Phil 140b **Spring 2016**

Assignment #5

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1. *Consider the theory Q given in class (this is the theory called R in section 16.4 of the book). Show by metatheoretical induction that if $i * j = k$, then $\models_Q$ $\boldsymbol{i} * \boldsymbol{j} = \boldsymbol{k}$.*

First, let's prove that $i + j = k \implies \overbrace{0''\cdots}^{\text{i times}} + \overbrace{0''\cdots}^{\text{j times}} = \overbrace{0''\cdots}^{\text{i+j times}}$. We'll do this through induction:

Base Case: If $j = 0$, then $i + j = k = i$, and by (Q3), we know that:

$$\mathbf{i} + \mathbf{0} = \mathbf{i}$$

$$\mathbf{i} + \mathbf{j} = \mathbf{k}$$

Inductive Case: Assume for all $n < j$ that $i + n = k \implies \overbrace{0''\cdots}^{\text{i times}} + \overbrace{0''\cdots}^{\text{n times}} = \overbrace{0''\cdots}^{\text{k times}}$. Then from (Q4) we have:

$$\overbrace{0''\cdots}^{\text{i times}} + (\overbrace{0''\cdots}^{\text{j-1 times}})' = (\overbrace{0''\cdots}^{\text{i times}} + \overbrace{0''\cdots}^{\text{j-1 times}})'$$

Using our inductive hypothesis:

$$= (\overbrace{0''\cdots}^{\text{i+j-1 times}})' = \overbrace{0''\cdots}^{\text{i+j times}}$$

Now we want to show that $\overbrace{0''\cdots}^{\text{i times}} * \overbrace{0''\cdots}^{\text{j times}} = \overbrace{0''\cdots}^{\text{k times}}$. And we'll do this through induction:

Base Case : If $j = 0$, then $i * j = k = 0$, and by (Q5), we know that:

$$\mathbf{i} * \mathbf{0} = \mathbf{0}$$

$$\mathbf{i} * \mathbf{j} = \mathbf{k}$$

Inductive Case : Assume for all $n < j$ that $i * n = k \implies \overbrace{0''\cdots}^{\text{i times}} * \overbrace{0''\cdots}^{\text{n times}} = \overbrace{0''\cdots}^{\text{k times}}$. Then from (Q6) we have:

$$\overbrace{0''\cdots}^{\text{i times}} *(\overbrace{0''\cdots}^{\text{j-1 times}})' = (\overbrace{0''\cdots}^{\text{i times}} * \overbrace{0''\cdots}^{\text{j-1 times}}) + \overbrace{0''\cdots}^{\text{i times}}$$

Using our inductive hypothesis:

$$= (\overbrace{0''\cdots}^{\text{i(j-1) times}}) + \overbrace{0''\cdots}^{\text{i times}} = \overbrace{0''\cdots}^{\text{ij - i times}} + \overbrace{0''\cdots}^{\text{i times}}$$

And using our last proof:

$$= \overbrace{0''\cdots}^{\text{ij - i + i times}} = \overbrace{0''\cdots}^{\text{ij times}}$$

2. *A formula $B(y)$ is called a truth-predicate for T if for any sentence G of the language T, $\models_T G \iff B(\ulcorner G \urcorner)$. Show that if T is a consistent theory in which diag is representable, then there is no truth-predicate for T.*

Assume there is such a truth-predicate, $B(y)$ in our consistent theory T. Let's construct a few preliminary sentences:

$$A(x) \iff \exists y(\text{Diag}(x, y) \wedge \neg B(y))$$
$$a = \ulcorner A(x) \urcorner$$

(This is using Diag as the representation of diag in T, as per the book's convention.)

$$G \iff \exists x(x = \mathbf{a} \wedge A(x))$$

$$g = \ulcorner G \urcorner$$

By construction G is equivalent to:

$$\iff \exists x(x = \mathbf{a} \wedge \exists y(\text{Diag}(x, y) \wedge \neg B(y))) \iff \exists y(\text{Diag}(\mathbf{a}, y) \wedge \neg B(y))$$

So then we have:

$$\models_T G \iff \exists y(\text{Diag}(\mathbf{a}, y) \wedge \neg B(y))$$

And since G is the diagonalization of $A(x)$ we also have:

$$\models_T \forall y(\text{Diag}(\mathbf{a}, y) \iff y = \mathbf{g})$$

Combining the last two formulas we get:

$$\models_T G \iff \exists y(y = \mathbf{g} \wedge \neg B(y))$$

Or:

$$\models_T G \iff \neg B(\mathbf{g}) \iff \neg B(\ulcorner G \urcorner)$$

But this is a contradiction with the original definition of $B(y)$, meaning that no such truth-predicate can exist.

3. *A set S of natural numbers is called recursively enumerable (r.e.) if there is a two-place recursive relation R such that $S = \{x : \exists y Rxy\}$. Show that for any set S, S is recursive iff both S and its complement $(\mathbb{N} - S)$ are recursively enumerable.*

If S is recursive:
then it must have some recursive characteristic function f_S s.t. $s \in S \iff f_S(s) = 1$ and $s \notin S \iff f_S(s) = 0$. Then we can easily define our relations:

$$R_S xy \iff (f_S(x) = y) \wedge (y = 1)$$

$$R_{\mathbb{N}-S} xy \iff (f_S(x) = y) \wedge (y = 0)$$

If S and $\mathbb{N} - S$ are r.e.:
then there must exist relations R_S and $R_{\mathbb{N}-S}$ as described above. We can construct a recursive characteristic function to show S is recursive:

$$f_S(x) = f'_S(x, 0)$$

$$f'_S(x, y) = \begin{cases} 1 & R_S xy \\ 0 & R_{\mathbb{N}-S} xy \\ f'_S(x, y+1) & \text{else} \end{cases}$$

$f'_S(x, 0)$ (and therefore f_S) is guaranteed to halt in a finite amount of time because for a given x either R_S or $R_{\mathbb{N}-S}$ is guaranteed to be true for some y. This means we have an effective procedure, and so f_S is a valid recursive characteristic function.

4. *Show that all r.e. sets are definable in arithmetic (i.e. the theory consisting of $L(Q)$ that are true in $\mathbb{N}$).*

All r.e. sets have some two-place recursive relation R such that $S = \{x : \exists y Rxy\}$. Furthermore, it was shown in the book (not the chapter PDFs) in Theorem 16.16 that every recursive relation is representable in Q. Say that $\phi(x, y, w)$ represents the recursive relation in Q. Then we can arithmetically define r.e. sets as:

$$F(x, y) \iff \phi(x, y, 1)$$