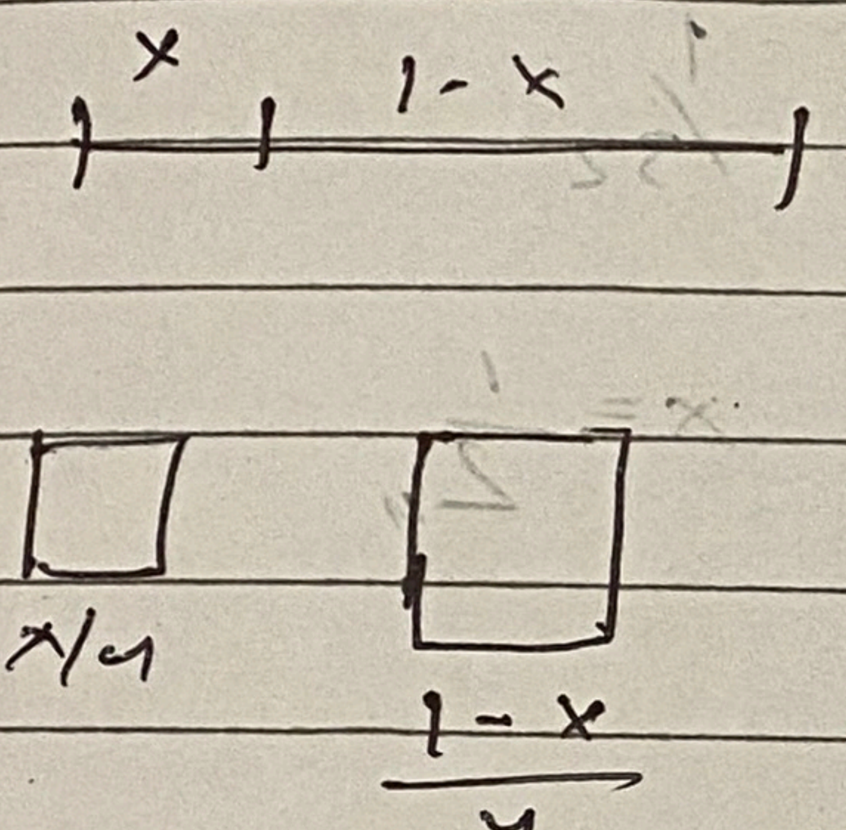
lecture 12

MAX/MIN

wire length 1 cut into 2 pieces.
Each wire
enclose a square.

Find largest area enclosed

Draw diagram
name variables.



$$A = \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2.$$

Find critical pts.

$A' = 0$

$A' = \frac{x}{8} - \frac{(1-x)}{8} = 0$

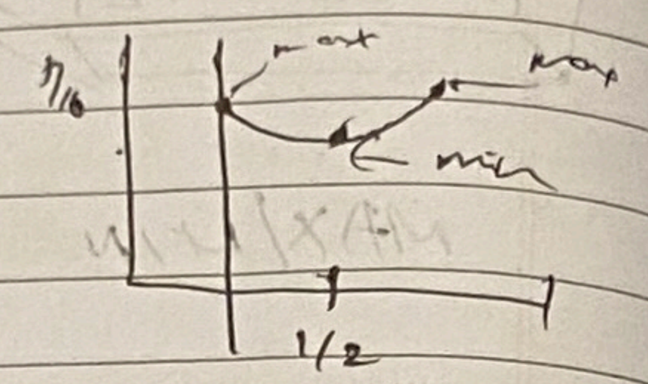
if $x = 1 - x \Leftrightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

$x = \frac{1}{2}$ crit pt

value

$A\left(\frac{1}{2}\right) = \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{1}{32}$

End point $0<x<1$



$A(0^+) = 0 + \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$A(1^-) = \left(\frac{1}{4}\right)^2 + 0 = \frac{1}{16}$

min point $= (1/2, 1/32)$

Least area enclosed $1/32$ when $x = 1/2$ (equal squares)

Most when only one square]

