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ORIGINAL ARTICLE

Gravity effects of polyhedral bodies with linearly varying density

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Abstract We extend a recent approach for computing the gravity effects of polyhedral bodies with uniform density by the case of bodies with linearly varying density and by consistently taking into account the relevant singularities. We show in particular that the potential and the gravity vector can be given an expression in which singularities are ruled out, thus avoiding the introduction of small positive numbers advocated by some authors in order to circumvent undefined operations. We also prove that the entries of the second derivative exhibit a singularity if and only if the observation point is aligned with an edge of a face of the polyhedron. The formulas presented in the paper have been numerically checked with alternative ones derived on the basis of different approaches, already established in the literature, and intensively tested by computing the gravity effects induced by real asteroids with arbitrarily assigned density variations.

Keywords Gravitational potential · Gradient · Singularities · Polyhedra · Eros

1 Introduction

The computation of the gravity effects (potential, gravity and tensor gradient fields) hinges upon the evaluation of integrals extended to the domain occupied by the given mass distribution. For this reason the first contributions on this fundamental topic of geodesy were restricted to bodies having a regular geometry such as cylinders, cones, etc., see e.g. Kellogg (1929), MacMillan (1930) and Heiskanen and Moritz (1967); actually, this allowed to carry out analytical integrations.

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When studying the local gravity field a further domain having a simple geometrical shape can be used, i.e. the right rectangular parallelepiped (prism) with constant density.

This has motivated, after the first fundamental contribution by Mader (1951), additional papers on the same issue aiming at improving the numerical efficiency and the generality of the resulting formulas, Koch (1965), Nagy (1966), Banerjee and DasGupta (1977), Nagy et al. (2000), Smith (2000), Tsoulis (2000), Jiancheng and Wenbin (2010), Tsoulis et al. (2003) and D'Urso (2012).

With the objective of computing analytically the gravity effects of bodies with complex geometrical shapes, what is particularly useful in space geodesy, the constant density polyhedron has been repeatedly analyzed in recent times, see e.g. Paul (1974), Barnett (1976), Okabe (1979), Waldvogel (1979), Pohanka (1988), Kwok (1991), Werner (1994), Holstein and Ketteridge (1996), Petroviè (1996), Werner and Scheeres (1997), Tsoulis and Petroviè (2001) and D'Urso (2013a).

However, in most geological structures the constant density assumption is not realistic. For instance, the geological evolution of a sedimentary basin is characterized by compaction for which density increases exponentially with depth. For this reason several authors have addressed the problem of computing the gravity effects produced by two- and three-dimensional bodies with non-uniform density contrast.

In perfect analogy with the case of constant density mass distributions the case of prisms has been first considered and increasing degrees of complexity in the modelling of the density variation have been taken into account, ranging from linear (Chai and Hinze 988), to quadratic (Garcìa-Abdeslem 1992; Gallardo-Delgado et al. 2003), to cubic (Garcìa-Abdeslem 2005), and exponential (Chai and Hinze 988).

Moreover, while the previous papers essentially deal with variations in the vertical direction, recent contributions consider not only more complex density distributions, such as rational or polynomial, but also different variations in vertical and horizontal direction (Zhou 2009, 2010).

In order to improve numerical efficiency and simplifying geometric modelling of real bodies, several authors have addressed the case of polyhedral bodies in which the geometrical nature of the body, more complex with respect to that of a prism, is countervailed by a simpler kind of density variation, usually linear.

In particular Pohanka (1998) was the first to introduce the expression of the attraction of an arbitrary polyhedral body having a linearly varying density, by means of line integrals, thus extending the sophisticated approach existing for the computation of the gravity effects of polyhedral bodies with constant density by the case of density variations. Alternative expressions for the potential and related first and second derivative were later obtained by Hansen (1999) and Holstein (2003).

More recently Hamayun et al. (2009) have derived the expression for the potential of a body with a linearly varying density, by extending the original approach by Pohanka (1988) limited to the constant density case, and proved that the computational algorithm exploited in Pohanka (1998) for the gravity attraction can be applied as well to the potential.

The aim of this paper is to extend a recent approach for computing the gravity effects of bodies with uniform density by the case of polyhedral bodies with linearly varying density and by consistently taking into account the relevant singularities. In particular, analytical formulas for the gravitational potential and for the related first- and second-order derivatives have been derived in D'Urso (2013a) as sums of quantities represented by one-dimensional integrals extending to the generic edge of the polyhedron.

Moreover by exploiting distribution theory (Tang 2006), the singularities in the computation of the gravity effects have been exactly computed independently from the position of the observation point. The same approach has been applied in D'Urso and Marmo (2013c) and Sessa and D'Urso (2013) to problems in geomechanics and in D'Urso and Marmo (2013d) to a problem in geophysics.

An additional objective of the present paper is to obtain, by applying the methodology illustrated in D'Urso (2014), formulas for computing the gravity effects of polyhedral bodies with linearly varying density which are expressed as explicit functions of the coordinates of the vertices of the relevant faces, i.e. the basic geometric data used to define the polyhedral body.

The proposed approach exploits differential identities which are recalled in the text by the symbol A# where # stands for the generic number of the formula and A reminds of the Appendix where all identities are collected. Please notice that the Appendix is linked to the paper in the form of Electronic Supplementary Material.

The paper is organized as follows. Sections 2, 3 and 4 are devoted to illustrate the application of the proposed approach in order to express, respectively, the gravitational potential, the first-order and the second-order derivatives as finite sums of quantities including 1D integrals extending to the edges of the polyhedron.

The specialization of these integrals to expressions depending solely upon the vertices of the faces of the polyhedron is reported in Sect. 5 together with a detailed discussion on the well-posedness of the final formulas.

It is thus proved that only the second derivatives of the potential can exhibit singularities and this happens if and only if the observation point is aligned with one of the edges of the polyhedron. In this case however, differently from the constant density polyhedron, the singularity is not limited to the off-diagonal entries of the second derivative.

Finally Sect. 6 gives a detailed account of the numerical experiments which have been carried out to validate the analytical results contributed in the paper and to intensively test the Matlab® code which has been programmed.

In this respect the formulas derived in the paper have been compared numerically with alternative formulas for evaluating the gravity effects of polyhedral bodies with linearly varying density derived on the basis of some alternative approaches existing in the literature.

2 Gravitational potential of a polyhedral body with linearly varying density

Let us consider an arbitrary bounded domain Ω whose continuous mass distribution has a density $\delta(\boldsymbol{s})$ varying linearly as function of the position vector $\boldsymbol{s}$ of an arbitrary point belonging to it. Hence

$$\delta(\boldsymbol{s}) = \delta_o + \boldsymbol{g} \cdot \boldsymbol{s} \tag{1}$$

where δ_o is a constant reference density evaluated at the origin O of a three-dimensional (3D) Cartesian reference frame (O, x, y, z) in which the coordinates of $\boldsymbol{s}$ are assigned; furthermore, the vector $\boldsymbol{g}$ represents the gradient of the linear law $\delta(\boldsymbol{s})$.

Denoting by $\boldsymbol{p}$ the position vector of an arbitrary point P, the gravitational potential U induced at P by the mass of Ω is defined by the Newton integral:

$$U(P) = U(\boldsymbol{p}) = G \int_{\Omega} \frac{\delta(\boldsymbol{s})}{\|\boldsymbol{p} - \boldsymbol{s}\|} dV(\boldsymbol{s}) = G \int_{\Omega} \frac{\delta(\boldsymbol{s})}{[(\boldsymbol{p} - \boldsymbol{s}) \cdot (\boldsymbol{p} - \boldsymbol{s})]^{1/2}} dV(\boldsymbol{s}) \tag{2}$$

where G is the gravitational constant. Substituting in the previous formula the expression (1) for $\delta(\boldsymbol{s})$ one has

$$\begin{aligned} U(P) &= G\delta_o \int_\Omega \frac{dV(\boldsymbol{s})}{\|\boldsymbol{p}-\boldsymbol{s}\|} + G\boldsymbol{g} \cdot \int_\Omega \frac{\boldsymbol{s}}{\|\boldsymbol{p}-\boldsymbol{s}\|} dV(\boldsymbol{s}) \\ &= G\delta_o U_c(P) + G\boldsymbol{g} \cdot \int_\Omega \frac{\boldsymbol{s}}{\|\boldsymbol{p}-\boldsymbol{s}\|} dV(\boldsymbol{s}) \end{aligned} \tag{3}$$

where

$$U_c(P) = \int_\Omega \frac{dV(\boldsymbol{s})}{\|\boldsymbol{p}-\boldsymbol{s}\|} \tag{4}$$

will be referred to in the sequel as the potential pertaining to a constant density body, whence the suffix $(\cdot)_c$ used in its expression.

Adding and subtracting the vector $\boldsymbol{p}$ in the last integral of (3) it turns out

$$U(P) = G(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) U_c(P) + G\boldsymbol{g} \cdot \int_\Omega \frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{p}-\boldsymbol{s}\|} dV(\boldsymbol{s}) \tag{5}$$

or equivalently, setting $\boldsymbol{r} = \boldsymbol{s} - \boldsymbol{p}$

$$U(P) = G(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) U_c(P) + G\boldsymbol{g} \cdot \int_\Omega \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} dV \tag{6}$$

It is apparent that the previous expression specializes to the one which characterizes the constant density case by setting $\boldsymbol{g} = \boldsymbol{o}$. In this case δ_o would represent the density of a constant density polyhedral model.

