Lec 4

$$\frac{d}{dt}(\text{constant}) = c\frac{du}{dt} \quad (cu)'$$

$$\frac{d}{dt}(u+v) = \frac{du}{dt} + \frac{dv}{dt}$$

$$(u+v)' = u' + v'$$

Product rule.

$$(uv)' = u'v + uv'$$

$$\frac{d}{dx}(x^n \sin(x)) = nx^{n-1}\sin(x) + x^n \cos(x)$$

proof

$$\Delta(uv)$$

$$= u(x+\Delta x)\,v(x+\Delta x) - u(x)\,v(x)$$

$$= (u(x+\Delta x) - u(x))\,v(x+\Delta x) + u(x)$$

$$+ (v(x+\Delta x) - v(x)).$$

$$= (\Delta u)\,v(x+\Delta x) + u(x)\Delta v$$

$$\frac{\Delta(uv)}{\Delta x} = \frac{\Delta u}{\Delta x}\,v(x+\Delta x) + u\frac{\Delta v}{\Delta x}$$

$$\Delta x \to 0 \quad \frac{d(uv)}{dx} = \frac{du}{dx}\cdot v + u\frac{du}{dx}$$

Quotient rule.

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

$$\frac{\Delta u}{\Delta v} = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$$

$$= \frac{uv + (\Delta u)v - uv - u\Delta v}{(v+\Delta v)v}$$

$$= \frac{(\Delta u)v - u\Delta v}{(v+\Delta v)v} \rightarrow$$

$$= \frac{\frac{\Delta u}{\Delta x}v - u\frac{\Delta v}{\Delta x}}{(v+\Delta v)v}$$

$$\Delta x \to 0 \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dv}\cdot v - u\frac{dv}{dx}}{v \cdot v}$$

$$\frac{\frac{du}{dx}\cdot v - u\frac{dv}{dx}}{v^2}$$

Ex:

$u = 1$

$$\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{-1 \cdot v'}{v^2} = -v^{-2}v'$$

Subeq.

$u = 1,\ v = x^n$

$\frac{d}{dx}x^{-n} = \frac{d}{dx}\left(\frac{1}{x^n}\right) = -x^{-2n} \cdot nx^{n-1}$

$= \cancel{\ }$

$= (-n)x^{(-n-1)}$

Composition Rule

$y = (\sin t)^{10}$

Method : Use new variable.

$x = \sin(t)$, $y = x^{10}$
Inside   outside

Proof

$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$

$\Delta t \to 0$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

chain rule   differentiation of a composition is a product.

Ex. $\frac{d}{dt}(\sin t)^{10}$

$= 10x^9 \cdot \cos(t)$

$= 10(\sin(t))^9 \cos(t)$

$= 10\sin^9 t \cos(t)$

Eg.

$\frac{d}{dt} \sin(10t)$

$x = 10t \qquad y = \sin(x)$

$\frac{dy}{dt} = \cos(x) \cdot 10$

$= 10\cos(10t)$

or briefly

$\frac{d}{dt}\sin(10t)$

$= \cos(10t) \cdot 10$

$= 10\cos(10t)$

Higher Derivatives (Differentiate over and over again)

$u = u(x) \qquad u' \qquad u'' = (u')' \qquad u''' = (u'')'$ third derivative

$u = \sin x, \quad u' = \cos(x) \quad u'' = -\sin(x) \quad u^{3} = -\cos(x)$

$u^4 = -(-\sin(x))$

$u^4 = +\sin(x)$

Other notation : $u' = \left(\frac{du}{dx}\right) = \frac{d}{dx}u$ operator applied to a function

$u'' = \frac{d}{dx}\frac{du}{dx} = \frac{d}{dx}\frac{d}{dx}u = \left(\frac{d}{dx}\right)^2 u = \frac{d^2}{(dx)^2}u = \frac{d^2u}{dx^2}$

$u''' = \frac{d^3u}{dx^3} = D^3u$

Ex. $D^n x^n = ?$ $n = 1, 2, 3 \ldots$

$Dx^n = nx^{n-1}$

$D^2x^n = n(n-1)x^{n-2}$

$D^3x^n = n(n-1)(n-2)x^{n-3}$

$\vdots$

$D^{n-1}x^n = n(n-1) \ldots 2x^1$

$D^n x^n = (n(n-1) \quad 2 \cdot 1 \cdot 1$ constant

n = factorial

$\boxed{D^n x^n = n!}$ constant

$\boxed{D^{n+1} x^n = 0}$

Lec 5

Implicit Diff

Eg

$\boxed{\frac{d}{dx} x^a = ax^{a-1}}$

so far: $a = 0, \pm 1, \pm 2 \ldots$

today $a = m/n$, m & n are integers

$y = x^{m/n}$

$y^n = x^m$

Apply dy/dx to eqn (2)

$\frac{d}{dx} y^n = \frac{d}{dx} x^m$

$\left(\frac{d}{dy} y^n\right) \frac{dy}{dx} = mx^{m-1}$

$ny^{n-1} \frac{dy}{dx} = mx^{m-1}$

$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}}$

$= \frac{m}{n} \frac{x^{m-1}}{(x^{m/n})^{n-1}}$ {law of expo

$= a\, x^{m-1-(n-1)\frac{m}{n}}$

$\boxed{= ax^{a-1}}$

$= m - 1 - (n-1)\frac{m}{n}$

$= m - 1 - m + \frac{m}{n}$

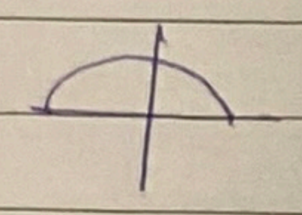
$= 0 - 1 + \frac{m}{n}$

$= -1 + \frac{m}{n}$

$= a - 1$

Eg. 2

$x^2 + y^2 = 1$



explicit

$y^2 = 1 - x^2$, $y = \pm\sqrt{1-x^2}$

$\downarrow y = +\sqrt{1-x^2}$ (positive branch)

$y = (1-x^2)^{1/2}$

$y' = \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-2x)$

$\frac{d}{d()}()^{1/2}$ $\quad a = \frac{1}{2}$ $\quad a - 1 = \frac{-1}{2}$

$= \frac{-x}{\sqrt{1-x^2}}$

Implicit.

$\frac{d}{dx}(x^2 + y^2 = 1)$

$2x + 2yy' = 0$

$y' = \frac{-2x}{2y} = \frac{-x}{y}$

$\cancel{y}$

$= \frac{-x}{\sqrt{1-x^2}}$ $\qquad \frac{-x}{y}$

$y = \sqrt{1-x^2}$

$= \frac{-x}{y}$ $\qquad \frac{-x}{y}$

Eg 3

$y^4 + xy^2 - 2 = 0$

explicit.

$y^2 = \frac{-x \pm \sqrt{x^2 - 4(-2)}}{2}$

$y = \pm\sqrt{\frac{-x \pm \sqrt{x^2+8}}{2}}$

Implicit.

$4y^3y' + y^2 + x(2yy') - 0 = 0$

$(4y^3 + 2xy)y' = -y^2$

$y' = \frac{-y^2}{4y^3 + 2xy}$

At $x=1, y=1$ solve.

$y^4 + xy^2 - 2 = 0$

At (1,1) along the curve

$\text{slope} = \dfrac{-1}{4 \cdot 1^3 + 2 \cdot 1 \cdot 1} = \dfrac{-1}{6}$

But at (say) $x=2$
we're struck using (-)
to find y

Derivatives.
Inverse function

Ex.

$y = \sqrt{x} \quad x>0 \quad y^2 = x$

$f(x) = \sqrt{x}, \quad g(y) = x$

$g(y) = y^2$

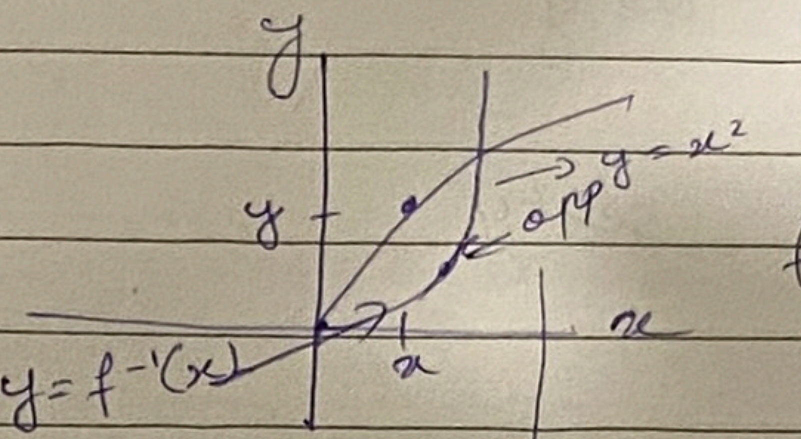
In general.

$y = f(x), \; g(y) = x$

$g(f(x)) = x, \quad g = f^{-1}$

$f = g^{-1}$

Picture of f and f^{-1} on the same graph



$g(y) = x$

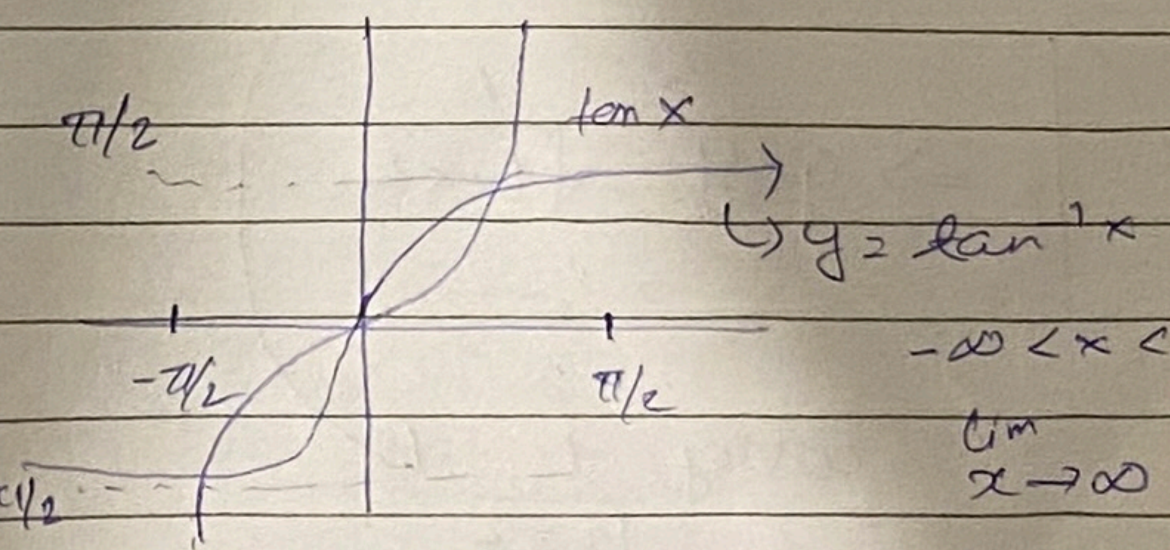
$g(x) =$
Ex change x & y

Implicit diff allows us to find the derivative of any inverse f'n provided we know the derivative of the function.

Example   inverse.

$y = \tan^{-1} x$ [arctan x]

use $\boxed{\tan y = x}$



$-\infty < x < \infty$

$\lim_{x \to \infty} \tan^{-1} x = \pi/2$

Recall.

$$\frac{d}{dy} \tan y = \frac{d}{dy} \frac{\sin y}{\cos y}$$

$$\text{after} = \frac{1}{\cos^2 y} = \sec^2 y$$

$$\frac{d}{dx} (\tan y = x)$$

$$\left(\frac{d}{dy} \tan y\right) \frac{dy}{dx} = 1$$

$$\frac{1}{\cos^2 y} y' = 1$$

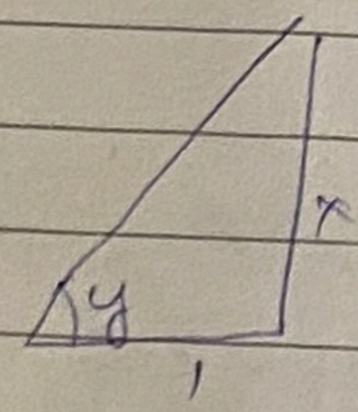
$$y' = \cos^2 y$$

→ correct, but way, too complicated.

$$\frac{d}{dx} \tan^{-1} x = \cos^2(\tan^{-1} x)$$

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}}$$

$\tan y = x$



$$\Rightarrow \cos y = \frac{1}{\sqrt{1+x^2}}$$

$$= \cos^2 y = \frac{1}{1+x^2}$$

$y = \sin^{-1} x$

$\sin y = x$

$(\cos y) y' = 1$

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{d} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$