| **Math 104** | **Spring 2016** |
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# Assignment #5

Nikhil Unni

10.2 Prove Theorem 10.2 for bounded decreasing sequences.

Copying the 10.2 proof almost exactly, we can get a similar proof for decreasing sequences:

Let $(s_n)$ be a bounded decreasing sequence. Let S denote the set $\{s_n : n \in \mathbb{N}\}$, and let $l = \sup S$. Since $S$ is bounded, $l$ represents a real number. We show $\lim s_n = l$. Let $\epsilon > 0$. Since $l + \epsilon$ is not a lower bound for S, there exists $N$ such that $s_N < l - \epsilon$. Since $(s_n)$ is decreasing, we have $s_N \geq s_n$ for all $n \geq N$. Of course, $s_n \geq l$ for all $n$, so $n > N$ implies $l + \epsilon > s_n \geq l$, which implies $|s_n - l| < \epsilon$. This shows $\lim s_n = l$.

10.3 For a decimal expansion $K.d_1d_2d_3\cdots$, let $(s_n)$ be defined as in Discussion 10.3. Prove $s_n < K + 1$ for all $n \in \mathbb{N}$. Hint: $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$ for all $n$.

If $K.d_1d_2d_3\cdots < K + 1$, then, subtracting K from both sides, we get:

$$\frac{d_1}{10} + \frac{d}{10^2} + \cdots \frac{d_n}{10^n} < 1, d_i \in \{0, \cdots, 9\}$$

We can easily show that for any index $i$, if we pick any digit other than 9, the resulting number will be less than the number resulting if we picked 9 as the digit. Pick any $n \in \{0, \cdots, 8\}$. Then, $\frac{9}{10^i} - \frac{n}{10^i} = \frac{9-n}{10^i} > 0$. So for a fixed $K$, the maximum number we can obtain for any decimal expansion is $K.999\cdots$.

And as the hint points out, $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$ for all $n$. And since $\frac{1}{10^n} > 0$ for all $n$, we know that $0.999\cdots = \frac{9}{10} + \cdots + \frac{9}{10^n} < 1$, for all $n$. And so adding back $K$ to both sides, we get that $K.d_1d_2d_3\cdots < K + 1$.

10.4 Discuss why Theorems 10.2 and 10.11 would fail if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

10.2: We know this will fail because we cannot always converge to rational numbers. For example, take the decimal expansion of $\sqrt{2}$ : $1.4, 1.41, \cdots$. This is bounded (by $\sqrt{2}$) and monotonically increasing (since $\frac{d_i}{10^n} > 0$ for all $n$). However, if we limit ourselves to just $\mathbb{Q}$, we know we cannot converge to $\sqrt{2}$ since it is not in our set. However, if we assume that there is some $q \in \mathbb{Q}$ limit of the sequence, by the denseness of $\mathbb{Q}$, Theorem 4.7, we know we can find another $q_1 \in \mathbb{Q}$ s.t. $q < q_1 < \sqrt{2}$, so there is no rational number that can be the limit.

10.11: This is just an application of 10.2 failing. Taking the last sequence (decimal expansion of $\sqrt{2}$) as an example: we see that it is still a Cauchy sequence, since terms **are** getting closer to $\sqrt{2}$ (which is a simple proof to show, with $N = -\log(\epsilon)$). However, as we showed earlier, the sequence does not converge. Thus, we have a Cauchy sequence that does not converge, breaking the iff statement.

10.5 Prove Theorem 10.4(ii).

Following the proof of 10.4(i):

Let $(s_n)$ be an unbounded decreasing sequence. Let $M < 0$. Since the set $\{s_n : n \in mathdsN\}$ is unbounded and it is bounded above by $s_1$, it must be unbounded below. Hence for some $N$ in $\mathbb{N}$ we have $s_N < M$. So then $n > N$ implies $s_n \leq s_N < M$, and so $\lim s_n = -\infty$.