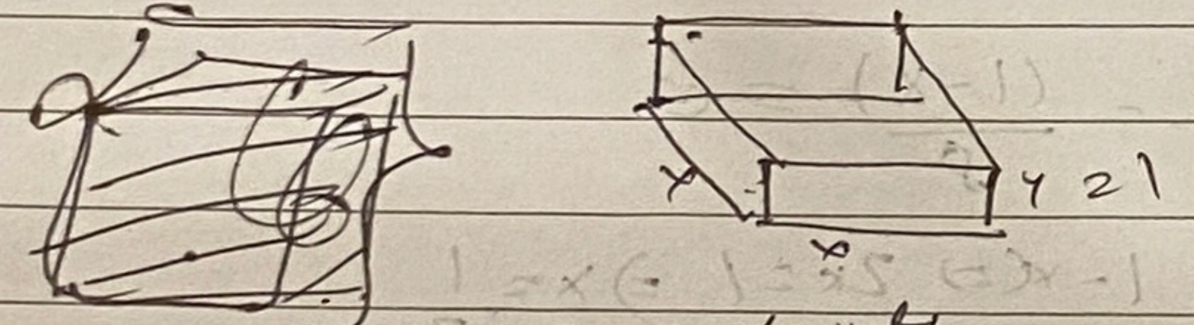
What is the minimum? $1/32$

minimum value

Where is the min point $x = \frac{1}{2}$

Ex2.

Consider the box without a top with least surface area for a fixed volume (square bottom)



$V = x^2 y$ ← constraint

$A = x^2 + 4xy$

bottom sides

$x = 2$

$y = \frac{V}{x^2}$

$A = x^2 + 4x\left(\frac{V}{x^2}\right)$

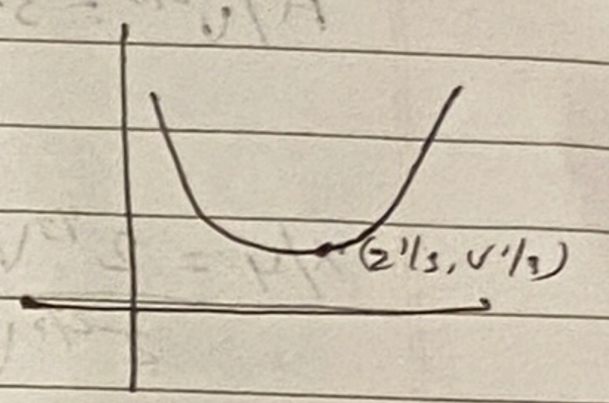
$= x^2 + 4V/x$

$A = x^2 + \frac{4V}{x}$

$A' = 2x - \frac{4V}{x^2} = 0$

$2x = \frac{4V}{x^2} \Rightarrow x^3 = 2V$

$\boxed{x = 2^{1/3} V^{1/3}}$ critical point



Ends: $0 < x < \infty$

$A(0^+) = x^2 + \frac{4V}{x}\Big|_{x=0^+} = \infty$

$A(\infty) = x^2 + \frac{4V}{x}\Big|_{x=\infty} = \infty$

Alternative to checking ends is 2nd derivative test.

$A' = 2x - \frac{4V}{x^2}, \quad A'' = 2 + \frac{8V}{x^3} > 0$

$x = 2^{1/3} V^{1/3}$

$y = \frac{V}{(2^{1/3} V^{1/3})^2} = 2^{-2/3} V^{1/3}$

$A^2 = \left(2^{4/3}V^{1/3}\right) + \frac{4V}{2^{1/3}V^{1/3}}$

$= 3 \cdot 2^{1/3} V^{2/3}$

More meaningful Answer

~~dimeslers~~
dimensionless variable

$A / V^{2/3} = 3 \cdot 2^{1/3}$

$x/y = \frac{2^{1/3}V^{1/3}}{2^{-2/3}V^{1/3}} = 2$

optimal shape

Ex by implicit diff.

$V = x^2 y$, $A = x^2 + 4xy$

goal min of A with V constant.

