GEOPHYSICS, VOL. 39, NO. 1 (FEBRUARY 1974), P. 85–92, 7 FIGS.

ADDITIONAL COMMENTS ON THE ANALYTIC SIGNAL OF TWO-DIMENSIONAL MAGNETIC BODIES WITH POLYGONAL CROSS-SECTION†

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In a previous paper (Nabighian, 1972), the concept of analytic signal of bodies of polygonal cross-section was introduced and its applications to the interpretation of potential field data were discussed.

The input data for the proposed treatment are the horizontal derivative $T(x)$ of the field profile, whether horizontal, vertical, or total field component. As it is known, this derivative curve can be thought of as being due to thin magnetized sheets surrounding the causative bodies.

From the Fourier transform $F(w)$ of $T(x)$, a new spectrum $\tilde{F}(w)$ was created as shown in Figure 1. The spectrum $\tilde{F}(w)$ is causal in the frequency domain, i.e., it is defined only for positive values of the argument, being nil otherwise.

As it is known, for such causal functions, their Fourier transform has the property that their real and imaginary parts are the Hilbert transform of each other. In our case, the real part of the inverse Fourier transform of $\tilde{F}(w)$ turns out to be the original input function $T(x)$, whereas the imaginary part turns out to be the vertical derivative of the field profile $H(x)$[1]. Let $T(x)-iH(x)$ be the inverse Fourier transform of $\tilde{F}(w)$ which, as shown previously, is an analytic function everywhere except at the corners of each polygon, where it has single poles.

For the case of a single infinite sheet, the amplitude of this analytic signal, or any of its horizontal derivatives, is a bell-shaped symmetric function maximizing exactly over the top of the magnetized sheet (Figure 1).

There is a one-to-one relationship between the depth h to the top of the sheet and the abscissa X_R of the point where the amplitude of the analytic signal (or of its derivatives) drops to $1/R$ of its maximum value:

$$X_R = h\sqrt{R^{1/n+1} - 1},$$

where n is the order of the derivative of the analytic signal ($n=0$ is the analytic signal itself). For $R=2$ and $n=0$, the half-maximum half-width is equal to h. The amplitude curves become narrower the higher the order of derivative used.

For the case of a superposition of many sheets, in a first-order approximation, the amplitude of the analytic signal (or of its derivatives) can be thought of as being made up of a combination of bell-shaped symmetric functions, each maximizing over the corresponding corner of each polygon.

Similar results can be obtained if we use the phase of the analytic signal. However, such an operation is less stable, since it involves a division ($\tan^{-1} H/T$) instead of sum of squares (T^2+H^2).

As a byproduct of the above developments, a simple integration of the analytic function yields immediately the field reduced to the pole

$$\Delta M\,\big|_{\text{pole}} = \text{Real part of} \int [T(x)-iH(x)]e^{i\phi_2}dx = \int [T(x)\cos\phi_2 + H(x)\sin\phi_2]dx, \tag{1}$$

where ϕ_2 is an angle which depends only on the inclination of the magnetic field. ($\phi_2=180-2I$ for total field measurements, $\phi_2=90-I$ for horizontal and vertical field measurements.)

Figures 2 through 5 show the amplitude of the

[1] For clarity, the Hilbert transform of $T(x)$ is denoted here by $H(x)$ instead of $T_1(x)$, as in the previous paper.

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$$\Delta M(x) \longrightarrow T(x) = \frac{\partial(\Delta M)}{\partial x} \longrightarrow F(\omega)$$

$$\tilde{F}(\omega) = \begin{cases} 2F(\omega) & \omega > o \\ F(\omega) & \omega = o \\ o & \omega < o \end{cases} \longrightarrow T(x) - iH(x) \text{ where } H(x) = \frac{\partial(\Delta M)}{\partial y}$$

For a single sheet

$$(\text{Amplitude})^2 = a(x) = T^2 + H^2 = \frac{\alpha^2}{h^2+x^2}$$

$$\text{Phase} = \theta(x) = -\tan^{-1}\frac{H}{T}$$

$$a_n(x) = \left(\frac{\partial^n T}{\partial x^n}\right)^2 + \left(\frac{\partial^n H}{\partial x^n}\right)^2 = \frac{(1^2 \,.\, 2^2 \,..n^2)\alpha^2}{(h^2+x^2)^{n+1}}$$

$$\text{with } a_o(x) = a(x)$$

FIG. 1. Summary of main results published previously (Nabighian, 1972). The horizontal axis is x, the vertical axis is y and is positive downward.

analytic signal and its first and third derivative for the previously studied trapezoidal body.

