Box 1 - Example DFA workflow

Model specification

For this example, we will assume that the observations at time t ($\mathbf{y}_t$) arise from a multivariate Gaussian distribution wherein the mean vector is the sum of a vector of intercepts ($\boldsymbol{\alpha}$) and latent factors ($\mathbf{L}\boldsymbol{\gamma}_t$), and the covariance matrix $\boldsymbol{\Phi}$ is diagonal and unequal. Eqn (1) then becomes:

$$\mathbf{y}_t \sim \text{MVN}(\boldsymbol{\alpha} + \mathbf{L}\boldsymbol{\gamma}_t, \boldsymbol{\Phi}).$$

The latent factors are themselves a multivariate random walk with Gaussian errors. Here we assume $\mathbf{Q}$ equals the identity matrix $\mathbf{I}$, such that Eqn (3) is

$$\boldsymbol{\gamma}_t \sim \text{MVN}(\boldsymbol{\gamma}_{t-1}, \mathbf{Q}).$$

Simulated data

Here are some simulated data for 5 time series with 30 measurements each based upon 2 latent factors. The observation variances are also unequal.

```r
set.seed(123)
## matrix dims
nn <- 5
mm <- 2
tt <- 30
## zero-mean latent factors
x1 <- cumsum(rnorm(tt, 0, 1))
x1 <- x1 - mean(x1)
x2 <- cumsum(rnorm(tt, 0, 1))
x2 <- x2 - mean(x2)
## factor loadings
zvals <- c(0.2, 0.9,
           0.4, 0.7,
           0.6, 0.5,
           0.8, 0.3,
           1.0, 0.1)
ZZ <- matrix(zvals, nn, mm, byrow=TRUE)
## covariance matrix for observation errors
RR <- diag(seq(1,5)/5)
## observation errors
vv <- t(MASS::mvrnorm(tt, matrix(0,nn,1), RR))
## simulated data
yy <- ZZ %*% rbind(x1,x2) + vv
## subtract mean
yy <- t(scale(t(yy), scale=FALSE))
```

Fitting the model

For this example, we make use of the **MARSS** package (Holmes et al. 2012) for **R** to estimate the parameters in the DFA model. The notation of Holmes et al. is somewhat different than ours, however, so we must write out our DFA model in a form based on the following specification of a state-space model:

$$
\begin{gathered}
\mathbf{y}_t \sim \text{MVN}(\mathbf{a} + \mathbf{Z}\mathbf{x}_t, \mathbf{R}) \\
\mathbf{x}_t \sim \text{MVN}(\mathbf{u} + \mathbf{B}\mathbf{x}_{t-1}, \mathbf{Q})
\end{gathered}.
$$

Thus, we will need to define $\mathbf{a} = \boldsymbol{\alpha}$, $\boldsymbol{\gamma}_t = \mathbf{x}_t$, $\boldsymbol{\Phi} = \mathbf{R}$, $\mathbf{u} = \mathbf{0}$, and $\mathbf{B} = \mathbf{I}$. Furthermore, estimating the mean vector $\mathbf{a}$ is often difficult, so we will instead substract the mean from the data before fitting and set $\mathbf{a} = \mathbf{0}$. Note that this will not affect our estimates of the factors and their loadings.

The `MARSS` function makes use of list matrices, which allow one to mix character and numeric classes within the same matrix. Any values of class `character` are interpreted as parameters to be estimated; anything `numeric` is fixed at its value.

```
library(MARSS)
## model list
mod_list <- list(
  ## observation eqn
  A = matrix(0,nn,1),         # zero vector
  Z = matrix(list(0),nn,mm), # all 0's for now
  R = matrix(list(0),nn,nn), # all 0's for now
  ## state eqn
  U = matrix(0,mm,1),         # zero vector
  B = diag(mm),               # identity matrix
  Q = diag(mm)                # identity matrix
)
## specify observation parameters
## upper right corner of Z stays zero
for(cc in 1:mm) {
  for(rr in cc:nn) {
    mod_list$Z[rr,cc] <- paste0(rr,cc)
  }
}
## R is diagonal and unequal
diag(mod_list$R) <- paste0(seq(nn),seq(nn))
## fit the model
dfa <- MARSS(yy, model=mod_list, inits=list(x0=matrix(0,2,1)))
```