$\frac{d}{dx}(V = x^2 y) = 0 = 2xy + x^2 y'$

$y' = \frac{-2xy}{x^2} \Rightarrow \frac{-2y}{x}$

$\frac{dA}{dx} = 2x - 4y + 4xy'$

$= 2x + 4y + 4x\left(\frac{-2y}{x}\right) = 0$

$2x + 4y - 8y = 0$

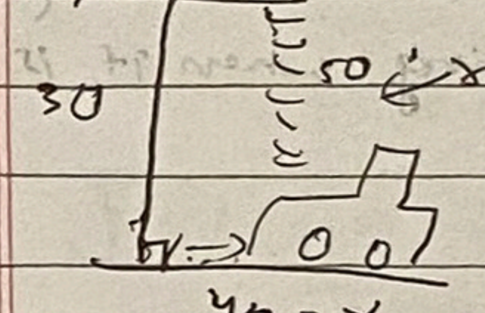
$2x = 4y$

$\boxed{x/y = 2 \text{ faster}}$

disadvantage: did not check whether this critical point is max point, min point

Related Rates

$\boxed{\text{Police}}$ $\frac{dD}{dt} = -80 \text{ ft/sec}$ approach



95 ft/sec (speed)
≈ 65 mph limit

t time in sec. $\frac{dx}{dt}$

$\boxed{\text{Lecture 13}}$

$x^2 + 30^2 = D^2$

$\frac{dD}{dt} = -80$

could solve for $x = \sqrt{D^2 - 30^2}$

Easiest, use implicit diff

$$x \frac{dv}{dt} = 2\pi \frac{dp}{dt}$$

$$2.40 \frac{dx}{dt} \cdot 2.50\left(\frac{dx}{dt}\right)$$

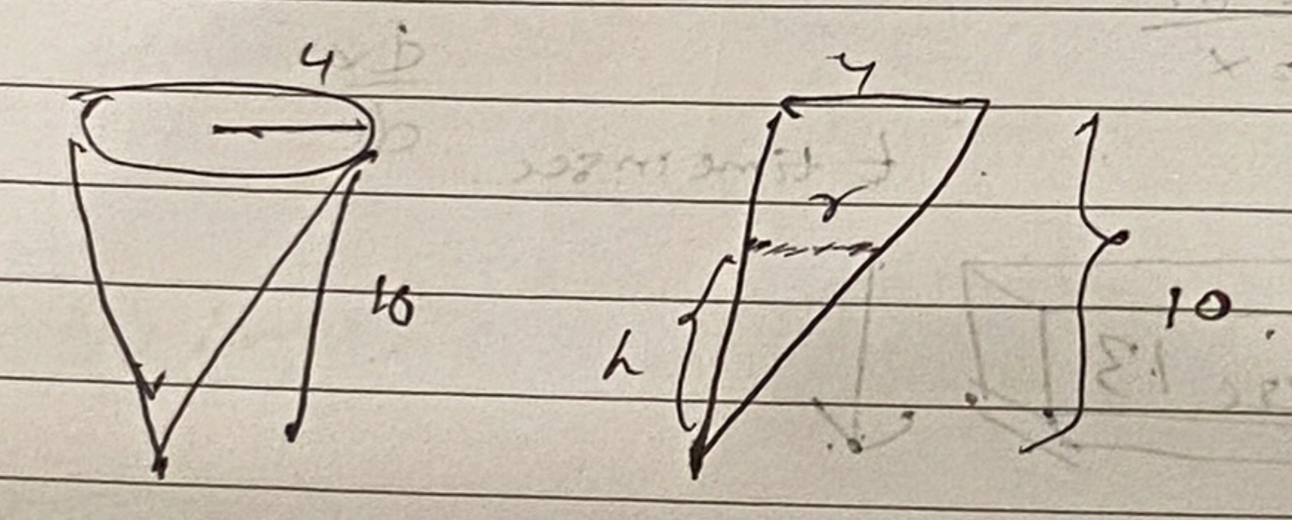
$$2 \cdot 40 \left(\cancel{\tfrac{dx}{dt}}\right) = 2.50(-80)$$

$$\Rightarrow \cancel{80 = 250} \quad \therefore \boxed{\frac{dx}{dt} = -100 \text{ ft/sec}}$$

$\frac{dx}{dt}$

(Ex) 2.

