On the application of Euler deconvolution to the analytic signal

Article *in* Geophysics · November 2006

DOI: 10.1190/1.2360204

3 authors:

Giovanni Florio
University of Naples Federico II

Maurizio Fedi
University of Naples Federico II

Roman Pasteka
Comenius University in Bratislava

GEOPHYSICS, VOL. 71, NO. 6 (NOVEMBER-DECEMBER 2006); P. L87–L93, 6 FIGS., 1 TABLE.
10.1190/1.2360204

On the application of Euler deconvolution to the analytic signal

G. Florio[1], M. Fedi[1], and R. Pasteka[2]

ABSTRACT

Standard Euler deconvolution is applied to potential-field functions that are homogeneous and harmonic. Homogeneity is necessary to satisfy the Euler deconvolution equation itself, whereas harmonicity is required to compute the vertical derivative from data collected on a horizontal plane, according to potential-field theory. The analytic signal modulus of a potential field is a homogeneous function but is not a harmonic function. Hence, the vertical derivative of the analytic signal is incorrect when computed by the usual techniques for harmonic functions and so also is the consequent Euler deconvolution. We show that the resulting errors primarily affect the structural index and that the estimated values are always notably lower than the correct ones. The consequences of this error in the structural index are equally important whether the structural index is given as input (as in standard Euler deconvolution) or represents an unknown to be solved for. The analysis of a case history confirms serious errors in the estimation of structural index if the vertical derivative of the analytic signal is computed as for harmonic functions. We suggest computing the first vertical derivative of the analytic signal modulus, taking into account its nonharmonicity, by using a simple finite-difference algorithm. When the vertical derivative of the analytic signal is computed by finite differences, the depth to source and the structural index consistent with known source parameters are, in fact, obtained.

INTRODUCTION

Since the first paper (Thompson, 1982) dealing with a practical interpretation scheme involving the use of the Euler homogeneity equation, the so-called "Euler deconvolution" has become a widespread, semiautomated method to estimate the 3D position of potential-field sources. The usual way in which Euler deconvolution is applied is with potential-field data measured on, or reduced to, a horizontal plane.

Euler deconvolution was originally implemented for the analysis of magnetic data. The effect of the complexity of the magnetic-anomaly shapes at mid-latitudes was claimed by Thompson (1982) to have a direct influence on the depth results, and for this reason he suggested reducing the field to the pole. Although this conclusion was refuted by Reid et al. (1990), some researchers found it useful to apply this algorithm to the analytic signal, both to the modulus, e.g., Huang et al. (1995), and to its real and imaginary parts, e.g., Cooper (2004). Note that the analytic signal does not depend on the magnetization inclination in the 2D case.

Later developments of Euler deconvolution involved formulations of the problem that accounted for the unknown background field. These developments allowed the structural index (N) to be transformed from an arbitrary input parameter into an unknown to be solved for, together with the source coordinates. Among others, Hsu (2002) and Fedi and Florio (2002) independently pointed out that the use of an adequate mth-order derivative of the field, instead of the field itself, allows one to solve for both N and source position (derivative Euler deconvolution, DED). For the same reason, Keating and Pilkington (2004) proposed the Euler deconvolution of the analytic signal modulus as a tool to obtain the source coordinates as well as the structural index. Another approach (Mushayandebvu et al., 2001; Nabighian and Hansen, 2001) considered instead the Hilbert transforms of the potential field. Thanks to Hilbert transform properties, it is possible to eliminate the constant background from the formula, and, by using two equations (i.e., the Hilbert transform components along x and y) for each data point, the system is solved for the source coordinates and N.

However, Euler deconvolution needs a function to be (1) homogeneous (gravity or magnetic fields of simple sources and their derivatives are homogeneous) and (2) harmonic (because it must be possible to compute the function's vertical derivative starting from data on a horizontal plane). This last property allows for upward continuation from the measurement plane to any level in the harmonic region and thus also calculation of the function's vertical derivative, according to potential-field theory (Green's theorem). It can be easily demonstrated that the analytic signal modulus is a homogeneous

Manuscript received by the Editor September 9, 2005; revised manuscript received May 15, 2006; published online November 3, 2006.
[1]Universitá Federico II, Dipartimento di Scienze della Terra, Largo S. Marcellino, 10, 80138 Napoli, Italia. E-mail: gflorio@unina.it; fedi@unina.it.
[2]Comenius University, Department of Applied and Environmental Geophysics, Faculty of Sciences, Mlynska Dol., Prifuk, Pav. G, 84215 Bratislava, Slovak Republic, E-mail: pasteka@fns.uniba.sk.

function (Huang et al., 1995). However, being a modulus, it is not harmonic. A well-known example of a nonharmonic function is the magnetic-field modulus (e.g., Grant and West, 1965), which, by Poisson's theorem, corresponds to the analytic signal of a gravity field. The nonharmonicity of the analytic signal modulus is an important aspect affecting Euler deconvolution but has received no attention in the literature. A direct consequence of the analytic signal nonharmonicity is that the application of Euler deconvolution to the analytic signal is not straightforward because the vertical derivative of the analytic signal cannot be computed by conventional methods for harmonic functions.

