Lec (5)

Differentials

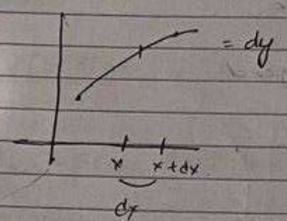
$y = f(x)$

diff of y: $dy = f'(x)dx$.

or $\frac{dy}{dx} = f'(x)$

Leibniz interpretation of derivative as a ratio of infinitesimals

Use in linear approximation.



dx replaces Δx

dy replaces Δy

Example.

$(64.1)^{1/3} \approx ?$

$y = x^{1/3}$, $dy = \frac{1}{3}x^{-2/3}dx$

At $x = 64$, $y = 64^{1/3} = 4$

$dy = \frac{1}{3}(64)^{-2/3}\,dx$

$= \frac{1}{3} \cdot \frac{1}{16}\,dx = \frac{1}{48}\,dx$

$x = 64$, $x + dx = 64.1$

$\Rightarrow dx = \frac{1}{10}$

$(64.1)^{1/3} \approx y + dy = 4 + \frac{1}{48}dx = 4 + \frac{1}{480} = \frac{1921}{480} \approx 4.002$

$\uparrow$ $y + \Delta y$. $\uparrow$ approx. $dy = \frac{1}{48}dx$

Compare to previous notation

(Review of linear Approx)

$f(x) = f(a) + f'(a)(x-a)$

$a = 64$, $f(x) = x^{1/3}$

$f(a) = f(64) = 4$

$f'(a) = \frac{1}{3}a^{-2/3} = \frac{1}{48}$

$x^{1/3} \approx 4 + \frac{1}{48}(x - 64)$

$64.1^{1/3} = 4 + \frac{1}{48}\left(\frac{1}{10}\right)$

$= 4 + \frac{1}{480}$

= same answer new notation.

ANTI Derivatives. ← Integral sign

$G(x) = \int g(x)\,dx$

antiderivative of g = indefinite integral of g

anti derivative
Integral

1. $\int \sin x\, dx = -\cos x$

$G(x) = -\cos x + c$ (c. constant.)

$G'(x) = \sin x$

2. $\int x^a dx = \frac{x^{a+1}}{a+1}$ ← $a \neq -1$.

$d(x^{a+1}) = (a+1)x^a dx$ (ok all

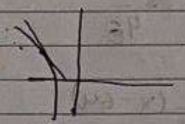
3. $\int \frac{dx}{x} = (\ln|x|) + c$

$x > 0$ ✓

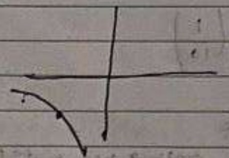
check $x < 0$ $\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x)$

$x < 0 = \frac{1}{-x} \frac{d}{dx}(-x) \cdot \frac{-1}{-x} = \frac{1}{x}$

$y = \ln(-x)$
$x < 0$



$y' = \frac{1}{x}$



$x < 0$.

$\int \sec^2 x\, dx = \tan x + c$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

$\int \frac{dx}{1+x^2} = \tan^{-1} + c$

Uniqueness of antiderivatives, up to a constant.
Thm
If $F' = G'$ then

$F(x) = G(x) + c$

Proof.
If $F' = G'$ the $(F-G)' = F' - G' = 0$.
$F(x) - G(x) = c$, $\boxed{\text{constant}}$

.) $F(x) = g(x) + c$

Ex $\int x^3 (x^4+2)^5 dx$

Method of subs, tailor made for diff notation

$u = x^4 + 2$, $du = 4x^3 dx$

$\int x^3 (x^4+2)^5 dx$

$\int (x^4+2)^5 x^3 dx$
$\int u^5 \cdot \frac{1}{4} du$

$= \int \frac{u^5 du}{4} = \frac{1}{24} u^6 + c = \frac{1}{24} (x^4+2)^6 + c$

Ex2.

$$\int \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2} + c.$$

subs:

$u = 1+x^2$

$du = 2x\,dx$

$\therefore \int u^{-1/2}\,du$

recommended method: advanced guessing

answer should be.

$$\frac{d}{dx}(1+x^2)^{1/2} = \frac{1}{2}(1+x^2)^{-1/2} 2x$$

$$= \frac{x}{\sqrt{1+x^2}}$$

Ex3

$$\int e^{6x}\,dx = \frac{1}{6}e^{6x} + C$$

guess e^{6x}

$\frac{d}{dx} e^{6x} = 6e^{6x}$ Also OK in subs

Ex

$$\int x e^{-x^2}\,dx = \frac{-1}{2}e^{-x^2} + c.$$

$$\frac{d}{dx}e^{-x^2} = e^{-x^2}(-2x)$$

Ex.

