_{(r-p)} \times r!$
9. Number of ways to permute r objects out of n objects in which $n_1$ are of identical of one type, $n_2$ are of identical of one type,…, $n_k$ are of identical of one type and rest are different is given by coefficient of $x^r$ in the expansion of

$$\left(1+\frac{x}{1!}+...+\frac{x^{n_1}}{n_1!}\right)\left(1+\frac{x}{1!}+...+\frac{x^{n_2}}{n_2!}\right)...\left(1+\frac{x}{1!}+...+\frac{x^{n_k}}{n_k!}\right)(1+x)^{n-(n_1+n_2+...+n_k)}\times r!$$

## Some important Tips and Tricks

1. The number of ways in which m (first kind of different) things and n (second kind of different) things $(m + 1 \geq n)$ can be arranged in a row so that no two things of second kind come together is $m!\,{}^{(m+1)}P_n$
2. The number of ways in which m (first kind of different) things and n (second kind of different) things can be arranged in a row so that all second type of things come together is $n!(m + 1)!$
3. The number of ways in which n (first type of different) things and n – 1(second type of different) things can be arranged in a row so that no two things of same type come together is $n!(n - 1)!$
4. The number of ways in which m (first type of different) things and n (second type of different) things can be arranged in a row alternatively is 2.n!.n!
5. The number of ways in which m (first kind of different) things and n (second kind of different) things, $(m \geq n)$ can be arranged in a circle so that no two things of second kind come together is $(m - 1)!\ {}^{m}P_n$
6. The number of ways in which m (first type of different) things and n (second type of different) things can be arranged in a circle so that all the second type of things come together is m!n!.
7. The number of ways in which m (first kind of different) things and n (second kind of different) things, $(m \geq n)$ can be arranged in the form of garland so that no two things of second kind come together is $(m-1)!\frac{{}^{m}P_n}{2}$.
8. The number of ways in which m (first kind of different) things and n (second type of different) things can be arranged in the form garland so that all the second type of things come together is $\frac{m!n!}{2}$

## Important results on Distribution in to groups

1. The number of ways in which m + n different things can be divided in to two groups which contain m and n ($m \neq n$) things respectively, is

   Case (i) if order of the groups is not to be taken in to account ${}^{(m+n)}C_n \times {}^{m}C_m = \dfrac{(m+n)!}{m!n!}$ .

   Case (ii) if order of the groups is to be taken in to account $\dfrac{(m+n)!}{m!n!} \times 2!$ .

2. The number of ways in which 2n different things can be divided into two equal groups, is

   Case (i) If order of the groups is not to be taken into account $\dfrac{{}^{2n}C_n \times {}^{n}C_n}{2!} = \dfrac{(2n)!}{2!(n!)^2}$ .

   Case (ii) If order of the groups is to be taken into account $\dfrac{(2n)!}{(n!)^2}$ .

3. The number of ways in which m + n + p different things can be divided into three groups which contain m, n and p ($m \neq n \neq p$) things respectively, is

   Case (i) if the order of the groups is not to be taken into account

   $$ {}^{(m+n+p)}C_m \times {}^{(n+p)}C_n \times {}^{p}C_p = \frac{(m+n+p)!}{m!n!p!} $$

   Case (ii) if order of the groups is to be taken in to account $\dfrac{(m+n+p)!}{m!n!p!} \times 3!$

4. The number of ways in which 3n different things can be divided into three equal groups is

   Case (i) if order of the groups is not to be taken in to account $\dfrac{(3n)!}{3!(n!)^3}$

   Case (ii) if order of the groups is to be taken in to account $\dfrac{(3n)!}{(n!)^3}$

   In general, the number of ways of dividing ($n_1 + n_2 + ... + n_k$) things into k different groups containing

   $n_1, n_2, ..., n_k$ thing respectively $= \dfrac{(n_1 + n_2 + ... + n_k)!}{n_1!n_2!...n_k!}$

   Similarly, the number of ways of dividing ‘mn’ things in to ‘m’ different groups containing ‘n’ things respectively $= \dfrac{(mn)!}{m!(n!)^m}$ .

5. Number of ways to distribute n identical objects in r distinct groups $= {}^{n+r-1}C_{r-1}$

6. Number of ways to distribute n identical objects in r distinct groups such that no group is vacant $= {}^{n-1}C_{r-1}$

7. Number of ways to distribute n distinct objects in r distinct groups $= r^n$.

## The Pigeon-Hole principle (PH.P.)

If more than n objects are distributed into n groups, at least one group must receive more than one object.

The PH.P. states that, if n pigeons are placed in m pigeon-holes, then at least one pigeon hole will contain more than $\left[\frac{n-1}{m}\right]$ pigeons, where $[x]$ denotes the greatest integer $\leq x$.

## Principle of Inclusion & Exclusion

Consider a set of N objects and r properties such that each object may or may not have each one of them.

Let the properties be $a_1, a_2, \ldots, a_r$.

Let $N(a_i)$ be the number of objects that have property $a_i$ and Let $N(\overline{a_i})$be the number of objects that do not have property $a_i$

Let $N(a_i a_j)$ be the number of objects that have both property $a_i$ and $a_j$ and Let $N(\overline{a_i}\,\overline{a_j})$be the number of objects that have neither property $a_i$ or $a_j$;

Let $N(a_i\,\overline{a_j})$be the number of objects that have property $a_i$ but not $a_j$.

