TRIGONOMETRIC RATIOS AND IDENTITIES

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1. ANGLES & UNITS OF MEASUREMENT

"*A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*" ~ Euclid, The Elements.

An **angle** is a measure of rotation.

Angle measure can be positive or negative, depending on the direction of rotation.

The angle measure is the amount of rotation between the two rays forming the angle. Rotation is measured from the **initial side** to the **terminal side** of the angle.

The two most common measures of an angle are DEGREE and RADIAN.

An angle is measured as positive when taken from initial to terminal ray i.e. anticlockwise and negative when taken from terminal to initial ray i.e clockwise.

RADIAN OR CIRCULAR SYSTEM

In circular system of measurement of an angle one radian (denoted as 1^c) is defined as the angle subtended by an arc of length equal to radius of a circle at the center of the circle as shown in the adjoining figure.

Now as the semicircumference is of length πr and it subtends a linear angle at centre hence in radian measure linear angle is π radian & hence one right angle equal to $\frac{\pi}{2}$ radian.

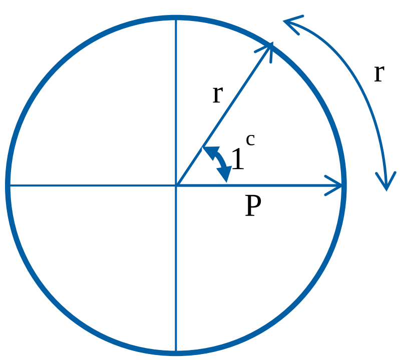

Figure 1.1

DEGREE OR RECTANGULAR SYSTEM

In the rectangular system of measurement of an angle, one right angle is of 90 degree and is denoted as 90^0. Also a linear angle is 180^0 & one complete angle is 360^0.

$$\textbf{number of radian} = \frac{\pi}{180} \times \textbf{number of degree}$$

From the perspective of trigonometry it will be explained later that an angle doesnot have a unique measure and $\theta, -2\pi+\theta, 2\pi+\theta$ etc. represent the same angles.

An Interesting Fact…

The system of measuring angles in degrees, such that 360° is one revolution, originated in ancient Babylonia. It is often assumed that the number 360 was used because the Babylonians supposedly thought that there were 360 days in a year (a year, of course, is one full revolution of the Earth around the Sun). However, there is another, perhaps more likely, explanation which says that in ancient times a person could travel 12 *Babylonian miles* in one day i.e. one full rotation of the Earth about its axis. The Babylonian mile was large enough, approximately 7 of our miles, to be divided into 30 equal parts for convenience, thus giving 12×30 = 360 equal parts in a full rotation.

2. PYTHAGOROS THEOREM

In a right angled triangle if the two mutually perpendicular sides are 'p' & 'b' & the hypotenuse is 'h', then according to the well known Pythagoros Theorem

$$\mathbf{p^2 + b^2 = h^2}$$

In the right triangle $\triangle$ABC in the adjoining figure, if we draw a line Segment from the vertex C to the point D on the hypotenuse such that CD is **perpendicular** to AB (that is, CD forms a right angle with AB), then this divides $\triangle$ABC into two smaller Triangles $\triangle$CBD and $\triangle$ACD, which are both similar to $\triangle$ABC.

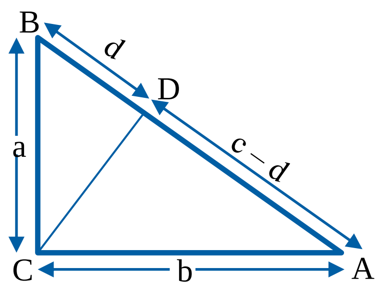

Figure 1.2

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional.

Thus, since $\triangle$ABC is similar to $\triangle$CBD, by proportionality of corresponding sides we see that AB is to CB (hypotenuses) as BC is to BD (vertical legs) $\Rightarrow \frac{c}{a} = \frac{a}{d} \Rightarrow cd = a^2$

Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d}=\frac{c}{b}\Rightarrow b^2=c^2-cd=c^2-a^2$$

Which gives $a^2+b^2=c^2$.

3. DEFINITION OF TRIGONOMETRIC RATIOS

Consider the right triangle in adjoining figure where θ denotes one of the two non-right angles.

The side of the triangle opposite the right angle (PM) is the **hypotenuse**.

The remaining two sides of the triangle can be uniquely identified by relating them to the angle θ as follows.

The **adjacent side** refers to the side (OP) that, along with the hypotenuse, forms the angle θ. The third side (OM) of the triangle let be referred as the **opposite side.**

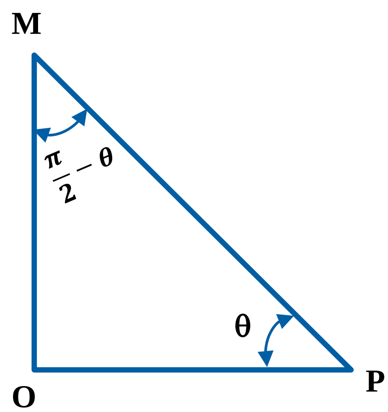

Figure 1.3

The values of the six trigonometric functions (Ratios) for the angle θ are given below.

THE TRIGONOMETRIC RATIOS FOR THE ∠OPM

$\frac{\text{opposite}}{\text{adjacent}}$ i.e. $\frac{OM}{OP}$ is called the **Tangent** of the angle OPM (tan θ) &

$\frac{\text{adjacent}}{\text{opposite}}$ i.e. $\frac{OP}{OM}$ is called the **Cotangent** of the angle OPM (cot θ)

$\frac{\text{opposite}}{\text{hypotenuse}}$ i.e. $\frac{OM}{PM}$ is called the **Sine** of the angle OPM (sin θ) &

$\frac{\text{adjacent}}{\text{hypotenuse}}$ i.e. $\frac{OP}{MP}$ is called the **Cosine** of the angle OPM (cos θ)

$\frac{\text{hypotenuse}}{\text{adjacent}}$ i.e. $\frac{PM}{OP}$ is called the **Secant** of the angle OPM (sec θ) &

$\frac{\text{hypotenuse}}{\text{opposite}}$ i.e. $\frac{PM}{OM}$ is called the **Cosecant** of the angle OPM (cosec θ)

4. BASIC TRIGONOMETRIC IDENTITIES

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ & $\sec\theta = \frac{1}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta - \tan^2\theta = 1$
- $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

TRIGONOMETRIC RATIOS FOR COMPLEMENTARY ANGLES

Here it can further be understood that as shown in the above figure $\angle OPM$ & $\angle OMP$ are complementary angles hence adjacent side for one angle will be opposite side for other angle.

Now this implies that Sine, Tangent & Secant of $\angle OPM$ will be equal to Cosine, Cotangent & Cosecant of $\angle OMP$ respectively.

This is an interesting fact that this is why names of three T – Ratios are Co-Sine, Co-Tangent & Co-Secant as these are Complementary or Co-functions of the other three T – Ratios.

Hence If A + B = 90^0, then sin A = cos B, tan A = cot B & sec A = cosec B.

5. Interrelations in T – Ratios for acute angles

The following relations may suitably be used to convert one T-Ratio in terms of the other for $0 < \theta < \frac{\pi}{2}$.

T-Ratios	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{1+\tan^2\theta}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{1+\cot^2\theta}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

6. VALUES OF T-RATIOS FOR SOME IMPORTANT ANGLES

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{10}$	$\frac{\pi}{6}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sqrt{\frac{\sqrt{5}+5}{8}}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	1
$\cos\theta$	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\sqrt{\frac{\sqrt{5}+5}{8}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	0
$\tan\theta$	0	$2-\sqrt{3}$	$\sqrt{\frac{5-2\sqrt{5}}{5}}$	$\sqrt{3}$	$\sqrt{5-2\sqrt{5}}$	1	$\frac{1}{\sqrt{3}}$	$\sqrt{5+2\sqrt{5}}$	$2+\sqrt{3}$	*

7. CHARACTERISTICS OF T – RATIOS FOR ANGLES OF ANY MAGNITUDE

Angles of a right triangle cannot be more than 90^o or negative, then how can we understand the existence of T - Ratios for angles of any magnitude???

