Some Comments on Likelihood Functions

The likelihood $\mathcal{L}$ is a function of the parameters of a statistical model. It is used to estimate the values of and uncertainties on those parameters for a given set of measurements. For an ensemble of n measurements, the likelihood is defined as

$$\mathcal{L}(x;\theta) = \prod_{i=1}^{n} \mathcal{L}_i(x;\theta) = \prod_{i=1}^{n} f(x;\theta)$$

where $f(x;\theta)$ is the probability density function for the statistical model of interest. The best value of the parameters θ can be determined by maximizing the likelihood function, or equivalently, the log of the likelihood function.

$$\begin{aligned} \frac{\partial \ln \mathcal{L}}{\partial \theta} &= \frac{\partial}{\partial \theta} \ln \prod_{i=1}^{n} \mathcal{L}_i \\ &= \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \ln \mathcal{L}_i \\ &= 0 \end{aligned}$$

Often this procedure is described instead as minimizing $-\ln \mathcal{L}$ (minimizing the minus log likelihood).

The log likelihood can be Taylor expanded about its minimum. Since $\partial \mathcal{L}/\partial \theta|_{\theta=\theta_{min}} = 0$:

$$\begin{aligned} \ln \mathcal{L} &= \ln \mathcal{L}_{min} + \frac{1}{2} \frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\bigg|_{\theta=\theta_{min}} (\theta - \theta_{min})^2 \\ 2\left(\ln \mathcal{L} - \ln \mathcal{L}_{min}\right) &= \frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\bigg|_{\theta=\theta_{min}} (\theta - \theta_{min})^2 \end{aligned}$$

In the limit of large n, the distribution $\mathcal{L}$ (due to the central limit theorem) becomes Gaussian. Since for a Gaussian distribution a change in $2 \ln \mathcal{L}$ of one unit corresponds to a 1-σ variation in the parameter θ, the uncertainty on θ is given by:

$$\sigma_\theta{}^2 \equiv \left\langle (\theta - \theta_{min})^2 \right\rangle = -\frac{1}{\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}}$$

Alternatively, the uncertainty on the estimated values of the parameters θ can be obtained by calculating the value of $\Delta\theta$ at which $-2 \ln \mathcal{L}$ increases by 1.0. In cases where $\ln \mathcal{L}$ is not parabolic, the uncertainties can be asymmetric.

The definitions above depend on the fact that the probability density function $f(x;\theta)$ is normalized over the region of x where measurements can occur

$$\int_{xmin}^{xmax} f(x;\theta)dx = 1$$

For this reason, likelihood fits are not sensitive to the value of n. It is possible to add a Poisson term to the likelihood function to include the number of events in the likelihood fit. When a fit includes such a term, it is called an *extended likelihood fit*.

An example

Suppose a set of measurements x_i are made in an experimental setup where the number of events as a function of x follows the distribution

$$N(x) = A + Bx \quad \text{for } 0 < x < 10$$

We would like to use the likelihood method to estimate the value of $\kappa \equiv A/B$. Let's see how to setup this problem.

The total number of events N_{Tot} can be determined

$$\begin{aligned} N_{Tot} &= \int_0^{10} A + Bx \\ &= \left(Ax + \frac{1}{2}Bx^2 \right) \Big|_0^{10} \\ &= 10A + \frac{1}{2}(100B) \\ &= 10A + 50B \end{aligned}$$

and normalized probability density function $f(x;\theta)$ is

$$\begin{aligned} f(x;A,B) &= \frac{1}{N_{Tot}} (A + Bx) \\ &= \frac{1}{10A + 50B} (A + Bx) \\ &= \frac{A}{10A + 50B} + \frac{Bx}{10A + 50B} \\ &= 0.1 \left(\frac{\kappa}{\kappa + 5} + \frac{x}{\kappa + 5} \right) \end{aligned}$$

The overall likelihood function is therefore

$$\mathcal{L}(x;\kappa) = \prod_{i=1}^{n} 0.1\left(\frac{\kappa}{\kappa+5} + \frac{x_i}{\kappa+5}\right)$$

