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Calculating velocity fields across plate boundaries from observed shear rates

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Summary. A general horizontal velocity field can be expressed uniquely in terms of the velocities at points along a line and an integral function of the 2-D shear rate. Consequently, the relative motion across a plate boundary can be calculated from observed shear rates by taking the velocities along a line within one of the plates to be zero.

Introduction

Observed shear rates in the South Island, New Zealand, can be approximately modelled by assuming that the relative motion of the Indian and Pacific plates produces a broad zone of uniform deformation 100 km wide (Walcott 1978). This illustrates that when the difference in horizontal velocity across a region is known the shear rate can be found.

The converse is also true. Because two linear, first-order differential equations relate the horizontal components of velocity to the components of the 2-D shear rate, the horizontal velocity field can be expressed uniquely in terms of the shear rate and the velocities at points along a line. Thus, when the velocities at one edge of a region are known (or are taken to be zero) the horizontal motion within the region is uniquely determined by the shear rates.

Theory

Consider the representation of the boundary zone between two plates shown in Fig. 1 where the directions x and y are parallel and normal to the trend of the zone respectively. Resolve the horizontal motion into a velocity u along x and v along y. The two components of the shear rate $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are given by

$$\dot{\gamma}_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

$$\dot{\gamma}_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

(Walcott 1978).

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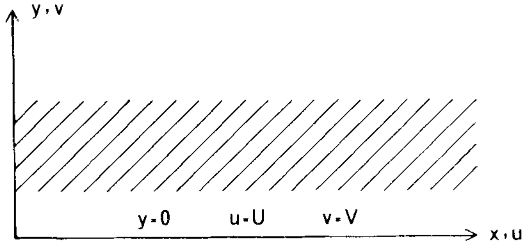

Figure 1. Plan view of the region between two plates.

These differential equations together with constraints

$u(x, 0) = U(x)$

$v(x, 0) = V(x)$

on the horizontal components of velocity at points along the x-axis uniquely define u and v once $\dot{\gamma}_1$, $\dot{\gamma}_2$, U and V are given. With u and v expressed as the real and imaginary components of a complex number, the solution is

$$u(x, y) + iv(x, y) = \bar{U}(x, iy) + i\bar{V}(x + iy)$$
$$-i \int_0^y \bar{\dot{\gamma}}_1 \, [x + i(y - y'), y'] \, dy'$$
$$+ \int_0^y \bar{\dot{\gamma}}_2 \, [x + i(y - y'), y'] \, dy' \tag{1}$$

where $\bar{U}$, $\bar{V}$, $\bar{\dot{\gamma}}_1$, $\bar{\dot{\gamma}}_2$ are the analytic complex-valued functions of complex arguments corresponding to the real-valued functions U, V, $\dot{\gamma}_1$, $\dot{\gamma}_2$ of real arguments (i.e. $\bar{U}(x + i0) = U(x)$ and if $U(x) = x^N$, $\bar{U}(x + iy) = (x + iy)^N$). In order to verify this solution we proceed as follows. Clearly, the values of $u(x, 0)$ and $v(x, 0)$ given by equation (1) are $U(x)$ and $V(x)$ respectively. To show that the differential equations are also satisfied, equation (1) is differentiated with respect to x and y:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \bar{U}'(x + iy) + i\bar{V}'(x + iy)$$
$$-i \int_0^y \frac{\partial}{\partial x} \bar{\dot{\gamma}}_1 [x + i(y - y'), y'] \, dy'$$
$$+ \int_0^y \frac{\partial}{\partial x} \bar{\dot{\gamma}}_2 [x + i(y - y'), y'] \, dy'$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\bar{U}'(x + iy) - \bar{V}'(x + iy)$$
$$-i\,\dot{\gamma}_1(x, y) + \int_0^y \frac{\partial}{\partial x} \bar{\dot{\gamma}}_1 [x + i(y - y'), y'] \, dy'$$
$$+ \dot{\gamma}_2(x, y) + i \int_0^y \frac{\partial}{\partial x} \bar{\dot{\gamma}}_2 [x + i(y - y'), y'] \, dy'.$$

This gives

$$\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=\dot{\gamma}_1+i\dot{\gamma}_2$$

whose real and imaginary parts are the required equations.

Finally we show that equation (1) is the only possible solution. Suppose that u_1+iv_1 and u_2+iv_2 are two solutions. Then their difference $u+iv$ satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

and thus $u+iv$ is an analytical function of $x+iy$. Since also $u+iv$ is zero at all points on the x-axis, then $u+iv$ is zero everywhere (Copson 1935, section 4.51). Therefore the difference between u_1+iv_1 and u_2+iv_2 must be identically zero and there is only one solution; in fact, the one given in equation (1).

Below are three artificial examples to illustrate the properties of the solution. (An actual problem is discussed later.) In the first two examples the horizontal velocity along the x-axis and then the shear rate are non-zero. In the third example the velocity from the first example and the shear rate from the second example are combined, resulting in a simpler solution.