In this paper, we theoretically and empirically determine the Euler deconvolution errors caused by such an incorrect determination of the vertical derivative of the analytic signal modulus. We then suggest a simple method to compute the vertical derivative by finite differences and show that, by using this method, correct depths and structural indexes can be obtained in either synthetic or real cases.

DERIVATIVE EULER DECONVOLUTION AND ANALYTIC SIGNAL EULER DECONVOLUTION

Since the first practical implementation of the Euler homogeneity equation as an interpretational formula (Hood, 1965; Thompson, 1982), it was clear that there was a need to include consideration of a background field to counteract a certain instability of the solution even when the regional field was a constant. The use of data windows of limited size on which to apply the formula allowed Thompson to approximate the unknown background field B by a constant:

$$\mathbf{d} \cdot \nabla T = -N(T - B), \tag{1}$$

where T is the total field $T = t + B$, with t as the anomalous component of T; $\mathbf{d} = \mathbf{r} - \mathbf{r}_0$ is the vector between measurement point and point source in the analyzed window; and N is the structural index. We refer to equation 1 as the Euler deconvolution equation. This equation is solved in each considered window for the three coordinates of the point source and for a constant background field by assuming a value for the structural index N, i.e., by assuming a model of the source. We refer to this case (Reid et al., 1990) as the standard Euler deconvolution (SED).

If the regional background field $B(x,y)$ can be approximated by a low-order polynomial, it is advantageous to apply the Euler deconvolution equation (equation 1) to the vertical derivative of order m of the total field ($T_z^{(m)}$) so that the corresponding derivative of order m of the background field B is zero:

$$\mathbf{d} \cdot \nabla T_z^{(m)} = -(N + m)T_z^{(m)}.$$

In this case, it is possible to modify the Euler deconvolution equation by neglecting the background field B and solving for x_0, y_0, z_0, and N. We refer to this case (Fedi and Florio, 2002; Hsu, 2002) as the derivative Euler deconvolution (DED) or, with expanded notation,

$$x_0 \frac{\partial T_z^{(m)}}{\partial x} + y_0 \frac{\partial T_z^{(m)}}{\partial y} + z_0 \frac{\partial T_z^{(m)}}{\partial z} - N_m T_z^{(m)} = x \frac{\partial T_z^{(m)}}{\partial x} + y \frac{\partial T_z^{(m)}}{\partial y} + z \frac{\partial T_z^{(m)}}{\partial z}, \tag{2}$$

where $N_m = N + m$.

Thus, the DED procedure overcomes the deconvolution instabilities caused by the presence of long-wavelength background fields. The direct estimation of N can help the interpretation with Euler deconvolution. In standard Euler deconvolution, the solution computation has to be repeated by using different N values, and, for a particular structure, the depth (and N) corresponding to the best-clustered solutions are selected. On the contrary, the direct estimation of N can help to solve the ambiguity between similar anomalies, such as those generated by a vertical line and a point source, those that result from a sheet and a horizontal line, or those caused by an infinite-in-depth fault and a limited-throw fault. One more advantage of using derivatives is that in cases of structures characterized by fractional N values (e.g., a limited-throw fault), the depth estimate is closer to the source top (as in cases with integer N) than the depth estimate obtainable with SED. This fact can simplify the interpretation of results. The use of derivatives of fields may, however, have the cost of lowering the signal-to-noise (S/N) ratio. Even when using stable procedures to determine them (e.g., Fedi and Florio, 2001; Pasteka and Richter, 2002), the derivatives may be characterized by some fragmentation of the anomalies, and the results are commonly limited to the shallowest sources. On the other hand, the derivatives — because of their increased resolution — are more suitably analyzed within a given window than the field because the effects of interference with nearby sources are strongly reduced.

The same considerations also apply to the Euler deconvolution of the analytic signal modulus (Keating and Pilkington, 2004). The analytic signal modulus A is defined as the square root of the sum of the squared horizontal and vertical derivatives of T. Because this function involves the spatial derivatives of the total field, a constant or low-order background field in the analyzed window can be removed; this procedure allows both the source coordinates and the structural index to be solved for. Also, in this case, the estimated N is increased by the order of the derivative used. Keating and Pilkington (2004) showed that parameters such as the susceptibility contrast and the dip could be determined by exploiting some relationships between the analytic signal and the orthogonal gradients of the magnetic fields, once the position and structural index of a source are known.

But as anticipated in the previous section, although the analytic signal modulus A of a potential field caused by a simple source is still a homogeneous function, the modulus is no longer harmonic, and this fact represents an obstacle to a correct implementation of this method. In Appendix A, we examine theoretically the homogeneity and harmonicity properties of the analytic signal modulus for a modeled thin, magnetic, dipping dike. The treatment in Appendix A shows that the A of such a dike is a homogeneous function characterized by a structural index having a value of $N = 2$. The fact that its Laplacian is not zero demonstrates that the A is not a harmonic function. If the A is not a harmonic function, its vertical derivative (needed to implement Euler deconvolution) cannot be computed by using the standard potential-field tools. When transforming the A with the vertical-derivative operator in the frequency domain, $-|\mathbf{k}|$ (where $\mathbf{k}$ is the wavenumber vector), the result will no longer be the true vertical derivative but a transformed function (which we call the k-function), very different from the true vertical derivative. The A and its k-function still satisfy the Euler deconvolution equation of a thin, magnetic dike but with $N = 1$. The fact that the k-function satisfy the equation means that an Euler deconvolution of the analytic signal modulus performed by neglecting the nonharmonicity of the A will lead to correct results on the depth but to wrong estimates of N.