Consider the adjoining figure. A point P moves on the circle about O from A_1 to A_1 via B_1, A_2 & B_2. As shown in the figure, in any of its position, OP as hypotenuse it completes a triangle with **abscissa** and **ordinate** of P as **adjacent** and **opposite**. To define T – Ratios for angles in $(0, 2\pi)$ we consider various such triangles and angle in each case will be taken as angular displacement traversed by OP.

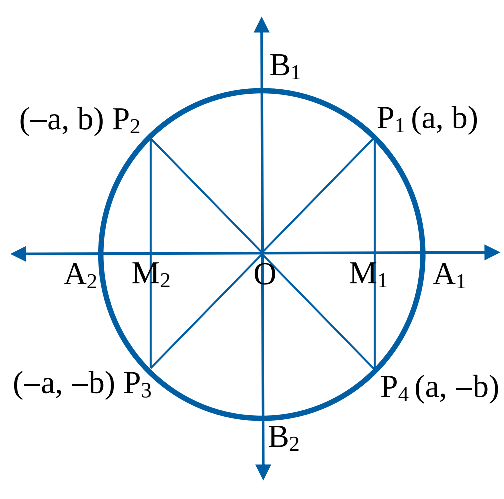

Figure 1.4

8. DOMAIN OF T – RATIOS

As we understood in the definition of a function, Domain of a function is that interval of the values of the independent variable for which the function is defined.

Now It can be observed that for 0, π, 2π…etc. cotangent and cosecant can't be defined and for $-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$…etc. tangent and secant can't be defined, as at these angles denominator in the definition of respective T – Ratios becomes zero.

Hence domains of various T – Ratios are as follows…

T - Ratio	Domain
$\sin\theta$ & $\cos\theta$	All Real Angles
$\tan\theta$ & $\sec\theta$	All Real Angles except $\left\{\frac{2n-1}{2}\pi, n\in Z\right\}$
$\operatorname{cosec}\theta$ & $\cot\theta$	All Real Angles except $\{n\pi, n\in Z\}$

9. SIGN OF T – RATIOS

First let us consider the ΔOM_1P_1. Here $\angle M_1OP_1=\theta$ and $\sin\theta=\frac{b}{r},\cos\theta=\frac{a}{r}$ & $\tan\theta=\frac{b}{a}$

Clearly all the T – Ratios will be positive.

Similarly in the ΔM_2OP_2, $\angle M_2OP_2=\alpha=\pi-\theta$ and $\sin\alpha=\frac{b}{r},\cos\alpha=-\frac{a}{r}$ & $\tan\alpha=-\frac{b}{a}$.

Clearly except sine & cosecant all the T – Ratios will be negative. This explains the existence of T – Ratios even for **angles beyond a right angled triangle** and why T – Ratios are of different **Sign** for angles of different magnitude.

Hence Sign of T – Ratios for various intervals will be as follows…

Quadrant $\rightarrow$ $\downarrow$ T – Ratios	$0<\theta<\frac{\pi}{2}$	$\frac{\pi}{2}<\theta<\pi$	$0<\theta<\frac{3\pi}{2}$	$\frac{3\pi}{2}<\theta<2\pi$
$\sin\theta$	POSITIVE	POSITIVE	NEGATIVE	NEGATIVE
$\cos\theta$	POSITIVE	NEGATIVE	NEGATIVE	POSITIVE
$\tan\theta$	POSITIVE	NEGATIVE	POSITIVE	NEGATIVE

10. RANGE OF T – RATIOS

While the revolving line OP (of length r) is in the first quadrant and is revolving from OA_1 to OB_1, abscissa of P_1 will vary from 'a' to zero and ordinate from zero to 'b'. Therefore values of sine & cosine will vary between 0 & 1.

While the revolving line OP is in the second quadrant and is revolving from OB_1 to OA_2, abscissa of P_2 will vary from zero to '–a' and ordinate from 'b' to zero.

Hence values of sine will vary between 0 & 1 and those of cosine will vary between –1 & 0. This clarifies how we arrived to the **Range** of values of the T – Ratios.

Hence Range of various T – Ratios is as follows…

Angles $\rightarrow$ $\downarrow$ T – Ratios	$0<\theta<\frac{\pi}{2}$	$\frac{\pi}{2}<\theta<\pi$	$\pi<\theta<\frac{3\pi}{2}$	$\frac{3\pi}{2}<\theta<2\pi$
$\sin\theta$	(0, 1)	(0, 1)	(–1, 0)	(–1, 0)
$\cos\theta$	(0, 1)	(–1, 0)	(–1, 0)	(0, 1)
$\tan\theta$	$(0, \infty)$	$(-\infty, 0)$	$(0, \infty)$	$(-\infty, 0)$
$\cot\theta$	$(0, \infty)$	$(-\infty, 0)$	$(0, \infty)$	$(-\infty, 0)$
$\sec\theta$	$(1, \infty)$	$(-\infty, -1)$	$(-\infty, -1)$	$(1, \infty)$
$\operatorname{cosec}\theta$	$(1, \infty)$	$(1, \infty)$	$(-\infty, -1)$	$(-\infty, -1)$

11. PERIODICITY OF T – RATIOS

Also it may be further understood that after completing one rotation the revolving line will start traversing the same points on the circumference as earlier though this time angular displacements will be $2\pi+\theta, 4\pi+\theta, 6\pi+\theta$...etc. Hence values of various T – Ratios will start repeating. This indicates to **Periodic Property** of T – Ratios.

Hence Periodic Property of various T – Ratios is as follows…

- $\sin(\theta \pm 2n\pi) = \sin\theta$
- $\cos(\theta \pm 2n\pi) = \cos\theta$
- $\tan(\theta \pm n\pi) = \tan\theta$

12. T – RATIOS FOR ALLIED ANGLES

The angles $\frac{n\pi}{2} \pm \theta$ where n is any integer, are known as allied or related angles. The trigonometric functions of these angles can be expressed as trigonometric functions of θ.
As we can observe from the figure 1.4, triangles OM_2P_2 , OM_2P_3 & OM_1P_4 can be drawn similar to ΔOM_1P_1 having same sides but with different signs for coordinates of P_2, P_3 & P_4 as well as $\angle M_1OP_1 = \theta \Rightarrow \angle M_2OP_2 = \pi - \theta, \angle M_2OP_3 = \pi + \theta$ & $\angle M_1OP_4 = 2\pi - \theta$.

Similarly $\angle M_1OP_1 = \theta \Rightarrow \angle M_1P_1O = \frac{\pi}{2} - \theta$.

It may be observe here that…

$\sin \angle M_1OP_1 = \sin \angle M_2OP_2, \cos \angle M_1OP_1 = -\cos \angle M_2OP_2$ & $\tan \angle M_1OP_1 = -\tan \angle M_2OP_2$.

Similarly we can define the values of various T – Ratios for $\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta, \pi + \theta$ & $2\pi - \theta$ using $\angle M_1OP_1, \angle M_1P_1O, \angle OM_2P_3$ & $\angle OM_1P_4$ respectively in terms of θ.

This concludes about relation in values of T – Ratios for an angle θ and the values of T – Ratios for **Allied Angles**.

