EE 126	HW11	**Spring 2016**

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1. If $X = 0, Y = U[-1, 1]$ and if $X = 1, Y = U[0, 2]$. Solve a hypothesis testing problem so that the probability of false alarm is less than or equal to β.

   First we can find the likelihood function:

$$L(-1 \leq y < 0) = \frac{0}{1} = 0$$

$$L(0 \leq y \leq 1) = \frac{1}{1} = 1$$

$$L(1 \leq y \leq 2) = \infty$$

   We want to pick a λ, γ such that $P(L(y) = \lambda | X = 0)(\gamma) = \beta$. Since we're conditioned on $X = 0$, $L(y)$ cannot be ∞. If we pick $\lambda = 0$ (or $\lambda = 1$, really), we get:

$$P(L(y) = \lambda | X = 0)(\gamma) = \beta$$

$$\frac{1}{2}(\gamma) = \beta$$

$$\gamma = 2\beta$$

2. The random variables X, Y, Z are i.i.d. $N(0, 1)$.

   First let's find a few probabilities:

$$E[X] = E[Y] = E[Z] = 0$$

$$E[X^2] = E[Y^2] = E[Z^2] = \mu^2 + \sigma^2 = 1$$

$$E[X^3] = E[Y^3] = E[Z^3] = \mu(\mu^2 + 3\sigma^2) = 0$$

   (a) Find $L[X^2 + Y^2 | X + Y]$

   From the definition of LLSE, we know:

$$L[X^2 + Y^2 | X + Y] = E[X^2 + Y^2] + \frac{\text{cov}(X^2 + Y^2, X + Y)}{\text{var}(X + Y)}(X + Y - E[X + Y])$$

   And $\text{cov}(X^2 + Y^2, X + Y) = E[(X^2 + Y^2)(X + Y)] - E[X^2 + Y^2]E[X + Y]$. The second expression is 0, since $E[X] = E[Y] = 0$, and so then we have:

$$\text{cov}(X^2 + Y^2, X + Y) = E[(X^2 + Y^2)(X + Y)] = E[X^3] + E[X^2Y] + E[XY^2] + E[Y^3] = 0 + 0 + 0 + 0$$

   So then we have:

$$L[X^2 + Y^2 | X + Y] = E[X^2] + E[Y^2] = 2$$

(b) Find $L[X + 2Y|X + 3Y + 4Z]$

From the definition of LLSE, we know:

$$L[X+2Y|X+3Y+4Z] = E[X+2Y]+\frac{\text{cov}(X + 2Y, X + 3Y + 4Z)}{\text{var}(X + 3Y + 4Z)}(X+3Y+4Z-E[X+3Y+4Z])$$

And:

$$\text{cov}(X+2Y, X+3Y+4Z) = E[X^2+3XY+4XZ+2XY+6Y^2+8YZ^2]-E[X+2Y]E[X+3Y+4Z] = (1+6)-0 = 7$$

And:

$$\text{var}(X + 3Y + 4Z) = \text{var}(X) + 9\text{var}(Y) + 16\text{var}(Z) = 1 + 9 + 16 = 26$$

So we have:

$$L[X^2 + Y^2|X + Y] = \frac{7}{26}(X + Y)$$

(c) Find $L[(X + Y)^2|X - Y]$

From the definition of LLSE, we know:

$$L[(X + Y)^2|X - Y] = E[(X + Y)^2] + \frac{\text{cov}((X + Y)^2, X - Y)}{\text{var}(X - Y)}(X - Y - E[X - Y])$$

$$= E[(X + Y)^2] + \frac{E[(X + Y)(X^2 - Y^2)] - E[(X + Y)^2]E[X - Y]}{E[(X - Y)^2] - E[X - Y]^2}$$

We first have to find the expectations:

$$E[(X + Y)^2] = E[X^2] + 2E[XY] + E[Y^2] = 1 + 0 + 1 = 2$$

$$E[(X + Y)^2(X - Y)] = E[X^3] + E[X^2Y] - E[XY^2] - E[Y^3] = 0 + 0 + 0 + 0 = 0$$

Since both $E[(X + Y)^2(X - Y)]$ and $E[X - Y]$ are 0, the entire fraction is 0. Meaning we have:

$$L[(X + Y)^2|X - Y] = 2$$

3. Consider a photodetector in an optical communications system that counts the number of photons arriving during a certain interval. A user conveys information by switching a photon transmitter on or off. Assume that the probability of the transmitter being on is p. If the transmitter is on, the number of photons transmitted over the interval of interest is a Poisson random variable θ with mean λ and if it is off, the number of photons transmitted is 0. Unfortunately, regardless of whether or not the transmitter is on or off, photons may be detected due to "shot noise". The number $N$of detected shot noise photons is a Poisson random variable N with mean μ. Given the number of detected photons, find the LLSE of the number of transmitted photons.

