**Phil 140b** **Spring 2016**

# Assignment #1

Nikhil Unni

1. *Let $L^* = \{0,\ ',\ +,\ .,\ <\ \}$. Taking for granted the inductive definition of the terms of $L^*$ provided in class, define the atomic formulas of $L^*$ as follows:*
*Atomic formulas: if $t_1, t_2$ are terms of $L^*$ then $= (t_1, t_2)$ and $< (t_1, t_2)$ are atomic formulas.*
*Now define inductively the class of well formed formulas of $L^*$ as follows:*
*Basis: all atomic formulas are well formed formulas;*
*Inductive clause: if $A$ and $B$ are well formed formulas, so are $(A\&B), \neg A$, and $\forall xA$;*
*External clause: nothing else is a well formed formula*

a. *Show by induction on the construction of the set of well formed formulas that all well formed formulas have the same number of left and right parentheses. [You can assume in your proof that all terms of $L^*$ have the same number of left and right parentheses ]*

Base Case: all atomic formulas are of the form: $= (t_1, t_2)$ or $< (t_1, t_2)$. Since it is assumed that all terms of $L^*$ have balanced parentheses, all atomic formulas must be balanced as well, since they just add a single left paren and right paren to the expression.

Inductive Case: Assuming that well-formed formulas $A$ and $B$ have balanced parentheses, then:

- $(A\&B)$ must have balanced parentheses, since it adds a single left and right paren to the already balanced expression (and the ampersand doesn't change anything).
- $\neg A$ must have balanced parentheses, since it doesn't add any left or right parentheses to the expression.
- $\forall xA$ must have balanced parentheses as well, since it doesn't add any left or right parentheses.

Since well-formed formulas are inductively defined this way, by induction we have shown that all possible well-formed expressions have balanced parentheses.

b. *Following the outline of the proof we did in class for terms of $L^*$, define a numerical measure of complexity for the well formed formulas ($f(w) = n$) and prove by induction on the natural numbers that "for all $n$, for all well formed formulas $w$, if $comp(w) = n$, then $w$ has the same number of left and right parentheses".*

We define a "complexity" function $f$ : well formed formulas $\rightarrow \mathbb{N}$, which just maps a well formed formula to the maximum recursive depth to atomic formulas. Concretely:

$$f(= (t_1, t_2)) = f(< (t_1, t_2)) = 0$$

$$f((A\&B)) = \max(f(A), f(B)) + 1$$

$$f(\neg A) = f(A) + 1$$

$$f(\forall xA) = f(A) + 1$$

Assuming all well-formed formulas are finitely long, they must have a finite complexity, and so for all $x \in$ well formed formulas, $f(x) = n \in \mathbb{N}$.

Base Case: All formulas of complexity 0 are atomic formulas. Since it's assumed that all terms in $L^*$ have equal parentheses, we know all atomic formulas have balanced left and right parentheses, since they just add a single left and a single right. In other words, if $t_1$ has $n$ of both ($\#_L(t_1) = \#_R(t_1) = n$), and $t_2$ has $m$ of both ($\#_L(t_2) = \#_R(t_2) = m$), then:

$$\#_L(=(t_1, t_2)) = \#_L(<(t_1, t_2)) = n + m + 1 = \#_R(=(t_1, t_2)) = \#_R(<(t_1, t_2))$$

Inductive Case: Assuming that all formulas of complexity $\leq n$ have balanced parentheses, then we can show that all formulas of complexity $n + 1$ have balanced parentheses. Say we have two well-formed formulas of complexity $<= n$ : A and B, where $\#_L(A) = \#_R(A) = x$, and $\#_L(B) = \#_R(B) = y$. Then:

$$\#_L((A\&B)) = 1 + x + y = \#_R((A\&B))$$

$$\#_L(\neg A) = x = \#_R(\neg A)$$

$$\#_L(\forall xA) = x = \#_R(\forall xA)$$

So by induction, all well-formed formulas of complexity $n \geq 0$ have balanced parentheses, and so all well-formed formulas must have balanced parentheses

2. *Show that:*

   (a) *If the sentence $E$ is implied by the set of sentences $\Delta$ and every sentence $D$ in $\Delta$ is implies by the set of sentences $\Gamma$, then $E$ is implied by $\Gamma$.*

   (b) *If the sentence $E$ is implied by the set of sentences $\Gamma \cup \Delta$ and every sentence $D$ in $\Delta$ is implied by the set of sentences $\Gamma$, then $E$ is implied by $\Gamma$.*

3. *Let $L = \{0,\ ',\ +,\ *\ \}$. Give an interpretation of $L$ with a finite domain that makes the sentences:*

$$\neg\forall x\forall y(x * y = y * x)$$

$$\neg\forall x\forall y(x + y = y + x)$$

4. *Show that the following sentences are invalid:*

   (a)

$$\forall x\exists yQ(x, y) \rightarrow \exists x\forall yQ(x, y)$$

   (b)

$$(\forall xQ(x, x)\&\forall x\forall y(Q(x, y) \implies Q(y, x))) \implies \forall x\forall y\forall z((Q(x, y)\&Q(y, z) \implies Q(x, z))$$

5. *Show that:*

   *(a)* *If $\Gamma \cup \{\sim (B\&C)\}$ is satisfiable, then either $\Gamma \cup \{\sim B\}$ is satisfiable or $\Gamma \cup \{\sim C\}$ is satisfiable.*

   *(b)* *If $\Gamma \cup \{\sim \forall xB(x)\}$ is satisfiable, then for any constant $c$ not occuring in $\Gamma$ or $\forall xB(x)$, $\Gamma \cup \{\sim B(c)\}$ is satisfiable.*