Hazard Rate of Lognormal Distribution: An Investigation

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Abstract: The behaviour of the hazard rate of the lognormal random variable, as has been reported in some recent publications, is quite misleading. This paper mainly attempts to put forth the true behaviour of the hazard rate of lognormal distribution, after carrying out analytical and numerical investigations. It has been shown, using method of calculus that the hazard rate is a unimodal function with convexity upwards. This typical behaviour is not acceptable in Reliability Engineering. The true behavior of the hazard rate has been numerically simulated in MATLAB, Version 7.0, with appropriate case studies. The findings of the investigation are discussed and conclusion drawn.

1. Introduction

Lognormal distribution, like Weibull distribution, can take on a variety of shapes, by virtue of the shape parameter. Therefore, the notion prevails that the lognormal distribution could be a serious competitor to the Weibull distribution, in Reliability Engineering [1]. This aspect is investigated here. The probability density function (pdf) for lognormal distribution is of the form:

$$f(t) = \frac{1}{t\sigma_x \sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\ln t - \mu_x}{\sigma_x}\right)^2}, \quad t \geq 0 \tag{1}$$

Salient reliability measures viz., reliability function R(t) and modal life (t_{mode}) of lognormal are as given in (2) and (3)

$$R(t) = 1 - \phi\left[\frac{\ln t - \mu_x}{\sigma_x}\right] \tag{2}$$

$$t_{mode} = e^{\mu_x - \sigma^2_x} \tag{3}$$

Hazard rate h(t), is an important reliability measure, which indicates the failure phenomenon. It is also known as instantaneous failure rate and is given as a ratio of f(t) to R(t). In failure data analysis, the hazard rate behaviour plays a key role and therefore, it is necessary to know its correct form. Misconception about the hazard rate behaviour widely prevails in literature.

2. Notation

T	: Random variable (r.v.)
μ_x	: Mean of ln T
σ_x	: Standard deviation of ln T

X   : Normal random variable
Z   : Standard normal random variable
h (t)   : Hazard rate
h (z)   : Hazard rate for standard normal variable
t*   : Location of Maximum of h(t)
z*   : Critical value of z corresponding to t*
$\Phi(z)$: Distribution function of standard normal
t_{mode} : Modal life
f(t)   : Probability density function (pdf)
R (t)   : Reliability function

3. Literature Review

In this section, the present understanding of hazard rate behaviour of lognormal distribution, in reported literature, is highlighted. Sinha and Kale [2] state that it can be proved, for lognormal distribution, that the hazard rate is a decreasing function of time. Wadsworth *et al* [3] relate hazard rate with parameter σ_x, stating that for σ_x approximately equal to 0.5, the hazard rate is constant; for σ_x less than 0.2, the failure rate is increasing, while for σ_x greater than 0.8, it is decreasing. They have also shown that the behaviour is an exponentially decreasing function for $\sigma_x = 1$. Lewis [4] reports that the failure rate can be increasing or decreasing function depending on σ_x. Kapur and Lamberson [5] report that the behaviour of lognormal hazard rate is a monotonically increasing function. Chang [6] used this distribution, to ascertain, effective burn-in time, for warranty modeling.

4. Problem on Hand

It is observed from the reviewed literature that the prevailing concept about hazard rate of lognormal distribution is a monotonically increasing function and its nature depends on σ_x. While dealing with a practical situation authors felt that the concept needs to be revisited and verified. Mariappan *et al* [7] report typical behavior of lognormal distribution on hazard rate. This work is a logical extension of the same.

5. Analytical Treatment

Hazard rate of a time to failure is given by

$$h(t) = \frac{f(t)}{R(t)} \Rightarrow h(0) = \frac{f(0)}{R(0)} = 0, \text{whereas}, h(\infty) = \frac{0}{0} \tag{4}$$

$h(\infty)$ in (4) gives an indeterminate form, which is evaluated as follows:
Differentiating (1) *w.r.t.* 't',

$$f'(t) = \frac{-f(t)}{t}\left[1 + \frac{\ln t - \mu_x}{\sigma_x^2}\right] \tag{5}$$

