Horizon
Date
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MIT
18.01
unit 1
Differentiation:

12/3/2025

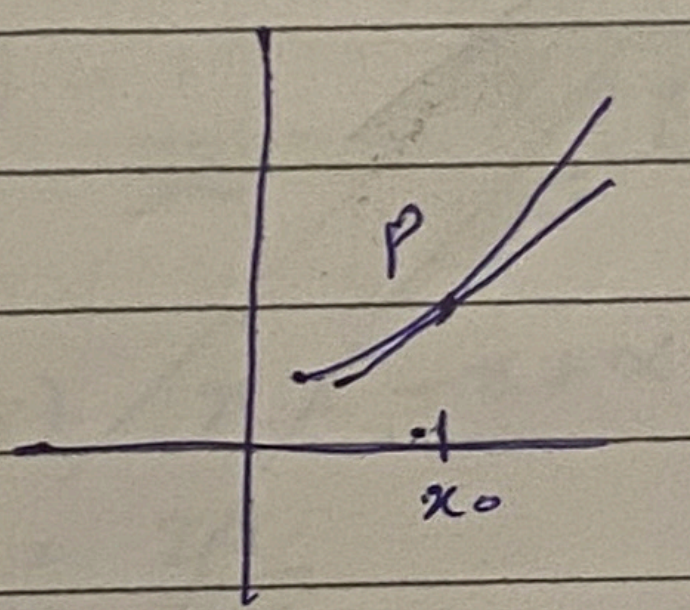
A What is a Derivative?
- geom interpretation
- physical interception
- important to all measurements.
(Science, Eng, Eco, political science)

B. How to Differentiate any $f'(x)$ you know

$\frac{d}{dx}$ exarctanx = ?

GEOM INTERPRETATION.

Find the tangent line to $y = f(x)$ at $P = (x_0, y_0)$



) for tangent of line

$y - y_0 = m(x - x_0)$

point $y_0 = f(x_0)$

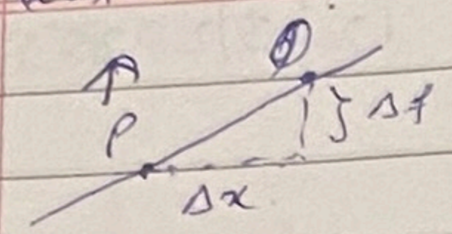
slope $m = f'(x_0)$

DEFIN.

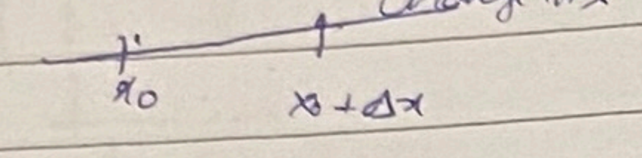
$f'(x_0)$, the derivative of f at x_0, is the slope of the tangent line to
$y = f(x)$ at P.

TANGENT LINE = LIMIT of SECANT LINES PQ AS Q→P
P (fixed)

$(x_0, f(x_0))$ $\to (x_0+\Delta x, f(x_0+\Delta x))$



delta x



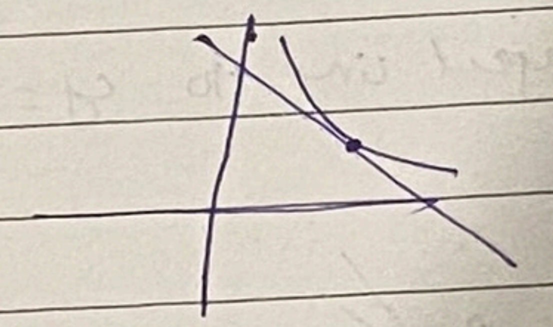
slope

$m = \lim_{x\to 0} \underbrace{\frac{\Delta f}{\Delta x}}_{\text{slope}}$ of secant PQ

$$\boxed{f'(x_0) = \lim_{\Delta x\to 0} \underbrace{\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}_{\text{difference quotient}} = m}$$

Eg. 1.