ALGORITHMS TO CORRECTLY COMPUTE THE VERTICAL DERIVATIVE OF THE ANALYTIC SIGNAL MODULUS

Being a nonharmonic function, the A cannot be vertically derived by a direct application of standard techniques, such as transformations based on convolution.

A direct expression for the vertical derivative of A is

$$\frac{\partial A}{\partial z} = \frac{1}{2}\frac{2T_x(\partial T_x/\partial z) + 2T_y(\partial T_y/\partial z) + 2T_z(\partial T_z/\partial z)}{\sqrt{T_x^2 + T_y^2 + T_z^2}}, \quad (3)$$

where T_x, T_y, and T_z are the horizontal and vertical derivatives of the total field.

Alternatively, the vertical derivative of A may be computed by a simple finite-difference algorithm, involving only first-order derivatives of the field: (1) computation of the A of the measured data at the measurement altitude; (2) upward continuation of the measured data to a slightly higher level; (3) computation of the A of the upward-continued data; and (4) computation of the vertical derivative of A by subtracting (1) and (3) and dividing the result by their altitude difference (i.e., a simple finite-difference relationship).

The results will be precise enough if the altitude difference used in the upward continuation is smaller than the sampling step (i.e., 1/10 or 1/100 of the sampling step). Conventionally, this vertical derivative should be assigned to an altitude halfway between the levels of the two analytic signal moduli; but actually, because this altitude difference is so small, we can refer the vertical derivative of A to the measurement level.

Equation 3 implies computation of three second-order field derivatives. For this reason, data errors may be enhanced in some cases, depending on the S/N ratio. The second method instead needs only computation of first-order derivatives, i.e., derivatives of the same order as the analytic signal modulus, and a stable transformation of the field, such as the upward continuation. So we decided to use this second approach. To test this finite-difference vertical derivative of the analytic signal modulus, we computed the vertical derivative for a dike model. We then compared the computed derivative with the theoretical one (equations A-3) and with the k-function (equation A-8). The derivative computed by using this finite-difference algorithm (Figure 1) is much closer to the true vertical derivative than to the k-function. In the next section, we use some synthetic examples to illustrate the application of this finite-difference vertical derivative to Euler deconvolution of A.

EULER DECONVOLUTION OF THE ANALYTIC SIGNAL MODULUS CAUSED BY SYNTHETIC SOURCES

In this section, we compare the results of the Euler deconvolution of the analytic signal modulus by using the finite-difference vertical derivative introduced before (the altitude-difference value set to 1/100 of the sampling step with the results obtained by using the k-function.

The first source considered is a vertical thin dike model below 1 m of overburden and with an ambient field inclination $I = 45°$. The analysis was conducted along a profile perpendicular to the source (Figure 2). As predicted, when using the k-function, the estimated

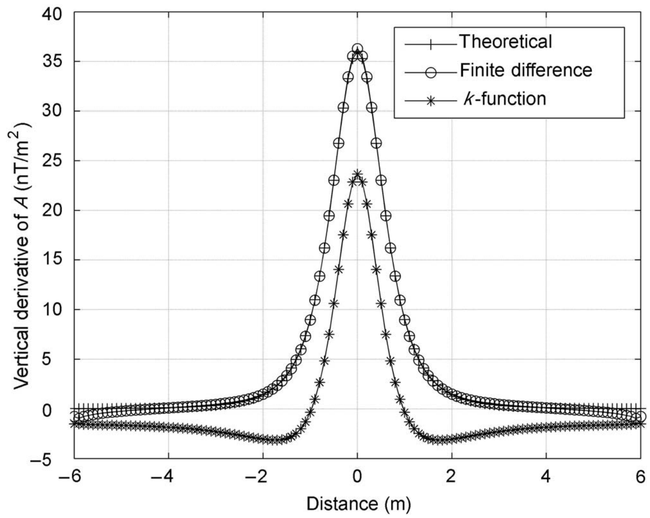

Figure 1. Comparison between the theoretical and finite-difference vertical derivatives of the analytic signal modulus for a dike model. Dike parameters: total magnetic intensity is 1 A/m, $z_0 = 1$ m, $t =$ 0.1 m, $i = 45°$, $d = 90°$, $a = 0°$ (see Appendix A for symbol explanation). Note that the curve of the theoretical values is almost perfectly superimposed by the curve of the finite-difference values; hence, the symbols are also superimposed.

