Phil 140b **Spring 2016**

Assignment #4

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1. *Show that the function $f(x)$ defined by:*

$$\begin{cases} x^2 & x \text{ is even} \\ x+1 & x \text{ is odd} \end{cases}$$

is primitive recursive.

First, let's define the (characteristic function of a) relation $R(x)$ to see if a number is odd. We can define it as:

$$R(x') = 1 - R(x)$$
$$R(0) = 0$$

So $R(x)$ is primitive recursive.

Then, we can simply define an $f(x)$ that is primitive recursive:

$$f(x) = R(x)(x+1) + (1 - R(x))(x * x)$$

Since f was constructed with only primitive recursive functions, it is also primitive recursive.

2. *Let $f(x_1, \cdots, x_n, y)$ be a function. Define $\sum_{y<z} f(x_1, \cdots, x_n, y)$ to be $f(x_1, \cdots, x_n, 0) + \cdots + f(x_1, \cdots, x_n, z-1)$ if $z \neq 0$ and 0 if $z = 0$. Morover, define $\Pi_{y<z} f(x_1, \cdots, x_n, y)$ to be equal to $f(x_1, \cdots, x_n, 0) \cdots f(x_1, \cdots, x_n, z-1)$ if $z \neq 0$ and equals 1 if $z = 0$. The class of elementary functions is the smallest class which contains $x+y$, xy, $|x-y|$, $id_i^n(x_1, \cdots, x_n)$, x/y, and is closed under composition, bounded sums, and bounded products. Show that the following functions are elementary:*

(i) $z(x)$

$$z(x) = |x - x|$$

(ii) $s(x)$

$$one(x) = \Pi_{y<0} z(x)$$
$$s(x) = x + one(x)$$

(iii) $sg(x)$

$$sg(x) = x/x$$

(iv) $sg^*(x)$

$$sg^*(x) = |one(x) - sg(x)|$$

(v) $C_k^n(x_1, \cdots, x_n) = k$

$$C_k^n(x_1, \cdots, x_n) = \sum_{y<k} one(id_1^n(x_1, \cdots, x_n))$$

(vi) $pred(x)$

$$pred(x) = sg(x)\,|x - one(x)|$$

3. *$R(x_1, \cdots, x_n)$ is elementary iff its characteristic function is elementary. Let $R_1(x_1, \cdots, x_n)$ and $R_2(x_1, \cdots, x_n)$ be elementary.*

(a) *Construct the characteristic functions for $\neg R_1(x_1, \cdots, x_n)$ and $R_1(x_1, \cdots, x_n) \wedge R_2(x_1, \cdots, x_n)$.*

$$C_{\neg R_1}(x_1, \cdots, x_n) = |one(id_1^n(x_1, \cdots, x_n)) - C_{R_1}(x_1, \cdots, x_n)|$$
$$C_{R_1 \wedge R_2} = C_{R_1}(x_1, \cdots, x_n) C_{R_2}(x_1, \cdots, x_n)$$

(b) *Show that if $R(x)$ is an arbitrary numerical relation and $\{x : R(x)\}$ is finite, then $R(x)$ is elementary.*

$R(x)$ is elementary iff $C_R(x)$ is elementary. First let's construct a relation: $E_k(x)$ holds iff $x = k$. We define its characteristic function as:

$$C_{E_k}(x) = sg^*(|x - k|)$$

Since its characteristic function is elementary, E_k is elementary.
Since there are a finite number of x such that $R(x)$ holds, let's say, without loss of generalization, that they are: $x_1, \cdots, x_n$. First, we define:

$$R_1 \vee R_2 \iff \neg(\neg R_1 \wedge \neg R_2)$$

Finally, we can define $R(x)$ as follows:

$$R(x) \iff E_{x_1}(x) \vee E_{x_2}(x) \vee \cdots \vee E_{x_n}(x)$$

Since both E_k and logical or are elementary, $R(x)$ must be elementary as well.

4. *Show that the function $J(a, b)$ given by $\frac{1}{2}(a + b)(a + b + 1) + a$ is onto.*

Let's consider all the terms of the sequence $(\frac{1}{2}(n)(n + 1))$. The first is $\frac{1}{2}(0)(1) = 0$. And the distance between subsequent terms is:

$$\frac{1}{2}(n + 1)(n + 2) - \frac{1}{2}(n)(n + 1) = n + 1$$

So inbetween subsequent terms, we can fit $n + 1$ varying values of a so that no values inbetween are left out. $a = 0, \cdots, n$, and we can define $b = n - a$. With this scheme, every possible natural number has an inverse mapping to a unique pair (a, b).

Alternatively, we can construct an inverse function $J^{-1} : \mathbb{N} \to \mathbb{N}^2$ graphically:

$$\begin{array}{cccc} 0 \mapsto (0,0) & 1 \mapsto (0,1) & 3 \mapsto (0,2) & \ldots \\ 2 \mapsto (1,0) & 4 \mapsto (1,1) & 7 \mapsto (1,2) & \ldots \\ 5 \mapsto (2,0) & 8 \mapsto (2,1) & \ddots & \end{array}$$

It is trivially verifiable that it is indeed the inverse function. And following this well-defined numbering scheme across the diagonals, we clearly include every natural number, and so the J must be onto.

Also, since $J \circ J^{-1} = id_{\mathbb{N}}$, J is necessarily surjective.