$f(x) = \frac{1}{x}$



$\cancel{\frac{\Delta f}{\Delta x} = f(x_0+\Delta x) -}$

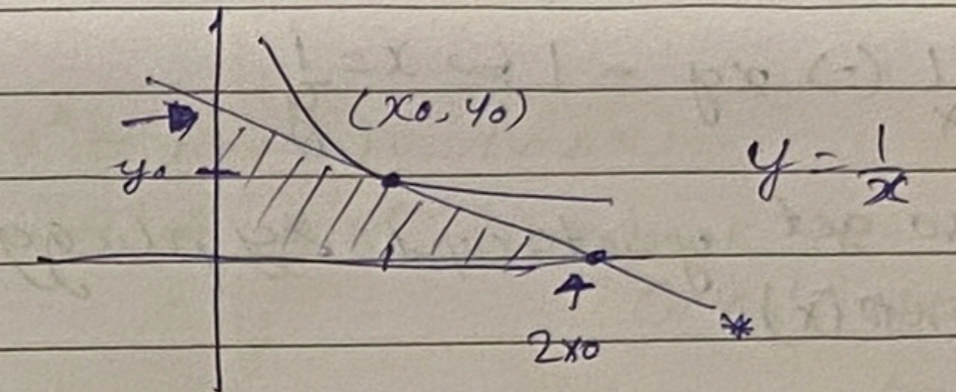
$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x}$

$\frac{1}{\Delta x}\left(\frac{x_0 - (x_0+\Delta x)}{(x_0+\Delta x)x_0}\right) = \frac{1}{\Delta x}\left(\frac{-\Delta x}{(x_0+\Delta x)x_0}\right) =$

$= \frac{-1}{(x_0+\Delta x)x_0} \xrightarrow{\Delta x\to 0} \frac{-1}{x_0^2}$

$f'(x_0) = \frac{-1}{x_0^2}$

FIND AREAS OF TRIANGLES Enclosed by Axes and tangent to $y = \frac{1}{x}$



$(*)$ $y - y_0 = \underbrace{\frac{-1}{x_0^2}}_{\text{only calculus}} (x - x_0)$

Find x-intercept.

$y = 0$

$0 - \frac{1}{x} = \frac{-1}{x^2}(x - x_0) =$

$\Rightarrow \cancel{-\frac{1}{x} = \frac{-x + x_0}{x^2}}$ $= \frac{-1}{x_0} = \frac{-x}{x_0^2} + \frac{1}{x_0} = \frac{x}{x_0^2} = \frac{1+1}{x_0}$

$\cancel{\Rightarrow x = -x + x_0}$

$\cancel{= \frac{2x}{x_0}}$ $= \frac{x}{x_0^2} = \frac{2}{x_0}$

$\Rightarrow x = 2x_0$

Shortcut to y-intercept.

use symmetry.

$y = 2y_0$

since the symmetry of the

$x, y \to y, x$

Symmetry explanation.

$y = \frac{1}{x} \Leftrightarrow xy = 1 \Leftrightarrow x = \frac{1}{y}$

Can also get y intercept by plugging $x=0$ into (x)

Area of triangle =

$\frac{1}{2} \times 2x_0 \times 2y_0 = 2x_0y_0 = 2$

MORE Notations

$y = f(x),\ \Delta y = \Delta f$

$$\underbrace{f'}_{\text{Newton}} = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f = \frac{d}{dx}y}_{\text{Leibniz}}$$

(omits x_0)

Example 2:

$f(x) = x^n,\ n = 1, 2, 3 \ldots$

$\frac{d}{dx} x^n = ?$ x is fixed, Δx moves

$$\frac{\Delta f}{\Delta x} = \frac{(x_0+\Delta x)^n - x^n}{\Delta x}$$

Power of sum
Binomial theorem.

$(x+\Delta x)^n = (x+\Delta x)\ldots(x+\Delta x)$

$= x^n + n x^{n-1}\Delta x + \text{junk}$

$O(\Delta x^2)$ terms of order $\Delta x^2, (\Delta x)^3$

$$\frac{\Delta f}{\Delta x} = \frac{1}{\Delta x}\left((x+\Delta x)^n - x^n\right)$$

$$= \frac{1}{\Delta x}\left(\cancel{x^n} + nx^{n-1}\Delta x + O((\Delta x)^2) - \cancel{x^n}\right)$$

$$= \frac{1}{\Delta x}\left(nx^{n-1}\Delta x + O((\Delta x)^2)\right) = nx^{n-1} + O(\Delta x)$$

$$\xrightarrow[\Delta x \to 0]{} nx^{n-1}$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

This extends to polynomials.

$$\frac{d}{dx}(x^3 + 5x^{10}) = 3x^2 + 50x^9$$

$\boxed{\text{Course 2: LAST TIME}}$

Derivative

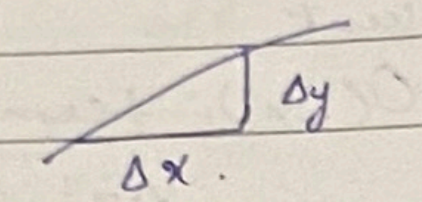
$$\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$n = 1, 2, \ldots$

What is a derivative?

