Math 104 **Spring 2016**

Assignment #12

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23.2 Repeat Exercise 23.1 for the following:

(a) $\sum \sqrt{n}x^n$

We know $|a_n| = \sqrt{n}$, so $|a_n|^{1/n} = n^{1/2n}$. We can simplify (in a manner of speaking) like:

$$y = n^{1/2n} \implies \ln(y) = \frac{1}{2n}\ln(n) \implies y = e^{\frac{\ln n}{2n}}$$

Then:

$$\lim_{n\to\ \infty} e^{\frac{\ln n}{2n}} = e^0 = 1$$

So since $\lim |a_n|^{1/n}$ exists, we know that the lim sup must be 1 as well, meaning $\beta = 1$ and $R = 1$. So we just need to test whether $x = -1$ and $x = 1$ converge or not.

For $x = 1$, we have $\sum \sqrt{n}$, and for $x = -1$, we have $\sum \sqrt{2n}$ which both clearly diverge, since the square root function is increasing and unbounded. So our interval of convergence is $(-1, 1)$.

(b) $\sum \frac{1}{n\sqrt{n}}x^n$

Using the same trick as part a, we have:

$$|a_n|^{1/n} = n^{-\sqrt{n}/n} = e^{\frac{-\sqrt{n}\ln n}{n}}$$

Taking the limit, we see that the exponent goes to 0, so we have

$$\lim |a_n|^{1/n} = \limsup |a_n|^{1/n} = 1$$

So then $\beta = 1$ and $R = 1$. So we just need to test whether $x = -1$ and $x = 1$ converge or not: For $x = 1$, we have $\sum \frac{1}{n\sqrt{n}}$, which converges, since $\sqrt{n} \geq 1$ for all n. And for $x = -1$, by the alternating series test, we see it converges as well. So we have an interval of convergence of $[-1, 1]$.

(c) $\sum x^{n!}$

For $|x| \geq 1$, the series clearly diverges, since the individual terms are monotonically increasing with n. But for $|x| < 1$, we know that the series will always be less than $\sum_{m=1}^{\infty} x^m$, since the terms of the former are a subset of the terms of the latter (the set of all factorial numbers is a subset of the natural numbers). So we have a radius of convergence of 1, and the interval of convergence is $(-1, 1)$.

(d) $\sum \frac{3^n}{\sqrt{n}}x^{2n+1}$

$$\beta = \lim |a_n|^{1/n} = \lim(\frac{3^n}{\sqrt{n}})^{\frac{1}{2n+1}}$$

$$= \lim \frac{3^{\frac{n}{2n+1}}}{n^{\frac{1}{4n+2}}}$$

The numerator's exponent tends to $\frac{1}{2}$, and the denominator's exponent tends to 0, so we end up with the limit being $\frac{\sqrt{3}}{1}$. So then $R = \frac{1}{\sqrt{3}}$. At $x = \frac{1}{\sqrt{3}}$, we end up with:

$$\sum \frac{3^n}{\sqrt{n}}\left(\frac{1}{3^{n+0.5}}\right) = \sum \frac{1}{\sqrt{3n}}$$

which we know diverges. And at $x = -\frac{1}{\sqrt{3}}$, we end up with:

$$\sum -\frac{1}{\sqrt{3n}}$$

which we also know diverges. So our interval of convergence becomes $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

23.5 Consider a power series $\sum a_n x^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many of them are nonzero, then $R \leq 1$.

Since all a_n are integers, then there are an infinite number of $|a_{n>N}| \geq 1$, for any finite $N \in \mathbb{N}$. So it follows that $\limsup |a_n|^{1/n} \geq 1$, so then $R \leq 1$.

(b) Prove that if $\limsup |a_n| > 0$, then $R \leq 1$.

Let c be some real number such that $0 < c < \limsup |a_n|$. We know that there must exist some subsequence a_{n_k} converging to some real number greater than c. So there must be some K such that $k > K \implies |a_{n_k}| \geq c$. Then:

$$|a_{n_k}|^{1/k} \geq c^{1/k}$$
$$\limsup |a_{n_k}|^{1/k} > \limsup c^{1/k} = 1$$

Since $\{a_{n_k}\}, k \in \mathbb{N}$ is a subset of $\{a_n\}, n \in \mathbb{N}$, we know that:

$$|a_n|^{1/n} \geq 1$$

And so we have that $R \leq 1$.

