Math 104 **Spring 2016**

Assignment #2

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2.3 Show $\sqrt{2+\sqrt{2}}$ is not a rational number.

First, we need to find an integer-cofficient polynomial such that $\sqrt{2+\sqrt{2}}$ is a solution.
Let $x = \sqrt{2+\sqrt{2}}$. Then:

$$x^4 = (2+\sqrt{2})(2+\sqrt{2}) = 4 + 4\sqrt{2} + 2$$

$$x^4 - 4x^2 = (4 + 4\sqrt{2} + 2) - 4(2+\sqrt{2}) = -2$$

$$x^4 - 4x^2 + 2 = 0$$

From Corollary 2.3, the only rational solutions can be $\pm 1, \pm 2$, and since $\sqrt{2+\sqrt{2}}$ is a solution, it cannot be rational.

2.6 Discuss why $4 - 7b^2$ is rational if b is rational.

If b is rational, it can be expressed as a fraction of two integers : $\frac{c}{d}$. Then:

$$4 - 7b^2 = 4 - 7(\frac{c}{d})^2 = 4 - 7\frac{c^2}{d^2}$$

$$= \frac{4d^2}{d^2} - \frac{7c^2}{d^2} = \frac{4d^2 - 7c^2}{d^2}$$

Since integers are closed under multiplication and subtraction, both the numerator and denominator are valid integers, making $4 - 7b^2$ a rational if b is rational.

2.7 Show the following irrational-looking expressions are actually rational numbers:

a. $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

Like the proof in 2.3, let $x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$.
Then, notice that $4 + 2\sqrt{3} = (1 + 2\sqrt{3} + \sqrt{3}^2) = (1+\sqrt{3})^2$.
So then, $x = \sqrt{(1+\sqrt{3})^2} - \sqrt{3} = 1 + \sqrt{3} - \sqrt{3} = 1$. 1 is clearly a rational number ($\frac{1}{1}$), so $x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$ is a rational number.

b. $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

Along the same lines:

$$6 + 4\sqrt{2} = 4 + 4\sqrt{2} + \sqrt{2}^2 = (2+\sqrt{2})^2$$

Then:

$$\sqrt{6+4\sqrt{2}} - \sqrt{2} = \sqrt{(2+\sqrt{2})^2} - \sqrt{2}$$

$$\sqrt{6+4\sqrt{2}}-\sqrt{2}=2+\sqrt{2}-\sqrt{2}=2$$

And since $2=\frac{2}{1}$ is a rational number, $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is a rational number as well.

2.8 Find all rational solutions of the equation $x^8-4x^5+13x^3-7x+1=0$.

Luckily, in the polyonmial, "c_0" is just 1, so by Corollary 2.3, the only possible rational solutions are ± 1. Let $f(x)$ be the evaluation homomorphism of the polynomial. Plugging in our candidates, we get:

$$f(1)=1-4+13-7+1=4\neq 0$$

$$f(-1)=1+4-13+7+1=0$$

So the only rational root to the polynomial is -1.

3.1 Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{N}$?

A4 "For each a, there is an element $-a$ such that $a+(-a)=0$."
This is not true, because for $1\in\mathbb{N}$, there is no $-1\in\mathbb{N}$ s.t. $1+(-1)=0$.

M4 "For each $a\neq 0$, there is an element a^{-1} such that $aa^{-1}=1$."
This is not true for any element in $\mathbb{N}$ except 1.

3.3 Prove (iv) and (v) of Theorem 3.1

(iv) $(-a)(-b)=ab$ for all a,b

First, we have to prove that $a+c=b+c$ implies $a=b$ (which is also (i) of 3.1). But from A4, we know that $a+c+(-c)=b+c+(-c)$ is equivalent to $a+0=b+0$. And with A3, we finally get $a=b$.

Next, we can easily prove that $-(-a)=a$. If there's an element $-a$ s.t. $a+(-a)=0(A4)$, we can just label $-a$ as b. Now there has to be a $-b$ s.t. $b+(-b)=0$. From the commutativity of addition from A2, we know that $(-b)+b=0=a+(-a)$. Since $b=-a$, from our previous proof, we know that $-b=a$, which shows that $-(-a)=a$.

Next, we can show that there exists an element -1 such that $-1*a=-a$. From DL and A4, we know that $0=a(1+(-1))=a*1+a*(-1)$. From M3, we know that $a*1=a$, and so we have $a+a*(-1)=0$. And from our first proof along with A4, we know that $a*(-1)=-a$. (And the commutative version $(-1)*a=-a$, because of M2.)

Next, we can show that there exists an element 0 such that $a*0=0$. From A4, this is equivalent to $a(b+(-b))=ab+a(-b)$. From our last proof, plus commutativity and associativity, we know this is equivalent to:

$$ab+a(-1*b)=ab+-1(ab)=ab+(-ab)$$

From M4, we know that $ab+(-ab)=0$. Hence $0a=0$.

