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Unification of Euler and Werner deconvolution in three dimensions via the generalized Hilbert transform

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ABSTRACT

The extended Euler deconvolution algorithm is shown to be a generalization and unification of 2-D Euler deconvolution and Werner deconvolution. After recasting the extended Euler algorithm in a way that suggests a natural generalization to three dimensions, we show that the 3-D extension can be realized using generalized Hilbert transforms. The resulting algorithm is both a generalization of extended Euler deconvolution to three dimensions and a 3-D extension of Werner deconvolution. At a practical level, the new algorithm helps stabilize the Euler algorithm by providing at each point three equations rather than one. We illustrate the algorithm by explicit calculation for the potential of a vertical magnetic dipole.

INTRODUCTION

Mushayandebvu et al. (1999) present a generalization of Euler deconvolution, called extended Euler deconvolution, which can be used to calculate physical property contrasts and dip information in addition to locations and depths. We show that extended Euler deconvolution also generalizes Werner deconvolution and so can be considered a unification and extension of the two algorithms. Furthermore, we show that the algorithm can be extended to three dimensions, providing both an extended 3-D Euler algorithm and a generalization of Werner deconvolution to three dimensions. This extension provides a unification of the two depth estimation schemes, an extension of both to the 3-D case, and a nontrivial application of the generalized Hilbert transforms of Nabighian (1984). We conclude our paper with an example that can be carried out in closed form, namely, a dipole source.

The term extended has several meanings in this paper. The first is that of Mushayandebvu et al. (1999), in which the 2-D Euler method for structural index zero is generalized in a way which makes it possible to calculate physical property contrasts and dip information in addition to locations and depths. The second, which is implicit in their work, is the addition of a second Euler equation for arbitrary structural index. The third meaning of extended is that additional Euler equations in three dimensions for an arbitrary structural index can be obtained if the field satisfying the Euler equation also satisfies Laplace's equation. The final meaning of the term is an application that we do not develop in detail but which follows from the existence of the additional Euler equations: the possibility of obtaining physical property contrast and dip information for two-dimensional, index zero structures from gridded (3-D) data. We believe the sense in which extended is used at any point in the paper is clear from the context and have avoided introducing pedantic terminology to distinguish these senses.

2-D CASE

Since Mushayandebvu et al. (1999) have already derived the necessary equations, we refer directly to their work. We first consider the case of Euler index zero, given in their equations (3) and (4):

$$(x - x_0)\frac{\partial M}{\partial x} + (z - z_0)\frac{\partial M}{\partial z} = \alpha, \tag{1}$$

$$(x - x_0)\frac{\partial M}{\partial z} - (z - z_0)\frac{\partial M}{\partial x} = \beta, \tag{2}$$

where x and z are spatial position and depth, respectively; x_0 and z_0 are source position and depth, respectively; and M is a homogeneous field. Here, we have made minor changes in notation to agree with our conventions in the 3-D case. Direct comparison of equations (1) and (2) with equation (4) of Hansen and Simmonds (1993), by writing out their equation in terms of real and imaginary parts, shows the equations are identical. Thus, extended Euler deconvolution for index zero is equivalent to Werner deconvolution (or at least equivalent to the version which uses the analytic signal), so extended Euler deconvolution generalizes Werner deconvolution.

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Mushayandebvu et al.'s (1999) equations for a nonzero Euler index appear to be completely new:

$$(x - x_0)\frac{\partial M}{\partial x} + (z - z_0)\frac{\partial M}{\partial z} + nM = 0, \quad (3)$$

$$(x - x_0)\frac{\partial M}{\partial z} - (z - z_0)\frac{\partial M}{\partial x} + n\mathcal{H}(M) = 0, \quad (4)$$

where n is the Euler index and $\mathcal{H}$ denotes the Hilbert transform. Their derivation requires that the field M satisfy Laplace's equation because they use a complex analytic transformation to obtain the second equation. We take some liberties in interpreting their second equation, but our interpretation agrees with theirs in the special case of a thin dike and furthermore can be shown to be correct. Since we effectively derive the 3-D equation later, we do not give the derivation here.

