Math 104 **Spring 2016**

Assignment #9

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17.1 Let $f(x) = \sqrt{4-x}$ for all $x \le 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

(a) Give the domains of $f+g$, fg, $f \circ g$, and $g \circ f$.

The domains of $f+g$ and fg are just the intersection of the two domains, or $(-\infty, 4]$. The domain of $f \circ g$ is $[-2, 2]$, and the domain of $g \circ f$ is $(-\infty, 4]$.

(b) Find the values of $f \circ g(0), g \circ f(0), f \circ g(1), g \circ f(1), f \circ g(2)$, and $g \circ f(2)$.

$$f \circ g(0) = 2$$
$$g \circ f(0) = 4$$
$$f \circ g(1) = \sqrt{3}$$
$$g \circ f(1) = 3$$
$$f \circ g(2) = 0$$
$$g \circ f(2) = 2$$

(c) Are the functions $f \circ g$ and $g \circ f$ equal?

No, since $f \circ g(0) \ne g \circ f(0)$.

(d) Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

$f \circ g(3)$ is not meaningful, since 3 is outside the domain, but $g \circ f(3)$ is meaningful since 3 **is** inside the domain.

17.2 Let $f(x) = 4$ for $x \ge 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x. Thus $\text{dom}(f) = \text{dom}(g) = \mathbb{R}$.

(a) Determine the following functions: $f+g, f \circ g, g \circ f$. Be sure to specify their domains.

$$(f+g)(x) = 4 + x^2 \text{ for } x \ge 0, (f+g)(x) = x^2 \text{ for } x < 0. \text{ dom}(f+g) = \mathbb{R}$$
$$(f \circ g)(x) = 4. \text{ dom}(f \circ g) = \mathbb{R}$$
$$(g \circ f)(x) = 16 \text{ for } x \ge 0, (g \circ f)(x) = 0 \text{ for } x < 0. \text{ dom}(g \circ f) = \mathbb{R}$$

(b) Which of the functions $f, g, f+g, fg, f \circ g, g \circ f$ is continuous?

Clearly $g, f \circ g$ are continuous, while $f, f+g$, and $g \circ f$ are not continuous. But I'll show that fg is continuous at $x = 0$, since it's clearly continuous everywhere else.

$$(fg)(x) = 0 \text{ for } x \le 0, (fg)(x) = 4x^2 \text{ for } x > 0. \text{ dom}(fg) = \mathbb{R}$$

We can prove its continuity with a $\delta - \epsilon$ proof. Let $\epsilon > 0$. If fg is continuous at 0, then:

$$|x| < \delta \implies |f(x)| < \epsilon$$

If $x <= 0$, then clearly any value of x will result in $|f(x)| < \epsilon$. So we can safely assume that $x > 0$, and with that we can also assume that $f(x) = 4x^2 > 0$. So setting $\delta = \frac{1}{2}\sqrt{\epsilon}$ will mean that $x < \delta \implies 4x^2 < \epsilon$. Since, for every $\epsilon > 0$, we can find a $\delta > 0$ such that the implication holds true, the function is continuous at 0. This means that fg is continuous everywhere.

17.5 (a) Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.

Going from the definition of continuity, suppose we have a sequence (s_n) whose limit is s_0. Then:

$$\lim f(x_n) = \lim(x_n^m) = \lim(x_n)^m = x_0^m = f(x_0)$$

Because all m are natural numbers, we know that the domains match up, and we won't get any imaginary numbers.