Two facts are worth mentioning here: First, as seen from Figures 3 and 4, we have a novel and unique way of recognizing the shape of causative bodies. Second, as seen from Figure 5, the use of higher order derivatives has the property of "spiking" the effect due to the shallowest poles at the expense of the effect of deeper poles.

Usually, spiking can be achieved by a downward continuation operation which, in frequency domain, is equivalent to multiplying the spectrum by e^{+zw}. This accounts for the noise amplification in such operations.

By contrast, a higher order derivative involves a multiplication by w^n which is always smaller compared to the exponential function. Thus, spiking achieved by taking the amplitude of higher order derivatives of the analytic function is inherently more stable than the spiking achieved by downward continuation.

Another interesting property of the analytic signal follows from the fact that the Hilbert transform and the Fourier transform are essentially at quadrature to each other. Indeed, as shown previously, if $F(w)$ is the Fourier transform of $T(x)$ then the Fourier transform $H(w)$ of $H(x)$ is given by

$$H(w) = i \operatorname{sgn}(w) F(w), \tag{2}$$

where sgn is the signum function and $i = \sqrt{-1}$.

From (2) it follows that the *finite* Fourier series representation of $T(x)$ and $H(x)$ is of the form

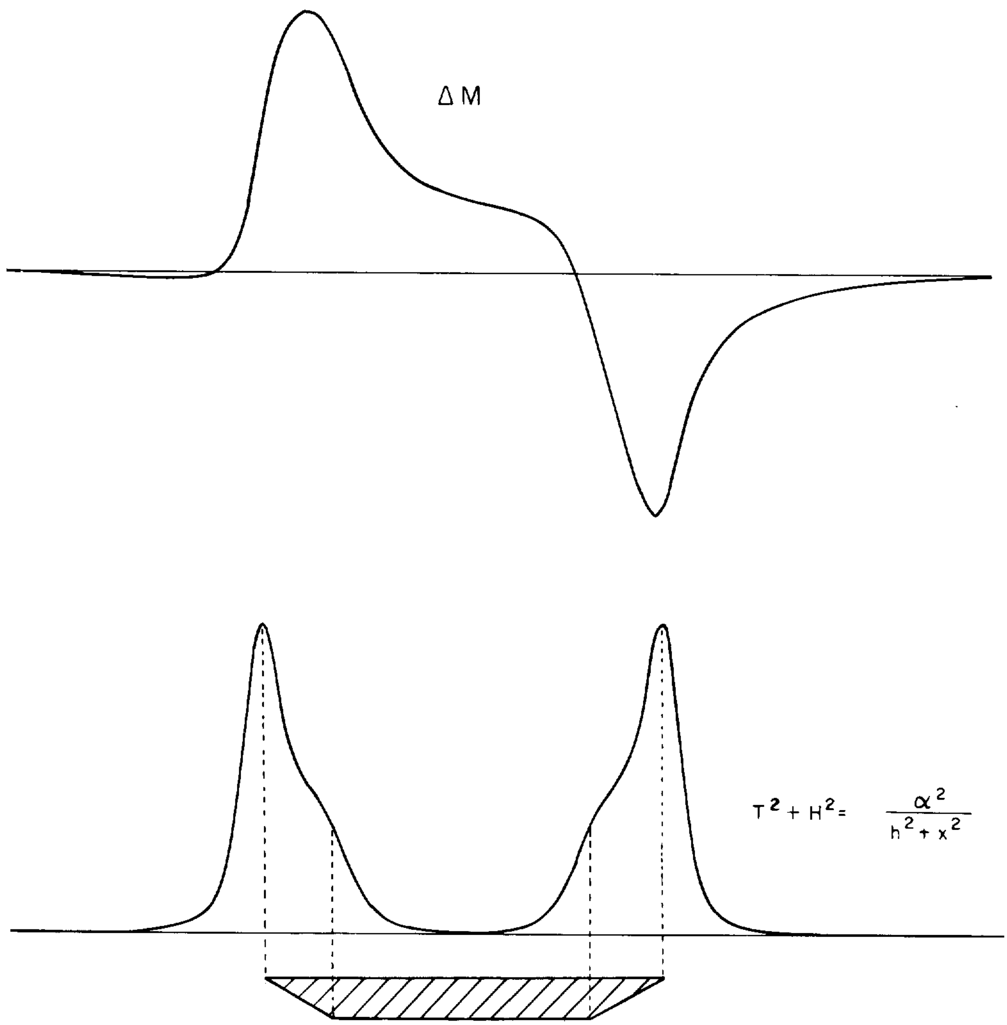

FIG. 2. The total field curve (ΔM) and the amplitude of the analytic signal (T^2+H^2) for a trapezoidal body.

$$
\begin{aligned}
T(x) &= \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx)e^{kz} \\
H(x) &= \sum_{k=1}^{N} (a_k \sin kx - b_k \cos kx)e^{kz}.
\end{aligned} \tag{3}
$$

Expression (3) has an oscillatory component, given by the trigonometric functions, and an amplification component given by the exponential term. In general, we can think of the trigonometric functions as the carrier, the remaining being the modulation.

When one continues downward ($z>0$), expression (3) yields increasingly oscillatory functions of higher and higher amplitudes. We can thus write approximately for large values of z and k:

$$
\begin{aligned}
T(x) &\simeq (a_N \cos Nx + b_N \sin Nx)e^{Nz} \\
H(x) &\simeq (a_N \sin Nx - b_N \cos Nx)e^{Nz},
\end{aligned} \tag{4}
$$

