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COMPRESSIVE BUCKLING OF SIMPLY SUPPORTED CURVED PLATES AND CYLINDERS OF SANDWICH CONSTRUCTION

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SUMMARY

Theoretical solutions are presented for the buckling in uniform axial compression of two types of simply supported curved sandwich plates: the corrugated-core type and the isotropic-core type. The solutions are obtained from a theory for orthotropic curved plates in which deflections due to shear are taken into account. Results are given in the form of equations and curves.

INTRODUCTION

The use of sandwich construction for compression-carrying components of aircraft will often require the calculation of the compressive buckling strength of curved sandwich plates.

In the present paper, therefore, a theoretical solution is given for the elastic buckling load, in uniform axial compression, of simply supported, cylindrically curved, rectangular plates and circular cylinders of two types of sandwich construction: the corrugated-core type and the isotropic type (e.g. Metalite).

The analysis is based on the small-deflection buckling theory of reference 1 which differs from ordinary curved-plate theory principally by the inclusion of the effects of deflections due to transverse shear. The curvature is assumed constant and the thickness small compared with the radius and axial and circumferential dimensions. The core modulus in the transverse direction is assumed to be infinite; thus, consideration of types of local buckling in which corresponding points on the upper and lower faces do not remain equidistant is eliminated. The corrugated-core sandwich is assumed to be symmetrical, on the average, about the middle surface, so that the force distortion relations are relatively simple (see reference 2), and is assumed to have infinite transverse shear stiffness in planes parallel to the corrugations. The core of the isotropic sandwich (flexural properties identical in axial and circumferential directions) is assumed to carry no face-parallel stresses.

The results of the solution are presented in nondimensional form through equations and graphs. The details of the analysis are given in two appendixes.

The results of the present investigation are compared with previous work on the subject of compressive buckling of curved sandwich plates and cylinders (references 3, 4, and 5). This previous work is confined to sandwich construction of the isotropic type only. The present paper includes results for the isotropic sandwich covering a larger curvature range and gives the same, or more conservative, results.

SYMBOLS

$\bar{A}_C$	cross-sectional area of corrugation per inch of width, inches
$\bar{A}_S$	cross-sectional area of faces per inch of width, inches $(2t_S)$
D_S	flexural stiffness of isotropic sandwich plate, inch-pounds $\left(\dfrac{E_S t_S h^2}{2(1 - \mu_S^2)}\right)$
D_x,D_y	beam flexural stiffnesses of orthotropic plate in axial and circumferential directions, respectively, inch-pounds
D_{xy}	twisting stiffness of orthotropic plate in xy-plane, inch-pounds
D_{Q_x},D_{Q_y}	transverse shear stiffnesses of orthotropic plate in axial and circumferential directions, respectively, pounds per inch (D_{Q_x} assumed infinite for corrugated-core sandwich)
D_Q	transverse shear stiffness of isotropic sandwich plate, pounds per inch
E_C	Young's modulus for corrugated-core material, psi
E_S	Young's modulus for face material, psi
E_x,E_y	extensional stiffnesses of orthotropic plate in axial and circumferential directions, respectively, pounds per inch

G_C	shear modulus of core material for isotropic sandwich plate, psi
G_{xy}	shear stiffness of orthotropic plate in xy-plane, pounds per inch
$\bar{I}_C$	moment of inertia of corrugation cross section per inch of width, inches^3
$\bar{I}_S$	moment of inertia of faces per inch of width about middle of surface of plate, inches^3 $\left(\frac{t_S h^2}{2}\right)$
L_E, L_E^{-1}, L_D; $\nabla^2, \nabla^4, \nabla^{-4}$	mathematical operators
N_x	middle-surface compressive force, pounds per inch
Q_x, Q_y	transverse shearing forces in yz- and xz-planes, respectively, pounds per inch
R	constant radius of curvature of plate or cylinder, inches
Z_a, Z_b	curvature parameters $\left(Z_a^2 = \frac{2t_S a^4}{R^2 \bar{I}_S};\ Z_b^2 = \frac{2t_S b^4}{R^2 \bar{I}_S}\right.$ for corrugated-core sandwich plate$\Big)$ $\left(Z_a^2 = \frac{2t_S a^4 \left(1 - \mu_S^2\right)}{R^2 \bar{I}_S};\ Z_b^2 = \frac{2t_S b^4 \left(1 - \mu_S^2\right)}{R^2 \bar{I}_S}\right.$ for isotropic sandwich plate$\Big)$
a	axial length of plate or cylinder, inches
b	circumference of cylinder or circumferential width of plate, inches
h	depth of sandwich plate measured between middle surfaces of faces, inches

k_{x_a}, k_{x_b} compressive load coefficients $\left(k_{x_a} = \frac{N_x a^2}{E_S \bar{I}_S \pi^2};\right.$

$$\left. k_{x_b} = \frac{N_x b^2}{E_S \bar{I}_S \pi^2} \text{ for corrugated-core sandwich plate} \right)$$

$$\left(k_{x_a} = \frac{N_x a^2}{D_S \pi^2};\ k_{x_b} = \frac{N_x b^2}{D_S \pi^2} \text{ for isotropic sandwich plate} \right)$$

m,n number of half-waves into which plate or cylinder buckles in axial and circumferential directions, respectively

2p pitch of corrugation, inches

l developed length of one corrugation leg, inches

r_a, r_b transverse shear stiffness parameters $\left(r_a = \frac{E_S \bar{I}_S \pi^2}{D_{Q_y} a^2};\right.$

$$\left. r_b = \frac{E_S \bar{I}_S \pi^2}{D_{Q_y} b^2} \text{ for corrugated-core sandwich plate} \right)$$

$$\left(r_a = \frac{D_S \pi^2}{D_Q a^2};\ r_b = \frac{D_S \pi^2}{D_Q b^2} \text{ for isotropic sandwich plate} \right)$$

t_C thickness of corrugation material, inches

t_S thickness of face material, inches

w radial displacement of point on middle surface of plate or cylinder, inches

x,y axial and circumferential coordinates, respectively

μ_C Poisson's ratio for corrugated-core material

μ_S Poisson's ratio for face material

μ_x, μ_y Poisson's ratios for orthotropic plate, defined in terms of curvatures

μ'_x, μ'_y Poisson's ratios for orthotropic plate, defined in terms of middle-surface strains

RESULTS

Corrugated-Core Sandwich

The theoretical compressive buckling load for a curved rectangular corrugated-core sandwich plate of axial length a and circumferential width b can be obtained from the following equation:

$$k_{x_b} = \frac{\eta}{\frac{1}{m^2}\left(\frac{a}{b}\right)^2} + \frac{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2}{\frac{1}{m^2}\left(\frac{a}{b}\right)^2} \frac{1}{1 - \mu_S{}^2 + \cfrac{n^2}{\cfrac{1}{r_b} + \cfrac{m^2}{2(1 + \mu_S)}\left(\cfrac{b}{a}\right)^2}} +$$

$$\frac{Z_b{}^2 \frac{1}{m^2}\left(\frac{a}{b}\right)^2}{\xi_1 + 2\xi_2\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + \xi_3\left(\frac{n}{m}\right)^4\left(\frac{a}{b}\right)^4} \tag{1}$$

where

$$k_{x_b} = \frac{N_x b^2}{E\bar{I}_S \pi^2}$$

m,n number of half-waves into which plate buckles in axial and circumferential directions

$$r_b = \frac{E_S \bar{I}_S \pi^2}{D_{Q_y} b^2}$$

$$Z_b^2 = \frac{2t_S b^4}{R^2 \bar{I}_S}$$

D_{Q_y} is the transverse shear stiffness, obtainable from reference 2, and

$$\eta = \frac{E_C \bar{I}_C}{E_S \bar{I}_S}$$

$$\xi_1 = \frac{1 + \left(1 - \mu_S^2\right)\frac{E_C \bar{A}_C}{E_S \bar{A}_S}}{1 + \frac{E_C \bar{A}_C}{E_S \bar{A}_S}}$$

$$\xi_2 = \frac{1 + \mu_S}{1 + \frac{1 + \mu_S}{1 + \mu_C}\left(\frac{p}{l}\right)^2 \frac{E_C \bar{A}_C}{E_S \bar{A}_S}}$$

$$\xi_3 = \frac{1}{1 + \frac{E_C \bar{A}_C}{E_S \bar{A}_S}}$$

The details in the derivation of equation (1) are presented in appendix A.