RATE of change



average change

$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$. instantaneous rate of change.

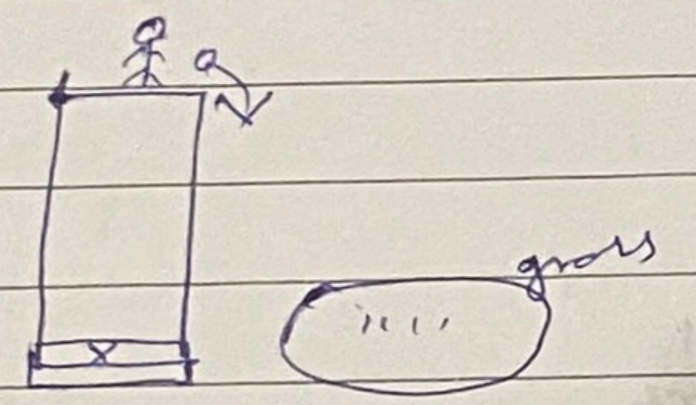
average

Examples.

1. q = charge, $\frac{dq}{dt}$ = current (rate of change)

2. s = distance $\frac{ds}{dt}$ = speed

Pumpkin drop



$h = 80 - 5t^2$

$t = 0,\ h = 80 \qquad t = 4,\ h = 0$

Average ~~change~~ speed; $\frac{\Delta h}{\Delta t} = \frac{0-80}{4-0}$

$= -20$ m/sec

instantaneous speed.

$80 - 5t^2$

$\frac{d}{dt} h = 0 - 10t$

$\frac{d}{dt} 80 = 0 \qquad \frac{d}{dt} t^2 = 2t$

$t = 4$, $h' = -40$ m/sec

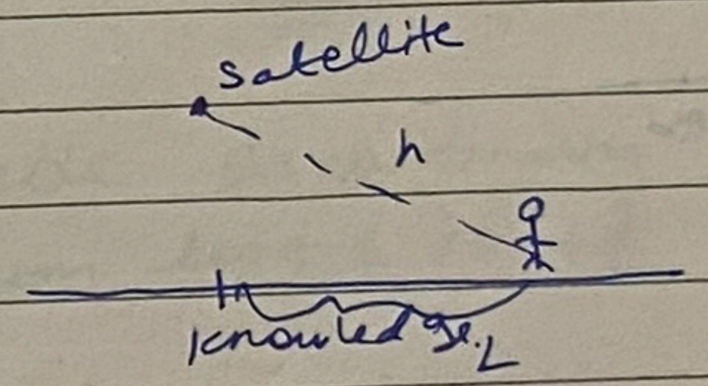
13 | 5 | 25

3. T = temperature.

$\frac{dT}{dx}$ = temp gradient

4. Sensitivity of measurements.

PS1. GPS



h measured by radio/clock

L deduced from h

Error in h: Δh

ΔL estimated by $\Delta L / \Delta h = \frac{dL}{dh}$ = easy

LIMITS and continuity.

1. Easy limits

$\lim_{x\to 4} \frac{x+3}{x^2+1} = \frac{4+3}{4^2+1} = \frac{7}{19}$

2. Derivatives are always harder:

$\lim_{x\to x_0} \frac{f(x_0+\Delta x - f(x_0))}{x-x_0}$ $x = x_0$ gives $\frac{0}{0}$

need cancelation

$\lim_{x\to x_0^+} f(x)$ = right hand limit.

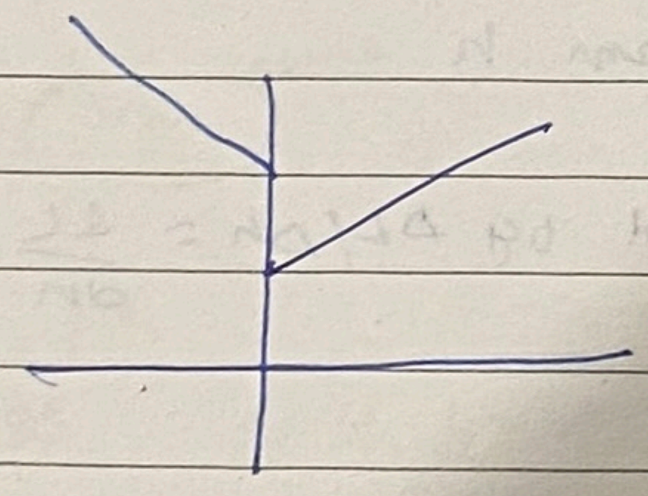
$\left[\begin{matrix} x\to x_0 \\ x > x_0 \end{matrix}\right.$ — x_0 — x