Factor rotations

For this example we will use a varimax rotation of the factors and loadings, where we seek an $m \times m$ non-singular rotation matrix $\mathbf{H}$, such that the following state-space forms of our DFA models are equivalent:

$$
\begin{gathered}
\mathbf{y}_t = \mathbf{L}\boldsymbol{\gamma}_t + \boldsymbol{\alpha} + \mathbf{v}_t \\
\boldsymbol{\gamma}_t = \boldsymbol{\gamma}_{t-1} + \mathbf{w}_t
\end{gathered}
$$

and

$$
\begin{gathered}
\mathbf{y}_t = \mathbf{L}\mathbf{H}^{-1}\boldsymbol{\gamma}_t + \boldsymbol{\alpha} + \mathbf{v}_t \\
\mathbf{H}\boldsymbol{\gamma}_t = \mathbf{H}\boldsymbol{\gamma}_{t-1} + \mathbf{H}\mathbf{w}_t
\end{gathered}.
$$

This is easily done in **R** via the `varimax` function:

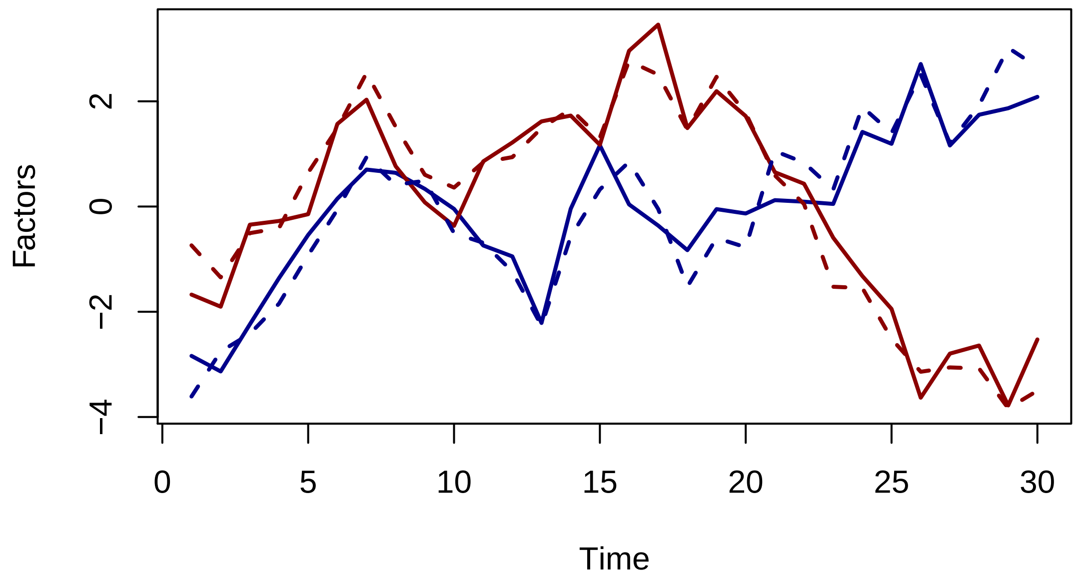
```
## inverse of the rotation matrix
H_inv <- varimax(coef(dfa, type="matrix")$Z)$rotmat
## rotate factor loadings
```

```
Z_rot <- coef(dfa, type="matrix")$Z %*% H_inv
## rotate trends
fac_rot <- solve(H_inv) %*% dfa$states
```

Estimated parameters

Here are plots of the true (solid) and estimated factors (dashed). Note that **MARSS** has no way of ordering the factors, so care should be used when examining them. In this case, the estimated factors have been interchanged.

```
matplot(cbind(x2,x1,t(fac_rot)), type="l", lty=c(1,1,2,2), col=c("darkblue", "darkred"), lwd=2,
        ylab="Factors", xlab="Time")
```



And here is a comparison of the factor loadings in **Z**.

```
## true values
ZZ
```

```
##      [,1] [,2]
## [1,]  0.2  0.9
## [2,]  0.4  0.7
## [3,]  0.6  0.5
## [4,]  0.8  0.3
## [5,]  1.0  0.1
```

```
## estimates
round(Z_rot[,2:1],2)
```

```
##      [,1] [,2]
## [1,] 0.27 0.86
## [2,] 0.44 0.63
## [3,] 0.64 0.47
## [4,] 0.78 0.28
## [5,] 0.91 0.34
```

Model fits

Here is a plot of the simulated data (numbers) and the fits (lines) from the estimated DFA model.

```
## model fits
yhat <- Z_rot %*% fac_rot + matrix(coef(dfa, type="matrix")$A, nn, tt)
## plots
matplot(t(yy), type="p", cex=0.8, col=rainbow(nn,start=0.55, end=0.85),
        ylab="Observations", xlab="Time")
matlines(t(yhat), type="l", lty="solid", lwd=2, col=rainbow(nn,start=0.55, end=0.85))
```