CHAPTER 4

Normal Earth Gravity Field Versus Gravity Effect of Layered Ellipsoidal Model

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4.1 INTRODUCTION

Loosely speaking, Bouguer anomaly is the measured gravity corrected for the known or modeled gravity effects at a planetary scale. The first of these corrections is subtracting the normal field or normal gravity. One can find several, slightly different, definitions of the normal field. For instance, the one by Jacoby and Smilde (2009) states, "Normal field would be observed on the oblate reference ellipsoid of rotation that approximates the equilibrium figure best fitted to Earth."

The well-known formula of Somigliana (1929) is commonly used to calculate the normal field:

$$\gamma = \gamma_E \frac{1 + k\sin^2\varphi}{\sqrt{1 - e^2\sin^2\varphi}} \tag{4.1}$$

where $k = \left(b\gamma_P / a\gamma_E\right) - 1, \quad e^2 = \left(a^2 - b^2\right)/a^2$, a is the major semiaxis, b is the minor semiaxis, γ_P and γ_E are theoretical gravity values at the equator and at the poles, respectively, e is the first eccentricity, and φ is the latitude. This formula is based on very important condition of constant potential on the surface of the ellipsoid. No information about density distribution within the model is known nor required (there are infinitely many density distributions to produce the same total mass). The (normal, reference) ellipsoid is defined by four parameters: major semiaxis a, flattening f, angular velocity ω, and GM value (product of the gravitational constant G and the mass of the ellipsoid M). These parameters are based on satellite measurements. The values of γ_P and γ_E are calculated from defining parameters.

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4.2 SOME PROBLEMS WITH THE NORMAL FIELD

We want to discuss two problems linked with using Eq. (4.1). The first one is the mass of the model. The surface of the ellipsoidal model represents the zero level of altitude, but its mass, represented by the GM value incorporated in Eq. (4.1), contains both the masses of topography and atmosphere. That means, these masses have been "pushed" into the ellipsoidal model interior. However, any moving of the masses from their correct or real positions is in general inadmissible in applied geophysics/gravimetry. The problem of topography is much more complicated, and it would require intensive and careful studies (and lots of hard discussions, because the "audience" is demanding), so it will be omitted in our model for now. On the other hand, the GM value with Earth's atmosphere excluded is presented in the specification NIMA (2000) and is used in presented calculations.

The second problem is the density distribution within the model. Formula (4.1) is based on idea of a constant potential on the surface. Yet the density distribution must be specific in order to produce a constant potential along the surface (see Fig. 4.1 and e.g., Moritz, 1968). One can find overly simple two-layered example consists of homogenous core enclosed by nearly homogenous mantle in Moritz (1990). Two layers are, of course, not enough. However, in the literature, one

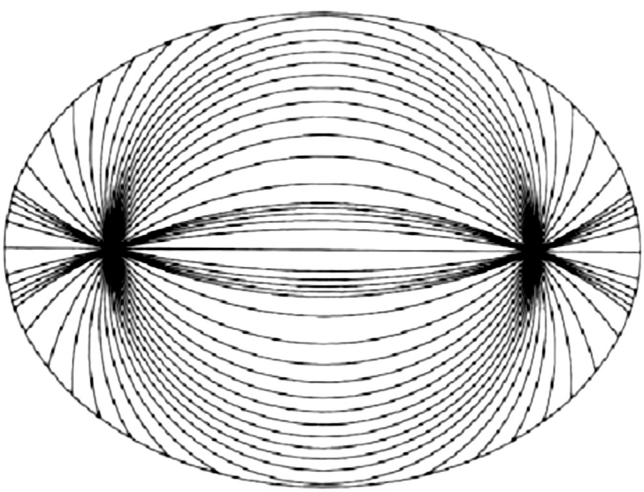

Figure 4.1 An example of one possible density distribution which would produce constant potential on the surface of ellipsoid, according to Conway (2000). Black lines stand for constant density surfaces which, in this cross-section, are represented by isodensity lines.

can find the statements which say that the question of density distribution is not important (e.g., Heiskanen and Moritz, 1967, p. 64). While this could be admissible for geodesy in the problem of the shape of the Earth, it is not acceptable for geophysics/gravimetry. Since the true inner structure of the Earth is close to a layered model (as we are convinced by seismic observations), the usefulness of the discussed condition of the constant potential should be critically revaluated.