Finally, we can prove the original question.

$$(-a)(-b)=(-1*a)(-1*b)$$

From M2, we can re-associate the multiplications:

$$=((-1*a)*-1)(b)$$

And since we proved that $-1 * a = -a$, and $-(-a) = a$, with commutativity of multiplication, we know that this is equivalent to:

$$= -(-a)(b) = ab$$

(v) $ac = bc$ and $c \neq 0$ imply $a = b$.

Since $c \neq 0$, there must exist a c^{-1} s.t. $cc^{-1} = 1$ from M4. This is equivalent to:

$$(ac)c^{-1} = (bc)c^{-1}$$

From M1 and M4, we can reassociate, and eliminate c:

$$a(cc^{-1}) = b(cc^{-1})$$

$$a * 1 = b * 1$$

Finally, from M3:

$$a = b$$

3.4 Prove (v) and (vii) of Theorem 3.2

(v) $0 < 1$

Suppose that $1 \leq 0$. For some $0 < a$, from O5, we know that $1a \leq 0a$, and from our previous proofs, we know this is equivalent to $a \leq 0$.
But from the definition of a, we know this is not true. So the statement $1 \leq 0$ cannot be true. From O1, this means that $0 \leq 1$.
But we know that $0 \neq 1$ (which we can derive from the fact that $0a = 0$ and $1a = a$, which are not equal for all values of a), so then we know that $0 < 1$.

(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

First, we show that a^{-1} and b^{-1} are greater than 0. If a and b are greater than 0, and from M4 we know that $aa^{-1} = 1$ and $bb^{-1} = 1$. If either a^{-1} or b^{-1} were negative, they could be represented as $-x$ and $-y$ respectively, for some $x > 0, y > 0$. But since a and b are positive, and x and y are positive, there's no way that $a(-1 * x)$ or $b(-1 * y)$ could be 1, a positive number. Thus, they must be nonnegative numbers. But they cannot be 0 either, since that goes against M4, so they must be positive. So $a^{-1} > 0$ and $b^{-1} > 0$.

Next, we show that $(ab)^{-1} = a^{-1}b^{-1}$. By definition, $(ab)(ab)^{-1} = 1$. But we also know that:

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a = a * 1 * a^{-1} = aa^{-1} = 1$$

. Since $(ab)(ab)^{-1} = (ab)(a^{-1}b^{-1})$, we know that $(ab)^{-1} = a^{-1}b^{-1}$ (by multiplying by $(ab)^{-1}$ on both sides).
Next, we show that $b^{-1} < a^{-1}$. This is more straightforward:

$$a < b$$

From our previous proof, we know that since $ab > 0$, $(ab)^{-1} > 0$, and with O5 we know that:

$$a(ab)^{-1} < b(ab)^{-1}$$

$$(aa^{-1})b^{-1} < (bb^{-1})a^{-1}$$

$$b^{-1} < a^{-1}$$

Since $0 < b^{-1}$ and $b^{-1} < a^{-1}$, by transitivity, $0 < b^{-1} < a^{-1}$.

3.5 (a) Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

If $|b| \leq a$: then $a \geq 0$, because $|x| \geq 0, x \in \mathbb{R}$. If b is positive, then $b \leq a$ implying $|b| \leq a$ is tautological, since $|x| = x$ for some $0 \leq x \in \mathbb{R}$. Conversely, if b is negative, then $b \leq |b|$. By transitivity, we know that $b \leq a$.

Similarly, if $b < 0$, then $-b \leq a$. If we add b and $-a$ to both sides, we get $-b + b + (-a) \leq a + (-a) + b$, and then $-a \leq b$.

Thus, if $|b| \leq a$, then $-a \leq b \leq a$.

Conversely, if $-a \leq b \leq a$, we can easily show $|b| \leq a$. Suppose b is positive, then, tautologically, $|b| \leq a$. Otherwise, if b is negative, b cannot be less than $-a$, since $-a \leq b$. Since the distance to b is less than the distance to $-a$ as well, $|b| \leq a$.

(b) Prove $|a| - |b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

If both a and b are positive, then the statement is either $a - b \leq a - b$ if $a - b \geq 0$, which is true since $a-b = a-b$. If $a-b$ is negative, then $a-b \leq 0 \leq |a-b|$, so by transitivity, that's true as well.

If both a and b are negative, the LHS becomes $b - a$, and the RHS becomes $|b - a|$. And by symmetry, we have the same scenario as when a and b were both positive, so we know that the statement is true.

If the signs of a and b differ, we see that we can just rearrange a and b, and the symmetry argument still holds.

3.8 Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Suppose the contrapositive, that $a > b$. However, for whichever value of a we pick, we can always pick a b_1 that is smaller a. For an arbitrarily small δ, let $b + \delta = a$. We can always find a $b_1 > b$ s.t. $b_1 = b + frac\delta 2$. This would mean $b_1 - a = -\frac{\delta}{2} < 0$, and so b_1 cannot be greater than a, thus disproving the contrapositive.

If $a > b$ cannot be true, then $a \leq b$.