Stability properties for MAR(1) models

Mark Scheuerell, NOAA Northwest Fisheries Science Center, Seattle, WA USA

Background

There is growing interest in the use of first-order vector autoregressive, or VAR(1), models in ecology where they are often referred to as multivariate autoregressive, or MAR(1), models (*e.g.*, Ives *et al*. 2003 *Ecological Monographs* 73:301–330).

Assume a MAR(1) model of the general form

$$\mathbf{x}_t = \mathbf{a} + \mathbf{B}(\mathbf{x}_{t-1} - \mathbf{a}) + \mathbf{w}_t$$

where $\mathbf{x}_t$ is an $n \times 1$ vector of state variates at time t, $\mathbf{a}$ is an $n \times 1$ vector of underlying levels (means) for each of the states, $\mathbf{B}$ is an $n \times n$ interaction matrix, and $\mathbf{w}_t$ is an $n \times 1$ vector of multivariate normal process errors; $\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$.

Of particular interest here is that the variance-covariance matrix of the stationary distribution for $\mathbf{x}_t$ as $t \to \infty$ gives an indication of the relative stability of the system.

State-space form

This MAR(1) model can be used within a state-space framework, wherein $\mathbf{a}$ is instead included as part of a second model for the observed data $\mathbf{y}$, such that

$$\begin{aligned}
\mathbf{x}_t &= \mathbf{B}\mathbf{x}_{t-1} + \mathbf{w}_t, \\
\mathbf{y}_t &= \mathbf{x}_t + \mathbf{a} + \mathbf{v}_t,
\end{aligned}$$

and $\mathbf{v}_t$ is an $n \times 1$ vector of observation errors, the statistical distribution of which does not affect the variance of the stationary distribution.

Variance of the stationary distribution

We will restrict this treatment to stationary models wherein all of the eigenvalues of $\mathbf{B}$ lie within the unit circle. Because $t = (t-1)$ as $t \to \infty$, under assumptions of stationarity we can write the process equation from the above state-space model as

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_t + \mathbf{w}_t.$$

From this, it follows that

$$\text{Var}(\mathbf{x}_\text{t}) = \mathbf{B}\text{Var}(\mathbf{x}_\text{t})\mathbf{B}^\top + \text{Var}(\mathbf{w}_\text{t}).$$

If we define $\mathbf{\Sigma} = \text{Var}(\mathbf{x}_{\text{t}})$, then

$$\mathbf{\Sigma} = \mathbf{B}\mathbf{\Sigma}\mathbf{B}^\top + \mathbf{Q}.$$

Thus, both the various community interactions contained in $\mathbf{B}$ and the variance in environmental forcing $\mathbf{Q}$ contribute to the variance of the stationary distribution. Unfortunately, however, there is no closed-form solution for $\mathbf{\Sigma}$ when written in this form.

The *vec* operator

It turns out that we can use the *vec* operator to derive an explicit solution for $\mathbf{\Sigma}$. The *vec* operator converts an $i \times j$ matrix into an $(ij) \times 1$ column vector. For example, if

$$\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},$$

then

$$vec(\mathbf{M}) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution

Thus, if $\mathbf{I}$ is an $n \times n$ identity matrix, and we define $\mathcal{I} = (\mathbf{I} \otimes \mathbf{I})$ and $\mathcal{B} = (\mathbf{B} \otimes \mathbf{B})$, then

$$vec(\mathbf{\Sigma}) = (\mathcal{I} - \mathcal{B})^{-1} vec(\mathbf{Q}).$$

Contribution of species interactions to stability

Among the many ways of classifying stability, we are interested in the extent to which community interactions, relative to environmental forcing, contribute to the overall variance of the stationary distribution. That is, when all of the community interactions are relatively weak, we would expect that the temporal evolution of $\mathbf{x}_t$ from t to $t-1$ is driven almost entirely by random environmental variation (*i.e.*, the variance in $\mathbf{w}_t$).

The matrices $\mathbf{\Sigma}$ and $\mathbf{Q}$ are the variances of the stationary distribution and process errors, respectively. As such, we can use the determinants of those matrices to estimate the hyperdimensional "volume" that they occupy.

The second method for comparing the variance of probability distributions uses determinants to measure the "volume" of covariance matrices.

Specifically, we want $\det(\mathbf{\Sigma} - \mathbf{Q})$.

Looking back to the equation for the matrix form of the variance of the stationary distribution, we see that

$$\begin{aligned}\boldsymbol{\Sigma} &= \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top + \mathbf{Q} \\ \boldsymbol{\Sigma} - \mathbf{Q} &= \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top.\end{aligned}$$

From this it follows that

$$\begin{aligned}\det(\boldsymbol{\Sigma} - \mathbf{Q}) &= \det(\mathbf{B})\det(\boldsymbol{\Sigma})\det(\mathbf{B}^\top) \\ &= \det(\boldsymbol{\Sigma})\det(\mathbf{B})^2.\end{aligned}$$

The proportion π of the volume of $\boldsymbol{\Sigma}$ attributable to species interactions is then

$$\frac{\det(\boldsymbol{\Sigma} - \mathbf{Q})}{\det(\boldsymbol{\Sigma})} = \det(\mathbf{B})^2.$$

Intra- versus inter-specific interactions

Now that we have a means for identifying the proportion of the volume of $\boldsymbol{\Sigma}$ attributable to species interactions, it would be nice to know what proportion