THE GRAVITATIONAL ATTRACTION OF VERTICAL TRIANGULAR PRISMS *

BY

D. J. WOODWARD **

Abstract

WOODWARD, D. J., 1975, The Gravitational Attraction of Vertical Triangular Prisms, Geophysical Prospecting 23, 526-532.

Formulae for the gravitational attraction, at a point, of vertical triangular prisms with oblique ends are presented. They are suited to computation on an electronic computer and are useful for determining the thickness of sediments in a sedimentary basin by iterative means.

Introduction

Many people (e.g. Bott 1960, Corbató 1965, Tanner 1967, Cordell and Henderson 1968) have given iterative methods for calculating the thickness of sediments in sedimentary basins from gravity anomalies. These methods have either treated the basin as "two-demensional" or have divided the area of the basin into square vertical prisms with horizontal ends. The two-dimensional analysis has the obvious disadvantage of treating the basin in sections while the division into square prisms usually requires many prisms to describe the basin adequately and hence instabilities are likely to occur in the computation of the solution.

By dividing the basin into triangular prisms with sloping ends the number of units to make up quite complex shapes is reduced considerably. The number of variables (the depth of individual corners of the prisms) is consequently reduced from that required for models using solely horizontal and vertical boundaries. Hence, although the formulae may be slightly more complex, the number of iterations and the computing time may be reduced and the instabilities in the computation of the solution minimized.

---

* Paper read at the 35th Meeting of the European Association of Exploration Geophysicists, Brighton June 1973.

** Geophysics Division, Department of Scientific and Industrial Research, Wellington, New Zealand now on leave at the Department of Geological Sciences, University of Durham, England.

The Gravitational Effect of a Vertical Triangular Prism

We shall use a rectangular coordinate system centred at the point at which we require the gravitational effect of the prism and with the z-axis positive vertically downward (fig. 1).

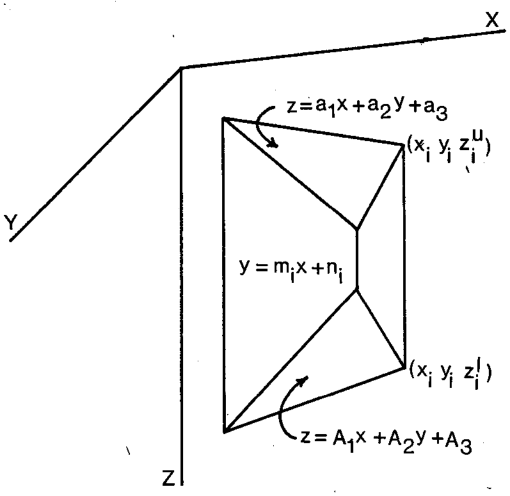

Fig. 1. The geometry of a vertical triangular prism with sloping ends.

A vertical triangular prism can be defined either by five planes (three vertical sides, and two ends) or by six points. Let the upper surface be defined by the three points (x_i, y_i, z_i^u); and the lower surface by three points vertically below, say (x_i, y_i, z_i^l) $i = 1, 2, 3$.

The planes defining the upper and lower surfaces can be determined from these points. Let the upper and lower surfaces have the equations

$$z = a_1x + a_2y + a_3$$

and

$$z = A_1x + A_2y + A_3, \quad \text{respectively.} \tag{1}$$

As the points (x_i, y_i, z_i^u) all lie in the upper surface, they must satisfy equation (1). Substitution gives a matrix equation for $a_1\, a_2\, a_3$:

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} z_1^u \\ z_2^u \\ z_3^u \end{bmatrix}$$

or

$$\mathbf{Pa} = \mathbf{z}^u. \tag{2}$$

Similarly, for the lower surface.

$$\mathbf{PA} = \mathbf{z}^l.$$

The vertical sides of the prism $y = m_i x + n_i$ contain the vertical lines $(x = x_j, y = y_j)$ and $(x = x_k, y = y_k)$ so that

$$\begin{bmatrix} x_j & 1 \\ x_k & 1 \end{bmatrix} \begin{bmatrix} m_i \\ n_i \end{bmatrix} = \begin{bmatrix} y_j \\ y_k \end{bmatrix},$$

where i, j, k are a cyclic permutation of 1, 2, 3.

