Lec 19

Fundamental Theorem of Calculus.

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

$F = \int f(x)\,dx$

Notation:

$F(b) - F(a) = F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b}$

Ex

$F(x) = x^3/3$

$F'(x) = x^2 \quad (= f(x))$

$\Rightarrow$ (FTC) $\int_a^b x^2\,dx = F(b) - F(a) = \frac{b^3}{3} - \frac{a^3}{3}$

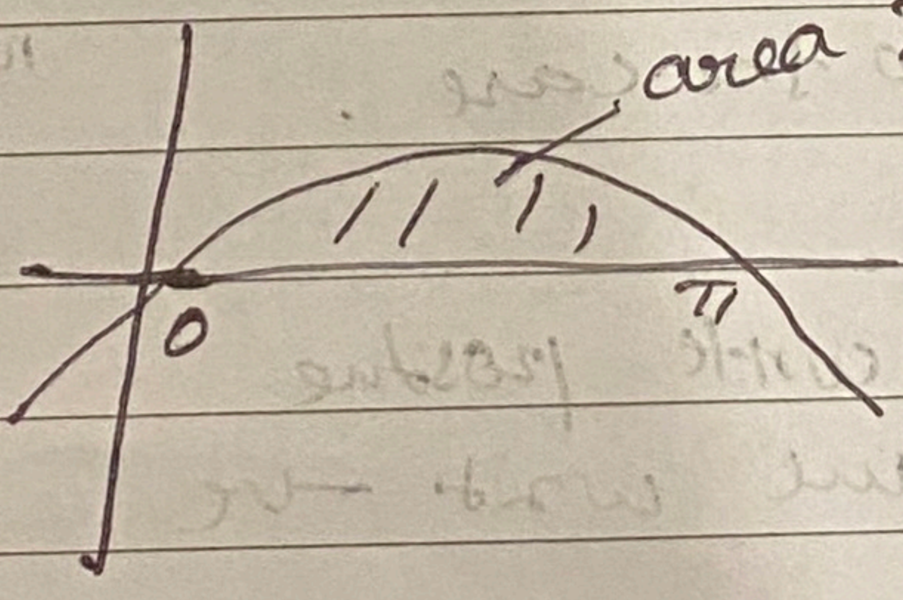
$\int_0^b x^2\,dx = \frac{x^3}{3}\Big|_0^b$

$a = 0$

$= \frac{b^3}{3} - \frac{0^3}{3} = \boxed{\frac{b^3}{3}}$

Ex 2

Area under one hump of $\sin x$.



$\int_0^{\pi} \sin x\,dx = (-\cos x)\Big|_0^{\pi}$

$= -\cos\pi - (-\cos 0)$

$= -(-1) - (-1) = \boxed{2}$

Horizon

Date
Page

Ex.

$$\int_0^1 x^{100}dx = \frac{x^{101}}{101}\Big|_0^1 = \frac{1}{101}$$

~~Interpret~~

Intuitive interpretation of Fund. Thm.

$x(t)$ position at time t.

$x'(t) = \frac{dx}{dt} = v(t)$ speed

It measures the net distance.

$$\underbrace{\int_a^b v(t)\,dt}_{\text{speedom}} = \underbrace{x(b) - x(a)}_{\text{distance travelled (odometer)}}$$

Riemann sum

$\uparrow \sum_{i=1} v(t_i)\Delta t \simeq \underbrace{\int_a^b v(t)\,dt}_{\text{exactly}} = \underbrace{x(b) - x(c)}_{\text{exactly}}$

Δt → in itself. 1sec

speed in that second

distance travelled in the ith second. approximation.

total distance $= \int_a^b |v(t)|\,dt$.

net distance $= \int_a^b v\,dt$

Extend integration to the case. $f<0$ (or $f>0$)

