Math 104	**Spring 2016**

Assignment #11

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19.1 Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.

(a) $f(x) = x^{17}\sin(x) - e^x\cos(3x)$ on $[0, \pi]$

Uniformly continuous. Since the function is continuous on the closed interval, by Theorem 19.2, it's uniformly continuous on the closed interval.

(b) $f(x) = x^3$ on $[0, 1]$

Uniformly continuous. Again, since x^3 is continuous on $[0, 1]$, it is uniformly continuous on the interval.

(c) $f(x) = x^3$ on $(0, 1)$
Uniformly continuous. Since it can be extended to $\tilde{f} = x^3$ which is continuous on $[0, 3]$, by Theorem 19.5, f is uniformly continuous.

(d) $f(x) = x^3$ on $\mathbb{R}$
Not uniformly continuous. Say that $\epsilon = 1$. Then for all $\delta > 0$, we want to show that $|x - y| < \delta$ but $\left|x^3 - y^3\right| \geq \epsilon$. Say that we have a sequence of δ like $\delta_n = \frac{1}{n}$, and a sequence of x and y values like $x_n = n$, and $y_n = n + \frac{1}{n+1}$. Then, for any $n \in \mathbb{N}$, we know that $|x_n - y_n| < \delta_n$. However, $\left|x_n^3 - y_n^3\right| = \left|n^3 - (n^3 + \frac{3n^2}{n+1} + \frac{3n}{(n+1)^2} + \frac{1}{(n+1)^3})\right|$, which clearly exceeds 1, for all $n \in \mathbb{N}$. So f cannot be uniformly continuous on $\mathbb{R}$.

(e) $f(x) = \frac{1}{x^3}$ on $(0, 1]$
Take the Cauchy sequence $s_n = \frac{1}{n}$. Clearly, it's in the domain $(0, 1]$. However, $(f(s_n)) = n^3$ is not a Cauchy sequence. And so by Theorem 19.4, f **cannot be uniformly continuous** on $(0, 1]$.

(f) $f(x) = \sin\frac{1}{x^3}$ on $(0, 1]$

Again, take the Cauchy sequence $s_n = \frac{1}{n}$. Since $(f(s_n)) = \sin(n^3)$ doesn't have a limit, it cannot be a Cauchy sequence. And so by Theorem 19.4, f **is not uniformly continuous** on $(0, 1]$.

(g) $f(x) = x^2\sin\frac{1}{x}$ on $(0, 1]$
Take $\tilde{f}(x) = x^2\sin\frac{1}{x}$ on $(0, 1]$, and $\tilde{f}(0) = 0$. We just need to show that $\tilde{f}$ is continuous at 0, since it is continuous in $(0, 1]$.

Let $\epsilon = 0$. We want to show that there exists some δ such that $|x - 0| < \delta \implies \left|x^2\sin\frac{1}{x}\right| < \epsilon$. Since sin is bound by $[0, 1]$, if $x^2 < \epsilon$, trivially, $x^2\sin(\frac{1}{x}) < \epsilon$. So if we let $\delta = \sqrt{\epsilon}$, then $x < \delta \implies x^2\left|\sin\frac{1}{x}\right| < \epsilon$.

Since $\tilde{f}$ is continuous on all $[0, 1]$, by Theorem 19.5, f **is uniformly continuous.**

19.2 Prove each of the following functions is uniformly continuous on the indicated set by directly veryfing the $\epsilon - \delta$ property in Definition 19.1

(a) $f(x) = 3x + 11$ on $\mathbb{R}$.

Let $\epsilon > 0$. We want to find a δ such that $|x - y| < \delta \implies |3x + 11 - 3y - 11| < \epsilon$. If we let $\delta = \frac{1}{3}\epsilon$ note that:

$$|x - y| < \delta \equiv |x - y| < \frac{1}{3}\epsilon$$

Then:

$$|x-y| < \frac{1}{3}\epsilon \implies 3|x-y| < \epsilon \implies |3x-3y| < \epsilon$$

(b) $f(x) = x^2$ on $[0,3]$

Let $\epsilon > 0$. We want to find a δ such that $|x-y| < \delta \implies \left|x^2-y^2\right| < \epsilon$. Note that since our domain is $[0,3]$, $|x+y|$ is upper-bounded by $3+3=6$. So if we let $\delta = \frac{1}{6}\epsilon$:

$$|x-y| < \frac{1}{6}\epsilon \implies |x-y||x+y| < \epsilon \implies \left|x^2-y^2\right| < \epsilon$$

