Math 104	**Spring 2016**

Assignment #10

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18.1 Let f be as in Theorem 18.1. Show that if $-f$ assumes its maximum at $x_0 \in [a, b]$, then f assumes its minimum at x_0.

Trivially, we know that if f is bounded, then $-f$ is bounded, since $|-f(x)| = |f(x)|$.

Say $M_0 = \sup\{f(x) : x \in [a, b]$ is the maximum value of the function. Then we know that $-M_0 = \inf\{-f(x) : x \in [a, b]$, which we proved from the Practice Midterm.

For each $n \in \mathbb{N}$ there exists $y_n \in [a, b]$ such that $M < f(y_n) < M + \frac{1}{n}$, and so $\lim f(y_n) = M$. Since (y_n) must contain a subsequence (y_{n_k}) converging to some $y_0 \in [a, b]$ by the Bolzano-Weierstrass theorem. We also know that $\lim_{k\to\infty} f(y_{n_k}) = \lim_{n\to\infty} f(y_n) = M$, and so $f(y_0) = M$. Since it was the infimum, we know the minimum is at y_0.

18.2 Reread the proof of Theorem 18.1 with $[a, b]$ replaced by (a, b). Where does it break down? Discuss.

The proof depends on the fact that an infinitely many s_n in $[a, b]$ means that $\lim s_n$ is in $[a, b]$. However, this is not true for (a, b). For example, every $s_n = \frac{1}{n+1} \in (0, 1)$. However, $\lim s_n = 0$, which is not in $(0, 1)$. This was needed to show that f is bounded. This property is explored in the next question, 18.4. Since there exists an unbounded continuous function on (a, b), we don't know if our function is bounded or not, and that a maximum or minimum even exists.

18.4 Let $S \subseteq \mathbb{R}$ and suppose that there exists a sequence (x_n) in S converging to a number $x_0 \subsetneq S$. Show there exists an unbounded continuous function on S.

We can disprove by example that all functions over non-closed sets are bounded. Take $S = (0, +\infty)$. Then the sequence $s_n = \frac{1}{n}$ is in the sequence, but the limit, 0, is not in the sequence. And the function $f(x) = \frac{1}{x}$ is in S, but is clearly unbounded (as we approach 0 from either side).

Since not all functions over non-closed sets are bounded, there must exist a function over a non-closed set that is unbounded.

18.5 (a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

Since $f(a) \geq g(a)$, $f(a) - g(a) \geq 0$. Similarly, $f(b) - g(b) \leq 0$. Since continuous functions are closed under addition and subtraction, $f - g$ is a continuous function as well. So then, from the Intermediate Value Theorem, there exists an $a < x < b$ such that $f(x) - g(x) = 0$, since it is inbetween the two values $f(b) - g(b)$ and $f(a) - g(a)$. So then at x, $f(x) = g(x)$.

(b) Show Example 1 can be viewed as a special case of part (a).

Example 1 is the same as part (a), except with $g(x) = x$, and with $a = 0$, and $b = 1$.

18.7 Prove $xe^x = 2$ for some $x \in (0, 1)$.

Call the function $f(x) = xe^x$. Then, $f(0) = 0$, and $f(1) = e^1 > 2.7$. Since $f(x)$ is continuous, by the Intermediate Value Theorem, there must be an $0 < x < 1$ such that $f(x) = 2$, since $0 < 2 < e$.

18.8 Suppose f is a real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove there exists x between a and b such that $f(x) = 0$.

Since $f(a)f(b) < 0$, either $f(a)$ is negative and $f(b)$ is positive, or vice versa. This is because the multiplication of two positive numbers is a positive number, and the multiplication of two negative numbers is a positive number. If either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$, then by the Intermediate Value Theorem, there exists an $a < x < b$ such that $f(x) = 0$. This is because either $f(a) < f(x) < f(b)$ or $f(b) < f(x) < f(a)$. Note that neither $f(a)$ nor $f(b)$ can be 0, since this would mean the product of the two would be 0.

18.9 Prove that a polynomial function f of odd degree has at least one real root. Hint: It may help to consider first the case of a cubic, i.e., $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_3 \neq 0$.

Let our polynomial be $f(x) = a_0 + a_1x + \cdots + a_{2n+1}x^{2n+1}$, for some $n \in (\mathbb{N} \cup 0)$. Since a_{2n+1} is nonzero, it is either positive or negative. In the limit, we know that the x^{2n+1} term dominates, and so: if $a_{2n+1} > 0$, then $\lim_{x \to +\infty} f(x) = +\infty$, and if $a_{2n+1} < 0$, then $\lim_{x \to +\infty} f(x) = -\infty$. Additionally, since we have an odd degree polynomial : if $a_{2n+1} > 0$, then $\lim_{x \to -\infty} f(x) = -\infty$, and if $a_{2n+1} < 0$, then $\lim_{x \to -\infty} f(x) = +\infty$.

If the limit is $+\infty$, from the definition, there must be some number N such that for all $n > N$, $f(x_n) > 0$, for all real sequences of x values. Similarly, if the limit is $-\infty$, there must be a number N, such that for all $n > N$, $f(x_n) < 0$. (This is further discussed in section 20, and we can treat the sequences as $\mathbb{N}$ or $-\mathbb{N}$ respectively, since all sequences in the domain of reals should diverge.)

Since our function diverges in the limit to $+\infty$ one one end, and $-\infty$ on the other, there must exist a value in the function that is less than 0, and there must exist a value in the function that is greater than 0. Since this is the case, from the Intermediate Value Theorem, there must be a value in the function that is equal to 0, meaning that at least one real root must exist.

18.10 Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist x, y in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. Hint: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

Looking at $g(x) = f(x + 1) - f(x)$, notice that if $g(x) = 0$, then we've satisfied the condition that $|x + 1 - x| = 1$, and $f(x) = f(x + 1)$. So if there's an x such that $0 \leq x \leq 1$ and $g(x) = 0$, then we've proved the theorem for $x = x, y = x + 1$.

$g(0) = f(1) - f(0)$, and $g(1) = f(2) - f(1) = f(0) - f(1)$. So then we know that $g(0) = -g(1)$. If either $g(0)$ or $g(1)$ are 0 (if one is 0, the other clearly is 0 as well), then we've proven the theorem. So let's suppose that neither are 0. Then one must be a positive number, and the other must be a negative number. Then, by the Intermediate Value Theorem, there must be an x such that $0 < x < 1$ and $g(x) = 0$, since 0 is inbetween all pairs of positive/negative numbers.

Since we've shown the existance of an $0 \leq x \leq 1$ such that $g(x) = 0$, we've proven the theorem.