EE 126 **Spring 2016**

HW7

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1. Michael misses shots with probability $\frac{1}{4}$, independent of other shots.

   (a) What is the expected number of shots that Michael will make before he misses three times?

   We want $E[\text{time until 3 misses}]$, which is just $E[3(\text{time until 1 miss})] = 3E[\text{time until 1 miss}]$. And this is just a Geometric R.V., which has expectation of $\frac{1}{p} = 4$. So $E[\text{time until 3 misses}] = 4(3) = 12$

   (b) What is the probability that the second and third time Michael makes a shot will occur when he takes his eigth and ninth shots, respectively?

   This is the probability that: (1) Michael makes exactly 1 ball in the first 7 shots, (2) Michael makes a ball on the 8th shot, and (3) Michael makes a ball on the 9th shot. Luckily, these are all independent, meaning:

$$P(X) = P(1 \text{ ball in first } 7)P(\text{makes 8th})P(\text{makes 9th})$$

$$P(X) = [7(\frac{1}{4})^6(\frac{3}{4})][\frac{3}{4}][\frac{3}{4}]$$

$$P(X) = \frac{189}{262144} \approx 0.072\%$$

   (c) What is the probability that Michael misses two shots in a row before he makes two shots in a row?

   Call the event A. We know that A occurs when Michael misses 2 shots in a row after a (potentially zero-long) string of alternating misses and shots. Noting X as a miss and Y as a score, an example event in A is "XYXYXY $\cdots$ YXX". The chain can be any length. With this in mind:

$$P(A) = (\frac{1}{4})^2 + \frac{3}{4}(\frac{1}{4})^2 + \frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + \frac{3}{4}\frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + \cdots$$

$$P(A) = [\frac{3}{4}(\frac{1}{4})^2 + \frac{3}{16}\frac{3}{4}(\frac{1}{4})^2 + (\frac{3}{16})^2\frac{3}{4}(\frac{1}{4})^2 + \cdots] + [(\frac{1}{4})^2 + \frac{3}{16}(\frac{1}{4})^2 + (\frac{3}{16})^2(\frac{1}{4})^2 + \cdots]$$

   Notice that these are just geometric series, which sum to:

$$P(A) = [\frac{\frac{3}{4}(\frac{1}{4})^2}{1 - \frac{3}{16}}] + [\frac{(\frac{1}{4})^2}{1 - \frac{3}{16}}]$$

$$P(A) = \frac{7}{52}$$

2. Starting at time 0, the F line makes stops at Cory Hall according to a poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

(a) Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus.

The number of students that enter the ith bus are going to be the number of students that queue up after $i-1$ arrives and before i arrives. This length of time is x. Because of the memoryless property of Poisson Processes, we know $N(x)$, the distribution of the number of jumps in $[0, x]$, is equal to the distribution of the number of jumps in $[y, y+x]$, where y is the time that bus $i-1$ arrived. And we know:

$$N(x) \sim \text{Poisson}(\lambda x)$$

(b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

Again, because of the memoryless property, we know that the time between 9:30AM and the next bus is exponentially distributed with parameter λ. So we have:

$$P(\text{n students on next bus}) = \int_{x=0}^{\infty} P(\text{bus arrives at time x})P(\text{n students on next bus given x time inbetween})dx$$

We already solved the second distribution in part (a) (which is Poisson with parameter μx), and we know the first distribution is just exponential, so we have:

$$= \int_{x=0}^{\infty} [\lambda e^{-\lambda x}][\frac{(\mu x)^n}{n!} e^{-\mu x}]dx$$

$$= \frac{\lambda \mu^n}{n!} \int_{x=0}^{\infty} e^{(-\lambda-\mu)x} x^n dx$$

Integrating by parts (with a little trickery to get factorial)

$$= \frac{\lambda \mu^n}{n!}[n!(\lambda+\mu)^{-n-1}]$$

$$= \lambda \mu^n (\mu+\lambda)^{-n-1}$$

(c) Find the distribution of the number of students getting on the next bus to arrive after 11:00 AM.

