Understanding Z^-1 in Digital Filters with an Exponential IIR Low-Pass Example

1. What does Z^-1 mean in digital filters?

In digital signal processing, the Z-transform is a mathematical tool used to analyze discrete-time systems. The term Z^{-1} represents a unit delay in this domain.

If $x[n]$ is the current input, then:
- $x[n-1] = Z^{-1} * x[n]$: one sample delay
- $x[n-2] = Z^{-2} * x[n]$: two samples delay, etc.

This notation allows us to write filter equations in a compact and manipulable algebraic form.

2. Example: First-Order Exponential IIR Low-Pass Filter

Time-domain difference equation:

$$y[n] = (1 - \alpha)x[n] + \alpha y[n-1]$$

Where:
- $y[n]$: current output
- $x[n]$: current input
- $y[n-1]$: previous output
- α: smoothing factor ($0 < \alpha < 1$)

Z-domain (using Z^{-1}):

$$Y(z) = (1 - \alpha)X(z) + \alpha Z^{-1}Y(z)$$

Transfer function:

$$H(z) = Y(z)/X(z) = (1 - \alpha) / (1 - \alpha Z^{-1})$$

3. Rewriting H(z) without Z^-1

To express the transfer function using positive powers of z, multiply numerator and denominator by z:

$$H(z) = (1 - \alpha)z / (z - \alpha)$$

This form is mathematically equivalent and often used for pole-zero analysis.

4. Summary of the Two Forms

Form	Expression	Use Case

Z^-1 form	(1 - α) / (1 - αZ^-1)	Implementation (difference eqns)
Positive powers	(1 - α)z / (z - α)	Analysis (poles and zeros)

- Pole: z = α

- Zero: z = 0

5. Behavior Example (α = 0.9)

Given initial conditions:
- x[0] = 1, x[1] = x[2] = ... = 0, y[-1] = 0

Calculated values:
- y[0] = 0.1
- y[1] = 0.09
- y[2] = 0.081
- ...

The output decays smoothly, demonstrating low-pass filtering.

This summary provides a foundational understanding of how Z^-1 represents delay in digital filters, using a first-order IIR low-pass filter as a practical example.