If v is +ve then we write positive and if v is -ve then we write -ve

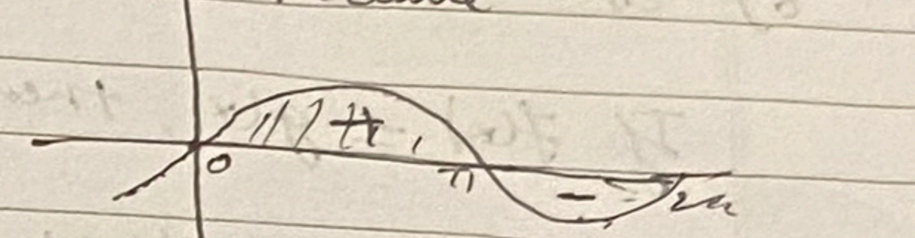
Horizon

Date
Page

Definite integral is not undercurve but it counts area when it is above the axis and it counts neg when it's below the curve

Example.

$$\int_0^{2\pi} \sin x\,dx$$



$= -\cos x\Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0)$
$= -1(-1) = 0$

True geometric integration of definite integral is
+ area above the x-axis
- minus area below the axis

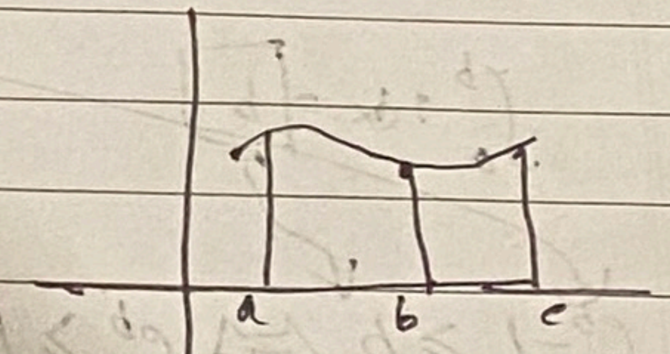
Properties of integrals

1. $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
cumulative sum $>$ a + b.

2. $\int_a^b c(f(x))\,dx = c\int_a^b f(x)$
constant out 'C'

3. $a < b < c$ not necessary (ok with (5))

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



4. $\int_a^a f(x)\,dx = 0 \quad (= F(a) - F(a))$

5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
$F(b) - F(a) = -(F(a) - F(b))$

Horizon
Date
Page

6) Estimation

If $f(x) \le g(x)$, then

$\int_a^b f(x)dx \le \int_a^b g(x)dx$ need $a<b$.

It says

If I am going more slower than you then you are going farther than I do.
- Neg mein bhi Karta se bola
Magar $\boxed{a<b}$ zaroori hai

Ex of estimation.
taking inequality
$e^x \geq 1, \; x \geq 0$. (start)

$\int_0^b e^x dx \geq \int_0^b 1\, dx$

$\int_0^b e^x dx = e^x\Big|_0^b = e^b - e^0 = \boxed{e^b - 1}$

$\int_0^b 1\,dx = \boxed{b}$

$e^b - 1 \geq b \Leftrightarrow e^b \geq 1 + b$ $b \geq 0$ works only
inequality.

Repeat
$e^x \geq 1 + x, \quad x \geq 0$

$\int_0^b e^x dx = \int_0^b (1+x)dx = \left(x + \frac{x^2}{2}\right)\Big|_0^b = b + \frac{b^2}{2}$
$e^b - 1$

Horizon
Date
Page

conclusion.

$e^b - 1 \geq b + b^2/2$

$e^b \geq 1 + b + b^2/2 \qquad (b \geq 0)$

Change of variables.

= substitution
$\int_{u_1}^{u_2} g(u)du = \int_{x_1}^{x_2} g(u(x))\, u'(x)\, dx$

only works when 'u' does not change sign.