A conical tank with top of radius 4ft depth 10ft is being filled at 2 cu ft/min. How fast is the water rising when it is at depth 5 feet.



$$\frac{r}{h} = \frac{4}{10} \quad \text{Similar triangles}$$

$$V = \frac{1}{3} \underbrace{\pi r^2}_{\text{base}} \underbrace{h}_{\text{ht.}} \qquad \frac{dV}{dt} = 2$$

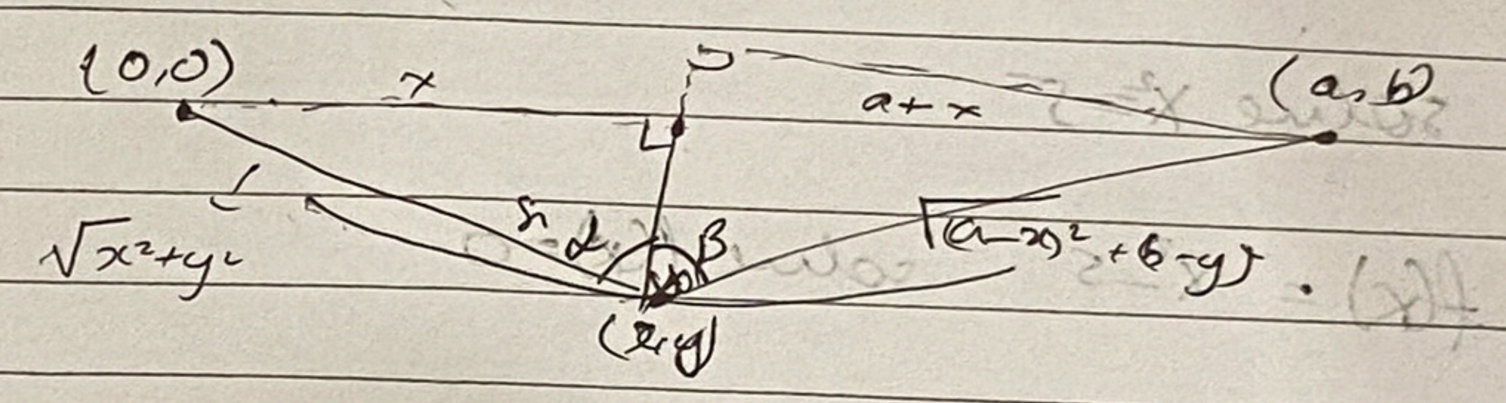
What is $\frac{dh}{dt}$ when $h = 5$?

$$r = \frac{2}{5}h, \quad V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$$

$$2 = \frac{dV}{dt} = \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2 \frac{dh}{dt}$$

$$2 = \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3 \cdot 5^2 \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{1}{2\pi} \text{ ft/min}}$$



$$\sqrt{x^2+y^2} + \sqrt{(a-x)^2+(b-y)^2} = L \text{ (constant)}$$

constant $\sin\alpha$ $\sin\beta$

Find the least y.

$y = y(x)$

From diagram bottom point is $y' = 0$ (critical point)

use Implicit diff on constant eq. $y' = 0$

$$\frac{2}{2}\frac{x + yy'}{(x^2+y^2)^{1/2}} - \frac{(a-x) + (b-y)y'}{\sqrt{(a-x)^2+(b-y)^2}} = 0$$

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{a-x}{\sqrt{(a-x)^2+(b-y)^2}}$$