Being interested to polyhedral bodies with linearly varying density, we shall concentrate on the evaluation of the last integral in (6) denoted as

$$\boldsymbol{U}_l(P) = \int_\Omega \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} dV \tag{7}$$

since the potential $U_c(P)$ and its derivatives can be evaluated efficiently by means of the theoretical approach outlined in D'Urso (2013a) and the numerical procedure illustrated in D'Urso (2014).

The evaluation of (7) has been already pursued in the literature, see e.g. Pohanka (1998), Hansen (1999), Holstein (2003) and Hamayun et al. (2009), by observing that

$$\boldsymbol{U}_l(P) = \int_\Omega \operatorname{grad}(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2} dV = \int_{Fr(\Omega)} (\boldsymbol{r} \cdot \boldsymbol{r})^{1/2} \boldsymbol{n} \, dA \tag{8}$$

where the last equality follows from (A10), $\boldsymbol{n}$ being the outward unit normal to the boundary $Fr(\Omega)$ of Ω.

However, in deriving the formula above, no mention is made in the literature of the singularity at $\boldsymbol{r} = \boldsymbol{o}$ intrinsic to the integrand in (7) and if this affects somehow the final result.

Thus, in order to provide an alternative procedure to the computation of the 3D integral in (7), we consider the expression

$$\operatorname{div} \frac{\boldsymbol{r} \otimes \boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} = \operatorname{div}\left[(\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{r} \otimes \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{3/2}}\right] \tag{9}$$

the symbol $\otimes$ denoting the tensor (or Kronecker) product of two tensors, (Bowen and Wang 2006). See (A1) for its definition.

Invoking the differential identity (A8) one has

$$\operatorname{div} \frac{\boldsymbol{r} \otimes \boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} = \operatorname{grad}[(\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{r}] \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{3/2}} + [(\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{r}] \operatorname{div} \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{3/2}} \tag{10}$$

Moreover, on account of (A4) it turns out

$$\mathrm{grad}[(\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{r}] = 2(\boldsymbol{r}\otimes\boldsymbol{r}) + (\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{I} \tag{11}$$

since $\mathrm{grad}(\boldsymbol{r}\cdot\boldsymbol{r}) = 2\boldsymbol{r}$ and $\boldsymbol{I}$ is the 3D identity operator.

Substitution of the previous expression in (10) yields finally

$$\begin{aligned}\mathrm{div}\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} &= \frac{3(\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} + [(\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{r}]\,\mathrm{div}\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}\\ &= \frac{3\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} + [(\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{r}]\,\mathrm{div}\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}\end{aligned} \tag{12}$$

Integrating over Ω the previous expression and recalling the definition of $\boldsymbol{U}_l$ in (7) one has

$$\begin{aligned}\int_{\Omega}\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}}dV &= \frac{1}{3}\int_{\Omega}\mathrm{div}\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}}dV - \frac{1}{3}\int_{\Omega}(\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{r}\,\Delta(\boldsymbol{o})dV\\ &= \frac{1}{3}\int_{Fr(\Omega)}\frac{(\boldsymbol{r}\otimes\boldsymbol{r})\boldsymbol{n}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}}dA = \frac{1}{3}\int_{Fr(\Omega)}\frac{(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}}dA\end{aligned} \tag{13}$$

since the second integral on the first row of the previous expression vanishes due to the very definition of the Dirac delta function Δ. See D'Urso (2013a, 2014) for further details.

For a polyhedral body characterized by N_F faces, the previous integral becomes

$$\boldsymbol{U}_l = \frac{1}{3}\sum_{i=1}^{N_F}\int_{F_i}\frac{(\boldsymbol{r}_i\cdot\boldsymbol{n}_i)\boldsymbol{r}_i}{(\boldsymbol{r}_i\cdot\boldsymbol{r}_i)^{1/2}}dA_i = \frac{1}{3}\sum_{i=1}^{N_F}d_i\int_{F_i}\frac{\boldsymbol{r}_i}{(\boldsymbol{r}_i\cdot\boldsymbol{r}_i)^{1/2}}dA_i \tag{14}$$

since the product between the vector $\boldsymbol{r}_i$ spanning the ith face F_i of the polyhedron and the unit vector $\boldsymbol{n}_i$, pointing outwards Ω, is constant over the face.

The last 2D integral above can be transformed to a line integral by a further application of Gauss theorem. To this end we denote by P_i the orthogonal projection on F_i of the observation point P and assume P_i as origin of a 2D reference frame local to the face.

Furthermore, we decompose the vector $\boldsymbol{r}_i$ in the form

$$\boldsymbol{r}_i = \boldsymbol{r}_i^{\perp} + \boldsymbol{r}_i^{\parallel} = (\boldsymbol{r}_i\cdot\boldsymbol{n}_i)\boldsymbol{n}_i + \boldsymbol{r}_i^{\parallel} = d_i\boldsymbol{n}_i + \boldsymbol{T}_{F_i}\boldsymbol{\rho}_i \tag{15}$$

i.e. as sum of a vector $\boldsymbol{r}_i^{\perp}$ orthogonal to the face F_i and a vector $\boldsymbol{r}_i^{\parallel}$ parallel to it. For a graphical illustration of the previous decomposition the reader is referred to D'Urso (2013a, 2014).

The vector $\boldsymbol{\rho}_i = (\xi_i, \eta_i)$ in the previous expression represents the position vector of a generic point of the ith face with respect to P_i and

$$\boldsymbol{T}_{F_i} = \begin{bmatrix}\boldsymbol{u}_{i1} & \boldsymbol{v}_{i1}\\ \boldsymbol{u}_{i2} & \boldsymbol{v}_{i2}\\ \boldsymbol{u}_{i3} & \boldsymbol{v}_{i3}\end{bmatrix} \tag{16}$$

the linear operator mapping the 2D vector $\boldsymbol{\rho}_i$ to the 3D one $\boldsymbol{r}_i^{\parallel}$. In turn $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ represent two arbitrary 3D unit vectors parallel to F_i.

We emphasize the use of roman and greek letters in (15) to denote, respectively, 3D and 2D vectors. The same notational distinction will be adopted throughout the paper.

Coming back to the evaluation of the integral (14), we infer from (15)

$$\begin{aligned}
\boldsymbol{U}_l = \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} dV &= \frac{1}{3}\sum_{i=1}^{N_F} d_i \int_{F_i} \frac{d_i \boldsymbol{n}_i + \boldsymbol{T}_{F_i}\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}} dA_i \\
&= \frac{1}{3}\sum_{i=1}^{N_F} \left\{ d_i^2 \boldsymbol{n}_i \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}} \right. \\
&\qquad \left. + d_i \boldsymbol{T}_{F_i} \int_{F_i} \frac{\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}} dA_i \right\} \\
&= \frac{1}{3}\sum_{i=1}^{N_F} \left\{ d_i^2 \boldsymbol{n}_i [I_{F_i} - |d_i|\alpha_i] + d_i \boldsymbol{T}_{F_i}\boldsymbol{\kappa}_{F_i} \right\}
\end{aligned} \tag{17}$$

where the quantity in square brackets stems from the results contributed in D'Urso (2013a, 2014) and α_i takes into account the singularity at $\boldsymbol{r} = \boldsymbol{o}$ of the field $(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}$ entering the Definition 7 of $\boldsymbol{U}_l$.

Due to the different approach exploited thus far, the previous expression has a rather different form from the analogous quantity considered by Pohanka (1998), Holstein (2003) and Hamayun et al. (2009). Thus, the equivalence of their expression with (17) can be proved only numerically, see e.g. D'Urso (2013b).

However, as shown in the sequel, the expression (17) is well defined for every position of P with respect to the polyhedral body in the sense that it does not need to be supplemented with the small positive number ε, introduced in the pionering papers on the subject by Pohanka (1988, 1998), and used as well by Hamayun et al. (2009), to avoid undefined operations when the computation point is near to $Fr(\Omega)$.

To give proper evidence of the computational features entailed by Eq. (17) let us express both surface integrals appearing in it in terms of line integrals extending to the boundary of the face and prove that the resulting expressions are well defined for every position of the point P with respect to the generic face F_i.

This is certainly true for the first integral in (17) since it has been proven in D'Urso (2013a) that

$$\int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}} = \sum_{j=1}^{N_{E_i}} \beta_{ij} \int_{l_j} \frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}}{\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i} ds_j - |d_i|\alpha_i \tag{18}$$

where N_{E_i} is the total number of edges belonging to the face F_i, l_j the length of the jth edge, s_j the relevant abscissa, $\boldsymbol{\nu}_j$ the outward unit normal and $\beta_{ij} = \boldsymbol{\rho}_i\cdot\boldsymbol{\nu}_j$.

As shown in D'Urso (2014) the previous relation can be equivalently expressed as function of the position vectors of the vertices of the face.

Furthermore, denoting by $\boldsymbol{\nu}$ the 2D unit normal belonging to F_i, one has

$$\begin{aligned}
\boldsymbol{\kappa}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}} dA_i &= \int_{F_i} \operatorname{grad}\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2} dA_i \\
&= \int_{Fr(F_i)} \left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2} \boldsymbol{\nu}\, ds_i
\end{aligned} \tag{19}$$

where use has been made of the identity (A10).

