Math 104	**Spring 2016**

Assignment #3

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4.3 For each set in Exercise 4.1, give its supremum if it has none. Otherwise write "NO sup."

(a) $[0, 1]$
1

(b) $(0, 1)$
1

(c) $\{2, 7\}$
7

(d) $\{\pi, e\}$
π

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$
$\frac{1}{1} = 1$

(f) $\{0\}$
0

(g) $[0, 1] \cup [2, 3]$
3

(h) $\cup_{n=1}^{\infty}[2n, 2n + 1]$
NO sup

(i) $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$
Since the bound approaches $(0, 1)$, the supremum is **1**

(j) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}$
1

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}$
NO sup

(l) $\{r \in \mathbb{Q} : r < 2\}$
2

(m) $\{r \in \mathbb{Q} : r^2 < 4\}$
2

(n) $\{r \in \mathbb{Q} : r^2 < 2\}$
$\sqrt{2}$

(o) $\{x \in \mathbb{R} : x < 0\}$
0

(p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$
10

(q) $\{0, 1, 2, 4, 8, 16\}$
16

(r) $\cap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})$
1

(s) $\{\frac{1}{n} : n \in \mathbb{N}$ and n is prime$\}$
$\frac{1}{2}$

(t) $\{x \in \mathbb{R} : x^3 < 8\}$
2

(u) $\{x^2 : x \in mathdsR\}$
NO sup

(v) $\{cos(\frac{n\pi}{3}) : n \in mathdsN\}$
1

(w) $\{sin(\frac{n\pi}{3}) : n \in mathdsN\}$
$\frac{\sqrt{3}}{2}$

4.7 Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Prove if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

* Let there be some element $x \in S$. Since $S \subseteq T$, $x \in T$. Because x is a member of T, $x \geq \inf T$, for all elements in S, making it a lower bound on S. And because $\inf S$ is the largest possible lower bound for S, $\inf T$ cannot be larger, meaning $\inf T \leq \inf S$.
* Let there be some element $x \in S$. Since $S \subseteq T$, $x \in T$. Because x is a member of T, $x \leq \sup T$, for all elements in S, making it an upper bound on S. And because $\sup S$ is the lowest possible upper bound for S, $\sup T$ cannot be smaller, meaning $\sup S \leq \sup T$.
* Let there be some element $x \in S$. Then we know that $\inf S \leq x \leq \sup S$ by the definition of infimum and supremum. By transitivity, $\inf S \leq \sup S$.

  Having shown all 3 inequalities, by the transitive law we know that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(b) Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$

Let there be some element $x \in S \cup T$. By definition we know that x must be a member of either S or T (or both). It follows that either $x \leq \sup S$ or $x \leq \sup T$, or both. Furthermore, we know that $\max\{\sup S, \sup T\}$ is greater than or equal to every element in $S \cup T$, precisely because x must belong to either S or T. But more than that, we know that the supremum of the intersection can't be a number smaller than $\max\{\sup S, \sup T\}$ because if there was a number smaller that was still greater than every element in S and T, then it must've been larger than every element in the set with the larger supremum, making **it** the supremum, so it cannot be larger than the suprmum itself.
Since there can't be another element smaller than max $\{\sup S, \sup T\}$ that is still bigger than every element in $S \cup T$, we know that max $\{\sup S, \sup T\}$ must be the supremum of $S \cup T$.

4.10 Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

By the Archimedean Property, we know that since $a > 0$ and $1 > 0$, there must be some $n_1 \in \mathbb{N}$ s.t. $n_1 a > 1$, and therefore $a > \frac{1}{n_1}$. Similarly, there must be some $n_2 \in \mathbb{N}$ s.t. $n_2 * 1 > a$. We know for sure that $a < \max\{n_1, n_2\}$. Let $n_{\max} = \max\{n_1, n_2\}$. We also know that $\frac{1}{n_{\max}} < \frac{1}{n_{\min}}$ and $\frac{1}{n_{\max}} = \frac{1}{n_{\max}}$. Thus, $\frac{1}{n_{\max}} \leq \frac{1}{n_1}$. By the same reasoning, we know that $n_{\max} \geq n_2$.