$\sin\alpha = \sin\beta$

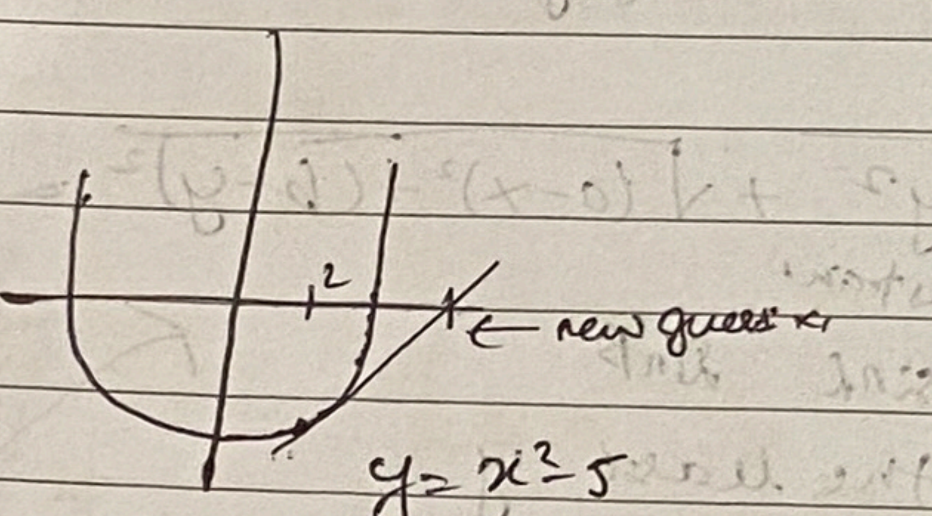
$\Rightarrow \alpha = \beta$

Newton Method

Example

solve $x^2 = 5$

$f(x) = x^2 - 5$ solve $f(x) = 0$



Start with initial guess with $x_0 = 2$.

Tangent line.

$y - y_0 = m(x - x_0)$

x_1 is the x-intercept.

$0 - y_0 = m(x_1 - x_0)$

$$\frac{-y_0}{m} = x_1 - x_0$$

$$x_1 = x_0 - \frac{y_0}{m}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_0 = 2$, $f(x) = x^2 - 5$

$f'(x) = 2x$

$$x_1 = x_0 - \frac{x_0^2 - 5}{2x_0} = x_0 - \frac{1}{2}x_0 + \frac{5}{2x_0}$$

$$x_1 = \frac{1}{2}x_0 + \frac{5}{2x_0}$$

$$x_1 = \frac{1}{2}\cdot 2 + \frac{5}{4} = \frac{9}{4}$$

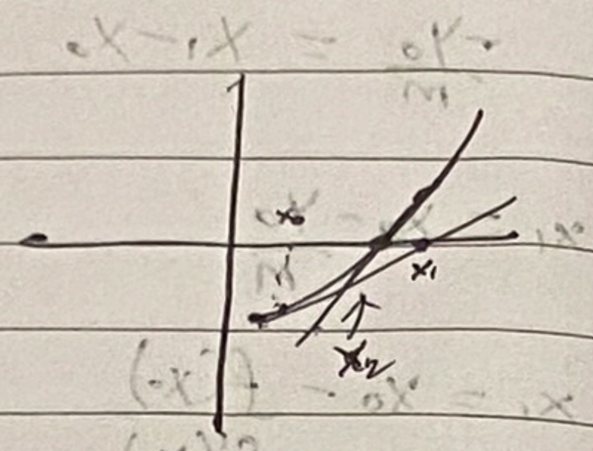
$$x_2 = \frac{1}{2}\cdot\frac{9}{4} + \frac{5}{2}\cdot\frac{4}{9} = \frac{161}{72}$$

$$x_3 = \frac{1}{2}\cdot\frac{161}{72} + \frac{5}{2}\cdot\frac{72}{161}$$

n	$\sqrt{5} - x_n$
0	2×10^{-1}
1	10^{-2}
2	4×10^{-5}
3	4×10^{-10}

[lec 14]