Equation (4) looks rather odd as it stands because two of its terms involve M but the Hilbert transform of M appears in the third. Using the Cauchy-Riemann equations (Nabighian, 1972),

$$\frac{\partial M}{\partial x} = -\frac{\partial}{\partial z}\mathcal{H}(M), \quad (5)$$

$$\frac{\partial M}{\partial z} = \frac{\partial}{\partial x}\mathcal{H}(M), \quad (6)$$

we can rewrite equation (4) as

$$(x - x_0)\frac{\partial}{\partial x}\mathcal{H}(M) + (z - z_0)\frac{\partial}{\partial z}\mathcal{H}(M) + n\mathcal{H}(M) = 0. \quad (7)$$

This shows that equation (4) is equivalent to the assertion that if a 2-D potential field is homogeneous with Euler index n, then so is its Hilbert transform. We furthermore see that the same transformation can be applied to the equations of index zero, so the general case can be written in the form

$$(x - x_0)\frac{\partial}{\partial x}M + (z - z_0)\frac{\partial}{\partial z}M + nM = \alpha,$$
$$(x - x_0)\frac{\partial}{\partial x}\mathcal{H}(M) + (z - z_0)\frac{\partial}{\partial z}\mathcal{H}(M) + n\mathcal{H}(M) = \beta, \quad (8)$$

where it is understood that α and β will generally vanish unlesss $n = 0$.

Equations (8) promise to be extremely useful in their own right as a unification and generalization of Euler and Werner deconvolution. However, they suggest more—that the algorithm might be extended to three dimensions, thereby providing both an extension of Werner deconvolution to the 3-D case and a generalization of the Euler equations. In the next section, we show this is the case.

3-D CASE

Suppose that M is a potential field,

$$\nabla^2 M = 0, \quad (9)$$

and that M also satisfies the generalized Euler equation with index n:

$$(\xi - x_0)\frac{\partial M}{\partial \xi} + (\eta - y_0)\frac{\partial M}{\partial \eta} + (\zeta - z_0)\frac{\partial M}{\partial \zeta} + nM = \alpha, \quad (10)$$

where ξ, η, and ζ are the x, y, and z spatial positions, respectively, and where α is a constant which normally vanishes except for $n = 0$. We carry it throughout both for generality and to handle this case along with that of $n > 0$. Let

$$r^2 = (\xi - x_0)^2 + (\eta - y_0)^2 \quad (11)$$

and define the 3-D Hilbert transforms by [Nabighian (1984), his equations (22) and (28)]

$$\mathcal{H}_x(M) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x - \xi}{r^3}M(\xi, \eta)\, d\xi d\eta,$$
$$\mathcal{H}_y(M) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y - \eta}{r^3}M(\xi, \eta)\, d\xi d\eta, \quad (12)$$
$$\boldsymbol{\mathcal{H}}(M) = \mathcal{H}_x(M)\mathbf{i} + \mathcal{H}_y(M)\mathbf{j}.$$

In the Appendix we show that the 3-D Hilbert transforms of M in the sense of Nabighian (1984) also satisfy equation (10) for all n but with different constants for the case $n = 0$. From equation (A-37) we obtain two equations which in compact form can be written as

$$(x - x_0)\frac{\partial}{\partial x}\boldsymbol{\mathcal{H}}(M) + (y - y_0)\frac{\partial}{\partial y}\boldsymbol{\mathcal{H}}(M)$$
$$+ (z - z_0)\frac{\partial}{\partial z}\boldsymbol{\mathcal{H}}(M) + n\boldsymbol{\mathcal{H}}(M) = \boldsymbol{\beta}. \quad (13)$$

Thus, if a solution of Laplace's equation satisfies an Euler equation with index n, so do its generalized Hilbert transforms.