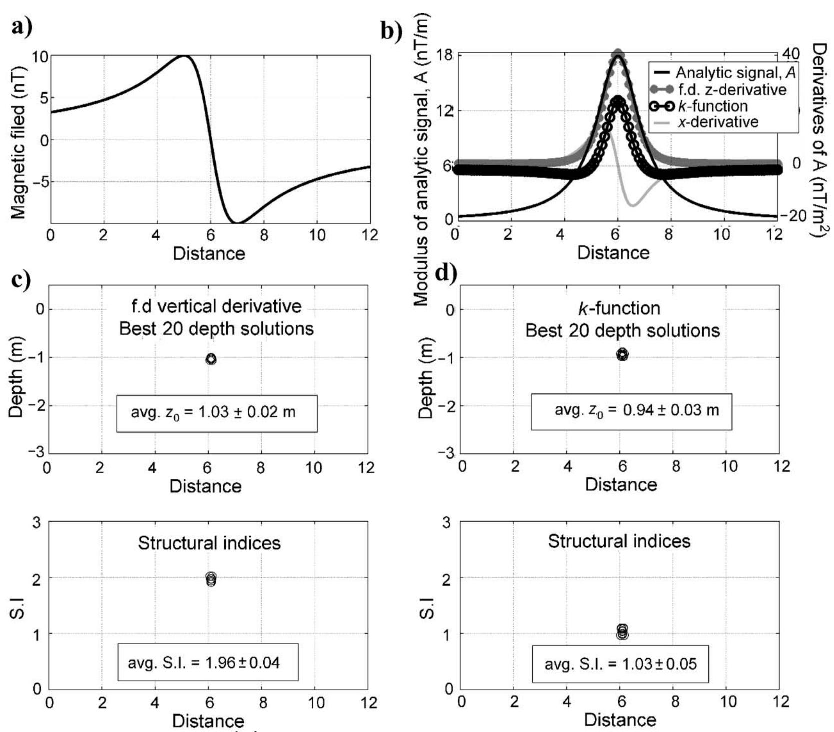

Figure 2. Euler deconvolution of the analytic signal modulus. Synthetic source is the same as in Figure 1. (a) Magnetic field. (b) Analytic signal modulus and its finite-difference (f.d.) z-derivative, k-function, and x-derivative. (c) Depth and structural index using a finite-difference vertical derivative. (d) Depth and structural index using the k-function. True depth is 1 m; theoretical structural index is $N = 2$.

structural-index values are approximately 1 instead of 2, whereas the depth estimates are close to the true value in both cases (i.e., using the finite-difference vertical derivative as well as using the *k*-function). However, the only correct estimates of the structural index are from using the finite-difference vertical derivative. Finally, the anomaly caused by a complex source is analyzed through the Euler deconvolution of *A*. The source is a rectangular prism having a horizontal square base with 10-km side lengths and a 5-km thickness at 1 km depth. The total magnetization intensity is 1 A/m, and the magnetization direction is vertical.

Neither the gravity field (e.g., Zhang et al., 2000) nor the magnetic field of a prism (Figure 3a) is homogeneous. In fact, this is the case for any finite extended source not describable as a one-point source. The analytic signal (Figure 3b) is therefore a nonharmonic and nonhomogeneous function. We analyze it through Euler deconvolution and, in spite of all these approximations, still find meaningful results. As before, we show results obtained by using the finite-difference vertical derivative and the *k*-function (Figure 4). In both cases, the estimated depth is correct (~ 1 km), but the estimated N is ~ 0.65 when using the *k*-function and ~ 1 when the finite-difference vertical derivative is used. However, in the first case, the value of $N = 0.65$ is clearly too low (see Table A-1), whereas when using the finite-difference vertical derivative, the estimated $N = 1$ is equal to the theoretical value for the analytic signal of a magnetic contact and the depth value is correctly related to the top of the structure.

APPLICATION TO REAL DATA

The Euler deconvolution of *A* was applied to the marine magnetic anomaly generated by the wreck of the military ship *Equa* (built in 1930) on the bottom of the Tyrrhenian Sea, 3 km off the Ligurian coast, Italy (F. Caratori Tontini, C. Carmisciano, M. Ciminale, O. Faggioni, M. Grassi, P. Lusiani, and S. Monti, personal communication, 2004). The ship is characterized by a 39.48-m length, a 6.83-m width, and a 5.61-m height, and its depth is expected to be between 34 and 40 m below sea level.

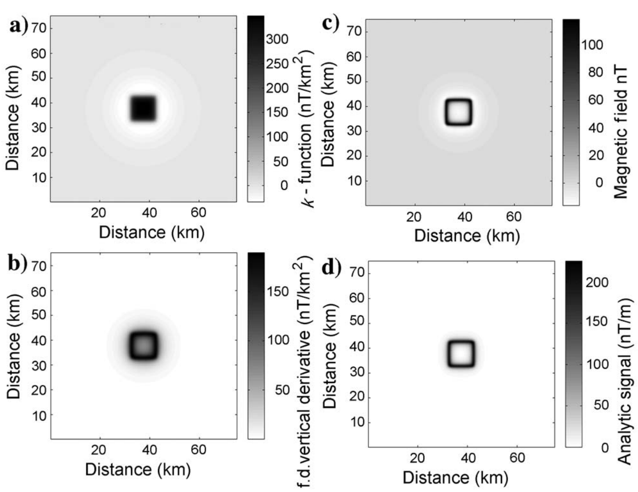

Figure 3. Euler deconvolution of the analytic signal modulus. The synthetic source is a rectangular prism having a horizontal square base with 10-km sides and a 5-km thickness at 1 km depth. The total magnetic intensity is 1 A/m, and the magnetization direction is vertical. (a) Magnetic field. (b) Analytic signal. (c) The *k*-function of the analytic signal modulus. (d) Finite-difference vertical derivative of the analytic signal modulus.