Whenever $d_i = 0$ the previous integral exhibits a singularity at $\boldsymbol{\rho}_i = \boldsymbol{o}$. This is associated with the singularity at $\boldsymbol{r} = \boldsymbol{o}$ in the Definition 7 of $\boldsymbol{U}_l$.

However, it will be shown in Sect. 5.2 that the contribution to $\boldsymbol{U}_l$ provided by a face with $d_i \to 0$ behaves as $d_i \ln |d_i|$; hence, a face containing P gives a null contribution to $\boldsymbol{U}_l$ and the relevant addend in formula (17) can be skipped.

In conclusion we infer from (19)

$$\boldsymbol{\kappa}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} dA_i = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_{l_j} \left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2} ds_j \right) \tag{20}$$

so that, invoking (18), $\boldsymbol{U}_l$ can be definitively expressed by means of line integrals as

$$\boldsymbol{U}_l(P) = \frac{1}{3} \sum_{i=1}^{N_F} \left\{ d_i^2 \boldsymbol{n}_i \left[\sum_{j=1}^{N_{E_i}} \beta_{ij} \int_{l_j} \frac{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}}{\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i} ds_j - |d_i| \alpha_i \right] \right.$$
$$\left. + d_i \boldsymbol{T}_{F_i} \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_{l_j} \left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2} ds_j \right) \right\} \tag{21}$$

On the other hand it has been proved in D'Urso (2013a) that

$$U_c(P) = \frac{1}{2} \sum_{i=1}^{N_F} d_i \left\{ I_{F_i} - |d_i| \alpha_i \right\} \tag{22}$$

Accordingly, Eq. (6) for the potential can be expressed solely as function of line integrals.

The expression of the gravitational potential as function of the 3D coordinates of the vertices of the polyhedral body will be illustrated in Sect. 5 and we anticipate that the resulting formulas do not suffer from any singularity.

3 First-order derivatives of the gravitational potential of a polyhedral body with linearly varying density

The gravitational vector, i.e. the gradient of the potential at P is defined as:

$$\text{grad}U(P) = d_{\boldsymbol{p}} U(\boldsymbol{p}) = G d_{\boldsymbol{p}}[(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) U_c(\boldsymbol{p})] + G d_{\boldsymbol{p}} \left[\boldsymbol{g} \cdot \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} dV \right] \tag{23}$$

where we have used the expression of $U(P)$ reported in Eq. (6).

In order to carry out the first derivative above we invoke the identity (A5) which yields

$$d_{\boldsymbol{p}} U(\boldsymbol{p}) = G[(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) d_{\boldsymbol{p}} U_c + U_c \boldsymbol{g}] + G \left[d_{\boldsymbol{p}} \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} dV \right]^t \boldsymbol{g} \tag{24}$$

The last integral represents the potential associated with a domain Ω of constant density; hence, according to Kellogg (1929), it is possible to move the derivative inside the integral and one has

$$d_{\boldsymbol{p}} \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} dV = - \int_{\Omega} d_{\boldsymbol{r}} \frac{\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{1/2}} dV \tag{25}$$

since $d_{\boldsymbol{p}}(\cdot) = -d_{\boldsymbol{r}}(\cdot)$ on account of the definition $\boldsymbol{r} = \boldsymbol{s} - \boldsymbol{p}$.

The derivative of the quantity in square brackets in (23) can be computed by invoking (A4) to get

$$d_{\boldsymbol{r}} \frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} = -\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} + \frac{\boldsymbol{I}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} \tag{26}$$

Recalling also (A3) one finally has

$$\left[d_{\boldsymbol{p}} \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} dV\right]^t \boldsymbol{g} = \left[\int_{\Omega} \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} dV\right]\boldsymbol{g} - \boldsymbol{g}\int_{\Omega} \frac{dV}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} \tag{27}$$

Introducing the previous formula in (24) and observing that the last integral above is U_c, see e.g. the definition (4), we finally arrive at

$$d_{\boldsymbol{p}}U(\boldsymbol{p}) = G\left\{(\delta_o + \boldsymbol{g}\cdot\boldsymbol{p})d_{\boldsymbol{p}}U_c + \left[\int_{\Omega} \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} dV\right]\boldsymbol{g}\right\} \tag{28}$$

For a polyhedral body Ω the derivative appearing in the first addend has been already computed in D'Urso (2013a) as

$$d_{\boldsymbol{p}}U_c = -\sum_{i=1}^{N_F}(I_{F_i} - |d_i|\alpha_i)\boldsymbol{n}_i \tag{29}$$

where $\boldsymbol{n}_i$ is the outward unit normal to the face while I_{F_i} and α_i have been previously specified, see also D'Urso (2014) for further details.

To express the volume integral in (28) by means of boundary integrals we set $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{r}$ and $\boldsymbol{c} = \boldsymbol{r}/(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}$ in the identity (A9) to arrive at

$$\operatorname{div}\left[\boldsymbol{r}\otimes\boldsymbol{r}\otimes\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}\right] = 2\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} + (\boldsymbol{r}\otimes\boldsymbol{r})\operatorname{div}\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} \tag{30}$$

Hence

$$\begin{aligned}\int\limits_{\Omega} \frac{(\boldsymbol{r}\otimes\boldsymbol{r})dV}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} &= \frac{1}{2}\int\limits_{Fr(\Omega)} \left[\boldsymbol{r}\otimes\boldsymbol{r}\otimes\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}\right]\boldsymbol{n}dA \\ &\quad - \frac{1}{2}\int_{\Omega}(\boldsymbol{r}\otimes\boldsymbol{r})\operatorname{div}\frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}dV\end{aligned} \tag{31}$$

where the first integral on the right-hand side follows from Gauss theorem.

On account of (A2) the expression (31) becomes

$$\int_{\Omega} \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} dV = \frac{1}{2}\int_{Fr(\Omega)} \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}(\boldsymbol{r}\cdot\boldsymbol{n})dA \tag{32}$$

Actually, recalling the very definition of Dirac's delta, see, e.g., D'Urso (2013a, 2014), the last integral in (31) amounts to evaluating the tensor field $\boldsymbol{r}\otimes\boldsymbol{r}$ at $\boldsymbol{r} = \boldsymbol{o}$ and this yields the rank-two null tensor $\boldsymbol{O}$.

In conclusion, for a polyhedral body, one has from (32)

$$\int_{\Omega} \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}} dV = \frac{1}{2}\sum_{i=1}^{N_F} d_i \int_{F_i} \frac{\boldsymbol{r}_i\otimes\boldsymbol{r}_i}{(\boldsymbol{r}_i\cdot\boldsymbol{r}_i)^{3/2}} dA_i = \frac{1}{2}\sum_{i=1}^{N_F} d_i \boldsymbol{F}_{F_i} \tag{33}$$

To express the integral on the right-hand side by means of line integrals extended to the edges of each face we invoke once more the decomposition (15) by setting

$$\begin{aligned} \boldsymbol{F}_{F_i} &= \int_{F_i} \frac{\boldsymbol{r}_i \otimes \boldsymbol{r}_i}{(\boldsymbol{r}_i \cdot \boldsymbol{r}_i)^{3/2}} dA_i \\ &= \int_{F_i} \frac{(d_i \boldsymbol{n}_i + \boldsymbol{T}_{F_i} \boldsymbol{\rho}_i) \otimes (d_i \boldsymbol{n}_i + \boldsymbol{T}_{F_i} \boldsymbol{\rho}_i)}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} dA_i \end{aligned} \tag{34}$$

Thus, invoking the following properties

$$(\boldsymbol{Q}\boldsymbol{c}) \otimes \boldsymbol{d} = \boldsymbol{Q}(\boldsymbol{c} \otimes \boldsymbol{d}) \quad \boldsymbol{c} \otimes (\boldsymbol{Q}\boldsymbol{d}) = (\boldsymbol{c} \otimes \boldsymbol{d})\boldsymbol{Q}^t \tag{35}$$

holding for an arbitrary second-order tensor $\boldsymbol{Q}$, one has

$$\begin{aligned} \boldsymbol{F}_{F_i} &= d_i^2(\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} \\ &\quad + d_i \left\{ \boldsymbol{n}_i \otimes \left[\boldsymbol{T}_{F_i} \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} \right] \right. \\ &\quad \left. + \left[\boldsymbol{T}_{F_i} \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} \right] \otimes \boldsymbol{n}_i \right\} \\ &\quad + \boldsymbol{T}_{F_i} \left[\int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} dA_i \right] \boldsymbol{T}_{F_i}^t \end{aligned} \tag{36}$$

Let us now observe that the first integral on the right-hand side of (36) has already been computed in D'Urso (2013a)

$$J_{F_i} = \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left(\beta_{ij} \int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i)(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right) \tag{37}$$

Seemingly the previous integral turns out to be undefined at $\boldsymbol{\rho}_i = \boldsymbol{o}$ when $d_i = 0$, i.e. when the observation point P does belong to a face F_i. However, as shown in (81), its computation is not needed when $d_i = 0$ since the integral above is multiplied by the factor $d_i^2(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)$ in (36).

