Assignment #3

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4.3 For each set in Exercise 4.1, give its supremum if it has none. Otherwise write "NO sup."

- (a) [0,1]
- (b) (0,1)
- (c) $\{2,7\}$
- (d) $\{\pi, e\}$
- (e) $\{\frac{1}{n} : n \in \mathbb{N}\}$ $\frac{1}{1} = 1$
- $(f) \{0\}$
- (g) $[0,1] \cup [2,3]$
- (h) $\bigcup_{n=1}^{\infty} [2n, 2n+1]$ NO sup
- (i) $\cap_{n=1}^{\infty}[-\frac{1}{n},1+\frac{1}{n}]$ Since the bound approaches (0,1), the supremum is **1**
- $(\mathbf{j}) \ \{1 \frac{1}{3^n} : n \in \mathbb{N}$
- (k) ${n + \frac{(-1)^n}{n} : n \in \mathbb{N} \atop \text{NO sup}}$
- $\begin{array}{c} \text{(l)} \ \{r \in \mathbb{Q} : r < 2\} \\ 2 \end{array}$
- $\text{(m)} \ \, \begin{cases} r \in \mathbb{Q} : r^2 < 4 \rbrace \\ 2 \end{cases}$
- $\text{(n) } \{r\in\mathbb{Q}: r^2<2\}$
- $\begin{array}{cc} \text{(o)} & \{x \in \mathbb{R} : x < 0\} \\ & 0 \end{array}$
- (p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$ 10
- $(q) \ \{0,1,2,4,8,16\}$
- (r) $\bigcap_{n=1}^{\infty} (1 \frac{1}{n}, 1 + \frac{1}{n})$
- (s) $\left\{\frac{1}{n}: n \in \mathbb{N} \text{ and } n \text{ is prime}\right\}$ (t) $\left\{x \in \mathbb{R}: x^3 < 8\right\}$

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- $\begin{array}{c} \text{(u)} \ \{x^2: x \in mathdsR\} \\ \text{NO sup} \end{array}$
- $\begin{array}{c} \cos p \\ \text{(v) } \{\cos(\frac{n\pi}{3}): n \in mathdsN\} \\ 1 \\ \text{(w) } \{\sin(\frac{n\pi}{3}): n \in mathdsN\} \\ \frac{\sqrt{3}}{2} \end{array}$
- 4.7 Let S and T be nonempty bounded subsets of \mathbb{R} .
 - (a) Prove if $S \subseteq T$, then inf $T \le \inf S \le \sup S \le \sup T$.
 - * Let there be some element $x \in S$. Since $S \subseteq T$, $x \in T$. Because x is a member of T, $x \ge \inf T$, for all elements in S, making it a lower bound on S. And because inf S is the largest possible lower bound for S, inf T cannot be larger, meaning inf $T \leq \inf S$.
 - * Let there be some element $x \in S$. Since $S \subseteq T$, $x \in T$. Because x is a member of T, $x \leq \sup T$, for all elements in S, making it an upper bound on S. And because $\sup S$ is the lowest possible upper bound for S, sup T cannot be smaller, meaning sup $S \leq \sup T$.
 - * Let there be some element $x \in S$. Then we know that inf $S \le x \le \sup S$ by the definition of infimum and supremum. By transitivity, inf $S \leq \sup S$.

Having shown all 3 inequalities, by the transitive law we know that $\inf T \leq \inf S \leq \sup S \leq$ $\sup T$.

(b) Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$

Let there be some element $x \in S \cup T$. By definition we know that x must be a member of either S or T (or both). It follows that either $x \leq \sup S$ or $x \leq \sup T$, or both. Furthermore, we know that $\max\{\sup S, \sup T\}$ is greater than or equal to every element in $S \cup T$, precisely because x must belong to either S or T. But more than that, we know that the supremum of the intersection can't be a number smaller than $\max\{\sup S, \sup T\}$ because if there was a number smaller that was still greater than every element in S and T, then it must've been larger than every element in the set with the larger supremum, making it the supremum, so it cannot be larger than the suprmum itself.

Since there can't be another element smaller than max $\{\sup S, \sup T\}$ that is still bigger than every element in $S \cup T$, we know that max $\{\sup S, \sup T\}$ must be the supremum of $S \cup T$.

