Variance of the stationary distribution for a first-order vector autoregressive model

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Background

There is growing interest in the use of first-order vector autoregressive, or VAR(1), models in ecology where they are often referred to as multivariate autoregressive, or MAR(1), models (*e.g.*, Ives *et al.* 2003 *Ecological Monographs* 73:301–330).

Assume a MAR(1) model of the general form

$$\mathbf{x}_t = \mathbf{a} + \mathbf{B}(\mathbf{x}_{t-1} - \mathbf{a}) + \mathbf{w}_t$$

where $\mathbf{x}_t$ is an $n \times 1$ vector of states at time t, $\mathbf{a}$ is an $n \times 1$ vector of underlying levels (means) for each of the states, $\mathbf{B}$ is an $n \times n$ interaction matrix, and $\mathbf{w}_t$ is an $n \times 1$ vector of multivariate normal process errors; $\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$.

I note here that MAR(1) models are often used for zero-mean processes, in which case $\mathbf{a} = \mathbf{0}$. Alternatively, MAR(1) models may be embedded within a state-space framework, which adds an observation model to account for noisy and/or missing data. In those so-called MARSS(1) models, any non-zero mean vector $\mathbf{a}$ is typically incorporated into the model for the observed data $\mathbf{y}_t$, such that

$$\begin{aligned}
\mathbf{x}_t &= \mathbf{B}\mathbf{x}_{t-1} + \mathbf{w}_t \\
\mathbf{y}_t &= \mathbf{a} + \mathbf{x}_t + \mathbf{v}_t,
\end{aligned}$$

and the observation errors, $\mathbf{v}_t$, are distributed as a multivariate normal with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{R}$.

Variance of the stationary distribution

Regardless of the specific form, the discussion here is restricted to stationary process models wherein all of the eigenvalues of $\mathbf{B}$ lie within the unit circle. One of the many appeals of stationary MAR(1) models is that the variance-covariance matrix of the stationary distribution for $\mathbf{x}_t$ as $t \to \infty$ gives an indication of the relative stability of the system.

Recognizing that $t = t - 1$ as $t \to \infty$, we can write

$$\mathbf{x}_\infty = \mathbf{a} + \mathbf{B}(\mathbf{x}_\infty - \mathbf{a}) + \mathbf{w}_\infty,$$

where

$$\text{Var}(\mathbf{x}_\infty) = \text{Var}(\mathbf{a}) + \mathbf{B}\big(\text{Var}(\mathbf{x}_\infty) - \text{Var}(\mathbf{a})\big)\mathbf{B}^\top + \text{Var}(\mathbf{w}_\infty).$$

If we define $\mathbf{\Omega} = \text{Var}(\mathbf{x}_\infty)$, then

$$
\begin{aligned}
\mathbf{\Omega} &= \mathbf{0} + \mathbf{B}\left(\mathbf{\Omega} - \mathbf{0}\right)\mathbf{B}^\top + \mathbf{Q} \\
&= \mathbf{B}\mathbf{\Omega}\mathbf{B}^\top + \mathbf{Q}.
\end{aligned}
$$

Unfortunately, however, there is no closed-form solution for $\mathbf{\Omega}$ when written in this form.

The *vec* operator

It turns out that we can use the *vec* operator to derive an explicit solution for $\mathbf{\Omega}$. The *vec* operator converts an $i \times j$ matrix into an $(ij) \times 1$ column vector. For example, if

$$
\mathbf{M} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},
$$

then

$$
vec(\mathbf{M}) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.
$$

Solution

Thus, if $\mathbf{I}$ is an $n \times n$ identity matrix, and we define $\mathcal{I} = (\mathbf{I} \otimes \mathbf{I})$ and $\mathcal{B} = (\mathbf{B} \otimes \mathbf{B})$, then

$$
vec(\mathbf{\Omega}) = (\mathcal{I} - \mathcal{B})^{-1} vec(\mathbf{Q}).
$$