4.3 THE GRAVITY POTENTIAL AND GRAVITY EFFECT OF THE EQUIPOTENTIAL MODEL

The first question is the magnitude of the difference between potentials and gravity effects of models with and without including the mass of the atmosphere. The defining parameters of the ellipsoidal model according to the specifications of NIMA (2000) are as follows:

- semimajor axis $a = 6378{,}137$ (m)
- flattening $f = 1/298.257223563$
- GM with the atmosphere included $GM = 3986{,}004.418 \cdot 10^{8}$ ($\mathrm{m}^3/\mathrm{s}^2$)
- angular velocity $\omega = 729{,}2115 \cdot 10^{-11}$ (rad/s).

The required derived parameters are as follows:

- semiminor axis $c = a(1 - f) \approx 6356{,}752.3142\ \ (\mathrm{m})$
- GM with the atmosphere excluded $GM' = 3986{,}000.9 \cdot 10^{8}\ \ (\mathrm{m}^3/\mathrm{s}^2)$

The value of gravitational constant G in this specification is: $G = 6.673 \cdot 10^{-11}\ \ \mathrm{m}^3/\mathrm{kg\ per\ s}^2$ what is a little bit less than the most recent update according to Mohr et al. (2012) which reads: $G = 6.67384 \cdot 10^{-11}\ \ \mathrm{m}^3/\mathrm{kg\ per\ s}^2$. However, we continue in using the NIMA value of G in order to stay consistent with defining constants (a, f, ω, GM) and because the last mentioned GM' value is still not common in other specifications of constants e.g., Petit and Luzum (eds.) (2010).

The formula for gravity potential, as a sum of the gravitational potential and the centrifugal component, in calculation points on its surface is (e.g., Heiskanen and Moritz, 1967, after some simplification)

$$U = \frac{GM}{e} \arctan\left(\frac{e}{c}\right) + \frac{1}{3}\omega^2 a^2, \tag{4.2}$$

where e is the linear eccentricity: $e = \sqrt{a^2 - c^2}$.

The value of potential for the GM value is $U = 62,636,851.7146\ \mathrm{m}^2/\mathrm{s}^2$. The solution for the case without atmosphere is (GM' is used): $U = 62,636,796.4957\ \mathrm{m}^2/\mathrm{s}^2$.

The gravity effects were obtained by using Eq. (4.1). First, we calculated the gravity values on the pole γ_P and on the equator γ_E. These values are derived from the defining parameters via the respective formulae (e.g., Heiskanen and Moritz, 1967):

$$\gamma_P = \frac{GM}{a^2}\left(1 + \frac{m}{3} \cdot \frac{e'q'}{q}\right), \tag{4.3}$$

$$\gamma_E = \frac{GM}{ac}\left(1 - m - \frac{m}{6} \cdot \frac{e'q'}{q}\right), \tag{4.4}$$

where $e' = e/c$ is the second eccentricity, $m = \left(\omega^2 a^2 c/GM\right)$, $q' = 3\left(a^2/e^2\right)\left[1 - \left(c/e\right)\arctan\left(e/c\right)\right] - 1$ and $q = 1/2\left[\left(1 + \left(3c^2/e^2\right)\right)\right.$ $\left.\arctan\left(e/c\right) - 3\left(c/e\right)\right]$. The formula of Somigliana (1929) (Eq. (4.1)) is then used to calculate the normal (theoretical) gravity as a function of φ. The results for GM (atmosphere included) and GM' (atmosphere excluded) are close to each other, and the difference between them is in the range from 0.863 to 0.868 mGal (Fig. 4.2).

