Math 104 **Spring 2016**

Assignment #4

Nikhil Unni

9.1 Using the limit theorems 9.2-9.7, prove the following. Justify all steps.

(a) $\lim \frac{n+1}{n} = 1$

From Theorem 9.4, the limit of products are the products of the limits:

$$= \frac{\lim n + 1}{\lim n}$$

$$= \frac{n^{-1} \lim n + 1}{n^{-1} \lim n}$$

From Theorem 9.2, product of a limit and nonlimit is the limit of the product of the two:

$$= \frac{\lim 1 + \frac{1}{n}}{\lim \frac{n}{n}}$$

From Theorem 9.3, the limit of a sum is the sum of the limits:

$$= \frac{\lim 1 + \lim \frac{1}{n}}{\lim 1}$$

From Theorem 9.7, we know that $\lim \frac{1}{n^1} = 0$, since $1 > 0$.

$$= \frac{1+0}{1} = 1$$

(b) $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$

Following the same steps as part a:

From Theorem 9.4:

$$= \frac{\lim 3n + 7}{\lim 6n - 5}$$

$$= \frac{n^{-1} \lim 3n + 7}{n^{-1} \lim 6n - 5}$$

From Theorem 9.2:

$$= \frac{\lim 3 + \frac{7}{n}}{\lim 6 - \frac{5}{n}}$$

From Theorem 9.3:

$$= \frac{\lim 3 + \lim \frac{7}{n}}{\lim 6 - \lim \frac{5}{n}}$$

From Theorem 9.7, we know that $\lim \frac{1}{n^1} = 0$, since $1 > 0$. We also know from Theorem 9.2 that $\lim 7\frac{1}{n} = 7 * 0 = \lim 5\frac{1}{n} = 5 * 0 = 0$. So:

$$= \frac{3+0}{6-0} = \frac{1}{2}$$

(c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

From Theorem 9.4:

$$= \frac{\lim 17n^5 + 73n^4 - 18n^2 + 3}{\lim 23n^5 + 13n^3}$$

$$= \frac{n^{-5} \lim 17n^5 + 73n^4 - 18n^2 + 3}{n^{-5} \lim 23n^5 + 13n^3}$$

From Theorem 9.2:

$$= \frac{\lim 17 + \frac{73}{n} - \frac{18}{n^3} + \frac{3}{n^5}}{\lim 23 + \frac{13}{n^2}}$$

From Theorem 9.3:

$$= \frac{\lim 17 + \lim \frac{73}{n} - \lim \frac{18}{n^3} + \lim \frac{3}{n^5}}{\lim 23 + \lim \frac{13}{n^2}}$$

From Theorem 9.7, all of those ($\lim \frac{1}{n^p}, p > 0$) evaluate to 0. And we can factor out the constants via Theorem 9.2. Evaluating all of the constant limits, we get:

$$= \frac{17 + 0 - 0 + 0}{23 + 0} = \frac{17}{23}$$

9.3 Suppose $\lim a_n = a, \lim b_n = b$, and $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

From Theorem 9.4, we know we can separate the limits of the numerator and denominator:

$$= \lim s_n = \frac{\lim a_n^3 + 4a_n}{\lim b_n^2 + 1}$$

From Theorem 9.3, we can separate the limit of a sum into the sum of limits:

$$= \frac{\lim a_n^3 + \lim 4a_n}{\lim b_n^2 + \lim 1}$$

From Theorem 9.4 again, we know we can separate the products:

$$= \frac{\lim a_n * \lim a_n * \lim a_n + \lim 4 * \lim a_n}{\lim b_n * \lim b_n + \lim 1}$$

Finally, we can start evaluating the limits. We know that $\lim a_n = a, \lim b_n = b$, and the limit of a constant is just the constant. So we finally get:

$$= \frac{a * a * a + 4a}{b * b + 1}$$

$$= \frac{a^3 + 4a}{b^2 + 1}$$

9.4 Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

$$s_1 = 1$$
$$s_2 = \sqrt{2}$$
$$s_3 = \sqrt{\sqrt{2}+1}$$
$$s_4 = \sqrt{\sqrt{\sqrt{2}+1}+1}$$

(b) It turns out that (s_n) converges. Assume this fact and prove the limit is $\frac{1}{2}(1+\sqrt{5})$.

