Lec 6.

Exp + log

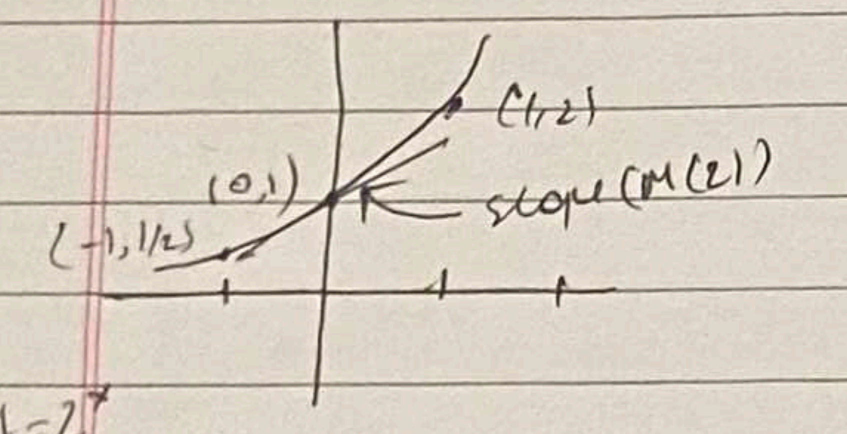
$\rightarrow a>0$, $a^0=1$, $a^1=a$, $a^2=a\cdot a$
(base)

$a^{x_1+x_2}=a^{x_1}a^{x_2}$

$(a^{x_1})^{x_2}=a^{x_1 x_2}$

$a^{m/n}=\sqrt[n]{a^m}$

a^x defined for all x by filling in by continuity



$y=2^x$

Goal $\frac{d}{dx}a^x=?$ $\quad a^{x+\Delta x}=a^x a^{\Delta x}$

$$\lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}=\lim_{x\to 0}\frac{a^x(a^{\Delta x}-1)}{\Delta x}$$

$$\boxed{\frac{d}{dx}a^x=a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}}$$

f' of something we don't know.

$$M(a)=\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$$

$\frac{d}{dx}a^x=M(a)a^x$

Plug in $x=0$

$\frac{d}{dx}a^x\Big|_{x=0}=M(a)a^0=M(a)$

$M(a)$ is the slope of a^x at $x=0$

What is $M(a)$?

Beg the question.
Define base e as the unique no. so that $M(e)=1$.
if $M(e)=1$

$\frac{d}{dx}e^x=e^x$

$\frac{d}{dx}e^x\Big|_{x=0}=1.$

slope 1. at $x=0$.

Why 'e' exists.

$f(x)=2^x \quad f'(0)=M(2)$ $\rightarrow$ derivative of tangent line.
stretch by k

$f(kx)=2^{kx}=(2^k)^x=b^x$ $\quad \rightarrow b=2^k$
explicitly numerically.

$\frac{d}{dx}b^x=\frac{d}{dx}f(kx)=kf'(kx)$

$\frac{d}{dx}b^x\Big|_{x=0}=kf'(0)=kM(2)$
$b=e$ when $k=1/M(2)$

$\frac{d}{dt} \sin(kt) = k\cos(kt)$

NATURAL LOG

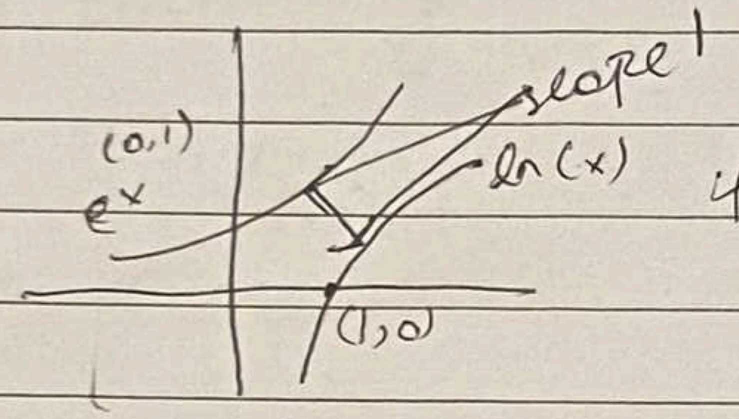
$w = \ln x$

$y = e^x \Leftrightarrow \ln y = x$

define 'ln'

$\ln(x_1 x_2) = \ln x_1 + \ln(x_2)$

$\ln 1 = 0; \ \ln e = 1$



for $x > 0$

To find $\frac{d}{dx} \ln x = \frac{1}{x}$

use implicit diff

$w = \ln x \Rightarrow e^w = x$

$e^{\ln x} = x$

$\frac{d}{dx} e^w = \frac{d}{dx} x = 1$

$\left(\frac{d}{dw} e^w\right)\left(\frac{dw}{dx}\right) = 1$

$e^w \frac{dw}{dx} = 1 \Rightarrow \frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$

diff any exp

To diff any exponential

2 methods

Method 1

$\frac{d}{dx} a^x = ?$

use base e $= a^x = (e^{\ln a})^x \quad e^{x \ln a}$

$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$ rate of change of this wrt x

$= (\ln a) e^{x \ln a}$

other way

$\boxed{\frac{d}{dx} a^x = (\ln a) a^x}$

$\frac{d}{dx} e^{3x} = 3e^{3x}$

called magic no.

