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Short Note

The gravitational attraction of a right rectangular prism with density varying with depth following a cubic polynomial

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INTRODUCTION

Over the last two centuries, the right rectangular prism has played an important role in both 3D forward and inverse gravity problems. This occurs because complex geometries are easy to build using a finite collection of prisms, and analytic solutions are available to compute the gravitational attraction of the prisms. In an article by Nagy (1966), a full derivation is presented for computing the gravitational attraction caused by a rectangular prism of constant density. The article by Nagy (1966) drew attention to earlier solutions of the same problem. Corbató (1966) pointed out that Everest (1830, 94-97) calculated closed-form expressions equivalent to those of Nagy (1966) for computing the horizontal and vertical gravitational effects of a prism, and he used them to estimate the deflection of the plumb bob caused by the Satpura Range in India. De Bremaecker (1966) called attention to the textbook by MacMillan (1930) where a formula for computing the gravitational potential caused by a prism is derived, along with a simple method to compute its derivatives along a coordinate axis. In addition, Nagy (1966) cited work by Sorokin (1951) and Haáz (1953) indicating that these authors had developed an equivalent solution, which turns out to be identical to the one later derived by Banerjee and Gupta (1977).

The uniform density prism has long been a useful tool in geophysical interpretation; however, in some geological settings, the constant density assumption does not hold. For example, one of the main physical processes occurring during the geological evolution of a sedimentary basin is compaction, where density increases exponentially with depth. Moreover, the geologic structure of a sedimentary basin can be complicated by other geologic processes such as chemical cementation, nonuniform stratigraphic layering, facies changes, and structural disruptions. Because of these processes, simple constant-density models cannot capture the complexities found in sedimentary basins. Figure 1 shows an excellent example of the variation of density versus depth from density logging of 46 wells in Green Canyon, located offshore Louisiana U.S., in the Gulf of Mexico (Li, 2001). The usefulness of the 3D prism in gravity computations may be significantly enhanced, therefore, by the ability to compute gravity effects caused by nonuniform density contrasts.

Only a few authors have treated the forward gravity problem for the case of nonuniform density variations. Chai and Hinze (1988) derived a hybrid method to compute the gravity anomaly caused by a right rectangular prism for which the density contrast decreases exponentially with depth. This method combines a solution in the space domain for uniform density with a wavenumber-domain approach based on the fast Fourier transform (FFT). A more general solution to the problem, which combines analytic and numerical methods of integration for the case of a rectangular prism whose density varies as a function of depth, was provided by García-Abdeslem (1992). More recently, Gallardo-Delgado et al. (2003) derived an analytic solution of the forward gravity problem for a right rectangular prism where density varies according to a quadratic polynomial law.

In the present work, I derive an analytic solution to compute the gravitational attraction caused by a rectangular prism with density that varies as a function of depth following a cubic polynomial law. The analytic solution described here is successfully validated using the numerical method derived in García-Abdeslem (1992). Both methods yield identical results.

FORMULATION OF THE PROBLEM

In a Cartesian coordinate system, the vertical component of the gravity field observed at a point $P(x_0, y_0, z_0)$ outside a 3D prismatic source body of density varying with depth is given

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$$g_P = G \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz \frac{\rho(z)(z-z_0)}{R^3}. \quad (1)$$

In equation 1, G is the universal gravitational constant, $R = [X^2 + Y^2 + Z^2]^{1/2}$, $X = x - x_0$, $Y = y - y_0$, and $Z = z - z_0$, where (x, y, z) represent the source coordinates. The prism is bounded by the planes $x = x_1, x_2$, $y = y_1, y_2$, and $z = z_1, z_2$. Its density varies with depth following a cubic polynomial law given by $\rho(z) = p + qz + rz^2 + sz^3$, where p, q, r, and s are constant coefficients.

