# A Gentle Introduction to Competition Calculus

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# 1 Introduction

Thank you for taking the time to look at my project! My name is Michael, and I'm a senior who enjoys competitive math (but has only taken AMC 12 once and missed cutoffs lolol). This book was not structurally planned, but instead is just a compilation of math handouts I have written throughout my senior year of high school. This is definitely a work in progress and I may finish it over the summer. Also, this isn't very rigorous, and almost every chapter (especially calculus) assumes prior knowledge of the subject. Enjoy!

# 2 Principles of Algebra

## 2.1 Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra tells us that for any polynomial of degree n, it will have exactly n complex roots. These roots can be real or imaginary, but regardless there are n of them. Before we get deep into this concept, let's define a few terms. What exactly is the degree of a polynomial? All polynomials can be written as

$$a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \dots a_{n+1}x^{n-n}$$

I'm sure many of you have seen this notation before, but maybe not quite understood it. All this is saying is that a polynomial with a degree of n can be written in this form, where every x term has a coefficient $a_i$. What is important to gain here is that a polynomial with degree n has a leading term with degree n.

While this notation can be useful for proofs involving polynomials, we are more concerned with what the Fundamental Theorem of Algebra implies. Specifically, it tells us that any polynomial with degree n can be rewritten as

$$a_1(x - r_1)(x - r_2)(x - r_3)\dots(x - r_n)$$

also known as factoring. This notation has important implications, most relevantly involving vieta's formulas (which we will cover later in this handout), factor theorem, and algebra in general. Another important fact that you may remember from Algebra 2 is that any polynomial with rational coefficients with a complex root will also have it's conjugate as a root. It is also true that and polynomial with rational coefficients that has an irrational root will have its conjugate as a root.

**Fact 2.1.1.** Any polynomial with rational coefficients with root $a + bi$ will also have $a - bi$ as a root

**Fact 2.1.2.** Any polynomial with rational coefficients with root $a + \sqrt{b}$ will also have $a - \sqrt{b}$ as a root

Intuitively, if we want our polynomial to have rational coefficients, we want our irrational roots and imaginary roots to "cancel out" when we multiply out our factors. Of course, this isn't rigorous by any means, but all that is important is that you know that this is true. Let's look at a very simple example:

**Example 2.1.3**
A real, fifth degree polynomial with rational coefficients has the roots $1 + 2i$ and $5 - \sqrt{2}$. Write a possible polynomial that fits this description

*Solution:* From what we went over earlier, we know that along with the given roots, the polynomial must also have $1 - 2i$ and $5 + \sqrt{2}$ as roots. However, this only gives us 4 total roots, and the question wants us to give a polynomial of degree 5. For the sake of simplicity, let 0 be the fifth root of our polynomial. It then suffices to write $x(x - (1 - 2i))(x - (1 + 2i))(x - (5 - \sqrt{2}))(x - (5 + \sqrt{2}))$ And we are done!

Of course, this problem is much simpler than what you will see on Mu Alpha Theta tests, but these examples are just to make sure you have a way to implement the topic.

## 2.2 Remainder Theorem

To understand what we are about to go over, let's take a brief look at long division. Say you wanted to divide 10 by 3. We find that our answer has a quotient of 3 and a remainder of 1. This can be written as:

$$\frac{10}{3} = 3 + \frac{1}{3}$$

If we multiply both sides of the equation by 3, we get that $10 = 9 + 1$, as desired. But why bother going over this? As it turns out, we can apply this idea of remainders to polynomials as well. Let's first review how to long divide using polynomials.

**Example 2.2.1**
Evaluate $\frac{2x^3 - 5x^2 + x - 10}{x^2 - 4x + 1}$

*Solution:* We will go over this in practice because I have no clue how to write long division in latex! =D

Notice that in this example, we had a remainder just like we did when dividing 10 by 3. This can be generalized in the following result:

**Theorem 2.2.2**

When dividing two polynomials $f(x)$ and $p(x)$, we have

$$\frac{f(x)}{p(x)} = q(x) + \frac{r(x)}{p(x)}$$

where f(x) is our dividend, p(x) is our divisor, q(x) is our quotient and r(x) is our remainder. By multiplying both sides of this equation by $p(x)$, we achieve the result

$$f(x) = p(x)q(x) + r(x)$$

Before we get into solving problems with this result, there are a few more things we must go over.

**Fact 2.2.3.** Consider a constant $r$ and a polynomial $P(x)$. r is a root of $P(x)$ if and only if $P(r) = 0$. Additionally, it follows that if $r$ is a root of $P(x)$, then $P(x)$ is divisible by $(x - r)$.

These results are easily shown from our first section on the Fundamental Theorem of Algebra. Let's tie this into what we just learned

**Fact 2.2.4.** If a polynomial $P(x)$ is divided by $x - a$, then the remainder is $P(a)$.

To prove this, look back to our first theorem of this section. We know that when dividing a polynomial (in this case $P(x)$) by another polynomial $(x - a)$, we can write it as $P(x) = (x - a)q(x) + r(x)$. We wish to solve for the remainder, and plugging in $x = a$ seems to do the trick. Doing so gives $P(a) = r(a)$, and we are done. This proof is very important in solving remainder theorem problems, and we will be doing more of this later in this section. It is also very important to note that our remainder is always at most one degree less than our divisor. So, since we were dividing by a polynomial of degree 1, our remainder is a constant. Let's use this information to solve an actual problem.

**Example 2.2.5 (Classic)**
Consider a polynomial $P(x)$. The remainder when it is divided by $(x-1)$ is 4 and the remainder when it is divided by $(x-2)$ is 8. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.

*Solution:* Remember that the remainder when dividing two polynomials is always at most 1 degree than the divisor. So when we set up our division equation, instead of the remainder being a constant, it will be a linear polynomial. Let's write it out:

$$P(x) = (x-1)(x-2)Q(x) + ax + b$$

*Remark 2.2.6.* It is important to note that the remainder is *at most* one degree less than the divisor. When we are dividing by a quadratic, it is entirely possible that our remainder is a constant, the a term in $ax + b$ would just be a 0. If you are wondering why this is true, think about what would happen if the numerator of the remainder was a higher degree than the denominator.