In using equation (1), in general, different combinations of integral values of m and n must be substituted until a minimum value of k_{x_b}

is obtained for given values of the other parameters. This minimum value of k_{x_b} determines the buckling load.

For the special case of an infinitely long curved plate $(a,m \rightarrow \infty)$, k_{x_b} must be minimized with respect to the axial wave length a/m and integral values of n.

For the special case of a cylinder, the width b is equal to $2\pi R$. Thus, R becomes involved in the parameters k_{x_b}, r_b, Z_b, and a/b. Equation (1), therefore, is not well-adapted to studying the effect of changes in R on the buckling strength of the cylinder. It is more convenient, for the special case of a cylinder, to consider equation (1) rewritten in terms of slightly different parameters as follows:

$$k_{x_a} = \eta m^2 + m^2\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2 \frac{1}{1 - \mu_S{}^2 + \dfrac{n^2\left(\frac{a}{b}\right)^2}{\dfrac{1}{r_a} + \dfrac{m^2}{2(1 + \mu_S)}}} +$$

$$\frac{\dfrac{Z_a{}^2}{\pi^4}\dfrac{1}{m^2}}{\xi_1 + 2\xi_2\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + \xi_3\left(\frac{n}{m}\right)^4\left(\frac{a}{b}\right)^4} \qquad (2)$$

where

$$k_{x_a} = \frac{N_x a^2}{E_S \bar{I}_S \pi^2}$$

$$r_a = \frac{E_S \bar{I}_S \pi^2}{D_{Q_y} a^2}$$

$$Z_a{}^2 = \frac{2t_S a^4}{R^2 \bar{I}_S}$$

This equation, unlike equation (1), can yield the limiting results for an infinitely wide flat plate ($R \rightarrow \infty$) compressed in the short direction.

For cylinders, different combinations of integral values of m and even integral values of n must be substituted until a minimum value for k_{x_a} is obtained. The values considered for n for the range of practical dimensions are: $n = 0$ for short cylinders (axisymmetrical buckling) and $n = 4, 6, 8 \ldots$ for medium or long cylinders. The case $n = 2$, which is not considered here, corresponds to column buckling. For medium or long cylinders, the same results or slightly conservative results are obtained if, instead of minimizing k_{x_a} with respect to even integral values of n, k_{x_a} is minimized with respect to the circumferential wave length b/n. The latter procedure is used in this paper.

Because of the large number of elastic and geometric parameters appearing in equations (1) and (2), the critical load coefficients can be readily obtained only for individual corrugated-core sandwich sections. For purposes of illustrating the application of equations (1) and (2) to a particular corrugated-core sandwich, the section shown in figure 1 has been selected. The required physical constants calculated for this section are also shown in figure 1. The value of D_{Q_y} is computed from the formulas and charts presented in reference 2.

The process of minimization is carried out for this particular section and buckling loads are obtained for infinitely long curved plates and cylinders of arbitrary over-all width and length, respectively, and of arbitrary radius. Nondimensional curves presenting these results are shown in figures 2 and 3 giving the load coefficients k_{x_a} and k_{x_b} and therefore the load as a function of the over-all dimensions. The dashed curves give the results obtained if transverse shear deformations are neglected. The equations for this case are presented in appendix A. It should be noted that the b/R and a/R curves are cut off where the dimensions of the plate or cylinder would no longer be consistent with the requirements of small-deflection theory.

Isotropic-Core Sandwich

The theoretical compressive buckling load for a curved, rectangular, isotropic-core sandwich plate of axial length a and circumferential width b can be obtained from the following equation:

$$k_{x_b} = \frac{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2}{\frac{1}{m^2}\left(\frac{a}{b}\right)^2 + r_b\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]} + \frac{\frac{Z_b^2}{\pi^4}\frac{1}{m^2}\left(\frac{a}{b}\right)^2}{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2} \tag{3}$$

where

$$k_{x_b} = \frac{N_x b^2}{D_S \pi^2}$$

$$r_b = \frac{D_S \pi^2}{D_Q b^2}$$

$$Z_b^2 = \frac{2t_S b^4(1 - \mu_S^2)}{R^2 \bar{I}_S}$$

and

$$D_Q = G_C \frac{h^2}{h - t_S} \quad \text{(see reference 6)}$$

The details of the derivation of equation (3) are presented in appendix B. The values of m and n to be used in equation (3) are obtained in the same manner as discussed in the previous section for the corrugated-core sandwich.

For the special case of an infinitely long curved plate $(a,m \rightarrow \infty)$, k_{x_b} can be minimized with respect to the axial wave length a/m and the number of circumferential half-waves n. The results are plotted in figure 4. The equations for the theoretical buckling load coefficients of infinitely long curved plates and the ranges in which they hold are

$$k_{x_b} \approx \frac{4}{(1 + r_b)^2} + \frac{Z_b^2}{\pi^4}\frac{1 - r_b}{4} \tag{4}$$

when $\frac{Z_b}{\pi^2} < \frac{4\sqrt{1 - r_b}}{(1 + r_b)^2}$,

$$k_{x_b} \approx \frac{\frac{Z_b}{\pi^2}}{\sqrt{1 - r_b}}\left(2 - \frac{\frac{Z_b}{\pi^2} r_b}{\sqrt{1 - r_b}}\right) \tag{5}$$

when $\frac{4\sqrt{1 - r_b}}{(1 + r_b)^2} < \frac{Z_b}{\pi^2} \leqq \frac{\sqrt{1 - r_b}}{r_b}$,

$$k_{x_b} = \frac{1}{r_b} \tag{6}$$

when $\frac{Z_b}{\pi^2} \geqq \frac{\sqrt{1 - r_b}}{r_b}$. Equations (4) and (5) are not exact but are quite accurate for the curvature-parameter ranges indicated.