The considerations illustrated above do apply as well to the second integral in (36) since its computation has been already carried out in D'Urso (2013a)

$$\boldsymbol{\iota}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right) \tag{38}$$

As proved in D'Urso (2014) the previous integral becomes singular if and only if the observation point P does belong to a straight line containing an edge of a face of Ω, a condition holding in turn if and only if $d_i = 0$. Also in this case, however, it is not necessary to compute the previous integral since the relevant factor $d_i \boldsymbol{T}_{F_i}$ in (36) vanishes.

In conclusion, in order to compute the integral in (28) we need to express, according to (33), only the last integral in (36) by means of line integrals. To this end we use the differential identity (A4) to obtain

$$\operatorname{grad} \frac{\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} = -\frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} + \frac{\boldsymbol{I}_{2D}}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} \tag{39}$$

where $\boldsymbol{I}_{2D}$ is the 2D identity tensor.

Hence, invoking the differential identity (A11) one infers

$$\begin{aligned}\boldsymbol{\Upsilon}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} dA_i \\ &= -\int_{Fr(F_i)} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\nu}}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} ds_i + \boldsymbol{I}_{2D} \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}}\end{aligned} \tag{40}$$

The last integral above has already been computed in (18) while the first one can be evaluated as

$$\begin{aligned}\boldsymbol{\Theta}_{F_i} &= -\int_{Fr(F_i)} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\nu}}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} ds_i \\ &= -\sum_{j=1}^{N_{E_i}} \left[\int_{l_j} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_j \otimes \boldsymbol{\nu}_j \right] = -\sum_{j=1}^{N_{E_i}} (\boldsymbol{\tau}_j \otimes \boldsymbol{\nu}_j)\end{aligned} \tag{41}$$

The well-posedness of the previous expression is discussed in formulas (83)–(85) of Sect. 5 where it is proved that a singularity can occur if and only if $d_i = 0$, a case which is ruled out by the fact that $\boldsymbol{\Upsilon}_{F_i}$ enters the expression of $\boldsymbol{F}_{F_i}$ in (36) which, in turn, is scaled by $d_i = 0$ in (33). Hence $\boldsymbol{F}_{F_i}$, as well as $\boldsymbol{\Upsilon}_{F_i}$, does not need to be computed for $d_i = 0$.

To sum up, formulas (28), (29) and (33) finally yield

$$d_{\boldsymbol{p}} U(\boldsymbol{p}) = G \left\{ -(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) \sum_{i=1}^{N_F} (I_{F_i} - |d_i| \alpha_i) \boldsymbol{n}_i + \frac{1}{2} \sum_{i=1}^{N_F} d_i \boldsymbol{F}_{F_i} \boldsymbol{g} \right\} \tag{42}$$

where $\boldsymbol{F}_{F_i}$ is obtained from (36)–(38) and (40)–(41).

The final expression is further detailed in Sect. 5 where the specialization of the previous integrals to expressions depending solely upon the 3D coordinates of the vertices of the polyhedron will be pursued; in addition the formula will be proved to exhibit no singularity.

4 Second-order derivatives of the gravitational potential of a polyhedral body with linearly varying density

The gravitational tensor, i.e. the second-order gradient of the potential at P is obtained by differentiating the expression of the first-order derivative

$$d_{\boldsymbol{p}}^2 U(\boldsymbol{p}) = G d_{\boldsymbol{p}} \left\{ (\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) d_{\boldsymbol{p}} U_c + \left[\int_{\Omega} \frac{\boldsymbol{r} \otimes \boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{3/2}} dV \right] \boldsymbol{g} \right\} \tag{43}$$

where use has been made of formula (28) for $d_{\boldsymbol{p}} U(\boldsymbol{p})$.

To carry out the derivatives in the previous formula we exploit the identity (A7) which can be applied to the first addend on the right-hand side of (43) to yield

$$d_{\boldsymbol{p}}[(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) d_{\boldsymbol{p}} U_c] = d_{\boldsymbol{p}} U_c \otimes \boldsymbol{g} + (\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) d_{\boldsymbol{p}}^2 U_c \tag{44}$$

since $\boldsymbol{g}$ is a constant vector field and grad $\boldsymbol{p} = \boldsymbol{I}$.

Before proceeding in our derivation it is useful to observe that the gravitational tensor, due to its very definition, is a symmetric tensor and that its expression turns out to be the sum of the quantities reported in (44) and of the quantity stemming from the differentiation of the last addend in (43).

This implies that the sum of the derivative

$$d_{\boldsymbol{p}}\left\{\left[\int_{\Omega}\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}dV\right]\boldsymbol{g}\right\}=d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})=\boldsymbol{A}_{\boldsymbol{g}} \tag{45}$$

and of the term $d_{\boldsymbol{p}}U_c\otimes\boldsymbol{g}$ in (44) has to be symmetric although both addends are non-symmetric.

Moreover, it has been proved numerically that

$$d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})\neq\boldsymbol{g}\otimes d_{\boldsymbol{p}}U_c=(d_{\boldsymbol{p}}U_c\otimes\boldsymbol{g})^t \tag{46}$$

where the last equality follows from (A3).

In conclusion, substitution of (44) and (45) in (43) yields

$$d_{\boldsymbol{p}}^2U(\boldsymbol{p})=G[(\delta_o+\boldsymbol{g}\cdot\boldsymbol{p})d_{\boldsymbol{p}}^2U_c+d_{\boldsymbol{p}}U_c\otimes\boldsymbol{g}+d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g}) \tag{47}$$

where the expression of $d_{\boldsymbol{p}}U_c$ is given by (29) and the second-order derivative has been derived in D'Urso (2013a) as

$$d_{\boldsymbol{p}}^2U_c=-\sum_{i=1}^{N_F}(J_{F_i}d_i\boldsymbol{n}_i\otimes\boldsymbol{n}_i+\boldsymbol{T}_{F_i}\boldsymbol{\iota}_{F_i}\otimes\boldsymbol{n}_i) \tag{48}$$

The symmetry of the previous expression has been proved numerically in D'Urso (2014).

The fulfillment of the symmetry condition

$$(d_{\boldsymbol{p}}U_c\otimes\boldsymbol{g}+\boldsymbol{A}_{\boldsymbol{g}})^t=\boldsymbol{g}\otimes d_{\boldsymbol{p}}U_c+\boldsymbol{A}_{\boldsymbol{g}}^t \tag{49}$$

in (47) can be proved only numerically by evaluating $d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})$ explicitly. To this end one has

$$\begin{aligned}
d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})&=-d_{\boldsymbol{r}}\left\{\left[\int_{\Omega}\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}dV\right]\boldsymbol{g}\right\}=-\left\{\int_{\Omega}d_{\boldsymbol{r}}\left[\frac{(\boldsymbol{r}\otimes\boldsymbol{r})\boldsymbol{g}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}\right]dV\right\}\\
&=-\int_{Fr(\Omega)}\frac{[(\boldsymbol{r}\otimes\boldsymbol{r})\boldsymbol{g}]\otimes\boldsymbol{n}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}dA=-\int_{Fr(\Omega)}\frac{(\boldsymbol{r}\otimes\boldsymbol{r})(\boldsymbol{g}\otimes\boldsymbol{n})}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}dA\\
&=-\sum_{i=1}^{N_F}\left\{\left[\int_{F_i}\frac{\boldsymbol{r}_i\otimes\boldsymbol{r}_i}{(\boldsymbol{r}_i\cdot\boldsymbol{r}_i)^{3/2}}dA_i\right](\boldsymbol{g}\otimes\boldsymbol{n}_i)\right\}=-\sum_{i=1}^{N_F}\left[\boldsymbol{F}_{F_i}(\boldsymbol{g}\otimes\boldsymbol{n}_i)\right]
\end{aligned} \tag{50}$$

where the identity $(50)_5$ follows from $(35)_1$.

In this respect, we remind that the last integral appearing in the expression $\boldsymbol{F}_{F_i}$, see e.g. Eq. (36), is undefined when the observation point P is aligned with any of the edges of the face F_i.

Remarkably, as proved in D'Urso (2014), this is exactly the case in which $d_{\boldsymbol{p}}^2U_c$ in (44) is undefined; hence the second derivative of the potential of a polyhedral body with linearly varying density exhibits the same kind of singularity which characterizes the constant density case.

However, differently from the latter case where the singular terms were confined to the off-diagonal entries, the singularity can now affect all entries of the matrix associated with the second derivative.

5 Computation of the gravity effects for a polyhedral body with linearly varying density

Due to the large number of formulas derived thus far it is convenient to group them together in order to give the reader a comprehensive account.

Specifically, we shall first present collectively the formulas expressed in terms of 1D integrals extended to the edges of each face of the polyhedron and, subsequently, we shall specialize them to expressions depending explicitly upon the 3D coordinates of the vertices of the polyhedron.