The formula of the principle of inclusion and exclusion says that

$$N\left(\overline{a_1}\,\overline{a_2}\ldots\overline{a_r}\right) = N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_k) + \ldots + (-1)^r N(a_1 a_2 \ldots a_r)$$

Let $S_0 = N,\ S_1 = \sum N(a_i),\ S_2 = \sum N(a_i a_j), \ldots, S_r = N(a_1 a_2 \ldots a_r)$ and let $e_m$ be the number of objects having exactly m properties then

$$e_m = S_m - {}^{m+1}c_1 S_{m+1} + {}^{m+2}c_2 S_{m+2} - {}^{m+3}c_3 S_{m+3} + \ldots + (=-1)^{r-m}\,{}^{r}c_{r-m} S_r$$

## Applications of PIE

### Derangements

If n things are arranged in a row, the number of ways they can be deranged so that r things occupy wrong places while (n – r) things occupy their original places, is

$$D_r = {}^nc_r\left\{r! - {}^rc_1(r-1)! + {}^rc_2(r-2)! - {}^rc_3(r-3)! + \ldots + (-1)^r\right\} = {}^nP_r\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r\frac{1}{r!}\right]$$

If n things are arranged in a row, the number of ways they can be deranged so that none of them occupies its original place, is $D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}\right]$

**Number of ways to distribute n distinct objects in r groups such that each group gets at least one object**

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \dots \quad (r \le n)$$

## Some more important results on permutations and combinations

### 1. Sum of the numbers

Sum of the numbers formed by taking all the given n digits is

$(\text{Sum of all the digits}) \times (1 + 10 + 10^2 + \dots + 10^{n-1}) \times (n-1)!$, where all digits are distinct.

### 2. Number of Divisors

The number of distinct positive integral divisors of $p_1^{k_1} \times p_2^{k_2} \times \dots \times p_r^{k_r}$ where $p_1, p_2, \dots, p_r$ are primes in ascending order, is $(k_1 + 1)(k_2 + 1)\dots(k_r + 1)$.

### 3. Sum of divisors

The sum of distinct positive integral divisors of $p_1^{k_1} \times p_2^{k_2} \times \dots \times p_r^{k_r}$ where $p_1, p_2, \dots, p_r$ are primes in ascending order, is $\frac{p_1^{k_1+1} - 1}{p_1 - 1} \times \frac{p_2^{k_2+1} - 1}{p_2 - 1} \times \dots \times \frac{p_r^{k_r+1} - 1}{p_r - 1}$

### 4. Number of Regions formed by n straight lines

Consider n lines drawn on a plane such that no two lines are parallel and no three lines are concurrent. One line divides the plane in to 2 regions.
A second line divides the two regions. That it passes through in to 4 regions, thereby increasing the number of regions by 2. Hence number of regions

$$\ell(n) = 2 + 2 + 3 + 4 + \dots + n = 1 + \frac{n(n+1)}{2}$$

### 5. Positive integral solutions

The number of integral solutions of the equation $x_1 + x_2 + \dots + x_k = n$, where each $x_i$ is a positive integer is ${}^{(n-1)}C_{k-1}$

The number of integral solutions of the equation $x_1 + x_2 + \dots + x_k \le n$ (where each $x_i$ is a positive integer) is equivalent to number of integral solutions of the equation

$$x_1 + x_2 + \dots + x_k + d = n$$

(where each $x_i$ is a positive integer and d is a non-negative integer); i.e. ${}^{n}C_k$

## 6. Non-negative integral solutions

The number of solutions of the equation $x_1 + x_2 + ...+ x_k = n$, with each $x_i$ a non-negative integer is ${}^{(n+k-1)}C_{k-1}$

The number of integral solutions of the equation $x_1 + x_2 + ...+ x_k \le n$ (where each $x_i$ is a non-negative integer) is equivalent to number of integral solutions of the equation

$$x_1 + x_2 + ...+ x_k + d = n$$

(where each $x_i$ is a non-negative integer and d is also a non-negative integer) i.e. ${}^{n+k}C_k$

## 7. Distribution in groups with restrictions

The number of ways in which n identical things can be distributed into r different groups if the groups contain a minimum of p things and a maximum of m things, is coefficient of $x^n$ in $(x^p + x^{p+1} + x^{p+2} + ...+ x^m)^r$.

## 8. Number of rectangles and squares

(i) The number of rectangles of any size that can be formed by m horizontal & n vertical lines, is ${}^{(m+1)}C_2 \times {}^{(n+1)}C_2$

(ii) The number of squares of any size that can be formed by m horizontal & n vertical lines, is $\sum_{r=1}^{m}(m-r+1)(n-r+1)$ , if $m<n$

$$\sum_{r=1}^{m} m^2 \text{ , if } m=n$$

## 9. Exponent of prime p in n!

The exponent of a prime p in n!, denoted by $E_p(n!)$, is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + .......... + \left[\frac{n}{p^s}\right]$$

where [ ] denotes the greatest integer function, and s is the largest number such that $p^s \le n < p^{s+1}$

## 10. Number of functions

If domain contains n elements and codomain contains r elements, then

Number of functions which can be formed = $r^n$

Number of one – one functions = ${}^{r}C_n \times n!\ (r \ge n)$

Number of onto functions = $r^n - {}^{r}C_1(r-1)^n + {}^{r}C_2(r-2)^n - {}^{r}C_3(r-3)^n + ...\ \ (r \le n)$