and

$$
T^2(x) + H^2(x) \simeq (a_N^2 + b_N^2)e^{2Nz}. \tag{5}
$$

In other words, the amplitude function of the

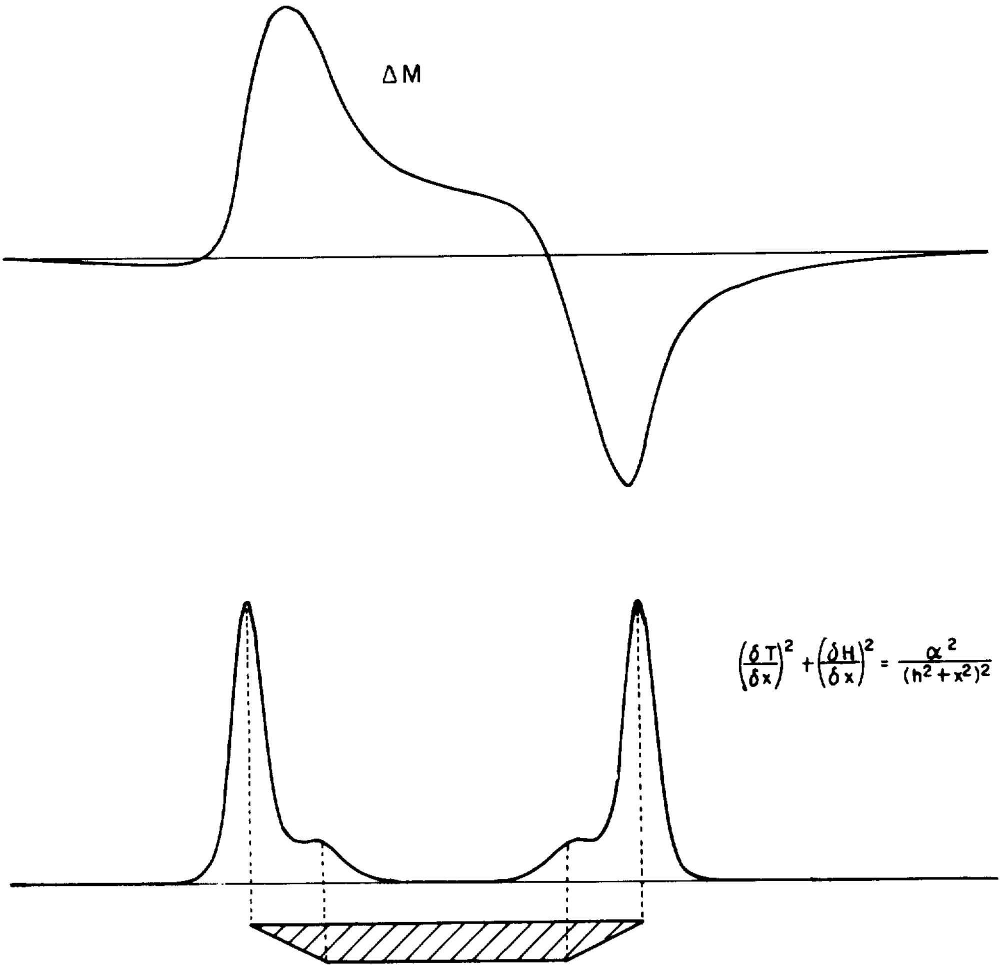

FIG. 3. The total field anomaly (ΔM) and the amplitude of the first derivative of the analytic signal ($T'^2+H'^2$) for a trapezoidal body.

analytic signal should cancel the oscillations, while retaining the modulation. The same is true, as can be easily seen, for the amplitude of the higher order derivatives of the analytic signal. Also, since the series is finite, the same is formally true even if one continues downward below the sources. Since for a single sheet we have

$$\frac{\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial H}{\partial x}\right)^2}{T^2(x)+H^2(x)}=\frac{1}{h^2+x^2}, \qquad (6)$$

it seems logical to use the above expression to provide a normalizing factor for the proposed technique.

The numerical procedure

From the ground surface, the analytic signal was continued downward to a depth of approximately one quarter of the depth of the sources ($h/4$). The downward continuation was accomplished by standard techniques and expression (6) was evaluated at this new level. Once this was accomplished, the continued $T(x)$ and $H(x)$ functions were normalized with respect to the maximum value of either $T(x)$ or $H(x)$ (in order to avoid overflow in the computer), and the data