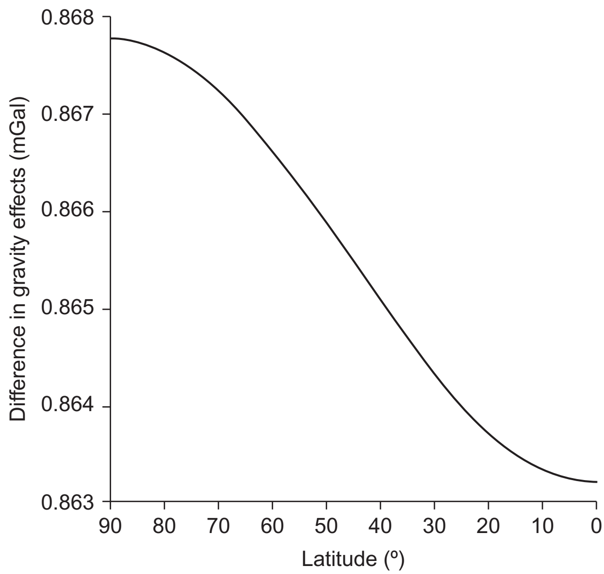

Figure 4.2 The difference in the normal (theoretical) gravity between the two discussed cases of the equipotential model (the Earth mass with and without atmosphere).

Please note that the commonly used atmospheric correction for $h = 0$ is 0.874 mGal which is a slightly higher figure than are those of our difference interval in Fig. 4.2, which is interesting. We can only speculate that this small-scale disagreement can be a result of the spherical approximation used by Wenzel while our values are based on the ellipsoidal model.

4.4 THE GRAVITY POTENTIAL AND GRAVITY EFFECT OF THE LAYERED MODEL

We attempted to address the previously mentioned problems (namely, the unwanted displacements of the existing masses, the realistic density distribution within the model, and the arbitrary position of the calculation point) by introducing the model with several layers in order to better approximation of the real density distribution within the Earth, without applying the limiting condition of a constant potential on the surface. There are, of course, some difficulties with this approach either. The first one is the shape of the layers. Generally, there are three possibilities how to construct the layered ellipsoidal model:

- to use a family of confocal ellipsoids,
- to take an advantage of a set of homeoids (linearly reduced ellipsoids), or
- to use ellipsoids with no specific mutual relations.

We have chosen the second option because of the important qualities of homeoids. The first one is the shape of the layers. The thickness of such layers is increasing from polar to equatorial areas and the eccentricity and sphericity of all layers remain the same (these properties do not hold for the confocal family). The second important advantage is the zero-gravitational effect within the inner hollow volume of a homeoid. This property is well known for the spherical shell, which represents, from this aspect, just a special case of homeoid. This property simplifies the calculations significantly.

The next problem is the density distribution itself. All the density tables we have come across were calculated for spherical approximation of the Earth (with the radius which provides equal volume to that of the ellipsoid). While the layers in our model are homeoids, we assigned the thicknesses of layers to the semimajor axis of the ellipsoidal model, and the thicknesses in semiminor axis were subsequently calculated. Unfortunately, the tabulated values of density are usually assigned to the upper and lower boundaries of each layer, and no exact information about density distribution within the layer is given, except of some graphical representations. As the first trial, we used the density table of Ochaba (1986) which is based on Bullen (1975), see Table 4.1. The density for each layer was set as the mean of the tabulated values for both the upper and the lower edge of individual layer, respectively. The density function within each layer could be, in the future work, partially replaced by a system of thin layers with a small density step (e.g., 0.01 g/cm^3 what is the common accuracy of the reported density values). It should be noted here that there are solutions for the gravitational potential of heterogeneous ellipsoid available, e.g., Dyson (1891) or Rahman (2001). These solutions are extraordinarily challenging, but they exceed the scope of this paper. Therefore, this possibility remains as another task for the future work for the time being.