From our earlier work, we showed that the remainder when $P(x)$ is divided by $(x-a)$ is equal to $P(a)$, and this seems like it will be helpful for us in this problem. Specifically, the problem tells us that the remainder when $P(x)$ is divided by $(x-1)$ is 4. This means that $P(1) = 4$. Following this logic, $P(2) = 8$. Let's plug in these values into our remainder equation and see what happens:

$$P(1) = a + b = 4$$

$$P(2) = 2a + b = 8$$

Remember the goal of this question: we want to find the value of $ax + b$, so if solve for a and b using these results as a system of equations, we should be good!

*Remark 2.2.7.* Always go back to the question to make sure you know what the goal of the question is. What good is work if you don't know what you're solving for?

By solving this system of equations, we find that a = 4, thus b = 0, and our answer is $\boxed{4x}$.

## 2.3 Factor Theorem

This section will go over a type of problem that shows up throughout all areas of FAMAT, and it was never really fully explained to me until recently when I decided to start going through AOPS Intermediate Algebra. Here is the basic idea:

**Fact 2.3.1.** A polynomial of degree n has exactly n roots. Thus, if you have an nth degree polynomial with n+1 zeroes, then that polynomial must equal zero for all values of x.

This may seem weird and not very useful at first, but it should immediately make sense when we go through some problems.

**Exercise 2.3.2.** Suppose that f(x) is a quadratic polynomial such that $f(2) = 4, f(3) = 9, and f(4) = 16$. Prove that $f(x) = x^2$

This sure seems to behave like $x^2$, but to be sure that this is the only second degree polynomial that satisfies this condition, let $f(x)$ be another second degree polynomial, and let $g(x) = f(x) - x^2$. Thus, we see that 2, 3, and 4 must be roots of $g(x)$, since $f(x) = x^2$ for $x = 2, 3, 4$. However, this implies that $g(x)$ has three roots, but $g(x)$ has a maximum degree of 2. Thus, $g(x) = 0 \implies f(x) = x^2$, and we are done.

**Exercise 2.3.3 (FAMAT Calculus).** A cubic polynomial f(x) with leading coefficient 1, satisfies $f(1) = 2, f(2) = 3, f(3) = 4$. Determine the value of $f'(0)$.

If you aren't in calculus don't worry, you can still follow most of the important steps of this problem. FAMAT often asks this type of question in all divisions, so it's very useful to know. First, we find the pattern that the given points have. We see that for these x values, $f(x)$ seems to be $x + 1$. We will know find $f(x)$ by defining a new function, $g(x)$, as

$$g(x) = f(x) - (x + 1)$$

We know that $x = 1, 2, 3$ must be roots of $g(x)$, and since $g(x)$ has a degree of at most 3, we know that these are the only roots it can have. Thus we can express $g(x)$ as

$$f(x) - (x + 1) = a(x - 1)(x - 2)(x - 3)$$

However, it is given that the leading coefficient of $f(x) = 1$, so $a = 1$. We can now subtract the x+1 to the other side and find that

$$f(x) = (x-1)(x-2)(x-3) + x + 1$$

And ta-da! We found f(x)! Now for the calculus part, we just expand and take the derivative, and find that $f'(0) = 12$.

This is all you need to know for basic FAMAT problems. I will leave harder (and cooler) problems as exercises.

## 2.4 Vieta's Formula

Let's experiment a bit. We saw in section 1 that any quadratic can be written as $a_1(x - r_1)(x - r_2)$. Why don't we see what happens when we multiply this out? We end up with

$$a_1(x-r_1)(x-r_2) = a_1(x^2 - (r_1+r_2)x + r_1r_2) = a_1x^2 - a_1(r_1+r_2)x + a_1r_1r_2$$

It may be easier to see what this implies by letting $a_1 = 1$. What we are seeing is a very peculiar relationship between the coefficients of a quadratic and it's roots. Specifically, if we write the quadratic as $ax^2 + bx + c$, we have that $\frac{b}{a} = -(r_1 + r_2)$ and that $\frac{c}{a} = r_1r_2$.

**Theorem 2.4.1**
For a quadratic in the form of $ax^2 + bx + c$ and roots $r_1$ and $r_2$, $r_1 + r_2 = -\frac{b}{a}$ and $r_1r_2 = \frac{c}{a}$

This allows us to create relationships between the coefficients of a quadratic and its roots! Let's look at a quick example that applies this idea:

**Example 2.4.2 (AMC 10A 2003/18)**
What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

*Solution:* The $\frac{1}{x}$ term encourages us to multiply the equation by x. We then get $\frac{2003}{2004}x^2 + x + 1 = 0$. Furthermore, let's multiply the equation by 2004 to get rid of the annoying fraction in the leading coefficient. We then have

$$2003x^2 + 2004x + 2004$$

Now that we have the polynomial in an easier form, let's look back at what the question is asking us. We want to find the sum of the reciprocals of the roots of this quadratic. We know that quadratics only have 2 roots, so let's make variables for these roots, r and t. Thus the sum of the reciprocals can be written as $\frac{1}{r} + \frac{1}{t}$. However, this doesn't look like anything we can solve with vieta's. Let's add the fractions together and see what happens. We see that $\frac{1}{r} + \frac{1}{t} = \frac{r+t}{rt}$ Aha! We can find both the numerator and the denominator using vieta's formulas, so let's go ahead and do that. We find that $r + t = \frac{-2004}{2003}$ (Remember that this is negative!) And that $rt = \frac{2004}{2003}$. Plugging these into the fraction we solved for earlier, we find that the sum of the reciprocal of the roots is equal to $\boxed{-1}$, and we are done.

While this does seem like a neat formula for quadratics, at this point you may be thinking that this is more of a novelty than anything, and that you can forget about it. Wrong! This formula not only applies to quadratics, but to *all* polynomials. Let's see how. We found our formula for the quadratic by multiplying out $a_1(x - r_1)(x - r_2)$, but what would happen if we made it a cubic polynomial instead? Let's find out! Through careful algebra, we find that:

$$a_1(x - r_1)(x - r_2)(x - r_3) = a_1(x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3)$$

I am not multiplying the $a_1$ term out because it will be harder to understand the result this way, and you already know why we need to divide by $a_1$ because of the quadratic result. Immediately, you should be noticing a few things. For starters, the positive and negative signs seem to be alternating, and this turns out to be true in all vieta results. In fact, it was also true for the quadratic result, it was just harder to see because there were only 3 terms. You may also notice that in each coefficient after the leading coefficient, the roots seem to be grouping closer together in a sense. They seem to be following a pattern, but how do we describe this pattern? Let's make a description for this ourselves. We will call this pattern "Symmetric Sums."