Say we arrive at some time t inside a bus interval $[L, U)]$. By the Random Incidence Paradox (and because time-reversed Poisson processes are Poisson processes as well), the time $t-L$ is exponentially distributed with λ and the time $U-t$ is exponentially distibuted with λ. Let A be the length of $t-L$ and let B be the time $U-t$. Let their sum be $X = A+B$. So the distribution of our total time interval is now:

$$f_X(x) = \int_{b=0}^{x} f_A(x-b) f_B(b) db = \int_{b=0}^{x} [\lambda e^{-\lambda(x-b)}][\lambda e^{-\lambda b}]db$$

$$= \lambda^2 \int_{b=0}^{x} e^{-\lambda x} db = \lambda^2 x e^{-\lambda x}$$

Now we can integrate like part (b) to find the distribution of the number of students:

$$P(\text{n students on next bus}) = \int_{x=0}^{\infty} P(\text{x time inbetween})P(\text{n students on next bus given x time inbetween})dx$$

$$= \int_{x=0}^{\infty} [\lambda^2 x e^{-\lambda x}][\frac{(\mu x)^n}{n!} e^{-\mu x}]dx$$

$$= \frac{\lambda^2\mu^n}{n!}\int_{x=0}^{\infty} e^{(-\lambda-\mu)x}x^{n+1}$$
$$= \frac{\lambda^2\mu^n}{n!}[(n+1)!(\lambda+\mu)^{-n-2}]$$
$$= \lambda^2 n\mu^n(\lambda+\mu)^{-n-2}$$

3. Consider a Poisson process $\{N_t, t \geq 0\}$ with rate $\lambda = 1$. Let random variable S_i denote the time of the i-th arrival.

   (a) Given $S_3 = s$, find the joint distribution of S_1 and S_2.

   For all future notation, note that $S_n = \sum_{i=1}^{n} X_i$, where X_i are the independent exponentially distributed interarrival times.
   As discussed in Bertsekas, the distribution of the nth arrival time is the Erlang Distribution:

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}e^{-\lambda t}}{(n-1)!} = \frac{t^{n-1}e^{-t}}{(n-1)!}$$

   Now note that $f_{X_1,S_2}(x_1,s_2) = f_{X_1}(x_1)f_{X_2}(s_2 - x_1) = e^{-x_1}e^{-s_2+x_1} = e^{-s_2}$, since X_i are i.i.d. This does **not** depend on x_1 at all. If we continue with this pattern, we note that: $f_{X_1,X_2,S_3}(x_1,x_2,s_3) = e^{-s_3}$. With this information we can solve the problem:c

$$f_{S_1,S_2|S_3=s}(s_1,s_2) = \frac{f_{S_1,S_2,S_3}(s_1,s_2,s)}{f_{S_3}(s)}$$
$$= \frac{e^{-s}}{\frac{s^2e^{-s}}{2!}} = \frac{2}{s^2}$$

   (b) Find $E[S_2|S_3 = s]$.

   First of all:

$$P(S_2 = s_2|S_3 = s) = \frac{P(S_2 = s_2, S_3 = s)}{P(S_3 = s)}$$

   Looking at the R.H.S:

$$P(S_2 = s_2, S_3 = s) = P(S_2 = s_2)P(X_3 = s - s_2) = (s_2e^{-s_2})(e^{-s+s_2}) = s_2e^{-s}$$

   We already solved for $P(S_3 = s)$ in part (a), so we have:

$$P(S_2 = s_2|S_3 = s) = \frac{s_2e^{-s}}{\frac{s^2e^{-s}}{2}} = \frac{2s_2}{s^2}$$

   Integrating for expectation, we get:

$$E[S_2|S_3 = s] = \int_{s_2=0}^{s} s_2\frac{2s_2}{s^2}ds_2 = \frac{2}{s^2}\int_{s_2=0}^{s} s_2^2 ds_2$$
$$= \frac{2}{s^2}(\frac{s^3}{3}) = \frac{2}{3}s$$

   (c) Find $E[S_3|N_1 = 2]$.

Since $N_1 = 2$, we know that at time $t = 1$, we've landed somewhere between the 2nd and the 3rd. Because of the "memorylessness", we've reset at time $t = 1$, so it's as if we just had the 2nd arrival at $t = 1$, even though that's not necessarily the case. So then, the interval between $t = 1$ and S_3 is exponentially distributed with $\lambda = 1$, meaning that the expected time in the interval is $\frac{1}{\lambda} = 1$. Thus:

$$E[S_3|N_1 = 2] = 1 + 1 = 2$$

4. Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drie by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for τ units of time. Let N be the number of police cars you see before you make a U-turn.