The first step was to calculate the GM constant for this layered model and compare it with the tabulated one (atmosphere excluded). This task was reduced to calculate the mass of our model, while the constant G is known. We considered our first result $GM_{\text{layered}} = 3989,853.8 \cdot 10^8\ \text{m}^3/\text{s}^2$ too high, so that each density value in Table 4.1 has been reduced by a constant value of 0.00261 g/cm^3. The new result was $GM_{\text{layered}} = 3986,000.97 \cdot 10^8\ \text{m}^3/\text{s}^2$ what we regarded as a good fit to GM'. Finally, the input parameters for our

Table 4.1 Earth Density Distribution According to Ochaba (1986)

Depth (km)		Density (g/cm^3)	
Upper edge	**Lower edge**	**Upper edge**	**Lower edge**
0	33	2.84	3.32
33	245	3.32	3.51
245	984	3.51	4.49
984	2000	4.49	5.06
2000	2700	5.06	5.40
2700	2886	5.40	5.69
2886	4000	9.95	11.39
4000	4560	11.39	11.87
4560	4710	11.87	12.30
4710	5160	12.30	12.74
5160	6378.137	12.74	13.03

Table 4.2 Input Parameters for the Layered Ellipsoidal Model

Layers			
No.	Thickness (km)		Density (kg/m³)
	Equatorial	Polar	
1	33	32.889	3002.39
2	212	211.289	3412.39
3	739	736.522	3997.39
4	1016	1012.594	4772.39
5	700	697.653	5227.39
6	186	185.376	5542.39
7	1114	1110.265	10,667.39
8	560	558.122	11,627.39
9	150	149.497	12,082.39
10	450	448.491	12,517.39
11	1218.137	1214.053	12,882.39

model are in Table 4.2. Please note that the density units were changed to kg/m^3.

Next, the gravity potential and gravity effect of the layered model are calculated. This is done in several steps (based on Kartvelishvili, 1982):

- Conversion of the calculation point coordinates to the Cartesian system. The conversion formulae are as follows:

$$x^2 = \left[\frac{a^2}{\sqrt{a^2\cos^2\varphi + c^2\sin^2\varphi}} + h\right]^2 \cos^2\varphi,$$

$$z^2 = \left[\frac{c^2}{\sqrt{a^2\cos^2\varphi + c^2\sin^2\varphi}} + h\right]^2 \sin^2\varphi,$$

  where h stands for altitude (positive, zero, or negative), φ is the latitude, and a and c are semimajor and semiminor axes, respectively. The calculation points are situated on the surface of the model ($h = 0$), and the step is 1° in latitude in the case of our example shown in Fig. 4.3.
- Calculation of the parameter λ, which "expresses" the change in the length of semiaxis. The gravitational effect of a given ellipsoid is

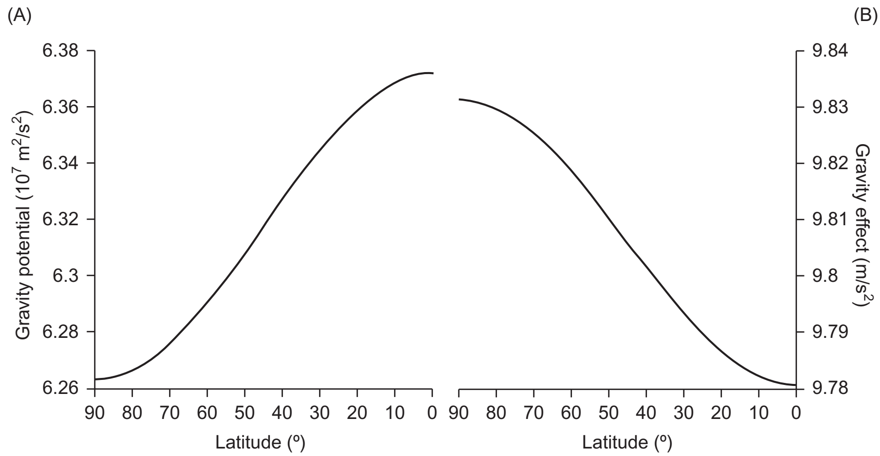

Figure 4.3 The gravity potential (A) and effect (B) of our layered ellipsoidal model.

