# Initial and Final Value Analysis using Laplace and Z-Transform

## Laplace Domain Analysis

Given Transfer Function:

H(s) = (2s + 1) / (s² + 2s + 1)

Input: Unit Step Function

U(s) = 1 / s

## Output in Laplace Domain

Y(s) = H(s) × U(s) = (2s + 1) / [s × (s² + 2s + 1)]

## Initial Slope (dy/dt at t = 0⁺)

Using the Initial Derivative Theorem:  
dy(t)/dt|ₜ₌₀⁺ = limₛ→∞ [s² × Y(s)] = limₛ→∞ [s²(2s + 1) / (s × (s² + 2s + 1))]

Simplifying:  
Numerator: s²(2s + 1) = 2s³ + s²  
Denominator: s(s² + 2s + 1) = s³ + 2s² + s  
As s → ∞, divide numerator and denominator by s³:  
limₛ→∞ [(2 + 1/s) / (1 + 2/s + 1/s²)] = 2

→ Initial slope at t = 0⁺ is \*\*2\*\*.

## Final Value (y(t) as t → ∞)

Using the Final Value Theorem:  
y(∞) = limₛ→₀ [s × Y(s)] = limₛ→₀ [s × (2s + 1) / (s × (s² + 2s + 1))]

Canceling s:  
limₛ→₀ (2s + 1) / (s² + 2s + 1) = 1 / 1 = 1

→ Final value as t → ∞ is \*\*1\*\*.

## Summary (Laplace)

• Initial Slope at t = 0⁺: 2

• Final Value as t → ∞: 1

## Discrete-Time (Z-domain) Equivalent Concepts

In the Z-domain, equivalent theorems exist for analyzing the initial and final behavior of discrete-time systems. These theorems mirror the Laplace-domain analysis but are adapted for sampled systems.

### Final Value Theorem (Z-domain)

For a stable causal system with output Y(z) and input X(z), the final value of y[n] is given by:

limₙ→∞ y[n] = lim\_{z→1} (1 - z⁻¹) Y(z)

This is analogous to taking the limit as s → 0 in the Laplace domain. It assumes all poles of (1 - z⁻¹)Y(z) lie inside the unit circle, except possibly a simple pole at z = 1.

### Initial Value Theorem (Z-domain)

The initial value of y[n] (i.e., y[0]) is given by:

limₙ→0⁺ y[n] = lim\_{z→∞} Y(z)

This matches the intuition that early behavior is governed by the high-frequency (large z) characteristics.

### About z = -1 (Nyquist Frequency)

The point z = -1 corresponds to e^{jπ}, i.e., the Nyquist frequency in discrete-time systems. This point is important in frequency response analysis but is not used for determining time-domain behaviors such as initial value, final value, or slope. Therefore, it is not applicable to initial/final value theorems.

### Discrete-Time Approximation of Slope

There is no direct analogue in the Z-domain for computing the initial slope as in Laplace. Instead, the slope at the start can be approximated using finite differences:

dy[n]/dt ≈ (y[1] - y[0]) / Tₛ

Where Tₛ is the sampling period. This gives an estimate of the rate of change at the beginning of the signal.