Noting that $dX = dx, dY = dy, dZ = dz$, and substituting $z = Z + z_0$ in the density function, equation 1 can be written as

$$g_P = G \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \int_{Z_1}^{Z_2} \frac{dZ}{R^3} \times (\rho_1 Z + \rho_2 Z^2 + \rho_3 Z^3 + \rho_4 Z^4), \quad (2)$$

where

$$\rho_1 = p + qz_0 + rz_0^2 + sz_0^3, \quad (3)$$

$$\rho_2 = q + 2rz_0 + 3sz_0^2, \quad (4)$$

$$\rho_3 = r + 3sz_0, \quad (5)$$

$$\rho_4 = s, \quad (6)$$

and the limits in the integrals $X_1 = x_1 - x_0$, $X_2 = x_2 - x_0$, $Y_1 = y_1 - y_0$, $Y_2 = y_2 - y_0$, $Z_1 = z_1 - z_0$, and $Z_2 = z_2 - z_0$ reflect the change in the variables of integration. For simplicity, equation 2 is written in the following form:

$$g_P = G \sum_{k=1}^{4} I_k, \quad (7)$$

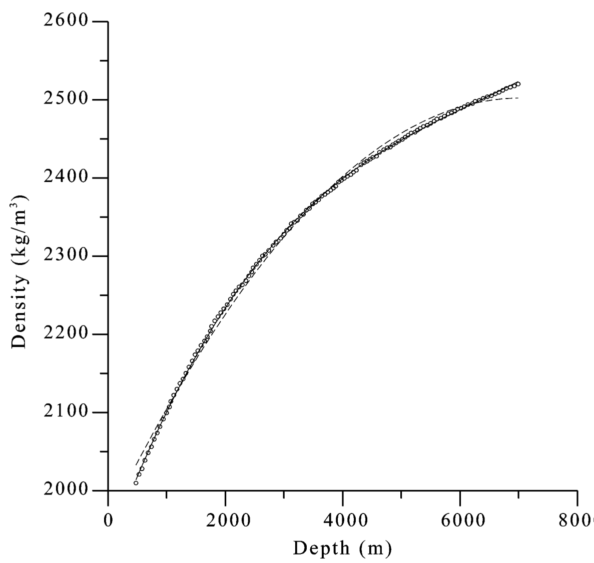

Figure 1. Open circles denote density versus depth data from Green Canyon, offshore Louisiana, U.S. Dashed and continuous lines are second- and third-degree best-fitting polynomials, respectively. The second-degree polynomial fails to fit the data on several portions of the curve.

where

$$I_k = \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \int_{Z_1}^{Z_2} dZ \left\{ \rho_k \frac{Z^k}{R^3} \right\}. \quad (8)$$

The following results were obtained using integrals found in Gradshtein and Ryzhik (1980) and by use of the partial-integration technique. The integration of equation 8 for $k = 1$, along the z, y, and x directions, respectively, yields the following results:

$$I_1 = -\rho_1 \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \left[\frac{1}{R} \right] \Bigg|_{Z_1}^{Z_2}, \quad (9)$$

$$I_1 = -\rho_1 \int_{X_1}^{X_2} dX \, [\ln(Y+R)]\big|_{Y_1}^{Y_2}\big|_{Z_1}^{Z_2}, \quad (10)$$

$$I_1 = \rho_1 \left\{ Z \arctan \frac{YX}{ZR} - X \ln(Y+R) - Y \ln(X+R) \right\} \Bigg|_{X_1}^{X_2} \Bigg|_{Y_1}^{Y_2} \Bigg|_{Z_1}^{Z_2}. \quad (11)$$