The following table gives the values of the T – Ratios for Allied Angles in terms of T – Ratios of the angle θ…

$\text{T} - \text{Ratios} \rightarrow$ $\downarrow$ **Allied Angles**	$\sin\theta$	$\cos\theta$	$\tan\theta$
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$
$90-\theta$	$\cos\theta$	$\cos\theta$	$\cot\theta$
$90+\theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$
$180-\theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$
$180+\theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$
$270-\theta$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$
$270+\theta$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$
$360-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$

13. T – Ratios for angles of any magnitude

Here note that as all the T – Ratios show periodic property hence the above relations may be generalized to be applicable to angles of any magnitude as follows…

$$\sin(2n\pi-\theta)=-\sin\theta \ \& \sin((2n-1)\pi\pm\theta)=\mp\sin\theta$$

$$\sin\left((4n+1)\frac{\pi}{2}\pm\theta\right)=\cos\theta \ \& \ \sin\left((4n-1)\frac{\pi}{2}\pm\theta\right)=-\cos\theta$$

$$\cos(2n\pi-\theta)=\cos\theta \ \& \cos((2n-1)\pi\pm\theta)=-\cos\theta$$

$$\cos\left((4n\pm1)\frac{\pi}{2}\mp\theta\right)=\sin\theta \ \& \cos\left((4n\pm1)\frac{\pi}{2}\pm\theta\right)=-\sin\theta$$

$$\tan(n\pi-\theta)=-\tan\theta \ \& \ \tan(n\pi+\theta)=\tan\theta$$

$$\tan\left((2n-1)\frac{\pi}{2}-\theta\right)=\cot\theta \ \& \ \tan\left((2n-1)\frac{\pi}{2}+\theta\right)=-\cot\theta$$

14. TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF ANGLES

- $\sin(\alpha+\beta)=\sin\alpha.\cos\beta+\cos\alpha.\sin\beta$
- $\sin(\alpha-\beta)=\sin\alpha.\cos\beta-\cos\alpha.\sin\beta$
- $\cos(\alpha+\beta)=\cos\alpha.\cos\beta-\sin\alpha.\sin\beta$
- $\cos(\alpha-\beta)=\cos\alpha.\cos\beta+\sin\alpha.\sin\beta$
- $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\ \tan\beta}$
- $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\ \tan\beta}$
- $\cot(\alpha+\beta)=\dfrac{\cot\alpha\ \cot\beta-1}{\cot\beta+\cot\alpha}$
- $\cot(\alpha-\beta)=\dfrac{\cot\alpha\ \cot\beta+1}{\cot\beta-\cot\alpha}$
- $\dfrac{\cos A+\sin A}{\cos A-\sin A}=\tan\left(\dfrac{\pi}{4}+A\right)$
- $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\tan\left(\dfrac{\pi}{4}-A\right)$

- $\sin(A_1+A_2+A_3+---+A_n)=S_1-S_3+S_5-S_7+---$
- $\cos(A_1+A_2+A_3+---+A_n)=S_0-S_2+S_4-S_6+---$

S_k = sum of all the products of sines of k of the n angles & cosines of the remaining (n - k) of the angles

- $\tan(A_1+A_2+A_3+---+A_n)=\dfrac{S_1-S_3+S_5-S_7+---}{S_0-S_2+S_4-S_6+---}$

In case of tangents, S_k = sum of all the products of tangents of k of the n angles

15. The "trigonometric form" of *Ptolemy's Theorem*

"***Ptolemy's Theorem states that a quadrilateral can be inscribed in a circle if and only if the sum of the products of its opposite sides equals the product of its diagonals***."

> Let A, B, C, and D be positive angles such that $A + B + C + D = 180^0$, then
>
> $$\sin A \sin B + \sin C \sin D = \sin (A + C) \sin (B + C)$$

Notice that the right side has no D term. So instead, we will expand the left side, since we can eliminate the D term on that side by using

$$D = 180^0 - (A + B + C) \ \& \ \sin (180^0 - \theta) = \sin \theta.$$

So since $\sin D = \sin (A + B + C)$, we get

LHS $= \sin A \sin B + \sin C \sin D = \sin A \sin B + \sin C \sin (A + B + C)$

$= \sin A \sin B + \sin C (\sin A \cos B \cos C + \cos A \sin B \cos C$

$\qquad + \cos A \cos B \sin C - \sin A \sin B \sin C)$

$= \sin A \sin B + \sin C \sin A \cos B \cos C + \sin C \cos A \sin B \cos C$

$\qquad + \sin C \cos A \cos B \sin C - \sin C \sin A \sin B \sin C$.

$= \sin A \sin B - \sin C \sin A \sin B \sin C + \sin C \cos A \sin B \cos C$

$\qquad + \cos B \sin C (\sin A \cos C + \cos A \sin C)$

$= \sin A \sin B (1 - \sin^2 C) + \sin C \cos A \sin B \cos C + \cos B \sin C \sin (A + C)$

$= \sin A \sin B \cos^2 C + \sin C \cos A \sin B \cos C + \cos B \sin C \sin (A + C)$.

$= \sin B \cos C (\sin A \cos C + \cos A \sin C) + \cos B \sin C \sin (A + C)$

$= \sin B \cos C \sin (A+C) + \cos B \sin C \sin (A+C)$

$= \sin (A+C) (\sin B \cos C + \cos B \sin C)$

Hence $\sin A \sin B + \sin C \sin D = \sin (A + C) \sin (B + C)$.

16. TRANSFORMATION OF SUM TO PRODUCT AND PRODUCT TO SUM

- $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
- $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$
- $\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta).\sin(\alpha-\beta) = \cos^2\beta - \cos^2\alpha$
- $\cos^2\alpha - \sin^2\beta = \cos(\alpha+\beta).\cos(\alpha-\beta) = \cos^2\beta - \sin^2\alpha$
- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

17. MULTIPLE ANGLE FORMULAE

- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$

- $\sin nA = \sum_{k=1}^{m+1} (-1)^{k+1}\, {}^{n}C_{2k-1} \sin^{2k-1} A.\cos^{n-2k+1} A$, where $m = \begin{cases} \dfrac{n-1}{2} & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even} \end{cases}$
- $\cos nA = \sum_{k=0}^{m} (-1)^{k}\, {}^{n}C_{2k} \sin^{2k} A.\cos^{n-2k} A$, where $m = \begin{cases} \dfrac{n-1}{2} & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even} \end{cases}$
- $\tan nA = \dfrac{\sum_{k=1}^{m+1} (-1)^{k+1}\, {}^{n}C_{2k-1} \tan^{2k-1} A}{\sum_{k=0}^{m} (-1)^{k}\, {}^{n}C_{2k} \tan^{2k} A}$, where $m = \begin{cases} \dfrac{n}{2} & \text{if n is even} \\ \dfrac{n-1}{2} & \text{if n is odd} \end{cases}$

Important results…

- $4\sin\left(\dfrac{\pi}{3}-\theta\right)\sin\theta\sin\left(\dfrac{\pi}{3}+\theta\right) = \sin 3\theta$
- $4\cos\left(\dfrac{\pi}{3}-\theta\right)\cos\theta\cos\left(\dfrac{\pi}{3}+\theta\right) = \cos 3\theta$
- $\tan\left(\dfrac{\pi}{3}-\theta\right)\tan\theta\tan\left(\dfrac{\pi}{3}+\theta\right) = \tan 3\theta$
- $\cot\theta - \tan\theta = 2\cot 2\theta$
- $\cot\theta + \tan\theta = 2\operatorname{cosec} 2\theta$

18. Euler's Identity

$$\cos\theta\cdot\cos 2\theta\cdot\cos 4\theta\ ...\ \cos 2^n\theta=\frac{\sin 2^{n+1}\theta}{2^{n+1}\sin\theta}$$

The above formula leads to the following formula

$$\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ ...\ \text{upto }\infty\text{ terms}=\frac{\sin x}{x}$$

It was discovered by ***Euler*** *and represents one of the very few examples of an infinite product in early days of elementary mathematics.*

Proof of this formula is as follows

$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}\Rightarrow\sin x=2^2\sin\frac{x}{4}\cos\frac{x}{4}\cos\frac{x}{2}$, which further implies

$\sin x=2^3\sin\frac{x}{8}\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2}$.

