Math 104	**Spring 2016**

Assignment #14

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25.7 Show $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$ to a continuous function.

Since $|\cos(nx)|$ is bounded by 1, we know that $\left|\frac{1}{n^2}\cos(nx)\right| = \frac{1}{n^2}|\cos(nx)| \leq \frac{1}{n^2}$ for all $x \in \mathbb{R}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$.

25.10 (a) Show $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0, 1)$.

We already know that $\sum x^n$ has an interval of convergence of $(-1, 1)$. And for all $x \in (-1, 1)$, we know that $\left|\frac{x^n}{1+x^n}\right| \leq |x^n|$, so it follows that $\sum \frac{x^n}{1+x^n}$ point-wise converges in $(-1, 1)$, and, trivially, in $[0, 1)$.

(b) Show that the series converges uniformly on $[0, a]$ for each a, $0 < a < 1$.

Since $a < 1$, we know that $\sum a^n$ converges (since it's a Geometric Series). We know that $\frac{x^n}{1+x^n} \leq a^n$ for all $x \in [0, a]$, so by the Weierstrass M-test, the series converges uniformly on $[0, a)$.

(c) Does the series converge uniformly on $[0, 1)$.

No. It was shown in example 5 that if a series $\sum g_n$ converges uniformly on S, then:

$$\lim_{n \to \infty} \sup\{|g_n(x)| : x \in S\} = 0$$

Looking at our series, $\sum \frac{x^n}{1+x^n}$, we know that $\frac{x^n}{1+x^n}$ is a strictly decreasing function in n, and a strictly increasing function in x. So it follows that the $\limsup$ is obtained at $n = 1, x = 1$, meaning:

$$\limsup\{\left|\frac{x^n}{1+x^n}\right| : x \in S\} = \frac{1}{2} \neq 0$$

So the series cannot converge uniformly on $[0, 1)$.

25.12 Suppose $\sum_{k=1}^{\infty} g_k$ is a series of continuous functions g_k on $[a, b]$ that converges uniformly to g on $[a, b]$. Prove:

$$\int_a^b g(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx$$

We can express $g(x)$ as the limit of partial sums :

$$g(x) = \lim_{n \to \infty} \sum_{k=1}^{n} g_k(x)$$

Then, from Theorem 25.2, we know that:

$$\int_a^b g(x)dx = \lim_{n \to \infty} \int_a^b (\sum_{k=1}^{n} g_k(x))dx$$

Since the integral of a (finite) sum is the sum of integrals:

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b g_k(x)dx$$

And since the limit of partial sums is the series:

$$= \sum_{k=1}^{\infty} \int_a^b g_k(x)dx$$

25.15 Let (f_n) be a sequence of continuous functions on $[a,b]$.

(a) Suppose that, for each x in $[a,b]$, $(f_n(x))$ is a decreasing sequence of real numbers. Prove that if $f_n \to 0$ pointwise on $[a,b]$, then $f_n \to 0$ on $[a,b]$. Hint: If not, there exists $\epsilon > 0$ and a sequence (x_n) in $[a,b]$ such that $f_n(x_n) \geq \epsilon$ for all n. Obtain a contradiction.

Following the hint, suppose not. Then there exists some $\epsilon > 0$ and a sequence (x_n) in $[a,b]$ such that $f_n(x_n) \geq \epsilon$ for all n. From Bolzano-Weierstrass, we know that there exist a convergent subsequence (x_{n_k}) of (x_n). Lets call that limit L.

Since $f_n(x)$ pointwise converges to 0, we know that $\lim_{n\to\infty} f_n(L) = 0$. This implies that there exists an N where:

$$f_{n\geq N}(L) < \epsilon$$

Since $(x_{n_k}) \to L$ and $f_N(L) < \epsilon$, there must exist some K such that:

$$f_N(x_{n_{k\geq K}}) < \epsilon$$

If we choose some $k > \max(N,K)$, and since $n_k \geq k$ (property of sequences and subsequences), we know that:

$$n_k \geq k > N$$

And since $(f_n(x))$ is a decreasing sequence, we know that $f_{n_k}(x_{n_k}) < f_N(x_{n_k})$. And since $k > K$, we know:

$$f_{n_k}(x_{n_k}) < f_N(x_{n_k}) < \epsilon$$

However, our original supposition was that $f_n \geq \epsilon$ for all n, so we have a contradiction. Thus, we know that $f_n \to 0$ on $[a,b]$.

