Phil 140b **Spring 2016**

Assignment #2

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1. *Let $L^* = \{0,\ ',\ +,\ .,\ <\ \}$. Taking for granted the inductive definition of the terms of L^* provided in class, define the atomic formulas of L^* as follows:*
Atomic formulas: if t_1, t_2 are terms of L^* then $=(t_1, t_2)$ and $<(t_1, t_2)$ are atomic formulas.
Now define inductively the class of well formed formulas of L^* as follows:
Basis: all atomic formulas are well formed formulas;
Inductive clause: if A and B are well formed formulas, so are $(A \wedge B), \neg A$, and $\forall x A$;
External clause: nothing else is a well formed formula

a. *Show by induction on the construction of the set of well formed formulas that all well formed formulas have the same number of left and right parentheses. [You can assume in your proof that all terms of L^* have the same number of left and right parentheses]*

Base Case: all atomic formulas are of the form: $=(t_1, t_2)$ or $<(t_1, t_2)$. Since it is assumed that all terms of L^* have balanced parentheses, all atomic formulas must be balanced as well, since they just add a single left paren and right paren to the expression.

Inductive Case: Assuming that well-formed formulas A and B have balanced parentheses, then:

- $(A \wedge B)$ must have balanced parentheses, since it adds a single left and right paren to the already balanced expression (and the ampersand doesn't change anything).
- $\neg A$ must have balanced parentheses, since it doesn't add any left or right parentheses to the expression.
- $\forall x A$ must have balanced parentheses as well, since it doesn't add any left or right parentheses.

Since well-formed formulas are inductively defined this way, by induction we have shown that all possible well-formed expressions have balanced parentheses.

b. *Following the outline of the proof we did in class for terms of L^*, define a numerical measure of complexity for the well formed formulas ($f(w) = n$) and prove by induction on the natural numbers that "for all n, for all well formed formulas w, if $comp(w) = n$, then w has the same number of left and right parentheses".*

We define a "complexity" function f : well formed formulas $\rightarrow \mathbb{N}$, which just maps a well formed formula to the maximum recursive depth to atomic formulas. Concretely:

$$f(=(t_1, t_2)) = f(<(t_1, t_2)) = 0$$

$$f((A \wedge B)) = \max(f(A), f(B)) + 1$$

$$f(\neg A) = f(A) + 1$$

$$f(\forall x A) = f(A) + 1$$

Assuming all well-formed formulas are finitely long, they must have a finite complexity, and so for all $x \in$ well formed formulas, $f(x) = n \in \mathbb{N}$.

Base Case: All formulas of complexity 0 are atomic formulas. Since it's assumed that all terms in L^* have equal parentheses, we know all atomic formulas have balanced left and right parentheses, since they just add a single left and a single right. In other words, if t_1 has n of both ($\#_L(t_1) = \#_R(t_1) = n$), and t_2 has m of both ($\#_L(t_2) = \#_R(t_2) = m$), then:

$$\#_L(=(t_1,t_2)) = \#_L(<(t_1,t_2)) = n+m+1 = \#_R(=(t_1,t_2)) = \#_R(<(t_1,t_2))$$

Inductive Case: Assuming that all formulas of complexity $\leq n$ have balanced parentheses, then we can show that all formulas of complexity $n+1$ have balanced parentheses. Say we have two well-formed formulas of complexity $<= n$: A and B, where $\#_L(A) = \#_R(A) = x$, and $\#_L(B) = \#_R(B) = y$. Then:

$$\#_L((A \wedge B)) = 1 + x + y = \#_R((A \wedge B))$$

$$\#_L(\neg A) = x = \#_R(\neg A)$$

$$\#_L(\forall x A) = x = \#_R(\forall x A)$$

So by induction, all well-formed formulas of complexity $n \geq 0$ have balanced parentheses, and so all well-formed formulas must have balanced parentheses

2. *Show that:*

(a) *If the sentence E is implied by the set of sentences Δ and every sentence D in Δ is implied by the set of sentences Γ, then E is implied by Γ.*

Without loss of generation, let γ be an arbitrary interpretation of the language. By definition, we know:

$$(\forall D \in \Delta(\gamma \models D)) \implies (\gamma \models E)$$

$$\forall D \in \Delta((\forall G \in \Gamma(\gamma \models G) \implies (\gamma \models D))$$

So then we know that:

$$\forall D \in \Delta, \forall G \in \Gamma((\gamma \models G) \implies (\gamma \models D) \implies (\gamma \models E))$$

which means that:

$$(\forall G \in \Gamma(\gamma \models G)) \implies (\gamma \models E)$$

Meaning that $\Gamma \models E$.

