Compact forms for MARSS models with covariates

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Standard MARSS model

There is growing interest in the use of first-order vector autoregressive, or VAR(1), models in ecology where they are often referred to as multivariate autoregressive, or MAR(1), models[1].

Process equation

The underlying process in a MAR(1) model is a discrete time version of a multivariate Gompterz equation for n "species", which is typically written as

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{c}_{t-h} + \mathbf{w}_t,$$

where $\mathbf{x}_t$ is an $n \times 1$ vector of state variates at time t, $\mathbf{B}$ is an $n \times n$ matrix of interaction strengths, $\mathbf{C}$ is an $n \times p$ matrix of covariate effects, and $\mathbf{c}_t$ is a $p \times 1$ vector of covariates at time $t-h$ where $0 \leq h < T$, and $\mathbf{w}_t$ is an $n \times 1$ vector of multivariate normal process errors; $\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$.

Observation equation

This process model can be used within a state-space framework wherein we add a second model for the observed data $\mathbf{y}$, such that

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t,$$

$\mathbf{a}$ is an $n \times 1$ vector of offsets, and $\mathbf{v}_t$ is an $n \times 1$ vector of observation errors; $\mathbf{v}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$. Note that if the variates in $\mathbf{y}$ have been scaled to have zero mean, then $\mathbf{a}$ can be dropped from the equation.

Compact MARSS model

Process equation

The above MARSS model can be rewritten in a more compact form. Beginning with the process equation, we can redefine the MAR(1) model as

$$\mathbf{x}_t^* = \mathbf{B}^*\mathbf{x}_{t-1}^* + \mathbf{w}_t^* \text{ with } \mathbf{w}_t^* \sim \text{MVN}(\mathbf{0}, \mathbf{Q}^*),$$

given

[1]For an extended treatment, see Ives et al. (2003) *Ecol Monogr* 73:301–330

$$\mathbf{x}_t^* = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{c}_{t-h} \end{bmatrix}$$

and

$$\mathbf{B}^* = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Here $\mathbf{I}_p$ is a $p \times p$ identity matrix. Thus, these vectors and matrices will have new dimensions, such that $\mathbf{x}_t^*$ is $(n+p) \times 1$, $\mathbf{B}_t^*$ is $(n+p) \times (n+p)$, and $\mathbf{Q}_t^*$ is also $(n+p) \times (n+p)$.

Option 1

If all of the covariates have been scaled to have zero mean and unit variance, then

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}.$$

Option 2

If, however, the covariates have not been scaled, then $\mathbf{Q}$ must be modified, such that

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix},$$

and the $p \times p$ matrix $\mathbf{S}$ contains the variances of each of the p covariates along the diagonal (*i.e.*, they are specified *a priori*) and 0's elsewhere. Specifically,

$$\mathbf{S} = \begin{bmatrix} \mathrm{Var}(c_1) & 0 & \cdots & 0 \\ 0 & \mathrm{Var}(c_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{Var}(c_p) \end{bmatrix}. \tag{1}$$

Observation equation

The observation equation is rewritten as

$$\mathbf{y}_t^* = \mathbf{Z}^* \mathbf{x}_t^* + \mathbf{a}^* + \mathbf{v}_t^* \text{ with } \mathbf{v}_t^* \sim \mathrm{MVN}(\mathbf{0}, \mathbf{R}^*),$$

with

$$\mathbf{y}_t^* = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{c}_t \end{bmatrix},$$

$$\mathbf{Z}^* = \begin{bmatrix} \mathbf{Z_y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z_C} \end{bmatrix},$$

and

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

These new vectors and matrices will have dimensions where $\mathbf{y}_t^*$ is an $(n+p) \times 1$ column vector, $\mathbf{Z}^*$ is an $(n+p) \times (n-k+p-l)$ matrix, and $\mathbf{R}_t^*$ is an $(n+p) \times (n+p)$ matrix. Note that $0 \leq k < n$ and $0 \leq l < p$, such that either $\mathbf{Z_y}$ or $\mathbf{Z_C}$ could range from a column vector of 1's if $k = n-1$ or $l = p-1$, respectively, to identity matrices if $k = 0$ or $l = 0$.

Option 1

If all of the covariates have been scaled to have zero mean and unit variance, then

$$\mathbf{a}^* = \mathbf{0}.$$

Option 2

If the covariates have not been scaled, then a $p \times 1$ vector $\mathbf{a}$ containing covariate means must be included in the $(n+p) \times 1$ column vector $\mathbf{a}^*$, such that

$$\mathbf{a}^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{a} \end{bmatrix}.$$

As with $\mathbf{S}$ above, the covariate-specific means are specified in $\mathbf{a}$ rather than estimated.