Continuing in the same manner for n times gives

$$\sin x=2^n\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ ...\ \cos\frac{x}{2^n}\sin\frac{x}{2^n}$$

or $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ ...\ \cos\frac{x}{2^n}=\dfrac{\sin x}{2^n\sin\frac{x}{2^n}}$

Hence $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ ...\ \text{upto }\infty\text{ terms}=\dfrac{\sin x}{x}\lim\limits_{n\to\infty}\dfrac{\frac{x}{2^n}}{\sin\frac{x}{2^n}}$

Setting $\frac{1}{2^n}=m$ reduces the limit to $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ ...\ \text{upto }\infty\text{ terms}=\dfrac{\sin x}{x}\lim\limits_{m\to 0}\dfrac{mx}{\sin mx}$

$\Rightarrow\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ ...\ \text{upto }\infty\text{ terms}=\dfrac{\sin x}{x}$

Now if we set $x=\frac{\pi}{2}$, then $\cos\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\cdot\cos\frac{\pi}{16}\ ...\ \text{upto }\infty\text{ terms}=\dfrac{2}{\pi}$

or $\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{2+\sqrt{2}}}{2}\cdot\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\ ..\ .\text{upto }\infty\text{ terms}=\dfrac{2}{\pi}$.

This formula was discovered by ***Viete*** *in 1593, in establishing it he used a geometric argument based on the ratio of areas of regular polygons of n and 2n sides inscribed in the same circle.*
Viete's formula marks a milestone in the history of mathematics. it was the first time an infinite process was explicitly written as a succession of algebraic operations.
Viete's formula is remarkable because it allows us to find the number π by repeatedly using four of the basic operations of arithmetic i.e. addition, multiplication, division, and square root extraction, all applied to the number 2.

19. SUB MULTIPLE ANGLE FORMULAE

If $\sqrt{\frac{1+\cos 60^0}{2}} = \cos 30^0$, *then why* $\sqrt{\frac{1+\cos 240^0}{2}}$ *is not equal to* $\cos 120^0$*???*

The entire concept of submultiple angle formulae is based on the characteristics of ***MODULUS FUNCTION***, as almost every formula for submultiple angles is obtained by taking square root and in general characteristics of square root function $\sqrt{x^2} = |x|$.

VALUES OF SINE, COSINE & TANGENT OF A/2 IN TERMS OF COSINE OF A

As we know $\cos 2\theta = 2\cos^2\theta - 1$ hence $\sqrt{\cos^2\theta} = \sqrt{\frac{1+\cos 2\theta}{2}}$

$\Rightarrow \sqrt{\frac{1+\cos 2\theta}{2}} = |\cos\theta|$.

Now if $\cos\theta > 0$, then $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$ & if $\cos\theta < 0$, then $\cos\theta = -\sqrt{\frac{1+\cos 2\theta}{2}}$.

Also $\cos\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ & $\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, hence

$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ &

$\cos\theta = -\sqrt{\frac{1+\cos 2\theta}{2}}$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

Conclusively we arrive to the following formula…

$$\cos\frac{A}{2} = \begin{cases} \sqrt{\frac{1+\cos A}{2}} & \text{if } -\frac{\pi}{2} \le \frac{A}{2} \le \frac{\pi}{2} \\ -\sqrt{\frac{1+\cos A}{2}} & \text{if } \frac{\pi}{2} < \frac{A}{2} \le \frac{3\pi}{2} \end{cases}$$

Here not that the above formula may be generalized by adding 2nπ to each of the end points of the above intervals.

Similarly $\cos 2\theta = 1 - 2\sin^2\theta$ Gives $\sqrt{\frac{1-\cos 2\theta}{2}} = |\sin\theta|$.

Now if $\sin > 0$, then $\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$ & if $\sin\theta < 0$, then $\sin\theta = -\sqrt{\frac{1-\cos 2\theta}{2}}$.

Also $\sin\theta > 0$ for $0 < \theta < \pi$ & $\cos\theta < 0$ for $\pi < \theta < 2\pi$, hence

$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$ for $0 \le \theta \le \pi$ &

$\sin\theta = -\sqrt{\frac{1-\cos 2\theta}{2}}$ for $\pi < \theta < 2\pi$

Conclusively we arrive to the following formula…

$$\sin\frac{A}{2} = \begin{cases} \sqrt{\frac{1-\cos A}{2}} & \text{if } 0 \le \frac{A}{2} \le \pi \\ -\sqrt{\frac{1-\cos A}{2}} & \text{if } \pi < \frac{A}{2} \le 2\pi \end{cases}$$

Here not that the above formula may be generalized by adding 2nπ to each of the end points of the above intervals.

Now $\tan x > 0$ for $n\pi < x < (2n+1)\frac{\pi}{2}$ & $\tan x < 0$ for $(2n+1)\frac{\pi}{2} < x < (n+1)\pi$

and $\tan x = \frac{\sin x}{\cos x}$ gives…

$$\tan\frac{A}{2} = \begin{cases} \sqrt{\frac{1-\cos A}{1+\cos A}} & \text{if } 0 \le \frac{A}{2} < \frac{\pi}{2} \\ -\sqrt{\frac{1-\cos A}{1+\cos A}} & \text{if } \frac{\pi}{2} < \frac{A}{2} \le \pi \end{cases}$$

Here not that the above formula may be generalized by adding nπ to each of the end points of the above intervals.

VALUES OF SINE, COSINE & TANGENT OF A/2 IN TERMS OF SINE OF A

Let us consider $(\sin x - \cos x)^2 = 1 - \sin 2x$. From this result we can have

$\sqrt{1-\sin 2x} = |\sin x - \cos x|$. Now

$$\sin x - \cos x < 0 \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

$$\sin x - \cos x > 0 \Rightarrow \frac{\pi}{4} < x < \frac{5\pi}{4}$$

Hence we get $\sin\frac{A}{2} - \cos\frac{A}{2} = \begin{cases} -\sqrt{1-\sin A} & \text{if } -\frac{3\pi}{4} \le \frac{A}{2} < \frac{\pi}{4} \\ \sqrt{1-\sin A} & \text{if } \frac{\pi}{4} \le \frac{A}{2} \le \frac{5\pi}{4} \end{cases}$

Similarly consider $(\sin x + \cos x)^2 = 1 + \sin 2x$. From this result we can have

$\sqrt{1+\sin 2x} = |\sin x + \cos x|$.

Now

$$\sin x + \cos x > 0 \Rightarrow -\frac{\pi}{4} < x < \frac{3\pi}{4}$$

$$\sin x + \cos x < 0 \Rightarrow \frac{3\pi}{4} < x < \frac{7\pi}{4}$$

Hence we get $\sin\frac{A}{2} + \cos\frac{A}{2} = \begin{cases} \sqrt{1+\sin A} & \text{if } -\frac{\pi}{4} \le \frac{A}{2} < \frac{3\pi}{4} \\ -\sqrt{1+\sin A} & \text{if } \frac{3\pi}{4} \le \frac{A}{2} \le \frac{7\pi}{4} \end{cases}$.