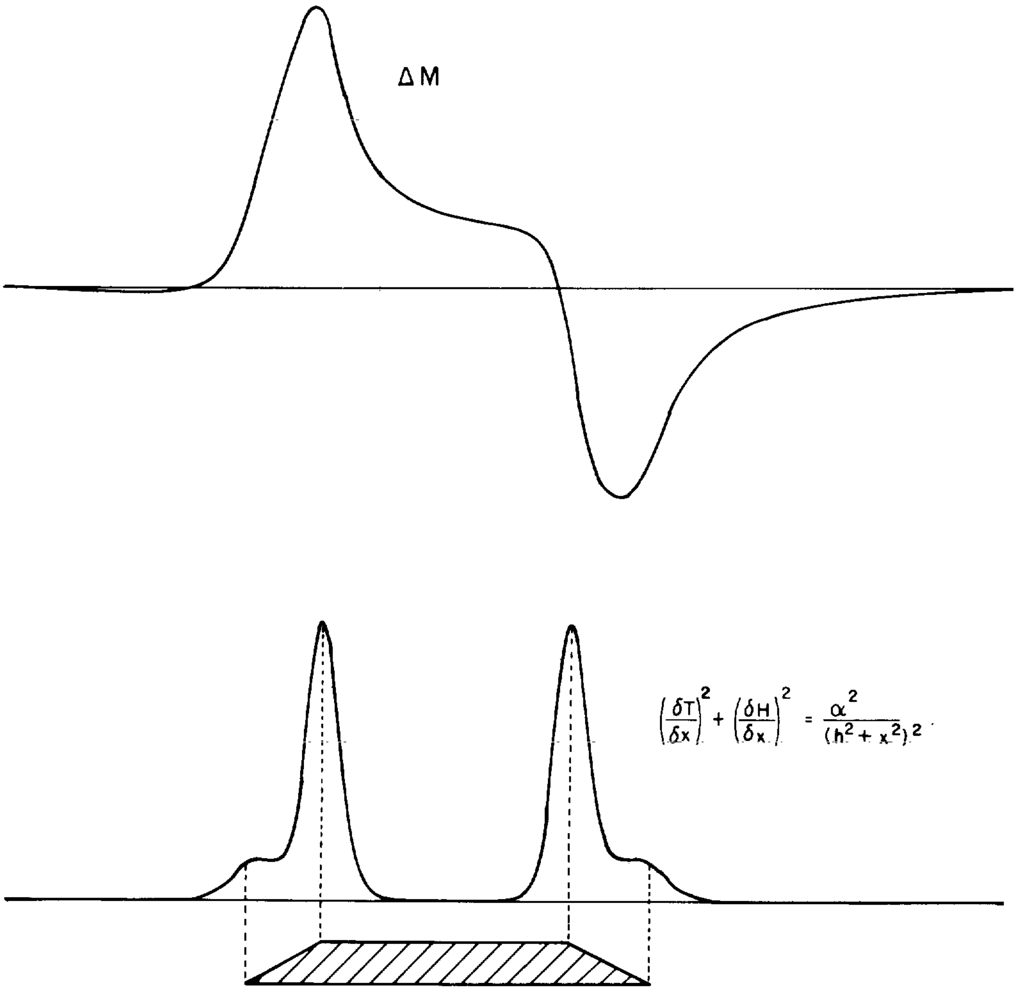

FIG. 4. The total field anomaly (ΔM) and the amplitude of the first derivative of the analytic signal ($T'^2+H'^2$) for an inverted trapezoidal body.

were continued downward another $h/4$ to a new level, where the procedure was repeated all over again.

If (5) is approximately true, (6) should also be approximately true and, as Figure 6 shows (for the case of a single sheet), the ratio (6) maximizes exactly over the top of the sheet and then decreases gradually in all directions.

The same holds true if there are many poles situated at more or less comparable depths. When the poles are situated at different depths, only the shallowest will appear by this procedure since the problem is inherently ill-posed.

The procedure is interesting in itself in that it could yield the depth to the top of the causative bodies. It also shows the possibilities of the concept of analytic signal.

Finally, an example of the use of the amplitude of higher order derivatives of the analytic signal is illustrated in Figure 7 on a noisy ground magnetic profile. Due to the high noise content, the data were first upward continued 400 ft with a standard upward continuation operator. The amplitude of the analytic signal and its third derivative were plotted at this level, together with the field reduced to the pole.