EXAMPLE: THE VERTICAL MAGNETIC DIPOLE

To illustrate the 3-D case, we calculate the generalized Hilbert transforms of the potential of a vertical magnetic dipole explicitly and show they satisfy Euler's equation with index two, as predicted.

The potential V for a unit dipole moment is given by

$$V = \frac{z - z_0}{R^3}, \quad (14)$$

where $R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$. The derivatives of V are

$$\frac{\partial V}{\partial x} = -\frac{3(x - x_0)(z - z_0)}{R^5},$$
$$\frac{\partial V}{\partial y} = -\frac{3(y - y_0)(z - z_0)}{R^5}, \quad (15)$$
$$\frac{\partial V}{\partial z} = \frac{R^2 - 3(z - z_0)^2}{R^5}.$$

By direct substitution from equations (15), we obtain

$$(x - x_0)\frac{\partial V}{\partial x} + (y - y_0)\frac{\partial V}{\partial y} + (z - z_0)\frac{\partial V}{\partial z}$$
$$= \frac{-3\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right](z - z_0)}{R^5}$$
$$+ \frac{z - z_0}{R^3}$$
$$= -\frac{2(z - z_0)}{R^3}. \quad (16)$$

This shows that V satisfies Euler's equation with index two.

For brevity, we omit here the rather trivial but lengthy derivations required to calculate the Hilbert transforms of $\partial V/\partial x$, $\partial V/\partial y$, and $\partial V/\partial z$ for the vertical magnetic dipole. The results

Table 1. Hilbert transform relations for the vertical magnetic dipole.

$V = \dfrac{\Delta z}{R^3}$	$\mathcal{H}_x(V) = \dfrac{\Delta x}{R^3}$	$\mathcal{H}_y(V) = \dfrac{\Delta y}{R^3}$
$\dfrac{\partial V}{\partial x} = -\dfrac{3\Delta x \Delta z}{R^5}$	$\mathcal{H}_x\left(\dfrac{\partial V}{\partial x}\right) = \dfrac{R^2 - 3\Delta x^2}{R^5}$	$\mathcal{H}_y\left(\dfrac{\partial V}{\partial x}\right) = -\dfrac{3\Delta x \Delta y}{R^5}$
$\dfrac{\partial V}{\partial y} = -\dfrac{3\Delta y \Delta z}{R^5}$	$\mathcal{H}_x\left(\dfrac{\partial V}{\partial y}\right) = -\dfrac{3\Delta x \Delta y}{R^5}$	$\mathcal{H}_y\left(\dfrac{\partial V}{\partial y}\right) = \dfrac{R^2 - 3\Delta y^2}{R^5}$
$\dfrac{\partial V}{\partial z} = \dfrac{R^2 - 3\Delta z^2}{R^5}$	$\mathcal{H}_x\left(\dfrac{\partial V}{\partial z}\right) = -\dfrac{3\Delta x \Delta z}{R^5}$	$\mathcal{H}_y\left(\dfrac{\partial V}{\partial z}\right) = -\dfrac{3\Delta y \Delta z}{R^5}$

are summarized in Table 1. The symmetry of the expressions obtained is remarkable. By applying Hilbert transforms, we have achieved a circular rotation of the coordinate axes. This is strongly reminiscent of the 2-D case (Nabighian, 1972) in which the Hilbert transform of $\partial M/\partial x$ was shown to be $\partial M/\partial z$.

DISCUSSION

The results just derived provide a generalization of both Euler deconvolution and Werner deconvolution in three dimensions. At a practical level, the new algorithm helps stabilize the Euler algorithm by providing at each point three equations for each measured field rather than one. The Hilbert transforms of a field are locally independent of the field; thus, we expect measurable reductions in the uncertainties of the depth estimates obtained using the new equations. Preliminary experiments confirm this expectation.

Furthermore, the additional equations should make it possible to compute the physical property contrast and the strike and dip directions in the context of a particular model, such as a dipping contact. In addition, since the Hilbert transform of a constant is zero, expression (13) does not include a constant regional value, usually inserted in the standard Euler equation (10).