These dimensions are compatible with a source of linear type. Thus, approximating the source as being two-dimensional, a structural index of ~ 3 is reasonably expected for a profile of *A* along a direction normal to the source. Figure 5 is a map of the magnetic anomaly of the ship (grid interval: 10 m) as well as the central profile of the observed total magnetic anomaly along the south-to-north direction at 3 m spacing (Figure 5b). From the shape of the mapped anomaly (Figure 5a), we can argue that the source is elongated approximately in the west-to-east direction. Because we had such a better resolution than the grid and also a direction approximately perpendicular to the structure, we chose to interpret this central profile at $x = 560{,}152$ m. In the frame of a study of the source-dimensionality effect in Euler deconvolution and in other semiautomated methods (Florio et al., 2004), we also interpreted the entire map with a 3D derivative Euler deconvolution. The results obtained were essentially the same as those found during the present study. Therefore, we confirmed that the source's strike length was large enough to justify the analysis of a profile passing through its center.

Results of Euler deconvolution of the analytic signal modulus, obtained with the finite-difference vertical derivative (by using an alti-

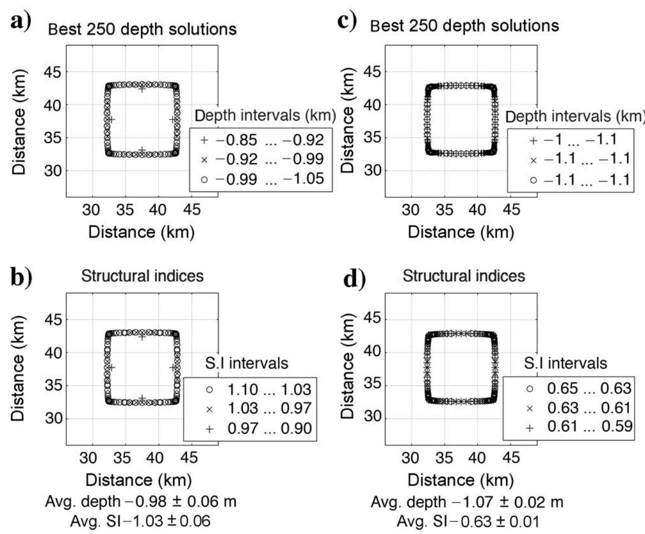

Figure 4. Euler deconvolution of the analytic signal modulus. Synthetic source is the same as in Figure 3. Depth and structural indices results were obtained (a) and (b) by using finite-difference vertical derivative and (c) and (d) by using the *k*-function.

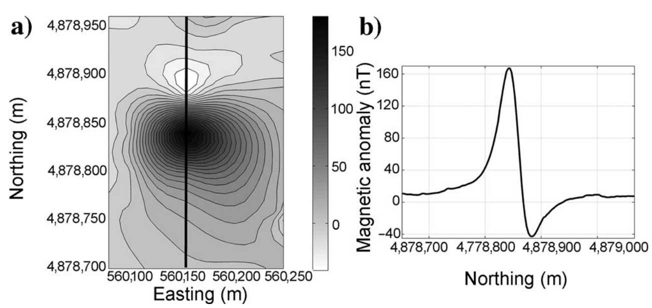

Figure 5. Total magnetic anomaly over the wreck of the military ship *Equa* (northern Tyrrhenian Sea, offshore of Italy). (a) Map of the magnetic anomaly at a 10-m grid interval; the black line corresponds to the profile selected for the interpretation. (b) Selected measured profile at $x = 560{,}152$ m; sampling interval: 3 m.