$M(a) = \ln a$

For eg

$\frac{d}{dx} 2^x = (\ln 2) 2^x$

$\frac{d}{dx} 10^x = (\ln 10) 10^x$

Method 2:

Log arithmetic differentiation

$\frac{d}{dx} u = ??$

$\frac{d}{dx} \ln u =$

$= \left(\frac{d}{du} \log u\right) \cdot \left(\frac{du}{dx}\right) = \frac{1}{u} \frac{du}{dx}$

chain rule

$(\ln u)' = u'/u$

$\frac{d}{dx} a^x = ? \quad \Big| \; u = a^x$

$\ln u = x \ln a$

$(\ln u)' = x \ln a \qquad (3x+1)' = 3$

$(\ln u)' = \ln a$

$u'/u = (\ln u)' = \ln a$

$\frac{d}{dx} a^x = (\ln a) a^x$

Ex2 (moving exp)

$v = x^x, \; \ln v = x \ln x$

$(\ln v)' = \ln x + x \cdot \frac{1}{x}$

$v'/v = 1 + \ln x$

$v' = v(1 + \ln x)$

$\frac{d}{dx} x^x = x^x (1 + \ln x)$

Ex 3 $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ → moving exp

$\Delta x = \frac{1}{n} \to 0$

$\ln\left(\left(1 + \frac{1}{n}\right)^n\right) = n \ln\left(1 + \frac{1}{n}\right) = \frac{1}{\Delta x} \left(\ln(1 + \Delta x) - \ln 1\right)$

$\downarrow \Delta x \to 0$

$\frac{d}{dx} \ln x \Big|_{x=1}$

$= \frac{1}{x} = 1$

$\lim \left(1 + \frac{1}{n}\right)^n = e^{\left[\lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)^n\right]}$

$= e^1 = e$

Lec 7

Exponents (CONT'D)

$a_k = \left(1 + \frac{1}{k}\right)^k$

$\lim_{k\to\infty} a_k = e \rightarrow e = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k$

last time

$\ln a_k \rightarrow 1$

$e^{\ln a_k} \rightarrow e^1 = e$

$a_k \; (e^{\ln a} = a)$

$\frac{d}{dx} x^r = r x^{r-1}$ all real

method 1
(base e)

$x^r = (e^{\ln x})^r = e^{r\ln x} = $

$\frac{d}{dx} x^r = (e^{r\ln x})' = e^{r\ln x}(r\ln x)'$

$= x^r \cdot \frac{r}{x}$

$= r x^{r-1}$

method 2. (log diff)

$u = x^r, \; \ln u = r\ln x$

$\frac{u'}{u} = (\ln u)' = \frac{r}{x}$

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$\boxed{\text{lec 7 continuation}}$

M 2.

$u' = u \cdot \frac{r}{x} = x^r \frac{r}{x} = r x^{r-1}$

Natural log is natural.

Example economics.

NSE 100 ↓ 27.9

$\frac{\Delta P}{P} = \frac{27.9}{6432} \approx .43\%$

$\boxed{\frac{P'}{P} = (\ln p)'}$ log differention in prices & stock market

REVIEW of unit 1
all the lectures.

General formula

$(U+V)' = U' + V'$

$(cU)', (UV)', \left(\frac{U}{V}\right)'$

$\frac{d}{dx} f(u) = f'(u)\, u'(x) \quad [u = u(x)]$
chainrule.

implicit diff.

for inverse, log diff

Specific formula

x^r, $\cos x$, $\sin x$, $\tan x$, $\sec x$

e^x, $\ln x$

$\tan^{-1} x$, $\sin^{-1} x$

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Chain Rule

$y = 10x + b, \quad \frac{dy}{dx} = 10$

$x = 5t + a, \quad \frac{dx}{dt} = 5$

$\left\} \frac{dy}{dt} = 10 \cdot 5 = 50 \right.$

$y = 10(5t+a) + b$
$= 50t + 10 + b$

$\left(\frac{1}{V}\right)' = (V^{-1})' = -V^{-2} \cdot V'$
chain rule.

$\left(\frac{U}{V}\right)' = (UV^{-1})' = U'V^{-1} + U(-V^{-2} \cdot V')$
$= (U'V - UV')/V^2$

Ex:

$\frac{d}{dx} \sec x = \frac{d}{dx} (\cos x)^{-1}$

$= -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$

Ex $\frac{d}{dx} \ln(\sec x) = \frac{(\sec x)'}{\sec x} = \frac{\sec x \tan x}{\sec} = \tan x$

Ex $\frac{d}{dx}(x^{10} + 8x)^6$

$= 6(x^{10} + 8x)(10x^9 + 8)$
chain rule

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Ex

$\frac{d}{dx} e^x \tan^{-1} x$

$= e^x \tan^{-1} \frac{d}{dx}(x \tan^{-1} x)$

$= e^x \tan^{-1} x \left(\tan^{-1} x + \frac{x}{1+x^2}\right)$

Defn of derivative.