and the log likelihood is

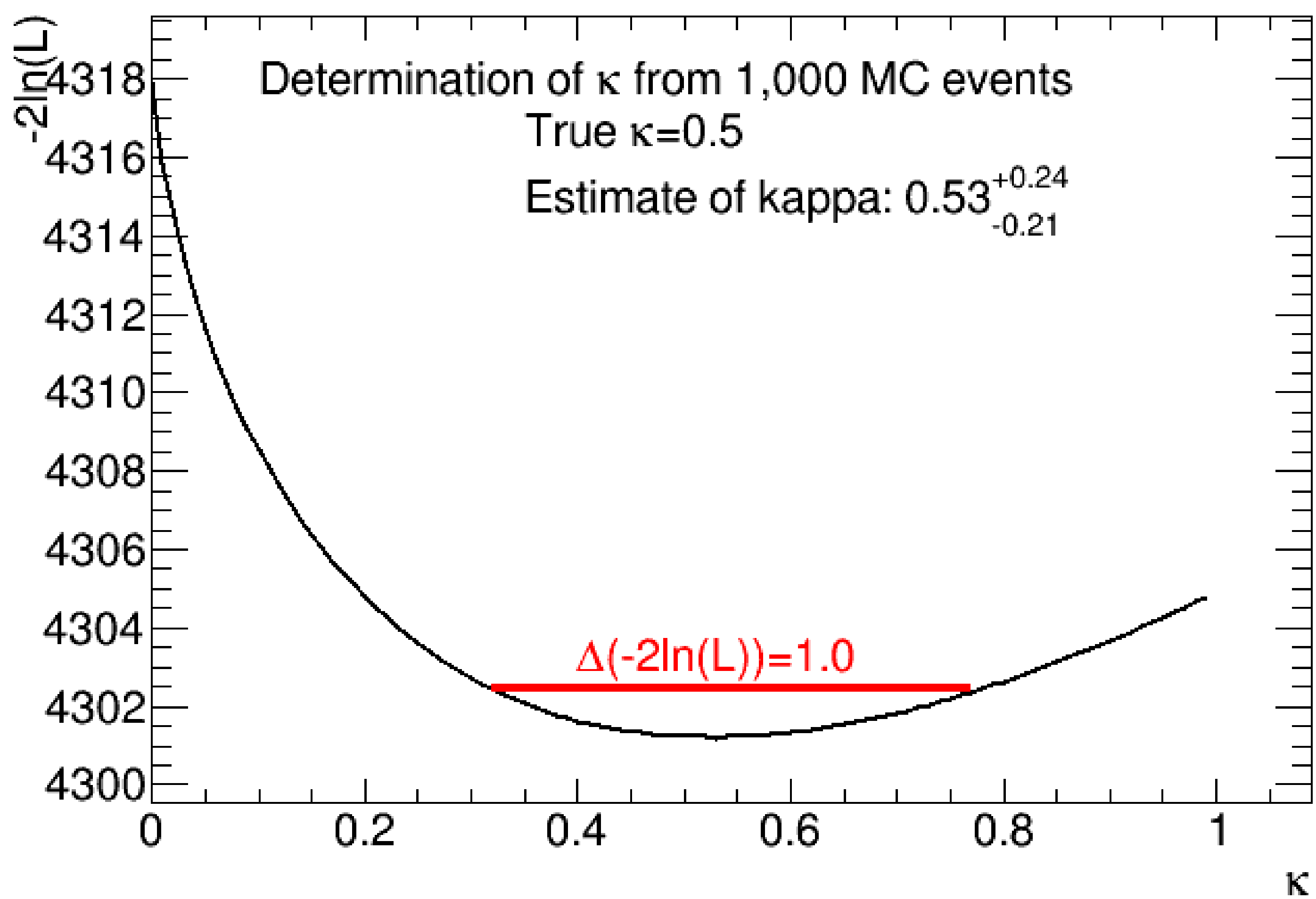
$$\begin{aligned}
\ln\left(\mathcal{L}(x;\kappa)\right) &= \sum_{i=1}^{n} \ln\left(0.1\left(\frac{\kappa}{\kappa+5} + \frac{x_i}{\kappa+5}\right)\right) \\
&= \sum_{i=1}^{n} \ln\left(\left(\frac{\kappa}{\kappa+5} + \frac{x_i}{\kappa+5}\right)\right) + n\ln(0.1)
\end{aligned}$$

If the values x_i are known, then $-\ln\left(\mathcal{L}(x;\kappa)\right)$ can be minimized with respect to κ. Because the last term in independent of x_i, it merely adds a constant term to the log likelihood and is irrelevant for the minimization.

There are many programs available to do such minimization However, it is sometimes useful to start with the simplest approach. It is possible to visualize the minimization process by seeing how $\ln\left(\mathcal{L}(x;\kappa)\right)$ changes when κ is varied. An example root macro to generate fake data for the case $A = 1$, $B = 2$ and to use these data to determine κ can be found here:

http://physics.lbl.gov/shapiro/Physics226/myLikelihoodFit.C

The output of this macro is provided on the next page:


Determination of κ from 1,000 MC events
True κ=0.5
Estimate of kappa: $0.53^{+0.24}_{-0.21}$
Δ(-2ln(L))=1.0
-2ln(L)
4300
4302
4304
4306
4308
4310
4312
4314
4316
4318
0
0.2
0.4
0.6
0.8
1
κ