$u = (ux)$ $\qquad u_1 = (ux_1)$
$du = u'(x)dx \qquad u_2 = u(x_2)$

~~student doubt~~

Ex

$\int_{x=1}^{x=2} (x^2+2)^5 x^2 dx = \int_3^{10} u^5 \frac{1}{3} du = \frac{1}{18} u^6 \Big|_{u=3}^{u=10} = \boxed{\frac{1}{18}(10^6 - 3^6)}$

$u_1 = x^3 + 2$
$= 1^3 + 2$
$= 3$

$u_2 = 2^3 + 2$
$= 10$

$u = x^3 + 2$
$du = 3x^2 dx$
$\frac{1}{3} du \; x^2 dx$

$u = x^2$; $u'(x) = 2x$
$>0 \quad x>0$
$<0 \quad x<0$

Warning. eg $\boxed{\text{student doubt}}$

$\int_{-1}^{1} x^2 dx \neq \int_1^1 \frac{u \cdot 1}{2\sqrt{u}} du = 0$ → not true.

$x = \pm\sqrt{u}$

$u = x^2$, $du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du = \frac{1}{2\sqrt{u}} du$ → mistake.
$u_1 = (-1)^2$, $u^2 = 1$

dec 20

FTC 1.

If $F' = f$, then

$\int_a^b f(x)dx = F(b) - F(a)$.

used to evaluate integrals.

use f' to understand F

Reverse

$F(b) - F(a) = \int_a^b f(x)\,dx$

INFO About $F' \Rightarrow$ INFO about F

Compare FTC1 with mean value theorem.

$\Delta F = F(b) - F(a)$; $\Delta x = b - a$.

$\Delta F = \int_a^b f(x)dx$.

In form of Inequalities

FTC1

$\Delta F = \text{Average}(F')\,\Delta x$

$$\frac{\Delta F}{\Delta x} = \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

MVT.

$\Delta F = F'(c)\,\Delta x \le (\max F')\,\Delta x$

vague

Average (f) why Avg.

$a = 0,\ b = n$

$\frac{f(1) \ldots + f(n)}{n} \approx \text{Avg.} \approx \int_0^n f(x)dx$

$\Delta x = 1$

nock.

increment by 1.

$\min \le \Delta F \overset{FTC}{=} \text{Ave}(F')\Delta x \le (\max F')\Delta x$

$(\min F')\Delta x \le \sim \Delta F = F'(c)\Delta x \le (\max F')\Delta x$

MVT

vague some c in btw

not clear

$a < c < b$

Exam 2:

$F'(x) = \frac{1}{1+x}$, $F(0) = 1$.

The mean implies

$A < F(4) < B$

for which A and B.

with MVT says.

$F(4) - F(0) = F'(c)(4 - 0)$

$= \frac{1}{1+c} \cdot 4$ — some c $0 < c < 4$

range.

$\frac{1}{1+0} \cdot 4$ to $\frac{1}{1+4} \cdot 4$

with FTC

$F(4) - F(0) = \int_0^4 \frac{dx}{1+x}$

biggest when $x = 0$

$\le \int_0^4 dx = 4$

$F(4) - F(0) = \int_0^4 \frac{dx}{1+x}$

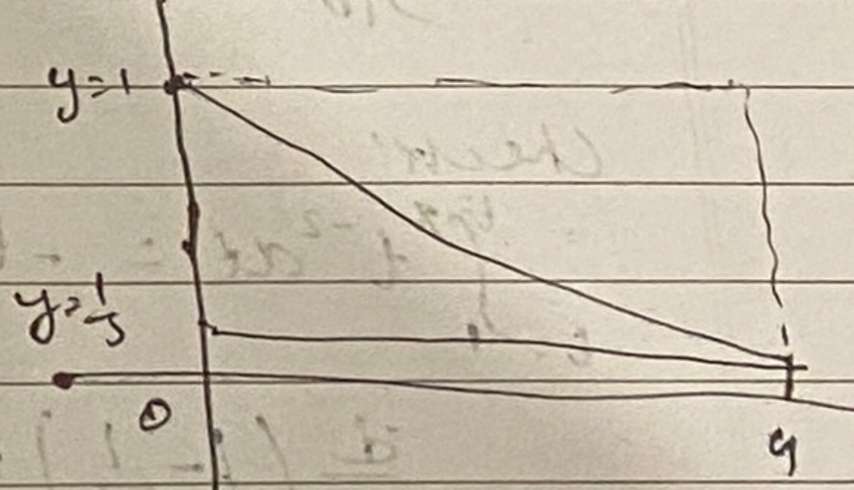
$> \int_0^4 \frac{1}{5}dx = \frac{4}{5}$

Same.

