2023 In Review

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Contents

1 FAMAT Problems

Here some of the interesting FAMAT problems I wrote this year.

1.1 2023 Jan Statewide P26

Let

$$x \lim_{x \to 0^+} \sum_{n=1}^{2023} \lfloor \frac{n}{x} \rfloor = S$$

where S is an integer. Find the remainder when S is divided by 6.

This problem has a very important takeaway: IIT-JEE prep books are excellent for calculus competitions! Let me elaborate: as I was looking for difficult problems to add to this test, I was an IIT-JEE prep book focusing on differential calculus, and there was a section on squeeze theorem which I had never seen before. By the time I wrote this test, I was already top 15 in Florida in FAMAT, and I always thought that squeeze theorem was just for "rigor", and that there was no way to write a problem relying on it, but I was wrong!

The first trick to simplify this problem is to make the substitution $x \mapsto \frac{1}{t}$. The fact that the limit is one-sided is a major indicator to do this, but also the fact that we probably need to use L'Hopital, which requires us to move the x which is outside the sum into the denominator. Doing so yields the new limit

$$\lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{2023} \lfloor nt \rfloor$$

The next step is to notice that in order to actually compute a limit involving floors, we must utilize squeeze theorem. In particular,

$$nt - 1 < \lfloor nt \rfloor \leq nt$$

It is easy to see that this inequality is true by subtracting each side by $\lfloor nt \rfloor$ then adding 1. Making this substitution yields the inequality chain

$$\lim_{t \to \infty} \sum_{n=1}^{2023} n - \frac{1}{t} < S \leq \lim_{t \to \infty} \sum_{n=1}^{2023} n$$

From this, it is clear by squeeze theorem that both sides of the inequality converge to the same sum, which is equal to $\frac{2023(2024)}{2}$, which has a remainder of 4 mod 6.

18.6 % of competitors solved this on competition day, which is a bit higher than I expected, but also it is probably easy to guess how the limit should behave.

1.2 2023 Jan Statewide P28

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(2^n)x^{(2^n-1)}}{1+x^{(2^n)}}$$

. Evaluate $f\left(\frac{1}{2023}\right)$

Surprisingly, this problem also gained inspiration from the same IIT-JEE calculus book, and I was quite proud of this problem, mainly because I thought it was really fun!

The first observation to be noticed is that the numerator is the derivative of the denominator. Hence, each term in the sum is of the form $\frac{g'(x)}{g(x)}$. However, this also appears from taking the derivative of $\ln(g(x))$, so we can rewrite the sum as

$$\left(\sum_{n=0}^{\infty} \ln(1+x^{(2^n)})\right)' = \left(\ln\left(\prod_{n=0}^{\infty} 1+x^{(2^n)}\right)\right)'$$

Now we need to simplify the product. Define the product to be $h(x)$. What happens if we multiply this by $(1-x)$? We see that everything collapses, and that

$$h(x)(1-x) = 1 - x^{2^{\infty}}$$

Note that we are evaluating f at a value of x with absolute value less than 1, so the $x^{2^{\infty}}$ term goes to 0, and thus the product simplifies to $\frac{1}{1-x}$, and hence

$$f(x) = (-\ln(1-x))' = \frac{1}{1-x}$$

Hence, the answer is $\frac{2023}{2022}$.

11.2 % of competitors solved this in competition, which seems about right.

1.3 2023 Jan Statewide P30

Evaluate

$$\lim_{x\to 0}\left(\frac{(1+\frac{x}{2023})^{2023}-1}{x}\right)^{\frac{1}{x}}$$

I believe the motivation of writing this problem was to create the hardest limit possible that was solvable strictly by elementary techniques. Of course, although elementary, it just goes to show that series expansions are always the best strategy no matter what limit you are doing.

Honestly, this problem dies from one observation, and that is to expand $(1+\frac{x}{2023})^{2023}$ using binomial theorem. Hence, the limit simplifies to

$$\lim_{x\to 0}\left(1+\frac{1011}{2023}x+\dots\right)^{\frac{1}{x}}$$

Note that this is actually very similar to the limit definition of e, just with extra terms which are $O(x^2)$ slapped inside. However, it can be shown by L'Hopital that those extra terms don't contribute to the limit, i.e.

$$\lim_{x\to 0}(1+ax+O(x^2))^{\frac{1}{x}}=e^a$$

Hence, our limit is equal to

$$e^{\frac{1011}{2023}}$$

9.2 % of competitors got this right in competition, quite difficult!