Let's call the number of transmitted photons X, and call the number of detected photons Y. Let T be the random variable denoting whether or not the transmitter is on. $T = 1$ with probability p, and $T = 0$ with probability $1 - p$. So then we know:

$$X = T\theta$$

$$Y = X + N$$

We want to find $L[X|Y]$. From our LLSE formula, we know this is:

$$L[X|Y] = E[X] + \frac{\text{cov}(X,Y)}{\text{var}(Y)}(Y - E[Y])$$
$$= E[X] + \frac{E[XY] - E[X]E[Y]}{E[Y^2] - E[Y]^2}(Y - E[Y])$$

Calculating all the necessary probabilities:

$$E[X] = E[T]E[\theta] = p\lambda$$
$$E[Y] = E[X] + E[N] = p\lambda + \mu$$
$$E[X^2] = pE[\theta^2] + (1-p) * 0 = p(\lambda^2 + \lambda)$$
$$E[XN] = E[X]E[N] = p\lambda\mu$$
$$E[Y^2] = E[X^2] + 2E[XN] + E[N^2] = p(\lambda^2 + \lambda) + 2p\lambda\mu + (\mu^2 + \mu)$$
$$E[XY] = E[X^2 + XN] = p(\lambda^2 + \lambda) + 2p\lambda\mu$$

So putting it all together, we get:

$$L[X|Y] = p\lambda + \frac{p\lambda^2 + p\lambda + 2p\lambda\mu - p\lambda(p\lambda + \mu)}{p(\lambda^2 + \lambda) + 2p\lambda\mu + \mu^2 + \mu - p^2\lambda^2 - 2p\lambda\mu - \mu^2}(Y - p\lambda - \mu)$$
$$= p\lambda + \frac{p\lambda^2 + p\lambda + p\lambda\mu - p^2\lambda^2}{p\lambda^2 + p\lambda + \mu - p^2\lambda^2}(Y - p\lambda - \mu)$$

4. Let $(V_n, n \geq 0)$ be i.i.d. $N(0, \sigma^2)$ and independent of $X_0 = N(0, u^2)$. Define

$$X_{n+1} = aX_n + V_n, n \geq 0$$

(a) What is the distribution of X_n for $n \geq 1$?

Looking at the terms, we have:

$$X_0 = N(0, u^2)$$
$$X_1 = N(0, a^2u^2) + N(0, \sigma^2) = N(0, a^2u^2 + \sigma^2)$$
$$\ldots$$

So we can create a recurrence relation on the variance at step n:

$$T(0) = u^2, T(n) = a^2T(n-1) + \sigma^2$$

Solving, we get:

$$T(n) = \frac{a^{2n}(a^2u^2 - u^2 + \sigma^2) - \sigma^2}{a^2 - 1}$$

So we have $X_n \sim N(0, \frac{a^{2n}(a^2u^2 - u^2 + \sigma^2) - \sigma^2}{a^2 - 1})$.

(b) Find $E[X_{n+m}|X_n]$ for $0 \leq n < n + m$.

Given X_n, we know that:

$$X_{n+1} = aX_n + N(0, \sigma^2) = N(aX_n, \sigma^2)$$
$$X_{n+2} = aN(aX_n, \sigma^2) + N(0, \sigma^2) = N(a^2X_n, a^2\sigma^2 + \sigma^2)$$
$$\ldots$$

In general, we know that that $X_{m \geq n}$ will be normally distributed with mean $a^{m-n}X_n$. So then, clearly:

$$E[X_{n+m}|X_n] = a^m X_n$$

(c) Find u so that the distribution of X_n is the same for all $n \geq 0$.