(meaning of Derivative

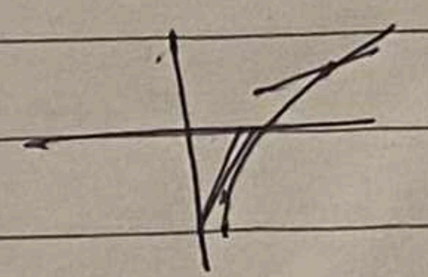
$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$1/x,\ x^n,\ \sin x,\ \cos x$, $a^x,\ (uv)',\ (u/v)',\ e^x,\ 1/x^2$
using $\cos x = 0$

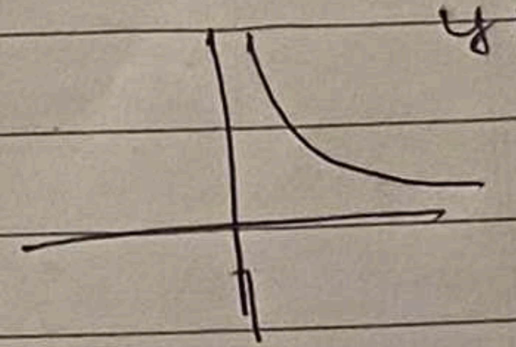
Read formula backwards

$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

$\lim_{u \to 0} \frac{e^u - 1}{u} = \frac{d}{du} e^u \Big|_{u=0} = 1$



Tangent lines.
- compute tangent lines
- graph y'
- recognize diff. fns
- left/right tangent should be equal



$y'=\ln x$

$y'=\frac{1}{x}$

$y=\tan^{-1}x$

$\tan y = x$

$(\sec^2 y)\cdot y' = 1$

$y' = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{\sqrt{1+x^2}}$

$\sqrt{1+x^2}$, y, x $\tan y = x$

$\boxed{\text{lec 8}}$

$\boxed{\text{unit 2}}$

App of Diff   curve $y=f(x)$

$\approx y = f(x) + f'(x_0)(x-x_0)$ tangent line

Linear Approx.

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x-x_0)} \quad x \approx x_0$$

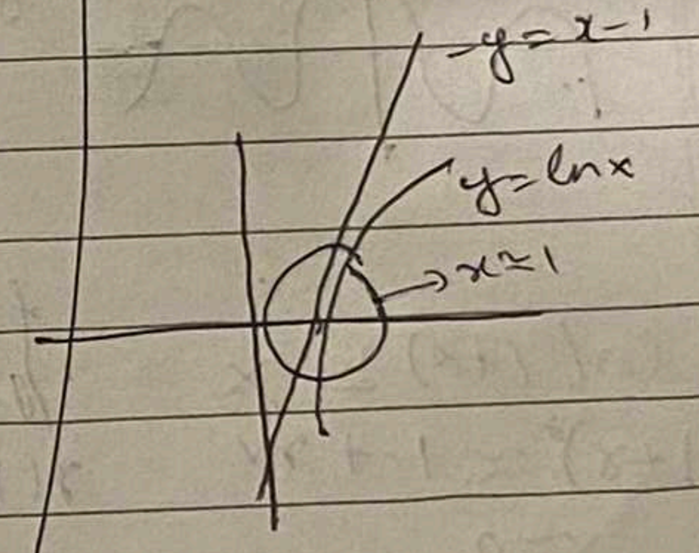
eg.

$f(x) = \ln x,\ f'(x) = \frac{1}{x}$

$x_0 = 1,\ f(1) = \ln 1 = 0,\ f'(1) = 1$ — same

$\ln x \approx 0 + 1\cdot(x-1)$

$\ln x \approx x - 1$



$f'(x_0) \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)$

$$\boxed{\frac{\Delta f}{\Delta x} \approx f'(x_0)} \quad \therefore x \approx x_0 \qquad \Delta x = \text{small}$$

$\square \Leftrightarrow \Delta f \approx f'(x_0)\,\Delta x \qquad \Delta x = x - x_0$

$\Leftrightarrow f(x) - f(x_0) \approx f'(x_0)(x-x_0)$

$\Leftrightarrow f(x) \approx f(x_0) + f'(x_0)(x-x_0) \leftarrow \boxed{\text{Same}}$

