Math 104	**Spring 2016**

Assignment #7

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13.4 Prove (iii) and (iv) in Discussion 13.7.

(iii) Given the union of open sets : $\bigcup\{E_i\}$, take any point in the union : $e \in \bigcup\{E_i\}$. We know that $e \in E_i$, for some i from the definition of set union. Since E_i is open, for some $r > 0, \{e_1 \in E : d(e, e_1) < r\} \subseteq E_i \subseteq \bigcup\{E_i\}$. Since any $e \in \bigcup\{E_i\}$ is interior to $\bigcup\{E_i\}$, $\bigcup\{E_i\}$ must be open.

(iv) Given the intersection of a finite number of open sets : $\bigcap_{i=1}^n\{E_i\}$, take any point $e \in \bigcap_{i=1}^n\{E_i\}$. So we have n "r" values, since e is interior to all $\{E_i\}$. If we pick $r = \min(r_1, \cdots, r_n)$, which has to be an actual real number, since n is finite, then we see that $\{e_1 \in \bigcap_{i=1}^n\{E_i\} : d(e, e_1) < r\} \in \bigcap_{i=1}^n$.

13.5 (a) Verify one of DeMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathbb{U}\} = S \setminus \bigcup\{U : U \in \mathbb{U}\}$$

For $S = [0, 1]$, $\mathbb{U} = \{[0, 0.25], [0.75, 1]\}$:

$$\bigcap\{S \setminus U : U \in \mathbb{U}\} = \{[0, 1] \setminus [0, 0.25]\} \cap \{[0, 1] \setminus [0.75, 1]\}$$

$$= (0.25, 1] \cap [0, 0.75) = (0.25, 0.75)$$

$$= [0, 1] \setminus ([0, 0.25] \cup [0.75, 1])$$

$$= S \setminus \bigcup\{U : U \in \mathbb{U}\}$$

(b) Show that the intersection of any collection of closed sets is a closed set.

A collection of closed sets is equivalent to a set of the total metric set S minus some open set U. So:

$$\bigcap\{C_i\} = \bigcap\{S \setminus U_i\}$$

From that DeMorgan Law:

$$= S \setminus \bigcup\{U_i\}$$

And the union of any collection of open sets is open from 13.4(iii). And S minus any open set is a closed set, meaning the intersection of any collection of closed sets is a closed set as well.

13.6 Prove Proposition 13.9.

(a) The set E is closed iff $E = E^-$.

If $E = E^-$: since E^- is the intersection of all closed sets containing E, from 13.5b, we know that E^- is closed. So then E must be closed.

If E is closed: note that the intersection of all closed sets containing E now contains E itself. So we know that the intersection is the smallest such set, which we know has to be E itself. So by definition, if E is the intersection of all closed sets containing E, meaning that $E = E^-$.

(b) The set E is closed iff it contains the limit of every convergent sequence of points in E.

(c) An element is in E^- iff it is the limit of some sequence of points in E.

(d) A point is in the boundary of E iff it belongs to the closure of both E and its complement.

13.10 Show that the interior of each of the following sets is the empty set.

For conciseness, I'll refer to each set as "E" in each problem.

(a) $\{\frac{1}{n} : n \in \mathbb{N}\}$

Suppose that the interior is not the empty set. Then there must be some $s_1 \in E$ s.t. for some $r > 0$, $\{s \in \mathbb{R} : |s_1 - s| < r\} \subseteq E$. We know that $s_1 = \frac{1}{n_1}$, for some n_0. We also know the closest point to s_1 is $\frac{1}{n_1+1}$. The smallest r that will contain another point in E has to be $r = \left|\frac{1}{n_1} - \frac{1}{n_1+1}\right|$, but all points $s \in \mathbb{R}$ s.t. $\frac{1}{n_1+1} < s < \frac{1}{n_1}$ are **not** in E, and we know that there are an infinite number of such points from the Denseness of $\mathbb{Q}$ theorem. Since there cannot be such a point, the interior is the empty set.

(b) $\mathbb{Q}$, the set of rational numbers

Again, let's prove that there cannot exist a point interior to $\mathbb{Q} = E$. If we pick some $q \in \mathbb{Q}$, example the interval $(q - r, q + r)$, for any $r > 0$. Since the set of all rationals in a nonempty interval is a strict subset of the interval itself, there must exist irrational numbers in the interval. Because there are elements in the neighborhood **not** in $\mathbb{Q}$, then, for any q, q cannot be interior to $\mathbb{Q}$, meaning the interior is the empty set.

(c) The Cantor set in Example 5.

13.11 Let E be a subset of $\mathbb{R}^k$. Show that E is compact if and only if every sequence in E has a subsequence converging to a point in E.

If every sequence in E has a subsequence converging to a point in E : from 13.6b, we know that E is closed. Also, we know that if E was unbounded, then it would have to contain a sequence s.t. $\lim d(s_n, 0)$ diverges, and obviously would not be a convergant sequence. So if E is closed and unbounded, by Theorem 13.12, we know E is compact.

If E is compact : by theorem 13.12, E is bounded and closed. By Theorem 13.5, we know that any sequence (s_n) in E will converge. Since E is closed, every such convergance point must be inside E.

13.12 Let (S,d) be any metric space.

(a) Show that if E is a closed subject of a compact set F, then E is also compact.

(b) Show that the finite union of compact sets in S is compact.

13.13 Let E be a compact nonempty subset of $\mathbb{R}$. Show $\sup E$ and $\inf E$ belong to E.

We know that E is closed and bounded from Theorem 13.12, and so must contain a sequence that converges to $\sup E$ and $\inf E$. And since E is closed, from 13.6b, we know that the limits of every convergant sequence is in E itself, and thus $\sup E, \inf E \in E$.

13.14 Let E be a compact nonempty subset of $\mathbb{R}^k$, and let $\delta = \sup\{d(x, y) : x, y \in E\}$. Show E contains points x_0, y_0 such that $d(x_0, y_0) = \delta$.

Again, from Theorem 13.12, we know E is closed and bounded. Since E is closed and bounded, we know that set $\{d(x, y) : x, y \in E\}$ is closed, and is bounded by δ. Since the set is closed and bounded, then δ must be an element of $\{d(x, y) : x, y \in E\}$, meaning that there must exist some x_0, y_0 s.t. $d(x_0, y_0) = \delta$.