With the above notation the gravitational effect g of a vertical triangular prism is given by

$$g = \rho G \sum_{i=1}^{3} \int_{x_i}^{x_j} \int_{y=y_0}^{m_k x + n_k} \int_{z=A_1 x + A_2 y + A_3}^{a_1 x + a_2 y + a_3} \frac{z\, dx \cdot dy \cdot dz}{(x^2 + y^2 + z^2)^{3/2}}, \tag{3}$$

where (i, j, k) takes the value (1, 2, 3), (2, 3, 1) and (3, 1, 2), G is the universal gravitational constant, ρ the density of the prism, and y_0 is arbitrary (usually taken as the y coordinate of the centroid). Consider one element of the sum and taking only the upper limit at each stage of integration, the integral becomes:

$$F(a_1, a_2, a_3, m, n, x) = \int^{x} \int^{mx+n} \int^{a_1 x + a_2 y + a_3} \frac{z\, dx \cdot dy \cdot dz}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= -\frac{1}{\sqrt{1 + a_2^2}} \int^{x} \ln\left(\alpha x + \beta + \sqrt{\gamma x^2 + 2\delta x + \eta}\right) dx, \tag{4}$$

where

$$\left.\begin{aligned} \alpha &= a_1 a_2 + (1 + a_2^2)\, m \\ \beta &= a_2 a_3 + (1 + a_2^2)\, n \\ \gamma &= (1 + a_2^2)\,[m^2 + 1 + (a_2 m + a_1)^2] \\ \delta &= (1 + a_2^2)\,[(a_2 m + a_1)(a_2 n + a_3) + mn] \\ \eta &= (1 + a_2^2)\,[n^2 + (a_2 n + a_3)^2] \end{aligned}\right\} \tag{5}$$

The next stage of integration is well behaved provided the plane of the end of the prism ($z = a_1 x + a_2 y + a_3$) does not pass through the origin, i.e. $a_3 \neq 0$.

When $a_3 \neq 0$ it can be shown that $\eta\gamma - \delta^2 > 0$
and so substituting $\tan\theta = (\gamma x + \delta)(\eta\gamma - \delta^2)^{-0.5}$ (6)

$$F = -\frac{1}{\sqrt{1 + a_2^2}} \cdot \frac{u}{\alpha} \int^{\theta} \ln\left(u \tan\theta + w + v \sec\theta\right) \sec^2\theta\, d\theta$$

where

$$u = \alpha(\eta\gamma - \delta^2)^{0.5}/\gamma$$
$$v = [(\eta\gamma - \delta^2)/\gamma]^{0.5}$$
$$w = (\beta\gamma - \alpha\delta)/\gamma.$$

Integrating by parts

$$F = -\frac{1}{\sqrt{1 + a_2^2}} \frac{u}{\alpha} [\tan\theta \cdot \ln(u \tan\theta + w + v \sec\theta) - f] \tag{7}$$

where

$$f(u, v, w) = \int \frac{u \sec\theta + v \tan\theta}{u \tan\theta + w + v \sec\theta} \tan\theta \sec\theta \, d\theta.$$

Substituting $t = \tan(\theta/2)$ gives

$$f = 4 \int^{t} \frac{ut^3 + 2vt^2 + ut}{(1 - t^2)^2 [(v - w) t^2 + 2ut + v + w]} dt. \tag{8}$$

When $a_3 \neq 0$ then $u^2 \neq v^2$ and $v^2 \neq w^2$ and the discriminant of the denominator in (8), $u^2 - v^2 + w^2$, is always negative. Dividing the integral of equation (8) into partial fractions and integrating yields

$$\begin{aligned} f = {} & \frac{w}{v + u} \cdot \ln(1 - t) - \frac{w}{v - u} \cdot \ln(1 + t) + \frac{2t}{1 - t^2} \\ & + \frac{uw}{v^2 - u^2} \cdot \ln[(v - w) t^2 + 2ut + v + w] \\ & - \frac{2v(v^2 - w^2 - u^2)^{0.5}}{v^2 - u^2} \cdot \tan^{-1}\left[\frac{(v - w) t + u}{(v^2 - w^2 - u^2)^{0.5}}\right]. \end{aligned} \tag{9}$$

Although the solution is long, it is easy to evaluate by computer.

The solution when $a_3 = 0$ is generally the same as for $a_3 \neq 0$ but two special cases can occur. When $a_3 = n = 0$, equation (4) becomes

$$F = -\frac{1}{\sqrt{1 + a_2^2}} \int^{x} \ln(\alpha x + \sqrt{\gamma x^2}) \, dx$$

Integrating gives

$$F = -\frac{1}{\sqrt{1 + a_2^2}} \, x \left[\ln\left(\left(\alpha + \frac{|x|}{x} \sqrt{\gamma}\right) \cdot x\right) - 1\right]$$

Secondly, when $a_3 = 0$ and a_1, a_2, and m are also zero then in equation (8) $u = 0$ and $v^2 = w^2$. The last two terms in (9) are zero and

$$F = -x \ln(n + \sqrt{n^2 + x^2}) + n \cdot \ln(\sqrt{n^2 + x^2} - x) + x.$$

In practice, however, it is easier to use the general form in the limit as a_3 tends to zero. Setting $a_3 = 1$ m has been found to give the answer for $a_3 \to 0$ to sufficient accuracy.