Conclusively we get…

$$\sin\frac{A}{2} = \begin{cases} \frac{1}{2}\left[\sqrt{1+\sin A} - \sqrt{1-\sin A}\right] & \text{if } -\frac{\pi}{4} \le \frac{A}{2} < \frac{\pi}{4} \\ \frac{1}{2}\left[\sqrt{1+\sin A} + \sqrt{1-\sin A}\right] & \text{if } \frac{\pi}{4} \le \frac{A}{2} < \frac{3\pi}{4} \\ -\frac{1}{2}\left[\sqrt{1+\sin A} - \sqrt{1-\sin A}\right] & \text{if } \frac{3\pi}{4} \le \frac{A}{2} \le \frac{5\pi}{4} \\ -\frac{1}{2}\left[\sqrt{1+\sin A} + \sqrt{1-\sin A}\right] & \text{if } \frac{5\pi}{4} \le \frac{A}{2} \le \frac{7\pi}{4} \end{cases}$$

$$\cos\frac{A}{2}=\begin{cases}\frac{1}{2}\left[\sqrt{1+\sin A}+\sqrt{1-\sin A}\right] & \text{if} \quad -\frac{\pi}{4}\le\frac{A}{2}<\frac{\pi}{4}\\ \frac{1}{2}\left[\sqrt{1+\sin A}-\sqrt{1-\sin A}\right] & \text{if} \quad \frac{\pi}{4}\le\frac{A}{2}<\frac{3\pi}{4}\\ -\frac{1}{2}\left[\sqrt{1+\sin A}+\sqrt{1-\sin A}\right] & \text{if} \quad \frac{3\pi}{4}\le\frac{A}{2}\le\frac{5\pi}{4}\\ -\frac{1}{2}\left[\sqrt{1+\sin A}-\sqrt{1-\sin A}\right] & \text{if} \quad \frac{5\pi}{4}\le\frac{A}{2}\le\frac{7\pi}{4}\end{cases}$$

$$\tan\frac{A}{2}=\begin{cases}\frac{1}{2}\left[\frac{\sqrt{1+\sin A}-\sqrt{1-\sin A}}{\sqrt{1+\sin A}+\sqrt{1-\sin A}}\right] & \text{if} \quad -\frac{\pi}{4}\le\frac{A}{2}<\frac{\pi}{4} \text{ or } \frac{3\pi}{4}\le\frac{A}{2}\le\frac{5\pi}{4}\\ \frac{1}{2}\left[\frac{\sqrt{1+\sin A}+\sqrt{1-\sin A}}{\sqrt{1+\sin A}-\sqrt{1-\sin A}}\right] & \text{if} \quad \frac{\pi}{4}\le\frac{A}{2}<\frac{3\pi}{4} \text{ or } \frac{5\pi}{4}\le\frac{A}{2}\le\frac{7\pi}{4}\end{cases}$$

Here not that the above formula may be generalized by adding 2nπ to each of the end points of the above intervals.

VALUE OF TANGENT OF A/2 IN TERMS OF TANGENT OF A

Consider $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$, rearranging the terms gives

$(\tan 2x)\tan^2 x+2\tan x-\tan 2x=0$.

Solving the above equation as quadratic equation in tan x gives

$$\tan x=\frac{-1\pm\sqrt{1+\tan^2 2x}}{\tan 2x}$$

Here the ambiguity in the values may be removed by comparing the two values of tan x by 1.

$$\tan\frac{A}{2}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$$

20. CONDITIONAL IDENTITIES

When the angles A, B and C satisfy a given relation, many interesting identities can be established connecting the trigonometric functions of these angles. In proving these identities, we require the properties of complementary and supplementary angles.

If $A+B+C=\pi$, then the following identities hold true…

- $\sin(B+C)=\sin A$, $\cos(B+C)=-\cos A$, $\tan(B+C)=-\tan A$, $\cot(B+C)=-\cot A$
- $\sin\frac{A+B}{2}=\cos\frac{C}{2}$, $\cos\frac{A+B}{2}=\sin\frac{C}{2}$, $\tan\frac{B+C}{2}=\cot\frac{A}{2}$ & $\tan\frac{B}{2}=\cot\frac{C+A}{2}$
- $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$
- $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$
- $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
- $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
- $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$
- $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Here note that these identities will be very useful in case of* A, B & C *being the angles of a Triangle*, *where* $(\mathbf{A+B+C=\pi})$ *is a default condition.

21. GRAPHS OF THE TRIGONOMETRIC RATIOS AS FUNCTIONS OF A VARIABLE

We will understand a geometrical way to create the graph, using the unit circle. This is the circle of radius 1 in the xy-plane consisting of all points (x, y) which satisfy the equation $x^2 + y^2 = 1$. We see in figure 1.6 that any point on the unit circle has coordinates (x, y) = (cos θ, sin θ), where θ is the angle that the line segment from the origin to (x, y) makes with the positive x-axis (by definition of Sine and Cosine). So as the point (x, y) goes around the circle, its y-coordinate is sin θ. We thus get a correspondence between the y-coordinates of points on the unit circle and the values f (θ) = sin θ, as shown by the horizontal lines from the unit circle to the graph of f (θ) = sin θ in Figure 5.1.2 for angles θ = 0, $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ the.

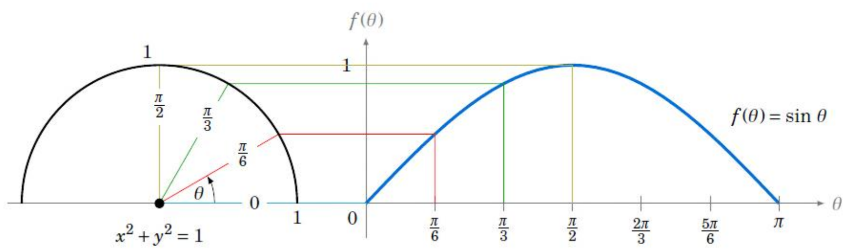

Figure 1.5

We can extend the above picture to include angles from 0 to 2 π radians, as in Figure 1.7. This illustrates what is sometimes called the *unit circle definition of the sine function*.