Using (5), the hazard rate at $t = \infty$ becomes

$$h(\infty) = t \to \infty \frac{\lim\left[1 + \dfrac{\ln t - \mu_x}{\sigma_x^2}\right]}{t} = t \to \infty \lim\left[\dfrac{\dfrac{1}{t\sigma_x^2}}{1}\right] = 0$$

Thus, hazard rate at $t = \infty$ is found to be zero and therefore, there has to be at least one mode. Investigation to ascertain this is as follows:

Taking first derivative of (4) w.r.t. 't' and substituting f'(t) from (5),

$$h'(t) = \frac{f(t)}{R^2(t)}\left[\frac{-R(t)}{t}\left\{1+\frac{\ln t - \mu_x}{\sigma_x^2}\right\}+f(t)\right] \tag{6}$$

Lognormal density is unimodal with convexity upwards; therefore f′(t) is positive in the closed interval [0,t_{mode}]. From this, it can be inferred that hazard rate h(t) is monotonically increasing function in this interval. Beyond this range, in the interval ($t_{mode,}$ ∞], f′(t) assumes negative value, revealing the possibility of h(t) assuming a stationary value, in between. As the first factor in (6), is non-negative, the second factor, in the equation, needs investigation. Let the second factor be Ψ(t), i.e.,

$$\psi(t) = \frac{-R(t)}{t}\left[1+\frac{\ln t - \mu_x}{\sigma_x^2}\right]+f(t) \Rightarrow \psi(t_{mode}) = f_{max} \tag{7}$$

From the above equation, Ψ(∞) will assume an indeterminate value. Evaluating this indeterminate form, using order of zero, Ψ(∞) is zero. It is further required to ascertain that Ψ(t) is a monotonically decreasing function in this region.

$$\psi'(t) = \frac{R(t)}{t^2\sigma_x^2}\left[-1+\sigma_x^2+\ln t - \mu_x\right]$$

Further investigation yields, $t = e^{1-\sigma^2 x + \mu_x}$, a stationary point. In order to classify this stationary point, a second derivative is evaluated.

$$\psi''(e^{1-\sigma_x^2+\mu_x}) = \frac{R(t)}{t^3\sigma_x^2}+0 > 0$$

As per the above equation, Ψ(t) is minimum at the stationary point and therefore, it must cross abscissa, at some point within the interval ($t_{mode,}$ ∞]. As seen from (6), h'(t) will continue to be positive, in some initial portion of this interval, till Ψ(t) is zero and therefore h(t) will have a mode at time t, where, Ψ(t) = 0. Substituting ψ (t) =0 in (7) gives

$$[1-\phi(z)]\left[1+\frac{z}{\sigma_x}\right] = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \Rightarrow h(z)-z = \sigma_x$$

Equation above, a transcendental equation, is solved numerically and the solution is plotted as shown in Fig. 1, verifying that the hazard rate of lognormal distribution is unimodal with convexity upwards.

6. Case Studies

To get a practical feel and statistical model adequacy of the entire investigation procedure, three cases are investigated. The details are given in the following sections.

6.1 Maintainability Analysis of Mechanical Pump

Data of repair time of a mechanical pump was collected as shown in Table 1. This data is then put under the data analysis for repair time. The mathematical model fitted for this repair time, using Lillietest (with a significance level of 5%) is found to be lognormal, with the parameters μ_x and σ_x as 4.7531 and 0.5755 respectively. Then the behaviour of hazard rate was investigated.

For this investigation, data were simulated with the obtained parameters in MATLAB, Version 7.0. For this simulated data, hazard rate was calculated using (4) and plotted. The plot obtained is shown in Fig. 2. Similarly two more cases are also investigated, whose details are given below.