For the special case of a cylinder ($b = 2\pi R$), it is convenient to rewrite equation (3) with parameters in terms of a rather than b as done in the preceding section; the resulting equation is

$$k_{x_a} = \frac{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2}{\frac{1}{m^2} + r_a\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]} + \frac{\frac{Z_a^2}{\pi^4}\frac{1}{m^2}}{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2} \tag{7}$$

where

$$k_{x_a} = \frac{N_x a^2}{D_S \pi^2}$$

$$r_a = \frac{D_S \pi^2}{D_Q a^2}$$

$$Z_a{}^2 = \frac{2t_S a^4 \left(1 - \mu_S{}^2\right)}{R^2 \overline{I}_S}$$

As is true of equation (2), equation (7) when applied to a cylinder must be restricted to values of n equal to 0 and even integers greater than 2. The minimization is again performed analytically so that m and n are eliminated. The results, which are plotted in figure 5, are given by the following equations:

$$k_{x_a} = \frac{1}{1 + r_a} + \frac{Z_a{}^2}{\pi^4} \tag{8}$$

when $\frac{Z_a}{\pi^2} \leqq \frac{1}{1 + r_a}$,

$$k_{x_a} = \frac{Z_a}{\pi^2}\left(2 - \frac{Z_a}{\pi^2} r_a\right) \tag{9}$$

when $\frac{1}{1 + r_a} \leqq \frac{Z_a}{\pi^2} \leqq \frac{1}{r_a}$,

$$k_{x_a} = \frac{1}{r_a} \tag{10}$$

when $\frac{Z_a}{\pi^2} \geqq \frac{1}{r_a}$.

In addition to the results presented for infinitely long curved plates and cylinders, calculations have been carried out for rectangular curved plates of dimensions such that the axial length is equal to the circumferential length and the axial length is equal to one-half the circumferential length. The theoretical buckling load coefficients for rectangular curved plates of these dimensions and, for comparison, the theoretical buckling load coefficient for cylinders $\left(\text{denoted by } \frac{a}{b} = 0\right)$ are presented in figure 6.

DISCUSSION

Effect of Transverse Shear Stiffness

The results of the investigation of the compressive buckling of corrugated-core, curved, sandwich plates having the particular cross section shown in figure 1 and isotropic curved sandwich plates indicate the following effects with regard to finite transverse shear stiffness:

For the infinitely long, corrugated-core, curved sandwich plates (see fig. 2), the effect of the finite transverse shear stiffness can be neglected for plates of high curvature ($Z_b > 100$) but should be considered for plates of lower curvature. For the corrugated-core cylinders (see fig. 3), the effect of transverse shear stiffness can be neglected when the cylinder is either very short $\left(\frac{a}{R} < 0.1\right)$ or extremely long $\left(\frac{a}{R} > 10\right)$. Equations are given in appendix A for the critical compressive load coefficients of infinitely long plates and cylinders when transverse shear stiffness is neglected. In the intermediate range, which corresponds to the range of practical dimensions, shearing deformations must be taken into account at least for lower values of Z_a $\left(\text{for } \frac{a}{r} = 1,\ Z_a < 10^3;\ \text{for } \frac{a}{r} = 10,\ Z_a < 10^4\right)$.

For isotropic curved sandwich plates and cylinders (figs. 4, 5, and 6) the results indicate that transverse shearing deformations have the greatest effect when

$$r_b \geqq \frac{\pi^2}{Z_b}\left(\sqrt{1 + \frac{\pi^4}{4{Z_b}^2}} - \frac{\pi^2}{2Z_b}\right) \qquad \left(\frac{a}{b} \geqq 1\right) \tag{11}$$

$$r_a \geqq \frac{\pi^2}{Z_a} \qquad \left(\frac{a}{b} < 1\right) \tag{12}$$

For these values, the critical compressive load coefficient k_{x_b} or k_{x_a} is given directly by the reciprocal of the shear stiffness parameter r_b or r_a, respectively, or in terms of the compressive load N_x and transverse shear stiffness D_Q,

$$N_x = D_Q$$

This equation represents the horizontal portion of the curves in figures 4, 5, and 6.

The results indicate that the effect of shear stiffness can be neglected when r_b and r_a are much smaller than the right-hand members of relations (11) and (12), respectively. Equations are given in appendix B for the critical compressive load coefficients of infinitely long plates and cylinders when transverse shear stiffness is neglected.

In the intermediate range (the range of practical dimensions), the effect of transverse shear deformation is always important.

Comparison with Previous Results

For the corrugated-core curved sandwich plate, no previous results are available for comparison with the present paper. The equations of the present paper, however, do reduce to the equation for the corrugated-core flat sandwich plate given in reference 7.

For the isotropic curved sandwich plate, previous results are available only for the infinitely long plate of small curvature and for the cylinder of medium or large curvature. Comparison of the present results with this previous work shows that the present results are the same or conservative.

Equation (4), which gives the theoretical buckling load coefficient for plates of small curvature, is similar to equation (16) of reference 3 and equation (48) of reference 4. Since the theory of reference 3 allows the core to carry compressive load and imposes the additional boundary condition of zero circumferential displacement along the unloaded edges, the results of the present paper are more conservative than those of reference 3. When the curvature is zero, equation (4) gives the same results as equation (48) of reference 4 but is increasingly more conservative as the curvature increases because of the approximate manner in which the theory of reference 4 takes the effects of curvature into account.

Equation (9) is equivalent to equation (49) of reference 4 and to equation (20) of reference 3, if the compressive load carried by the core is neglected, and can be shown to be the result of minimizing with respect to the wave length the buckling coefficient appearing in equations (15) and (16) of reference 5.

Equation (10), which gives the buckling coefficient for zero-wave-length shear buckling of the core, is equivalent to equation (18) of reference 5. It is of interest to note here that, in reference 5, tests conducted on thin isotropic-core sandwich cylinders (cellular-cellulose-acetate cores) indicate apparent correlation with equation (18) of reference 5 and equation (10) of the present paper.

Empirical Reduction Factors

It is well-known that the small-deflection buckling solutions for cylinders and curved plates of ordinary homogeneous construction give compressive buckling stresses that may be much higher than the experimentally observed values; empirical "knockdown" or reduction factors have therefore been proposed for ordinary curved plates in compression (see, for example, reference 8). The same shortcoming may be expected of the present buckling solution for curved sandwich plates; the reductions required, however, will probably not be so severe as in the case of homogeneous cylinders, partly because the greater thickness of the sandwich plate reduces, relatively, the importance of initial irregularities (see discussion in reference 3). Some experimental data (reference 5) seem to indicate that no reduction is required for the isotropic sandwich with a core of sufficiently low shear stiffness. Further investigations are required, however, before this conclusion may be taken as generally valid for all curved sandwich plates.

CONCLUDING REMARKS

A theoretical solution is presented for the elastic buckling load in uniform axial compression, of simply supported, cylindrically curved, rectangular plates and circular cylinders of two types of sandwich construction: the corrugated-core type and the isotropic type.

Like the results for flat sandwich plates, the results for curved sandwich plates show, in general, that the effect of finite transverse shear stiffness is to lower the buckling load. For given cross-sectional dimensions and properties, this effect diminishes as curvature increases. In the range of practical dimensions, however, the effect of transverse shear deformations on the buckling load is always important.