We know that the mean of the X_n are all 0. So we want the variances to be all the same. Looking at the closed formula for the variance of X_n, we have:

$$\frac{a^{2n}(a^2u^2 - u^2 + \sigma^2) - \sigma^2}{a^2 - 1}$$

The only way to make this a constant is to remove the a^{2n} term, by finding a value of u such that:

$$a^2u^2 - u^2 + \sigma^2 = 0$$

Rearranging and solving for u:

$$u^2(a^2 - 1) + \sigma^2 = 0$$
$$u = \frac{\sigma}{\sqrt{1 - a^2}}$$

5. The difficulty of an EE126 exam, θ, is uniformly distributed on $[0, 100]$, and Alice gets a score X that is uniformly distributed on $[0, \theta]$. Alice gets her score back and wants to estimate the difficulty of the exam.

(a) What is the LLSE for θ?

$$\begin{aligned} LLSE[\theta|X] &= E[\theta] + \frac{\text{cov}(X, \theta)}{\text{var}(X)}(X - E[X]) \\ &= E[\theta] + \frac{E[X\theta] - E[X]E[\theta]}{E[X^2] - E[X]^2}(X - E[X]) \end{aligned}$$

First, we have to solve for those expectations:

$$E[X] = E[E[X|\theta]] = E[\frac{\theta}{2}] = 25$$
$$E[\theta^2] = \text{var}(\theta) + E[\theta^2] = \frac{1}{12}(100^2) + 50^2 = \frac{10000}{3}$$
$$E[X\theta] = E[E[X\theta|\theta]] = E[\theta(\frac{\theta}{2})] = \frac{1}{2}E[\theta^2] = \frac{5000}{3}$$
$$E[X^2] = E[E[X^2|\theta]] = E[\frac{1}{12}(\theta)^2 + (\frac{\theta}{2})^2] = \frac{1}{12}E[\theta^2] + \frac{1}{4}E[\theta^2] = \frac{10000}{9}$$

So putting it all together:

$$LLSE[\theta|X] = 50 + \frac{5000/3 - 25 * 50}{10000/9 - 25 * 25}(X - 25) = 50 + \frac{6}{7}(X - 25)$$

(b) What is the MAP of θ?

$$\begin{aligned} MAP[\theta|X] &= \arg\max_{\theta \geq X} P(X|\theta) \\ &= \arg\max_{\theta \geq X} \frac{1}{\theta} \\ &= \arg\min_{\theta \geq X} \theta = X \end{aligned}$$

So the MAP decision would be $\min X, 100$, since θ is upper bounded by 100.

(c) Find the mean squared error of each estimate as a function of the score X.

6. The situation is the same as in the previous problem.

(a) What is the MMSE of θ given X?

$$MMSE[\theta|X] = E[\theta|X] = \int_\theta \theta \frac{f_\theta(\theta) f_{X|\theta}(X|\theta)}{f_X(X)}\, d\theta$$

Finding $f_X(x)$:

$$f_X(x) = \int_\theta P(\theta)P(x|\theta)\, d\theta$$

Since $\theta \geq X$, we change the limits of integration accordingly:

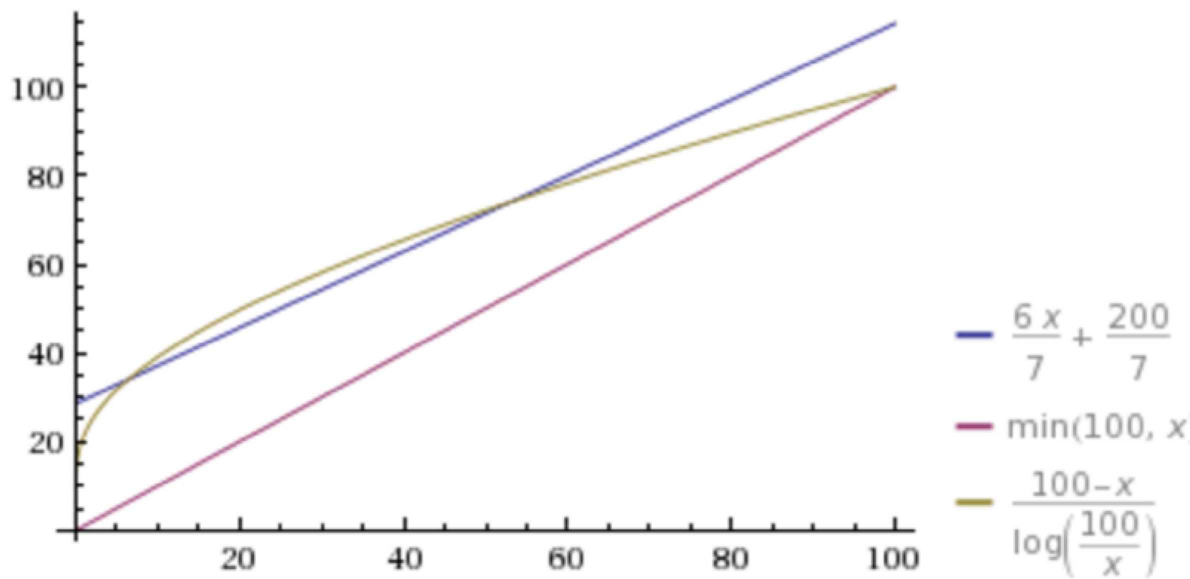
$$f_X(x) = \int_{\theta=X}^{100} \frac{1}{100}\frac{1}{\theta}\, d\theta = \frac{1}{100}(\ln(100/x))$$