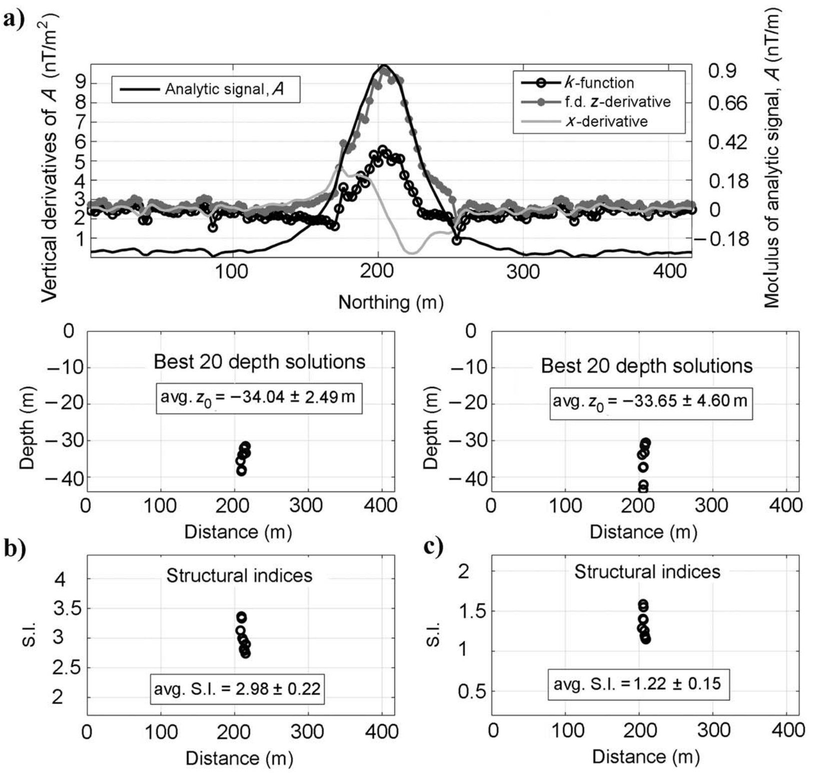

Figure 6. Euler deconvolution of the analytic signal modulus for the magnetic anomaly over the wreck of the military ship *Equa* (northern Tyrrhenian Sea, offshore of Italy). (a) Analytic signal modulus, and its k-function, finite-difference (f.d.) z-derivative, and x-derivative. (b) Depth and structural index using a finite-difference vertical derivative. (c) Depth and structural index using the k-function.

tude difference of 1/100 of the sampling step) and with the k-function, are presented in Figure 6. As predicted by the theory and also observed in synthetic cases, the depths estimated in the two cases are very similar ($\sim$34 m) and also correct, being in the expected depth range.

The estimated structural index obtained by using the finite-difference vertical derivative is about 3. This result suggests an elongated shape for the source and is consistent with the source dimensions as well as with the shape of the mapped anomaly. On the contrary, the structural index obtained by using the k-function ($N = 1.22$) is very different from the theoretical one. This result, close to 1, would imply a source close to a contact, which is, of course, not realistic.

Only by recognizing the fact that, for a given source, the homogeneity degree changes significantly when the k-function is used (instead of the true vertical derivative) may this result not be misunderstood. To this end, a table of the structural-index values expected when the k-function is used should be built. In Appendix A, we have analytically shown the value of $N_{k\text{-function}}$ only for the thin dike and have merely reported the values found empirically for other sources. In this sense, the value of 1.22 for the wrecked ship is not far from the one expected for an elongated source ($N_{k\text{-function}} = 1.5$).

CONCLUSIONS

We have shown that to correctly implement the Euler deconvolution of the analytic signal modulus, the nonharmonicity of this function must be taken into account. In fact, this statement implies that the vertical derivative needed in the Euler deconvolution equation should be computed by a nonstandard method. To this aim, we proposed a simple yet effective finite-difference algorithm. We have shown that the errors on the evaluation of the vertical derivative of the analytic signal (computed as if the analytic signal were harmonic) affect the structural index, so that it appears lower than the correct value. The consequences of this error are equally important either if the structural index is given as input (as in standard Euler deconvolution) or if it represents an unknown to be solved for. The estimated depth results are, on the contrary, independent of the use of a correct vertical derivative or the k-function because also with the k-function there is homogeneity in equation A-9 for simple sources.

Analysis of a case history confirms strong errors in the estimation of the structural index if the analytic signal modulus is treated as a harmonic function. On the other hand, results consistent with known source parameters are obtained when a finite-difference vertical derivative of the analytic signal is used.

Final consideration should be given to the most appropriate degree of the analytic signal to be used. In fact, if the A is built with the first derivatives of the field, only a constant background is effectively removed. Exactly as in the case of the DED, in some cases it may be necessary to compute a higher-order A, starting from the first or perhaps the second vertical derivative, to really remove higher-degree regional fields. So, the Euler deconvolution of the analytic signal modulus also has exactly the same limitations caused by noise enhancement and attenuation of deeper-source effects. However, we showed that Euler deconvolution of the analytic signal modulus has an additive disadvantage with respect to the DED because of also needing implementation of a vertical-derivative algorithm that takes into account the nonharmonicity of the analytic signal modulus.

ACKNOWLEDGMENTS

We thank F. Caratori Tontini, C. Carmisciano, M. Ciminale, O. Faggioni, M. Grassi, P. Lusiani, and S. Monti for making available the marine magnetic data used in this study. We also thank two anonymous reviewers for useful comments. R. Pasteka expresses thanks for the financial support obtained in the frame of the projects APVT-51-002804 and VEGA 1/3066/06.

APPENDIX A

AN EXAMPLE OF HOMOGENEITY AND NONHARMONICITY OF *A* USING A THIN, MAGNETIC, DIPPING DIKE AS A MODEL

Here we show that the A of a thin dike is a homogeneous function of the distance between source and observer of order $n = -2$ ($N = 2$), but its Laplacian is not zero, so demonstrating that it is not a harmonic function. The structural index $N = 2$ is consistent with what is expected when a derivative of the total field of a dike is considered (see Table A-1).