The new algorithm also generalizes immediately to the multiple-source case (Hansen and Suciu, 2000). This should have the immediate benefit of increasing the number of equations at each point, thereby stabilizing the depth estimates. Unfortunately, physical property and strike and dip angle information cannot be extracted in any obvious way except in the single-source case.

The new algorithm suggests some possible variations of the Euler deconvolution method which use the fact that there are three equations at each point rather than one. For example, for $n > 0$, it should be possible to solve for x_0, y_0, and z_0 pointwise using the three equations in three unknowns. Another possibility for $n > 0$ is to eliminate the structural index n between pairs of equations, yielding a system of two equations at each point which are still linear in x_0, y_0, and z_0, do not contain n explicitly, but are bilinear in the field variables. This approach might prove useful when carrying out calculations over a large area where various bodies with different structural indices n might be encountered.

On a more theoretical level, the new algorithm proves the appropriateness of the generalized Hilbert transform operators (Nabighian, 1984). The rather delicate interaction between the Euler equation, Laplace's equation, and the generalized Hilbert transform could not plausibly be preserved for any other transformation. It also strengthens the notion of the 3-D analytic signal, although it may be that some modification may yet be found which is invariant under changes in magnetization direction, as in the 2-D case.

One outstanding issue is whether the constants which appear on the right-hand side of the zero-index equations can in fact be calculated from invariants of the field. In that case, the number of unknowns in the zero-index equations could be reduced to match that in the $n > 0$ case. This issue is still under active investigation.

CONCLUSIONS

We have shown that the extended Euler deconvolution of Mushayandebvu et al. (1999) generalizes both Euler deconvolution and Werner deconvolution. A reformulation of their equations suggests a generalization to three dimensions, which we have carried out completely. This generalization unifies Euler and Werner deconvolution in a general 3-D setting. Explicit calculations for the potential of a vertical magnetic dipole verify that the equations are valid in this case.

REFERENCES

Hansen, R. O., and Simmonds, M., 1993, Multiple-source Werner deconvolution: Geophysics, **58**, 1792–1800.

Hansen, R. O., and Suciu, L., 2002, Multiple-source Euler deconvolution: Geophysics, in press.

Mushayandebvu, M. F., van Driel, P., Reid, A. B., and Fairhead, J. D., 1999, Magnetic imaging using extended Euler deconvolution: Presented at the 69th Ann. Internat. Mtg., Soc. Expl. Geophys.

Nabighian, M. N., 1972, The analytic signal of two-dimensional magnetic bodies with polygonal cross-section: Its properties and use for automated interpretation: Geophysics, **37**, 507–517.

——— 1984, Toward a three-dimensional automatic interpretation of potential field data via generalized Hilbert transforms: Fundamental relations: Geophysics, **49**, 780–786.

APPENDIX

DERIVATION OF 3-D HILBERT TRANSFORM RELATIONS

Suppose that M is a potential field satisfying the generalized Euler equation with index n:

$$(\xi - x_0)\frac{\partial M}{\partial \xi} + (\eta - y_0)\frac{\partial M}{\partial \eta} + (\zeta - z_0)\frac{\partial M}{\partial \zeta} + nM = \alpha, \tag{A-1}$$

where α is a constant which normally vanishes except for $n = 0$. Let

$$r^2 = (\xi - x_0)^2 + (\eta - y_0)^2, \tag{A-2}$$

$$\mathbf{D} = \frac{\partial}{\partial \xi}\mathbf{i} + \frac{\partial}{\partial \eta}\mathbf{j}, \tag{A-3}$$

and define the 3-D Hilbert transforms by [Nabighian (1984), equations (22) and (28)]

$$\begin{aligned}\mathcal{H}_x(M) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-\xi}{r^3}M(\xi,\eta)\,d\xi\,d\eta,\\ \mathcal{H}_y(M) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y-\eta}{r^3}M(\xi,\eta)\,d\xi\,d\eta,\\ \boldsymbol{\mathcal{H}}(M) &= \mathcal{H}_x(M)\mathbf{i} + \mathcal{H}_y(M)\mathbf{j}.\end{aligned} \tag{A-4}$$