Example (1): if $U(x)=0$, $V(x)=V_0\exp(kx)$, $\dot{\gamma}_1(x,y)=0$, $\dot{\gamma}_2(x,y)=0$, then

$$u(x,y)+iv(x,y)=iV_0\exp(kx+iKy)=-V_0\exp(kx)\sin ky+iV_0\exp(kx)\cos ky.$$

Consequently,

$$u(x,y)=-V_0\exp(kx)\sin ky.$$

$$v(x,y)=V_0\exp(kx)\cos ky.$$

Example (2): if $U(x)=0$, $V(x)=0$, $\dot{\gamma}_1(x,y)=0$, $\dot{\gamma}_2(x,y)=kV_0\exp(kx)$

$$u(x,y)+iv(x,y)=\int_0^y kV_0\exp\left[kx+ik(y-y')\right]dy'=iV_0\exp(kx)\left[1-\exp(iky)\right].$$

Consequently,

$$u(x,y)=V_0\exp(kx)\sin ky$$

$$v(x,y)=V_0\exp(kx)(1-\cos ky).$$

Example (3): $U(x)=0$, $V(x)=V_0\exp(kx)$, $\dot{\gamma}_1(xy)=0$, $\dot{\gamma}_2(x,y)=kV_0\exp(kx)$. Then superimposing the solutions in the above examples

$$u(x,y)=0$$

$$v(x,y)=V_0\exp(kx).$$

The rates of dilation $\dot{\sigma}$ and rotation $\dot{\omega}$ given by

$$\dot{\sigma}=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\dot{\omega}=\frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$$

(Walcott 1978) are uniquely defined when u and v are known. In example (1)

$$\dot{\sigma}(x, y) = -kV_0 \exp(kx) \sin ky$$

$$\dot{\omega}(x, y) = -kV_0 \exp(kx) \cos ky.$$

$\dot{\sigma}$ and $\dot{\omega}$ can be used instead of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ to obtain u and v, in which case

$$v(x, y) + iu(x, y) = \bar{V}(x + iy) + i\,\bar{U}(x + iy) + 2\int_0^y \bar{\dot{\sigma}}\,[x + i(y - y'), y']\,dy' + 2i\int_0^y \bar{\dot{\omega}}\,[x + i(y - y'), y']\,dy',$$

and $\dot{\gamma}_1$ and $\dot{\gamma}_2$ can be calculated from u and v.

Practice

Average values of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ for regions several kilometres wide can be deduced from geodetic data (e.g. Bibby 1976). Smooth surfaces have to be fitted to such values before equation (1) is used – otherwise $\bar{\dot{\gamma}}_1$ and $\bar{\dot{\gamma}}_2$ cannot be calculated. I have found that it may be best to fit simple functions involving few parameters (see below).

Nth order polynomials in x and y

$$\sum_{p=0}^{N} \sum_{q=0}^{p} A\,pq\, x^{p-q} y^q$$

are probably the easiest functions to use. If the functions $U(x)$ and $V(x)$ are zero and

$$\dot{\gamma}_1(x, y) = \sum_{p=0}^{N} \sum_{q=0}^{p} G_{pq}^1 x^{p-q} y^q$$

$$\dot{\gamma}_2(x, y) = \sum_{p=0}^{N} \sum_{q=0}^{p} G_{pq}^2 x^{p-q} y^q$$

then

$$u(x, y) = \sum_{p=0}^{N+1} \sum_{q=0}^{p} U_{pq} x^{p-q} y^q$$

$$v(x, y) = \sum_{p=0}^{N+1} \sum_{q=0}^{p} V_{pq} x^{p-q} y^q$$

where the coefficients U_{pq} and V_{pq} are defined by the equations

$$U_{p0} = 0, p = 0, \ldots, N+1$$

$$V_{p0} = 0, p = 0, \ldots, N+1$$

$$U_{(p+1)\,(q+1)} = [G_{pq}^2 - (p + 1 - q)\; V_{(p+1)q}]/(q+1), p = 0, \ldots, N, q = 0, \ldots, p$$

$$V_{(p+1)\,(q+1)} = [(p + 1 - q)\; U_{(p+1)\,q} - G_{pq}^1]/(q+1), p, 0, \ldots, N, q = 0, \ldots, p.$$