For isotropic, sandwich, curved plates and cylinders of low transverse shear stiffness (weak cores), the critical compressive load N_x

becomes independent of curvature and equal to the transverse shear stiffness D_Q, a result found also for flat sandwich plates of low shear stiffness.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 5, 1951

APPENDIX A

ANALYSIS OF CORRUGATED-CORE SANDWICH PLATE

A corrugated-core sandwich plate with the corrugations oriented parallel to the x-axis may be considered to be an orthotropic plate having infinite transverse shear stiffness in the axial direction; that is, $D_{Q_x} \rightarrow \infty$. For such a plate loaded in axial compression (N_x positive in compression), the general equations of equilibrium for orthotropic plates developed in reference 1 reduce to

$$L_D w + \frac{G_{xy}}{R^2} L_E^{-1} \frac{\partial^4 w}{\partial x^4} + N_x \frac{\partial^2 w}{\partial x^2} -$$

$$\frac{1}{D_{Q_y}}\left[\frac{D_y}{1 - \mu_x\mu_y} \frac{\partial^3 Q_y}{\partial y^3} + \left(\frac{\mu_y D_x}{1 - \mu_x\mu_y} + D_{xy}\right) \frac{\partial^3 Q_y}{\partial x^2 \partial y}\right] = 0 \qquad \text{(A1)}$$

and

$$Q_y + \frac{D_y}{1 - \mu_x\mu_y}\left(\frac{\partial^3 w}{\partial y^3} - \frac{1}{D_{Q_y}} \frac{\partial^2 Q_y}{\partial y^2} + \mu_x \frac{\partial^3 w}{\partial x^2 \partial y}\right) + \frac{1}{2} D_{xy}\left(2 \frac{\partial^3 w}{\partial x^2 \partial y} - \frac{1}{D_{Q_y}} \frac{\partial^2 Q_y}{\partial x^2}\right) = 0 \qquad \text{(A2)}$$

where L_D is the linear differential operator defined by

$$L_D = \frac{D_x}{1 - \mu_x\mu_y} \frac{\partial^4}{\partial x^4} + \left(\frac{\mu_y D_x}{1 - \mu_x\mu_y} + 2D_{xy} + \frac{\mu_x D_y}{1 - \mu_x\mu_y}\right)\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{D_y}{1 - \mu_x\mu_y} \frac{\partial^4}{\partial y^4}$$

L_E is the linear differential operator defined by

$$L_E = \frac{G_{xy}}{E_y}\frac{\partial^4}{\partial x^4} + \left(1 - \mu'_x \frac{G_{xy}}{E_x} - \mu'_y \frac{G_{xy}}{E_y}\right)\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{G_{xy}}{E_x}\frac{\partial^4}{\partial y^4}$$

and L_E^{-1} is the inverse operator defined by

$$L_E^{-1}\left(L_E w\right) = L_E\left(L_E^{-1} w\right) = w$$

For a simply supported orthotropic sandwich plate, since deflections due to transverse shear are considered, three boundary conditions must be imposed at a plate edge. Two of these are the usual boundary conditions specified on the displacement w and the bending moment M_x (if edge is parallel to the y-axis), that is,

$$w = 0$$

$$M_x = 0$$

The third condition is imposed on the transverse shearing force Q_y, which requires that the transverse shear strain in the plane of the boundary be zero; hence,

$$\frac{Q_y}{D_{Q_y}} = 0$$

A similar set of conditions exists along an edge parallel to the x-axis.

For a corrugated-core sandwich plate, the boundary conditions are satisfied if the displacement and transverse shear functions are taken as

$$\left.\begin{aligned} w &= A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ Q_y &= B \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned}\right\} \tag{A3}$$

since the assumption that $D_{Q_x} \rightarrow \infty$ for the corrugated-core plate automatically satisfies the boundary condition $\dfrac{Q_x}{D_{Q_x}} = 0$.

Substitution of relations (A3) into equations (A1) and (A2) leads to two linear homogeneous equations in the arbitrary constants A and B. If nonzero solutions of these equations are to exist, the determinant of the coefficients of A and B must vanish. This condition is represented by

$$N_x = \left(\frac{m\pi}{a}\right)^2\left[\frac{D_x}{1 - \mu_x\mu_y} + \left(\frac{\mu_y D_x}{1 - \mu_x\mu_y} + 2D_{xy} + \frac{\mu_x D_y}{1 - \mu_x\mu_y}\right)\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + \right.$$

$$\left.\frac{D_y}{1 - \mu_x\mu_y}\left(\frac{n}{m}\right)^4\left(\frac{a}{b}\right)^4\right] - \frac{\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\left\{\frac{D_y}{1 - \mu_x\mu_y}\left[\mu_x + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right] + D_{xy}\right\}^2}{D_{Q_y} + \left(\frac{m\pi}{a}\right)^2\left[\frac{D_y}{1 - \mu_x\mu_y}\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + \frac{1}{2}D_{xy}\right]} +$$

$$\frac{E_xE_y}{\left(\frac{m\pi}{a}\right)^2 R^2\left[E_x - \left(\mu'_yE_x - \frac{E_xE_y}{G_{xy}} + \mu'_xE_y\right)\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + E_y\left(\frac{n}{m}\right)^4\left(\frac{a}{b}\right)^4\right]} \tag{A4}$$

where the reciprocal relationship $\mu_xD_y = \mu_yD_x$ (see reference 9) has been used to simplify the numerator of the second term.

By replacing the physical constants D_x, D_y, D_{xy}, E_x, E_y, G_{xy}, μ_x, μ_y, μ'_x, and μ'_y with their respective formulas, as developed in reference 2 for the symmetrical corrugated-core sandwich plate, and defining

$$k_{x_b} = \frac{N_xb^2}{E_S\overline{I}_S\pi^2}$$

$$Z_b^2 = \frac{2t_S b^4}{R^2 \bar{\bar{I}}_S}$$

$$r_b = \frac{E_S \bar{\bar{I}}_S \pi^2}{D_{Q_y} b^2}$$

equation (A4) becomes

$$k_{x_b} = \frac{\eta}{\frac{1}{m^2}\left(\frac{a}{b}\right)^2} + \frac{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2}{\frac{1}{m^2}\left(\frac{a}{b}\right)^2} \frac{1}{1 - \mu_S^2 + \cfrac{n^2}{\cfrac{1}{r_b} + \cfrac{m^2}{2(1 + \mu_S)}\left(\cfrac{b}{a}\right)^2}} +$$

$$\frac{Z_b^2 \frac{1}{m^2}\left(\frac{a}{b}\right)^2}{\xi_1 + 2\xi_2\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + \xi_3\left(\frac{n}{m}\right)^4\left(\frac{a}{b}\right)^4} \tag{A5}$$

where

$$\eta = \frac{E_C \bar{\bar{I}}_C}{E_S \bar{\bar{I}}_S}$$

$$\xi_1 = \frac{1 + \left(1 - \mu_S^2\right)\frac{E_C \bar{A}_C}{E_S \bar{A}_S}}{1 + \frac{E_C \bar{A}_C}{E_S \bar{A}_S}}$$

$$\xi_2 = \frac{1 + \mu_S}{1 + \frac{1 + \mu_S}{1 + \mu_C}\left(\frac{p}{l}\right)^2 \frac{E_C\bar{A}_C}{E_S\bar{A}_S}}$$

$$\xi_3 = \frac{1}{1 + \frac{E_C\bar{A}_C}{E_S\bar{A}_S}}$$

Equation (A5) is applicable to infinitely long curved plates, to finite rectangular curved plates, and to cylinders. For the special case of a cylinder ($b = 2\pi R$), however, it is more convenient, for reasons discussed in the section entitled "Results," to consider this equation written in terms of slightly different parameters as follows:

$$k_{x_a} = \eta m^2 + m^2\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2 \frac{1}{1 - \mu_S^2 + \frac{n^2\left(\frac{a}{b}\right)^2}{\frac{1}{r_a} + \frac{m^2}{2(1 + \mu_S)}}} +$$

$$\frac{\frac{Z_a^2}{\pi^4}\frac{1}{m^2}}{\xi_1 + 2\xi_2\left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2 + \xi_3\left(\frac{n}{m}\right)^4\left(\frac{a}{b}\right)^4} \tag{A6}$$

where

$$k_{x_a} = \frac{N_x a^2}{E_S\bar{I}_S\pi^2}$$

$$Z_a{}^2 = \frac{2t_S a^4}{R^2 \overline{I}_S}$$

$$r_a = \frac{E_S \overline{I}_S \pi^2}{D_{Q_y} a^2}$$

The process of minimization of k_{x_b} and k_{x_a} with respect to the arbitrary parameters m and n is discussed in the section entitled "Results."