Equation 11 is the solution to the forward gravity problem for the case of a prism with constant density, attributed first to Sorokin (1951), Haáz (1953), and later to Banerjee and Gupta (1977). For $k = 2$, the integration of equation 8 yields the following expressions:

$$I_2 = \rho_2 \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \left[\ln(Z+R) - \frac{Z}{R} \right] \Bigg|_{Z_1}^{Z_2}, \quad (12)$$

$$I_2 = \rho_2 \int_{X_1}^{X_2} dX \left\{ Y \ln(Z+R) - X \arctan \frac{ZY}{XR} \right\} \Bigg|_{Y_1}^{Y_2} \Bigg|_{Z_1}^{Z_2}, \quad (13)$$

$$I_2 = \rho_2 \left[\frac{Z^2}{2} \arctan\left(\frac{XY}{ZR}\right) - \frac{Y^2}{2} \arctan\left(\frac{ZX}{YR}\right) - \frac{X^2}{2} \arctan\left(\frac{ZY}{XR}\right) + YX \ln(Z+R) \right] \Bigg|_{X_1}^{X_2} \Bigg|_{Y_1}^{Y_2} \Bigg|_{Z_1}^{Z_2}. \quad (14)$$

The following expressions are obtained by integrating equation 8 for $k = 3$ as

$$I_3 = \rho_3 \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \left[R + \frac{(X^2+Y^2)}{R} \right] \Bigg|_{Z_1}^{Z_2}, \quad (15)$$

$$I_3 = \rho_3 \int_{X_1}^{X_2} dX \, [YR + X^2 \ln(Y+R)]\big|_{Y_1}^{Y_2}\big|_{Z_1}^{Z_2}, \quad (16)$$

$$I_3 = \rho_3 \left[\frac{Z^3}{3} \arctan \frac{XY}{ZR} + \frac{X^3}{3} \ln(Y+R) + \frac{Y^3}{3} \ln(X+R) + \frac{2}{3} XYR \right] \Bigg|_{X_1}^{X_2} \Bigg|_{Y_1}^{Y_2} \Bigg|_{Z_1}^{Z_2}. \quad (17)$$

Finally, the integration of equation 8 for $k = 4$ yields the following results:

$$I_4 = \rho_4 \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \left[\frac{1}{2} ZR + \frac{(X^2+Y^2)}{R} Z - \frac{3}{2}(X^2+Y^2)\ln(Z+R) \right] \Bigg|_{Z_1}^{Z_2}, \quad (18)$$

$$I_4 = \rho_4 \int_{X_1}^{X_2} dX \left[X^3 \arctan \frac{YZ}{XR} + \frac{1}{2} YZR - \frac{1}{2} Y^3 \ln(Z+R) - \frac{3}{2} X^2 Y \ln(Z+R) \right] \Bigg|_{Y_1}^{Y_2} \Bigg|_{Z_1}^{Z_2}, \quad (19)$$

$$I_4 = \rho_4 \left[\frac{X^4}{4} \arctan \frac{YZ}{XR} + \frac{Y^4}{4} \arctan \frac{ZX}{YR} + \frac{Z^4}{4} \arctan \frac{XY}{ZR} + \frac{XYZR}{4} - \frac{XY}{2}(X^2+Y^2) \times \ln(Z+R) \right] \Bigg|_{X_1}^{X_2} \Bigg|_{Y_1}^{Y_2} \Bigg|_{Z_1}^{Z_2}. \quad (20)$$

Substituting the results obtained in equations 11, 14, 17, and 20 into equation 7, the gravity anomaly caused by a prism of density varying with depth following a cubic polynomial is finally obtained.

Figure 2 shows the gravity anomaly caused by a prism that extends along the x- and y-directions between 10 and 20 km, and between 0 and 8 km along the z-direction. To avoid a singularity in the analytic solution, occurring when the observation and the source coordinates coincide, the observer location was set –15 cm off the plane $z = 0$ (the top of the prism). The density function used in this example was obtained by subtracting a reference density of 2670 kg/m^3 from the cubic polynomial that best fits the density data (Li, 2001) in Green Canyon, located offshore Louisiana, U.S., in the Gulf of Mexico:

$$\rho(z) = -747.7 + 203.435\, z - 26.764\, z^2 + 1.4247\, z^3, \quad (21)$$

where density is in kg/m^3 and z is in km.

