Math 113 **Fall 2015**

Homework #5

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1. This problem deals with S_7 and U_{10}.

   (a) Find an element of S_7 that has order 10. Call it x. List the elements of $G = \langle x \rangle$, the subgroup of S_7 generated by your element x.

   $x = (12345)(67)$

$$x^1 = (12345)(67)$$
$$x^2 = (13524)$$
$$x^3 = (14253)(67)$$
$$x^4 = (15432)$$
$$x^5 = (67)$$
$$x^6 = (12345)$$
$$x^7 = (13524)(67)$$
$$x^8 = (14253)$$
$$x^9 = (15432)(67)$$
$$x^{10} = e$$

   The only generators for my group G are x^1, x^3, x^7, and x^9. Because G is a cyclic group of order 10, it's isomorphic to Z_{10}, which only has 4 generators, so there cannot be any more.

   (b) Determine the number of isomoprhisms $\phi : G \to U_{10}$.

   As we just demonstrated, Z_{10}, which is isomorphic to U_{10} and G, only has 4 possible generators. As we showed in class with U_6, there are no other ways to possibly map two groups, so there are only 4 isomorphisms.

2. This problem deals with dihedral groups and symmetry groups.

(a) The groups D_{12} and S_4 both have order 24. Prove that they are both nonabelian, but they are not isomorphic to each other.

We can show D_{12} is nonabelian by showing that two of the operations do not commute.

$$s * (rs) \stackrel{?}{=} s * (sr)$$

Geometrically speaking, the left hand side is reflecting, rotating forwards, then reflecting. Intuitively, this is just rotating backwards. The right hand side can be reassociated:

$$r^{-1} \stackrel{?}{=} (ss) * r$$

$$r^{-1} \neq r$$

Similarly, elements in S_4 do not commute, and we can show it with an example:

$$(134)(124) \stackrel{?}{=} (124)(134)$$

$$(12)(34) \neq (13)(24)$$

However, the two groups are not isomorphic since their elements do not match. Intuitively, we know that elements of order 2 are their own inverse in their generated cyclic subgroup. This is only the case for elements with cycles of 2. The only 9 elements that have an order 2 are therefore : $(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)$ and $(12)(34),(14)(24),(13)(24)$. So there are only 9 elements with order 2 in S_4. However in D_{12}, all of the elements of the form $r^n s$ are their own inverse. And there's an additional element r^6 which is its own inverse. So in total, D_{12} has 13 elements of order 2, and S_4 only has 9, so they cannot be isomorphic.

(b) Does D_{12} have a subgroup which is isomorphic to the Klein-4 group V? If so, find it and write out its group table.

Yes, the subgroup of $\{e, r^6, s, r^6 s\}$ is isomorphic to V.

	e	r^6	s	$r^6 s$
e	e	r^6	s	$r^6 s$
r^6	r^6	e	$r^6 s$	s
s	s	$r^6 s$	e	r^6
$r^6 s$	$r^6 s$	s	r^6	e

Table 1: Subgroup of D_{12}

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Table 2: Cayley table of Klein-4 group

(c) Find a group of D_{12} that is isomorphic to S_3.
The group of $\{e, s, r^4, r^8, r^4s, r^8s\}$ is isomorphic to S_3.

Let our mapping be of the generators : $\phi : s \to (12), r^4 \to (23)$. If we step through all the possible multiplications, we'll see that (12) and (23) generate 6 distinct elements, so the mapping is one-to-one and onto.
(If s is not in the term $r^n s$, then it is a trivial mapping, where $\phi(r^n s^0) = \phi(r^n)$). Stepping through the other 3 terms (r^4s, r^8s, s), we see that the mapping is consistent.

3. (a) In the group D_6 what is the subgroup L generated by $\{r^2, s\}$?

Geometrically, we can see that this is just D_3. Everything about D_6 is the same, except now all the rotations are 2x as far, meaning that there are half the rotations. This means that there are 3 equally spaced rotations of $\frac{2\pi}{3}$, which is geometrically identical to D_3 (and S_3 for that matter). The reflections don't change anything geometrically.

(b) In S_{10} what is the subgroup K generated by the two-element set $\{(18)(29), (37)(56)\}$? Is K a subgroup of A_{10}?

The elements of K are $\{(), (18)(29), (37)(56), (18)(29)(37)(56)\}$
If we try multiplying the elements, we see that both of the elements (call them a and b) squared yields the identity. Also, $a * (ab) = b$ and $b * (ab) = a$, which makes this group, element-for-element, isomorphic to the Klein-4 group, V.

It is indeed a subgroup of A_{10}, since both elements are even permutations, and even permutations are closed under composition.