When the shear-stiffness parameters r_b and r_a are zero (deformations due to transverse shear neglected), the minimum values of k_{x_b} and k_{x_a} are found to be as follows: For an infinitely long curved plate

$$k_{x_b} \approx \eta + \frac{4}{1 - \mu_S{}^2} + \frac{Z_b{}^2/\pi^4}{\xi_1 + 2\xi_2 + \xi_3} \tag{A7}$$

when $\dfrac{Z_b}{\pi^2} < \sqrt{\xi_1\left(\eta + \dfrac{1}{1 - \mu_S{}^2}\right)\left(1 + 2\dfrac{\xi_2}{\xi_1} + \dfrac{\xi_3}{\xi_1}\right)}$,

$$k_{x_b} = 2\frac{Z_b}{\pi^2}\sqrt{\frac{\eta + \dfrac{1}{1 - \mu_S{}^2}}{\xi_1}} \tag{A8}$$

when $\dfrac{Z_b}{\pi^2} > \sqrt{\xi_1\left(\eta + \dfrac{1}{1 - \mu_S{}^2}\right)\left(1 + 2\dfrac{\xi_2}{\xi_1} + \dfrac{\xi_3}{\xi_1}\right)}$; for a cylinder

$$k_{x_a} = \eta + \frac{1}{1 - \mu_s^2} + \frac{Z_a^2/\pi^4}{\xi_1} \tag{A9}$$

when $\frac{Z_a}{\pi^2} \leqq \sqrt{\xi_1\left(\eta + \frac{1}{1 - \mu_s^2}\right)}$,

$$k_{x_a} = 2\,\frac{Z_a}{\pi^2}\sqrt{\frac{\eta + \frac{1}{1 - \mu_s^2}}{\xi_1}} \tag{A10}$$

when $\frac{Z_a}{\pi^2} \geqq \sqrt{\xi_1\left(\eta + \frac{1}{1 - \mu_s^2}\right)}$.

APPENDIX B

ANALYSIS OF ISOTROPIC SANDWICH PLATE

The equation of equilibrium for an isotropic sandwich plate loaded in axial compression (N_x positive in compression) obtained from reference 1 is

$$D_S \nabla^4 w + \left(1 - \frac{D_S}{D_Q} \nabla^2\right)\left(\frac{2t_S E_S}{R^2} \nabla^{-4} \frac{\partial^4 w}{\partial x^4} + N_x \frac{\partial^2 w}{\partial x^2}\right) = 0 \qquad \text{(B1)}$$

The boundary conditions for simply supported edges can be satisfied if the deflection function is taken as

$$w = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{(B2)}$$

Substitution of equation (B2) into equation (B1), defining

$$k_{x_b} = \frac{N_x b^2}{D_S \pi^2}$$

$$Z_b^2 = \frac{2t_S b^4 (1 - \mu_S^2)}{R^2 \overline{I}_S}$$

$$r_b = \frac{D_S \pi^2}{D_Q b^2}$$

and simplifying yields the theoretical load coefficient of an isotropic curved sandwich plate or cylinder,

$$k_{x_b} = \frac{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2}{\frac{1}{m^2}\left(\frac{a}{b}\right)^2 + r_b\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]} + \frac{\frac{Z_b^2}{\pi^4}\frac{1}{m^2}\left(\frac{a}{b}\right)^2}{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2} \tag{B3}$$

For the reason mentioned in the section entitled "Results," it is convenient to obtain a separate equation for cylinders. By defining

$$k_{x_a} = \frac{N_x a^2}{D_S \pi^2}$$

$$Z_a^2 = \frac{2t_S a^4\left(1 - \mu_S^2\right)}{R^2 \bar{I}_S}$$

$$r_a = \frac{D_S \pi^2}{D_Q a^2}$$

the equation becomes

$$k_{x_a} = \frac{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2}{\frac{1}{m^2} + r_a\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]} + \frac{\frac{Z_a^2}{\pi^4}\frac{1}{m^2}}{\left[1 + \left(\frac{n}{m}\right)^2\left(\frac{a}{b}\right)^2\right]^2} \tag{B4}$$

The process of minimization of these equations with respect to the arbitrary parameters is the same as that for the corrugated-core plates and is discussed in the section entitled "Results." For infinitely long isotropic-core sandwich plates and cylinders, since the equations are relatively simple, the results can be presented in more advanced form. The following equations give the minimum load coefficients: For infinitely long plates,

$$k_{x_b} \approx \frac{4}{(1 + r_b)^2} + \frac{Z_b{}^2}{\pi^4}\,\frac{1 - r_b}{4} \tag{B5}$$

when $\dfrac{Z_b}{\pi^2} < \dfrac{4\sqrt{1 - r_b}}{(1 + r_b)^2}$,

$$k_{x_b} \approx \frac{\dfrac{Z_b}{\pi^2}}{\sqrt{1 - r_b}}\left(2 - \frac{\dfrac{Z_b}{\pi^2}\,r_b}{\sqrt{1 - r_b}}\right) \tag{B6}$$

when $\dfrac{4\sqrt{1 - r_b}}{(1 + r_b)^2} < \dfrac{Z_b}{\pi^2} \leqq \dfrac{\sqrt{1 - r_b}}{r_b}$,

$$k_{x_b} = \frac{1}{r_b} \tag{B7}$$

when $\dfrac{Z_b}{\pi^2} \geqq \dfrac{\sqrt{1 - r_b}}{r_b}$; for cylinders,

$$k_{x_a} = \frac{1}{1 + r_a} + \frac{Z_a{}^2}{\pi^4} \tag{B8}$$

when $\dfrac{Z_a}{\pi^2} \leqq \dfrac{1}{1 + r_a}$,

$$k_{x_a} = \frac{Z_a}{\pi^2}\left(2 - \frac{Z_a}{\pi^2}\,r_a\right) \tag{B9}$$

when $\frac{1}{1 + r_a} \leqq \frac{Z_a}{\pi^2} \leqq \frac{1}{r_a}$,

$$k_{x_a} = \frac{1}{r_a} \tag{B10}$$

when $\frac{Z_a}{\pi^2} \geqq \frac{1}{r_a}$.

Equations (B5) and (B6) are very accurate for the curvature ranges indicated and give exact minimums for $Z = 0$. The remaining equations give the exact minimums for all values of Z. These results and results for a rectangular, curved, isotropic-core, sandwich plate are presented in figures 4, 5, and 6.