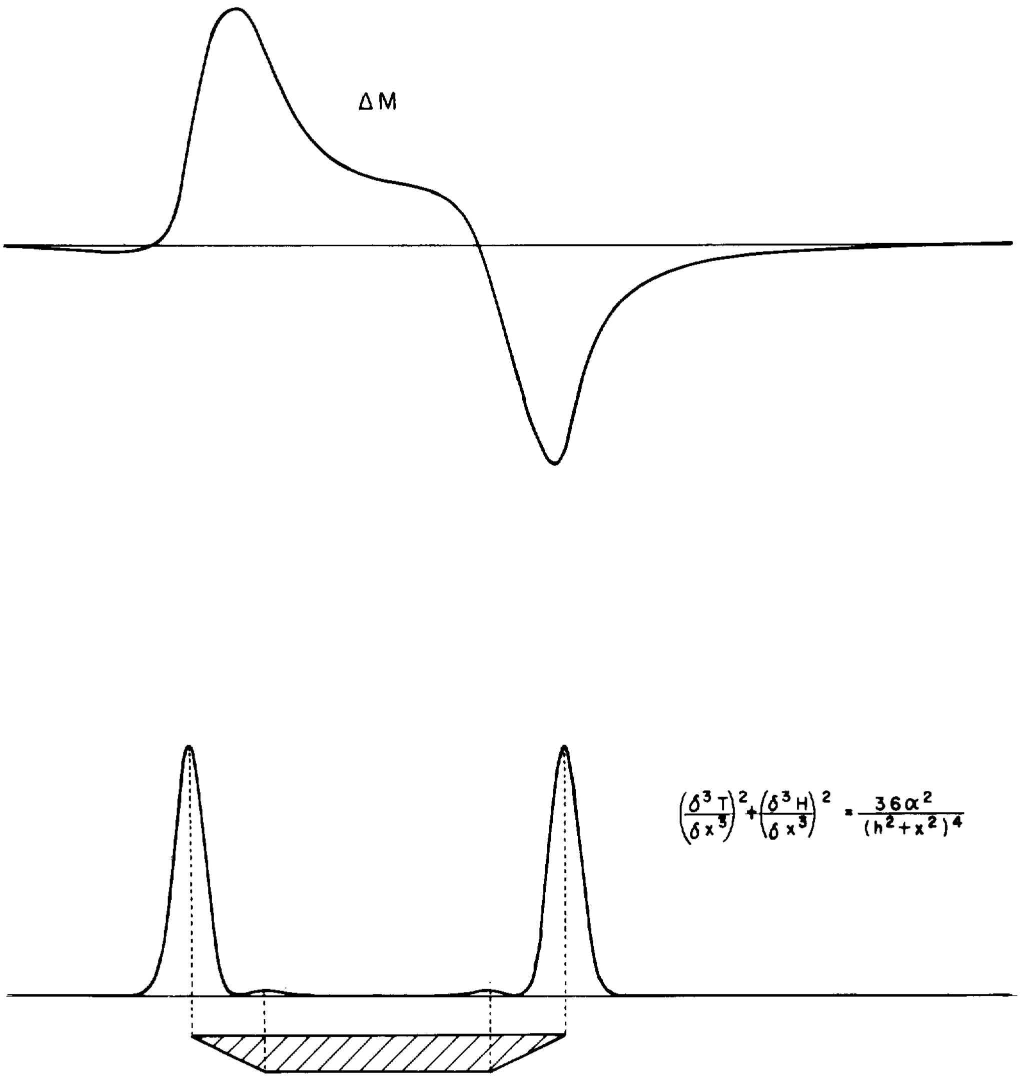

FIG. 5. The total field anomaly (ΔM) and the amplitude of the third order derivative of the analytic function ($T'''^2+H'''^2$) for a trapezoidal body.

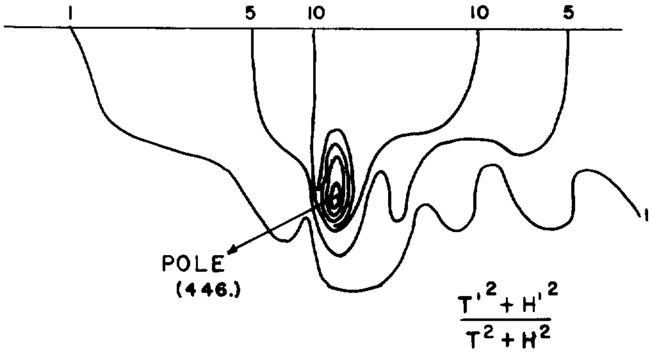

FIG. 6. Contours in a vertical plane of the normalized ratio $(T'^2+H'^2)/(T^2+H^2)$ for a single sheet with the top at the point marked pole. Contours are independent of the dip of the magnetized sheet.