Multiply equation (A-1) by $(1/2\pi)\mathbf{D}(1/r)$ and integrate over ξ and η from $-\infty$ to ∞:

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(\xi - x_0)\frac{\partial M}{\partial \xi} + (\eta - y_0)\frac{\partial M}{\partial \eta}\right.\\ &\quad\left. + (\zeta - z_0)\frac{\partial M}{\partial \zeta} + nM\right]\mathbf{D}\left(\frac{1}{r}\right)d\xi\,d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha\mathbf{D}\left(\frac{1}{r}\right)d\xi\,d\eta.\end{aligned} \tag{A-5}$$

We consider the terms in equation (A-5) one by one, working from the right. For the right-hand side we have

$$\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{D}\left(\frac{1}{r}\right)d\xi\,d\eta = 0. \tag{A-6}$$

The last term on the left-hand side is

$$\frac{n}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}M\mathbf{D}\left(\frac{1}{r}\right)d\xi\,d\eta = n\boldsymbol{\mathcal{H}}(M) \tag{A-7}$$

by equation (22b) of Nabighian (1984). This is essentially the spatial-domain definition of the generalized Hilbert transform. Similarly, the second-last term on the left side of equation (A-5) is

$$\frac{1}{2\pi}(\zeta - z_0)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial M}{\partial \zeta}\mathbf{D}\left(\frac{1}{r}\right)d\xi d\eta = (\zeta - z_0)\frac{\partial}{\partial \zeta}\boldsymbol{\mathcal{H}}(M). \tag{A-8}$$

The remaining two terms are a little more difficult:

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\eta - y_0)\frac{\partial M}{\partial \eta}\mathbf{D}\left(\frac{1}{r}\right)d\xi\,d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\eta - y_0)\frac{\partial M}{\partial \eta}\left(\frac{x-\xi}{r^3}\mathbf{i} + \frac{y-\eta}{r^3}\mathbf{j}\right)d\xi\,d\eta.\end{aligned} \tag{A-9}$$

The x-component of this equation is

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-\xi}{r^3}(\eta - y_0)\frac{\partial M}{\partial \eta}d\xi\,d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-\xi}{r^3}[(\eta - y) + (y - y_0)]\frac{\partial M}{\partial \eta}d\xi\,d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta - y}{r^3}(x - \xi)\frac{\partial M}{\partial \eta}d\xi\,d\eta\\ &\quad + (y - y_0)\mathcal{H}_x\left(\frac{\partial M}{\partial \eta}\right)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta - y}{r^3}[(x - x_0) - (\xi - x_0)]\frac{\partial M}{\partial \eta}d\xi\,d\eta\\ &\quad + (y - y_0)\mathcal{H}_x\left(\frac{\partial M}{\partial \eta}\right)\\ &= -(x - x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial \eta}\right) + \mathcal{H}_y\left(\frac{\partial F}{\partial \eta}\right)\\ &\quad + (y - y_0)\mathcal{H}_x\left(\frac{\partial M}{\partial \eta}\right),\end{aligned} \tag{A-10}$$

where $F = (\xi - x_0)M$. The y-component is equal to

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\eta - y_0)\frac{y-\eta}{r^3}\frac{\partial M}{\partial \eta}d\xi\,d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y-\eta}{r^3}\left(\frac{\partial G}{\partial \eta} - M\right)d\xi d\eta\\ &= \mathcal{H}_y\left(\frac{\partial G}{\partial \eta}\right) - \mathcal{H}_y(M),\end{aligned} \tag{A-11}$$

where $G = (\eta - y_0)M$.

