Math 113 **Fall 2015**

Homework #7

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1. Let F denote the additive group of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

   (a) Let $\phi : F \to \mathbb{R}$ be defined by $f \to \int_0^1 f(x)dx$ for all $f \in F$. Prove that ϕ is a group homomorphism.

$$f, g \in F$$
$$\phi(f+g) \stackrel{?}{=} \phi(f) + \phi(g)$$
$$\int_0^1 (f+g)(x)dx \stackrel{?}{=} \int_0^1 f(x)dx + \int_0^1 g(x)dx$$
$$\int_0^1 (f(x)+g(x))dx \stackrel{?}{=} \int_0^1 f(x)dx + \int_0^1 g(x)dx$$
$$\int_0^1 f(x)dx + \int_0^1 g(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$$

   (b) Find the kernel K of ϕ.

   K is the set of all continuous functions where the integral from 0 to 1 equals 0. Algebraically, from the Fundamental Theorem of Calculus, this is the set of functions f with antiderivative F, where $F(1) = F(0)$. Geometrically, this is the set of functions where the average value between 0 and 1 is 0 (like $sin(2\pi x)$, or $2x - 1$).

   (c) Describe the coset $x + K$ of the kernel, with both algebraic and geometric descriptions. Is there a "nicer" representative of this coset?

   It is the set of functions where the integral from 0 to 1 equals some constant, c. This means the average value from 0 to 1 is c. This can be represented as : $\{f_c | \int_0^1 f_c dx = c, f_c \in F, c \in \mathbb{R}\}$

   (d) Note that the cosets of K are in bijection with $\mathbb{R}$. Does Lagrange's Theorem apply here? Explain very briefly.

Because the cosets of K form the quotient group of $G/ker(\phi)$, and since ϕ is a surjective function, by the Fundamental Homomorphism Theorem $\mathbb{R}$ is isomorphic to $G/ker(\phi)$, meaning the mapping is bijective. Lagrange's Theorem does not help us here because it can only speak of cardinalities for infinite sets.

2. (a) Suppose G is a finite group of order m. Prove that $g^m = e$ for all $g \in G$.

   From Lagrange's Theorem, we know that the order of any $\langle g \rangle$ (let's call it x), for some $g \in G$ divides m. Meaning that $xy = m$, for some $y \in \mathbb{Z}$. Then:

$$g^m = g^{x+y} = (g^x)^y = e^y = e$$

   (b) Suppose G is a finite group, and $N \trianglelefteq G$. If there are k cosets of N in G, prove that $g^k \in N$ for all $g \in G$.

   Because N is a normal subgroup, the set of cosets becomes the quotient group G/N of order k, where $(aN)(bN) = (ab)N$ for the binary operation, and $eN = N$ is the identity. From part (a), we know that for any quotient group, q, $q^k = N$.
   By the definition of the binary operation, this means for any element g:

$$(gN)^k = N$$
$$(g^k N) = eN$$
$$g^k = e$$

3. This problem will deal with the group $G = D_4 \times S_3$.

   (a) How many elements of each order do the groups D_4 and S_4 have? Using this info, determine how many elements of each order G has.

For D_4:

Order Number	Count
1	1
2	5
4	2

For S_3:

Order Number	Count
1	1
2	3
3	2

For G:

Order Number	Count
1	1
2	29
3	2
4	2
6	10
12	4

(b) Find all subgroups of G that are isomorphic to $Z_2 \times Z_2$. Be sure to explain why you have found all of them.

All sets are of the form $\{(e, ()), ds\}$ for all $d \in \{r^2, s, rs, r^2s, r^3s\}$, and $s \in \{(1,2), (2,3), (1,3)\}$. All of those elements in d and s are their own inverse in their respective group. So paired with the group's inverse, they become isomorphic to Z_2.