5.1 Gravity effects of a polyhedral body with linearly varying density in terms of 1D integrals

By combining formulas (17) and (22), the expression (6) for the potential becomes

$$U(\boldsymbol{p}) = \frac{G}{2}(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) \sum_{i=1}^{N_F} d_i [I_{F_i} - |d_i| \alpha_i] + \frac{G}{3} \boldsymbol{g} \cdot \sum_{i=1}^{N_F} \left\{ d_i^2 \boldsymbol{n}_i [I_{F_i} - |d_i| \alpha_i] + d_i \boldsymbol{T}_{F_i} \boldsymbol{\kappa}_{F_i} \right\} \tag{51}$$

where $\boldsymbol{T}_{F_i}$ is defined in (16)

$$I_{F_i} = \sum_{j=1}^{N_{E_i}} \beta_{ij} \int_{l_j} \frac{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}}{\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i} ds_j, \tag{52}$$

β_{ij} denotes the scalar products between $\boldsymbol{\rho}_i$ and the unit normal $\boldsymbol{\nu}_j$ to the jth side of F_i, see e.g. Eq. (18), and

$$\boldsymbol{\kappa}_{F_i} = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_{l_j} \left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2} ds_j \right) \tag{53}$$

Furthermore, (42) reads

$$d_{\boldsymbol{p}} U(\boldsymbol{p}) = G \left\{ -(\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) \sum_{i=1}^{N_F} (I_{F_i} - |d_i| \alpha_i) \boldsymbol{n}_i + \frac{1}{2} \sum_{i=1}^{N_F} d_i \boldsymbol{F}_{F_i} \boldsymbol{g} \right\} \tag{54}$$

where

$$\boldsymbol{F}_{F_i} = d_i^2 (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) J_{F_i} + d_i [\boldsymbol{n}_i \otimes (\boldsymbol{T}_{F_i} \boldsymbol{\iota}_{F_i}) + (\boldsymbol{T}_{F_i} \boldsymbol{\iota}_{F_i}) \otimes \boldsymbol{n}_i] + \boldsymbol{T}_{F_i} \boldsymbol{\Upsilon}_{F_i} \boldsymbol{T}_{F_i}^t \tag{55}$$

on account of (36); in addition

$$J_{F_i} = - \sum_{j=1}^{N_{E_i}} \left(\beta_{ij} \int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i)(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right) + \frac{\alpha_i}{|d_i|} \tag{56}$$

from (37),

$$\boldsymbol{\iota}_{F_i} = - \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right) \tag{57}$$

from (38),

$$\boldsymbol{\Upsilon}_{F_i} = \boldsymbol{\Theta}_{F_i} + (I_{F_i} - |d_i|\alpha_i)\boldsymbol{I}_{2D} \tag{58}$$

from (40) and

$$\boldsymbol{\Theta}_{F_i} = -\sum_{j=1}^{N_{E_i}} \left[\int_{l_j} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_j \otimes \boldsymbol{\nu}_j \right] \tag{59}$$

from (41).

Finally, substitution of (29), (48) and (50) in (47) yields

$$\begin{aligned} d_{\boldsymbol{p}}^2 U(\boldsymbol{p}) = -G \Bigg\{ (\delta_o + \boldsymbol{g} \cdot \boldsymbol{p}) \left[\sum_{i=1}^{N_F} \Big(J_{F_i} d_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i + (\boldsymbol{T}_{F_i} \boldsymbol{\iota}_{F_i}) \otimes \boldsymbol{n}_i \Big) \right] \\ + \sum_{i=1}^{N_F} \Big[(I_{F_i} - |d_i|\alpha_i) \boldsymbol{n}_i \otimes \boldsymbol{g} + \boldsymbol{F}_{F_i} \boldsymbol{g} \otimes \boldsymbol{n}_i \Big] \Bigg\} \end{aligned} \tag{60}$$

It is apparent that, for $\boldsymbol{g} = \boldsymbol{o}$, formulas (51), (54) and (60) trivially specialize to those pertaining to the constant-density polyhedron.

Some of the integrals appearing in the previous formulas, namely I_{F_i} and J_{F_i}, have been already expressed in D'Urso (2014) as function of the 3D coordinates of the vertices of the polyhedron. For this reason, in the next subsection, we shall address only the computation of the remaining integrals and discuss the singularities they can exhibit.

5.2 Gravity effects of a polyhedral body with linearly varying density in terms of 3D coordinates of the vertices

Aim of this subsection is to provide formulas for the gravity effects of polyhedral bodies with linearly varying density which are expressed as functions of the 3D coordinates of the vertices which define the faces of the polyhedron.

To this end we need to further specialize the line integrals, appearing in the definitions of the quantities (51), (54) and (60), extended to the edges of each face.

The rationale of our approach is to first parameterize the generic edge as function of the 2D coordinates of the end vertices of the edges by means of the relation

$$\boldsymbol{\rho}_i(s_j) = \boldsymbol{\rho}_j + \lambda_j(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \tag{61}$$

where $\lambda_j = s_j / l_j$ is the non-dimensional abscissa along the jth edge; notice that we have simplified the notation by writing λ_j instead of the more correct symbol λ_{ij}.

The same has been done for the symbols $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ which denote the position vectors of the end vertices of the jth edge of the face F_i. In particular $\boldsymbol{\rho}_j = (\xi_j, \eta_j)$ is a 2D vector whose coordinates ξ_j, η_j are relevant to the reference frame local to each face.

They are obtained from the 3D coordinates, which represent the basic input data for assigning the polyhedron, by inverting the relation (15) as follows

$$\boldsymbol{\rho}_i = \boldsymbol{T}_{F_i}^t (\boldsymbol{r}_i - d_i \boldsymbol{n}_i) \tag{62}$$

The outward unit normal $\boldsymbol{\nu}_j$ appearing in the definition of β_{ij}, see e.g. formula (18) as well as formulas (52) and (56), can be computed as

$$\boldsymbol{\nu}_j = \frac{(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j)^\perp}{l_j} = \frac{\boldsymbol{\rho}_{j+1}^\perp - \boldsymbol{\rho}_j^\perp}{l_j} \tag{63}$$

where $(\cdot)^\perp$ stands for a vector orthogonal to $(\cdot)$. Assuming a counter-clockwise circulation sense along $Fr(F_i)$, it turns out $\boldsymbol{\rho}_j^\perp = (\eta_j, -\xi_j)$.

Finally, the scalar product $\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i$ is expressed as function of the position vectors of the end vertices of each edge as follows

$$\boldsymbol{\rho}_i(s_j) \cdot \boldsymbol{\rho}_i(s_j) = p_j \lambda_j^2 + 2 q_j \lambda_j + u_j \tag{64}$$

where it has been set

$$\begin{aligned} p_j &= (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \\ q_j &= \boldsymbol{\rho}_j \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \qquad u_j = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j \end{aligned} \tag{65}$$

Denoting also $v_j = u_j + d_i^2$, let us now detail the specialization of all the integrals appearing in the expressions (51), (54) and (60) by defining in advance the following quantities

$$AT1_j = \arctan \frac{|d_i|(p_j + q_j)}{\sqrt{p_j u_j - q_j^2}\sqrt{p_j + 2q_j + v_j}} \tag{66}$$

$$AT2_j = \arctan \frac{|d_i| q_j}{\sqrt{p_j u_j - q_j^2}\sqrt{v_j}} \tag{67}$$

$$LN_j = \ln k_j = \ln \frac{p_j + q_j + \sqrt{p_j}\sqrt{p_j + 2q_j + v_j}}{q_j + \sqrt{p_j v_j}} \tag{68}$$

where the suffix $(\cdot)_j$ has been added to remind that they all refer to the jth edge of the generic face F_i.

The quantity I_{F_i} has already been expressed in D'Urso (2014) by means of an integral which is defined in the sense of Riemann provided that the null vector $\boldsymbol{o}$ does not belong to the edge defined by the vectors $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$. In this case one has

$$I_{F_i} = \sum_{j=1}^{N_{E_i}} (\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^\perp) \left\{ \frac{|d_i|}{\sqrt{p_j u_j - q_j^2}} \left[AT1_j - AT2_j \right] + \frac{1}{\sqrt{p_j}} LN_j \right\} \tag{69}$$

Conversely, should the null vector belong to an edge, the contribution to I_{F_i} provided by this edge is represented by a (weakly) singular integral which has been assumed to be zero on the basis of several claims, detailed in D'Urso (2014).

In this last paper the well-posedness of the integral I_{F_i} has been proved on the basis of some considerations which are reported hereafter in a simpler form. They are also required, with some modifications, for discussing the well-posedness of the additional integrals appearing in the computation of the gravity effects.

The expression (69) is well defined since the radicands are positive and the argument of the logarithm function does not vanish unless the null vector $\boldsymbol{o}$ belongs to the segment defined by the two vectors $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$, i.e. to the jth edge of a face.

To prove our statement on the well-posedness of the expression (69), we first notice that

$$\beta_{ij} l_{ij} = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^\perp \tag{70}$$

and that

$$p_j u_j - q_j^2 = (\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1})(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j) - (\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1})^2 \geq 0 \tag{71}$$

by the Cauchy–Schwarz inequality, see e.g. Tang (2006).

The equality in the previous expression holds if and only if one of the two vectors is zero or, alternatively, if the two vectors are parallel and point in opposite direction. In both cases the factor $\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}$ in (69) is zero so that the singularity associated with a null value of $p_j u_j - q_j^2$ in (66)–(68) becomes inessential from the computational point of view (D'Urso 2014).

Moreover

$$\begin{aligned} p_j &= (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) = l_j^2 > 0 \\ p_j + q_j &= \boldsymbol{\rho}_{j+1} \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \\ v_j &= u_j + d_i^2 = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j + d_i^2 = |\boldsymbol{r}_j|^2 \geq 0 \\ p_j + 2q_j + v_j &= \boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1} + d_i^2 = |\boldsymbol{r}_{j+1}|^2 \geq 0 \end{aligned} \tag{72}$$

The last two quantities vanish if and only if $d_i = 0$ and, at the same time, either $\boldsymbol{\rho}_j$ or $\boldsymbol{\rho}_{j+1}$ is zero. In both cases, however, $\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} = 0$ so that v_j and $p_j + 2q_j + v_j$ in (66) and (67) can be assumed to be positive.