The final integral to be evaluated is

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi - x_0)\frac{\partial M}{\partial \xi}\mathbf{D}\left(\frac{1}{r}\right)d\xi d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi - x_0)\frac{\partial M}{\partial \xi}\left(\frac{x-\xi}{r^3}\mathbf{i} + \frac{y-\eta}{r^3}\mathbf{j}\right)d\xi\,d\eta.\end{aligned} \tag{A-12}$$

The x-component is

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-\xi}{r^3}(\xi - x_0)\frac{\partial M}{\partial \xi}d\xi\,d\eta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-\xi}{r^3}\left(\frac{\partial F}{\partial \xi} - M\right)d\xi\,d\eta\\ &= \mathcal{H}_x\left(\frac{\partial F}{\partial \xi}\right) - \mathcal{H}_x(M),\end{aligned} \tag{A-13}$$

and the y-component is

$$
\begin{aligned}
\frac{1}{2\pi}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y-\eta}{r^3}(\xi-x_0)\frac{\partial M}{\partial\xi}\,d\xi\,d\eta \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y-\eta}{r^3}[(\xi-x)+(x-x_0)]\frac{\partial M}{\partial\xi}\,d\xi\,d\eta \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\xi-x}{r^3}(y-\eta)\frac{\partial M}{\partial\xi}\,d\xi\,d\eta \\
&\quad+(x-x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial\xi}\right) \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\xi-x}{r^3}[(y-y_0)-(\eta-y_0)]\frac{\partial M}{\partial\xi}\,d\xi\,d\eta \\
&\quad+(x-x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial\xi}\right) \\
&=-(y-y_0)\mathcal{H}_x\left(\frac{\partial M}{\partial\xi}\right)+\mathcal{H}_x\left(\frac{\partial G}{\partial\xi}\right) \\
&\quad+(x-x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial\xi}\right). \qquad \text{(A-14)}
\end{aligned}
$$

Adding the x-components of the last two terms, i.e., the right-hand sides of equations (A-10) and (A-13), we obtain

$$
\begin{aligned}
-(x-x_0)\mathcal{H}_y&\left(\frac{\partial M}{\partial\eta}\right)+\mathcal{H}_y\left(\frac{\partial F}{\partial\eta}\right)+(y-y_0)\mathcal{H}_x\left(\frac{\partial M}{\partial\eta}\right) \\
&+\mathcal{H}_x\left(\frac{\partial F}{\partial\xi}\right)-\mathcal{H}_x(M). \qquad \text{(A-15)}
\end{aligned}
$$

To proceed further, we must eliminate the terms involving F. Because M satisfies Laplace's equation, we have

$$\nabla^2 F = 2\frac{\partial M}{\partial\xi}. \qquad \text{(A-16)}$$

Taking Fourier transforms in ξ and η yields

$$-(p^2+q^2)\tilde{F}+\frac{\partial^2\tilde{F}}{\partial\zeta^2}=2ip\tilde{M}. \qquad \text{(A-17)}$$

Equation (A-17) can be solved as an ordinary differential equation in ζ (i.e., treating p and q as constants), subject to the boundary conditions that $\tilde{F}$ vanishes, or at least is a function of slow growth in the case $n=0$, at $\zeta=-\infty$ (for ζ positive downward) and that $\tilde{F}(p,q,z)$ is a given function on the observation surface $\zeta=z$.

We first give the corresponding well-known solution for $\tilde{M}$, which we need in any case. Since M satisfies Laplace's equation (9), we obtain for the Fourier transform $\tilde{M}$

$$-(p^2+q^2)\tilde{M}+\frac{\partial^2\tilde{M}}{\partial\zeta}=0. \qquad \text{(A-18)}$$