In a 2D space, let us denote the horizontal distance with x and the depth with z. Through the use of the formula from Nabighian (1972), we can write the total magnetic-field anomaly of a thin, infinite, dipping dike as

$$T = \alpha \frac{(x - x_0)\sin\phi + (z - z_0)\cos\phi}{r^2}, \quad \text{(A-1)}$$

where $\alpha = 2C_m\kappa Fct \sin d$, $\phi = 2I - d - (\pi/2)$, $r^2 = (x - x_0)^2 + (z - z_0)^2$ is the distance squared between a measurement point at (x,z) and the top of the dike at (x_0,z_0), $C_m = \mu_0/4\pi$, μ_0 is the magnetic permeability of the free space, κ is the susceptibility contrast between the dike and the background, F is the magnitude of the earth's magnetic field, $c = 1 - \cos^2 i \sin^2 a$, t is the horizontal thickness of the dike ($\ll z_0$), d is the dip of the dike, i is the inducing-field inclination, a is the angle between magnetic north and the positive x-axis, and $\tan I = \tan i/\cos a$.

The A is defined as the square root of the sum of the squared x and z derivatives of T. From now on, let us consider that the source coordinates $(x_0,z_0) = (0,0)$ so that $r^2 = x^2 + z^2$. For 2D sources, the A is a symmetric function with respect to the source x coordinate [the point $(x - x_0) = 0$] and does not depend on ϕ (Nabighian, 1972):

$$A = \frac{\alpha}{r^2} = \frac{\alpha}{x^2 + z^2}. \quad \text{(A-2)}$$

To test this function for homogeneity, we have to compute the derivatives of equation A-2 with respect to x and z,

$$\frac{\partial A}{\partial x} = -\alpha \frac{2x}{(x^2 + z^2)^2}$$

and

$$\frac{\partial A}{\partial z} = -\alpha \frac{2z}{(x^2 + z^2)^2}, \quad \text{(A-3)}$$

and insert them into the 2D form of the Euler deconvolution equation:

$$x\frac{\partial A}{\partial x} + z\frac{\partial A}{\partial z} = -2\frac{\alpha}{x^2 + z^2} = -2A \Rightarrow N = 2. \quad \text{(A-4)}$$

Table A-1. Structural indices for magnetic sources.

Source	Magnetic field	First derivative A
Contact	0	1
Step	0 to 1	1 to 2
Sill or dike	1	2
Cylinder or pipe	2	3
Barrel	2 to 3	3 to 4
Sphere	3	4

Equation A-4 demonstrates that for a thin, magnetic dike, the A is a homogeneous function characterized by a structural index having the value $N = 2$.

However, to make use of the Euler deconvolution equation for the A, its x and z derivatives must be computed from the A. The horizontal derivative is readily obtained by stable space-domain algorithms; but to compute a correct vertical derivative with standard potential-field tools, we must verify that A is actually a harmonic function, satisfying the Laplace equation:

$$\nabla^2(A) = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} = 0.$$

We take the second derivatives of equation A-2 with respect to x and z,

$$\frac{\partial^2 A}{\partial x^2} = 2\frac{\alpha(3x^2 - z^2)}{(x^2 + z^2)^3}$$

and

$$\frac{\partial^2 A}{\partial z^2} = -2\frac{\alpha(x^2 - 3z^2)}{(x^2 + z^2)^3}, \quad \text{(A-5)}$$

so that we obtain

$$\nabla^2(A) = \frac{4\alpha}{(x^2 + z^2)^2} \neq 0. \quad \text{(A-6)}$$

The fact that the Laplace equation of the A (equation A-6) is not zero demonstrates that A is not a harmonic function. Thus, it is not possible to compute its vertical derivative (needed to implement Euler deconvolution) by using the standard potential-field tools. However, we may still transform the field with the vertical-derivative operator in the frequency domain, $-|\mathbf{k}|$ (e.g., Naidu and Mathew, 1998), where k is the wavenumber vector. For a nonharmonic function, the result is no longer the true vertical derivative but a transformed function (which we call the k-function) that is very different from the vertical derivative itself. Note that in the 2D case, this k-function forms a Hilbert pair with the horizontal derivative of the nonharmonic function considered.

In the case of A, we next show that the Euler homogeneity equation is still fulfilled with the k-function. However, the structural index when the k-function is used is $N = 1$ for the thin dike model.

To demonstrate this fact, we need an expression for the k-function to enter in the Euler deconvolution equation. The Fourier transform of A (equation A-2) is (Gradshteyn and Ryzhik, 1980, p. 1147):

$$\mathrm{FT}\{A\} = \alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{-ikx}}{x^2 + z^2}dx = \sqrt{\frac{\pi}{2}}\frac{\alpha}{z}e^{-|k|z}. \quad \text{(A-7)}$$

If the z-axis is assumed to be positive upward, the k-function (K) is the inverse Fourier transform of $\mathrm{FT}\{A\}$ multiplied by $-|k|$,

$$K = \mathrm{FT}^{-1}\left\{-\sqrt{\frac{\pi}{2}}\frac{\alpha}{z}|k|e^{-|k|z}\right\},$$

so that (Gradshteyn and Ryzhik, 1980, p. 1147)

$$K = -\frac{1}{\sqrt{2\pi}}\frac{\alpha}{z}\int_{-\infty}^{+\infty}\sqrt{\frac{\pi}{2}}|k|e^{-|k|z}e^{ikx}dk = \frac{\alpha}{z}\frac{x^2 - z^2}{(x^2 + z^2)^2}. \quad \text{(A-8)}$$