10.6 (a) Let $(s_n)$ be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n}, \forall n \in \mathbb{N}$$

Prove $(s_n)$ is a Cauchy sequence and hence a convergent sequence.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be a natural number s.t. $2^{-N} < \epsilon$. In other words, let $N > -log_2(\epsilon)$. And we know that, for some $n, m \in \mathbb{N}$ :

$$s_n - s_m = (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m) < \Sigma_{i=m+1}^{n} 2^{-i} = 2^{-m} - 2^{-n}$$

So:

$$|s_n - s_m| < \left|2^{-m} - 2^{-n}\right|$$

So we know that $n, m > N$ implies $|s_n - s_m| < \epsilon$ for our choice of some $N > -log_2(\epsilon)$.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

It is not true, since to do the calculation, we have to calculate $\Sigma_{i=m+1}^{n} \frac{1}{i}$, which approaches $\infty$ as $n - m$ approaches $\infty$, so $n, m > N$ cannot imply that $|s_n - s_m| < \epsilon$, since $|s_n - s_m|$ can be arbitrarily large.

10.7 Let $S$ be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in $S$. Prove there is a sequence $(s_n)$ of points in $S$ such that $\lim s_n = \sup S$. See also Exercise 11.11.

Since $\sup S$ is the least upper bound, $\sup S - \frac{1}{n^2}$ must be less than some $s_n$ for all $n \in \mathbb{N}$ (or else **it** would be the supremum of the set). Since there's always an $s_n$ s.t. $\sup S - \frac{1}{n^2} < s_n < \sup S$. And the limit $\lim_{n \to \infty} \sup S - \frac{1}{n^2} = \sup S$, and clearly $\lim \sup S = \sup S$. By the squeeze theorem, we know that the limit of the sequence is $\sup S$.

10.8 Let $(s_n)$ be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove $(\sigma_n)$ is an increasing sequence.

From Definition 10.1, we know that $(\sigma_n)$ is an increasing sequence if $\sigma_n \leq \sigma_{n+1}$ for all $n$. So we have:

$$\sigma_n = \frac{1}{n}(s_1 + \cdots s_n)$$

$$\sigma_{n+1} = \frac{1}{n}(s_1 + \cdots s_{n+1}) = \sigma_n + \frac{1}{n} s_{n+1}$$

Since $(s_n)$ is an increasing sequence of positive numbers, $\frac{1}{n} s_{n+1}$ for all $n \in \mathbb{N}$ is nonnegative, and thus $\sigma_n \leq \sigma_{n+1}$ for all $n$.

10.10 Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1}) s_n^2$ for $n \geq 1$.

(a) Find $s_2, s_3$ and $s_4$.

$$s_2 = \frac{1}{2}(1)^2 = \frac{1}{2}$$
$$s_3 = \frac{2}{3}(\frac{1}{2})^2 = \frac{1}{6}$$
$$s_4 = \frac{3}{4}(\frac{1}{6})^2 = \frac{1}{48}$$

(b) Show $\lim s_n$ exists.

We can first easily show that the sequence $(s_n)$ is decreasing. For any number $0 \leq n \leq 1$, $n^2 \leq n$. It is clear that multiplying any positive number by a number $0 \leq n \leq 1$ will yield a smaller number than the original positive number. If we pick that positive number to be $n$ itself, then we see that $n^2 \leq n$. Similarly, $0 < \frac{n}{n+1} < 1$ for all $n$, so this yields a smaller number still. So if $\frac{n}{n+1}(s_n)^2 < s_n$ for all $n$, $s_{n+1} \leq s_n$ for all $n$, meaning it is a decreasing sequence.

Similarly, we can show that $(s_n)$ is bounded by 0. $s_1 = 1 \geq 0$. Assume that $s_n \geq 0$. Then: $s_n^2 \geq 0$, since $s_n$ is positive. Similarly, $s_{n+1} = \frac{n}{n+1}s_n^2 \geq 0$, since $\frac{n}{n+1}$ is positive as well, for all $n$. So by induction, $s_n > 0$ for all $n \geq 1$, and so $(s_n)$ is bounded by 0.

Since the sequence is decreasing and bounded, $(s_n)$ converges, and so $\lim s_n$ exists.

(c) Prove $\lim s_n = 0$.

For all $\epsilon > 0$ we can find an $N$ such that $n > N$ implies $|s_n - 0| < \epsilon$. Since $(s_n)$ is a decreasing sequence and bounded by 0 (as proved in part b), we can always find a smaller $s_n$ until finally $s_N < \epsilon$. Since all following terms are smaller than or equal to $s_N$, we know that $n > N$ implies that $s_n < \epsilon$ and clearly that $|s_n - 0| < \epsilon$. Then by Definition 7.1, $\lim s_n = 0$.