So then we have:

$$MMSE[\theta|X] = \int_{\theta=X}^{100} \theta \frac{\frac{1}{100}\frac{1}{\theta}}{\frac{1}{100}\ln(100/x)}\, d\theta$$

$$= \int_{\theta=X}^{100} \frac{1}{\ln(100/x)}\, d\theta = \frac{100-X}{\ln(\frac{100}{X})}$$

(b) Plot the MAP, LLSE, and MMSE as a function of the score X.



(c) Find the mean squared error of the MMSE as a function of the score X. Plot it along with the mean squared error of the MAP and LLSE.

7. Let the joint density of two random variables X and Y be

$$f_{X,Y}(x,y) = \frac{1}{4}(2x+y), 0 \leq x \leq 1, 0 \leq y \leq 2$$

First show that this is a valid joint distribution. Suppose you observe Y drawn from this joint density. Find MMSE$[X|Y]$.

We can show it's a valid distribution by integrating through x and y:

$$\int_{x=0}^{1}\int_{y=0}^{2}\frac{1}{4}(2x+y)\,\mathrm{d}y\,\mathrm{d}x$$

$$=\frac{1}{4}\int_{x=0}^{1}[2xy+\frac{y^2}{2}]_{y=0}^{2}\,\mathrm{d}x=\frac{1}{4}\int_{x=0}^{1}4x+2\,\mathrm{d}x$$

$$=\frac{1}{4}[2x^2+2x]_{x=0}^{1}=\frac{1}{4}(2+2)=1$$

To find $E[X|Y]$, we first must find $f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}$. And:

$$f_Y(y)=\int_{x=0}^{1}\frac{1}{4}(2x+y)dx=\frac{1}{4}[x^2+xy]_{x=0}^{1}=\frac{1}{4}(1+y)$$

And so:

$$f_{X|Y}(x|y)=\frac{\frac{1}{4}(2x+y)}{\frac{1}{4}(1+y)}=\frac{2x+y}{1+y}$$

Finally:

$$E[X|Y]=\int_{x=0}^{1}\frac{2x+y}{1+y}x=\frac{1}{1+y}[\frac{2}{3}x^3+\frac{y}{2}x^2]_{x=0}^{1}=\frac{3y+4}{6y+6}$$

8. Let X,Y,Z be three random variables. Prove formally that

$$E[|X-E[X|Y]|^2]\geq E[|X-E[X|Y,Z]|^2]$$

What is the intuition behind the inequality?

$$E[(X-E[X|Y])^2]\geq E[(X-E[X|Y,Z])^2]$$

Using the proof of Lemma 7.6(a) from Walrand:

$$E[X^2]-2E[X|Y]E[X]+E[X|Y]^2\geq E[X^2]-2E[X|Y,Z]E[X]+E[X|Y,Z]^2$$

$$E[X|Y]^2-2E[X|Y]E[X]\geq E[X|Y,Z]^2-2E[X|Y,Z]E[X]$$

The intution is that, with more information, Z, that our "educated guess" $E[X|Y,Z]$ can only be closer to the actual value of X than our "educated guess" with only the information of Y, $E[X|Y]$. Z may not help getting closer to X, but it cannot possibly move us farther away from the actual value of X.