Now let us check the value of the structural index N by using the k-function (K, equation A-8) in a 2D form of the Euler deconvolution equation:

$$\begin{aligned} x\frac{\partial A}{\partial x} + z(K) &= -x\alpha\frac{2x}{(x^2 + z^2)^2} + z\frac{\alpha}{z}\frac{x^2 - z^2}{(x^2 + z^2)^2} \\ &= -\alpha\frac{x^2 + z^2}{(x^2 + z^2)^2} = -\alpha\frac{1}{x^2 + z^2} \\ &= -A \Rightarrow N = 1. \end{aligned} \quad \text{(A-9)}$$

From equation A-9, it appears that the k-function is homogeneous but with a homogeneity degree $n = -1$, i.e., $N = 1$. This result is significantly different from that theoretically expected for A of a thin, magnetic dike (equation A-4). Nevertheless, equation A-9 implies that using such a value for the structural index in a standard Euler deconvolution (i.e., when N is given as an input) would result in a correct depth estimation. On the other hand, if we were looking for dikes, we would input the theoretical value of $N = 2$, thus obtaining wrong z estimates (too deep). Moreover, Euler deconvolution of the analytic signal modulus performed to solve for both source coordinates and structural index, and using a k-function in place of a correct vertical derivative, would result in a correct depth estimate but in a wrong structural-index estimate. The structural index found will be 1 instead of 2, leading to relevant interpretation errors.

Similar analytic formulas could be obtained for other simple 2D magnetic sources. For a horizontal cylinder and a contact, we empirically evaluated the structural-index values that apply when using the k-function. We assumed different values for N and estimated the best results, in terms of position and solution clustering, of a standard Euler deconvolution of A. It resulted in a relationship of the type $N_{k\text{-function}} = N_{\mathrm{A}}/2$, where $N_{k\text{-function}}$ is the best value of the structural index when the k-function is used in place of a correct vertical derivative and N_{A} is the theoretical structural index for the analytical signal or the first derivative of the magnetic field (Table A-1).

REFERENCES

Cooper, G. R. J., 2004, Euler deconvolution applied to potential field gradients: Exploration Geophysics, **35**, 165–170.

Fedi, M., and G. Florio, 2001, Detection of potential fields source boundaries by enhanced horizontal derivative method: Geophysical Prospecting, **49**, 40–58.

———, 2002, Euler deconvolution with no a priori definition of structural index: European Geophysical Society, 27th General Assembly, Abstract 3610, 2002EGSGA.27.3610F.

Florio, G., M. Fedi, and F. Cella, 2004, On the use of two- vs. three-dimensional transformation and interpretation of potential field anomalies: Near Surface 2004, Abstract P011.

Gradshteyn, I. S., and I. M. Ryzhik, 1980, Table of integrals, series, and products: Academic Press.

Grant, F. S., and G. F. West, 1965, Interpretation theory in applied geophysics: McGraw-Hill Book Co.

Hood, P., 1965, Gradient measurements in aeromagnetic surveying: Geophysics, **30**, 891–902.

Hsu, S. K., 2002, Imaging magnetic sources using Euler's equation: Geophysical Prospecting, **50**, 15–25.

Huang, D., D. Gubbins, R. A. Clark, and K. A. Whaler, 1995, Combined study of Euler's homogeneity equation for gravity and magnetic field: 57th Conference and Exhibition, EAGE, Extended Abstracts, P144.

Keating, P., and M. Pilkington, 2004, Euler deconvolution of the analytic signal and its application to magnetic interpretation: Geophysical Prospecting, **52**, 165–182.

Mushayandebvu, M. F., P. van Driel, A. B. Reid, and J. D. Fairhead, 2001, Magnetic source parameters of two-dimensional structures using extended Euler deconvolution: Geophysics, **66**, 814–823.

Nabighian, M. N., 1972, The analytic signal of two-dimensional magnetic bodies with polygonal cross-section: Its properties and use for automated anomaly interpretation: Geophysics, **37**, 507–517.

Nabighian, M. N., and R. O. Hansen, 2001, Unification of Euler and Werner deconvolution in three dimensions via the generalized Hilbert transform: Geophysics, **66**, 1805–1810.

Naidu, P. S., and M. P. Mathew, 1998, Analysis of geophysical potential fields — A digital signal processing approach: Elsevier Science Publ. Co., Inc.

Pasteka, R., and P. Richter, 2002, A simple approach to regularized gradients calculation in gravimetry and magnetometry: 64th Annual Conference and Exhibition, EAGE, Extended Abstracts, P118.

Reid, A. B., J. M. Allsop, H. Granser, A. J. Millett, and I. W. Somerton, 1990, Magnetic interpretation in three dimensions using Euler deconvolution: Geophysics, **55**, 80–91.

Thompson, D. T., 1982, EULDPH: A new technique for making computer-assisted depth estimates from magnetic data: Geophysics, **47**, 31–37.

Zhang, C., M. F. Mushayandebvu, A. B. Reid, J. D. Fairhead, and M. E. Odegard, 2000, Euler deconvolution of gravity tensor gradient data: Geophysics, **65**, 512–520.