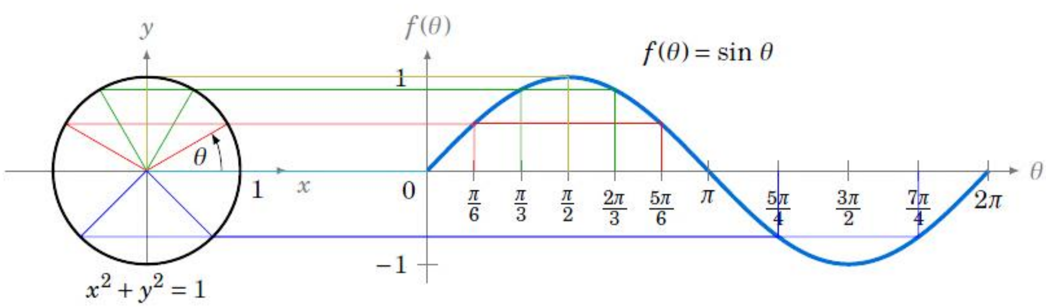

Figure 1.6

Graph of y = sin x

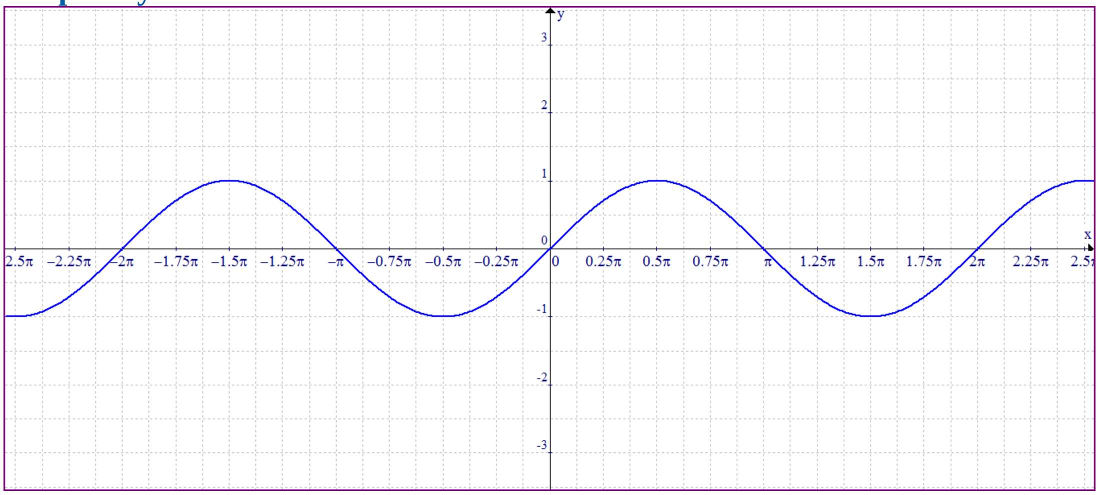

Figure 1.7

Graph of y = cos x

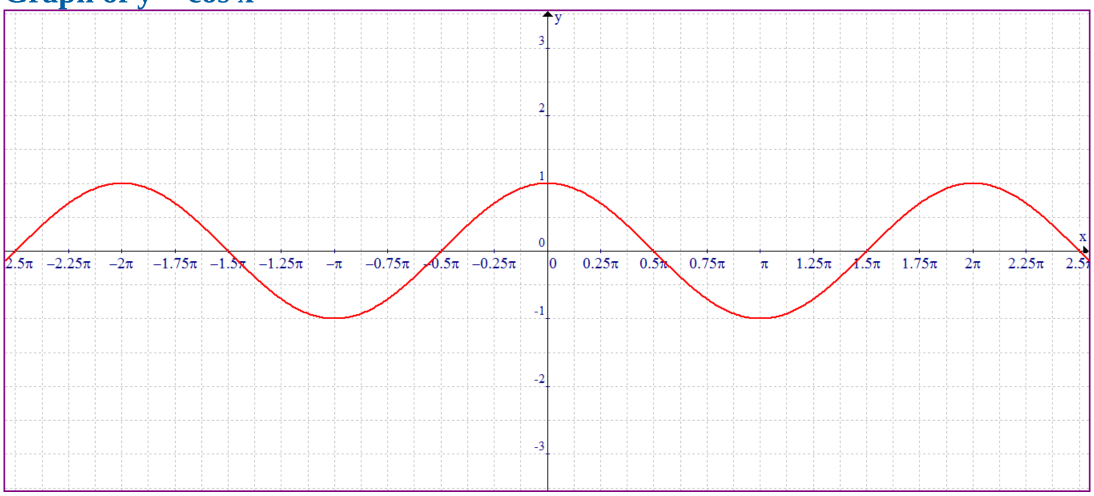

Figure 1.8

Here note that graph of Cosine is similar to that of Sine if the latter is translated along x-axis by a right angle. This is because of the fact that Sine & Cosine are complementary i.e.

$$\sin\left(\frac{\pi}{2}+x\right)=\cos x$$

Graph of y = tan x

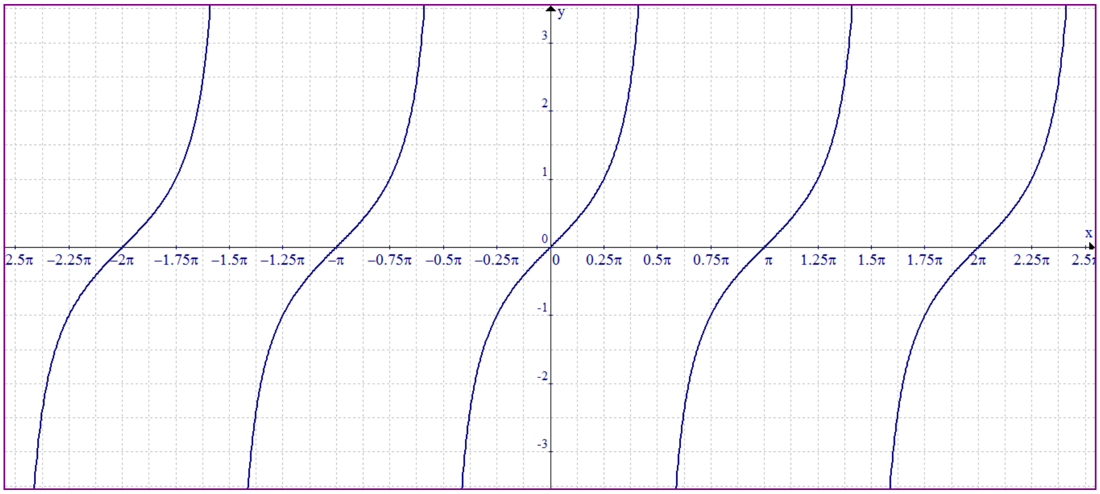

Figure 1.9

Graph of y = cot x

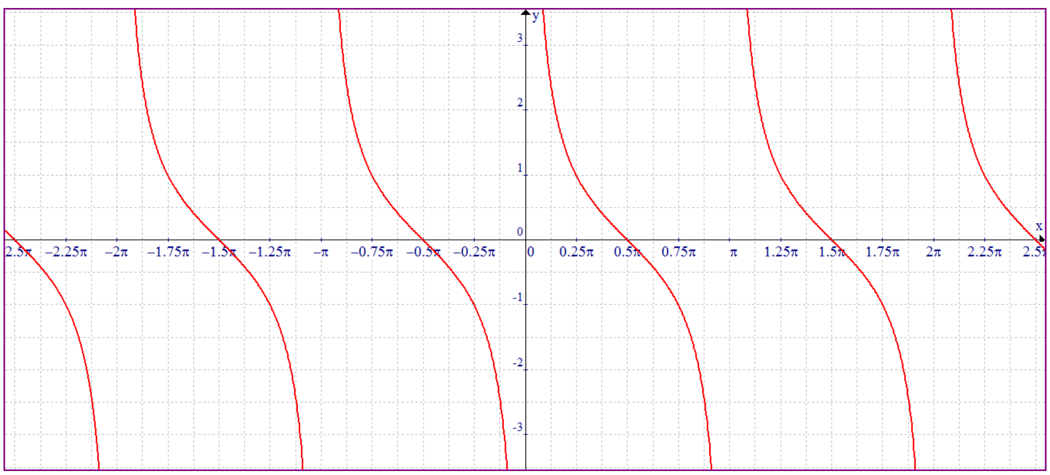

Figure 1.10

The graph of the tangent function has a vertical asymptote at $x = \pi/2$. This is because the tan x approaches infinity as x approaches $\pi/2$. (Actually, it approaches minus infinity as x approaches $\pi/2$ from the right as you can see on the graph).

You can also see that tangent has period "π" i.e. the graph replicates after every π interval.

Graph of y = sec x

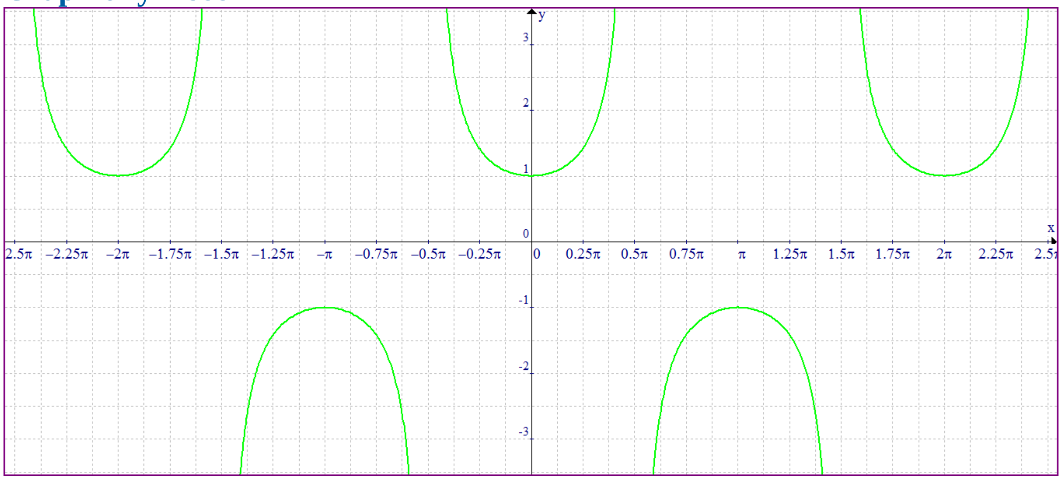

Figure 1.11

Graph of y = cosec x

Figure 1.12

The secant is the reciprocal of the cosine, and as the cosine only takes values between –1 and 1, therefore the secant only takes values above 1 or below –1, as shown in the graph. Also secant has a period of 2. As you would expect by now, the graph of the cosecant looks much like the graph of the secant.