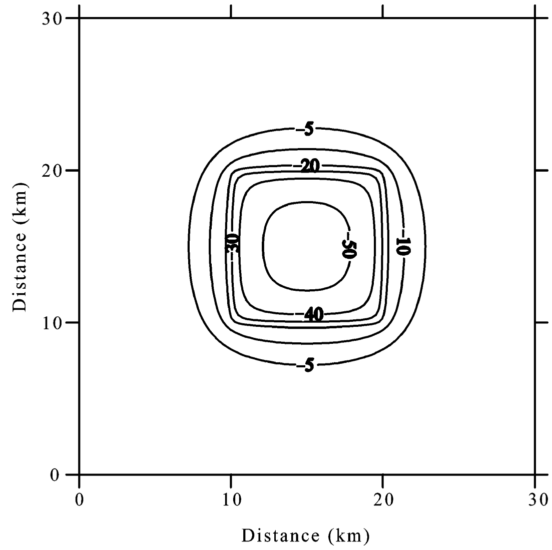

Figure 2. Gravity anomaly computed using the derived analytic solution. Contours are shown in mGal.

Validation of method

To validate the analytic solution, the gravity anomaly caused by the prism described above was computed using a method (García-Abdeslem, 1992) that combines analytic and numeric methods of integration. In this method, the density is defined analytically as a function of depth, the integration of equation 1 along the x- and y-direction is carried out analytically, and the resulting expression is numerically integrated along the z-direction.

For this comparison, I separate computation of the gravity anomalies caused by the constant, linear, quadratic, and cubic terms of density versus depth function as given in equation 21. The results are shown in Figure 3. In all of these particular

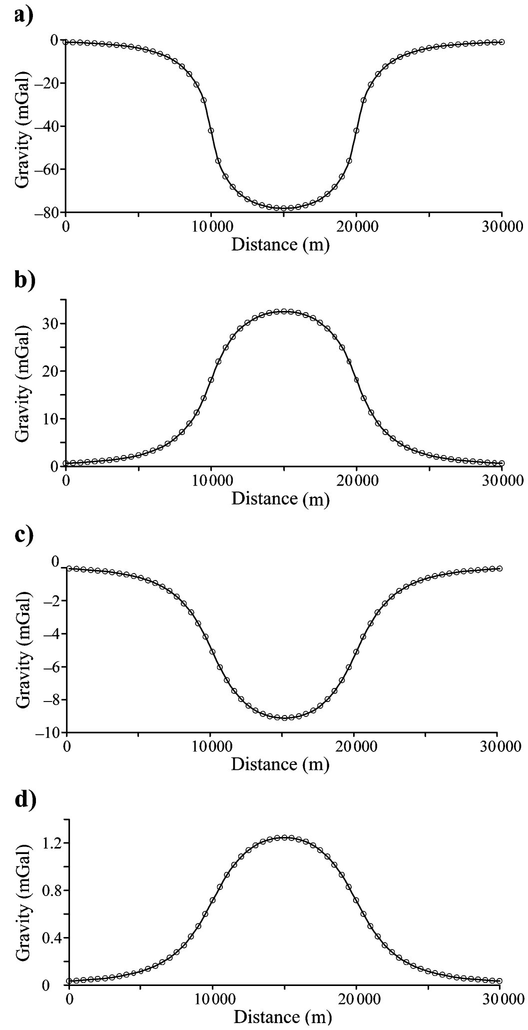

Figure 3. Profiles showing the gravitational attraction caused by each term of the density function: (a) constant, (b) linear, (c) quadratic, and (d) cubic variation of density with depth. In all cases, the continuous line corresponds to the analytic solution and open circles to the numerical solution. The results are identical up to the fifth decimal place.

cases, the gravity anomalies computed using the analytic solution and the numerical method are identical up to the fifth decimal place.

CONCLUSION

A new analytic solution has been obtained for computing the gravity anomaly caused by a right rectangular prism of density varying with depth following a cubic polynomial law. The analytic solution was successfully validated using a previously reported numerical method (García-Abdeslem, 1992).

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