4.10 Prove that if a > 0, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

By the Archimedean Property, we know that since a>0 and 1>0, there must be some $n_1\in\mathbb{N}$ s.t. $n_1a>1$, and therefore $a>\frac{1}{n_1}$. Similarly, there must be some $n_2\in\mathbb{N}$ s.t. $n_2*1>a$. We know for sure that $a<\max\{n_1,n_2\}$. Let $n_{\max}=\max\{n_1,n_2\}$. We also know that $\frac{1}{n_{\max}}<\frac{1}{n_{\min}}$ and $\frac{1}{n_{\max}}=\frac{1}{n_{\max}}$. Thus, $\frac{1}{n_{\max}}\leq\frac{1}{n_1}$. By the same reasoning, we know that $n_{\max}\geq n_2$.

Finally, this means that $\frac{1}{n_{\text{max}}} \leq \frac{1}{n_1} < a < n_2 \leq n_{\text{max}}$, and therefore there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$ for any $a \in \mathbb{R}$.

4.11 Consider $a, b \in \mathbb{R}$ where a < b. Use Denseness of \mathbb{Q} 4.7 to show there are infinitely many rationals between a and b.

By theorem 4.7, we can find a rational number r_0 inbetween a and b s.t. $a < r_0 < b$. This means we can find a **distinct** number $r_0 < b$. Now, if we recursively "enter" the problem again. We have r as Nikhil Unni 3

a, and b as b. And so repeatedly, we'll be able to find a new distinct rational number r_n greater than r_{n-1} and less than b. Since this process can never end, n will approach infinity, producing an infinite amount of rational numbers.

4.15 Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

Suppose the opposite. That if $a \le b + \frac{1}{n}$: $\forall n \in \mathbb{N}$, a > b. a > b, so we know that a - b > 0. And since 1 > 0, by the Archimedean Property, there is some $n \in \mathbb{N}$ s.t. n(a - b) > 1, or $a > b + \frac{1}{n}$. But this contradicts our original assumption and so, by proof by contradiction, the opposite is true.

4.16 Show $\sup\{r \in \mathbb{Q} : r < a\} = a \text{ for each } a \in \mathbb{R}.$

For the rest of the problem, I'll refer to the set as S_a , for some arbitrary $a \in \mathbb{R}$. By definition of the set, we know that a is larger than every element $x \in S_a$, so it is a valid upper bound. Now, suppose that there is some $b \in \mathbb{R}$ s.t. b < a and $b \ge x, \forall x \in S_a$, that is, that there's a smaller upper bound than a.

However, by theorem 4.7, there exists a rational number inbetween b and a, meaning that b is not a valid upper bound on S_a for all b < a. So for any b < a we can find a rational number inbetween b and a. Since there cannot exist a smaller upper bound on S_a , we know that a is the supremum.

7.1 Write out the first five terms of the following sequences.

(a)
$$s_n = \frac{1}{3n+1}$$

$$\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{17}$$

(b)
$$b_n = \frac{3n+1}{4n-1}$$

$$\frac{4}{3}$$
, 1, $\frac{10}{11}$, $\frac{13}{15}$, $\frac{17}{19}$

(c)
$$c_n = \frac{n}{3^n}$$

$$\frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \frac{5}{243}$$

(d)
$$sin(\frac{n\pi}{4})$$

$$\begin{split} sin(\frac{\pi}{4}), sin(\frac{2\pi}{4}), sin(\frac{3\pi}{4}), sin(\frac{4\pi}{4}), sin(\frac{5\pi}{4}) \\ &= \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2} \end{split}$$

7.2 For each sequence in 7.1, determine whether it converges. If it converges, give its limit.

(a)
$$s_n = \frac{1}{3n+1}$$

(b)
$$b_n = \frac{3n+1}{4n-1}$$

$$\frac{3}{4}$$

(c)
$$c_n = \frac{n}{3^n}$$

0

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(d) $sin(\frac{n\pi}{4})$

Does not converge.

7.4 Give examples of

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

The sequence $x_n = \frac{\pi}{n}$ has a limit of 0, a rational number. And we can easily prove that $\frac{\pi}{n}$ is irrational, for all natural numbers $n \in \mathbb{N}$. Assume that it's rational. Then, we can say:

$$\frac{\pi}{n} = \frac{p}{a}$$

But then we can represent π as a fraction of integers:

$$\pi = \frac{pn}{q}$$

And this is impossible since π is irrational, so $\frac{\pi}{n}$ must be irrational.

(b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

The decimal expansion of any irrational number approaches the irrational number, while having rational number terms of the series. For example:

$$s_{\pi} = 3, 3.1, 3.14, 3.141, \cdots$$