Math 104 **Spring 2016**

Assignment #8

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14.1 Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{n^4}{2^n}$

$$\limsup |a_{n+1}/a_n| = \limsup \left| \frac{(n+1)^4}{2^{n+1}} \frac{2^n}{n^4} \right|$$

$$\lim \left| \frac{(n+1)^4}{2^{n+1}} \frac{2^n}{n^4} \right| = \frac{1}{2} \lim \frac{(n+1)^4}{n^4} = \frac{1}{2}$$

Since the limit exists and is 1/2, lim sup is 1/2, meaning the series converges by the Ratio Test.

(b) $\sum \frac{2^n}{n!}$

$$\limsup |a_{n+1}/a_n| = \limsup \left| \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \right|$$

$$\lim \left| \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \right| = \lim \frac{2}{n+1} = 0$$

Since the limit exists and is 0, lim sup is 0, meaning the series converges by the Ratio Test.

(c) $\sum \frac{n^2}{3^n}$

$$\limsup |a_{n+1}/a_n| = \limsup \left| \frac{(n+1)^2}{3^{n+1}} \frac{3^n}{n^2} \right|$$

$$\lim \left| \frac{(n+1)^2}{3^{n+1}} \frac{3^n}{n^2} \right| = \frac{1}{3} \lim \frac{(n+1)^2}{n^2} = \frac{1}{3}$$

Since the limit exists and is 1/3, lim sup is 1/3, meaning the series converges by the Ratio Test.

(d) $\sum \frac{n!}{n^4+3}$

Since $n!$ grows faster than n^4, we know that $\lim a_n = +\infty \neq 0$, meaning that the series cannot converge.

(e) $\sum \frac{\cos^2 n}{n^2}$

Since $\left|cos^2(n)\right|$ is bounded by 0 and 1, we know that every term $\left|\frac{cos^2(n)}{n^2}\right| < \frac{1}{n^2}$, by the Comparison Test, the series converges.

(f) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

Since $\log n < n$, we know that $\frac{1}{\log n} > \frac{1}{n}$, meaning that each term is larger than $\frac{1}{n}$, which is a divergent series. By the Comparison Test, the series diverges.

14.6 (a) Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. (Hint : Use Theorem 14.4.)

Since (b_n) is bounded, we know that $\{|b| \leq x : b \in (b_n)$, for some $x \in \mathbb{R}$. Take the absolute value of any term, $|a_n b_n| = |a_n| \, b_n$. Since $b_n < x$, we know that $|a_n| \, |b_n| \leq x \, |a_n|$.

But note that $\sum M \, |a_n| = M \sum |a_n|$, which is absolutely convergent. So if every term is less than the term of a convergent series, and $\lim a_n b_n = 0$ (since $\lim b_n$ is some bound, and $\lim a_n = 0$ since absolutely convergent series are convergent), by the Comparison Test, the limit converges.

(b) Observe that Corollary 14.7 is a special case of part (a).

If we set $b_n = 1$, then we arrive at Corollary 14.7, but part (a) works for **any** bounded sequence.

14.7 Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Since $\sum a_n$ is a convergence series, by definition 14.3, there exists a number N s.t. $n \geq m > N$ implies $|\sum_{k=m}^{n} a_k| < 1$. Since all the terms are nonnegative, this is equivalent to saying $\sum_{k=m}^{n} a_k < 1$. Again, since all the terms are nonnegative, this means each a_k for $k > N$, $0 \leq a_k < 1$. (This is also evident by the fact that all terms are nonnegative and $\lim a_n = 0$ from 14.5.)

After N, since all the terms are less than 1, we know that $a_n^p < a_n$, since $p > 1$, and by the Comparison Test, the sequence starting from $N + 1$ converges. Since $\sum_{k=N+1}^{\infty} a_k$ converges, and the finite sum $\sum_{k=1}^{N}$ clearly converges, the entire sum must converge.

14.12 Let $(a_n)_{n \in \mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

14.13 We have seen that it is often a lot harder to find the value of an infinite sum than to show it exists. Here are some sums that can be handled.

(a) Calculate $\sum_{n=1}^{\infty} (\frac{2}{3})^n$ and $\sum_{n=1}^{\infty} (-\frac{2}{3})^n$

Applying the standard geometric series formula, we get $\sum_{n=1}^{\infty} (\frac{2}{3})^n = 2$, and $\sum_{n=1}^{\infty} (-\frac{2}{3})^n = -\frac{2}{5}$.