$\lim_{x\to x_0^-} f(x)$ = left hand limit

$\left[\begin{matrix} x\to x_0 \\ x < x_0 \end{matrix}\right]$ — x — x_0

Example

$f(x) = \begin{cases} x+1, & x>0 \\ -x+2, & x<0 \end{cases}$



$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x+1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} (-x+2) = 2$

DEF'N f is continuous at x_0 means

~~cts~~ at x_0. means

$\lim_{x\to x_0} f(x) = f(x_0)$ ($\leftarrow$ easy limits)

continuous at x_0:

1. $\lim_{x\to x_0} f(x)$ exists (from left & right) L + R

$x \neq x_0$

2. $f(x_0)$ is defined.

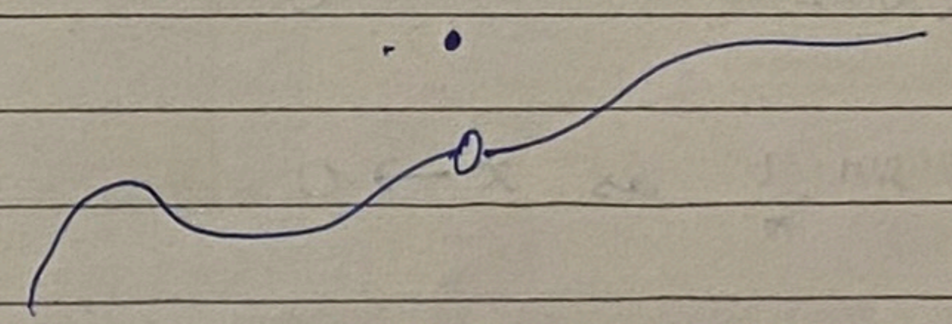
Discontinuous $f(x)$

JUMP Discontinuity.

lim from left right exists but are not equal (as in eg)

Removable Discontinuity

lim from left & right are equal.

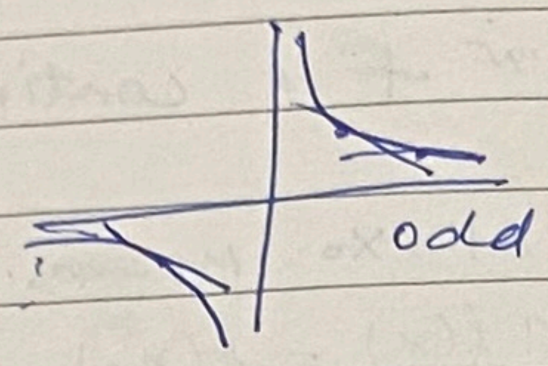


Eg $g(x) = \frac{\sin x}{x}$ $h(x) = \frac{1-\cos x}{x}$

$g(0) = ?$ $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

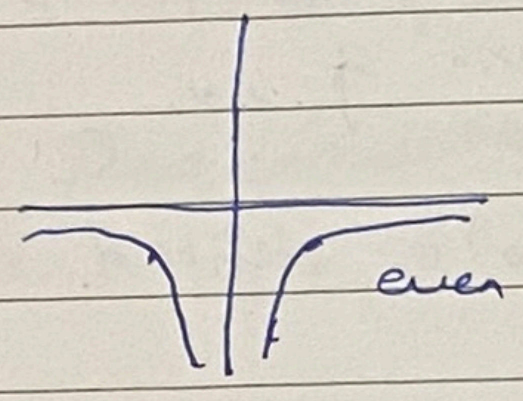
$y = \frac{1}{x}$



$\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$y' = \frac{-1}{x^2}$

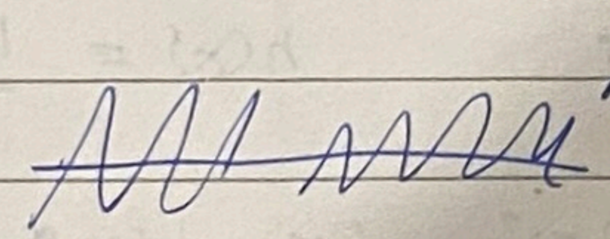


$\lim_{x \to 0} \frac{-1}{x^2} = -\infty$

$x \to 0^+$ and
$x \to 0^-$
both

OTHER (UGLY) Discontinuity)

$y = \sin \frac{1}{x}$ as $x \to 0$



no left or limit.

THEORM (DIFF = Dis continous)

If f is differential at x_0, then f is continous at x_0.

