Math 104 **Spring 2016**

Assignment #6

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11.1 Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.

(a) List the first eight terms of the sequence (a_n).

$$a_1 = 1, a_2 = 5, a_3 = 1, a_4 = 5, a_5 = 1, a_6 = 5, a_7 = 1, a_8 = 5$$

(b) Give a subsequence that is constant [takes a single value]. Specify the selection function σ.

The selection function $\sigma(k) = 2k$ will be the subsequence of only even-indexed terms, so that $t(\sigma(n)) = 5$, for all $n \in \mathbb{N}$.

11.2 Consider the sequences defined as follows:

$$a_n = (-1)^n, b_n = \frac{1}{n}, c_n = n^2, d_n = \frac{6n+4}{7n-3}$$

(a) For each sequence, give an example of a monotone subsequence.

For (a_n), with the selection function $\sigma(k) = 2k$, we get a constant sequence of 1, which is monotonic, since $1 \geq 1$.

For (b_n), the trivial subsequence, $\sigma(k) = k$ is monotonic, since each term is decreasing.

For (c_n), is monotonic as well, the trivial subsequence is monotonic. For (d_n), like $\frac{1}{n}$, this sequence is monotonically decreasing, so we can, again, select the original sequence as the subsequence.

(b) For each sequence, give its set of subsequential limits.

For (a_n), the only limits it can possibly have are 1 and -1, and both are possible, if you just select even or odd indeces. So $\{-1, 1\}$.

For (b_n), since the limit of the entire sequence is 0, by Theorem 11.3, all subsequences converge to 0, so the set is just $\{0\}$.

For (c_n), again, since the limit is $+\infty$, the set is $\{+\infty\}$.

For (d_n), the limit of the sequence is $\frac{6}{7}$, so the set is $\{\frac{6}{7}\}$.

(c) For each sequence, give its lim sup and lim inf.

For (a_n), regardless of how large N is, the sup is 1, since the only possible values of $a_{n>N}$ are -1 and 1, with an arbitrarily large N. Similarly, the lim sup is -1.

For (b_n), by Theorem 10.7, both the $\liminf b_n = \limsup b_n = \lim b_n = 0$.

For (c_n), by Theorem 10.7, lim inf c_n = lim sup c_n = lim c_n = $+\infty$. For (d_n), by Theorem 10.7, lim inf d_n = lim sup d_n = lim d_n = $\frac{6}{7}$.

(d) Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?

$$(a_n) \text{ is not convergent, and does not diverge to } +\infty \text{ or } -\infty.$$

$$(b_n) \text{ is convergent to 0, and thus not divergant to anything.}$$

$$(c_n) \text{ is not convergant, but is divergant to } +\infty.$$

$$(d_n) \text{ is convergant to } \frac{6}{7}, \text{ and is not divergant to anything.}$$

(e) Which of the sequences is bounded?

(a_n) is bounded, (b_n) is bounded, (c_n) is unbounded, and (d_n) is bounded.

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.

(a) Give the set of subsequential limits for (q_n).

Since there are an infinite amount of rationals between real numbers, by the Denseness of $\mathbb{Q}$ Theorem, the set includes every value between 0 and 1. Even though 0 is not in (q_n), there are an infinite amount of rationals close to 0, so 0 is a valid limit. Concretely, the set is $[0, 1]$.

(b) Give the values of lim sup q_n and lim inf q_n.

$$\limsup q_n = 1$$

$$\liminf q_n = 0$$

The reasoning being that, if you include enough terms in the subsequence (selected by indices $n > N$), you'll include rational numbers arbitrarily close to 0 and 1.

11.6 Show every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence.

Let's define a subsequence $t(k)$ as the composition of the given sequence s_n with some selection function $\sigma(k)$ so that:

$$t(k) = s(\sigma_1(n)), \text{ for } n \in \mathbb{N} \text{ (from Definition 11.1).}$$

Then, the subsequence of a subsequence would be:

$$u(k) = t(\sigma_2(n)) = s(\sigma_1(\sigma_2(n)))$$

But note that $\sigma_3 = \sigma_1 \circ \sigma_2$ is a valid function as well. Then, the subsequence's subsequence, $u(k)$ is a subsequence of the given sequence $s(k)$ by selection function $\sigma_3 = \sigma_1 \circ \sigma_2$. In other words:

$$u = s \circ \sigma_3(k) = s \circ (\sigma_1 \circ \sigma_2)(k)$$

11.9 (a) Show the closed interval $[a, b]$ is a closed set.

If we can find a sequence (s_n) where the set of subsequential limits is $[a, b]$, then we've showed that $[a, b]$ is closed. But we've already found this in 11.5 part (a). Let (q_n) be an enumeration of all the rationals in the interval $(a, b]$. Then, as we showed in 11.5, the set of subsequential limits is $[a, b]$. Thus $[a, b]$ is a closed set.

(b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

There cannot possibly be one, since $(0, 1)$ is an open set.

11.10 Let (s_n) be the sequence of numbers in Fig. 11.2 listed in the indicated order.

(a) Find the set S of subsequential limits of (s_n).

All the values are $\frac{1}{n}, n \in \mathbb{N}$. And there are a countably infinite amount of each $\frac{1}{n}$. We can also just select $t_n = \frac{1}{n}$ as a valid subsequence. So the set of subsequential limits is given by:

$$\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$$

(b) Determine lim sup s_n and lim inf s_n.

$$\limsup s_n = 1$$

$$\liminf s_n = 0$$

This is evident by the same reasoning as 11.5.

11.11 Let S be a bounded set. Prove there is an increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$. Compare Exercise 10.7. **Note:** if $\sup S$ is in S, it's sufficient to define $s_n = \sup S$ for all n.

As shown in the hint, if $\sup S$ is in S, $s_n = \sup S$ is a valid increasing sequence. So, assume $\sup S$ is not in S. But the set is upper bounded by $\sup S$, so there must be an infinite number of points less than $\sup S$.

So let s_1 be some arbitrary element in S. And let every following s_k be some element larger than all $s_{n<k}$. Since it's the supremum, there has to be an infinite number of points between s_1 and $\sup S$. So just call a countably finite **ordered** subset of them (s_n).