23.7 For each $n \in \mathbb{N}$, let $f_n(x) = (\cos x)^n$. Each f_n is a continuous function. Nevertheless, show

(a) $\lim f_n(x) = 0$ unless x is a multiple of π

If x is not a multiple of π, then we know that $0 \leq \cos x < 1$. So then $\lim_{n\to\infty} (\cos x)^n = 0$.

(b) $\lim f_n(x) = 1$ if x is an even multiple of π

If x is an even multiple of π, then $\cos x = 1$, so then $f_n(x) = 1^n = 1$, and a sequence of 1 will clearly have a limit of 1.

(c) $\lim f_n(x)$ does not exist if x is an odd multiple of π

If x is an odd multiple of π, then $\cos x = -1$, so our sequence becomes $f_n(x) = (-1)^n$, which is an alternating series that does not converge, so the limit does not exist.

23.9 Let $f_n(x) = nx^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Show

(a) $\lim f_n(x) = 0$ for $x \in [0, 1)$.

Since $cx < c$ for any $x \in [0, 1)$ and $c > 0$, we know that the limit of x^n must be 0. And since exponents grow faster than linear terms, we know that the limit of nx^n must be 0 for any $x \in [0, 1)$.

(b) However, $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$

$$\int_0^1 f_n(x)dx = \int_0^1 nx^n dx = n[\frac{x^{n+1}}{n+1}]_0^1 = \frac{n}{n+1}$$

And:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

24.2 For $x \in [0, \infty)$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.

Since $\frac{1}{n}$ converges to 0, we know that any constant multiple $\frac{x}{n}$ will converge to 0. So:

$$f(x) = \lim f_n(x) = 0$$

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$

Let $\epsilon > 0$. If we pick $N = \lceil \frac{1}{\epsilon} \rceil$, then

$$n > N \implies |f_n(x) - 0| = \frac{x}{n} < \epsilon$$

And this is because $n\epsilon > \frac{1}{\epsilon}\epsilon = 1$, and all $x \leq 1$. So then we know f_n uniformly congerges to 0 on $[0, 1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

We can prove that f_n does not uniformly converge to 0 on $[0, \infty)$. Let $\epsilon = 1$. Then, for all N, if we choose $x = n$ for any $n > N$:

$$|f_n(x) - 0| = \frac{n}{x} = 1 \geq 1$$

So f_n does not converge uniformly to 0.

24.11 Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}$.

(a) Observe $f_n \to f$ uniformly on $\mathbb{R}$ [obvious!] and $g_n \to g$ uniformly on $\mathbb{R}$ [almost obvious].

For any $\epsilon > 0$, we can choose $N = 0$, and then $|f_n(x) - f(x)| = 0 < \epsilon$ for all $x \in \mathbb{R}$, and for all $n > N$.

For any $\epsilon > 0$, if we choose $N = \lceil \frac{1}{\epsilon} \rceil$, then $|g_n(x) - g(x)| = \frac{1}{n} < \epsilon$, since $n\epsilon > \frac{1}{\epsilon}\epsilon = 1$. And this is true for all x, since the inequality doesn't depend on x.

(b) Observe the sequence $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. Compare Exercise 24.2.

$(f_n g_n)(x) = \frac{x}{n}$, and $fg(x) = 0$. We showed in 24.2 that this function does not converge uniformly on $[0, \infty)$. It's clear that the function does not conform uniformly on $\mathbb{R}$ either, since the set of all $|f_n g_n(x)|$ over $\mathbb{R}$ is the same set as $\left|\frac{x}{n}\right|$ over $[0, \infty)$. So it is the exact same proof as 24.2, with $\epsilon = 1$ and $x = n$, for all $n > N$, for any $N \in \mathbb{N}$.

24.13 Prove that if (f_n) is a sequence of uniformly continuous functions on an interval (a, b) and if $f_n \to f$ uniformly on (a, b), then f is uniformly continuous on (a, b). Hint : Try an $\frac{\epsilon}{3}$ argument as in the proof of Theorem 24.3.

Using the same $\frac{\epsilon}{3}$ argument as 24.3, we note that:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

Then, since $f_n \to f$ uniformly, then there must exist some N such that:

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in (a, b), n > N$. Similarly, since all (f_n) are uniformly continuous, for a given n, there must exist some $\delta_n > 0$ such that:

$$|x - y| < \delta_n \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$$

Finally, let $\epsilon > 0$. Then there must exist a sequence of (δ_n) values and a number N, such that $\delta = \min(\delta_{n>N})$ implies:

$$|f(x) - f(y)| \leq |f_{n>N}(x) - f(x)| + |f_{n>N}(x) - f_{n>N}(y)| + |f_{n>N}(y) - f(y)| < \epsilon$$