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1) $f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$$\nabla f_{x_1} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}_{x = x_0} = 0 \Rightarrow \quad x_1 = x_2, \quad x_1 = 1 = x_2$$

Stationary point = (1,1)

$$H(x) = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \Rightarrow H(x) \neq 0$$

Taylor

$$f(x) = f(x_0) + \cancel{\nabla_x f(x_0)^T (x - x_0)}^{\to 0} + \frac{1}{2}(x - x_0)^T H(x_0)(x - x_0)$$

$$= f(1,1) + 0 + \frac{1}{2}\underbrace{\begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix}}_{1\times 2}\underbrace{\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}}_{2\times 2}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$= f(1,1) + \frac{1}{2}\begin{bmatrix} \partial x_1 & \partial x_2 \end{bmatrix}\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$= f(1,1) + \frac{1}{2}\begin{bmatrix} 4\partial x_1 - 4\partial x_2 & -4\partial x_1 + 3\partial x_2 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$= f(1,1) + \frac{1}{2}\left[4\partial x_1^2 - 4\partial x_2 \partial x_1 - 4\partial x_1 \partial x_2 + 3\partial x_2^2\right]$$

$$= f(1,1) + \frac{1}{2}\left[4\partial x_1^2 - 8\partial x_1 \partial x_2 + 3\partial x_2^2\right]$$

$4 - 6 - 2 + 3$

$2(2-3) - \cancel{2}(2-3)$

$(2-1)(2-3)$

$$= f(1,1) + \frac{1}{2}\left[(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2)\right]$$

$$(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2) < 0$$

$\Rightarrow \quad 2\partial x_1 - \partial x_2 < 0 \qquad\qquad 2\partial x_1 - 3\partial x_2 < 0$

$\quad\;\; 2\partial x_1 - 3\partial x_2 > 0 \qquad\qquad 2\partial x_1 - \partial x_2 > 0$

2(a) $x_1+2x_2+3x_3=1$ $\in \mathbb{R}^3$ close to $(-1,0,1)^T$.

min $(x_1+1)^2 + x_2^2 + (x_3-1)^2$ — f

s.t $x_1+2x_2+3x_3=1$ — g

$$\frac{x_1+1}{1} = \frac{x_2}{2} = \frac{x_3-1}{3} = \lambda$$

$\Rightarrow x_1 = \lambda - 1$

$x_2 = 2\lambda$ $\Rightarrow$ This satisfies $x_1+2x_2+3x_3=1$

$x_3 = 3\lambda+1$

$\Rightarrow \lambda - 1 + 4\lambda + 9\lambda + 3 = 1$

$\Rightarrow 14\lambda = -1$

$\Rightarrow \lambda = -\frac{1}{14}$

$\Rightarrow x_1 = \lambda - 1 = \frac{-15}{14}$

$x_2 = 2\lambda = \frac{-1}{7}$

$x_3 = 3\lambda+1 = \frac{11}{14}$

Point in plane $x_1+2x_2+3x_3=1$ nearest to point $(-1,0,1)$ is $\left(\frac{-15}{14}, \frac{-1}{7}, \frac{11}{14}\right)$

Converting into an <u>unconstrained problem</u>:

Substituting $x_1 = 1-2x_2-3x_3$

$\Rightarrow (x_1+1)^2 + x_2^2 + (x_3-1)^2$

$= (1-2x_2-3x_3+1)^2 + x_2^2 + (x_3-1)^2$

$f(x_2,x_3) = (2-2x_2-3x_3)^2 + x_2^2 + (x_3-1)^2$

$$\nabla f = \begin{bmatrix} 2(2-2x_2-3x_3)(-2)+2x_2 \\ 2(2-2x_2-3x_3)(-3)+2(x_3-1) \end{bmatrix}$$

$-8+8x_2+12x_3+2x_2$

$-12+12x_2+18x_3+2x_3-2$

$-8+10x_2+12x_3$

$-14+12x_2+20x_3$

$H(f) = \Big[\ -4(-2)+2 \quad -4(-3)$

$$\begin{bmatrix} -6(-2) & -6(-3)+2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix} = 200 - 144$$
$$= 56$$

$H|_{f(x_2,x_3)} > 0$ — positive definite everywhere

$\Rightarrow$ Problem is convex

b) See Python Code

3 To prove: Hyperplane is a convex set.

Hyperplane: $H = \{x \mid x \in R^n, a^T x = c\}$ a = normal direct^n of hyperplane
c = constant

Let there be any 2 points in hyperplane x_1, x_2

As per def^n of convex set, $\forall x_1, x_2 \in H$

To prove: $\lambda x_1 + (1-\lambda)x_2 \in H$ when $\lambda \in [0,1]$

$a^T x_1 = c$ $\quad$ $a^T x_2 = c$

$$a^T(\lambda x_1 + (1-\lambda)x_2) = \lambda a^T x_1 + (1-\lambda)a^T x_2$$
$$= \lambda c + (1-\lambda)c$$
$$= c$$

$\Rightarrow a^T(\lambda x_1 + (1-\lambda)x_2) = c$

$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in H$ $\rightarrow$ So, H is convex set