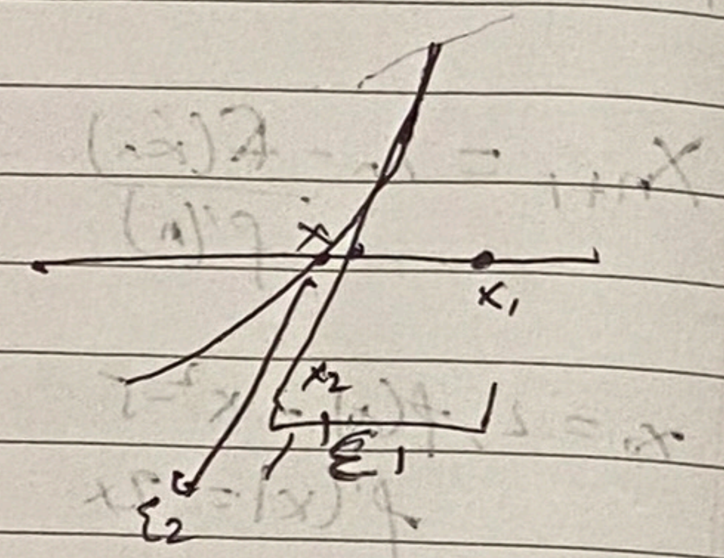
Newton's Method Review



$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

repeat / iterate.

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$



$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Error Analysis. $\varepsilon_2 \sim \varepsilon_1^2$

$\varepsilon_1 = |x - x_1|$

ε_0	ε_1	ε_2	ε_3	ε_4
10^{-1}	10^{-2}	10^{-4}	10^{-8}	10^{-16}

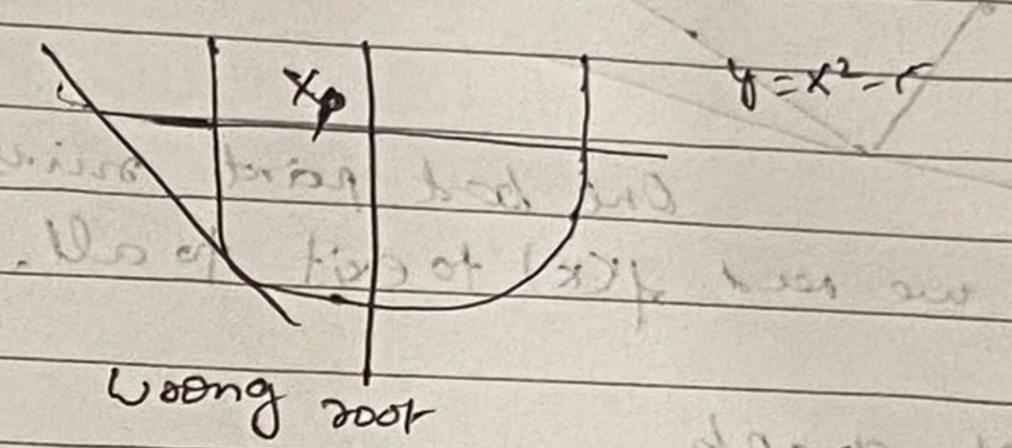
$\varepsilon_2 = |x - x_2|$

digits of accuracy doubles

lastly. $\varepsilon_n = |\sqrt{5} - x_n|$

Newton's method works (very) well if f' (not small) and $|f''|$ not too big and x_0 ~~not nearby~~ start nearby x.

Ways the method can fail



MEAN VALUE THEORM (MVT)

If you go from Boston to LA (3000 mi) in 6 hrs then at some time you are going at average speed.

