

Compact forms for MARSS models with covariates

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Standard MARSS model

There is growing interest in the use of first-order vector autoregressive, or VAR(1), models in ecology where they are often referred to as multivariate autoregressive, or MAR(1), models¹.

Process equation

The underlying process in a MAR(1) model is a discrete time version of a multivariate Gompertz equation for n “species”, which is typically written as

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{c}_{t-h} + \mathbf{w}_t,$$

where \mathbf{x}_t is an $n \times 1$ vector of state variates at time t , \mathbf{B} is an $n \times n$ matrix of interaction strengths, \mathbf{C} is an $n \times p$ matrix of covariate effects, and \mathbf{c}_t is a $p \times 1$ vector of covariates at time $t - h$ where $0 \leq h < T$, and \mathbf{w}_t is an $n \times 1$ vector of multivariate normal process errors; $\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$.

Observation equation

This process model can be used within a state-space framework wherein we add a second model for the observed data \mathbf{y} , such that

$$\mathbf{y}_t = \mathbf{x}_t + \mathbf{a} + \mathbf{v}_t,$$

\mathbf{a} is an $n \times 1$ vector of offsets, and \mathbf{v}_t is an $n \times 1$ vector of observation errors; $\mathbf{v}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$. Note that if the variates in \mathbf{y} have been scaled to have zero mean, then \mathbf{a} can be dropped from the equation.

Compact MARSS model

Process equation

The above MARSS model can be rewritten in a more compact form. Beginning with the process equation, we can redefine the MAR(1) model as

¹For an extended treatment, see Ives et al. (2003) *Ecol Monogr* 73:301–330

$$\mathbf{x}_t^* = \mathbf{B}^* \mathbf{x}_{t-1}^* + \mathbf{w}_t^* \text{ with } \mathbf{w}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^*),$$

given

$$\mathbf{x}_t^* = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{c}_{t-h} \end{bmatrix},$$

$$\mathbf{B}^* = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

Here \mathbf{I}_p is a $p \times p$ identity matrix. Thus, these vectors and matrices will have new dimensions, such that \mathbf{x}_t^* is $(n+p) \times 1$, \mathbf{B}_t^* is $(n+p) \times (n+p)$, and \mathbf{Q}_t^* is also $(n+p) \times (n+p)$.

Option 1

If all of the covariates have been scaled to have zero mean and unit variance, then

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}.$$

Option 2

If, however, the covariates have not been scaled, then \mathbf{Q} must be modified, such that

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix},$$

and the $p \times p$ matrix \mathbf{S} contains the variances of each of the p covariates along the diagonal (*i.e.*, they are specified *a priori*) and 0's elsewhere. Specifically,

$$\mathbf{S} = \begin{bmatrix} \text{Var}(c_1) & & 0 \\ & \ddots & \\ 0 & & \text{Var}(c_p) \end{bmatrix}. \quad (1)$$

Observation equation

The observation equation is rewritten as

$$\mathbf{y}_t^* = \mathbf{x}_t^* + \mathbf{a}^* + \mathbf{v}_t^* \text{ with } \mathbf{v}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{R}^*),$$

with

$$\mathbf{y}_t^* = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{c}_t \end{bmatrix},$$

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

These new vectors and matrices will have dimensions where \mathbf{y}_t^* is an $(n+p) \times 1$ column vector and \mathbf{R}_t^* is an $(n+p) \times (n+p)$ matrix.

Option 1

If all of the covariates have been scaled to have zero mean and unit variance, then

$$\mathbf{a}^* = \mathbf{0}.$$

Option 2

If the covariates have not been scaled, then a $p \times 1$ vector \mathbf{a} containing covariate means must be included in the $(n+p) \times 1$ column vector \mathbf{a}^* , such that

$$\mathbf{a}^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{a} \end{bmatrix}.$$

As with \mathbf{S} above, the covariate-specific means are specified in \mathbf{a} rather than estimated.