(b)

If the sentence E is implied by the set of sentences $\Gamma \cup \Delta$ and every sentence D in Δ is implied by the set of sentences Γ, then E is implied by Γ.

For future shorthand, say that for interpretation γ and set of sentences χ, $\gamma \models \chi$ means $\forall x \in \chi(\gamma \models x)$.

Without loss of generlization, let γ be an arbitrary interpretation of the language. We know:

$$((\gamma \models \Gamma) \wedge (\gamma \models \Delta)) \implies (\gamma \models E)$$

$$\forall D \in \Delta((\gamma \models \Gamma) \implies (\gamma \models D))$$

So then we know:

$$(\gamma \models \Gamma) \implies (\gamma \models \Gamma) \wedge (\forall D \in \Delta(\gamma \models D)) \implies (\gamma \models \Gamma) \wedge (\gamma \models \Delta) \implies (\gamma \models E)$$

3. *Let L = {0, ', +, * }. Give an interpretation of L with a finite domain that makes the following sentences true:*

$$\neg\forall x\forall y(x * y = y * x)$$
$$\neg\forall x\forall y(x + y = y + x)$$

This is equivalent to the two statements:

$$\exists x\exists y(x * y \neq y * x)$$
$$\exists x\exists y(x + y \neq y + x)$$

So let's define an interpretation γ with finite set-theoretic domain $\{0, 1\}$, where $0^\gamma = 0$, and $(0')^\gamma = 1$, $(0'')^\gamma = 0$, etc. And we define our operators as:

$$*^\gamma = \{(0,0,0),(0,1,1),(1,0,0),(1,1,1)$$
$$+^\gamma = \{(0,0,1),(0,1,0),(1,0,1),(1,1,0)\}$$

Then, we have $x = 0$ and $y = 1$ as examples where $(x * y \neq y * x) \wedge (x + y \neq y + x)$.

4. *Show that the following sentences are invalid:*

(a)

$$\forall x\exists y Q(x,y) \implies \exists x\forall y Q(x,y)$$

Let's use $\mathcal{Z}$ to be the interpretation of the symbols as integers, and define:

$$Q(x,y) \iff x < y$$

Then, the left side of the implication always holds, but the right side of the implication never holds, meaning that the sentence is invalid.

(b)

$$(\forall x Q(x,x) \wedge \forall x\forall y(Q(x,y) \implies Q(y,x))) \implies \forall x\forall y\forall z(Q(x,y) \wedge Q(y,z) \implies Q(x,z))$$

Let's use $\mathcal{N}$ to be the interpretation of the symbols as natural numbers, and define:

$$Q(a,b) \iff (a = b) \vee (a + b < 100)$$

Then, we just need to show that:

$$(\forall x Q(x,x) \wedge \forall x\forall y(Q(x,y) \implies Q(y,x))) \wedge (\exists x\exists y\exists z(Q(x,y) \wedge Q(y,z) \wedge \neg Q(x,z)))$$

The left side clearly holds for our defined function (by construction), and if we define $x = 0, y = 150, z = 50$, then the right side is also true, since:

$$Q(0,150) \wedge Q(150,50) \wedge \neg Q(0,50)$$

5. Show that:

(a) *If $\Gamma \cup \{\neg(B \wedge C)\}$ is satisfiable, then either $\Gamma \cup \{\neg B\}$ is satisfiable or $\Gamma \cup \{\neg C\}$ is satisfiable.*

If $\Gamma \cup \{\neg(B \wedge C)\}$ is satisfiable, then we know that $\Gamma \cup \{\neg B \vee \neg C\}$ is satisfiable. This denotes adding either $\neg B$ or $\neg C$ to Γ. This means that either $\Gamma \cup \{\neg B\}$ is satisfiable or $\Gamma \cup \{\neg C\}$ is satisfiable (from (a) of the satisfiability principles from the textbook).

(b) *If $\Gamma \cup \{\neg\forall x B(x)\}$ is satisfiable, then for any constant c not occuring in Γ or $\forall x B(x)$, $\Gamma \cup \{\neg B(c)\}$ is satisfiable.*

If $\Gamma \cup \{\neg\forall x B(x)\}$ is satisfiable, then $\Gamma \cup \{\exists x \neg B(x)\}$ is satisfiable. This means that for any constant c not occuring in Γ or $\forall x B(x)$, $\Gamma \cup \{\neg B(c)\}$ is satisfiable (from (b) of the satisfiability principles).