SOLUTION BY LEAST SQUARES

Let each triangular prism be specified by the coordinates of the vertices (x_i, y_i, z_i^l), (x_i, y_i, z_i^u) where $i = 1, 2, 3$. It is required to determine the coordinates z_i^u and z_i^l so as to minimize the departure of the computed gravitational effect of the prisms from a set of observed gravity anomalies.

For a given trial set of depth coordinates $\bar{z}_j$, let $\bar{g}_m$ be the computed anomaly at the mth observation station. If g_m is the measured anomaly at that station then

$$g_m = \bar{g}_m + \sum \frac{\partial \bar{g}_m}{\partial z_j} \Delta z_j + \nu_m, \tag{10}$$

where Δz_j is the small correction to the trial depth $\bar{z}_j$, and ν_m a residual, the sum of squares of which is to be minimized.

The derivative of the computed gravity effect with respect to the depth of the vertex $(\partial \bar{g}_m/\partial z_j)$ can be computed as follows:

$$\frac{\partial \bar{g}_m}{\partial z_j} = \sum_{k=1}^{3} \frac{\partial \bar{g}_m}{\partial a_k} \cdot \frac{\partial a_k}{\partial \bar{z}_j}. \tag{11}$$

From equation (2)

$$\mathbf{Pa} = \mathbf{z},$$

it can be shown that

$$\frac{\partial a_k}{\partial z_j} = P_{kj}^{-1} \tag{12}$$

where P_{kj}^{-1} is the element kj of the inverse of matrix $\mathbf{P}$.

Taking coordinates relative to the mth observation station

$$\bar{g}_m = G\rho \int_\Delta \int_{A_1x+A_2y+A_3}^{a_1x+a_2y+a_3} \frac{z\, dS \cdot dz}{(x^2 + y^2 + z^2)^{1.5}}$$

where the integral dS is over the triangle $(x_i\, y_i)$, $i = 1, 2, 3$,

i.e. $$\bar{g}_m = -G\rho \int_\Delta \left. \frac{1}{(x^2 + y^2 + z^2)^{0.5}} \right]_{z=A_1x+A_2y+A_3}^{a_1x+a_2y+a_3} dS,$$

we interchange the order of integration and differentiation:

$$\frac{\partial \bar{g}_m}{\partial a_1} = \int_\Delta x \cdot I \cdot dS,$$

$$\frac{\partial \bar{g}_m}{\partial a_2} = \int_\Delta y \cdot I \cdot dS,$$

$$\frac{\partial \bar{g}_m}{\partial a_3} = \int_\Delta I \cdot dS, \tag{13}$$

where

$$I = G\rho \frac{a_1x + a_2y + a_3}{[x^2 + y^2 + (a_1x + a_2y + a_3)^2]^{1.5}}.$$

Since the three integrals (13) only need to be known approximately, they can be evaluated numerically by, say, a seven point quintic integration formula for triangles.

Thus $(\partial \bar{g}_m/\partial z_j)$ can be evaluated using equation (11) and the results (12) and (13).

If the number of gravity observations M exceeds the number of unknown depths of vertices then the set of M observation equations (10) in the unknown Δz_j form an overdetermined system which can be solved by the method of least squares. From the solution, a new approximation can be made for the depth coordinates, which can then be taken as a second set of trial coordinates. This process is repeated until no improvement is made in the fit of the computed and measured anomalies.

Instabilities in the solution are unlikely to occur when the vertices being changed are separated by distances greater than half their depth and in general, because of the ease with which complex shapes may be specified by few prisms, distances much greater than this can be used. However, care needs to be taken to ensure that each iteration does not cause the variables to contravene certain criteria such as $z_i^u \leqslant z_i^l$. If these bounds are going to be broken during an iteration, the solution to the equations may be adjusted by scaling all the Δz_i so that the criteria are met. Alternatively, only the offending variables may be scaled to bring them into the permitted range.

Discussion

The advantages of the above method over previous methods of calculating the thickness of sedimentary basins using gravity data are that the body described by the triangular prisms has a continuous shape, that there are fewer variables in the solution thus making it more stable, and that the influence of fixing any one point where it may be known from say drill-hole data has a wider effect.

A comparison between this method and that of using vertical prisms is analogous to comparing a two-dimensional body described by right vertical prisms to a model using a polygonal cross-section.

Acknowledgements

I wish to thank Dr. H. Bibby, Geophysics Division, for his many helpful suggestions which have made the text of this paper more lucid.

References

Bott, M. H. P., 1960, The use of rapid digital computing methods for direct gravity interpretation of sedimentary basins. Geophysical Journal of the Royal Astronomical Society, 3, 63-70.

Corbató, C. E., 1965, A least-squares procedure for gravity interpretation. Geophysics 30, 228-33.

Cordell, L. and Henderson, R. G., 1968, Iterative three-dimensional solution of gravity anomaly data using a digital computer. Geophysics 33, 596-601.

Tanner, J. G., 1967, An automatic method of gravity interpretation. Geophysical Journal of the Royal Astronomical Society, 13, 339-47.