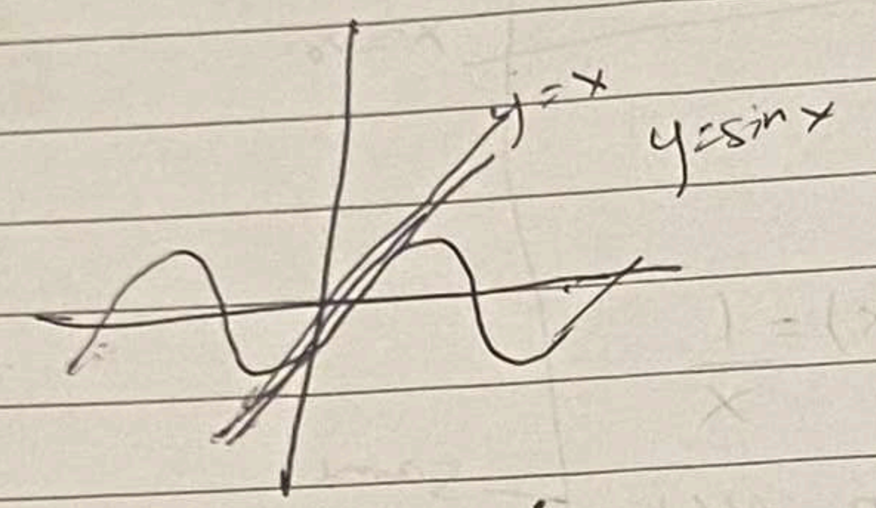
Systematic approach.

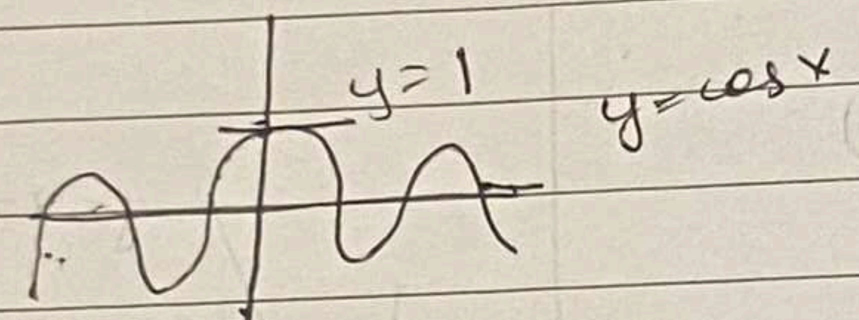
$x = 0$

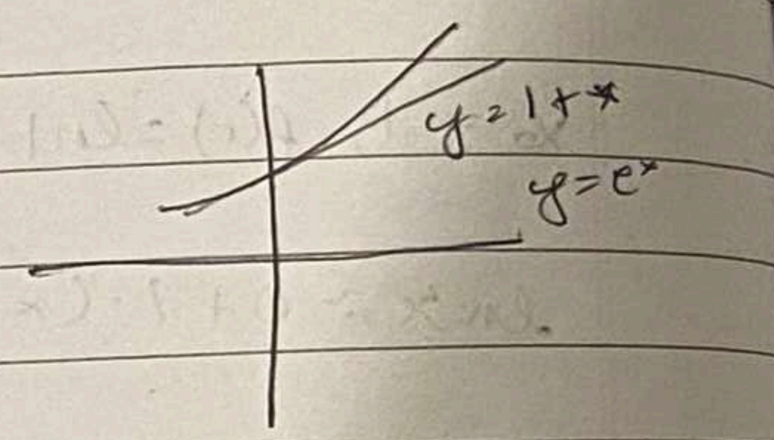
$f(x) \approx f(0) + f'(0)x.$

linear, quadratic

	f'	$f(0)$	$f'(0)$	f''	$f''(0)$
$\left.\begin{array}{l}\sin x \approx x \\ \cos x \approx 1, 1-\frac{1}{2}x^2\end{array}\right\} x \approx 0$	$\cos x$	0	1	$-\sin x$	0
	$-\sin x$	1	0	$-\cos x$	-1
$e^x \approx 1 + x + \frac{1}{2}x^2$	e^x	1	1	e^x	1







	f'	$f(0)$	$f'(0)$
$\ln(1+x) \simeq x$	$1/(1+x)$	0	1
$(1+x)^r \approx 1 + rx$	$r(1+x)$	1	r

$x \approx 0$

connects to ~~...~~

$\ln u \approx u - 1, \; u = 1 + x, \; u - 1 = x$

Ex

$\ln 1.1 \approx \frac{1}{10}$

Easy — hard.

$\ln(1+x), x \approx \frac{1}{10}$ — $\ln(1+4), x \approx \frac{1}{10}$

Ex 3

Find linear near $x = 0$ ($x \approx 0$)

of $\dfrac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}$

$= (1 - 3x)\left(1 - \frac{1}{2}x\right)$

$= 1 - 3x - \frac{1}{2x} + \frac{3}{2}x^2$ we should eliminate quadratic x^2 terms since we're dealing with linear.

$\approx 1 - 3x - \frac{1}{2x}$

$= -\frac{7x}{2}$

x^2 drop terms x^3 and higher

Ex 4 (real life)

satellite — time dilation

T time — special relativity.