Furthermore, both the numerator and the denominator of the logarithm in (68) are positive when the observation point does not belong to the jth edge. In the opposite case at least one of them can vanish but, as stated above, the computation of the logarithm is not required.

To prove this result we observe that

$$k_j = \frac{p_j + q_j + \sqrt{p_j}\sqrt{p_j + 2q_j + v_j}}{q_j + \sqrt{p_j v_j}} = \frac{\boldsymbol{\rho}_{j+1} \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) + l_j |\boldsymbol{r}_{j+1}|}{\boldsymbol{\rho}_j \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) + l_j |\boldsymbol{r}_j|} \tag{73}$$

according to (72).

The scalar products on the right-hand side of the expression above can be manipulated by invoking the characteristic property, see e.g. D'Urso (2013a), $\boldsymbol{T}_{F_i}^t \boldsymbol{T}_{F_i} = \boldsymbol{I}_{2D}$ where $(\cdot)^t$ denotes transpose and $\boldsymbol{I}_{2D}$ is the identity operator on the two-dimensional space. To fix the ideas we make reference to the numerator of (73) to get

$$\begin{aligned} \boldsymbol{\rho}_{j+1} \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) &= \boldsymbol{T}_{F_i} \boldsymbol{\rho}_{j+1} \cdot \boldsymbol{T}_{F_i} (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) \\ &= (\boldsymbol{T}_{F_i} \boldsymbol{\rho}_{j+1} + d_i \boldsymbol{n}_i - d_i \boldsymbol{n}_i) \cdot [\boldsymbol{T}_{F_i} (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j)] \\ &= (\boldsymbol{r}_{j+1} - d_i \boldsymbol{n}_i) \cdot (\boldsymbol{r}_{j+1} - \boldsymbol{r}_j) \end{aligned} \tag{74}$$

according to (15).

Lying in the plane of the face, the vector $\boldsymbol{r}_{j+1} - \boldsymbol{r}_j$ is orthogonal to $\boldsymbol{n}_i$ so that one finally has

$$\boldsymbol{\rho}_{j+1} \cdot (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) = \boldsymbol{r}_{j+1} \cdot (\boldsymbol{r}_{j+1} - \boldsymbol{r}_j) \tag{75}$$

and a similar relation holds for the denominator of (73).

Observing that $l_j = |\boldsymbol{r}_{j+1} - \boldsymbol{r}_j|$ one finally has

$$k_j = \frac{|\boldsymbol{r}_{j+1}|(1 + cos\alpha_j)}{|\boldsymbol{r}_j|(1 + cos\delta_j)} \tag{76}$$

where α_j and δ_j denote the angles between $\boldsymbol{r}_{j+1} - \boldsymbol{r}_j$ and $\boldsymbol{r}_{j+1}$ or $\boldsymbol{r}_j$, respectively.

The previous expression shows that both the numerator and the denominator are non-negative. In particular they can become zero if and only if $d_i = 0$ and the observation point is aligned with the jth edge. Actually, under the assumption $d_i = 0$, either $|\boldsymbol{r}_{j+1}|$ or $|\boldsymbol{r}_j|$ can vanish while the alignment of P with the jth edge implies that either $cos\alpha_j$ or $cos\delta_j$, or both, can become equal to -1.

All the previous cases imply that either $\boldsymbol{\rho}_j$ or $\boldsymbol{\rho}_{j+1}$ is the null vector or that $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ are parallel. Accordingly, LN_j in (68) tends to $+\infty$ or $-\infty$ with an infinitesimally low rate. Since $\beta_{ij} = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} = 0$, the product $\beta_{ij}(LN_j)$ in (69) tends to zero, what excludes the necessity of computing LN_j.

To sum up, on account of the previous specifications, the expression (69) for I_{F_i} is well-posed. Moreover we remind that the singularity term α_i in (51) can be computed efficiently as shown in D'Urso and Russo (2002) and D'Urso (2014).

The integral in the expression (53) for $\boldsymbol{\kappa}_{F_i}$ can be computed by setting

$$\int_{l_j} \left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2} ds_j = l_j \int_0^1 \left(p_j \lambda_j^2 + 2q_j \lambda_j + v_j\right)^{1/2} d\lambda_j \tag{77}$$

Being

$$\begin{aligned}\int_0^1 (px^2 + 2qx + v)^{1/2} dx = \frac{1}{2p}\Big[(px+q)\sqrt{px^2+2qx+v}\Big]_0^1 \\ + \frac{pv - q^2}{2p^{3/2}}\Big[\ln\big(px + q + \sqrt{p}\sqrt{px^2+2qx+v}\big)\Big]_0^1\end{aligned} \tag{78}$$

and $l_j = \sqrt{p_j}$ it turns out

$$\begin{aligned}\int_{l_j} \left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2} ds_j = \frac{p_j v_j - q_j^2}{2p_j} LN_j \\ + \frac{1}{2\sqrt{p_j}}\Big[(p_j + q_j)\sqrt{p_j + 2q_j + v_j} - q_j\sqrt{v_j}\Big]\end{aligned} \tag{79}$$

where LN_j is defined in (68).

The non-negativity of the radicands in the second addend on the right-hand side follows from $(72)_3$ and $(72)_4$.

Furthermore, the logarithm in (79) enters the expression of $\boldsymbol{\kappa}_{F_i}$ in (53) which, in turn, is scaled by d_i in (51).

Thus, the role played by β_{ij} for ensuring the well-posedness of I_{F_i} is played by d_i for $\boldsymbol{\kappa}_{F_i}$. Actually, $k_j \to 0$ or $k_j \to +\infty$ depending on the fact that the numerator or the denominator in (68) vanishes; in both cases $LN_j = \ln k_j$ is an infinite of arbitrarily low degree. This means that

$$\lim_{d_i \to 0} d_i \, LN_j = \lim_{d_i \to 0} d_i \, \ln k_j = 0 \tag{80}$$

stating that the jth edge of F_i gives a null contribution to the quantity $\boldsymbol{\kappa}_{F_i}$ in (51).

Thus, independently from the value of k_j, one can skip the evaluation of the contribution of the edges belonging to a face characterized by $d_i = 0$.

Finally, we remark that the parameter k_j is reminiscent of the parameter $\Lambda = l_j/(|\boldsymbol{r}_{j+1}| + |\boldsymbol{r}_j|)$ introduced by Strakhov et al. (1986) and extensively used in Holstein (2003).

In conclusion, holding an analogous property for I_{F_i}, the potential $U(P)$ of a polyhedral body with linearly varying density is well-defined irrespectively of the position of the observation point P with respect to the polyhedron.

Let us now address the computation of the integrals (56), (57) and (59) required for computing the tensor $\boldsymbol{F}_{F_i}$ in (55) which contributes to $d_{\boldsymbol{p}}U(\boldsymbol{p})$ in (54) and to $d^2_{\boldsymbol{p}}U(\boldsymbol{p})$ in (60). The integral in (56), see e.g. D'Urso (2014), is given by

$$\beta_{ij} \int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i)(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{|d_i| \sqrt{p_j u_j - q_j^2}} (AT1_j - AT2_j) \tag{81}$$

where $AT1_j$ and $AT2_j$ are supplied in (66) and (67). Accordingly, the expression above for the computation of J_{F_i} is well defined analogously to formula (69).

Furthermore, the integral in (57) becomes

$$\int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = LN_j \tag{82}$$

The logarithm above is scaled by d_i in (55) so that the discussion on its well-posedness is analogous to that carried out for the logarithm in (79).

Finally, the integral $\boldsymbol{\tau}_j$ appearing in the definition of the tensor $\boldsymbol{\Theta}_{F_i}$ in (41) and (59) can be computed as follows

$$\begin{aligned} \boldsymbol{\tau}_j &= \int_{l_j} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_j \\ &= \boldsymbol{\rho}_j \int_{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} + \frac{(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j)}{l_j} \int_{l_j} \frac{s_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_j \end{aligned} \tag{83}$$

on account of (61).

The first integral has been computed in (82) while for the second one we need the result

$$\begin{aligned} \int_0^1 \frac{x}{(px^2 + 2qx + v)^{1/2}} dx &= \frac{1}{p} \left[\sqrt{px^2 + 2qx + v} \right]_0^1 \\ &\quad - \frac{q}{p^{3/2}} \left[\ln \left((px + q + \sqrt{p} \sqrt{px^2 + 2qx + v} \right) \right]_0^1 \end{aligned} \tag{84}$$

yielding finally

$$\begin{aligned} \frac{1}{l_j} \int_{l_j} \frac{s_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_j &= l_j \int_0^1 \frac{\lambda_j}{(p\lambda_j^2 + 2q_j \lambda_j + v_j)^{1/2}} d\lambda_j \\ &= \frac{1}{\sqrt{p_j}} \left[\sqrt{p_j + 2q_j + v_j} - \sqrt{v_j} \right] - \frac{q_j}{p_j} LN_j \end{aligned} \tag{85}$$

The quantities above, which are needed for the computation of the tensor $\boldsymbol{\Theta}_{F_i}$, are scaled by the factor d_i in (54) since they enter the expression of $\boldsymbol{\Upsilon}_{F_i}$ which, in turn, appears in the definition of the tensor $\boldsymbol{F}_{F_i}$.