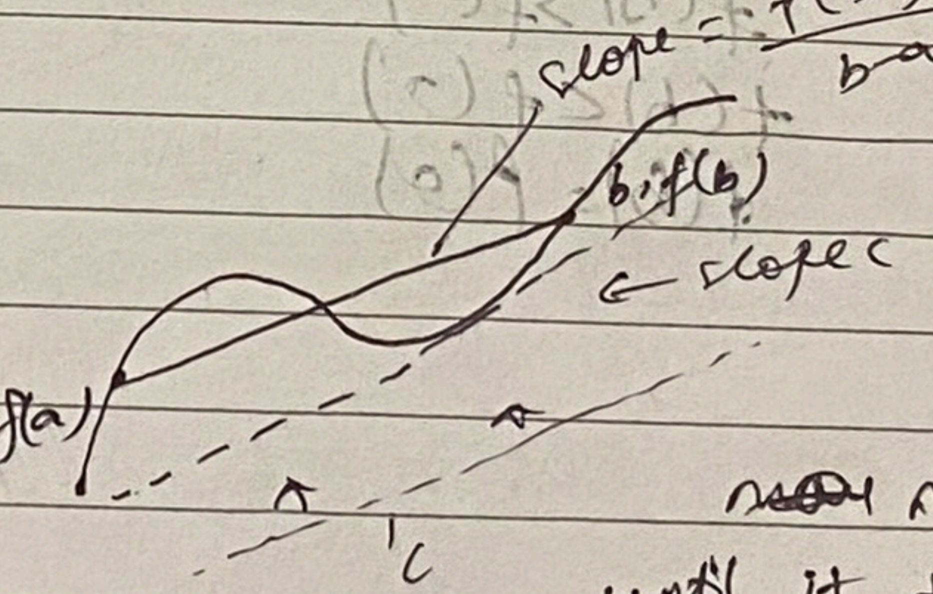
$\frac{5000}{6} = 300$

THEORM

$\frac{f(b) - f(a)}{b-a} = f'(c)$ provided f is differentiable in $a < x < b$ and continuous

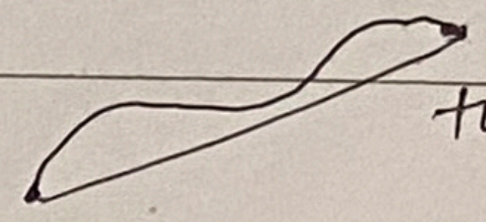
for some c, $a < c < b$

PROOF (MVT)

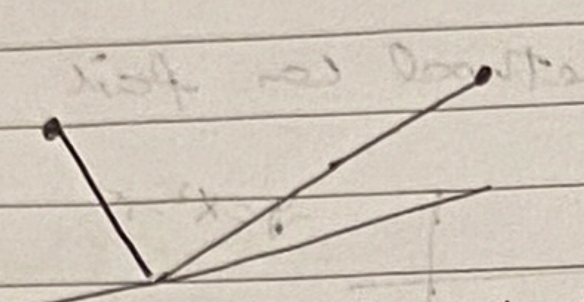


now move up in parallel until it touches.

If it does not touch.



then being parallel.



One bad point ruins the proof we need $f(x)$ to exist to all.

App of graph.

1. If $f'>0$, then f is increasing
2. If $f'<0$ then f is decreasing
3. If $f'=0$ then f is constant.

$\boxed{\text{PROOF}}$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$f(b)-f(a) = f'(c)(b-a)$ $\quad a<b$, $(b-a)>0$

$f(b) = f'(c)(b-a) + f(a)$

1. $f'(c)>0 \Rightarrow$ $\quad f(b)>f(a)$
2. $f'(c)<0 \Rightarrow$ $\quad f(b)<f(a)$
3. $f'(c)=0$ $\quad f(b)=f(a)$

Comparison w/ Liner Approx.

$\frac{\Delta f}{\Delta x} \approx f'(a)$ $\quad$ b near a

$\frac{\Delta f}{\Delta x} = f'(c)$ $\quad$ some c between a and b.

$$\min f' \le \frac{f(b)-f(a)}{b-a} = f'(c) \le \max f'$$

on $a \le x \le b$.

MVT.

min $\le$ avg speed $\le$ max

linear approx.

avg $\approx$ initial speed.

INEQUALITIES

1. $e^x \geq 1+x \quad (x>0)$

Proof $f(x) = e^x - (1+x)$

Start $f(0) = e^0 - (1+0) = 0$
and $\quad f'(x) = e^x - 1, \; x>0$

$\therefore f(x) > f(0) \quad$ for $x>0$

$e^x - (1+x) > 0 \Rightarrow e^x > 1+x$