Table 1: Repair Times in Minutes of a Mechanical Pump

47.1	84.8	151.9	122.5	218.2	99.6	59.8	138.8	231.5	53.4
102.4	100.8	230.1	104.6	61.5	122.1	186.2	498.4	77.0	78.7
112.3	44.0	151.3	151.3	222.8					

## 6.2 Failure Analysis of Electrical Insulation

The case of electrical insulation is taken up here, for investigation. Life testing data of insulation, at temperature 190°F and 240° F, are as given in Tables 2 and 3 respectively. The similar analysis was carried out, giving the hazard rate behaviour as shown in Fig. 3 and   Fig. 4.
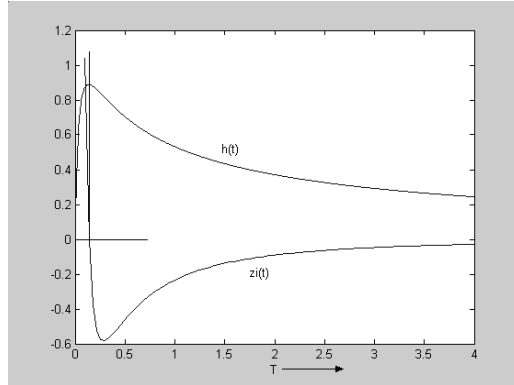
Fig. 1: Hazard Rate of Lognormal Distribution
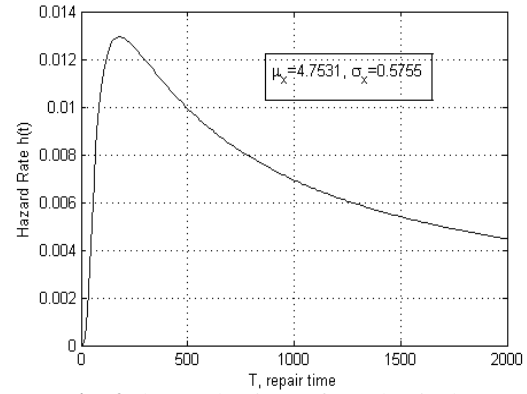
Fig. 2: h(t) Behaviour of Mechanical Pump

Table 2: Failure Data of Electrical Insulation at 190° F

7228	7228	7228	8448	9167	9167	9167	9167	10511	10511

Table 3: Failure Data of Electrical Insulation at 240° F

1175	1175	1521	1576	1617	1665	1665	1713	1761	1953
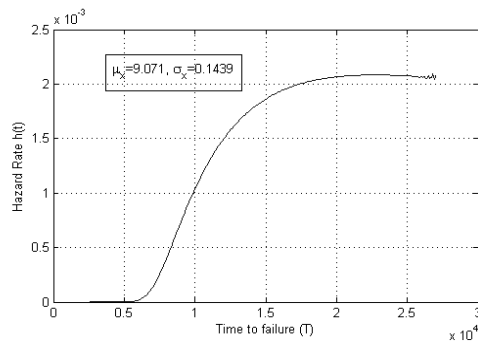
Fig. 3: Hazard Rate for Insulation at 190°F
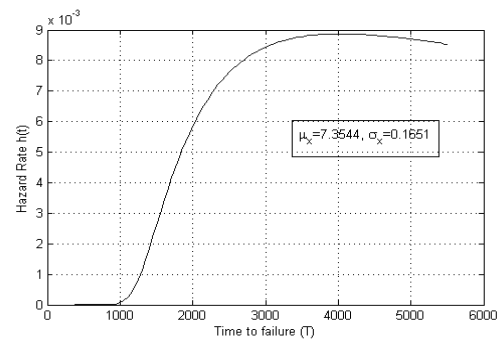
Fig. 4: Hazard Rate for Insulation at 240°F

6.3 Failure Analysis of Ball Bearings

Life data of 23 ball bearings considered by Jiang [8] is shown in Table 4. This was also analyzed, in a similar way, giving hazard rate behaviour, as shown in Fig. 5.

Parameters μ_x and σ_x, obtained in case of ball bearings, are 4.7631 and 0.5755 respectively. They compare well with the values obtained by Jiang [8], $\mu_x= 4.15$ and $\sigma_x= 0.522$. Thus, the procedure was validated.