**Theorem 2.4.3**

Let's create an operator called a "symmetric sum". We shall notate this as $s_n$. Let's take a look at a set of 4 terms: a, b, c, and d. The n in $s_n$ will dictate how many terms we will have in each group of our sum. This is how it will look:

$$s_1 = a + b + c + d$$

$$s_2 = ab + ac + ad + bc + bd + cd$$

$$s_3 = abc + abd + acd + bcd$$

$$s_4 = abcd$$

Note that we can only have the n of $s_n$ be as great as the number of terms in the set. This notation will allow us to make a generalization of vietas to all real polynomials. Note that some problems may call $s_2$ "Taking the sum of the roots 2 at a time", etc.

Let's apply this formula to our cubic result of vietas. Let $r_1, r_2 and r_3$ be members of the set we wish to apply symmetric sums to, and let our cubic polynomial be $ax^3 + bx^2 + cx + d$. We then have

$$r_1 + r_2 + r_3 = s_1 = -\frac{b}{a}$$

$$r_1r_2 + r_1r_3 + r_2r_3 = s_2 = \frac{c}{a}$$

$$r_1r_2r_3 = s_3 = -\frac{d}{a}$$

Finally, let's do some nasty notation to generalize this result to all real polynomials.

**Theorem 2.4.4**

For a real polynomial $P(x)$ with degree n, we can write this as

$$a_1x^n + a_2x^{n-1} + \ldots a_{n+1}$$

Now, let the roots of this polynomial be in a set for which we can apply our symmetric sums operator. It is then true that:

$$s_1 = -\frac{a_2}{a_1}$$

$$s_2 = \frac{a_3}{a_1}$$

$$s_3 = -\frac{a_4}{a_1}$$

$$\vdots$$

$$s_n = (-1^n)\frac{a_{n+1}}{a_1}$$

Now in all honesty, this generalization is super scuffed and doesn't use "real math" notation, but it is just a way for you guys to see how vieta's can be applied to all real polynomials. To wrap up this section, let's look at another vieta's problem.

**Example 2.4.5 (AIME)**

Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t$ are a+b, b+c, and c+a. Find t

*Solution:* When tackling vieta problems, I always like to write down all the information I can get, and then work from there. Let's start with the first polynomial. We see that $a + b + c = -3$, $ab + bc + ac = 4$, and $abc = 11$. From the second polynomial, everything is irrelevant except t, and we find that $t = -(a + b)(a + c)(b + c)$. It is entirely possible to just multiply t out, and then factor a bunch to arrive at the answer, but the most efficient way to approach this problem–and many other problems like it– is to use substitutions to make life easier. We know that $a + b + c = -3$, which means that $a + b = -3 - c$. We will make a similar substitution for each term in t to get that $t = -(-3 - a)(-3 - b)(-3 - c) = (3 + a)(-3 - b)(-3 - c)$. From here, we can simply multiply each term out and factor, obtaining $t = 27 + 9(a + b + c) + 3(ab + bc + ac) + abc$. By substituting in known values we find that $t = \boxed{23}$, and we are done. Furthermore, think back to our first section. Can't we write the first polynomial as $(x - a)(x - b)(x - c)$? The leading coefficient is 1, meaning that we can ignore the $a_1$ that would usually be in front of our roots. Now look at what t was equal to: $-(-3 - a)(-3 - b)(-3 - c)$! If we let the first polynomial be $P(x)$, this is the same as $-P(-3)$! This saves us a lot of time and makes the problem rather trivial.

## 2.5 Useful Factorizations

We will now look at a few factorizations that will come in handy no matter what level of math you are competing in. The most important one and most well known is difference of squares.

**Theorem 2.5.1**
$a^2 - b^2 = (a + b)(a - b)$

The importance of this is self explanatory. It shows up in trig, in limits, pretty much everywhere. However, how would you factor the sum of two squares? Is it even important? Let's see what it looks like:

**Theorem 2.5.2**
$a^2 + b^2 = (a + b)^2 - 2ab$

This doesn't look anything like difference of squares! How did we think of doing this? The method used for this factorization is important in many problems. Basically, you want to think of what you could do to come up with an $a^2 + b^2$ term. The most obvious is $(a + b)^2$, but the only problem is that it comes with an extra $2ab$. By setting up the equation $(a + b)^2 = a^2 + 2ab + b^2$, we can see that the above factorization is found by subtracting $2ab$ from the RHS.

Let's look at 2 factorizations that may show up on statewides that could be useful

**Theorem 2.5.3 (Difference of Powers)**
$x^n - y^n = (x - y)(\sum_{i=0}^{n-1} x^{n-1-i}y^i)$

**Theorem 2.5.4 (Sum of Odd Powers)**
$x^{2n+1} + y^{2n+1} = (x+y)(\sum_{i=0}^{2n}(-1)^i(x^{2n-i}y^i)$

When learning these, don't try to memorize the formula. This is a generalization and is used only for reference. Instead, write out a few examples, and remember the pattern.

**Example 2.5.5**
Factor $x^5 - y^5$

*Solution:* Using the given formula for Difference of Powers, we find that the factorization is $(x-y)(x^3 + x^2y + xy^2 + y^3)$. If you want to try and understand why the formula works, multiply this out and you will see what is happening.

The last factorization I want to go over won't be extremely important, but the way it is derived is good practice for previous factorizations in this section.

**Theorem 2.5.6 (Sophie-Germain)**
$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$

While this factorization seems complicated, the steps to deriving it are actually quite simple. First, let's look at our terms. Both the a term and the b term are perfect squares, which inspires us to try to factor this as a sum of squares. We start doing this by evaluating $(a^2 + 2b^2)^2 = a^4 + 4b^4 + 4a^2b^2$. We then isolate our original expression, and end up with $a^4 + 4b^4 = (a^2 + 2b^2)^2 - 4a^2b^2$. While this may look like we are making the expression more complicated, we are actually at our final step. We see that each term on the RHS is a perfect square, so we can simply factor by difference of squares and we will be done.