The above graphs are very useful in those questions where we need to decide about behavior of trigonometric functions or compare two trigonometric functions over a set of general angles in which these are supposed to repeat periodically.

22. GRAPHICAL TRANSFORMATIONS

***To plot the graph of* y = – f(x)…**

Plot the graph of y = f(x) and plot its reflection in X– Axis.

***To plot the graph of* y = f(–x)…**

Plot the graph of y = f(x) and plot its reflection in Y– Axis.

The following figure shows graphs of y = sin x {in blue}, graph of y = –sin x {in red}.

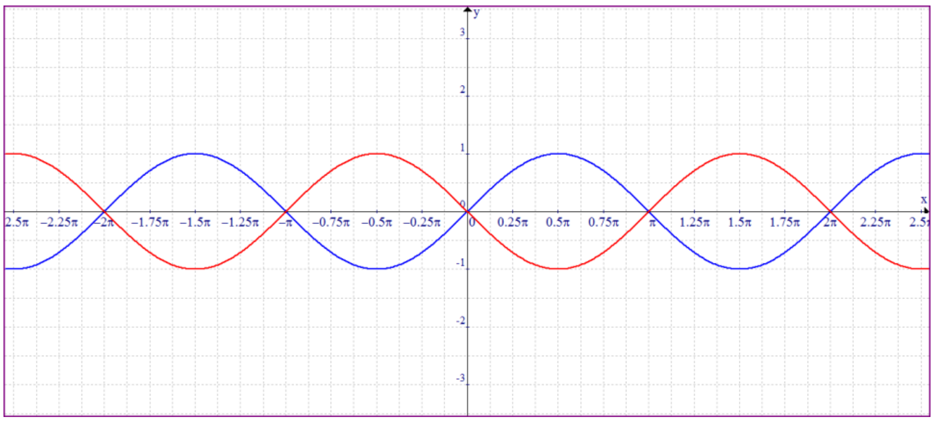

Figure 1.13

To plot the graph of* y = f(x + α), *where* 'α' *is a constant…

Plot the graph of y = f(x) and translate the Y–Axis α units backward or forward as according a is negative or positive. The following figure shows graphs of y = sec x {in blue}, graph of y = sec (x + 90^0) {in red} and the graph of y = sec (x – 90^0) {in green}

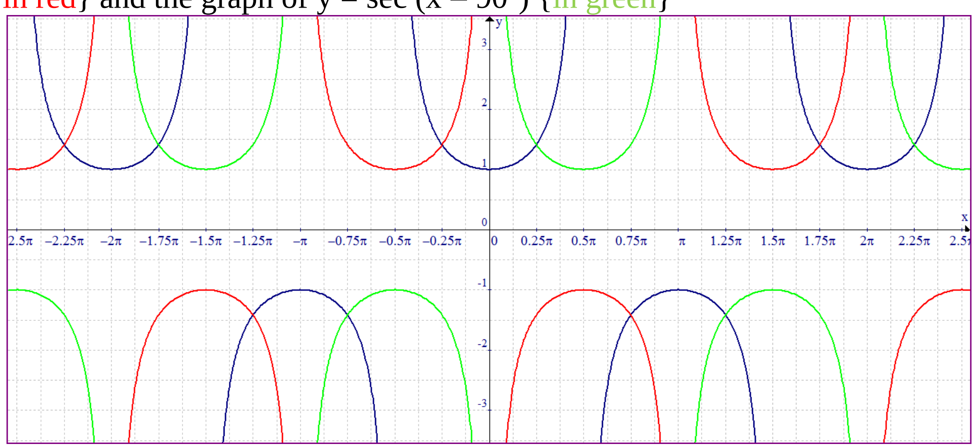

Figure 1.14

To plot the graph of* y = f(x) + a, *where* 'a' *is a constant…

Plot the graph of y = f(x) and translate the X–Axis a units upward or downward as according a is negative or positive.

The following figure shows graphs of y = tan x {in yellow}, graph of y = tan x + 1 {in green} and the graph of y = tan x – 1 {in blue}

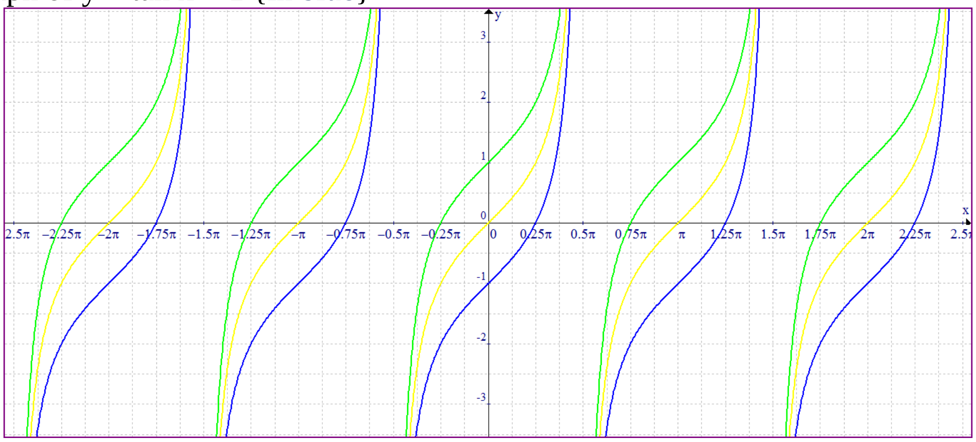

Figure 1.15

To plot the graph of* y = a f(x), *where* 'a' *is a positive constant…

Plot the graph of y = f(x) and compress or expand the graph about Y– Axis in ratio of 'a' as according a < 1 or a > 1. The following figure shows graphs of y = sin x {in yellow}, graph of y = 2sin x {in blue} and the graph of y = $\frac{1}{2}$ sin x {in red}

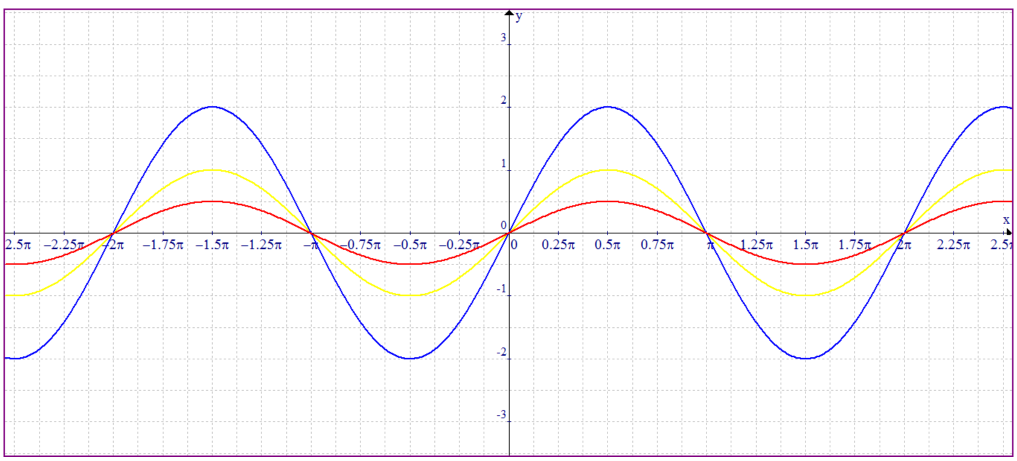

Figure 1.16

To plot the graph of* y = f(ax), *where* ‘a’ *is a nonzero constant…

Plot the graph of y = f(x) and compress or expand the graph about X – Axis in ratio of ‘a’ as according a > 1 or a < 1.

