Phil 140b **Spring 2016**

Assignment #3

Nikhil Unni

1. *Write out the sentences G and the sentence D for the following machine (not pictured) with input $n = 2$.*

We have the same "background information":

(1)
$$\forall u \forall v \forall w(((\text{Suv} \wedge \text{Suw}) \implies v = w) \wedge ((\text{Svu} \wedge \text{Swu}) \implies v = w))$$

(2)
$$\forall u \forall v(\text{Suv} \implies u < v) \wedge \forall u \forall v \forall w((u < v \wedge v < w) \implies u < w)$$

(3)
$$\forall u \forall v(u < v \implies u \neq v)$$

(4)
$$\forall u \forall v \forall w(((S_m\text{uv} \wedge S_m\text{uw}) \implies v = w) \wedge ((S_m\text{vu} \wedge S_m\text{wu}) \implies v = w))$$

(5)
$$\forall u \forall v \forall w(((S_m\text{uv} \implies u < v) \text{ if } m \neq 0$$

(6)
$$\forall u \forall v \forall w(((S_m\text{uv} \implies u \neq v) \text{ if } m \neq 0$$

(7)
$$\forall u \forall v \forall w((S_m\text{wu} \wedge \text{Suv}) \implies S_k\text{wv}) \text{ if } k = m + 1$$

(8)
$$\forall u \forall v \forall w((S_k\text{wv} \wedge \text{Suv}) \implies S_m\text{wu}) \text{ if } m = k - 1$$

(9)
$$p \neq q \text{ if } p \neq q$$

(10)
$$\forall v(\text{Smv} \implies v = k) \text{ where } k = m + 1$$

(11)
$$\forall u(\text{Suk} \implies u = m) \text{ where } m = k - 1$$

Now, the initial state of our machine with input $n = 2$ can be described as:

(12)
$$Q_1 0 \wedge @00 \wedge \text{M00} \wedge \text{M01} \wedge \forall x((x \neq 0 \wedge x \neq 1) \implies \neg\text{M0x})$$

And, finally, the encoding of our instructions:

(13)
$$\forall t \forall x((Q_1 t \wedge @\text{tx} \wedge M_1\text{tx}) \implies \exists u(\text{Stu} \wedge (\exists y(\text{Sxy} \wedge @\text{uy})) \wedge Q_1 u \wedge$$
$$\forall y(M_0\text{ty} \implies M_0\text{uy} \wedge M_1\text{ty} \implies M_0\text{uy})))$$

(14)
$$\forall t \forall x((Q_1 t \wedge @\text{tx} \wedge M_0\text{tx}) \implies \exists u(\text{Stu} \wedge (\exists y(S_{-1}\text{xy} \wedge @\text{uy})) \wedge Q_2 u \wedge$$
$$\forall y((M_0\text{ty} \implies M_0\text{uy} \wedge M_1\text{ty} \implies M_0\text{uy}))))$$

(15)

$$\forall t \forall x((Q_2 t \wedge @\text{tx} \wedge M_1\text{tx}) \implies \exists u(\text{Stu} \wedge (@\text{ux} \wedge M_0\text{ux}) \wedge Q_j u \wedge \forall y((y \neq x \wedge M_1\text{ty}) \implies M_1\text{uy}) \wedge \forall y((y \neq x \wedge M_0\text{ty}) \implies M_0\text{uy})))$$

Now for the parts of D:

(16)

$$\exists t \exists x(Q_2 t \wedge @\text{tx} \wedge M_0\text{tx})$$

So then G consists of the set of equations (1) to (15), and D is just equation (16).

2. *Show that*

$$\forall w \forall v(Twv \iff \exists y(Rwy \wedge Syv))$$

and

$$\forall u \forall v \forall y((Suv \wedge Syv) \implies u = y)$$

together imply

$$\forall u \forall v \forall w((Twv \wedge Suv) \implies Rwu)$$

Assume that for some u, v, w, Twv and Suv. Then, from the first formula, we know that there exists some y such that Rwy and Syv. But now we have Suv and Syv, so from the second formula, we know $y = u$. So then we know that since Rwy is true, Rwu is true as well.

3. *Show that*

$$\forall x(\neg Ax \implies \neg \exists t(Bt \wedge Rtx))$$

and

$$\neg \exists x(Cx \wedge Ax)$$

together imply

$$\neg \exists t \exists x(Bt \wedge Cx \wedge Rtx)$$

We can rephrase our assumptions as:

$$\forall x(\neg Ax \implies \forall t(\neg Bt \vee \neg R\text{tx}))$$

$$\forall x(\neg Cx \vee \neg Ax)$$

and the final implication as:

$$\forall t \forall x(\neg Bt \vee \neg Cx \vee \neg R\text{tx})$$

We know that, for some x, either $\neg Cx$ or $\neg A$. If we assume $\neg Cx$, then $\forall t \forall x(\neg Bt \vee \neg Cx \vee \neg R\text{tx})$ is trivially true, since $\neg Cx$ is true. So let's assume that $\neg A$ is true. Then, we know that $\forall t(\neg Bt \vee \neg R\text{tx})$. This makes $\forall t \forall x(\neg Bt \vee \neg Cx \vee \neg R\text{tx})$ true as well, since $\forall x(\forall t(\neg Bt \vee \neg R\text{tx}))$ is true.

4. *Verify that $G \models D$ for $n = 2$ where G and D are as defined in exercise 1. (You can appeal to facts about S and $<$ that have been established in the book, i.e. equations (1) to (11).)*

Let's start with the initial state of the machine, given by (12). From this, we know that at time 0, we're at Q_1, and the only marked squares are at $x = 0$ and $x = 1$. So then from equation (13), we know that this together implies that there is a successor time (which can be expressed as $t = 1$, thanks to our background formulae), and a successor square ($x = 1$), such that at time $t = 1$, we're at square $x = 1$, and all of the squares' markings are unchanged. From equation (13) again, we move another square to the right, meaning we're at $x = 2$ and $t = 2$, and all squares remain unchanged. Now, from (12), we know that $\neg M02$, and since we know that the markings haven't changed from $t = 0$ to $t = 2$, we know that $\neg M22$.

So at this point, we have:

$$Q_1 2 \wedge @22 \wedge \text{M20} \wedge \text{M21} \wedge \forall x(x \neq 0 \wedge x \neq 1 \implies \neg M2\text{x})$$

Then, from (14), we move to the left by 1, move to state Q_2, and all the squares remain unchanged. This means we have:

$$Q_2 3 \wedge @31 \wedge \mathrm{M30} \wedge \mathrm{M31} \wedge \forall x(x \neq 0 \wedge x \neq 1 \implies \neg M3\mathrm{x})$$

From equation (15), we know this all implies that there's a successor time $u = 4 = t + 1$ such that @41 with a blank square 1: $\neg M41$, with all other squares besides x remaining unchanged. This leaves us at:

$$Q_2 4 \wedge @41 \wedge \mathrm{M40} \wedge \neg\mathrm{M41} \wedge \forall x(x \neq 0 \wedge x \neq 1 \implies \neg M4\mathrm{x})$$

Since we have $Q_2 4 \wedge @41 \wedge \mathrm{M40} \wedge \neg\mathrm{M41}$, this makes statment (16) :

$$\exists t \exists x(Q_2 t \wedge @\mathrm{tx} \wedge M_0\mathrm{tx})$$

a true statement, with $t = 4, x = 1$ (since M_0tx is equivalent to $\neg M$tx).