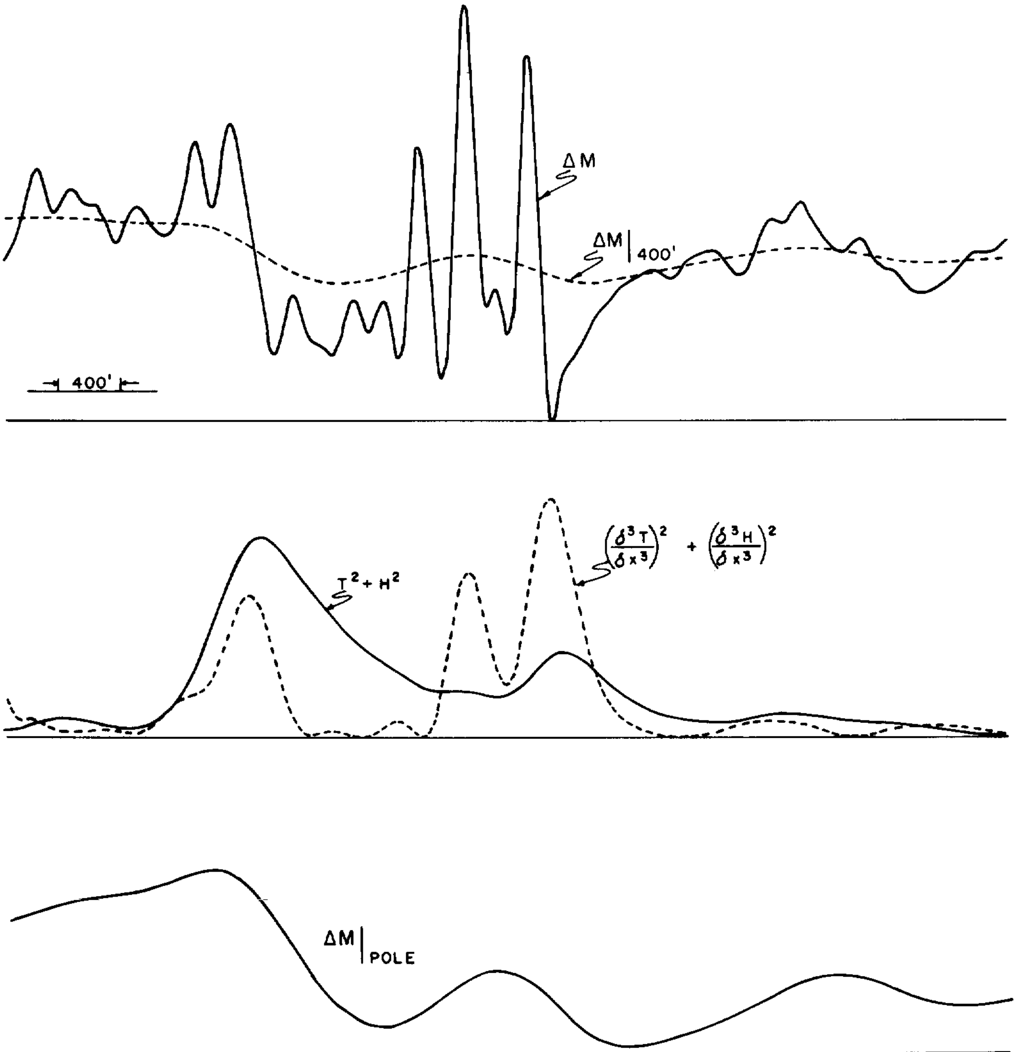

FIG. 7. Ground magnetic data (vertical component) processed with the proposed method.

Although of poor quality, the data still seem to be tractable by the proposed technique. The spiking, due to the third derivative, is evident and the field reduced to the pole seems to indicate three main magnetized units. The depths can be read directly from the half-maximum width of the curves. Nevertheless, it is acknowledged that with such poor data, any interpretation is highly questionable.

In this particular case, depths determined from the amplitude of the analytic signal and from the amplitude of its third derivative agree well with each other, both yielding outcropping poles. Experience shows that this is indeed the case if the two-dimensionality assumption is satisfied. If such is not the case, the higher order derivatives usually yield less reliable results.

Again, experience shows that higher order derivatives could sometimes yield additional poles, due to the inherent noise introduced by such an operation. Only by the introduction of geologic constraints can these artificial poles be filtered out.

As previously mentioned, these concepts can be applied to gravity data. Also, the concept of a causal function in the frequency domain can be extended to a function of two variables leading to the possible interpretation of three-dimensional anomalies by similar techniques. This will form the object of a future paper.

ACKNOWLEDGMENT

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REFERENCE

Nabighian, M. N., 1972, The analytic signal of two-dimensional magnetic bodies with polygonal cross-section: Its properties and use for automated anomaly interpretation: Geophysics, v. 37, p. 507–517.