Math 104	**Spring 2016**

Assignment #12

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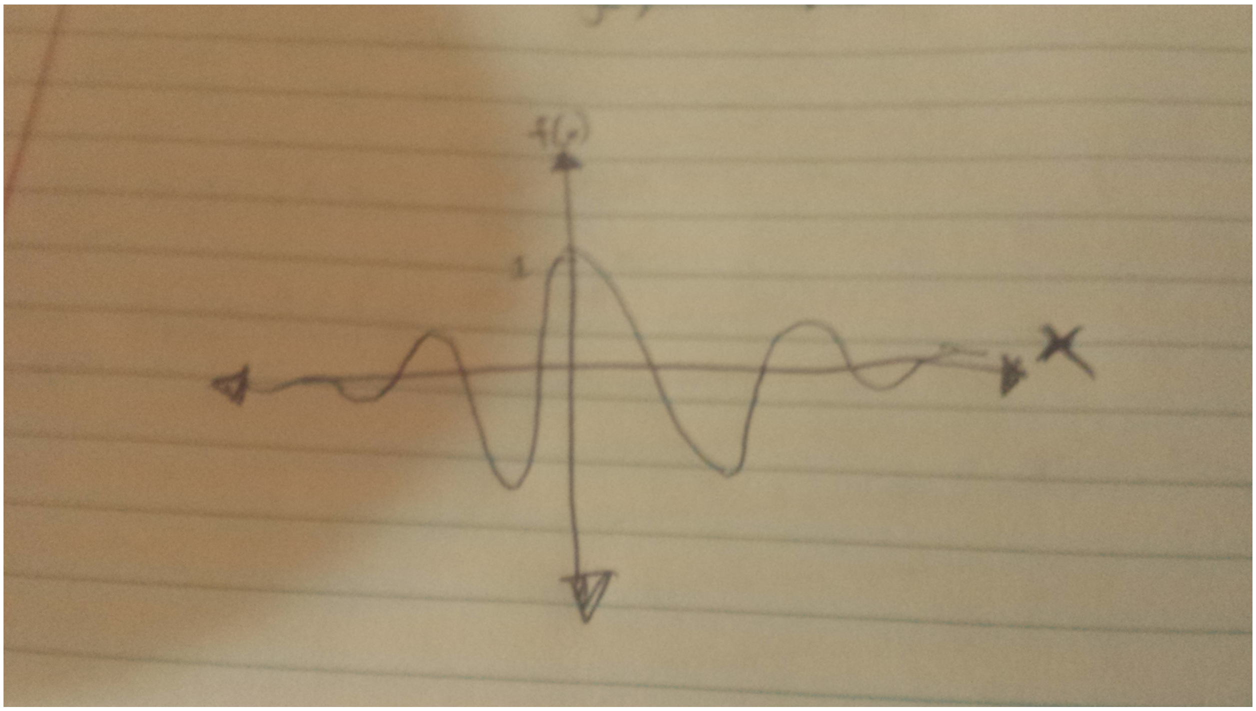
20.3 Repeat Exercise 20.1 for $f(x) = \frac{\sin x}{x}$. See Example 9 section 19.



$$\lim_{x \to \infty} f(x) = 0$$
$$\lim_{x \to 0^+} f(x) = 1$$
$$\lim_{x \to 0^-} f(x) = 1$$
$$\lim_{x \to -\infty} f(x) = 0$$
$$\lim_{x \to 0} f(x) = 1$$

20.7 Prove the limit assertions in Exercise 20.3.

Let (x_n) be some sequence in $(0, \infty)$ with limit $+\infty$. Since $\lim(\frac{1}{x_n}) = 0$, and since $\sin x$ is bounded between -1 and 1, we know that $\lim(\frac{\sin x}{x_n}) = 0$. The same is clearly true for some sequence in $(-\infty, 0)$.

We can prove that $\lim_{x \to 0} f(x) = 1$ with L'Hospital's rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

And from Theorem 20.10, we know that since $\lim_{x \to 0} f(x)$ exists, then it must be equal to both $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$, which both must be 1.

20.16 Suppose the limits $L_1 = \lim_{x \to a^+} f_1(x)$ and $L_2 = \lim_{x \to a^+} f_2(x)$ exist.

(a) Show if $f_1(x) \leq f_2(x)$ for all x in some interval (a, b), then $L_1 \leq L_2$.

Suppose that $L_1 > L_2$. Let $\epsilon = \frac{L_1 - L_2}{2}$. Since L_1 exists we know that there must exist some δ_1 such that:

$$0 < x < a + \delta_1 < b \implies |f_1(x) - L_1| < \epsilon \implies f_1(x) > L_1 - \epsilon \implies f_1(x) > \frac{L_1 + L_2}{2}$$

Similarly, there must exist some δ_2 such that:

$$0 < x < a + \delta_2 < b \implies |f_2(x) - L_2| < \epsilon \implies f_2(x) < L_2 + \epsilon \implies f_2(x) < \frac{L_1 + L_2}{2}$$

So then we know that, for some $\delta = \min\{\delta_1, \delta_2\}$ that:

$$a < x < a + \delta < b \implies f_2(x) < \frac{L_1 + L_2}{2} < f_1(x)$$

Or that $f_2(x) < f_1(x)$, which is a contradiction.