calculated as the effect of confocal ellipsoid which passes through calculation point (if located outside of the given ellipsoid). If the original semiaxes are a and c, the semiaxes of the larger ellipsoid are $a' = \sqrt{a^2 + \lambda}$ and $c' = \sqrt{c^2 + \lambda}$. The formula for the parameter λ is as follows:

$$\lambda = \frac{x^2 + z^2 - a^2 - c^2 + \sqrt{e^4 - 2e^2(x^2 - z^2) + (x^2 + z^2)^2}}{2},$$

where $e = \sqrt{a^2 - c^2}$.

- Calculation of the parameter p: $p = \mathrm{sign}(\lambda)\left(\frac{\mathrm{sign}(\lambda)+1}{2}\right)\cdot\lambda = \begin{cases} \lambda & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda \leq 0 \end{cases}$.
- The centrifugal components: $W = 1/2\omega^2 x^2$ for the potential and $a_x = \omega^2 x$ for the effect
- The gravity potential of a single homogenous ellipsoid:

$$U = \frac{\pi G \sigma a^2 c}{e^3}\left[(2z^2 + 2e^2 - x^2)\operatorname{arctg}\frac{e}{\sqrt{c^2+p}} + e\left(x^2\frac{\sqrt{c^2+p}}{a^2+p} - 2z^2\frac{1}{\sqrt{c^2+p}}\right)\right] + \frac{1}{2}\omega^2 x^2 \tag{4.5}$$

where σ is the density, and G is the gravitational constant.

- Calculating of the x- and z-components of the gravitational attraction vector:

$$g_x = -\frac{2x\pi a^2 c\sigma G}{e^3}\left[\operatorname{arctg}\frac{e}{\sqrt{c^2+p}} - \frac{e\cdot\sqrt{c^2+p}}{(a^2+p)}\right] \tag{4.6}$$

$$g_z = \frac{4z\pi a^2 c\sigma G}{e^3}\left[\operatorname{arctg}\frac{e}{\sqrt{c^2+p}} - \frac{e}{\sqrt{c^2+p}}\right] \tag{4.7}$$

(e.g., MacMillan, 1930—integral form, or Kartvelishvili, 1982—closed form)

- The gravity effect of a single homogenous ellipsoid:

$$|\vec{g}| = \sqrt{(g_x + a_x)^2 + (g_z)^2} \tag{4.8}$$

The gravitational potential/effect of a single layer (homeoid) is obtained as a difference of the potentials/effects of two ellipsoids. The parameter p was added to the formula by us to avoid "if else" statements in our MATLAB code. These formulae allow us to calculate gravitational or gravity potential/effect of a homogenous ellipsoid for an arbitrary position of the calculation point (even inside the model). This is an obvious practical advantage in comparison with the formula of Somigliana (1929) which allows calculating these values only on the surface of the model. However, the formula for gravity potential and gravity effect of equipotential model extended to positive heights can be found in e.g., Li and Götze (2001), but still, such formulae do not work for negative heights.

The results for gravity potential and gravity effect are depicted in Fig. 4.3A and B.

The important thing is comparison of this solution with the effect of the homogenous ellipsoid with the same size and GM. Those effects are identical in the case of spherical model, and since the flattening of our layered ellipsoidal model can be regarded as insignificant ($f = 1/298.257223563 = 0.003352810664747$), one could expect similar solutions for ellipsoidal model, too. The result of this experiment is presented in Fig. 4.4. It is obvious that the difference is significant. The expected equality of effects will be true for models based on a confocal family of ellipsoids, but, as we already indicated above, such model is not suitable for modeling the layered structure of the real Earth.