(b) Prove every polynomial function $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

Again, going from the definition of continuity, suppose we have a sequence (s_m) whose limit is s_0. Then:

$$\lim f(x_m) = \lim(a_0 + a_1 x_m + \cdots + a_n x_m^n)$$

$$= a_0 + a_1(\lim x_m) + \cdots + a_n(\lim x_m)^n = a_0 + a_1 x_0 + \cdots + a_n x_0^m = f(x_0)$$

17.9 Prove each of the following functions is continuous at x_0 by verifying the $\epsilon - \delta$ property of Theorem 17.2

(a) $f(x) = x^2, x_0 = 2$;

Let $\epsilon > 0$. We want to find a δ such that $|x-2| < \delta$ implies $\left|x^2 - 4\right| < \epsilon$. Notice this also implies:

$$|(x-2)(x+2)| < \epsilon$$

Or

$$|x-2|\,|x+2| < \epsilon$$

Suppose that $\delta < 1$. Even if there exists a $\delta_0 \geq 1$ that satisfies the same inequality, we know that all $0 < \delta_1 \leq \delta_0$ must satisfy the same inequality. So the constraint is a valid one. Continuing, this means:

$$|x-2| < \delta < 1$$

$$-1 < x - 2 < 1$$

$$3 < |x+2| < 5$$

Now we need $5\,|x-2| < \epsilon$. So set $\delta = \min\{1, \frac{\epsilon}{5}\}$. Now:

$$\left|x^2 - 4\right| = |x-2|\,|x+2| < 5\delta \leq \epsilon$$

(b) $f(x) = \sqrt(x), x_0 = 0$;

Let $\epsilon > 0$. We want to find a δ such that $x \in [0, +\infty), |x| < \delta$ implies $|\sqrt{x}| < \epsilon$. Since the domain and image of the square root function is the set of nonnegative real numbers, this is equivalent to $x \in [0, +\infty), x < \delta$ implying $\sqrt{x} < \epsilon$.

If we pick $\delta = \epsilon^2$, then:

$$x < \delta = \epsilon^2 \implies \sqrt{x} < \epsilon$$

(c) $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0, x_0 = 0$;

Let $\epsilon > 0$. We want to find a δ such that $|x| < \delta \implies \left|x\sin(\frac{1}{x})\right| < \epsilon$. The right-side implication is equivalent to:

$$|x|\left|\sin(\frac{1}{x})\right|$$

We know that the sin function is bounded by -1 and 1, so $|x| < \delta \implies \left|x\sin(\frac{1}{x})\right| < \delta * 1$. So any $0 < \delta \leq \epsilon$ will work. For the sake of the proof, say $\delta = \epsilon$. Then:

$$|x| < \delta \implies \left|x\sin(\frac{1}{x})\right| < \epsilon$$

(d) $g(x) = x^3, x_0$ arbitrary. Hint : $x^3 - x_0^3 = (x - x_0)(x^2 + x_0x + x_0^2)$.

Let $\epsilon > 0$. We want to find a δ such that $|x - x_0| < \delta \implies \left|x^3 - x_0^3\right| < \epsilon$. Following the tip, the right-side implication is equivalent to:

$$|x - x_0|\left|x^2 + x_0x + x_0^2\right| < \epsilon$$

Say $\delta < 1$. Then $|x - x_0| < \delta \implies |x| < |x_0| + 1$, and:

$|x - x_0|\left|x^2 + x_0x + x_0^2\right| < \delta(\left|x^2\right| + |x_0x| + \left|x_0^2\right|) < \delta[(|x_0|+1)^2 + (|x_0|+1)|x_0| + x_0^2] = \delta(3x_0^2 + 3|x_0| + 1)$

So if we set $\delta = \min\{1, \epsilon/(3x_0^2 + 3|x_0| + 1)\}$, we satisfy the implication.

17.10 Prove the following functions are discontinuous at the indicated points. You may use either Definition 17.1 or the $\epsilon - \delta$ property in Theorem 17.2.

(a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0, x_0 = 0$;

Suppose we have the sequence $s_n = \frac{1}{n}$. Clearly, for any $n \in \mathbb{N}$, $f(s_n) = \frac{1}{n} > 0$, so $\lim_{n \to \infty} f(s_n) = 1$. However, $f(0) = 0$, so by the definition of continuity, the function is discontinuous at 0.

(b) $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0, x_0 = 0$;

Again, let's work with the sequence $s_n = \frac{1}{n}$. Then:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(n)$$

Since this limit is undefined, it cannot be $g(0) = 0$, meaning the function is discontinuous at 0.