(b) Suppose that, for each x in $[a,b]$, $(f_n(x))$ is an increasing sequence of real numbers. Prove that $f_n \to f$ pointwise on $[a,b]$ and if f is continuous on $[a,b]$, then $f_n \to f$ uniformly on $[a,b]$. This is Dini's Theorem.

Let $g_n = f - f_n$. Since $(f_n(x))$ is an increasing sequence, we know that $(g_n(x))$ must be a decreasing sequence (negative of an increasing sequence is a decreasing sequence, as we've proven in class before). We also know that $g_n \to f - f = 0$ pointwise. From part (a), we know this means that $g_n \to 0$ uniformly on $[a,b]$. And this is equivalent to saying that $f_n \to f$ uniformly on $[a,b]$, by definition.

26.2 (a) Observe $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$; see Example 1.

We can factor out x from the summation, giving us $x\sum_{n=1}^{\infty} nx^{n-1}$. As shown in Example 1, $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, so the actual summation is clearly $\frac{x}{(1-x)^2}$.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Compare with Exercise 14.13(d).

Suppose $x = \frac{1}{2}$. Then, our summation can be expressed as:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} = \frac{0.5}{0.5^2} = 2$$

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

We can repeat the same trick as part (b). The first summation is $x = \frac{1}{3}$, and the second summation is $x = -\frac{1}{3}$. The first summation is then:

$$\frac{x}{(1-x)^2} = \frac{1/3}{(2/3)^2} = \frac{3}{4}$$

And the second summation is:

$$\frac{x}{(1-x)^2} = \frac{-1/3}{(4/3)^2} = -\frac{3}{16}$$

26.4 (a) Observe $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for $x \in \mathbb{R}$, since we have $e^x = \sum_{n=1}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$.

First of all, we know that:

$$e^x = \sum_{n=1}^{\infty} \frac{1}{n!} x^n = x + \frac{1}{2!} x^2 + \cdots$$

So we know that:

$$e^x = \frac{d}{dx} e^x = 1 + \frac{1}{1!} x + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Since we have:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

We know that:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n$$

The term $(-x^2)^n$ is positive and equal to x^{2n} if n is even (since we can factor out a 2 which will negate the negative sign), and it's negative and equal to $-x^{2n}$ if n is odd. We can factor out this negative-or-not property with $(-1)^n$, which has the same property of its sign – leaving just $(-1)^n x^{2n}$. So then we have:

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

(b) Express $F(x) = \int_0^x e^{-t^2} dt$ as a power series.

We have:

$$F(x) = \int_0^x e^{-t^2} dt = \lim_{n \to \infty} \int_0^x [\sum_{k=1}^{n} \frac{(-1)^k}{k!} t^{2k}] dt$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_0^x t^{2k} dt$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{(-1)^k}{k!} [\frac{x^{2k+1} - 0^{2k+1}}{2k+1}]$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(2k+1)} x^{2k+1}$$

26.6 Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.

(a) Prove $s' = c$ and $c' = -s$.

$$s' = 1 - 3(\frac{x^2}{3!}) + 5(\frac{x^4}{5!}) + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = c$$

Proving $c' = -s$ is equivalent to proving $-(c') = s$. With this:

$$-(c') = -[-2(\frac{x}{2!}) + 4(\frac{x^3}{4!}) - \cdots] = -[-\frac{1}{1!}x + \frac{x^3}{3!} - \cdots] = x - \frac{x^3}{3!} - \cdots = s$$

(b) Prove $(s^2 + c^2)' = 0$.

First, we have:

$$\frac{d}{dx}(s(x)^2 + c(x)^2) = \frac{d}{dx}s(x)^2 + \frac{d}{dx}c(x)^2 = 2s(x)s'(x) + 2c(x)c'(x)$$

The last equality comes from the chain rule. Then, from part (a) we have:

$$= 2sc + 2c(-s) = 0$$

(c) Prove $s^2 + c^2 = 1$.

First of all, we know that $s^2 + c^2$ is a function. We also know that its derivative is 0, meaning that $s^2 + c^2$ is a constant function. Trivally plugging in 0 into $(s^2 + c^2)(x)$ gives us $0^2 + 1^2 = 1$. Since its a valid constant function, it cannot represent more than one value, and so it follows that all values of $(s^2 + c^2)(x)$ must also be 1.