Conclusion

$\frac{4}{5} < F(4) - F(0) < 4$

$\frac{9}{5} < F(4) < 5$



$y = \frac{1}{1+x}$ Riemann sums with rectangle lower RS $< \int_0^1 \frac{dx}{1+x} <$ upper R.S

FT2:

If f is continuous and $G(x) = f$

$\boxed{G(x) = \int_a^x f(t)\,dt \;;\; \boxed{a < t < x}}$

if this is true. don't mix 'x' upper limit e

then $\boxed{G'(x) = f(x)}$

$G(x)$ solves the differential eqn

$\begin{cases} y' = f \\ y(a) = 0 \end{cases}$ It solves.

Ex

$\frac{d}{dx}\left[\int_1^x \frac{dt}{t^2}\right] = \frac{1}{x^2}$

$\underset{G(x)}{}$, $G'(x) = f(x)$

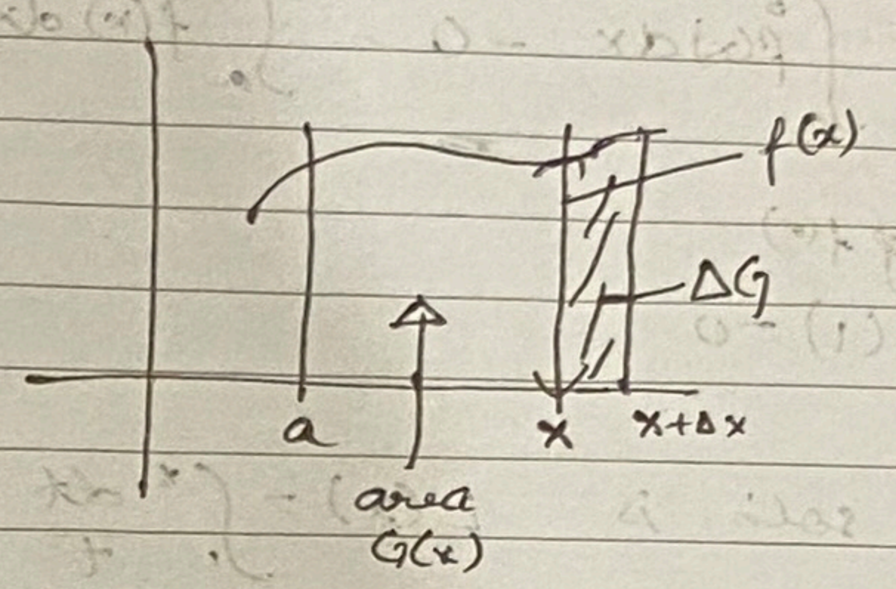
$\int_1^x f(t)\,dt \Rightarrow f(t) = \frac{1}{t^2} = \frac{1}{x^2}$

Check:

$\int_{t=1}^{t=x} t^{-2}\,dt = -t^{-1}\Big|_1^x = \frac{-1}{x} - (-1).$

$\frac{d}{dx}\underbrace{\left(1 - \frac{1}{x}\right)}_{G(x)} = \frac{1}{x^2}$

PROOF OF FTC2



$\Delta G \approx \underset{\text{base}}{\Delta x} \cdot \underset{\text{ht}}{f(x)}$

$\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = f(x)$ f continuous should be.

PROOF of FTC 1

START with F'

Define new $G(x) = \int_a^x f(t)\,dt$

$FTC2 \rightarrow G'(x) = f(x)$

WK $F'(x) = G'(x) \boxed{\Rightarrow} F(x) = G(x) + C$ (C - constant)

uses MVT

Hence.