The general solution of this equation is

$$\tilde{M}(p,q,\zeta)=Ae^{\sqrt{p^2+q^2}\zeta}+Be^{-\sqrt{p^2+q^2}\zeta}, \qquad \text{(A-19)}$$

where A and B are arbitrary functions of p and q. Since $\tilde{M}$ must be bounded as $\zeta\to-\infty$, $B=0$. Using the boundary condition that $\tilde{M}(p,q,z)$ is given, we then have

$$A=\tilde{M}(p,q,z)e^{-\sqrt{p^2+q^2}z} \qquad \text{(A-20)}$$

and thus

$$\tilde{M}(p,q,\zeta)=\tilde{M}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)}. \qquad \text{(A-21)}$$

We now return to equation (A-17) for $\tilde{F}$. As usual, we solve the equation in two steps. First, we construct a solution of the corresponding homogeneous equation satisfying the required boundary conditions. Then we find a solution of the inhomogeneous equation satisfying null boundary conditions.

The homogeneous equation is the same as that for M and has analogous boundary conditions. Thus,

$$\tilde{F}_h=Ce^{\sqrt{p^2+q^2}\zeta}+De^{-\sqrt{p^2+q^2}\zeta}, \qquad \text{(A-22)}$$

where C and D are arbitrary functions of p and q. The boundary condition that $\tilde{F}$ be bounded as $\zeta\to-\infty$ implies that $D=0$. The function C is determined by the given value of $\tilde{F}$ when $\zeta=z$, so we obtain for the solution of the homogeneous equation

$$\tilde{F}_h(p,q,\zeta)=\tilde{F}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)}. \qquad \text{(A-23)}$$

We now turn to the inhomogeneous equation (A-17). We use the method of variation of coefficients, i.e., we now allow C and D in equation (A-22) to be functions of ζ. We first note that we can again set $D=0$ because the solution is required to vanish as $\zeta\to-\infty$. Substituting in equation (A-17) then yields

$$
\begin{aligned}
\frac{\partial^2 C}{\partial\zeta^2}&e^{\sqrt{p^2+q^2}(\zeta-z)}+2\sqrt{p^2+q^2}\frac{\partial C}{\partial\zeta}e^{\sqrt{p^2+q^2}(\zeta-z)} \\
&=2ip\tilde{M}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)}. \qquad \text{(A-24)}
\end{aligned}
$$

Canceling the exponential factors, a solution which vanishes at $\zeta=z$ can be found by inspection:

$$C=\frac{ip}{\sqrt{p^2+q^2}}\tilde{M}(p,q,z)(\zeta-z) \qquad \text{(A-25)}$$

for $p^2+q^2\neq 0$. We deal with the case where $p^2+q^2=0$ separately.

Adding the homogeneous and inhomogeneous solutions, we obtain for $\tilde{F}$

$$
\begin{aligned}
\tilde{F}(p,q,\zeta)&=\tilde{F}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)} \\
&\quad+(\zeta-z)\frac{ip}{\sqrt{p^2+q^2}}\tilde{M}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)}. \qquad \text{(A-26)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial\tilde{F}}{\partial\zeta}(p,q,\zeta)&=\sqrt{p^2+q^2}\tilde{F}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)} \\
&\quad+\frac{ip}{\sqrt{p^2+q^2}}\tilde{M}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)} \\
&\quad+(\zeta-z)ip\tilde{M}(p,q,z)e^{\sqrt{p^2+q^2}(\zeta-z)}. \qquad \text{(A-27)}
\end{aligned}
$$

Finally, evaluating equation (A-27) on the observation surface $\zeta = z$, we obtain

$$\frac{\partial \tilde{F}}{\partial z}(p, q, z) = \sqrt{p^2 + q^2}\tilde{F}(p, q, z) + \frac{ip}{\sqrt{p^2 + q^2}}\tilde{M}(p, q, z). \quad \text{(A-28)}$$

Equation (A-28) can be rewritten in terms of generalized Hilbert transforms using equation (17) of Nabighian (1984):

$$\frac{\partial F}{\partial z}(p, q, z) = \tilde{\mathcal{H}}_x ip\tilde{F}(p, q, z) + \tilde{\mathcal{H}}_y iq\tilde{F}(p, q, z) - \tilde{\mathcal{H}}_x\tilde{M}(p, q, z). \quad \text{(A-29)}$$