If (s_n) converges, then as $n \to \infty$, $s_{n+1} - s_n \to 0$. So for an arbitrarily high n, we can say that $s_{n+1} = s_n = x$, where x is just a temporarily variable. Then, from our recurrence relation we have:

$$x = \sqrt{x+1}$$
$$x^2 = x + 1$$
$$x^2 - x - 1 = 0$$

Solving with the quadratic formula, we get:

$$x = \frac{1}{2}(1 \pm \sqrt{5})$$

But we know that since $s_1 > 0$, and each following term is generated by a sqrt, which only generates positive values, we know that $\frac{1}{2}(1-\sqrt{5})$ cannot possibly be a value, since it's negative ($5 > 1$, so $\sqrt{5} > \sqrt{1}$). So then, the value of s_{n+1} as $n \to \infty$ approaches $\frac{1}{2}(1+\sqrt{5})$.

9.12 Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$.

If the ratio of subsequent terms is approaching some $|L| < 1$, then for an aribtrarily high value of n, we know that $|s_{x+k}| \to L^k\,|s_x|$, where you can think of s_x as engulfing all the $\frac{s_{n+1}}{s_n}$ values, combined with s_1, until the ratio has converged to the constant L. And since $|L| < 1$, by Theorem 9.7, as k continues to grow (i.e. as n starts to grow, since n = x+k, for some arbitrarily high x), $L^k \to 0$. So the entire $|s_{x+k}| \to 0\,|s_x| = 0$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.

By Theorem 9.5, we know that:

$$\lim \left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{L}$$

Since $L > 1$, $\frac{1}{L} < L$. And now, using $\frac{1}{L}$ as our new "L", by part a, we know $\lim s_{n+1} = 0$. Since the limit of the reciprical of the fraction we are looking for evaluates to 0, we know from Theorem 9.10, that the original fraction evaluates to $+\infty$.

9.14 Let $p > 0$. Use Exercise 9.12 to show:

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & |a| \leq 1 \\ +\infty & a > 1 \\ \text{does not exist} & a < -1 \end{cases}$$

Let $s_n = \frac{a^n}{n^p}$. Then:

$$\frac{s_{n+1}}{s_n} = \frac{a^{n+1}n^p}{a^n(n+1)^p} = \frac{an^p}{(n+1)^p}$$
$$= a(\frac{n}{n+1})^p$$

Finally, evaluating $L = \lim \frac{s_{n+1}}{s_n}$, we get:

$$\lim a(\frac{n}{n+1})^p$$

From 9.2:

$$= a\lim(\frac{n}{n+1})^p$$

From 9.4:

$$= a(\Pi_{i=1}^{p} \lim \frac{n}{n+1})$$

From our proof in 9.1a (altering the steps only slightly, since it's the reciprocal), we know this just:

$$L = a(\Pi_{i=1}^{p} 1) = a$$

So if $|a| \leq 1$, then $L \leq 1$. If $a = 1$ or $a = -1$, from theorems 9.7 and 9.2, $\lim_{n\to\ \infty} \frac{1}{n^p} = 0$, and $\lim_{n\to\ \infty} -1\frac{1}{n^p} = 0$. If $|a| < 1$, then $L < 1$, and from 9.12, we know that $\lim s_n = \lim \frac{a^n}{n^p} = 0$.

Similarly, if $a > 1$, then $L > 1$, and by 9.12, we know that $\lim s_n = +\infty$, since s_n can only be a positive value.