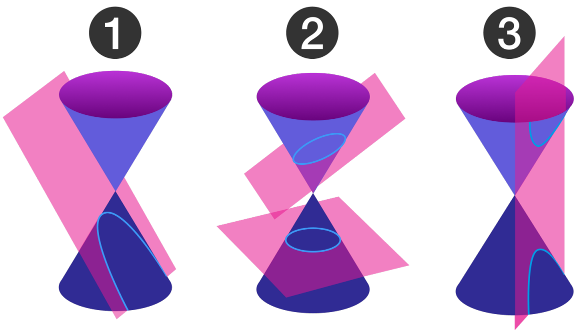
This factorization is a perfect example of how we can take advantage of squares in expressions to factor them. I believe that this section is sufficient enough for MAO calculus, however if you wish to indulge more into factorizations then primeri.org has an excellent free handout covering many factorization results.

# 3 Conics

## 3.1 Introduction

This handout will only cover the fundamentals of conic sections and important formulas used in Calculus. Thus, this won't go over rotations of conics or anything fancy that shows up in Analytical Geometry. If you are taking that test, the best way to prepare is going through past states tests and taking notes on different topics that come up in questions, and do research on them. Otherwise, enjoy.

When we talk about conic sections, what we mean is the different graphs we can make by taking a plane and intersecting a pair of congruent cones that open in opposite directions with it, as shown below.




It is also possible for the plane to intersect parallel to the base of the cones right between them, creating a single point. We call such special cases degenerate, and we will go over these later in more detail.

A key vocabulary term to understand is a *locus*. A locus is a collection of points that meet a certain criteria. For example, a circle is the locus of points that are a set distance away from a central point, where this distance is the radius of the circle, and the central point is the circle's center. This is how we will define the conic sections throughout this handout.

## 3.2 Parabolas

Let's start this section with a definition of a parabola.

**Fact 3.2.1 (Parabolas).** A parabola is the locus of points in a plane that are equidistant from a given line and a point not on the line.

The line we use is called the *directrix*, and the point we use is called the *point*. When we draw a line perpendicular to the directrix that passes through the focus, we call this the parabolas axis of symmetry (although this is somewhat obvious when graphed). The point where the axis of symmetry intersects the parabola is called the vertex. You can also think of the vertex as the critical point of the parabola, or where it has a local extrema.

By our definition of a parabola, if we are given only the focus and directrix, how would we find our vertex? We know that for any point on the parabola, the distance from the focus to this point is the same as the distance it is away from the directrix. Since the vertex is on the axis of symmetry, this implies that the vertex is at the midpoint of the line segment connecting the focus to the point where the axis of symmetry intersects the directrix.

**Theorem 3.2.2**
The vertex of a parabola is "in between" the focus and the directrix. More specifically, if we make a line segment connecting the focus to the point where the axis of symmetry intersects the directrix, the vertex is the midpoint of this line segment.

Let's derive the equation for a parabola. We will use a very simple case to work out the standard form of the equation, and then we can use what we know about transformations of graphs to find the standard form. While working through these sections, try drawing images in your notes to help you visualize what is going on here.

**Exercise 3.2.3.** Suppose a parabola on the Cartesian plane has focus (0,p) and vertex (0,0), where $t \neq 0$. Find an equation for this parabola.

First we must find our directrix. We found earlier that the vertex is in between the focus and the directrix, which means that in this case our directrix is the line $y = -p$. Now we must relate the distance from some point on the parabola (x,y) to each of these. We can write the distance from the focus to this point as $\sqrt{x^2+(y-p)^2}$. For the distance from the point to the directrix, we can write it as $y+p$. Since these are equal, we have

$$\sqrt{x^2+(y-p)^2} = y+p$$

We then square both sides to obtain

$$x^2+y^2-2py+p^2 = y^2+2py+p^2$$

This simplifies to

$$x^2 = 4py$$
$$\frac{1}{4p}x^2 = y$$

And here is our parabola! Using our knowledge of function transformations, we can see that our standard form of a parabola is

$$y = \frac{1}{4p}(x-h)^2+k$$

Where the vertex of the parabola is at (x,h). This isn't a rigorous proof by any means, but for now take it for granted so we don't spend too much time on it.

**Theorem 3.2.4**

The standard form of a parabola which opens vertically is

$$y = \frac{1}{4p}(x-h)^2+k$$

If we want our parabola to open upside down, we just have to use the parent function $y=-x^2$. Notice that in our equation, p is the distance from the focus to the vertex, and additionally the distance from the directrix to the vertex.

Parabolas can also open up horizontally, and the equation becomes slightly different.

**Theorem 3.2.5**

The standard form of a parabola that opens horizontally is

$$x = \frac{1}{4p}(y-k)^2 + h$$

Notice that as well as flipping the x and y, we also flip our h and k. (h,k) is still our vertex, but we move the h and k to be next to their x and y, respectively.

**Exercise 3.2.6.** Find an equation whose graph is a parabola with vertex (4,-1) and focus (0,-1)

Drawing a quick graph, we see that our vertex is to the right of our focus, and our parabola is going to open horizontally. Since the distance between our vertex and focus is 4, we know p is 4. We already know what our vertex is, so we can just plug this information into our standard form equation to get $x = -\frac{1}{16}(y+1)^2 + 4$.

The last thing to learn about for parabolas is the latus rectum. The latus rectum is the line that goes through the focus that is parallel to the directrix.

**Exercise 3.2.7.** Find a formula for the length of the latus rectum.

This exercise will show the power of a parabolas symmetry. WLOG, assume the center of our parabola is $(0,0)$ (We can do this because any other vertex is just a shift of the parabola, which won't change the length of the latus rectum). Additionally, let the parabola open upwards, such that the focus is at $(0,p)$. The two points where the latus rectum intersects the parabola is $(x,p)$ and $-x,p$. The equation of this parabola is thus

$$y = \frac{1}{4p}x^2$$

We want to find out what x is, so we plug in y=p to get

$$p = \frac{1}{4p}x^2$$

$$4p^2 = x^2$$

By taking the square root of both sides, we find that

$$2p = x$$

Finally, since this is the distance from one point of intersection of the latus rectum and parabola to the focus, we simply double this value to find that the length of the latus rectum is 4p.

**Theorem 3.2.8**
The latus rectum of a parabola is the line segment that goes through the focus of a parabola and is parallel to the parabolas directrix, such that it has both endpoints on the parabola. The distance of this line segment is 4p.

## 3.3 Circles

There isn't much to this section, so let's start with the definition of a circle.

**Fact 3.3.1.** The locus of points in a plane that are a fixed distance from a given point in that plane is called a circle. The fixed distance is the radius of the circle, and the given point is the center of the circle.

Let's derive the equation for a circle.