Proof:

$\lim_{x \to x_0} f(x) - f(x_0) = 0$

$\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) = f'(x_0) \cdot 0 = 0$

$\boxed{\text{lect } 3}$

Specific $f'(x)$ & $f(x) = x^n, \frac{1}{x}$

General = $(u+v)' = u' + v'$
$(cu)' = cu'$ constant

Ex: need both kinds for polynomials

$\frac{d}{dx} \sin x = \cos x$	$\sin(a+b) = \sin a \cos b + \cos a \sin b$
$\frac{d}{dx} \cos x = -\sin x$ in radians	$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$

$\cos x$

$= \sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \cos x \left(\frac{\sin \Delta x}{\Delta x} \right)$

A $\downarrow$ 0 $\Delta x \to 0$ B $\downarrow$ 1

$$\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \frac{\cos x\cos\Delta x \overset{-\sin x\sin\Delta x}{\wedge} - \cos x}{\Delta x}$$

$$= \cos x\left(\frac{\cos\Delta x - 1}{\Delta x}\right) + (-\sin x)\left(\frac{\sin\Delta x}{\Delta x}\right)$$

$$= \underset{0}{\downarrow (A)} \qquad \underset{1}{\downarrow}$$

$= -\sin x$

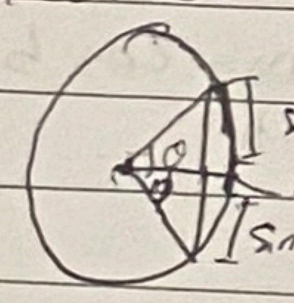
Remarks.

$$\left.\frac{d}{dx}\cos x\right|_{x=0} = \lim_{\Delta x\to 0}\frac{\cos(\Delta x)-1}{\Delta x} \overset{(A)}{=} 0$$

$$\left.\frac{d}{dx}\sin x\right|_{x=0} = \lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} \overset{(B)}{=} 1$$

Derivative of sine and cosine at $x=0$ give all values of $\frac{d}{dx}\sin x$, $\frac{d}{dx}\cos x$

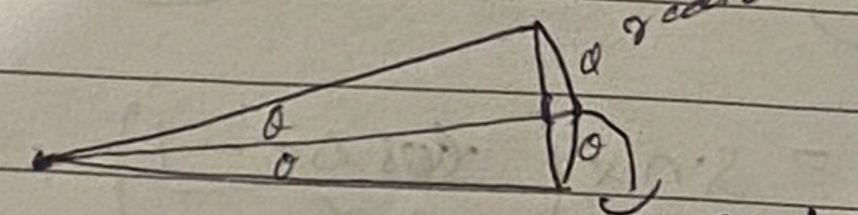
(B) Geometric proof: $\Delta x \approx p\ \theta$



arc length θ $= \frac{2\sin\theta}{2\theta}$ $\leftarrow$ counting $\frac{\sin\theta}{\theta} \to 1$ $\uparrow$ how $\theta\to 0$

Principle: Short curves are nearly straight

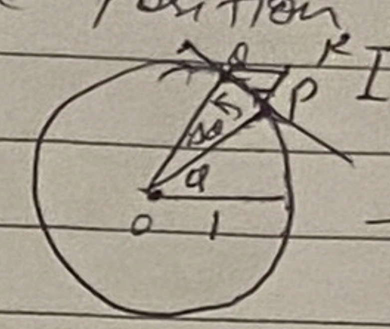
(A) $\frac{1-\cos\theta}{\theta} \to 0$ as $\theta\to 0$ (θ in radians)



gap $= 1-\cos\theta$

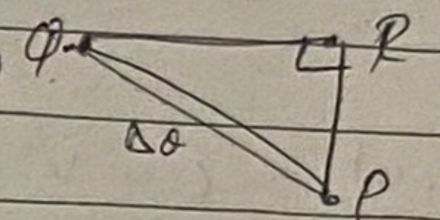
Geom proof of $\frac{d}{d\theta}\sin\theta = \cos\theta$.

$y = \sin\theta$
vertical position of circular motion.



$\longrightarrow$ So here we are trying to turn way to 90° of OP

$\Delta y = PQ \simeq (\Delta\theta)\cos\theta$



$PQ \simeq \Delta\theta$
$\angle QPR = ?$
$PQ \perp OP$
PR vertical
$\angle QPR = \theta$

$\widehat{PQ} \simeq \widehat{PQ}$

$\frac{\Delta y}{\Delta\theta} \approx \cos\theta$

$\lim_{\Delta\theta\to 0}\frac{\Delta y}{\Delta\theta} = \cos\theta$.

General Derivative Rules.

PRODUCT rule.

$\boxed{(uv)' = u'v + uv'}$ $\leftarrow$ change one at time

Quotient Rule

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

$v \neq 0$