Compact forms for MARSS models with covariates

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Standard MARSS model

There is growing interest in the use of first-order vector autoregressive, or VAR(1), models in ecology where they are often referred to as multivariate autoregressive, or MAR(1), models[1].

Process equation

The underlying process in a MAR(1) model is a discrete time version of a multivariate Gompterz equation for n "species", which is typically written as

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{w}_t,$$

where $\mathbf{x}_t$ is an $n \times 1$ vector of state variates at time t, $\mathbf{B}$ is an $n \times n$ matrix of interaction strengths, and $\mathbf{w}_t$ is an $n \times 1$ vector of multivariate normal process errors; $\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$.

Often we would like to include the effects of some number p of external drivers of the system, whether they be environmental (*e.g.*, temperature) or anthropogenic (*i.e.*, harvest). In those cases, the model is expanded to

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{c}_{t-h} + \mathbf{w}_t,$$

where $\mathbf{C}$ is an $n \times p$ matrix of covariate effects, and $\mathbf{c}_t$ is a $p \times 1$ vector of covariates at time $t-h$ where $0 \leq h < T$.

Observation equation

This process model can be used within a state-space framework wherein we add a second model for the observed data $\mathbf{y}$, such that

$$\begin{aligned}\mathbf{x}_t &= \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{c}_{t-h} + \mathbf{w}_t,\\ \mathbf{y}_t &= \mathbf{x}_t + \mathbf{a} + \mathbf{v}_t,\end{aligned}$$

$\mathbf{a}$ is an $n \times 1$ vector of offsets, and $\mathbf{v}_t$ is an $n \times 1$ vector of observation errors; $\mathbf{v}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$.

[1] See Ives et al. (2003) *Ecol Monogr* 73:301–330)

Compact MARSS model: Option 1

Process equation

The above MARSS model can be rewritten in a more compact form if we make some assumptions about the time lag h for the covariates. Beginning with the process equation, if all of the covariates have been scaled to have zero mean and unit variance, then we can redefine the MAR(1) model as

$$\mathbf{x}_t^* = \mathbf{B}^* \mathbf{x}_{t-1}^* + \mathbf{w}_t^* \text{ with } \mathbf{w}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^*),$$

given

$$\mathbf{x}_t^* = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{c}_{t-h} \end{bmatrix},$$

$$\mathbf{B}^* = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}.$$

Here $\mathbf{I}_p$ is a $p \times p$ identity matrix. Thus, these vectors and matrices will have new dimensions, such that $\mathbf{x}_t^*$ is $(n+p) \times 1$, $\mathbf{B}_t^*$ is $(n+p) \times (n+p)$, and $\mathbf{Q}_t^*$ is also $(n+p) \times (n+p)$.

Observation equation

The observation equation is then rewritten as

$$\mathbf{y}_t^* = \mathbf{x}_t^* + \mathbf{v}_t^* \text{ with } \mathbf{v}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{R}^*),$$

and

$$\mathbf{y}_t^* = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{c}_t \end{bmatrix},$$

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

These new vectors and matrices will have dimensions where $\mathbf{y}_t^*$ is an $(n+p) \times 1$ column vector and $\mathbf{Q}_t^*$ is an $(n+p) \times (n+p)$ matrix.

Compact MARSS model: Option 2

If the covariates have not been scaled, then the state-space model must instead be written as

$$\begin{aligned} \mathbf{x}_t^* &= \mathbf{B}^*\mathbf{x}_{t-1}^* + \mathbf{w}_t^* \text{ with } \mathbf{w}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^*) \\ \mathbf{y}_t^* &= \mathbf{x}_t^* + \mathbf{a}^* + \mathbf{v}_t^* \text{ with } \mathbf{v}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{R}^*), \end{aligned}$$

wherein

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \text{Var}(c_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \text{Var}(c_p) \end{bmatrix}.$$

The $p \times p$ matrix $\mathbf{S}$ contains the variances of each of the p covariates along the diagonal and 0's elsewhere (*i.e.*, they are specified *a priori* and not estimated). To allow for non-zero means in the covariates, a $p \times 1$ vector $\mathbf{a}$ containing covariate means must be included in the $(n + p) \times 1$ column vector $\mathbf{a}^*$, such that

$$\mathbf{a}^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{a} \end{bmatrix}.$$

As with $\mathbf{S}$ above, the covariate-specific means are specified rather than estimated.