**Exercise 3.3.2.** Show that the circle with center $(h, k)$ and radius r, where $r > 0$, is the graph of the equation $(x - h)^2 + (y - k)^2 = r^2$.

We know by the definition of a circle that any point on the circle is a distance of r away from (h,k). Applying the distance formula, this implies

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

We can square both sides to get

$$(x-h)^2 + (y-k)^2 = r^2$$

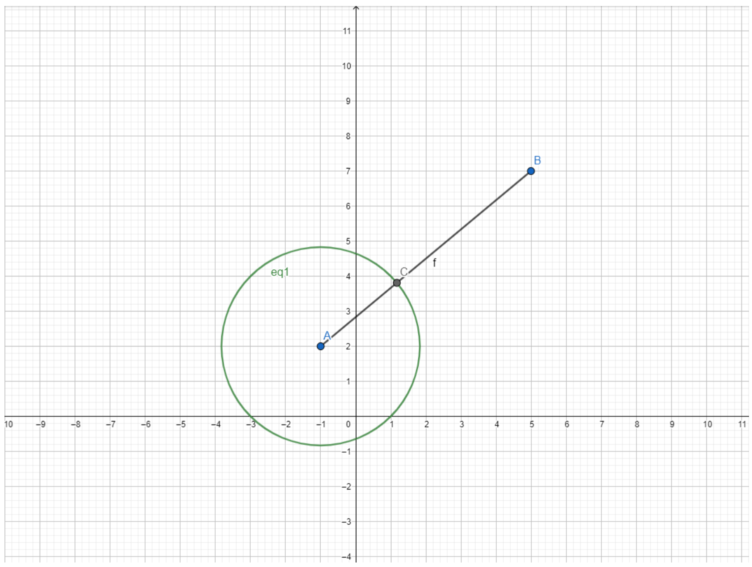
And that's it!
Let's go over some ideas that are good to know for FAMAT

**Exercise 3.3.3.** Find the distance from (5,7) to the nearest point on the graph of $2x^2 + 4x + 2y^2 - 8y = 6$

From the equation given, we know that we are dealing with a conic section because we have a $y^2, y, x^2$, and $x$ term. To put this into our standard form, we will perform what you guys are (hopefully) used to doing by now, and that is completing the square. We find that the standard form of our conic (which coincidentally happens to be a circle!) is

$$(x + 1)^2 + (y - 2)^2 = 8$$

Whenever you have a problem involving graph able functions that is not the easiest to visualize, it's always a good idea to make a quick sketch. Here is a visual of what we are doing




Intuitively, the shortest distance from the given point to our circle should be co-linear to the line segment containing the center of our circle and the given point. From here, we can see that if we find the distance between the center of our circle to the given point, and then subtract by the radius of the circle, we will have our answer. Doing so gives us a distance of $\boxed{\sqrt{61} - 2\sqrt{2}}$.

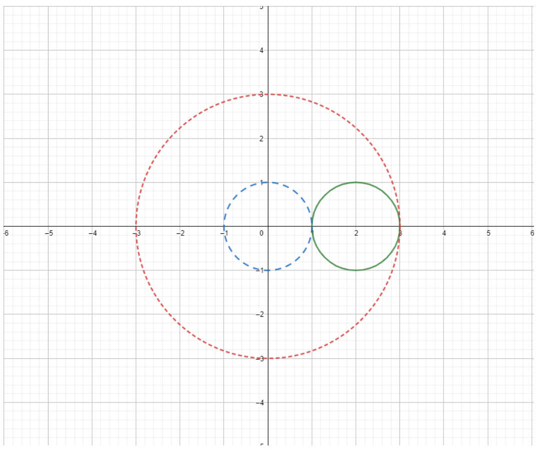
Let's look at another problem

**Exercise 3.3.4.** Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. Find the maximum and minimum values of $x^2 + y^2$. (NYSML)

The first obvious step should be completing the square of the first equation, and see what type of graph we have. By doing so we obtain

$$(x-2)^2 + y^2 = 1$$

We now have a circle with center (2,0) and a radius of 1, but we need to think back to what the question is asking us. We know that x and y must be on this circle, and we have to find the maximum value of $x^2 + y^2$. We recognize $x^2 + y^2$ as the graph of a circle with center at (0,0), and this expression equals the radius of the circle squared.



In this figure, the green circle is our graph of $x^2 - 4x + y^2 + 3 = 0$. We know that at least one point in our graph of $x^2 + y^2 = r^2$ must be in the green circle, thus our smallest circle we can make is $x^2 + y^2 = 1$ and our largest circle we can make is $x^2 + y^2 = 9$. Thus, our minimum is 1 and our maximum is 9.

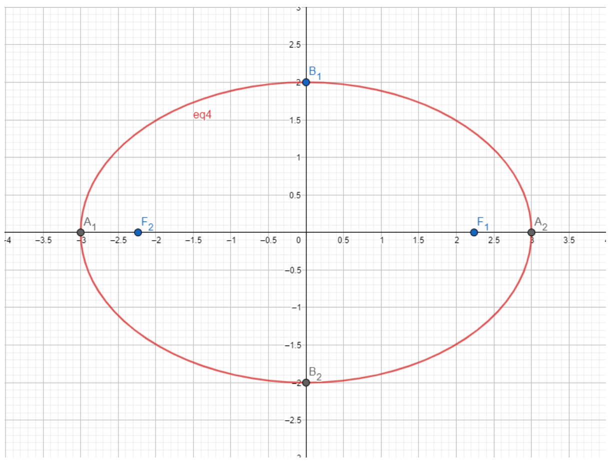
Hopefully you are a bit more comfortable with circles now. We will go over more problems with circles at the end of this handout.

## 3.4 Ellipses

Ellipses may seem intimidating at first, but once you understand its definition and how to create them, they aren't as bad as the initially look. Let's first read the definition of an ellipse.

**Fact 3.4.1.** Suppose we have points $F_1$ and $F_2$ in a plane. For any positive constant $k$ greater than the distance $F_1F_2$, the locus of all points $P$ in the plane such that $PF_1 + PF_2$ equals $k$ is called an ellipse.

This definition will be easier to understand when combined with a diagram, shown below.




Points $F_1$ and $F_2$ are our foci. What our definition of an ellipse tells us is that for any point we choose on the ellipse, say point $P$, if we connect it to each of our foci, and compute the distance of each line segment, the sum of these distances will be the same for any point we choose on the ellipse. We will explore this idea later in this handout in the problems section.