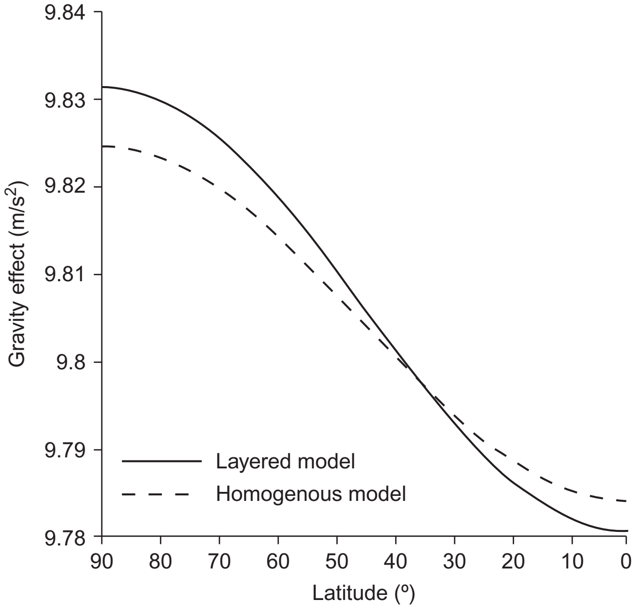

Figure 4.4 The comparison of gravity effects of the layered and homogenous ellipsoidal models.

4.5 COMPARISON OF THE RESULTS FOR THE EQUIPOTENTIAL AND THE LAYERED MODELS

The obtained results are depicted in Fig. 4.5A, while their difference is in Fig. 4.5B. This difference varies from −40.2 to 76.6 mGal. It is evident that there is nothing like a constant shift involved in this difference, what was, in fact, expected, and that the difference is not symmetric either around its zero or around 45° of latitude. The perfect fit is not possible, except of a situation when a thin layer with negative density is placed on the surface of layered model, as described in Pizzeti (1894), what is of course completely unrealistic. The goal was not to obtain the fit, but to show that a layered model can produce an acceptable output, with some previously mentioned advantages in comparison with the equipotential model.

If we take a closer look at the latitudes that limit e.g., Slovakia (from ~47.7°N to ~49.7°N), the discussed difference gets the shape of a linear trend with the range close to 4 mGal (Fig. 4.6). This will be true for many countries, while only the value of the range will be changed depending on the meridional extent of the particular country.

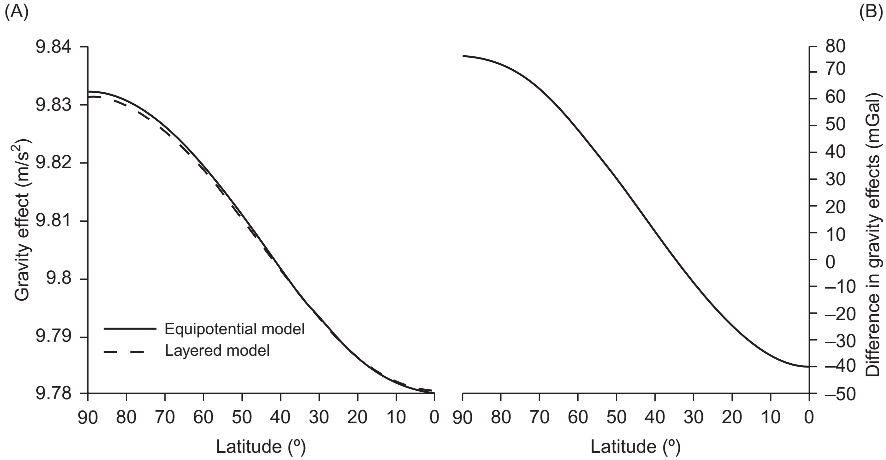

Figure 4.5 (A) The comparison of the solutions for equipotential and layered models; (B) the difference equipotential minus layered solution.