$T' = \dfrac{T}{\sqrt{1 - v^2/c^2}}$

me. T'

$v = 4\,\text{km/sec}$

$c = 3 \times 10^5\,\text{km/sec}$

$T' \approx T\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$

$(1 - u)^{-1/2} \approx 1 + \frac{1}{2}u$

$u = v^2/c^2 \approx 10^{-10}$

$u^2 \sim 10^{-20}$

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Quadratic Approx.

$$f(x_0) = \boxed{f(x_0) + f'(x_0)(x-x_0)} + \frac{f''(x_0)}{2}(x-x_0)^2$$

linear

Ex 2 $\ln(1.1) \approx \frac{1}{10}$

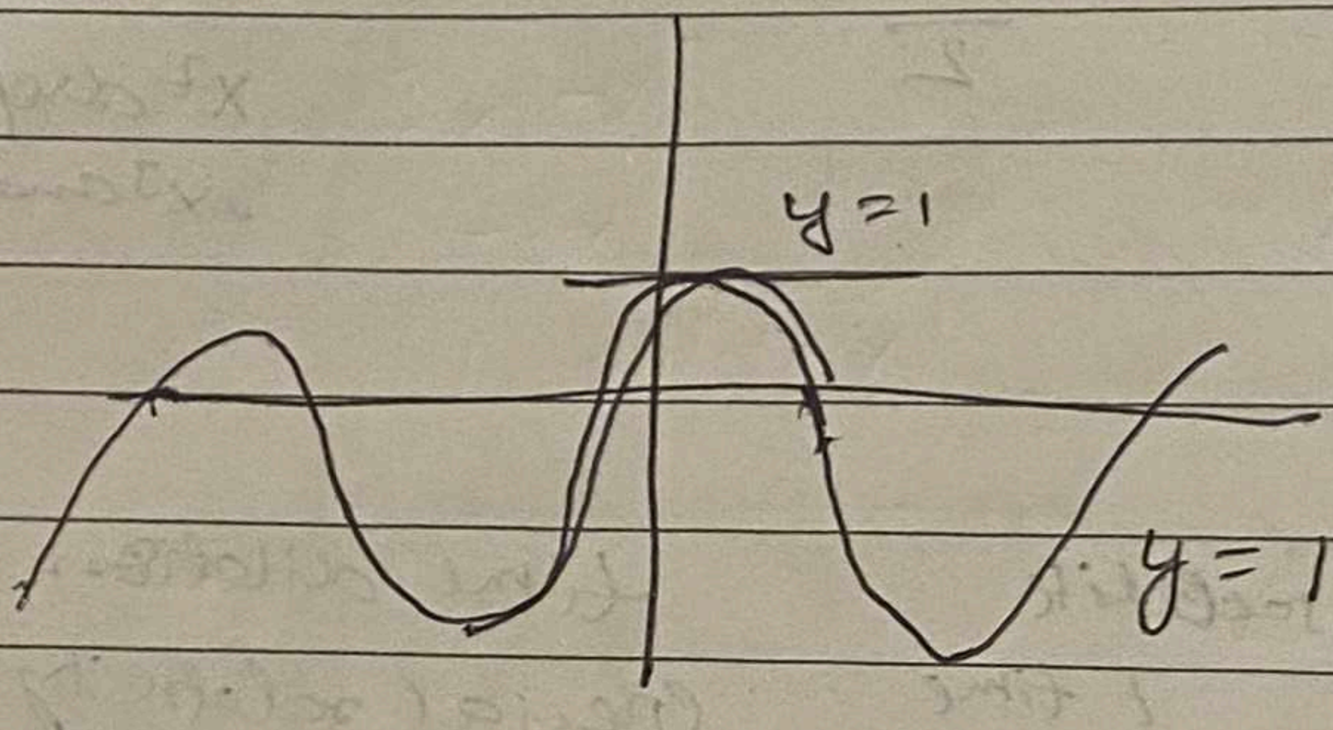
$\ln(1+x) \approx x, x = \frac{1}{10}$
linear

$\ln(1+x) \approx x - x^2/2$

$\ln(1.1) = \ln\left(1 + \frac{1}{10}\right)$

$\approx \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2 = .095$

Geometric significance of quadratic.



$y = \cos x$

$y = 1 - \frac{1}{2}x^2$, 'best fit' ~~para~~ parabola'

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Lec 10

Approx

$T' = T(1 - V^2/c^2)^{1/2}$

$\frac{\Delta T}{T} \simeq \frac{1}{2} \frac{V^2}{c^2}$

Error function is proportional to V^2/c^2
with factor $\frac{1}{2}$

Quadratic Approx.

Use these when linear is not enough

$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$x \approx 0$

why $\frac{1}{2} f''(0)$

$f(x) = a + bx + cx^2$

$f'(x) = b + 2cx$

$f''(x) = 2c$

Hence, $f(0) = a$

$f'(0) = b$

$\frac{1}{2} f''(0) = c$

x near 0

$\sin x \approx x$

$\cos x \approx 1 - \frac{1}{2}x^2$

$e^x \approx 1 + x + \frac{1}{2}x^2$

$\ln(1+x) \approx x - \frac{1}{2}x^2$

$(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$

$a_k = \left(1+\frac{1}{k}\right)^k \xrightarrow[k\to\infty]{} e$

$\ln a_k = k\left(\ln\left(1+\frac{1}{k}\right)\right)$

$\approx k\left(\frac{1}{k}\right)$

$\ln(1+x) \approx$ $\quad x = \frac{1}{k} \approx 0$

rate of convergence.