(a) Find $E[N]$.

Notice this is just a Bernoulli Trial, and we want the number of failures before a first success, making this a Geometric distribution. We know the expected number of trials total in a Geometric distribution is $\frac{1}{p}$. p in this case is the probability that a time interval between cop cars is greater than τ. So:

$$p = 1 - (1 - e^{-\lambda\tau}) = e^{-\lambda\tau}$$

Then:

$$E[N] = \frac{1}{e^{-\lambda\tau}} - 1 = e^{\lambda\tau} - 1$$

(b) Find the conditional expectation of the time elapsed between police cars $n - 1$ and n, given that $N \geq n$.

We want to know $E[X_n|N \geq n]$. Remember that if $N \geq n$, then n is "too short" and is less than τ. So we want to find $E[X_n|X_n < \tau]$. From total expectation we have:

$$E[X_n] = E[X_n|X_n < \tau]P(X_n < \tau) + E[X_n|X_n \geq \tau]P(X_n \geq \tau)$$

$E[X_n]$ is $\frac{1}{\lambda}$, since it's exponentially distributed. And from the "memorylessness" property of the exponential distribution we know that $E[X_n|X_n > \tau] = \tau + \frac{1}{\lambda}$, since how long we waited doesn't determine how much more we have to wait. So now we have:

$$\frac{1}{\lambda} = E[T_n|T_n < \tau](1 - e^{-\lambda\tau}) + (\tau + \frac{1}{\lambda})(e^{-\lambda\tau})$$

$$E[T_n|T_n < \tau] = \frac{\frac{1}{\lambda} - (\tau + \frac{1}{\lambda})(e^{-\lambda\tau})}{1 - e^{-\lambda\tau}}$$

(c) Find the expected time that you wait until you make a U-turn.

Call the amount of time I have to wait X. Then:

$$X = X_1 + X_2 + \cdots + X_N + \tau$$

$$E[X] = \tau + \sum_{n=0}^{\infty} E[X_1 + \cdots + X_n|N = n]P(N = n) = \tau + \sum_{n=0}^{\infty} P(N = n)(nE[X_n|X_n < \tau])$$

Remember that $E[T_n|T_n < \tau]$ doesn't depend on n. So we can rearrange this to:

$$E[X] = \tau + E[T_n|T_n < \tau]\sum_{n=0}^{\infty} P(N = n)n = \tau + E[T_n|T_n < \tau]E[N]$$

Combining with our previous answers:

$$E[X] = \tau + \frac{\frac{1}{\lambda} - (\tau + \frac{1}{\lambda})(e^{-\lambda\tau})}{1 - e^{-\lambda\tau}}(e^{\lambda\tau} - 1)$$

5. Team A and Team B are playing an untimed basketball game in which the two team score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team. Find the probability that A wins.

   Notice that this the Gambler's ruin problem from Bertsekas. Suppose we start with k points. At each point scored, if A scores, we get $+1$ points, and if B scores, we get -1 points. We win if we reach $2k$, and we lose if we reach 0. Let A be the probability of winning, let w_k be the probability of winning from k points, and let F denote the event that we win the very next point. Then:

$$P(A) = w_k$$

$$w_k = P(A|F)P(F) + P(A|F^C)P(F^C)$$

   Now we need $P(F)$, the probability of Team A scoring the first point. As noted in the book, since the merging of Poisson processes are Poisson processes themselves, we know the probability that Team A scoring first is $\frac{\lambda_A}{\lambda_A+\lambda_B}$. So we have:

$$w_k = w_{k+1}\frac{\lambda_A}{\lambda_A + \lambda_B} + w_{k-1}\frac{\lambda_B}{\lambda_A + \lambda_B}$$

   Solving the recurrence, we get:

$$P(A) = w_k = \frac{1 - (\frac{\lambda_A}{\lambda_B})^k}{1 - (\frac{\lambda_A}{\lambda_B})^{2k}}$$

   In the special case of $\lambda_A = \lambda_B$:

$$P(A) = \frac{1}{2}$$