Hence, being composed of quantities identical to those appearing in (79), the integral above is well defined whatever is the position of the point P with respect to the polyhedron.

This definitively proves that the first derivative of the potential is well defined independently from the position of P.

As a final remark we further point out that the quantity $d_{\boldsymbol{p}}(\boldsymbol{Ag})$ in (47) exhibits, according to (50), a singularity whenever the observation point P is aligned with one of the edges of the polyhedron. This happens because the tensor $\boldsymbol{\Upsilon}_{F_i}$ defined in (58), representing the last integral in the expression (55) of $\boldsymbol{F}_{F_i}$, depends upon the integral $\boldsymbol{\Theta}_{F_i}$ in (59) and hence upon $\boldsymbol{\tau}_j$ in (83), which is not scaled by d_i as in (54).

This further proves that the only singularity which can affect the formulas (51), (54) and (60) for the gravity effects of polyhedral bodies with linearly varying density is confined to the last formula.

Similarly to the constant density polyhedron a singularity in the expression of the second derivative of the potential is experienced if and only if the observation point P is aligned with an edge of the polyhedron; however, now the singularity affects not only the off-diagonal entries, as for the constant density case, but also the entries on the main diagonal since the matrix associated with the tensor $d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})$ is full.

6 Numerical examples

In order to check the validity of the formulas derived in the previous sections two sets of numerical examples have been carried out by means of a Matlab® code.

The first set concerned the validation of the code against implementation errors as well as the robustness and numerical efficiency of the formulas with respect to alternative ones discussed in the sequel.

The second set of numerical examples was instead conceived with the aim of assessing the computational performance of the code for polyhedral models of real bodies.

For both sets of examples it has been checked that the solution obtained for a linearly varying density smoothly tended to that characterizing the constant density case as $|\boldsymbol{g}| \to 0$.

6.1 Validation tests

The first set of numerical examples is concerned with the prism considered by Tsoulis (2012) and D'Urso (2014), which the interested reader is referred to for further details.

This simple example has been considered for comparing the results obtained by means of formulas (51), (54) and (60) with those obtained by using the approach first contributed by Pohanka (1998), Holstein (2003) and Hamayun et al. (2009).

This has already been described in D'Urso (2014) for the potential so that we limit hereafter to address the case of the first and second-order derivatives.

To derive a formula for the first-order derivative as an alternative to that reported in (28), let us fix our attention on formula (25) and observe that, instead of carrying out the derivative, as detailed in (26), we can invoke the identity (A11) to get

$$d_{\boldsymbol{p}} \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} dV = -\int_{Fr(\Omega)} \frac{\boldsymbol{r}\otimes\boldsymbol{n}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} dA \tag{86}$$

Hence, setting

$$\boldsymbol{H} = \left[d_{\boldsymbol{p}} \int_{\Omega} \frac{\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} dV \right]^t = -\int_{Fr(\Omega)} \frac{\boldsymbol{n}\otimes\boldsymbol{r}}{(\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}} dA \tag{87}$$

one has from (24)

$$d_{\boldsymbol{p}} U(\boldsymbol{p}) = G[(\delta_o + \boldsymbol{g}\cdot\boldsymbol{p}) d_{\boldsymbol{p}} U_c + U_c \boldsymbol{g} + \boldsymbol{H}\boldsymbol{g}] \tag{88}$$

instead of (28).

The second derivatives of the potential can now be derived from the previous formula thus obtaining the expression

$$d^2_{\boldsymbol{p}} U(\boldsymbol{p}) = G\Big\{(\delta_o + \boldsymbol{g}\cdot\boldsymbol{p}) d^2_{\boldsymbol{p}} U_c + d_{\boldsymbol{p}} U_c \otimes \boldsymbol{g} + \boldsymbol{g}\otimes d_{\boldsymbol{p}} U_c + d_{\boldsymbol{p}}(\boldsymbol{H}\boldsymbol{g})\Big\} \tag{89}$$

which replaces formula (47).

The numerical results obtained from the implementation of formulas (88) and (89) have been successfully compared with those resulting from (54) and (60) although space limitations do not allow us to express formulas (88) and (89) as function of the 3D coordinates of the vertices of the polyhedron.

Here we simply limit ourselves to point out that application of formula (88) leads to an expression exhibiting a singularity when the observation point P is aligned with one of the edges of the polyhedron.

Actually, expressing the second integral in (87) as a sum extended to the faces of the polyhedron, one has finally to consider a quantity of the kind $\boldsymbol{n}_i \otimes (\boldsymbol{T}_{F_i} \boldsymbol{\kappa}_{F_i})$ so that, recalling (53), the relevant singularity is discussed according to formulas (77)–(79).

However, this singularity is not filtered out by d_i in formula (88) so that this last one, differently from (28) and its computational counterpart (54), can exhibit a singularity for special positions of the observation point.

For this reason the numerical comparison between formulas (28) and (88), as well as between formulas (47) and (89), has been carried out for positions of the prism with respect to the observation point which did not induce any kind of singularity.

Finally, concerning the second derivative, it is apparent that formula (89) is computationally more expensive than formula (47), though yielding the same final result whenever singularities, which affect both expressions, do not come into play.

Actually, the expression of $d_{\boldsymbol{p}}(\boldsymbol{H}\boldsymbol{g})$ in (89) as sum extending to the faces of the polyhedron is characterized by a number of addends which is superior to that appearing in the expression of $\boldsymbol{F}_{F_i}$ entering $d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})$ in (50).

This can be explained by the fact that $d_{\boldsymbol{p}}(\boldsymbol{H}\boldsymbol{g})$, differently from $d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})$, is symmetric so that formula (89) returns a symmetric expression as sum of the symmetric tensors $d_{\boldsymbol{p}} U_c \otimes \boldsymbol{g} + \boldsymbol{g} \otimes d_{\boldsymbol{p}} U_c$ and $d_{\boldsymbol{p}}(\boldsymbol{H}\boldsymbol{g})$.

Conversely, formula (47) yields the same result by summing two non-symmetric tensors, i.e. $d_{\boldsymbol{p}} U_c \otimes \boldsymbol{g}$ and $d_{\boldsymbol{p}}(\boldsymbol{A}\boldsymbol{g})$, what increases the computational efficiency.

6.2 Performance tests

The second set of numerical examples is concerned with the polyhedral models of the asteroid 433 Eros. They are managed by the Planetary Science Institute (USA) and are available, together with models of further asteroids, at the address http://www.psi.edu/pds/archive/shape.html. In particular, we dispose of models characterized by 1,708, 7,790, 10,152 and 22,540 faces.

Data organization for all models of asteroids can be downloaded at the address http://sbn.psi.edu/pds/asteroid/NEAR_A_5_COLLECTED_MODELS_V1_0/data/msi/. In particular the topology of the model is based on the classical BREP approach, see e.g. Hoffmann (2002).

The gravitational potential as well as its first- and second-order derivatives are reported in Tables 1 and 2 for the four polyhedral models detailed above. The results are referred to a computation point coincident with the origin of the reference frame and have been obtained by assuming progressively decreasing values of $|\boldsymbol{g}|$.

This has been made to check that the results smoothly tended to those pertaining to the constant density polyhedron whose properties are characterized by G and δ_o. Specifically, the values $G = 6.67259 \times 10^{-11} \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ and $\delta_o = 2.670\,\mathrm{kg}\,\mathrm{m}^{-3}$, already used in D'Urso (2014) have been considered.

Table 1 Values of the potential, of its first- and second-order derivatives for the asteroid 433 Eros for different numbers N_F of faces and values of $\boldsymbol{g} = (a, a, a)$