(c) $sgn(x) = -1$ for $x < 0$, $sgn(x) = 1$ for $x > 0$, and $sgn(0) = 0, x_0 = 0$. Note $sgn(x) = \frac{x}{|x|}$ for $x \neq 0$.

Yet again, let's use the sequence $s_n = \frac{1}{n}$. For any $n \in \mathbb{N}$, $0 < \frac{1}{n}$, so $\lim_{n\to\infty} sgn(\frac{1}{n}) = 1$. However, $f(0) = 0$, so by the definition of continuity, the function is discontinuous at 0.

17.11 Let f be a real-valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. Prove f is continuous at x_0 if and only if, for every monotonic sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$. Hint : Don't forget Theorem 14.4.

If f is continuous at x_0, of course, for every monotonic sequence (x_n) in the domain converging to x_0, $\lim f(x_n) = f(x_0)$, from the definition of continuity. (The set of all monotonic sequences in the domain converging to x_0 is a subset of the set of all sequences in the domain converging to x_0, clearly.)

Conversely, suppose that if a monotonic subsequence (s_n) in the domain converges to x_0, then $\lim_n f(s_n) = f(s_0)$. Suppose that f is **not** continuous at x_0. Then there must be a subsequence (y_{n_k}) s.t. $|f(y_{n_k}) - f(x_0)| \geq 0$. However, we know from Theorem 14.4, that y_{n_k} must have a monotonic subsequence $(y_{n_{k_l}})$. However, since its a monotonic sequence, $\lim_n f(y_{n_{k_l}}) = f(s_0)$, and so we have a contradiction. This means that f must be continuous at x_0.

17.12 (a) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

Suppose $f(y) \neq 0$ for all irrational y. As explained in 17.13(a), for any $x \in \mathbb{R}$, we can find both irrational and rational sequences, (r_n) and (q_n) respectively, such that their limit is x. Then, $\lim_n f(q_n) = 0$, but $\lim_n f(r_n) \neq 0$, since f at every rational number is not 0. Thus, we have a contradiction, since f is continuous at every point, and so $f(y) : y \in \mathbb{R} \setminus \mathbb{Q}$ can't be any value other than 0. Since $f(x) = 0$ for all rational and irrational numbers, and $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$, $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$.

This is mostly the same as part (a). Suppose $f(y) \neq g(y)$ for each irrational number q in (a, b). For any x, we can find irrational and irational sequences (r_n) and (q_n) whose limits are x. Then, $\lim_n f(q_n) = g(y)$, yet $\lim_n f(r_n) \neq g(y)$, so we have a contradiction, since f is continuous across the entire interval. Thus, $f(y)$ has to be $g(y)$ for all irrational numbers in (a, b). This means that $f(x) = g(x)$ for all $x \in (a, b)$.

17.13 (a) Let $f(x) = 1$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show f is discontinuous at every $x \in \mathbb{R}$.

For each $x \in \mathbb{R}$, there exists at least one rational sequence (q_n) whose limit is x, and at least on irrational sequence (r_n) whose limit is x. This follows from the denseness of both the rational and irrational numbers. (Concretely, there are an infinite number of irrational and rational numbers between 0 and x, so we can find at least one monotonic sequence converging to x for both.)

However, $\lim_n f(q_n) = 1 \neq \lim_n f(r_n) = 0$. Thus, not all sequences in the domain converging to x converge to the same $f(x)$, and by definition is not continuous at x.

Since this holds true for an arbitrary $x \in \mathbb{R}$, f is discontinuous at every $x \in \mathbb{R}$.

(b) Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for irrational numbers. Show h is continuous at $x = 0$ and at no other point.

Again, for each point x, let's take a rational sequence (q_n) converging to x, and an irrational sequence (r_n) converging to x.

Then, $\lim_n h(q_n) = x$, since the sequence of $h(q_n)$ is the same sequence as (q_n), which converges to x. However, $\lim_n h(r_n) = 0$, since every term in the sequence is 0. Since, trivially, $x \neq 0$ for every point except for $x = 0$, h is discontinuous at every point other than $x = 0$.