If $a < -1$, then $L < -1$, meaning that consequent values of a_n will be alternating signs, and increasing in value with each iteration. Because the values of s_n get larger and larger in magnitude, while also alternating sign, we know that the limit will not converge to a value, and it will also not converge to either $+\infty$ or $-\infty$ since the sign flips with every iteration. Thus, the limit does not exist.

9.15 Show $\lim_{n\to\ \infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Let $s_n = \frac{a^n}{n!}$. Then, $L = \frac{s_{n+1}}{s_n} = \frac{a^{n+1}n!}{a^n(n+1)!} = \frac{a}{n+1}$. Using the same theorems/reasoning as problem 9.1 (divide numerator/denominator by n, and evaluate), the limit of L is equivalent to $\frac{\lim \frac{a}{n}}{1+\lim \frac{1}{n}} = \frac{0}{1} = 0$.

Since $L = 0$, using the proof from 9.12, we know that $\lim \frac{a^n}{n!} = \lim s_n = 0$.

9.17 Give a formal proof that $\lim n^2 = +\infty$ using only Definition 9.8.

For some arbitrary $M > 0$, if $n^2 > M$, then $n > \sqrt{M}$. So we can pick $N = \sqrt{M}$, for any value M. This way, for each $M > 0$, there is a number $N = \sqrt{M}$, so that both $n > N$ and $n^2 > M$. So by definition 9.8, $\lim n^2 = +\infty$.

9.18 (a) Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

For $n = 0$:

$$\frac{1 - a^{0+1}}{1 - a}$$
$$= \frac{1 - a}{1 - a} = 1$$

Assume that the equality holds for some n. Then we can show that it holds for $n + 1$:

$$1 + a + a^2 + \cdots + a^n + a^{n+1}$$
$$= (\frac{1 - a^{n+1}}{1 - a}) + a^{n+1}$$
$$= \frac{1 - a^{n+1} + a^{n+1}(1 - a)}{1 - a}$$
$$= \frac{1 - a^{n+2}}{1 - a}$$

Since the equality holds for $n = 0$, and we can show that it holds for $n + 1$ given it holds for n, by induction, the equality holds for all $n \geq 0$.

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

$$\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$$

From part a:

$$= \lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - a}$$

From Theorem 9.7, we know that $\lim_{n\to\infty} a^n = 0$. So we get:

$$= \lim_{n\to\infty} \frac{1 - 0a}{1 - a} = \frac{1}{1 - a}$$

(c) Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \cdots + \frac{1}{3^n})$.

Let $a = \frac{1}{3}$. Then, the sequence becomes:

$$= \lim_{n\to\infty}(1 + a + \cdots a^n)$$

Since $\left|\frac{1}{3}\right| < 1$, we've already solved this problem in part b. The result of the limit is:

$$\frac{1}{1 - a} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

(d) What is $\lim_{n\to\infty}(1 + a + \cdots + a^n)$ for $a \geq 1$?

From part a, we know this limit is equivalent to:

$$\lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - a}$$

From definition 9.8: for any $M > 0$, if $a^{n+1} > M$, then $n > log_a(M) - 1$. So for any M, we can pick $N = log_a(M)$. Since the log base $a \geq 1$, we know that N is positive. Thus, $n > N$ implies $a^{n+1} > M$. So $\lim_{n\to\infty} a^{n+1} = +\infty$ for $a \geq 1$.

Then, the limit of the numerator evaluates to $-\infty$, and the limit of the denominator evalutes to $1 - a$. So from theorem 9.9, we know that the limit evaluates to ∞, since the denominator is a negative value (since $a > 1$).

10.1 Which of the following sequences are increasing? decreasing? bounded?

(a) $\frac{1}{n}$: **Both bounded and decreasing**

(b) $\frac{(-1)^n}{n^2}$: **Bounded**

(c) n^5 : **Increasing**

(d) $\sin(\frac{n\pi}{7})$: **Bounded**

(e) $(-2)^n$: **None of the choices**

(f) $\frac{n}{3^n}$: **Both bounded and decreasing**