Math 104 **Spring 2016**

Assignment #1

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1.1 Prove $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

We can prove this with the principle of mathematical induction.

Let P_n denote whether or not the statement is true for some positive integer n.

Then $P_1 \equiv 1^2 = \frac{1}{6} * 1(1+1)(2*1+1) = \frac{6}{6} = 1$.

So next we have to show that for any P_{n+1}, if P_n is true, that P_{n+1} must be true as well. So we can make the following substitution:

$$(1^2 + 2^2 + \cdots + n^2) + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

Then, grouping together with a common denominator:

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)[(n(2n+1) + 6(n+1))]}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

Since we've proved the statement for P_1, and that any P_{n+1} is true when P_n is true, by the principle of mathematical induction, the statement is true for all positive integers n.

1.8 The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \cdots$ of prepositiions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.

a Prove $n^2 > n + 1$ for all integers $n \geq 2$.

Let P_k be the statement that $k^2 > k + 1$ for some $k \geq 2$.

- $2^2 > 2 + 1 \equiv 4 > 3$
  So the statement P_2 is correct.
- If P_n is true, we can show that P_{n+1} is true, for some $n \geq 2$.

$$(n+1)^2 = n^2 + 2n + 1 > n + 1 + 2n + 1 = 3n + 2 > n + 2$$

  $2 = 2$ and $3n$ must be greater than n for a positive number n (which it is because $n \geq 2$).

By the extension of induction, the statement is true for all $n \geq 2$.

b Prove $n! > n^2$ for all integers $n \geq 4$.

- $4! = 24 > 4^2 = 16$. So the statement P_4 is correct.
- If P_n is true, we can show that P_{n+1} is true, for some $n \geq$. So we know that:

$$(n+1)! = n!(n+1) > n^2(n+1)$$

From part (a) we know that $x^2 > x+1, x \geq 2$, and since $n \geq 4 \geq 2$, we know this is true for any of our n.

$$(n+1)! > n^2(n+1) > (n+1)(n+1) = (n+1)^2$$

1.11 For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

a Prove P_{n+1} is true whenever P_n is true.

If P_n is true, then $n^2 + 5n + 1$ is even and can be represented as $2k$, for some $k \in \mathbb{Z}$. Then:

$(n+1)^2 + 5(n+1) + 1 = n^2 + 2n + 1 + 5n + 5 + 1 = (n^2 + 5n + 1) + 2n + 6 = 2k + 2n + 6 = 2(k + n + 6)$

Since $k + n + 6$ is an integer (since addition of integers is closed), $(n+1)^2 + 5(n+1) + 1$ is even, and so P_{n+1} is true if P_n is true.

b For which n is P_n actually true? What is the moral of this exercise?

It's not true for any integer. (For any odd integer, you have the addition of 3 odd numbers, and for any even number you have the addition of 2 even numbers and an odd number, always yielding an odd sum). The moral of the exercise is that you cannot inductively prove anything without a base case, even if you can prove the recursive case.

1.12 a Verify the binomial theorem for $n = 1$, 2, and 3.

$$(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = \frac{1!}{0!1!}a + \frac{1!}{1!0!} = a + b$$

$$(a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = \frac{2!}{0!2!}a^2 + \frac{2!}{1!1!}ab + \frac{2!}{2!0!}b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = \frac{3!}{0!3!}a^3 + \frac{3!}{1!2!}a^2b + \frac{3!}{2!1!}ab^2 + \frac{3!}{3!0!}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

b Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \cdots, n$.

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)(n-k)!}$$

Grouping together with a common denominator we get:

$$\frac{n!(n-k+1)}{k(k-1)!(n-k)!(n-k+1)} + \frac{n!k}{k(k-1)!(n-k+1)(n-k)!}$$

$$= \frac{n!(n-k+1+k)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!}$$
$$= \binom{n+1}{k} \blacksquare$$

c Prove the binomial theorem using mathematical induction and part (b).

Let P_k be the assertion that the binomial theorem is valid for some $k \in \mathbb{N}$.
From part (a), we've already proven P_1, P_2, and P_3. So we have to show that P_{n+1} is true given that P_n is true.

In summation notation, P_n states that $(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
So then:

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)(\Sigma_{k=0}^{n} \binom{n}{k} x^k y^{n-k})$$

Distributing the x and y we get:

$$= \Sigma_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \Sigma_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}$$

Changing around the indexing, this is equal to:

$$= (\Sigma_{k=1}^{n} \binom{n}{k-1} x^k y^{n-(k-1)}) + \binom{n}{n} x^{n+1} + (\Sigma_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1}) + \binom{n}{0} y^{n+1}$$

Then, if we group together the summations:

$$= (\Sigma_{k=1}^{n} x^k y^{n+1-k} (\binom{n}{k-1} + \binom{n}{k})) + \binom{n}{n} x^{n+1} + \binom{n}{0} y^{n+1}$$

Now, from part (b), we know that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.
Also, from the definition, we know that $\binom{n}{n} = \frac{n!}{n!0!} = 1$ and that $\binom{n}{0} = \frac{n!}{0!n!} = 1$. So we can say that $\binom{n}{n} = \binom{n+1}{n+1}$, and $\binom{n}{0} = \binom{n+1}{0}$. This gives us :

$$= (\Sigma_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k}) + \binom{n+1}{n+1} x^{n+1} + \binom{n+1}{0} y^{n+1}$$

Finally, we can group all the terms together to get:

$$= \Sigma_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

So, given that P_n is true, we can show that P_{n+1} is true as well, and we've already proven P_1. So by the principle of mathematical induction, the binomial theorem is true for $n = 1, 2, \cdots$.