When the shear-stiffness parameters r_b and r_a are zero (deformations due to transverse shear neglected) the minimum values of k_{x_b} and k_{x_a} in equations (B3) and (B4) are found to be: For infinitely long plates,

$$k_{x_b} = 4 + \frac{Z_b^2}{4\pi^4} \tag{B11}$$

when $\frac{Z_b}{\pi^2} \leqq 4$,

$$k_{x_b} = 2\frac{Z_b}{\pi^2} \tag{B12}$$

when $\frac{Z_b}{\pi^2} \geqq 4$; for cylinders,

$$k_{x_a} = 1 + \frac{Z_a}{\pi^2} \tag{B13}$$

when $\frac{Z_a}{\pi^2} \leqq 1$,

$$k_{x_a} = 2 \frac{Z_a}{\pi^2} \qquad \text{(B14)}$$

when $\frac{Z_a}{\pi^2} \geqq 1.$

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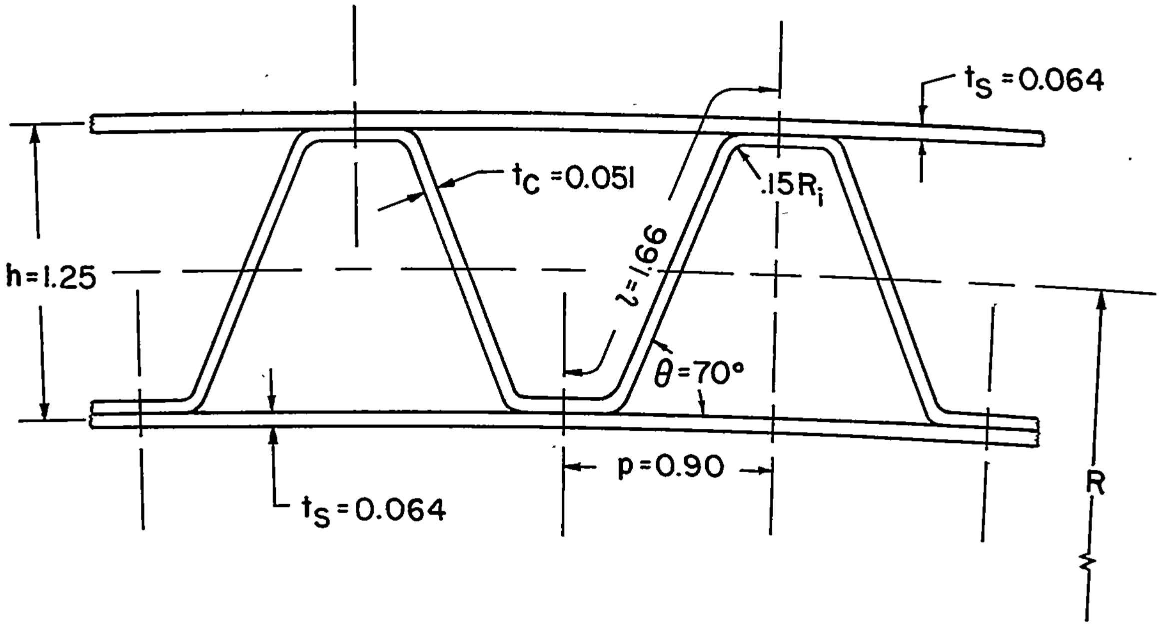

$\bar{I}_S = 0.050$ in.^{4}/in. $\qquad$ $\bar{I}_C = 0.015$ in^4/in.

$\bar{A}_S = 0.128$ in.^{2}/in. $\qquad$ $\bar{A}_C = 0.094$ in^2/in.

$E_S = 10.5 \times 10^6$ lb/in.^{2} $\qquad$ $E_C = 10.5 \times 10^6$ lb/in.^{2}

$\mu_S = \mu_C = \frac{1}{3}$ $\qquad$ $D_{Q_y} = 8500$ lb/in.

Figure 1.- Corrugated-core sandwich section.

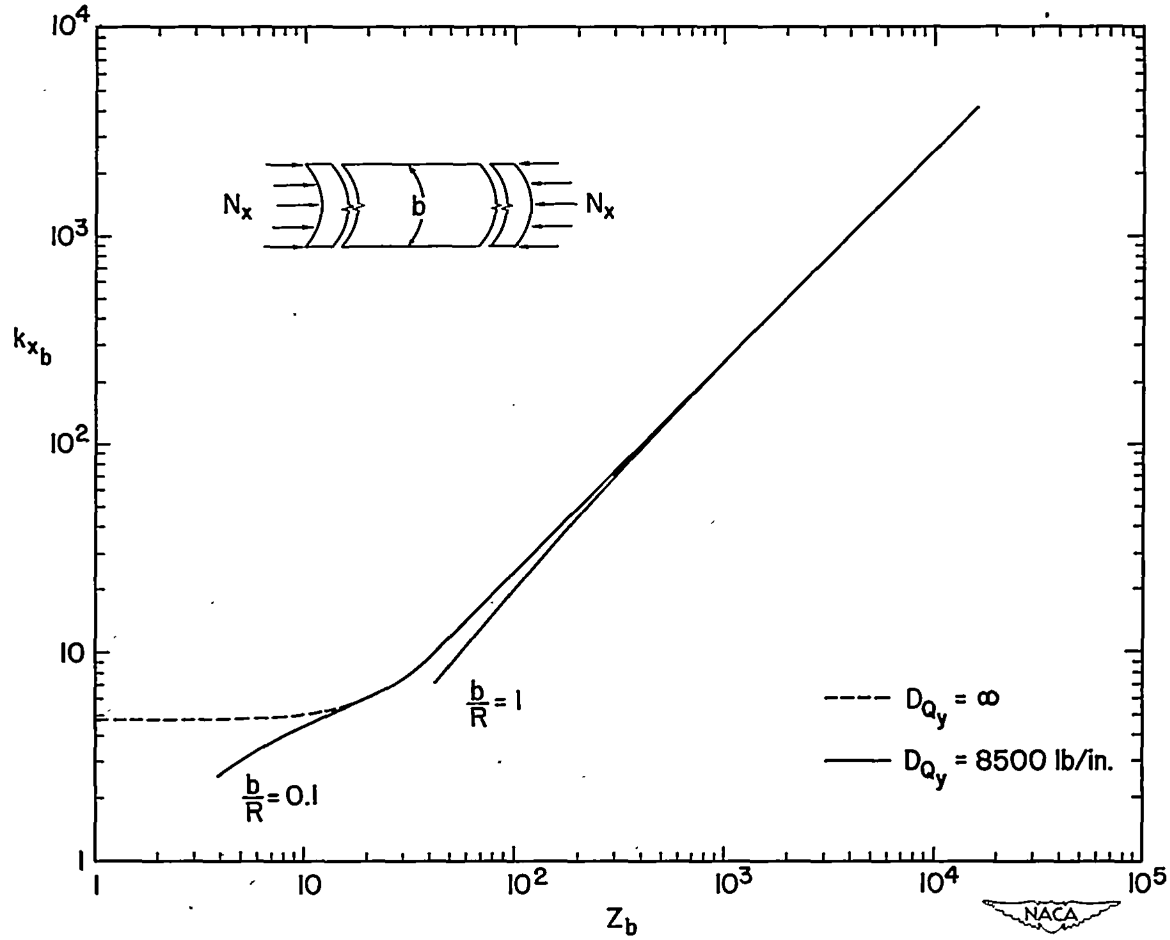

Figure 2.- Critical axial compressive-load coefficients for simply supported infinitely long curved plates with corrugated-core sandwich section shown in figure 1.