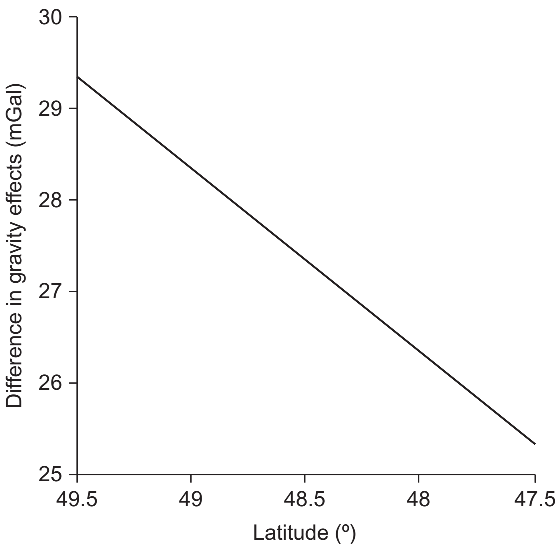

Figure 4.6 The difference between equipotential and layered solutions within the limiting latitudes of Slovakia.

Several pros and cons of both approaches were already mentioned above. The most important problem with the equipotential model is the ignorance of any possible density distribution within the real Earth, and the usage of the improper *GM* constant value what is a result of the unwarranted displacement of the atmospheric and especially the topographic masses. The advantage, on the other hand, is very simple and fast calculation and requires only four input parameters. The problem of the layered model is in the specific values of density in each layer, which come mostly from indirect measurements (they are prevailingly estimated from the velocities of seismic waves), and they are usually estimated for a spherical approximation of the Earth. From the other side, the advantages are the possibility of an arbitrary position of the calculation point or the generally realistic density distribution within the model.

4.6 RELATION OF THE NORMAL FIELD TO THE FREE-AIR CORRECTION

As it is commonly understood and as we already wrote above, the Bouguer anomaly is, roughly speaking, the measured gravity minus the expected gravity. This expected gravity is deemed to be caused by some "normal Earth" as its gravity effect. For our convenience, this effect is standardly composed of some partial effects which, within the process of the data processing, are usually applied as corrections to the measured gravity. The first effect or correction (and the largest one) is the normal field and the second is usually the free-air correction:

$$\Delta g_B = g_m - g_n + 0.3086h + \cdots \tag{4.9}$$

where Δg_B is the Bouguer anomaly, g_n is the normal field defined on the ellipsoid, $0.3086h$ is the free-air (or sometimes Faye's) correction, and h is the (unspecified) elevation. Please note that the literature is not in accord regarding the definition of the free-air correction. Some works consider that it contains also the normal field as such, while in other sources the free-air correction is understood solely as the $-0.3086h$ term. We have to mention here that certain part of the geophysical community still interpret the $-0.3086h$ term as the reduction (relocation) of the measured field to the surface of the ellipsoidal model. This approach, however, has been abandoned within the applied geophysics (LaFehr, 1991; Li and Götze, 2001; Hinze et al., 2005), and recently, Bouguer anomaly has been considered strictly as a measuring point or station quantity.

The examples of the discussed vertical gradient for the layered ellipsoidal model are listed in Table 4.3 along with the "standard" values (obtained from the second order formula for GRS80 according to Hinze et al. 2013, p. 133, Eq. 6.13 (Please note that there is a minus

Table 4.3 The Comparison of The Possible Free-Air Correction Values

Latitude (°)	GRS80 (mGal)		Layered ellipsoidal model (mGal)		Homogenous sphere (mGal)	
	$h = 0$ m	$h = 2000$ m	$h = 0$ m	$h = 2000$ m	$h = 0$ m	$h = 2000$ m
0	−0.308769	−0.308481	−0.308666	−0.308376	−0.308811	−0.308521
45	−0.308549	−0.308261	−0.308407	−0.308118	−0.308544	−0.308254
90	−0.308329	−0.308041	−0.308152	−0.307863	−0.308279	−0.307989

sign missing in front of the linear term in the cited source.) or NIMA, 2000, p. 4−2, Eq. 4.3). The gradient for homogenous sphere (with the "radius of equal volume") is presented for comparison, too.

If we rewrite Eq. (4.9) to

$$\Delta g_B = g_m - \left(g_n - 0.3086h\right) + \cdots \tag{4.10}$$

the whole term in the parentheses can be replaced either by the equation for the equipotential model in height h (Li and Götze, 2001) (and then to face the problem with the negative heights) or by the equation for the gravitational effect of the layered ellipsoidal model as described in this paper (any height). However, the latter will be more complicated.