(c) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$

Let $\epsilon > 0$. We want to find a δ such that $|x-y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon$. We know that:

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \left|\frac{1}{y}\right|\left|\frac{1}{x}\right||x-y|$$

And since $x, y \geq \frac{1}{2}$, we know $0 < \frac{1}{x}, \frac{1}{y} \leq 2$. So if we let $\delta = \frac{\epsilon}{4}$:

$$|x-y| < \frac{\epsilon}{4} \implies \left|\frac{1}{y}\right|\left|\frac{1}{x}\right||x-y| < 2*2*\frac{\epsilon}{4} \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon$$

19.3 Repeat Exercise 19.2 for the following:

(a) $f(x) = \frac{x}{x+1}$ on $[0,2]$

Let $\epsilon > 0$. We want to find a δ such that $|x-y| < \delta \implies \left|\frac{x}{x+1} - \frac{y}{y+1}\right| < \epsilon$. We know that:

$$\left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \left|\frac{xy+x-xy-y}{(x+1)(y+1)}\right| = |x-y|\left|\frac{1}{x+1}\right|\left|\frac{1}{y+1}\right|$$

And since $0 \leq x, y \leq 2$, we know $\frac{1}{3} < \frac{1}{x}, \frac{1}{y} \leq 1$. So if we let $\delta = \epsilon$:

$$|x-y| < \epsilon \implies |x-y|\left|\frac{1}{x+1}\right|\left|\frac{1}{y+1}\right| < \epsilon*1*1 \implies \left|\frac{x}{x+1} - \frac{y}{y+1}\right| < \epsilon$$

(b) $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$

Let $\epsilon > 0$. We want to find a δ such that $|x-y| < \delta \implies \left|\frac{5x}{2x-1} - \frac{5y}{2y-1}\right| < \epsilon$. We know that:

$$\left|\frac{5x}{2x-1} - \frac{5y}{2y-1}\right| = \left|\frac{10xy-5x-10xy+5y}{(2x-1)(2y-1)}\right| = 5|x-y|\left|\frac{1}{2x-1}\right|\left|\frac{1}{2y-1}\right|$$

And since $1 \leq x, y$, we know $\frac{1}{2x-1}, \frac{1}{2y-1} \leq 1$. So if we let $\delta = \frac{\epsilon}{5}$:

$$|x-y| < \frac{\epsilon}{5} \implies 5|x-y|\left|\frac{1}{2x-1}\right|\left|\frac{1}{2y-1}\right| < 5*\frac{\epsilon}{5}*1*1 \implies \left|\frac{5x}{2x-1} - \frac{5y}{2y-1}\right| < \epsilon$$

19.4 (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. Hint : Assume not. Use Theorems 11.5 and 19.4.

Assume not. Then there must exist a uniformly continuous function f on a bounded set S that is an unbounded function on S. Define some sequence $(x_n) \in S$, such that for some $n \in \mathbb{N}$, $f(x_n) > n$ (since it is an unbounded function). From Bolzano-Weierstrass, there must be a Cauchy sequence (x_{k_n}), since x_n is bounded (since S is bounded). And since (x_{k_n}) is Cauchy, then $f(x_{k_n})$ is Cauchy as well from Theorem 19.4, and therefore bounded. However, by definition, $f(x_{k_n}) > k_n$ for all $n \in \mathbb{N}$, meaning we have a contradiction. Therefore, if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Since $\frac{1}{x^2}$ is not bounded on the bounded set, $(0, 1)$, it is unot uniformly continuous by 19.4(a).

19.6 (a) Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show f' is unbounded on $(0, 1]$ but f is nevertheless uniformly continuous on $(0, 1]$. Compare with Theorem 19.6.

$f'(x) = \frac{1}{x}$ is unbounded on $(0, 1]$. Let $\epsilon > 0$. We want to find a δ such that $|x - y| < \delta \implies \left|\sqrt{x} - \sqrt{y}\right| < \epsilon$. Note that for $0 < x, y \leq 1, |\sqrt{x} - sqrty| \leq \sqrt{|x - y|}$. So if we let $\delta = \epsilon^2$:

$$|x - y| < \epsilon^2 \implies \left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|} < \epsilon$$

So f is uniformly continuous on $(0, 1]$.