Table 4: Bearing Life Time (millions of revolutions)

|								
17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84	51.96
54.12	55.56	66.80	68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40				

7. Results and Discussions

Lognormal distribution, perhaps, has the origin in Quality Management, where manufacturing process, with large number of factors with multiplicative property, is to be dealt with. However, it has established its place in Reliability Engineering.
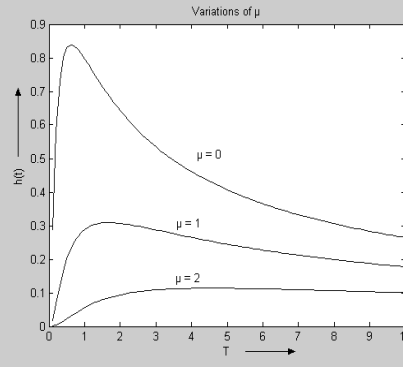
It is found that the hazard rate of lognormal distribution increases first, peaks and then decreases. This is true for all the values of the parameters σ_x and μ_x as has been analytically shown in § 5. The same has been verified numerically and the corresponding plots are shown in Figures 6 and 7.
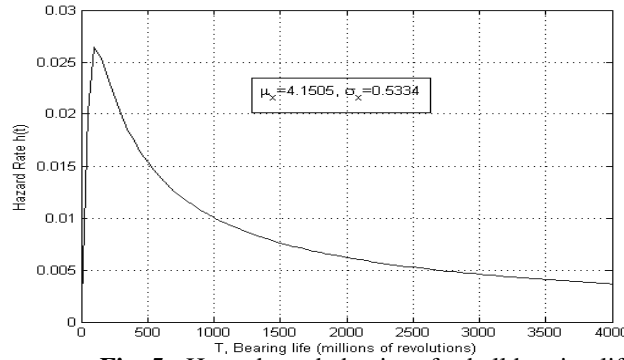
Fig. 5: Hazard rate behaviour for ball bearing life
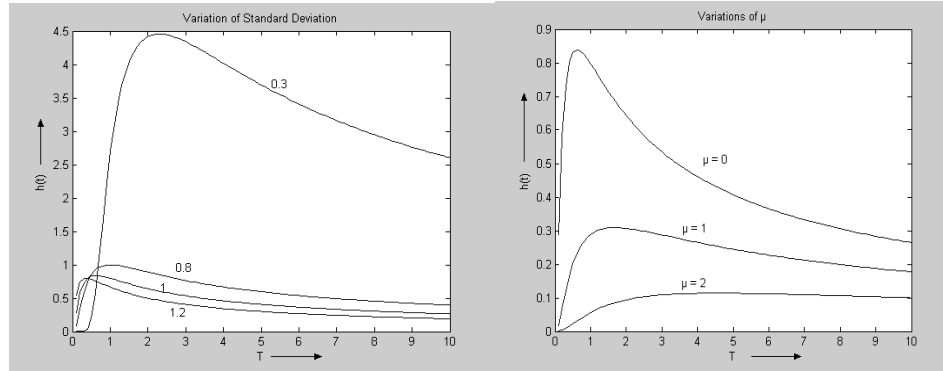
Fig. 6: Effect of σ_x on h(t)   **Fig. 7:** Effect of μ_x on h(t)

During numerical investigation for sensitivity of hazard rate behaviour with respect to change in parameters, it is observed that the occurrence of t_{mode} depends on both σ_x and μ_x as seen in Figures 6 and 7. Occurrence of t_{mode} shifts to the right with the increase in μ_x whereas; it shifts to the left with the increase in σ_x. With the increase in σ_x, t_{mode} comes very close to $t = 0$, perhaps because of this, a notion prevails that the lognormal

distribution has a negative exponential behaviour. Further, it can be observed, from these figures, that the hazard rate decreases with the increase in μ_x and σ_x. With this typical behaviour, lognormal distribution can model corrosive failure with parameter values $\sigma_x < 1.5$ and $\mu_x < 2$. As the severity of corrosion increases, both the parameter values decrease proportionately, from the given values here. This can also model repair time distribution, for a well-evolved repair procedure, with the parameter values σ_x and μ_x close to 1.5 and 2 respectively. For values of σ_x less than 0.15, the nature of hazard rate versus T curve, beyond t_{mode}, was found to be peculiar, while carrying out sensitivity analysis. This phenomenon was also observed while dealing with the case studies.

8. Conclusion

The hazard rate of lognormal distribution has been investigated, both analytically and numerically, to establish clearly its pattern. It has been shown that hazard rate first increases with time to failure, reaches its peak and then decreases, thus, making it suitable for modeling repair time and failures due to corrosion. However, one has to be careful while using this distribution for warranty modeling and maintenance policy decisions.

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