4.7 CONCLUSION

We have tested the gravity effect of a layered ellipsoidal model as a possible alternative to the standard estimation of the so-called normal or theoretical gravity using the formula of Somigliana (1929). We also briefly comment some of the difficulties related to the two approaches. Our main goal however, was to initiate a broader discussion rather than to give any definitive solution.

We also briefly discuss the question of the free-air correction, which, in our approach, becomes an organic part of the normal gravity calculation and gets straightforward physical background.

While applied gravimetry does allow some approximations and has to work with the inherent ambiguity, it, on the other hand, does not allow any rearrangement of the Earth masses since that would be incompatible with the (geological) data interpretation. Although this may look rather peculiar, yet, it corresponds to the physical reality. From this aspect, the layered ellipsoidal model and its gravity effect could be an alternative way how to look at, and how to understand to the normal gravity despite the fact that there still are some important unanswered questions, both mentioned and unmentioned in our present contribution.

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REFERENCES

Bullen, K.E., 1975. The Earth's Density. Chapman & Hall, London.

Conway, J.T., 2000. Exact solutions for gravitational potential of a family of heterogeneous ellipsoids. Mon. Not. R. Astron. Soc. 316, 555–558.

Dyson, F.D., 1891. The potentials of ellipsoids of variable densities. Q. J. Pure Appl. Math. 25, 259–288.

Heiskanen, W.A., Moritz, H., 1967. Physical Geodesy. W. H. Freeman and Company.

Hinze, W.J., Aiken, C., Brozena, J., Coakley, B., Dater, D., Flanagan, G., et al., 2005. New standards for reducing gravity data: The North American gravity database. Geophysics 70, J25–J32.

Hinze, W.J., von Frese, R.R.B., Saad, A.H., 2013. Gravity and Magnetic Exploration. Cambridge University Press, 512 pp.

Jacoby, W., Smilde, P.L., 2009. Gravity Interpretation – Fundamentals and Application of Gravity Inversion and Geological Interpretation. Springer-Verlag, Berlin Heidelberg.

Kartvelishvili, K.M., 1982. Planetary Density Model and Normal Gravity Field of the Earth (in Russian). Publishing House "Nauka", Moscow.

LaFehr, T.R., 1991. Standardization in gravity reduction. Geophysics 56, 1170–1178.

Li, X., Götze, H.-J., 2001. Ellipsoid, geoid, gravity, geodesy and geophysics. Geophysics 66 (6), 1660–1668.

MacMillan, W.D., 1930. The Theory of The Potential. Dover Publications Inc, New York.

Mohr, P.J., Taylor, B.N., Newell, D.B., 2012. CODATA recommended values of the fundamental constants: 2010. Rev. Modern Phys. 84.

Moritz, H., 1968. Mass distributions for the equipotential ellipsoid. Bollettino di Geofisica Teorica ed Applicata 10 (37), 59–66.

Moritz, H., 1990, The Figure of the Earth: Wichman Verlag, Karlsruhe.

Nima Agency, 2000, TR 8350.2, Third Edition, Amendment 1, January 3, 2000, e-report.

Ochaba, Š., 1986, Geofyzika (in Slovak): SPN.

Petit, G., Luzun, B. (Eds.), 2010. IERS Conventions. IERS Conventions Centre, Frankfurt am Main.

Pizzeti, P., 1894. Sulla espressione della gravità alla superficie del geoide, supposto ellissoidic. Atti R. Accad. Lincei, Ser. V 3.

Rahman, M., 2001. On the Newtonian potentials of heterogeneous ellipsoids and elliptical disc. Proc. R. Soc. Lond. 457, 2227–2250.

Somigliana, C., 1929. Teoria generale del campo gravitazionale dell' ellissoide di rotazione. Mem. Soc. Astron. Ital. 4, 47 pp.