MAE-598

Mohammed Abu Talha

ASU ID-1220897145

2. x & $b \in R^n$ - vectors

$A \in R^{n \times n}$

$f: R^n \to R \qquad f(x) = b^T x + x^T A x$

a) $\nabla f(x) = \frac{\partial}{\partial x}(b^T x) + \frac{\partial}{\partial x}(x^T A x)$

Now,

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$b^T = [b_1 \ b_2 \ \cdots \ b_n]_{1\times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$b^T x = [b_1 x_1 + b_2 x_2 + \cdots + b_n x_n]_{1\times n}$$

$$\frac{\partial}{\partial x}(b^T x) = \begin{bmatrix} \frac{\partial}{\partial x_1}(b_1 x_1 + \cdots + b_n x_n) \\ \frac{\partial}{\partial x_2}(b_1 x_1 + \cdots + b_n x_n) \\ \vdots \\ \frac{\partial}{\partial x_n}(b_1 x_1 + \cdots + b_n x_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad x^T = [x_1 \ x_2 \ \cdots \ x_n]_{1\times n}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

$x^T A x$

$$= [x_1 \ x_2 \ \cdots \ x_n]_{1\times n} \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix}_{n\times 1}$$

$$= x_1 \sum_{i=1}^{n} a_{1i} x_i + \cdots x_n \sum_{i=1}^{n} a_{ni} x_i$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} x_i$$

$$x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

Differentiating w.r.t. x_k (k^{th} element)

$$\frac{\partial}{\partial x_k}(x^T A x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_k}\sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_k}\sum_{i=1}^{n} a_{ij} x_i$$

$$= \sum_{i=1}^{n} x_i a_{ik} + \sum_{j=1}^{n} x_j a_{kj}$$

Generalizing it for all x:

$$\frac{\partial}{\partial x}(x^T A x) = (A + A^T)x$$

$\Rightarrow \nabla f(x) = b + (A+A^T)x$

Hessian : $\nabla^2 f(x) = A + A^T$

b) Taylor expansion

$f(x)|_{x=0} =$

$f(x) \approx f(x_0) + \nabla_x f(x_0)^T (x - x_0) + \frac{1}{2}(x-x_0)^T H(x_0)(x-x_0)$

$f(x)|_{x=0} = f(0) + (b^T + (A+A^T)0)(x-0) + \frac{1}{2}(x-0)^T(A+A^T)(x-0)$

$= 0 + b^T x + \frac{1}{2} x^T (A+A^T) x$

$f(x)|_{x=0} = b^T x + \frac{1}{2} x^T (A+A^T) x$ — Second order approximation

$f(x)|_{x=0} = b^T x$ — First order approximation

The second order approximation is ~~a~~ good approximation while first order approximation is not

c) A is positive definite if $x^T A x > 0$ for every non-zero vector x

d) A is a $n \times n$ matrix.

For it be a full rank, when its rows and columns are linearly independent.

$\det|A| \neq 0$

e) $y \in \mathbb{R}^n$ and $y \neq 0$ s.t. $A^T y = 0$

Value of b for $Ax = b$ to have solⁿ for x.

b is in column space of A & y needs to be orthogonal to b.

3 Let x_i be the purchased amount of food type $i = 1, 2, \ldots, N$

Objective function: $J = c^T x$

Total of nutrition type j in purchased quantity of food i can be represented as $\sum_{i=1}^{N} a_{ij} x_i \quad \forall \; j = 1, 2, \ldots M$

Optimization problem is:

$$\min_{x} \; c^T x$$

$$\text{s.t.} \quad \sum_{i=1}^{N} a_{ij} \, x_i \geq b_j \quad \forall \; j = 1, 2, \ldots M$$

The constraints can be written in matrix format:

s.t. $Ax \geq b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \in \mathbb{R}^{M \times n}$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix} \in \mathbb{R}^{M}$$

$$\& \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{N}$$