Compact forms for MARSS models with covariates

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Standard MARSS model

There is growing interest in the use of first-order vector autoregressive, or VAR(1), models in ecology where they are often referred to as multivariate autoregressive, or MAR(1), models[1].

Process equation

The underlying process in a MAR(1) model is a discrete time version of a multivariate Gompterz equation for n "species", which is typically written as

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{w}_t,$$

where $\mathbf{x}_t$ is an $n \times 1$ vector of state variates at time t, $\mathbf{B}$ is an $n \times n$ matrix of interaction strengths, and $\mathbf{w}_t$ is an $n \times 1$ vector of multivariate normal process errors; $\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$.

Often we would like to include the effects of some number p of external drivers of the system, whether they be environmental (*e.g.*, temperature) or anthropogenic (*i.e.*, harvest). In those cases, the model is expanded to

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{c}_{t-h} + \mathbf{w}_t,$$

where $\mathbf{C}$ is an $n \times p$ matrix of covariate effects, and $\mathbf{c}_t$ is a $p \times 1$ vector of covariates at time $t-h$ where $0 \leq h < T$.

Observation equation

This process model can be used within a state-space framework wherein we add a second model for the observed data $\mathbf{y}$, such that

$$\begin{aligned}\mathbf{x}_t &= \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{c}_{t-h} + \mathbf{w}_t, \\ \mathbf{y}_t &= \mathbf{x}_t + \mathbf{a} + \mathbf{v}_t,\end{aligned}$$

$\mathbf{a}$ is an $n \times 1$ vector of offsets, and $\mathbf{v}_t$ is an $n \times 1$ vector of observation errors; $\mathbf{v}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$.

[1] See Ives et al. (2003) *Ecol Monogr* 73:301–330)

Compact MARSS model: Option 1

Process equation

The above MARSS model can be rewritten in a more compact form, if we make some assumptions about the time lag h for the covariates. Beginning with the process equation, if $h = 1$ and all of the covariates have been scaled to have zero mean and unit variance, then we can redefine the MAR(1) model as

$$\mathbf{x}_t^* = \mathbf{B}^* \mathbf{x}_{t-1}^* + \mathbf{w}_t^* \text{ with } \mathbf{w}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^*),$$

with the following definitions:

$$\mathbf{x}_t^* = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{c}_t \end{bmatrix},$$

$$\mathbf{B}^* = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ and}$$

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}.$$

Here $\mathbf{I}_p$ is a $p \times p$ identity matrix. Thus, these vectors and matrices will have new dimensions, such that $\mathbf{x}_t^*$ is $(n+p) \times 1$, $\mathbf{B}_t^*$ is $(n+p) \times (n+p)$, and $\mathbf{Q}_t^*$ is also $(n+p) \times (n+p)$.

Observation equation

The observation equation is then rewritten as

$$\mathbf{y}_t^* = \mathbf{x}_t^* + \mathbf{v}_t^* \text{ with } \mathbf{v}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{R}^*),$$

and

$$\mathbf{y}_t^* = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{c}_t \end{bmatrix} \text{ and}$$

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

These new vectors and matrices will have dimensions where $\mathbf{y}_t^*$ is $(n+p) \times 1$ and $\mathbf{Q}_t^*$ is $(n+p) \times (n+p)$.

Compact MARSS model: Option 2

If the covariates have not been scaled, then the state-space model must instead be written as

$$
\begin{aligned}
\mathbf{x}_t^* &= \mathbf{B}^*\mathbf{x}_{t-1}^* + \mathbf{w}_t^* \text{ with } \mathbf{w}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^*) \\
\mathbf{y}_t^* &= \mathbf{x}_t^* + \mathbf{a}^* + \mathbf{v}_t^* \text{ with } \mathbf{v}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{R}^*),
\end{aligned}
$$

wherein

$$
\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix}, \text{ and}
$$

$$
\mathbf{S} = \begin{bmatrix} \text{Var}(c_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \text{Var}(c_p) \end{bmatrix}.
$$

The $p \times p$ matrix $\mathbf{S}$ contains the variances of each of the p covariates along the diagonal and 0's elsewhere (*i.e.*, they are input and not estimated). To allow for non-zero means in the covariates, an $n \times 1$ vector $\mathbf{a}$ containing covariate means must be included in $\mathbf{a}^*$, such that

$$
\mathbf{a}^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{a} \end{bmatrix}.
$$