The following figure shows graphs of y = cos x {in blue}, graph of y = cos 2x {in red} and the graph of y = cos $\frac{x}{2}$ {in green}

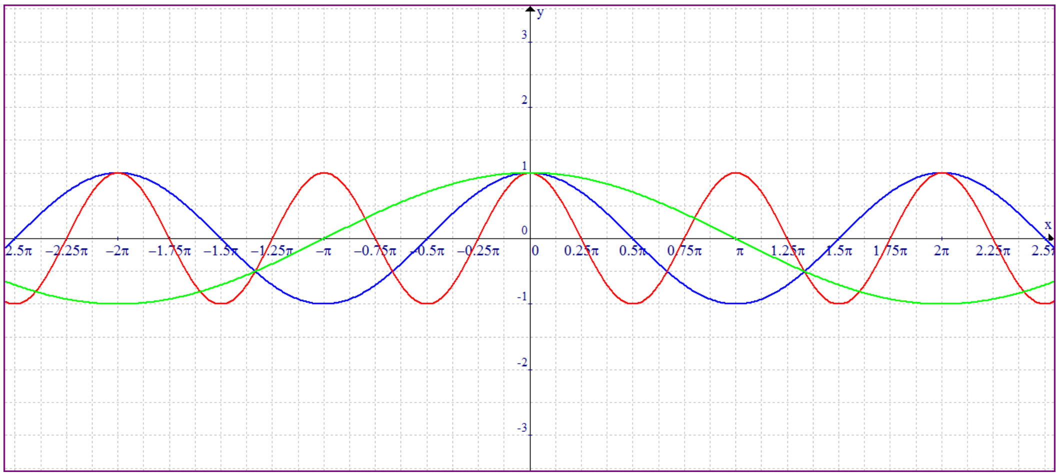

Figure 1.17

***To plot the graph of* y = |f(x)|…**

Plot the graph of y = f(x), take image of the part of graph which lies below x – axis in x – axis and delete the graph for y < 0. The following figure shows graphs of y = cos x {in red}, graph of y = |cos x| {in green}.

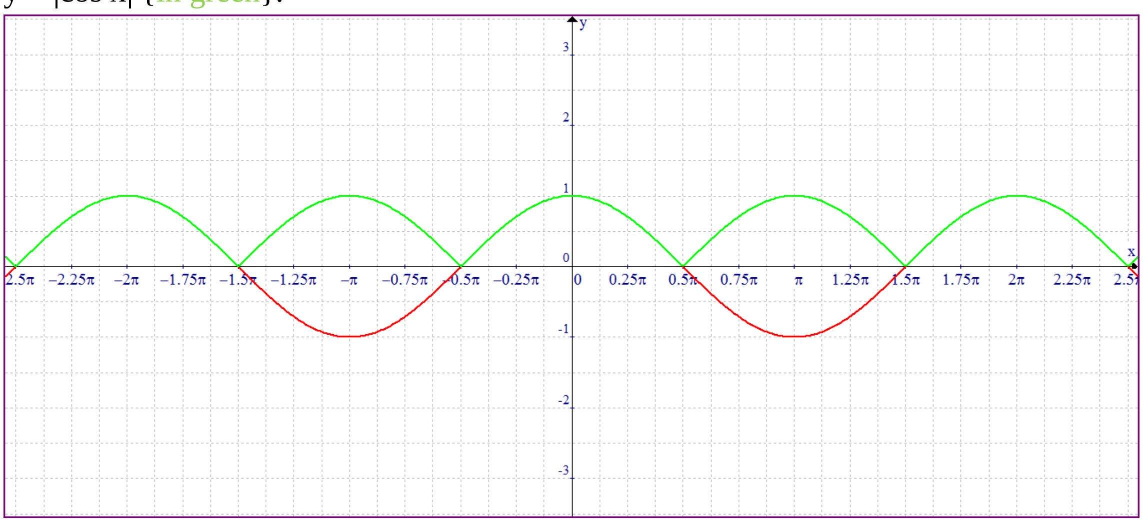

Figure 1.18

23. TRIGONOMETRIC INEQUALITIES

Inequalities of type $f(x) \le a$ & $f(x) \ge a$***, where f(x) is one of the T – Ratios & 'a' is a constant…***

Step I : Draw the graph of respective T – Ratio in (0, π) for tan x & cot x and in (0, 2π) for rest.

Step II : Identify the interval in which the inequality is being satisfied.

Step III : Generalize the identified interval by adding nπ in case of tan x & cot x and 2nπ in case of other T – Ratios to the extreme value.

Inequalities of type $f(x) \le g(x)$***, where f(x)& g(x) are the T – Ratios…***

Step I : Draw the graph of respective T – Ratios on the same scale in (0, π) for tan x & cot x and in (0, 2π) for rest.

Step II : Identify the interval in which the inequality is being satisfied by finding points of intersection.

Step III : Generalize the identified interval by adding nπ in case of tan x & cot x and 2nπ in case of other T – Ratios to the extreme value.

24. RANGE OF TRIGONOMETRIC EXPRESSIONS

If the least value of sin x is −1, then why cannot we obtain least value of $\sin^2 x - 2\sin x$ by substituting sin x = −1???

Here we need to combine the concept of range of rational algebraic expressions with the fact that range of values of x in an algebraic function of x and that of a trigonometric function in a function of a trigonometric function are not identical.

For example finding the range of $ax^2 + bx + c$ where x may take all real numbers is different from finding the range of $ax^2 + bx + c$ when $x = \sin\theta$ as $\sin\theta$ can't take all real values but only those between -1 & 1, whereas it is not at all different for $x = \tan\theta$ as $\tan\theta$ can take all real values

Range of expressions of type $a\sin^2 x + b\sin x + c$ & $a\cos^2 x + b\cos x + c$ ***…***

Step I : Rewrite the given expression as $a\left(t + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$, where $t = \sin x$ or $\cos x$

Step II : Now (i) if $-1 \le -\frac{b}{2a} \le 1$, then one extreme value will be $\frac{4ac-b^2}{4a}$ and other extreme value will occur at t = –1 or t = 1.

(ii) otherwise one extreme value will occur for t = –1 and other extreme for t = 1.

Range of expressions of type $\frac{a\sin^2 x + b\sin x + c}{p\sin^2 x + q\sin x + r}$ & $\frac{a\cos^2 x + b\cos x + c}{p\cos^2 x + q\cos x + r}$ **…**

Step I : Put the given expression equal to say y and rearrange as a quadratic equation in respective T – Ratio.

Step II : Apply conditions from theory of equations for at least one root to lie in [–1, 1].

Here note that…

For similar type of expressions in tan x or cot x no restrictions are required and theory of rational algebraic expressions may completely be applied.

Range of expressions of type $\frac{a\sin^2 x + b\sin x\cos x + c\cos^2 x}{p\sin^2 x + q\sin x\cos x + r\cos^2 x}$ **…**

Step I: Divide the numerator and the denominator by $\sin^2 x$ and transform the given expression in this form: $\frac{a\tan^2 x + b\tan x + c}{p\tan^2 x + q\tan x + r}$.

Step I: Put the given expression equal to say y and rearrange as a quadratic equation in respective T – Ratio.

Step II: Apply conditions from theory of equations for at least one root to be real.

25. SUM OF TRIGONOMETRIC SERIES

To find sum of Sines or Cosines of 'n' angles in A.P.:

Step I: Multiply and divide by $2\sin\theta$, where θ is half of common difference of A.P. of angles.

Step II: Use Product to Sum Transformation formulae to get cancellation of all intermediate terms.

To find sum & Product of a trigonometric series by forming a polynomial equation in respective T – Ratio:

Step I: Use Demoiver's theorem or multiple angle formulae to form a polynomial equation in respective T – Ratio.

Step II: Use sum and product of roots of a polynomial equation to get desired result.

Two Important Series…

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\left(\alpha+\left(n-1\right)\beta\right)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right)$$

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\left(\alpha+\left(n-1\right)\beta\right)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha+\frac{n-1}{2}\beta\right)$$