4 Illumination Problem

$$\min_{P} \max_{K} \{h(a_K^T P, I_t)\}$$

subject to: $0 \leq P_i \leq P_{max}$

where: $P: = [P_1, \dots, P_n]^T$ – power output of n lamps

a_k – fixed parameter for m mirrors $(k = 1, \dots, m)$

I_t – target intensity level

$$h(I, I_t) = \begin{cases} I_t / I, & \text{if } I \leq I_t \\ I / I_t, & \text{if } I \geq I_t \end{cases}$$

$$\max_k \{h(a_k^T P, I_t)\} \quad \text{subject to} \quad 0 \leq P_i \leq P_{max}$$

$$\Rightarrow \max\{h(a_1^T P, I_t), h(a_2^T P, I_t) \dots h(a_3^T P, I_t)\}$$

If a function $f(x)$ is convex then $\max f(x)$ is also convex

Also, the constraints are linear $\Rightarrow$ It lies in a convex space.

Function

$h(a_i^T p, I_t)$: $$\frac{dh}{dp} = \frac{dh}{dI} \cdot \frac{dI}{dp} \qquad (I = a^T p)$$

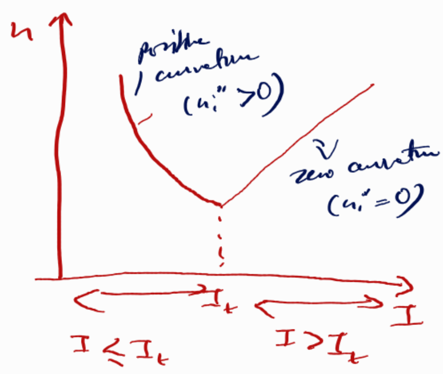
$$= h_i' \cdot a_i$$

$$\frac{d^2h}{dp^2} = \frac{dh'}{dI} \cdot \frac{dI}{dp} \cdot a^T$$

$$= h_i'' a_i a_i^T$$

$$h' = \frac{dh}{dI} = \begin{cases} -I_t / I^2 & I \leq I_t \\ 1/I_t & I > I_t \end{cases}$$

$$h'' = \frac{dh'}{dI} = \begin{cases} \frac{2I_t}{I^3} & I \leq I_t \\ 0 & I > I_t \end{cases} \qquad I = a_i^T p$$



$h_i'' > 0$ & $a a_T \geq 0$

The function is a convex ~~fn~~ in p.

b) Given: Constraint: Overall power output of any of 10 lamps less than p^*

– This leads to a ~~set~~ of linear functions

$$\Rightarrow {}^{n}C_{10} \begin{cases} P_1 + P_2 + \dots + P_{10} \le P^* \\ P_2 + P_3 + \dots + P_{11} \le P^* \end{cases}$$

Reformulated to $\sum_{i=1} a_i P_i \le P^*$ $a \in \{0,1\}$ — depending on whether lamp is on or off.

So, the original function was convex but imposing a constraint to the function, does it still remain convex or not?

As these are linear constraint and we know $a^T P \le P^*$ is convex which means the feasible set is convex

Function is convex
Feasible set is convex } $\Rightarrow$ There can be a unique solution.

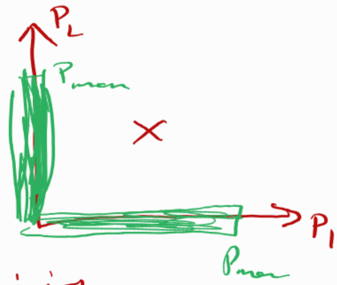
c) Linear constraint: No more than 10 lamps to be switched on $(P>0)$

$\Rightarrow$ So it can be that maybe 1 lamps is ON or 2 lamps is ON or 3 lamps is ON and so on.

Imposing a constraint on number of lamps to be turned ON, we are restricting our feasible set to be a union of certain ones & that makes it non-convex.

Ex: If we take 2 lamps P_1, P_2 as the case

Feasible set is the two green line segment which is not convex.



So, As for constraint only 1 of 2 lamps can be turned ON at a time. But the line segment joining them (red cross X, for example) doesn't lie in convex set

So, non-convex $\Rightarrow$ Not-unique sol^n

5 $c(x)$ = cost of producing x amt. of product A

$c(x)$ – $diff^n$ everywhere

y – price for product

Profit: $c^*(y) = \max_x \{xy - c(x)\}$

To prove: $c^*(y)$ is convex function wrt. y

Function is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ $x, y \in R^n$ $\in [0,1]$

$$f(y_1) = xy_1 - c(x)$$

$$f(y_2) = xy_2 - c(x)$$

$$f(\lambda y_1 + (1-\lambda)y_2) = x(\lambda y_1 + (1-\lambda)y_2) - c(x)$$

$$\lambda f(y_1) + (1-\lambda) f(y) = \lambda x y_1 - \lambda c(x) + (1-\lambda) x y_2 - (1-\lambda) c(x)$$

$$= x[\lambda y_1 + (1-\lambda) y_2] - c(x)$$

$$= f(\lambda y_1 + (1-\lambda) y_2)$$

$$\Rightarrow f[\lambda y_1 + (1-\lambda) y_2] \leq \lambda f(y_1) + (1-\lambda) f(y_2)$$

So, it is a convex function

Also, if a function is convex $\Rightarrow$ Its maximum is convex

So $c^*(y)$ is convex function wrt. y