Math

cat ≡ Catalan

$cat(0) = 1$

$cat(n) = \frac{\binom{2n}{n}}{n+1}$

$cat(m) = \frac{4m-2}{m+1} \cdot cat(m-1)$

$cat(k) = \sum_{i=0}^{k-1} cat(i) * cat(k-1-i)$

cat(n):
- number of distinct binary trees with n vertices.
- number of expressions with n pairs of matched parentheses.
- number of different ways n+1 factors can be completely parenthesized.

cat(n):
- counts the number of ways a convex polygon of n+2 sides can be triangulated.
- counts the number of monotonic paths along the edges of an n×n grid, which do not pass above the diagonal.

- starts in the lower left corner, finishes in the upper right corner, and consists of edges pointing rightwards or upwards.

Counting Derangements

$A_n = (n-1) \times (A_{n-1} + A_{n-2})$

$A_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

$A_n = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$

Number of odd numbers in n^*-th row of pascal triangle $= 2^{(\text{number of 1's in binary of } n)}$

$(1^b + 2^b + \dots + a^b) \% a = \begin{cases} 0 & a \text{ is odd} \\ \left(\frac{a}{2}\right)^b & \text{o.w.} \end{cases}$

b is odd

$gcd(x, y) = gcd(x, x-y)$

$\Rightarrow gcd(x+k, y+k) = gcd(x+k, x-y) \Rightarrow gcd(x_1+k, x_2+k, \dots, x_n+k) = gcd(x_1+k, x_1-x_2, \dots, x_1-x_n)$

DP → Speed-up with Matrix Power

① $fib(n) + fib(n+1) = fib(n+2)$
$fib(n-1) + fib(n) = fib(n+1)$

$\Rightarrow$ ② $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix} = \begin{bmatrix} fib(n+2) \\ fib(n+1) \end{bmatrix}$

$a = b = c = 1, \; d = 0$

$\boxed{a^n + b^n = (a+b)(a^{n-1} + b^{n-1}) - (a \times b)(a^{n-2} + b^{n-2})}$

$\Rightarrow$ ③ $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^p}_{O(\log(p))} \times \underbrace{\begin{bmatrix} fib(n+1) \\ fib(n) \end{bmatrix}}_{\text{set } n=0} = \begin{bmatrix} fib(n+1+p) \\ fib(n+p) \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^p = \begin{bmatrix} fib(p+1) & fib(p) \\ fib(p) & fib(p-1) \end{bmatrix}$