The line $A_1A_2$ is our major axis, while the line $B_1B_2$ is our minor axis. As we will see shortly, our major axis is just whatever line passes through both foci. This means that it can be horizontal, vertical, or even diagonal. The minor axis is whatever is perpendicular to our major axis that also passes through the center of the ellipse. Alright, now that we have the basic definitions down, let's derive the equation of an ellipse.

**Exercise 3.4.2.** Suppose the ellipse $E$ has foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, for some positive $c$. Moreover, suppose that for every point $P$ on $E$, we have $PF_1 + PF_2 = 2a$ for some nonzero constant $a$ such that $a > c$. Find the equation whose graph is this ellipse.

First we will experiment with some easy points. Refer back to our initial diagram. Let $A_1 = (x_1, 0)$. Based off our definition of an ellipse, we know that $A_1F_1 + A_1F_2 = 2a$. We can then find the individual lengths of the line segments. $A_1F_1 = x_1 - c$, and $A_1F_2 = x_1 + c$. Thus, $A_1F_1 + A_1F_2 = 2x_1 = 2a$. This means that $PF_1 + PF_2$ equals the length of the ellipses major axis!

**Theorem 3.4.3**
If $P$ is on an ellipse with foci $F_1$ and $F_2$, then $PF_1 + PF_2$ equals the length of the major axis of the ellipse.

Experimenting with the endpoints of the major axis seemed to give us nice results, so let's see what happens when we focus on $B_1$. We see that this point creates a triangle with our foci. Thus by pythagorean theorem, we know that

$$B_1F_1 + B_1F_2 = \sqrt{b^2+c^2} + \sqrt{b^2+c^2} = 2\sqrt{b^2+c^2}$$

However, we know that this must be equal to the length of the major axis, and thus our next major idea is presented.

**Theorem 3.4.4**
For ellipses, $c^2 = a^2 - b^2$.

Now we will attempt to find the equation for the entire ellipse. Let $P = (x, y)$ be a point on the ellipse. We then have

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

In order to simplify this equation, we have no choice but to square it. But before we do that, let's move one of the radicals onto the other side to make our lives easier.

$$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2$$

And after simplifying we obtain

$$4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4xc$$

After dividing by 4 and squaring yet again, we get

$$a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^2 + 2a^2xc + x^2c^2$$

After canceling like terms and moving the $x^2$ and $y^2$ terms to one side, we get

$$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2 = a^2(a^2 - c^2)$$

The terms in the parenthesis looks familiar...oh yea! We already found a relation ship between a, b, and c. Thus we know that $a^2 - c^2 = b^2$, so by making that substitution, we get

$$b^2x^2 + a^2y^2 = a^2b^2$$

And finally, after dividing both sides by $a^2b^2$, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**Theorem 3.4.5**
The graph of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a > b$, is that of an ellipse with major axis of length 2a and minor axis of length 2b that is horizontally aligned.

"Horizontally aligned" just refers to the fact that the major axis is a horizontal line.

We know that the value of $a$ corresponds to the length of the major axis, and the value of $b$ corresponds to the length of the minor axis, so what happens if $b > a$? It turns out that if this is the case, then the major axis becomes vertical, and it correlates to the value of $b$. Note that our equation for an ellipse does not change, it behaves exactly the same. Also, what happens when $a = b$? It becomes clear that this special case graphs a circle, which means that a circle is a type of ellipse!

**Fact 3.4.6.** Circles are ellipses

One last thing to note about our values of $a$ and $b$, we have a way of defining how "stretched out" a conic section is, and we find it by calculating its eccentricity. The eccentricity for any conic section is calculated by dividing $\frac{c}{a}$. The eccentricity for a parabola is always 1, the eccentricity for a circle is always 0, the eccentricity for a conic is $0 < e < 1$, and the eccentricity for a hyperbola is $e > 1$.

The last thing to note about ellipses (in this section at least) is that the area of an ellipse is $ab\pi$. We will show why this is true later in this handout.

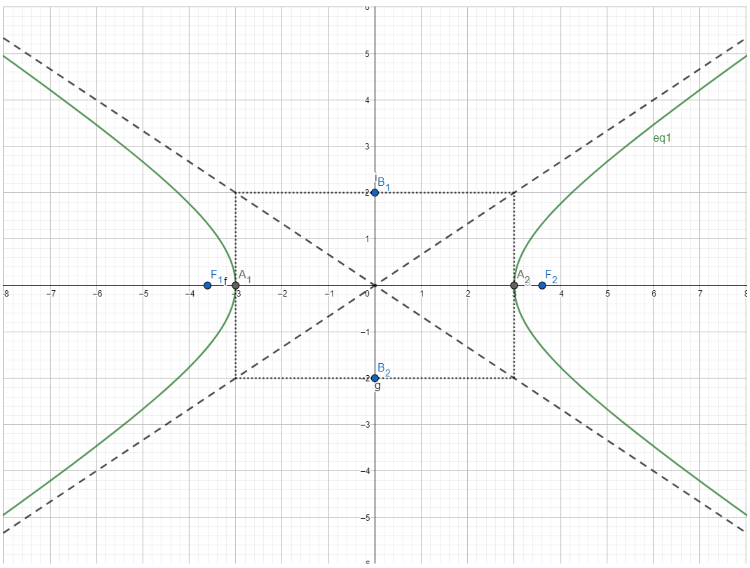
If all of this feels intimidating, don't worry. The problems at the end of this handout will give you the chance to play around with these new formulas

## 3.5 Hyperbolas

Let's start off with the definition of a hyperbola

**Fact 3.5.1.** Supose we have points $F_1$ and $F_2$ in a plane. For any positive constant k, the locus of all points $P$ in the plane such that $|PF_1 - PF_2| = k$ is called a hyperbola.

Let's look at a graph of a hyperbola.