Quantity	$N_F = 1,708$	$N_F = 7,790$	$N_F = 10,152$	$N_F = 22,540$
$a = 0\ kgm^{-4}$				
$U(m^2s^{-2})$	6.86887867707387E-05	6.92842481400130E-05	6.93271268377282E-05	6.93982176669190E-05
$U_{,x}(ms^{-2})$	-1.76825455806395E-07	-1.75812975733089E-07	-1.75537974761583E-07	-1.75262758962081E-07
$U_{,y}(ms^{-2})$	-7.77589850085898E-07	-7.78077155728463E-07	-7.78713449883631E-07	-7.78984963349048E-07
$U_{,z}(ms^{-2})$	1.35387195350492E-07	1.38468353379429E-07	1.39237443923765E-07	1.39508051491509E-07
$U_{,xx}(s^{-2})$	-1.65555066637347E-07	-1.63038064763321E-07	-1.62714879023875E-07	-1.62462008020327E-07
$U_{,yy}(s^{-2})$	-1.09567457938927E-06	-1.09985783423449E-06	-1.10032568022212E-06	-1.10093869945156E-06
$U_{,zz}(s^{-2})$	-9.77571732540457E-07	-9.75905479569893E-07	-9.75760819321694E-07	-9.75400671095918E-07
$U_{,xy}(s^{-2})$	-1.99693089627594E-07	-2.00090602844709E-07	-1.99999504484800E-07	-2.00006733615375E-07
$U_{,yx}(s^{-2})$	-1.99693089627593E-07	-2.00090602844710E-07	-1.99999504484799E-07	-2.00006733615372E-07
$U_{,xz}(s^{-2})$	-7.98974456293249E-09	-8.11786871968376E-09	-8.14636885483864E-09	-8.17479181774823E-09
$U_{,zx}(s^{-2})$	-7.98974456293250E-09	-8.11786871968375E-09	-8.14636885483863E-09	-8.17479181784828E-09
$U_{,yz}(s^{-2})$	2.56495326728823E-08	2.71763638018798E-08	2.73729390845432E-08	2.76631302645930E-08
$U_{,zy}(s^{-2})$	2.56495326728824E-08	2.71763638018799E-08	2.73729390845431E-08	2.76631302645927E-08
$a = 0.01\ kgm^{-4}$				
$U(m^2s^{-2})$	6.86888657842561E-05	6.92843268161264E-05	6.93272054972610E-05	6.93982962655924E-05
$U_{,x}(ms^{-2})$	-1.76947523601149E-07	-1.75936156602627E-07	-1.75661240728568E-07	-1.75386151316269E-07
$U_{,y}(ms^{-2})$	-7.77633417887935E-07	-7.78121049094904E-07	-7.78757370694887E-07	-7.79028928523441E-07
$U_{,z}(ms^{-2})$	1.35327595186142E-07	1.38408211682296E-07	1.39177261219073E-07	1.39447796910420E-07
$U_{,xx}(s^{-2})$	-1.65553742102120E-07	-1.63036747812190E-07	-1.62713564132670E-07	-1.62460695190660E-07
$U_{,yy}(s^{-2})$	-1.09566875474620E-06	-1.09985200594119E-06	-1.10031984716257E-06	-1.10093286435820E-06
$U_{,zz}(s^{-2})$	-9.77572746676750E-07	-9.75906516786030E-07	-9.75761862298800E-07	-9.75401716100040E-07
$U_{,xy}(s^{-2})$	-1.99689515038430E-07	-2.00087030222490E-07	-1.99995930509420E-07	-2.00003159653862E-07
$U_{,yx}(s^{-2})$	-1.99689515038429E-07	-2.00087030222491E-07	-1.99995930509419E-07	-2.00003159653860E-07
$U_{,xz}(s^{-2})$	-7.98958936345423E-09	-8.11772885218685E-09	-8.14623289778943E-09	-8.17465790498175E-09
$U_{,zx}(s^{-2})$	-7.98958936345421E-09	-8.11772885218687E-09	-8.14623289778945E-09	-8.17465790498177E-09
$U_{,yz}(s^{-2})$	2.56519379262715E-08	2.71787593404635E-08	2.73753341257516E-08	2.76655253092312E-08
$U_{,zy}(s^{-2})$	2.56519379262714E-08	2.71787593404634E-08	2.73753341257517E-08	2.76655253092314E-08

Table 2 Values of the potential, of its first- and second-order derivatives for the asteroid 433 Eros for different numbers N_F of faces and values of $\boldsymbol{g} = (a, a, a)$

Quantity	$N_F = 1,708$	$N_F = 7,790$	$N_F = 10,152$	$N_F = 22,540$
$a = 0.1\ kgm^{-4}$				
$U(m^2s^{-2})$	6.86895769059140E-05	6.92850349011404E-05	6.93279134330556E-05	6.93990036536528E-05
$U_{,x}(ms^{-2})$	-1.78046133753754E-07	-1.77044784428543E-07	-1.76770634431454E-07	-1.76496682503983E-07
$U_{,y}(ms^{-2})$	-7.78025528106348E-07	-7.78516089392793E-07	-7.79152657996291E-07	-7.79424615093019E-07
$U_{,z}(ms^{-2})$	1.34791193706942E-07	1.37866936408691E-07	1.38635616877254E-07	1.38905505680454E-07
$U_{,xx}(s^{-2})$	-1.65541821284870E-07	-1.63024895252030E-07	-1.62701730111975E-07	-1.62448879723770E-07
$U_{,yy}(s^{-2})$	-1.09561633295855E-06	-1.09979955130148E-06	-1.10026734962663E-06	-1.10088034851798E-06
$U_{,zz}(s^{-2})$	-9.77581873903421E-07	-9.75915851731235E-07	-9.75771249092770E-07	-9.75411121137220E-07
$U_{,xy}(s^{-2})$	-1.99657343735980E-07	-2.00054876622550E-07	-1.99963764731069E-07	-1.99970940031701E-07
$U_{,yx}(s^{-2})$	-1.99657343735979E-07	-2.00054876622551E-07	-1.99963764731070E-07	-1.99970940030169E-07
$U_{,xz}(s^{-2})$	-7.98819256816273E-09	-8.11647004469815E-09	-8.14500928438416E-09	-8.17345269012552E-09
$U_{,zx}(s^{-2})$	-7.98819256816275E-09	-8.11647004469813E-09	-8.14500928438418E-09	-8.17345269012550E-09
$U_{,yz}(s^{-2})$	2.56735852067742E-08	2.72003191877357E-08	2.73729688949674E-08	2.76878071073921E-08
$U_{,zy}(s^{-2})$	2.56735852067743E-08	2.72003191877356E-08	2.73729688949674E-08	2.76878071073922E-08
$a = 1\ kgm^{-4}$				
$U(m^2s^{-2})$	6.86966881224931E-05	6.92921157513462E-05	6.93349927910015E-05	6.94060775342575E-05
$U_{,x}(ms^{-2})$	-1.89032235279799E-07	-1.88131062687700E-07	-1.87864571460313E-07	-1.87601994381119E-07
$U_{,y}(ms^{-2})$	-7.81946630290483E-07	-7.82466492371684E-07	-7.83105531010331E-07	-7.83381480788847E-07
$U_{,z}(ms^{-2})$	1.29427178914942E-07	1.32454183672644E-07	1.33219173459062E-07	1.33482593380792E-07
$U_{,xx}(s^{-2})$	-1.65422613112420E-07	-1.62906369650410E-07	-1.62583389904190E-07	-1.62330725054800E-07
$U_{,yy}(s^{-2})$	-1.09509211508209E-06	-1.09985783423449E-06	-1.09974237426716E-06	-1.10035519011572E-06
$U_{,zz}(s^{-2})$	-9.77673146169930E-07	-9.76009201182920E-07	-9.75865117032490E-07	-9.75505171509010E-07
$U_{,xy}(s^{-2})$	-1.99335630711520E-07	-1.99733340623184E-07	-1.99642106947481E-07	-1.99643374646936E-07
$U_{,yx}(s^{-2})$	-1.99335630711521E-07	-1.99733340623183E-07	-1.99642106947479E-07	-1.99643374646935E-07
$U_{,xz}(s^{-2})$	-7.98974456293250E-09	-8.11786871968376E-09	-8.13277315039512E-09	-8.16140054154627E-09
$U_{,zx}(s^{-2})$	-7.98974456293250E-09	-8.11786871968375E-09	-8.13277315039514E-09	-8.16140054154625E-09
$U_{,yz}(s^{-2})$	2.58900580117315E-08	2.74159176604357E-08	2.76124432066242E-08	2.79026347259675E-08
$U_{,zy}(s^{-2})$	2.58900580117314E-08	2.74159176604355E-08	2.76124432066241E-08	2.79026347259676E-08

Differently from the previous values, the modulus and the direction of the gradient $\boldsymbol{g}$ of the density variation are completely arbitrary and have been considered only for numerical purposes.

As a final remark we observe that, being the origin of the reference frame located within the interior of the body, a further check of the results for the second-order derivatives is made possible via the Poisson's equation (Heiskanen and Moritz 1967)

$$d_{xx}^2 U_c + d_{yy}^2 U_c + d_{zz}^2 U_c = -4\pi G \delta_o \tag{90}$$

In particular, the right-hand side of the previous equation is equal to $-22.377 \times 10^{-7}\ \mathrm{s}^{-2}$, a value substantially equivalent to the sum of the quantities $U_{xx} + U_{yy} + U_{zz}$ appearing in Table 1 corresponding to the value $a = 0$. Actually, making reference to the column $N_F = 22,540$ it turns out $U_{xx} + U_{yy} + U_{zz} = -22.388 \times 10^{-7}\ \mathrm{s}^{-2}$

7 Conclusions

The approach presented in D'Urso (2014), based on distribution theory and differential calculus, has been extended here to consistently address the singularities affecting the integrals required for computing the gravity effects of polyhedral bodies with linearly varying density.

We have shown in particular that the potential and the gravity vector can be given an expression in which singularities are ruled out, thus avoiding the introduction of small positive numbers advocated by some authors in order to avoid undefined operations.

In addition the second derivatives of the potential exhibit the same source of singularities as for the constant density polyhedron, i.e. the case of the observation point aligned with one of the edges of the polyhedron.

For the case with linearly varying density, however, the singularity affects all the entries of the second derivative and not only the off-diagonal terms as for the constant density polyhedron.

The analytical results derived in the paper concerning the range of validity of the derived formulas have been validated numerically by a Matlab® code which has also been used to evaluate the gravity effects induced by polyhedral models of a real asteroid.

Forthcoming papers will be devoted to specialize formulas (51), (54) and (60) to the case of a prism, due to its importance for applications in Digital Terrain Modelling, and to extending the formulation presented in this paper to gravity models in which the density varies with a polynomial or an exponential law.

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