(b) Prove $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Looking at an individual term, a_n:

$$a_n = \frac{1}{n(n+1)} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So now our summation looks like:

$$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots$$

Regrouping, we get:

$$\begin{aligned} &= 1 + (-\frac{1}{2} + \frac{1}{2}) + (-\frac{1}{3} + \frac{1}{3}) + (-\frac{1}{4} + \cdots \\ &= 1 + 0 + 0 + \cdots \end{aligned}$$

Since, for every $\frac{1}{n}$ in our summation $-\frac{1}{n}$ exists as well, **except** for 1, the total must be 1.

(c) Prove $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$.

Looking at the term a_n, we have:

$$a_n = \frac{n-1}{2^{n+1}} = \frac{2k}{2^{k+1}} - \frac{k+1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$$

So now our series becomes:

$$1/2 + (-2/4 + 2/4) + (-3/8 + \cdots)$$

Since every term is cancelled except the first, $\frac{1}{2}$, we know that must be the sum of the series.

(d) Use (c) to calculate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$
$$= \sum_{n=2}^{\infty} \frac{n-1}{2^{n-1}}$$
$$= 4\sum_{n=2}^{\infty} \frac{n-1}{2^{n+1}}$$
$$= 4([\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}}] - \frac{1-1}{2^{1+1}})$$
$$= 4(\frac{1}{2} - 0) = 2$$

14.14 Prove $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by comparing with the series $\sum_{n=2}^{\infty} a_n$ where (a_n) is the sequence

$$(1/2, 1/4, 1/4, 1/8, 1/8, 1/8, 1/8, \cdots)$$

15.1 Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{(-1)^n}{n}$

Since $\frac{1}{n} \geq \frac{1}{n+1} \geq \cdots \geq 0$, and $\lim \frac{1}{n} = 0$, by the Alternating Series Theorem, the series converges.

(b) $\sum \frac{(-1)^n n!}{2^n}$

Clearly $\lim \frac{n!}{2^n} = +\infty$ ($n!$ grows almost as quickly as n^n), so $\lim a_n \neq 0$, and so it diverges.

15.4 Determine which of the following series converge. Justify your answers.

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$

First, let's prove that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges. We know that since $y = \frac{1}{x \log x}$ is a decreasing function past $x = 0$, we know:

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \geq \int_2^{\infty} \frac{1}{x \log x} dx$$

And:

$$\int_2^{\infty} \frac{1}{x \log x} dx = \log \log x|_2^{\infty}$$

And that integral diverges, so we know that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ must diverge.

And since $\frac{1}{\sqrt{n} \log n} > \frac{1}{n \log n}$ (since $\sqrt{n} < n$), we know it must diverge as well by the Comparison Test.

(b) $\sum_{n=2}^{\infty} \frac{\log n}{n}$

The series diverges. After $n = 10$, for all $n > 10$, $\log n > 1$, and so by the Comparison Test with $\frac{1}{n}$, $\sum_{n=10}^{\infty} \frac{\log n}{n}$ diverges. And if that sum diverges, then $\sum_{n=2}^{\infty} \frac{\log n}{n}$ diverges as well.

(c) $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)}$

Again, we have a decreasing function, and so we have $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)} > \int_{x=4}^{\infty} \frac{1}{x(\log x)(\log \log x)} dx$ And:

$$\int_{x=4}^{\infty} \frac{1}{x(\log x)(\log \log x)} dx = \log \log \log x|_{x=4}^{\infty}$$

And that integral diverges, meaning $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)}$ diverges.

(d) $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$

Again, we can compare with $\int_{n=2}^{\infty} \frac{\log x}{x^2} dx$, and taking "right sums", we get : $\sum_{n=2}^{\infty} \frac{\log n}{n^2} \leq 1 + \int_{n=2}^{\infty} \frac{\log x}{x^2} dx$. Integrating by parts we get:

$$\int_{n=2}^{\infty} \frac{\log x}{x^2} dx = \frac{-1 - \log x}{x}|_{x=2}^{\infty}$$

which converges, so we know $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges.

15.6 (a) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

The obvious example of $a_n = \frac{1}{n}$ diverges, while $a_n^2 = \frac{1}{n^2}$ converges.

(b) Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges. See Exercise 14.7.

We already proved this in 14.7. The gist being that after a finite N, $a_{n>N} < 1$, and so all subsequent $a_{n>N}^2$ are smaller than $a_{n>N}$, making the infinite series past N converge by the Comparison Test.

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

$(a_n) = \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Theorem, since $\frac{1}{\sqrt{1}} \geq \frac{1}{\sqrt{2}} \geq \cdots 0$, and $\lim \frac{1}{\sqrt{n}} = 0$. But its square, $(a_n^2) = \frac{(-1)^{2n}}{n} = \frac{(-1^2)^n}{n} = \frac{1}{n}$ is the harmonic series, and thus diverges.

15.7 (a) Prove if (a_n) is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$.

(b) Use (a) to give another proof that $\sum \frac{1}{n}$ diverges.