Also

$$\dot{\sigma}(x, y) = \sum_{p=0}^{N} \sum_{q=0}^{p} S_{pq}\, x^{p-q} y^{q}$$

$$\dot{\omega}(x, y) = \sum_{p=0}^{N} \sum_{q=0}^{p} W_{pq}\, x^{p-q} y^{q}$$

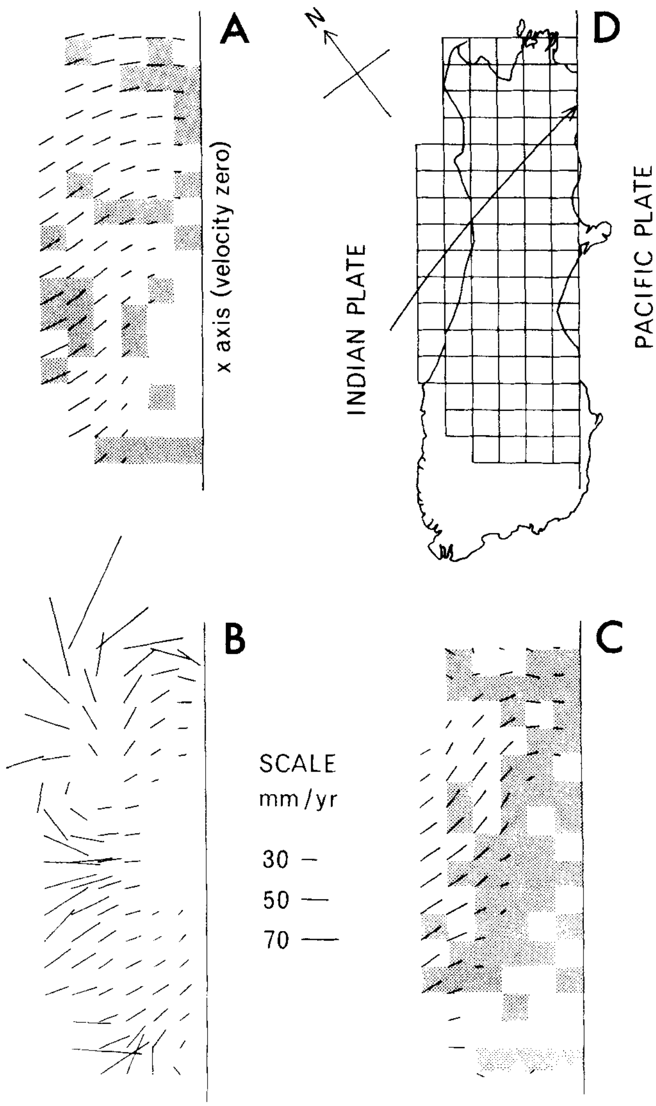

Figure 2. Motion of the South Island, New Zealand, relative to the Pacific plate. (A) From shear rates alone using second-order polynomials showing the squares where the average shear rates are known (shaded). (B) From shear rates alone using fifth-order polynomials. (C) From shear rates and rates of dilatation using fifth-order polynomials, showing the squares with constraints on the average rate of dilatation (shaded). (D) Map showing the grid of 40 × 40 km squares and the direction in which the Indian plate is moving at 45 mm yr^{-1} relative to the Pacific plate. The scales are such that the distance between neighbouring grid-points is the same as the length of 50 mm yr^{-1} velocity vectors. Velocities less than 10 mm yr^{-1} have not been plotted.

where

$$S_{pq} = \frac{1}{2}\left[(p+1-q)\, U_{(p+1)q} + (q+1)\, V_{(p+1)\,(q+1)}\right]$$

$$W_{pq} = \frac{1}{2}\left[(q+1)\, U_{(p+1)\,(q+1)} - (p+1-q)\, V_{(p+1)q}\right].$$

When modelling motion where one plate is assumed to be fixed the x-axis can be chosen to lie in that plate – then $U(x)$ and $V(x)$ are zero. If, however, the x-axis cannot be placed so that $U(x)$ and $V(x)$ are known accurately, the solutions u and v will depend strongly on the forms chosen for $U(x)$ and $V(x)$ as can be seen by comparing the results in examples (2) and (3) above where the shear rates are the same.

The present method has been used to reconstruct the horizontal motion of the South Island, New Zealand, which lies between the Indian and Pacific plates, relative to the Pacific plate. R. I. Walcott kindly provided estimates of the average values of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ in 28 squares 40 × 40 km. In Fig. 2(A) second-order polynomials, with six coefficients G^1pq and six coefficients G^2pq, are used to approximate these values. The coefficients were calculated by least squares with the weight assigned to each observation inversely proportional to the square of its standard error. The surfaces do not fit all the values of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ to within three standard errors because the pattern of deformation is too complicated to be modelled precisely using such simple functions.

In Fig. 2(B) fifth-order polynomials, with 21 coefficients G^1pq and 21 coefficients G^2pq, are used. Now, the fitted surfaces closely reproduce all the values of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ but the observations of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are too sparse to constrain the values of the rates of dilatation and rotation. That is, the first derivatives of u and v with respect to x and y have impossibly large values (with even larger standard errors) even though

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

are close to the observed values of $\dot{\gamma}_1$ and $\dot{\gamma}_2$.

Thus, the derivatives of u and v had to be further constrained before a realistic model of the motion could be obtained which also fitted all the values of $\dot{\gamma}_1$ and $\dot{\gamma}_2$. Imprecise estimates of the average rates of dilatation (with large standard errors) have been deduced from the rates of uplift in 49 of the 40 × 40 km squares in Fig. 2 (R. I. Walcott, private communication). Fifth-order polynomials, with the same set of coefficients as in Fig. 2(B), were used to fit these values simultaneously and the observed shear rates by weighted least squares, giving good results. The solution (Fig. 2C) is similar to that in Fig. 2(A), showing that much of the detailed motion could be inferred from the shear rates alone even though the observations are thinly scattered. In both cases the velocities at the west coast are in general agreement with both the direction and the 45 mm yr^{-1} rate with which the Indian and Pacific plates are converging (Walcott 1978 who used the pole given by Minster *et al.* 1974).

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