1.4 2023 Area and Volume P25

Consider the set

$$S=\{(x,y)\in\mathbb{R}^2 : x^2+5xy+y^2=42, xy>0\}$$

Define $A_{(x,y)}$ to be the area of the right triangle formed by the coordinates $(0,0),(x,0),(x,y)$, where $(x,y)\in S$. Evaluate the maximum of $A_{(x,y)}$.

I don't remember where I got the inspiration to write this problem, but the trick behind it is somewhat neat.

Basically, $A_{(x,y)} = \frac{1}{2}xy$, so we need to maximize xy. However, note that we can rewrite the condition imposed by our set as

$$(x-y)^2 + 7xy = 42 \implies 7xy = 42 - (x-y)^2$$

When $x = y$, we see that $7xy = 42 \implies xy = 6$, and this is maximum since $(x-y)^2 \geq 0$. Hence, the maximum area of the triangle is 3.

9 % of competitors solved this in competition. Mainly because this is one of the hardest and poor-judgement tests I've ever written, and will ever write, since it was the first test I wrote.

1.5 2023 Area and Volume P27 (Shortlist?)

> Define a normal segment as the line segment formed by the two intersection points of a graph and a line normal to the graph. Let $f(x) = x^2$, and let $g(x)$, where defined, be the locus of the midpoints of all normal segments of $f(x)$. If $g(1) = \frac{a}{b}$, where a, b are relatively prime, find $a + b$.

Loci in calculus are cool, especially since you can basically take an arbitrary curve and consider random normals/tangents, and find a cool construction. In particular, the fact that (most) normals of a parabola intersect the parabola exactly twice motivated the problem.

We seek to find $g(x)$ explicitly, then just plug in 1 to finish. So to start, suppose we take the normal line to f at the point (a, a^2). Then the normal line is equal to $-\frac{1}{2a}(x-a) + a^2$. Setting this equal to x^2, we can move everything to one side and factor to get that $(x-a)(x+a+\frac{1}{2a}) = 0$, hence the second point of intersection is $(-a-\frac{1}{2a}, (a+\frac{1}{2a})^2)$. The midpoint of this line segment is hence $(-\frac{1}{4a}, a^2 + \frac{1}{8a^2} + \frac{1}{2})$. Now, consider the substitution $a \mapsto -\frac{1}{4t}$. This transforms the midpoint to be $(t, \frac{1}{16t^2} + 2t^2 + \frac{1}{2})$, hence $g(x) = \frac{1}{16x^2} + 2x^2 + \frac{1}{2}$, and our answer is $\frac{41}{16}$.

The editor removed this problem from the test, weirdge.

1.6 2023 Area and Volume P28

> For every real t in the interval $(0, \infty)$, draw the line segment connecting $\left(0, \frac{1}{t}\right)$ and $(t, 0)$. Doing so bounds a curve, $f(x)$, in the first quadrant. Evaluate $\int_1^e f(x)\mathrm{d}x$.

This problem was inspired by a CMUMC problem of the day!

The idea is to find an explicit form for the curve $f(x)$. To see how, think about what it means for the curve to be bounded. In particular, for a given $x = a$, there must be some maximum value of y which is contained by one of the line segments, and we take this value to be $f(a)$. Note that each line segment is of the form $-\frac{1}{t^2}(x-t) = \frac{1}{t} - \frac{x}{t^2}$. Now, fix $x = a$, and we differentiate with respect to t to get a derivative of $-\frac{1}{t^2} + \frac{2a}{t^3}$. Setting this equal to 0, we find that $2a - t = 0$ yields a critical point. I.e., when $t = 2a$, we get our maximum value of y for $x = a$, hence $f(a) = -\frac{1}{4a^2}(-a) = \frac{1}{4a}$. Integrating this from 1 to e yields our answer of $\frac{1}{4}$.

Only 6.2 % of competitors got this right in competition, which equates to 4 people.

2 CMWMC

2.1 CMWMC Indiv P13

There exist two complex numbers z_1, z_2 such that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 338$$

Find the length of the hypotenuse of the right triangle with legs of length $|z_1|$, $|z_2|$.

This problem is very simple, I'm actually surprised it made the test. The motivation came from reading "Complex Numbers and Beyond" by Titu Andreescu.

Using the fact that $|a|^2 = a\bar{a}$, we find that the equation simplifies to

$$2|z_1|^2 + 2|z_2|^2 = 338$$

Hence, the length of the hypotenuse we seek is $\sqrt{169} = 13$.

Honestly, kind of cute! It's just Parallelogram Law though :pensive: