Phil 140b **Spring 2016**

Assignment #3

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1. *Show that the function $f(x)$ defined by:*

$$\begin{cases} x^2 & x \text{ is even} \\ x+1 & x \text{ is odd} \end{cases}$$

is primitive recursive.

First, let's define the (characteristic function of a) relation $R(x)$ to see if a number is odd. We can define it as:

$$R(x') = 1 - R(x)$$
$$R(z(x)) = 0$$

So $R(x)$ is primitive recursive.

Then, we can simply define an $f(x)$ that is primitive recursive:

$$f(x) = R(x)(x+1) + (1 - R(x))(x * x)$$

Since f was constructed with only primitive recursive functions, it is also primitive recursive.

2. *Let $f(x_1, \cdots, x_n, y)$ be a function. Define $\sum_{y<z} f(x_1, \cdots, x_n, y)$ to be $f(x_1, \cdots, x_n, 0) + \cdots + f(x_1, \cdots, x_n, z-1)$ if $z \neq 0$ and 0 if $z = 0$. Morover, define $\Pi_{y<z} f(x_1, \cdots, x_n, y)$ to be equal to $f(x_1, \cdots, x_n, 0) \cdots f(x_1, \cdots, x_n, z-1)$ if $z \neq 0$ and equals 1 if $z = 0$.*
The class of elementary functions is the smallest class which contains $x + y$, xy, $|x - y|$, $id_i^n(x_1, \cdots, x_n)$, x/y, and is closed under composition, bounded sums, and bounded products.
Show that the following functions are elementary:

(i) $z(x)$

$$z(x) = |x - x|$$

(ii) $s(x)$

$$one(x) = \Pi_{y<0} z(x)$$
$$s(x) = x + one(x)$$

(iii) $sg(x)$

$$sg(0) = 0$$
$$sg(s(y)) = 1$$

(iv) $sg^*(x)$

$$sg^*(x) = |one(x) - sg(x)|$$

(v) $C_k^n(x_1, \cdots, x_n) = k$

$$C_k^n(x_1, \cdots, x_n) = s(C_{|k-1|}^n(x_1, \cdots, x_n))$$
$$C_0^n(x_1, \cdots, x_n) = 0$$

(vi) *$pred(x)$*

$$pred(x) = sg(x)\,|x-1|$$

3. *$R(x_1, \cdots, x_n)$ is elementary iff its characteristic function is elementary. Let $R_1(x_1, \cdots, x_n)$ and $R_2(x_1, \cdots, x_n)$ be elementary.*

(a) *Construct the characteristic functions for $\neg R_1(x_1, \cdots, x_n)$ and $R_1(x_1, \cdots, x_n) \wedge R_2(x_1, \cdots, x_n)$.*

$$C_{\neg R_1}(x_1, \cdots, x_n) = |one(x) - C_{R_1}(x_1, \cdots, x_n)|$$
$$C_{R_1 \wedge R_2} = C_{R_1}(x_1, \cdots, x_n) C_{R_2}(x_1, \cdots, x_n)$$

(b) *Show that if $R(x)$ is an arbitrary numerical relation and $\{x : R(x)\}$ is finite, then $R(x)$ is elementary.*

$R(x)$ is elementary iff $C_R(x)$ is elementary. First let's construct a relation: $E_k(x)$ holds iff $x = k$. We define its characteristic function as:

$$C_{E_k}(x) = sg^*(|x-k|)$$

Since its characteristic function is elementary, E_k is elementary.
Since there are a finite number of x such that $R(x)$ holds, let's say, without loss of generalization, that they are: $x_1, \cdots, x_n$. First, we define:

$$R_1 \vee R_2 \iff \neg(\neg R_1 \wedge \neg R_2)$$

Finally, we can define $R(x)$ as follows:

$$R(x) \iff E_{x_1}(x) \vee E_{x_2}(x) \vee \cdots \vee E_{x_n}(x)$$

Since both E_k and logical or are elementary, $R(x)$ must be elementary as well.

4. *Show that the function $J(a, b)$ given by $\frac{1}{2}(a+b)(a+b+1) + a$ is onto.*

First, we can show that J is one-to-one. Say we have some (a, b) and (c, d) such that $J(a, b) = J(c, d)$. We can partition the set of all (a, b, c, d) into 3 cases: either $(a+b) < (c+d)$, or $(a+b) = (c, d)$ or $(a+b) > (c+d)$.

Let's examine the first case. Suppose that $(c+d)$ is x larger than $(a+b)$. We can show that it's impossible that $J(a, b) = J(c, d)$ if $x > 0$:

$$\frac{1}{2}(a+b+x)(a+b+x+1) - \frac{1}{2}(a+b)(a+b+1)$$
$$= \frac{1}{2}x(2a + 2b + x + 1)$$

If $x > 0$, then there is no way that a can be large enough to make $J(a, b) = J(c, d)$ because the difference of the first monomials includes a itself (from the $\frac{1}{2}(2a + \cdots)$). Because we've hit a contradiction, there's no way that $(a+b) < (c+d)$. And through symmetry, this also means that there's no way that $(c+d) < (a+b)$. So we now know that:

$$J(a, b) = J(c, d) \implies (a+b) = (c+d)$$

And if the sums are equal, then the first monomials must be equal ($\frac{1}{2}(a+b)(a+b+1) = \frac{1}{2}(c+d)(c+d+1)$), which then means that $a = c$. And if $a = c$ and $a + b = c + d$, then $b = d$.
This means that $J(a, b) = J(c, d) \implies a = b \wedge c = d$, which is the definition of one-to-one functions.

Let's consider all the terms of the sequence ($\frac{1}{2}(n)(n+1)$). The first is $\frac{1}{2}(0)(1) = 0$. And the distance between subsequent terms is:

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}(n)(n+1) = n + 1$$

So inbetween subsequent terms, we can fit $n+1$ varying values of a so that no values inbetween are left out. $a = 0, \cdots, n$, and we can define $b = n - a$.