Week 1: Polynomials

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1 Problems

1.1 Problems

[1 ♟] Problem 1. Consider the quadratic

$$3x^2 + 5x + 7$$

- What is the sum of the roots?
- What is the product of the roots?
- What is the sum of the reciprocals of the roots?
- What is the sum of the squares of the roots?

[1 ♟] Problem 2. Consider the polynomial

$$x^n + 2x^{n-1} + 3x^{n-2} + \cdots + nx + (n+1)$$

- What is the sum of the roots?
- What is the product of the roots?
- What is the sum of the roots taken two at a time?
- What is the sum of the roots taken three at a time?
- What is the sum of the squares of the roots?

[1 ♟] Problem 3. Consider the polynomial

$$x^{2022} - x$$

which has roots $r_1, r_2, \ldots r_{2022}$. Compute

$$r_1^{2022} + r_2^{2022} + \cdots + r_{2022}^{2022}$$

> *Remark 1.1.1.* Problems that require Viete's Formulas may have the application of Viete's more hidden, or require the solver to compute a difficult sum of product. Not all problems will be this straight forward.

[1 ♟] Problem 4. Suppose $P(x)$ is a monic polynomial of degree 3, such that $P(0) = 0$, $P(1) = 1$, and $P(2) = 4$.

- Find $P(3)$
- In fact, just go ahead and find $P(x)$
- Find $P'(x)$, just for fun.
- Just for practice, find the sum of the roots taken 2 at a time of $P(x)$
- What happens if we make the extra condition that $P(3) = 9$?

[1 ♟] Problem 5. Suppose we have that

$$P(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$$

- What polynomial is equal to $x^4 P(\frac{1}{x})$?
- Let $R(x) = x^4 P(\frac{1}{x})$. What can we say about the roots of $R(x)$?
- Find the sum of the reciprocals of the roots of $P(x)$, by using $R(x)$.

[2 ♟] Problem 6. Suppose we have that

$$P(x) = x^3 + x^2 + 5x + 2$$

- What can we say about the roots of $P(x+1)$ compared to the roots of $P(x)$?
- Let r, s, t be the roots of $P(x)$. Compute

$$\frac{1}{r+2} + \frac{1}{s+2} + \frac{1}{t+2}$$

[1 ♟] Problem 7 (SMT 2022). If

$$f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2022$$

compute $f'(3)$.

[1 ♟] Problem 8 (SMT 2011). If

$$f(x) = (x-1)^4(x-2)^3(x-3)^2$$

Compute $f'''(1) + f''(2) + f'(3)$.

[1 ♟] Problem 9 (HMMT 2005). Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose that the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

[1 ♟] Problem 10 (HMMT 2010). Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

[1 ♟] Problem 11 (BMT 2020). Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function defined by

$$f(x) = f'(2)x^2 + x$$

Find $f(x)$.

[1 ♟] Problem 12 (BMT 2022). Let $f(x)$ be a degree-4 polynomial such that $f(x)$ and $f'(x)$ both have 20 and 22 as roots. Given that $f(21) = 21$, compute $f(23)$.

[1 ♟] Problem 13 (BMT 2021). Let $f(x) = (x+3)(2x+5)(3x+7)(x+1)$. Compute $f^{(4)}(5)$, i.e. the fourth derivative of f evaluated at 5.

[2 ♟] Problem 14 (BMT 2020). If a is a positive real number such that the region of finite area bounded by the curve $y = x^2 + 2020$, the line tangent to that curve at $x = a$, and the y-axis has area 2020, compute a^3.

[2 ♟] Problem 15 (SMT 2022). The straight line $y = ax + 16$ intersects the graph of $y = x^3$ at 2 distinct points. What is the value of a? (You can actually solve this without any calculus, but the calculus solution is also nice.)

[2 ♟] Problem 16 (SMT 2014). There is a unique positive real number a such that the tangent line to $y = x^2 + 1$ at $x = a$ goes through the origin. Compute a.

[2 ♟] Problem 17 (HMMT 2023). Suppose $P(x)$ is a cubic polynomial such that $P(\sqrt{5}) = 5$ and $P(\sqrt[3]{5}) = 5\sqrt[3]{5}$. Compute $P(5)$.

[2 ♟] Problem 18 (HMMT 2007). Find the real number a such that the curve $f(x) = e^x$ is tangent to the curve $g(x) = ax^2$.

[2 ♟] Problem 19 (FAMAT Feb 2022). For $0 \leq a < 1$, the tangent line to the graph of $y = (x-1)^2$ at $x = a$ forms a non-degenerate right triangle with the coordinate axes. Find the maximum possible area of such a triangle.

[2 ♟] Problem 20 (HMMT 2002). Determine the positive value of a such that the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0,0)$, $(a,0)$, $(0, a^2+1)$, and (a, a^2+1).

[2 ♟] Problem 21 (HMMT 2007). The elliptic curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4,0)$ at the point (x_0, y_0). Determine the sum of all possible values of x_0.

[2 ♟] Problem 22 (HMMT 2006). A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

[2 ♟] Problem 23 (HMMT 2009). Let P be a fourth degree polynomial, with derivative P', such that $P(1) = P(3) = P(5) = P'(7) = 0$. Find the real number $x \neq 1, 3, 5$ such that $P(x) = 0$.

[3 ♟] Problem 24 (HMMT 2000). An envelope of a set of lines is a curve tangent to all of them. What is the envelope of the family of lines $y = \frac{2}{x_0} + x(1 - \frac{1}{x_0^2})$, with x_0 ranging over the positive real numbers?

[3 ♟] Problem 25 (SMT 2019). Consider the parabola $y = ax^2 + 2019x + 2019$. There exists exactly one circle which is centered on the x-axis nd is tangent to the parabola at exactly two points. It turns out that one of these tangent points is $(0, 2019)$. Find a.

[3 ♟] Problem 26 (FAMAT Feb 2022). Let $p(x)$ be the polynomial of least degree with leading coefficient 1 such that

$$|(x-1)(x-2)^2(x-3)(x-4)^3| \cdot p(x)$$

is differentiable everywhere. Evaluate $p(6)$.

[3 ♟] Problem 27 (FAMAT Feb 2022). Let the complex roots of the polynomial $x^3 - x^2 + 4x + 1$ be a, b, and c. Compute the following:

$$\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c}$$

[3 ♟] Problem 28 (FAMAT Feb 2022). The normal lines to the graph of $y = x^2$ at $x = a$, $x = b$, and $x = c$, where a, b, c are distinct real numbers, intersect at $(20, 22)$. Find $a^2 + b^2 + c^2$.

[3 ♟] Problem 29 (SMT 2023). Let $r_1(t) \leq r_2(t) \leq r_3(t)$ be the roots of

$$x^3 + tx + 2$$

When $t = -3$, compute $r_1'(t)$

[4 ♟] Problem 30 (Navid Safaei). Consider the polynomial

$$P(x) = dx^d - x^{d-1} - x^{d-2} - \cdots - x - 1$$

- Prove that $P(x)$ has d distinct roots
- Prove that all roots of $P(x)$ have absolute value less than or equal to 1.