So we know that $L_1 \leq L_2$.

(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all x in some interval (a, b). Can you conclude that $L_1 < L_2$?

No. Take for example $f_1(x) = x, f_2(x) = 2x$ with the interval $(0, 1)$. In this case, $L_1 = L_2 = 0$.

20.17 Show that if $\lim_{x \to a^+} f_1(x) = \lim_{x \to a^+} f_3(x) = L$ and if $f_1(x) \leq f_2(x) \leq f_3(x)$ for all x in some interval (a, b), then $\lim_{x \to a^+} f_2(x) = L$. This is called the squeeze lemma. (Warning : This is not immediate from Exercise 20.16(a).)

Let $\epsilon > 0$. Since $\lim_{x \to a^+} f_1(x) = \lim_{x \to a^+} f_3(x) = L$, by 20.8, we know there must exist some δ_1, δ_3 such that:

$$0 < x < a + \delta_1 \implies |f_1(x) - L| < \epsilon \implies L - \epsilon \leq f_1(x) \leq L + \epsilon$$
$$0 < x < a + \delta_3 \implies |f_3(x) - L| < \epsilon \implies L - \epsilon \leq f_3(x) \leq L + \epsilon$$

Let $\delta = \min\{\delta_1, \delta_3\}$. Then we know that:

$$0 < x < a + \delta \implies L - \epsilon \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq L + \epsilon$$

So by 20.8, we know that $\lim_{x \to a^+} f_2(x) = L$.

20.19 The limits defined in Definition 20.3 do not depend on the choice of the set S. As an example, consider $a < b_1 < b_2$ and suppose f is defined on (a, b_2). Show that if the limit $\lim_{x \to a^+} f(x)$ exists for either $S = (a, b_1)$ or $S = (a, b_2)$, then the limit exists for the other choice of S and these limits are identical. Their common value is what we write as $\lim_{x \to a^+} f(x)$.

Say that $\lim_{x \to a^{S_2}} f(x) = L$ for $S_2 = (a, b_2)$. Since every sequence in $S_1 = (a, b_1)$ is in S_2, (since the former is a subset of the latter,) from Definition 20.1, we get for free that $\lim_{x \to a^{S_1}} = L$.

Now suppose that $\lim_{x \to a^{S_1}} f(x) = L$ for $S_1 = (a, b_1)$. Let (x_n) be an arbitrary sequence in $S_2 = (a, b_2)$ with limit a. We know that for some N, $n > N \implies x_n < b_1$. So the sequence $(y_n) = (x_{n>N})$ is in S_1, and so $\lim f(y_n) = L$. Since the limit of a sequence starting from a finite n is the same as the limit of a sequence starting at 1, $\lim_{n \to \infty} f(x_n) = L$, and so $\lim_{x \to a^{S_2}} = L$

21.5 Let E be a noncompact subset of $\mathbb{R}^k$.

(a) Show there is an unbounded continuous real-valued function on E.

From Heine-Borel we know that E is either not closed or unbounded.

If E is unbounded, then we can construct $f : \mathbb{R}^k \to \mathbb{R}^k$, where $f(x) = x$ (identity function). Since E is unbounded, clearly f must be an unbounded function on E.

If E is not closed, then its complement is not open, and its closure E^- must contain some x_0 that is not in E. As noted in the hint, we can construct a function $f : \mathbb{R}^k \to \mathbb{R}$ where:

$$f(x) = \frac{1}{d(x, x_0)}$$

(b) Show there is a bounded continuous real-valued function on E that does not assume its maximum on E.

The function : $g(x) = \frac{|f(x)|}{1+|f(x)|}$ has supremum of 1, but cannot possibly achieve it in E.

21.6 For metric spaces (S_1, d_1), (S_2, d_2), (S_3, d_3), prove that if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are continuous, then $g \circ f$ is continuous from S_1 into S_3.

If g is continuous, $\forall s_0 \in S_2, \forall \epsilon_g > 0, \exists \delta_g > 0$ such that:

$$d_2(s, s_0) < \delta_g \implies d_3(g(s), g(s_0)) < \epsilon_g$$

And if g is continuous, for $\epsilon_f = \delta_g, \forall s_1 \in S_1, \exists \delta_f > 0$ such that:

$$d_1(s, s_1) < \delta_f \implies d_2(f(s), f(s_1)) < \delta_g$$

So then, for some $\epsilon_g > 0$, there exists some δ_f, δ_g such that $d_1(s, s_1) < \delta_f \implies d_2(f(s), f(s_1)) < \delta_g \implies d_3(g(s), g(s_0)) < \epsilon_g$, meaning that $f \circ g$ is continuous by definition.

21.8 Let (S, d) and (S^*, d^*) be metric spaces. Show that if $f : S \to S^*$ is uniformly continuous, and if (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence in S^*.

Let $\epsilon > 0$. Since f is uniformly continuous, there exists some $\delta > 0$ such that $s, t \in S$ and $d(s, t)$ imply $d^*(f(s), f(t)) < \epsilon$.

And since (s_n) is a Cauchy sequence, there must exists some N such that $m, n > N \implies d(s_m, s_n) < \delta \implies d^*(f(s_m), f(s_n)) < \epsilon$. And so by the (metric space) definition of a Cauchy sequence, $(f(s_n))$ is Cauchy.