$\underbrace{(\ln a_k) - 1}_{\text{how big}} \to 0$

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$\boxed{\text{lec } 10}$ continuation.

Find quadratic approx.

f x x near 0 to

$e^{-3x}(1+x)^{-1/2} \approx$ quad

as per the quad approx in lec 9.

$\approx \left(1+(-3x)+\frac{(-3x)^2}{2}\right)\left(1-\frac{1}{2}x+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2\right)$

$\approx 1-3x-\frac{1}{2}+\frac{3}{2}x^2+\frac{9}{2}x^2-\frac{3}{8}x^2$ drop x^3, x^4 etc terms.

$\approx \underbrace{1-\frac{7}{2}x}_{\text{linear}}+\frac{51}{8}x^2$

at $x=0$

(Ex) $f = \ln(1+x)$ — 0

$f' = 1/(1+x)$ — 1

$f'' = \frac{-1}{(1+x)^2}$ — -1

at $x=0$

$(1+x)^r$ — 1

$r(1+x)^{r-1}$ — r

$r(r-1)(x+1)^{r-2}$ — $r(r-1)$

CURVE SKETCHING

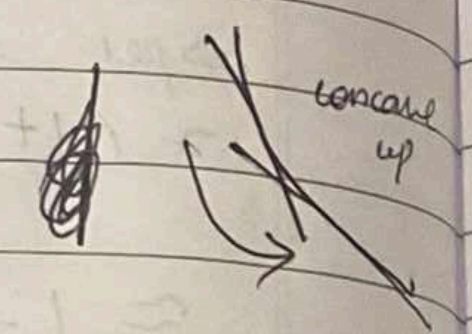
Goal: Draw graph of f
using f', f'' positive/negative

Warning
don't abandon ur precalculus skill and
common sense.

$f' > 0 \Rightarrow \Delta f$ is increasing

$f' < 0 \Rightarrow \Delta f$ is decreasing

$f'' > 0 \Rightarrow \Delta f'$ is increasing $\Longleftrightarrow$



$f'' < 0 \Rightarrow f$ concave down.

Ex).

$f(x) = 3x - x^3$

$= 3 - 3x^2$

$= 3(1-x)(1+x)$

$-1 < x < 1 \Rightarrow f'(x) > 0$

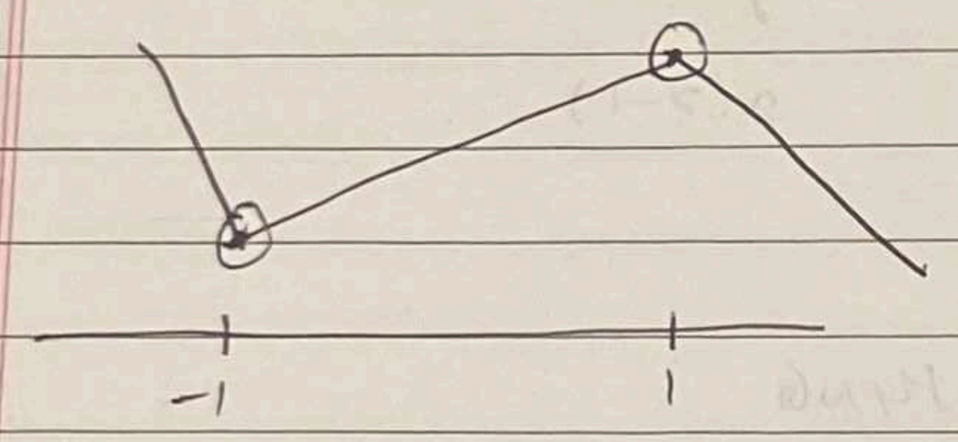
f is increasing

$1 < x \Rightarrow f'(x) < 0$

f is decreasing

$x < -1$ ↗

Schematic.



turning point

Defn: If $f'(x_0) = 0$

x_0 a critical point

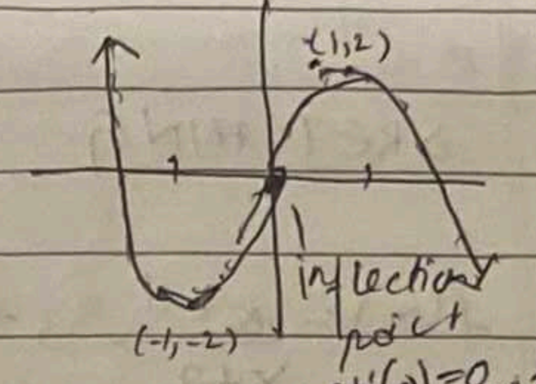
$y_0 = f(x_0)$ is a critical ~~point~~ value.

$f'(x) = 0 = 3(1-x)(1+x) = 0 \Rightarrow x = \pm 1$

Plot critical points/ends.

$f(1) = 3.1 - 1^3 = 2$

$f(-1) = 3(-1) - (-1)^3 = -2$



$f''(0) = 0, f(0) = 0$

$f(x) = 3x - x^3$

$(f(0) = 0$

f is odd

$f(0)$ is 0.