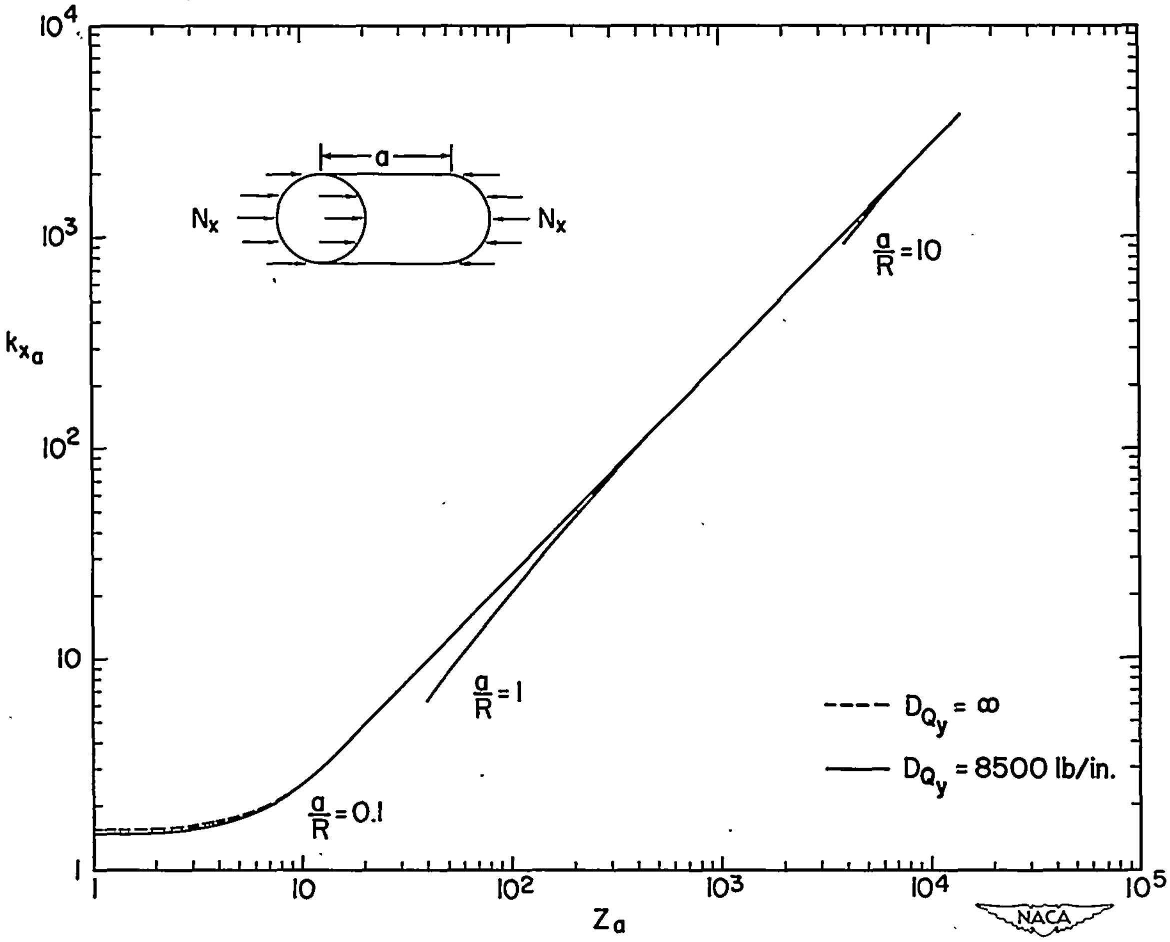

Figure 3.- Critical axial compressive-load coefficients for simply supported cylinders with corrugated-core sandwich section shown in figure 1.

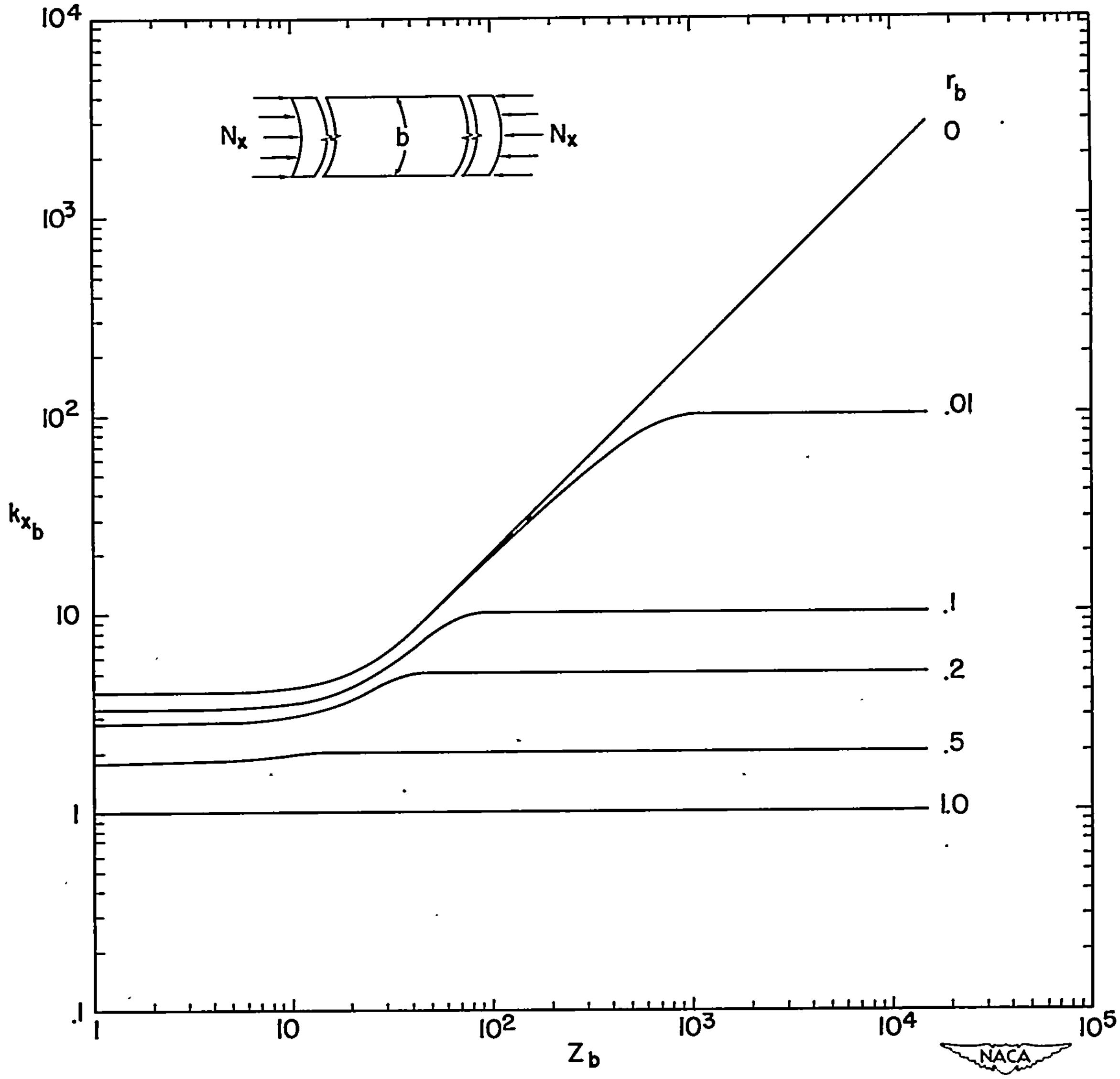

Figure 4.- Critical axial compressive-load coefficients for simply supported infinitely long curved isotropic sandwich plates.