Taking the inverse Fourier transform yields

$$\frac{\partial F}{\partial z}(x, y, z) = \mathcal{H}_x\left(\frac{\partial F}{\partial x}\right)(x, y, z) + \mathcal{H}_y\left(\frac{\partial F}{\partial y}\right)(x, y, z) - \mathcal{H}_x(M)(x, y, z). \quad \text{(A-30)}$$

However,

$$\frac{\partial F}{\partial z} = (x - x_0)\frac{\partial M}{\partial z}. \quad \text{(A-31)}$$

Rearranging this last equation to put the remaining terms in F on the left-hand side we obtain

$$\mathcal{H}_x\left(\frac{\partial F}{\partial x}\right) + \mathcal{H}_y\left(\frac{\partial F}{\partial y}\right) = (x - x_0)\frac{\partial M}{\partial z} + \mathcal{H}_x(M) = (x - x_0)\mathcal{H}_x\left(\frac{\partial M}{\partial x}\right) + (x - x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial y}\right) + \mathcal{H}_x(M). \quad \text{(A-32)}$$

Finally, we return to the case where $p^2 + q^2 = 0$, i.e., the dc term. For Euler index $n > 0$, we expect that equation (A-32) will stand as written because in that case M falls off as $O(1/r)$ at infinity and any dc coefficient will be bounded. This precludes the addition of any extra constant term to the equation. For $n = 0$, the situation is not as clear. In that case, the dc term may be undefined and an additional constant may appear in the inverse Fourier transform. Thus, expression (A-32) should be modified to

$$\mathcal{H}_x\left(\frac{\partial F}{\partial x}\right) + \mathcal{H}_y\left(\frac{\partial F}{\partial y}\right) = (x - x_0)\mathcal{H}_x\left(\frac{\partial M}{\partial x}\right) + (x - x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial y}\right) + \mathcal{H}_x(M) - \beta_x, \quad \text{(A-33)}$$

where β_x is a constant which can be expected to vanish unless $n = 0$ and the negative sign is simply for later convenience.

Substituting for the left-hand side of equation (A-33) in expression (A-15), we are left with

$$(x - x_0)\mathcal{H}_x\left(\frac{\partial M}{\partial x}\right) + (y - y_0)\mathcal{H}_x\left(\frac{\partial M}{\partial y}\right) - \beta_x. \quad \text{(A-34)}$$

An exactly similar argument for G yields the y-component:

$$(x - x_0)\mathcal{H}_y\left(\frac{\partial M}{\partial x}\right) + (y - y_0)\mathcal{H}_y\left(\frac{\partial M}{\partial y}\right) - \beta_y. \quad \text{(A-35)}$$

Collecting all terms from the original Hilbert transforms and returning to vector notation, we have

$$(x - x_0)\boldsymbol{\mathcal{H}}\left(\frac{\partial M}{\partial x}\right) + (y - y_0)\boldsymbol{\mathcal{H}}\left(\frac{\partial M}{\partial y}\right) + (z - z_0)\boldsymbol{\mathcal{H}}\left(\frac{\partial M}{\partial z}\right) + n\boldsymbol{\mathcal{H}}(M) = \boldsymbol{\beta}. \quad \text{(A-36)}$$

A straightforward integration by parts argument, which we omit, shows that the generalized Hilbert transform commutes with differentiation for functions with reasonable behavior at infinity, which will be the case for M. We finally obtain

$$(x - x_0)\frac{\partial}{\partial x}\boldsymbol{\mathcal{H}}(M) + (y - y_0)\frac{\partial}{\partial y}\boldsymbol{\mathcal{H}}(M) + (z - z_0)\frac{\partial}{\partial z}\boldsymbol{\mathcal{H}}(M) + n\boldsymbol{\mathcal{H}}(M) = \boldsymbol{\beta}. \quad \text{(A-37)}$$