Ends

$x \to \pm\infty$

$f(x) = 3x - x^3 - x^3 \to -\infty$

$x \to +\infty$

$f(x) = +\infty$ if $x \to -\infty$

$f''(x) = (3 - 3x^2)' = -6x$

$f''(x) < 0$ if $x > 0$ (concave down)

$f''(x) > 0$ if $x < 0$ (concave up)

$\boxed{\text{lec 11}}$

SKETCHING

$f(x) = \frac{x+1}{x+2}$, $f'(x) = \frac{1}{(x+2)^2} \neq 0$

No critical points.

Plot points : $x = -2^+$
when $f(x)$ is not defined.

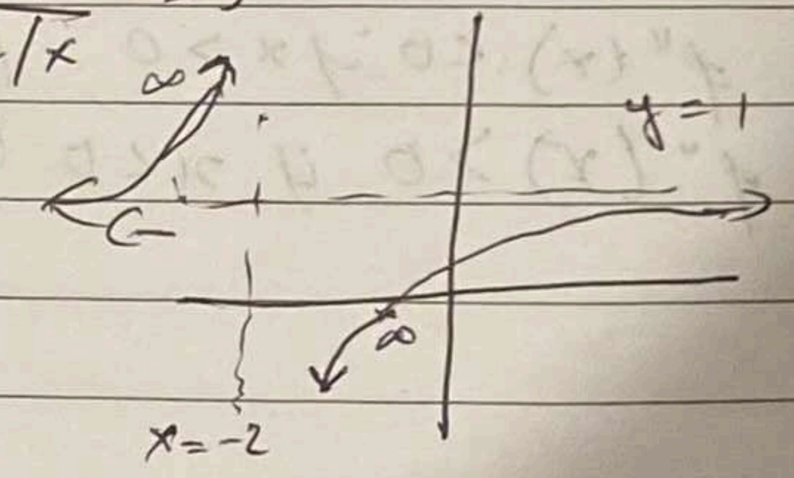
$f(-2^+) = \frac{-2+1}{-2^+ + 2} = \frac{-1}{0^+} = -\infty$

$f(-2^-) = \frac{-2+1}{(-2)^- + 2} = \frac{-1}{0^-} = +\infty$

Ends: $x \to \pm\infty$

$f(x) = \frac{x+1}{x+2} = \frac{1 + 1/x}{1 + 2/x} \Rightarrow 1$

$f(\pm\infty) = 1$



Double check

$\frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{(x+2)}$ $f' \neq 0$ f can't backtrack.

$f'(x) = \frac{1}{(x+2)^2} > 0$ ✓ $x \neq -2$ so it won't go to 0.

2) Δf is increasing on $-\infty < x < \infty$

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$f''(x) = \frac{-2}{(x+2)^3}$ $(x \neq -2)$

$f''(x) < 0$ $-2 < x < \infty$ concave down) — no wiggle in graph of f.

$f''(x) > 0$ $-\infty < x < -2$ (concave up)

General Strategy.

1. Plot a) discontinuities
   (especially infinite)
   b) endpoints for $x \to \pm\infty$
   c) easy pts (optional)

2) a) Solve $f'(x) = 0$
   b) Plot crit pts and values.

3) Decide whether $f' < 0$ or > 0 on btw critical points / discont.
(consistent w/ 1 & 2)

4) f'' >0 / <0 concave up/down

$f''(x) = 0 \Leftrightarrow x_0$ inflection pt

To continue.

Ex $f(x) = \frac{x}{\ln x}$, $x > 0$

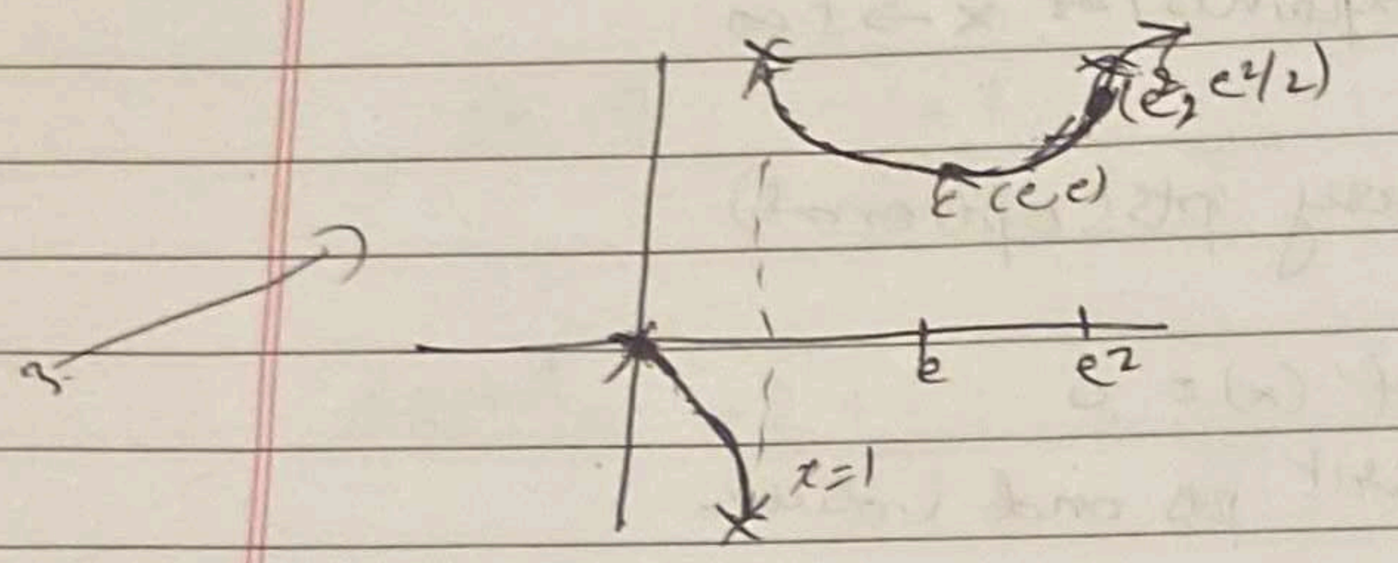
1a) $f(1^+) = \frac{1}{\ln(1^+)} = \frac{1}{0^+} = \infty$