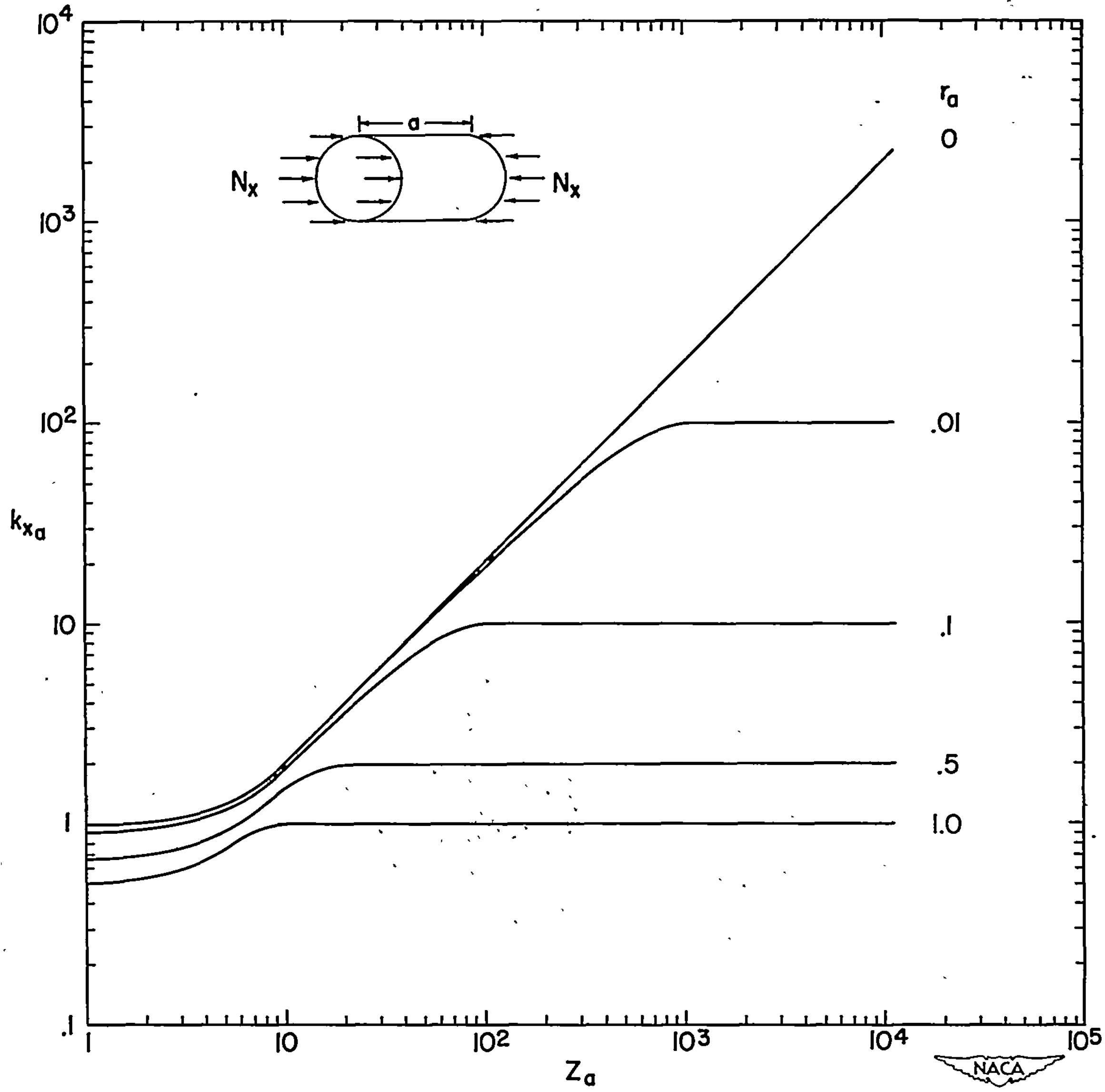

Figure 5.- Critical axial compressive-load coefficients for simply supported isotropic sandwich cylinders.

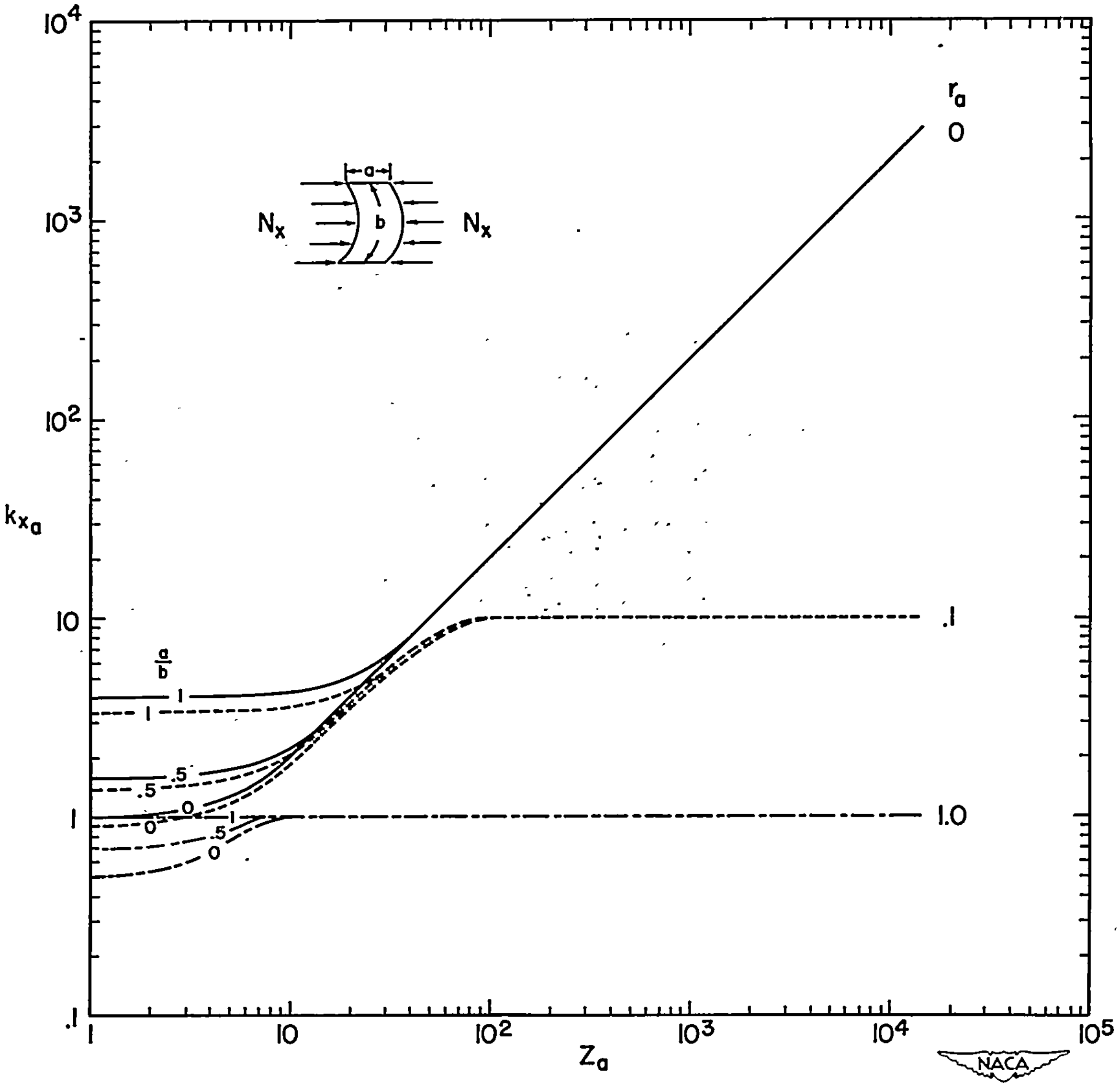

Figure 6.- Critical axial compressive-load coefficients for simply supported rectangular curved isotropic sandwich plates.