Approximating the error function erf by analytical functions

The Error function

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$

shows up in many contexts, but can't be represented using elementary functions.

I compared it with another function f which also starts linearly, has $f(0) = 0$ and converges against the constant value 1 fast, namely

$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Astoningishly to me, I found that they never differ by more than $|\Delta f| = 0.0812$ and converge against each other exponentially fast!

I consider $\tanh(x)$ to be the somewhat prettyier function, and so I wanted to find an approximation to erf with "nice functions" by a short expression. I "naturally" tried

$f(x) = A \cdot \tanh(k \cdot x^a - d)$

Changing $A = 1$ or $d = 0$ on it's own makes the approximation go bad and the exponent a is a bit difficult to deal with. However, I found that for $k = \sqrt{\pi} \log(2)$ the situation gets "better". I obtained that k value by the requirement that "norm" given by

$\int_0^\infty \mathrm{erf}(x) - f(x) dx,$

i.e. the difference of the functions areas, should valish. With this value, the maximal value difference even falls under $|\Delta f| = 0.03$. And however you choose the integration bounds for an interval, the area difference is no more than 0.017.

```
k := √π Log[2];

Plot[{Erf[x] - Tanh[k x]}, {x, 0, 5}, PlotRange → {-.05, .05}]
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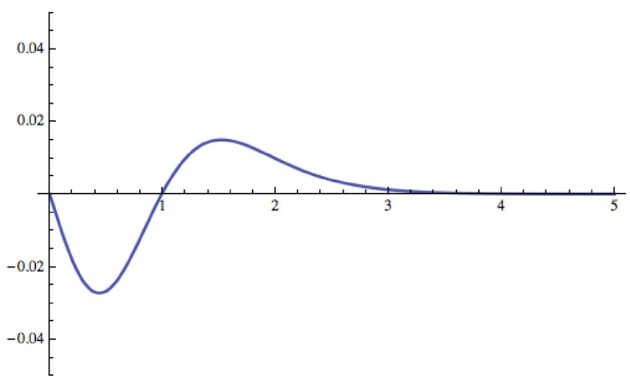


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Integrate[Erf[x] - Tanh[k x], {x, 0, ∞}]
0
```

Numerically speaking and relative to a unit scale, the functions erf and $\tanh(\sqrt{\pi} \log(2) x)$ are essentially the same.

My question is if I can find, or if there are known, substitutions for this non-elementary function in terms of elementary ones. In the sense above, i.e. the approximation is compact/rememberable while the values are even better, from a numerical point of view.

The purpose being for example, that if I see somewhere that for a computation I have to integrate erf, that I can think to myself "oh, yeah that's maybe complicated, but withing the bounds of 10^{-3} usign e.g. $\tanh(k \cdot x)$ is an incredible accurate approximation."

(approximation) (elementary-functions)

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