$= F(b) - F(a)$

$= (G(b) + C) - (G(a) + C)$

$$= G(b) - G(a) = \int_a^b f(x)\,dx - 0 = \int_a^b f(x)\,dx$$

$\int_a^a \to 0.$

log f(x)

~~Use~~ $L'(x) = \frac{1}{x}$; $L(1) = 0$

antideri.

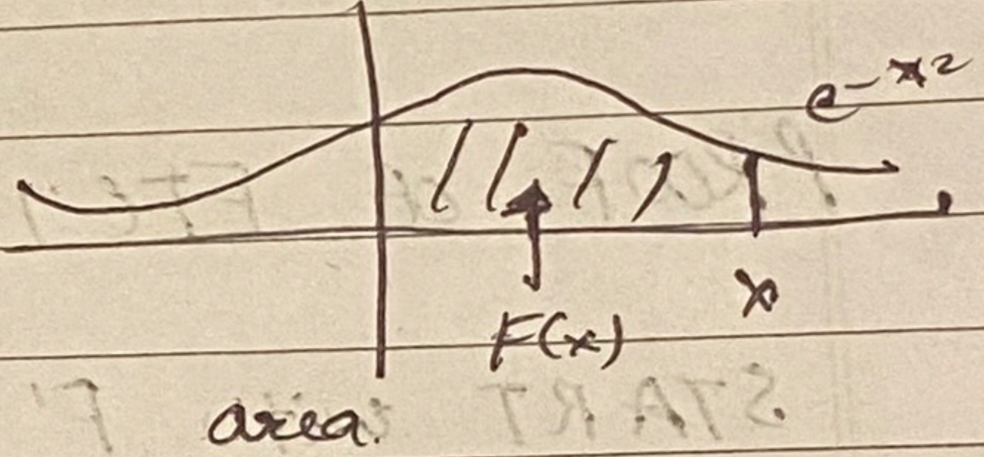
FTC 2: says the soln is $L(x) = \int_1^x \frac{dt}{t}$

$$L(x) = \int_1^x \frac{dt}{t}.$$

We can get "New" funtion.

$y' = e^{-x^2}$, $y(0) = 0$

$F(x) = \int_0^x e^{-t^2}\,dt.$



F(x) cannot be expressed in terms of log, exp, trig....

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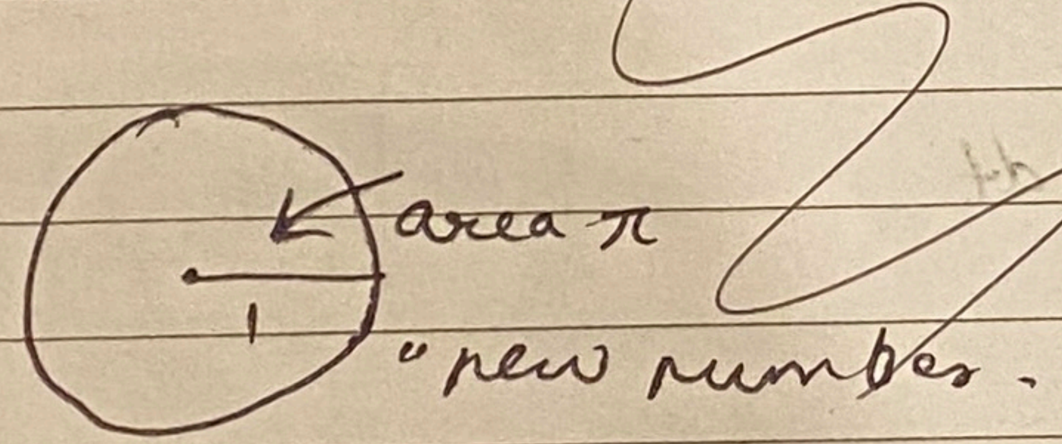

π: not the root of an algebraic eqn with any ratio coeff.
It is not transidental