$$\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + c_1$$

$$\frac{d}{dx}\sin^2 x = 2\sin x \cos x$$

But also

$$\frac{d}{dx}\cos^2 x = 2\cos x(-\sin x)$$

Another ans.

$$\int \sin x \cos x\,dx = -\frac{1}{2}\cos^2 x + c_2$$

for both answer

$$\frac{1}{2}\sin^2 x - \left(-\frac{1}{2}\cos^2 x\right)$$

$$\frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$$

$$c_1 - c_2 = \frac{-1}{2}$$

Ex

$$\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\frac{dx}{x} = \int \frac{du}{u} = \ln u + c$$

$$\int \frac{1}{u}\,du = \ln[\ln x] + c$$

sub $u = \ln x$

$$du = \frac{dx}{x}$$

Horizon Date Page

$\boxed{\text{Lec 16}}$

Differential Equation.

Ex: $\frac{dy}{dx} = f(x)$ ← solved before

$y = \int f(x)\,dx$

$\boxed{\text{Ex 2}}$

$\left(\frac{d}{dx} + x\right) y = 0$

annihilator operator in quantum mechanism

$\frac{dy}{dx} = -xy$

$\frac{dy}{y} = -x\,dx$

$\int \frac{dy}{y} = -\int x\,dx$

$\ln y = -x^2/2 + c$

$e^{\ln y} = e^{-x^2/2 + c}$

$y = Ae^{-x^2/2}$ $(A = e^c)$

soln

$y = ae^{-x^2/2}$

$\frac{dy}{dx} = a \frac{d}{dx} e^{-x^2/2}$

$= a(-x)e^{-x^2/2}$

$= -xy$

Horizon Date Page

Separation of variables.

$\frac{dy}{dx} = \underbrace{f(x)}_{x} \cdot \underbrace{g(y)}_{y}$.

$\frac{dy}{g(y)} = f(x)\,dx$. $H(y) = \int \frac{dy}{g(y)}$; $F(x) = \int f(x)\,dx$

any

$-x^2/2$

implicit

$H(y) = F(x) + c$

$y = H^{-1}(F(x) + c)$.

Remarks.

could have written.

$\ln|y| = -x^2/2 + c$ $(y \neq 0)$

$|y| = Ae^{-x^2/2}$

$y = \pm Ae^{-x^2/2}$ $a = \pm A$

Still leaves out

$y = 0$

not surprising

• $\ln y + c_1 = -\frac{x^2}{2} + c_2$

rewrite.

$\ln y = -\frac{x^2}{2} + c_2 - c_1$

two constants.

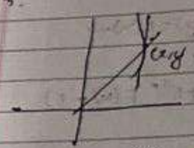
can always be combined.

Ex

$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)dx$

$y = \int dy = \int f(x)dx$

Ex 3. slope of tangent line = twice slope of ray from origin.



$\frac{dy}{dx} = 2\frac{y}{x}$

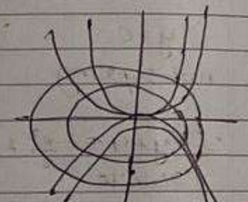
$\int \frac{dy}{y} = \int \frac{2dx}{x}$

$\ln y = 2\ln x + c$

$e^{\ln y} = e^{2\ln x + c}$ Solution

$y = Ax^2$

$(e^{\ln x})^2 = x^2$. $A = e^c$



$y = ax^2$

$\frac{dy}{dx} = 2ax = \frac{2ax^2}{x} = \frac{2y}{x}$ $y = a^2$ a any constant.

work for $a>0$, $a<0$, $a=0$

Warning $\frac{dy}{dx} = \frac{2y}{x}$

Ex: 4

Find curves $\perp$ to the parabolas.

New diff eq.

$\frac{dy}{dx} = \frac{-1}{\text{slope of tangent to parabola}} = \frac{-1}{2ky(x)}$

$= -\frac{x}{2y}$

Separate variables

$2y\,dy = -x\,dx$
antiderivative

$y^2 = -\frac{x^2}{2} + c$

Solution are

$\frac{x^2}{2} + y^2 = c$
$= a^2$

ellipses.

a
$\sqrt{2}a$

explicit.

$y = +\sqrt{a^2 - x^2/2}$ - top half
$y = -\sqrt{a^2 - x^2/2}$ - bottom half

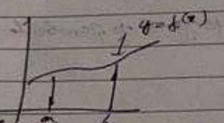
problem at $y=0$ solution.