$f(1^-) = \frac{1}{\ln(1^-)} = \frac{1}{0^-} = -\infty$

1b) ends. $0 < x < \infty$

$f(0^+) = \frac{0^+}{\ln(0^+)} = \frac{0^+}{-\infty} = 0$

$f(10^{10}) = \frac{10^{10}}{\ln(10^{10})} = \frac{10^{10}}{10\ln 10} >> 1$ ($f(\infty) = \infty$)



& $x \to$ can be max/mino - s of them.

2) $f'(x) = \frac{1 \cdot \ln x - x(1/x)}{(\ln x)^2}$

$= \frac{(\ln x) - 1}{(\ln x)^2} = 0$

$f'(x) = 0 \Rightarrow x = e$ $\quad x = 0^+$ slope is 0

$f(e) = e/\ln e = e$

3) Double check (?)

f is decreasing on $0 < x < 1$

" " on $1 < x < e$

f is increasing on $e < x < \infty$

$f'(x) = \frac{(\ln x) - 1}{(\ln x)^2} = \frac{1}{(\ln x)} - \frac{1}{(\ln x)^2}$

$0 < x < 1$

$\frac{-}{+} < 0$; $0 < x < 1$

$\frac{-}{+} < 0$; $1 < x < e$

$\frac{+}{+} > 0$ $e < x$

4). $f'(0^+) = \frac{1}{\ln 0} + \frac{1}{(\ln 0)^2} = \frac{1}{-\infty} - \frac{1}{(\infty)^2} = 0$

means we are going down to 0.

$f''(x) = -(\ln x)^{-2}\frac{1}{x} + 2(\ln x)^{-3}\frac{1}{x}$

$f''(x) = \frac{2 - \ln x}{x(\ln x)^3}$

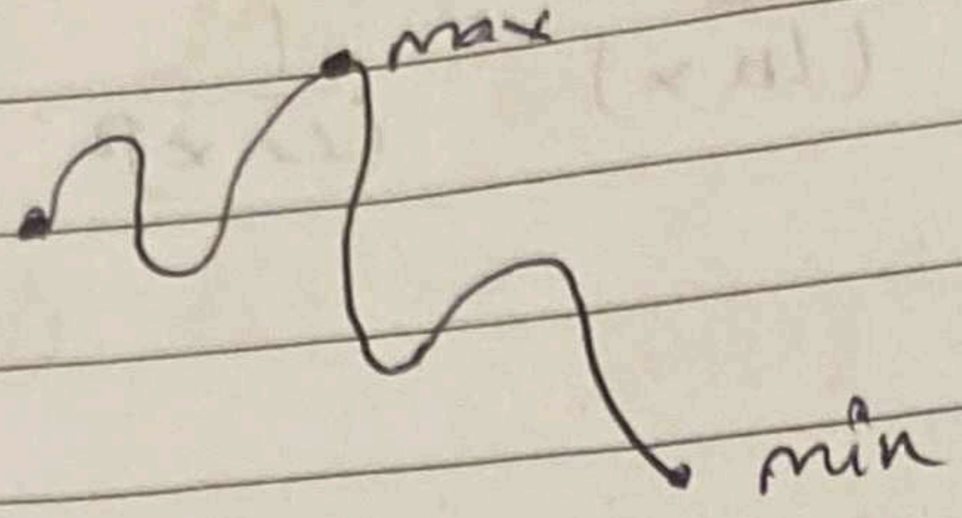
$0 < x < 1$ $\quad f'' = \frac{+}{-} < 0$ concave ↓

$1 < x < e^2$ $\quad f'' = \frac{+}{+} > 0$ concave ↑

$e^2 < x < \infty$ $\quad f'' = \frac{-}{+} < 0$ concave ↓

Horizon
Date
Page

Maxima & minima.



Easy to find max/min with sketch.

goal is to use shortcut.

KEY to finding max + min only need to look at critical points and Endpoints and points of Discontinuity.