This is in contrast to Theorem 19.6 since we have an unbounded f' on the modified interval without endpoints. However, since the theorem is not a "if and only if" implication, we haven't disproven anything.

(b) Show f is uniformly continuous on $[1, \infty)$.

Proven in part (a).

19.7 (a) Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some k, then f is uniformly continuous on $[0, \infty)$.

Let $\epsilon > 0$. Since f is continuous on $[0, \infty]$, for all $y \in [0, \infty]$, there must exist a δ_C such that $|x - y| < \delta_C \implies |f(x) - f(y)| < \epsilon$.

Using the same ϵ, since f is uniformly continuous on $[k, \infty]$, there must exist a δ_U such that $|x - y| < \delta_U \implies |f(x) - f(y)| < \epsilon$.

So if we select $\delta = \min\{\delta_C, \delta_U, k\}$, then: $x, y \in [0, \infty), |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$, since we are always in the range of one of the original domains.

(b) Use (a) and Exercise 19.6(b) to prove $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Since we showed that $\sqrt{x}$ is uniformly continuous on $[1, \infty)$, by 19.7(a), if we let $k = 1$, then we know $\sqrt{x}$ is uniformly continuous on $[0, \infty)$, since $\sqrt{x}$ is continuous on $[0, \infty)$.

19.9 Let $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$, and $f(0) = 0$.

(a) Observe f is continuous on $\mathbb{R}$. See Exercises 17.3(f) and 17.9(c).

f is clearly continuous for all $\mathbb{R} \setminus \{0\}$. We can show its continuous at 0 with a $\delta - \epsilon$ proof. Let $\epsilon > 0$. Then there must exist a δ such that:

$$|x| < \delta \implies \left|x\sin(\frac{1}{x})\right| < \epsilon$$

Since sin is bounded by $[-1, 1]$, if we let $\delta = \epsilon$:

$$|x| < \epsilon \implies |x|\left|\sin(\frac{1}{x})\right| < \epsilon$$

And so f is continuous at 0.

(b) Why is f uniformly continuous on any bounded subset of $\mathbb{R}$?

Because f is continuous on all closed intervals $[a, b]$, it is uniformly continuous on all $[a, b]$. Because all bounded subsets of $\mathbb{R}$ are also subsets of closed intervals, we know that f must be uniformly continuous on any bounded subset of $\mathbb{R}$.

(c) Is f uniformly continuous on $\mathbb{R}$?

Let's first show that f is uniformly continuous on $[10, \infty)$. We know that $|\sin x - \sin y| \leq |x - y|$. We also know:

$$\left|x\sin(\frac{1}{x}) - y\sin(\frac{1}{y})\right| = \left|(x-y)\sin(\frac{1}{x}) + y(\sin(\frac{1}{y}) - \sin(\frac{1}{x}))\right| \leq |x-y|\left|\sin(\frac{1}{x})\right| + |y|\frac{|x-y|}{|x|\,|y|} \leq \frac{11}{10}|x-y|$$

So letting $\delta = \frac{10}{11}\epsilon$, we find that the implication holds. Since we've shown that f is continuous on all $\mathbb{R}$, and its uniformly continuous on $[10, \infty)$, by 19.7(a) (with a slight addition to account for $(-\infty, k]$), we know that it is uniformly continuous on all $\mathbb{R}$.

19.10 Repeat Exercise 19.9 for the function g where $g(x) = x^2\sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$.

(a) Again, g is clearly continuous for all $x \neq 0$. We can prove g is continuous at 0 with a $\delta - \epsilon$ proof. So for any $\epsilon > 0$, if we let $\delta = \sqrt{\epsilon}$:

$$|x| < \sqrt{\epsilon} \implies \left|x^2\right|\left|\sin(\frac{1}{x})\right| < \epsilon$$

So g is continuous for all $x \in \mathbb{R}$.

(b) Same explanation as 19.9(b)

(c) Unlike 19.9, $x^2\sin(\frac{1}{x})$ is **not uniformly continuous** on $\mathbb{R}$.