The steps to deriving the equation of a hyperbola is very similar to the steps we took for ellipses, so I'll leave that as an exercise. Here are the equations for a hyperbola:

**Theorem 3.5.2**

The equation of a hyperbola that opens horizontally is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The equation of a hyperbola that opens vertically is given by

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Where the distance from the foci to the center is given by c, where $c^2 = a^2 + b^2$

These equations are very similar to the equations of an ellipse, so make sure to understand the differences. The graph above shows us how a, b, and c correspond to points on our graph. The a values are the vertexes of the hyperbola, and it follows that the distance between the endpoints of a hyperbola is 2a. However, the b points are slightly perplexing. What exactly do they do? Think of the a and b points as points on a rectangle. This rectangle helps as sort of a guide to graph the hyperbola, because the asymptotes of the hyperbola are the diagonals of this rectangle. We can write the asymptotes as follows:

**Theorem 3.5.3**

The lines $\frac{x-h}{a} = \pm\frac{y-k}{b}$ are the asymptotes of the graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

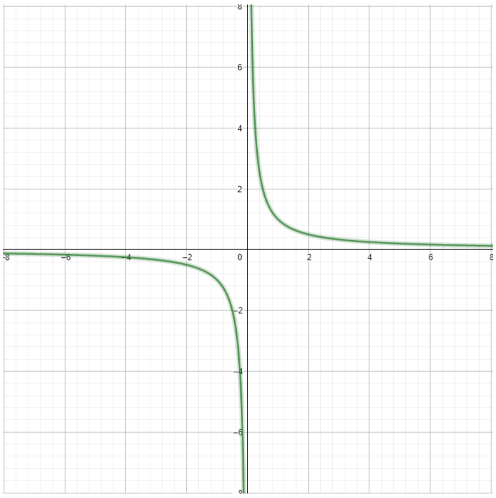
Additionally, the lines $\frac{y-k}{a} = \pm\frac{x-h}{b}$ are the asymptotes of the graph of

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Notice that these formulas are very easy to remember, as they highly resemble their respective hyperbola equations. However, if you understand how these formulas are derived from the rectangle, then you shouldn't need to memorize these. Let's look at a special form of a hyperbola.

**Exercise 3.5.4.** Examine the graph of$xy = 1$. Assuming it is a hyperbola, what are it's asymptotes and vertices?

Let's first examine this equations graph, which is $y = \frac{1}{x}$



Looking at the graph shown, we can see that the asymptotes are $x = 0, y = 0$, but we can also find these by thinking about what it means with respect to the equation. If $x = 0$, then there is no corresponding y value for the equation to equal 1. Applying this logic, we know that neither x or y can be 0. Thus $x = 0, y = 0$ must be asymptotes. Note that the asymptotes for this hyperbola are perpendicular. A hyperbola with perpendicular asymptotes is called a *Rectangular Hyperbola*

For the vertices, we know that the graph of $xy = 1$ is symmetric about the line $x = y$. (There are several ways we can notice this, one by seeing for every point $(x_1, y_1)$, $(y_1, x_1)$ is also on the graph). We also know by graphing the line that this line must cross through both vertices (This is true for any line that divides the hyperbola's branches into two symmetric pieces). Thus, the vertices must occur when $x = y$. Substituting, we have $y^2 = 1 \implies y = \pm 1$. Thus the vertices occur at $(1, 1), (-1, -1)$.

As it turns out, it holds true that any graph in the form of $(x - h)(y - k) = t$ is a hyperbola.

## 3.6 Degenerate Cases

We can describe (almost) all conic sections as

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

I say almost because rotated conic sections will have an xy term, but this is not the focus of this handout. For our purposes of non-rotated conics, we can describe all of them as such. We can see that if either A or B is 0, then it should graph a parabola. Similarly, if both A and B are positive, it should graph either an ellipse or circle, and if one of A or B is negative, then it should graph a hyperbola. However, it turns out that there are a few more cases of curves that we can obtain from conics. Let's look at a few.

**Exercise 3.6.1.** What is the graph of

$$\frac{x^2}{4} + \frac{(y-2)^2}{9} = 0$$

We see that this closely resembles the graph of an ellipse, besides the fact that it equals 0 instead of 1. When can this equation even equal zero? We know that both terms are $\geq 0$ since they are squared, so we can only have this equal zero when both terms equal zero. This occurs at the point $(0, 2)$. Thus this graph is a single point.

**Exercise 3.6.2.** What is the graph of

$$9x^2 + 4y^2 - 16y + 52 = 0$$

By completing the square, we find that this is the graph of

$$\frac{x^2}{4} + \frac{(y-2)^2}{9} = -1$$

Much like the last example, we want to think about when this equation can equal -1. We already know that both terms are non-negative, so this graph is just an empty set.

These examples have shown that even if A and B are both positive, then we can have either an ellipse, a point, or an empty set. (Note that a circle is an ellipse!) Let's look at some more examples

**Exercise 3.6.3.** Find the graph of

$$\frac{x^2}{4} - \frac{(y-2)^2}{9} = 0$$

Unlike the first example we did, these are not all non-negative terms. Thus we can rearrange this as

$$\frac{x^2}{4} = \frac{(y-2)^2}{9}$$

We can then take the square root of both sides to obtain

$$\frac{x}{2} = \pm\frac{y-2}{3}$$

We recognize this as the graph of a hyperbolas asymptotes! Thus this is the graph of two lines. Let's look at one more

**Exercise 3.6.4.** Find the graph of

$$\frac{x^2}{4} - \frac{(y-2)^2}{9} = -1$$

This one seems a little more complicated, since we have a negative term, it seems that there should be an infinite amount of points satisfying this condition. This becomes more clear if we multiply the whole equation by negative 1. Doing so gives us

$$\frac{(y-2)^2}{9} - \frac{x^2}{4} = 1$$

Thus this is just a regular hyperbola! Let's summarize everything we found in this section.

**Fact 3.6.5.** Graphs of equations in the form of

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

where A, B, C, D, and E are constants such that not both A and B are zero, can be summarized as follows:

- If either A or B is 0, then it is a parabola
- If $AB > 0$, then it is either an ellipse, a point, or an empty set

- If $AB < 0$, then it is either a hyperbola or a pair of lines

## 3.7 Important Formula(s) Pertaining to Calculus

I don't really know how much I want to add to this section, so I guess it subject to change in the future. Here is one theorem that shows up consistently in Mu Area and Volume:

**Fact 3.7.1.** Scaling a closed figure in some direction by a factor k produces a figure with area k times the area of the original figure.