Info: about f' (of f'')?
tells about info about f

lec 18

Unit III

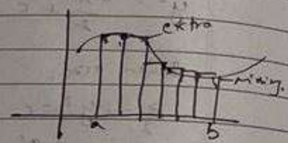
Definite Integrals.
Find area under a curve.



Find the area under curve.

$\int_a^b f(x)\,dx$.

To compute the area.



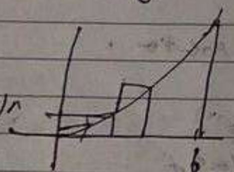
1. divide into rectangles
2. add up areas.
3. take limit as rectangles get thin

Ex (1)

$f(x) = x^2$; $a = 0$

b arbitrary

2b/n



divide into pieces

base length b/n (all equal intervals)

x	$f(x)$
b/n	$(b/n)^2$
$2b/n$	$(2b/n)^2$
$3b/n$	$(3b/n)^2$

Sum of areas of $\square$s $\quad \frac{b^3}{n^3}\sum_{i=1}^{n} i^2$

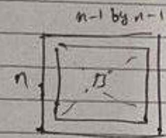
$(b/n)(b/n)^2 + \frac{b}{n}\left(\frac{2b}{n}\right)^2 \cdots + \frac{b}{n}\left(\frac{nb}{n}\right)^2 = \sum_{i=1}^{n} \frac{b}{n}\left(\frac{ib}{n}\right)^2$

$= \left(\frac{b}{n}\right)^3 (1 + 2^2 + 3^2 + \dots + n^2) = b^3\left(\frac{1^2 + 2^2 \dots + n^2}{n^3}\right)$

$\approx \frac{1}{3}n^3$

$\frac{1}{3}n^3 < 1^2 + 2^2 + 3^2 + \dots + n^2 < \frac{1}{3}(n+1)^2(n+1)$

n-1 by n-1



top view

vol inside $\frac{1}{3}(b)(h^2)$

$= \frac{1}{3}n^2 \cdot n$

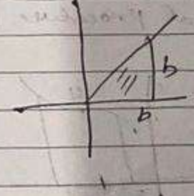
Divide by n^3: $\frac{1}{3} < \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3} < \frac{1}{3}\frac{(n+1)^3}{n^3}$

$= \frac{1}{3}\left(1 + \frac{1}{n}\right)^3$

Total area under x^2:

$\int_a^b x^2\,dx = \frac{1}{3}b^3$

Ex $\quad f(x) = x$.



Area $= \frac{1}{2} b \cdot b = \frac{1}{2}b^2$

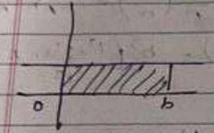
Ex: 3

$f(x) = 1$

$f(x) = 1$

Area $= b \cdot 1$

$= b.$



Pattern:

$f(x)$	$\int_0^b f(x)dx$
x^2	$b^3/3$
$x = x^1$	$b^2/2$
$1 = x^0$	$b = b^1/1$

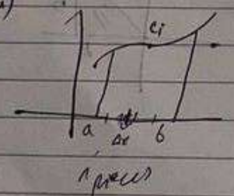
guess $f(x) = x^3$

$$\int_0^b x^3 dx = b^4/4$$

Notations (Riemann Sums)

General procedure for def integrals.

$f(c_i)$



$$\Delta x = \frac{b-a}{n}$$

Pick any height of f in each.

$$\sum_{i=1}^{n} \underbrace{f(c_i)}_{ht} \underbrace{\Delta x}_{base} \xrightarrow[\Delta x \to 0]{} \int_a^b f(x)dx$$

Integrals as commulative sums.

t time in years.

$f(t)$ \$/yr borrowing rate.

$$\Delta t = \frac{1}{365}$$

In day 45 $\left(t = \frac{45}{365}\right)$

$$f\left(\frac{45}{365}\right)\Delta t = f\left(\frac{45}{365}\right)\frac{1}{365}$$

$$\sum_{i=1}^{365} f\left(\frac{i}{365}\right)\Delta t = \int_0^1 f(t)dt$$

total borrowed.

Owe the bank.

borrow interest rate.

P after time T

You owe. Pe^{rT} r rate. (0.05/yr)

$$= f\left(\frac{i}{365}\right)\Delta t)\, e^{r\left(1-\frac{i}{365}\right)} \quad T = 1 - \frac{i}{365}.$$

$$= \int_0^1 e^{r(1-t)} f(t)\, dt$$