Finally, this means that $\frac{1}{n_{\max}} \leq \frac{1}{n_1} < a < n_2 \leq n_{\max}$, and therefore there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$ for any $a \in \mathbb{R}$.

4.11 Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $\mathbb{Q}$ 4.7 to show there are infinitely many rationals between a and b.

By theorem 4.7, we can find a rational number r_0 inbetween a and b s.t. $a < r_0 < b$. This means we can find a **distinct** number $r_0 < b$. Now, if we recursively "enter" the problem again. We have r as

a, and b as b. And so repeatedly, we'll be able to find a new distinct rational number r_n greater than r_{n-1} and less than b. Since this process can never end, n will approach infinity, producing an infinite amount of rational numbers.

4.15 Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

Suppose the opposite. That if $a \leq b + \frac{1}{n} : \forall n \in \mathbb{N}$, $a > b$. $a > b$, so we know that $a - b > 0$. And since $1 > 0$, by the Archimedean Property, there is some $n \in \mathbb{N}$ s.t. $n(a - b) > 1$, or $a > b + \frac{1}{n}$. But this contradicts our original assumption and so, by proof by contradiction, the opposite is true.

4.16 Show $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

For the rest of the problem, I'll refer to the set as S_a, for some arbitrary $a \in \mathbb{R}$. By definition of the set, we know that a is larger than every element $x \in S_a$, so it is a valid upper bound. Now, suppose that there is some $b \in \mathbb{R}$ s.t. $b < a$ and $b \geq x, \forall x \in S_a$, that is, that there's a smaller upper bound than a.

However, by theorem 4.7, there exists a rational number inbetween b and a, meaning that b is not a valid upper bound on S_a for all $b < a$. So for any $b < a$ we can find a rational number inbetween b and a. Since there cannot exist a smaller upper bound on S_a, we know that a is the supremum.

7.1 Write out the first five terms of the following sequences.

(a) $s_n = \frac{1}{3n+1}$

$$\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{17}$$

(b) $b_n = \frac{3n+1}{4n-1}$

$$\frac{4}{3}, 1, \frac{10}{11}, \frac{13}{15}, \frac{17}{19}$$

(c) $c_n = \frac{n}{3^n}$

$$\frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \frac{5}{243}$$

(d) $sin(\frac{n\pi}{4})$

$$sin(\frac{\pi}{4}), sin(\frac{2\pi}{4}), sin(\frac{3\pi}{4}), sin(\frac{4\pi}{4}), sin(\frac{5\pi}{4})$$
$$= \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}$$

7.2 For each sequence in 7.1, determine whether it converges. If it converges, give its limit.

(a) $s_n = \frac{1}{3n+1}$

0

(b) $b_n = \frac{3n+1}{4n-1}$

$\frac{3}{4}$

(c) $c_n = \frac{n}{3^n}$

0

(d) $sin(\frac{n\pi}{4})$

Does not converge.

7.4 Give examples of

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

The sequence $x_n = \frac{\pi}{n}$ has a limit of 0, a rational number. And we can easily prove that $\frac{\pi}{n}$ is irrational, for all natural numbers $n \in \mathbb{N}$. Assume that it's rational. Then, we can say:

$$\frac{\pi}{n} = \frac{p}{q}$$

But then we can represent π as a fraction of integers:

$$\pi = \frac{pn}{q}$$

And this is impossible since π is irrational, so $\frac{\pi}{n}$ must be irrational.

(b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

The decimal expansion of any irrational number approaches the irrational number, while having rational number terms of the series. For example:

$$s_\pi = 3, 3.1, 3.14, 3.141, \cdots$$