Math 113 **Fall 2015**

Homework #5

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1. Determine whether each of the following statements are true. If so give a proof, if not give a counterexample.

   (a) If G and H are groups suck that G has exactly k different subgroups, and H has exactly l different subgroups, then the group $G \times H$ has exactly kl different subgroups.

   False. If G and H are both Z_2, then $G \times H$ is isomorphic to the Klein-4 group. And G and H both have 2 subgroups – the trivial subgroup and themselves. But the Klein-4 group, Z, has 5 – $\{e, \{e, a\}, \{e, b\}, \{e, c\}\}$ plus itself.

   (b) Each dihedral group D_n, $n \geq 3$ is isomorphic to the direct product of a group of order n and a group of order 2.

   False for D_3.

   Every group of order 2 has to be abelian because it can only contain $\{e, a = a^{-1}\}$
   And every group of order 3 also has to be abelian because it can only contain $\{e, a, a^{-1}\}$. And the product of two abelian groups is also abelian because the product is done component-wise (trivial proof), so the product of any group of 3 and 2 has to be abelian as well. But D_3 is not abelian, so the two cannot be isomorphic.

2. In the group D_6, let H be the subgroup $\{e, r^2, r^4\}$

   (a) Do the left cosets and right cosets partition G the same way?

   Yes:

$$eH = \{e, r^2, r^4\}$$
$$rH = \{r, r^3, r^5\}$$
$$sH = \{s, sr^2, sr^4\} = \{s, r^2s, r^4s\}$$
$$rsH = \{rs, rsr^2, rsr^4\} = \{rs, r^3s, r^5s\}$$

$$He = \{e, r^2, r^4\}$$
$$Hr = \{r, r^3, r^5\}$$
$$Hs = \{s, r^2s, r^4s\}$$
$$Hrs = \{rs, r^3s, r^5s\}$$

The two partitions are the same.

(b) Write out the group table for the group over cosets. What group is it isomorphic to?

	eH	rH	sH	rsH
eH	eH	rH	sH	rsH
rH	rH	eH	rsH	sH
sH	sH	rsH	eH	rH
rsH	rsH	sH	rH	eH

It's isomorphic to the Klein-4 group, V.

3. Try to find an example of each of the following. Show that your example meets the criterion, or prove no example is possible.

(a) A subgroup S_5 which has order 10

The dihedral group D_5 is a subgroup of S_5. It is the group generated by the set $\langle(1,2,3,4,5),(2,5)(3,4)\rangle$. Geometrically, we can see that $(1,2,3,4,5)$ is a single rotation on the vertices, and $(2,5)(3,4)$ is the reflection over vertex 1. Since we know D_5 is just $\langle r, s\rangle$, and these are our two operations, the generated group is indeed D_5. And those elements (r and s) are of course members of S_5.

(b) An abelian subgroup of S_5 which has order 6.

The cyclic group generated by $(1,2,3)(4,5)$ is isomorphic to Z_6, which is abelian. We can see the isomorphism by listing out the elements generated:

$(1,2,3)(4,5),(1,3,2),(4,5),(1,2,3),(1,3,2)(4,5),(),(1,2,3)(4,5),\ldots$

(c) A nonabelian group which has subgroups of order 3, 6, and 7.

The dihedral group D_{42} has subgroups of order 3, 6, and 7. Ignoring s generated elements in D_{42}, we can generate these subgroups just with rotations:

The cyclic group generated by $\langle r^{14} \rangle$ has order 3, since it's just 3 distinct rotations.
The cyclic group generated by $\langle r^7 \rangle$ has order 6, and the cyclic group generated by $\langle r^6 \rangle$ has order 7.

(d) A subgroup of $GL(2, \mathbb{C})$ which has order 11

Similar to part c, we can just generate this with rotations. The rotation matrix about the origin for some angle θ is:

$$R = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

which is in $GL(2, \mathbb{C})$.

So the group generated by the rotation of $\frac{2\pi}{11}$ is a cyclic group that's isomorphic to Z_{11}, which has order 11. This is just:

$$H = \left\langle \begin{pmatrix} cos(\frac{2\pi}{11}) & -sin(\frac{2\pi}{11}) \\ sin(\frac{2\pi}{11}) & cos(\frac{2\pi}{11}) \end{pmatrix} \right\rangle$$