We will now use this fact to show why the area of an ellipse is $ab\pi$ by considering the fact that the area of a circle is $\pi r^2$. Let rewrite this as

$$r * r * \pi$$

Now let's consider what happens if we scale this circle horizontally by k such that $rk = a$. Intuitively, this creates an ellipse with a minor axis length of 2r (since we didn't scale the circle vertically) and a major axis of length 2a. Using the fact given at the beginning of this chapter, we see that the area of this figure must be

$$r * r * k * \pi = a * r * \pi$$

Similarly, if we were to then apply a vertical shift of t such that $rt = b$ then our area would be

$$ab\pi$$

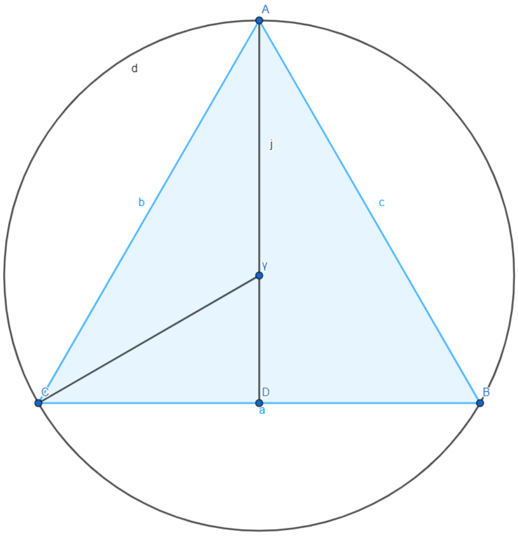
And we are done. Let's look at how we can apply this to a real problem

**Exercise 3.7.2 (2019 States Mu Individual).** Find the maximum area of a triangle inscribed in the ellipse $4x^2 + 25y^2 = 900$

Our first step should be to complete the square and get this thing into standard form. We then get

$$\frac{x^2}{15^2} + \frac{y^2}{6^2} = 1$$

Thus a = 15 and b = 6. Now let's think through what to do next. We know that the ratio of areas is conserved when scaling, so if we can somehow scale this ellipse into a circle, the ratio of the triangles area will be preserved by the same scale factor that we used to scale our ellipse. This is beneficial because it is much easier to find the maximum area of a triangle inscribed in a circle. Thus, let's create a circle with radius 15. To do this, we must have $6k = 15 \implies k = \frac{5}{2}$, so k is our vertical scaling factor. Our next step will be to find the area of this triangle in our circle. Intuitively, this triangle should be equilateral. Let's make a quick graph to see how we find its area




(Sorry if the figure isn't centered :b) In our figure, the center of our circle is labeled $y$. Since the radius is 15, we know that line segment $CY = 15$. Since our triangle is equilateral, we know that this is an angle bisector (For equilateral triangles, the median and angle bisector are on the same line). Thus triangle $CYD$ is a 30-60-90 triangle, and we can easily find that $CD = \frac{15\sqrt{3}}{2}, DY = \frac{15}{2}$. We also know that $AY$ is a radius of the circle, so $AY = 15$. From here, we have all the info we need to find the area of the triangle, and we find that it is $\frac{675\sqrt{3}}{4}$. Now we must scale this back to an ellipse, so we have $(\frac{675\sqrt{3}}{4})(\frac{2}{5}) = \boxed{\frac{135\sqrt{3}}{2}}$.

Just like all things in famat, this is much faster if you memorize the formula, but you should mostly never do that in my opinion.

Another important tactic for Mu is that the shortest distance between a parabola and a line is when their derivatives are the same. This is a similar idea to problems that ask for the shortest distance between a function and its inverse. Since the inverse function is just the original reflected about the line $x = y$, you just have to find the distance between the function and the line, and double that distance. This shortest distance commonly occurs when the derivative of the function is 1. This handout is too long to have an exercise problem on this, but you can play with this idea on your own (and also in the exercise problems!)

## 3.8 Problems

Let's do some problems to get comfortable with conics!

**Exercise 3.8.1.** A parabolic arch has a height of 20 feet and a span of 30 feet. How high is the arch 6 feet from its center?

A common strategy for dealing with problems involving conic sections is to plot them on a coordinate axis. In this example, we can choose where we want to plot it, as long as it meets the conditions. We will graph it such that it has zeroes at $(-15, 0), (15, 0)$. By doing it this way, we know that the center is at $(0, 20)$ and to solve the problem we must find the corresponding y value to $x = 6$. To create the equation of this parabola, we will look at it in standard form:

$$y = \frac{1}{4p}(x)^2 + 20$$

We know that (h,k) = (0,20) because of how we chose to plot our parabola. From here, we know 2 other points on our parabola, so we can solve for p as follows

$$0 = \frac{1}{4p}(15)^2 + 20 \implies 20 = -\frac{225}{4p}$$

$$\frac{1}{4p} = -\frac{4}{45}$$

Thus the equation of our parabola is

$$y = -\frac{4}{45}x^2 + 20$$

To solve to problem, we simply plug in x = 6 to get $\boxed{y = \frac{84}{5}}$.

It is important to note that sometimes we want to solve for parabolas given a set of points, but using standard form gives us some nasty $h^2$ or $k^2$ terms. In these cases, you can simply solve using the form $y = ax^2 + bx + c$ and solve for the coefficients.

**Exercise 3.8.2 (Mandelbrot).** If the graph of the equation $y = ax^2 + 6$ is tangent to the graph of $y = x$, then what is a?

While it seems that calculus is the way to go for this question, there is actually an easier solution if we take a second to think about the information the problem gives us. We know that we have a parabola with a vertex at $(0, 6)$, and we want it to touch the graph of $x = y$ once. Thus we know that for one point, $ax^2 + 6 = x$. We can rewrite this as

$$ax^2 - x + 6 = 0$$

And we now have a quadratic. We know that we only want one point to touch the line, so this quadratic must only have one solution, or in other words, have a double root. The easiest way to proceed from here is to note that if the discriminant of a quadratic is 0, then it has a double root. (The discriminant is what's inside the square root of the quadratic formula: $b^2 - 4ac$). Thus

$$1 - 24a = 0 \implies 1 = 24a \implies \frac{1}{24} = a$$

And we are done!

**Exercise 3.8.3 (HMMT Calculus).** A parabola is inscribed in equilateral triangle ABC of side length 1 in the sense that AC and BC are tangent to the parabola at A and B. Find the area between AB and the parabola.

If you aren't in calculus, you can skip this problem. If you are, this should be pretty helpful! In this problem, drawing a picture will be useful. We see that A and B are points on the parabola, and we know that the derivative at these points will be the same as the slope of the sides of the triangle. (If you are reading this outside of practice, draw it with the triangle "upside down"). Naturally, the first step is to find the slope of the sides of the triangle. There are many ways to do this (I created a right triangle with one side of the equilateral triangle as the hypotenuse), but we find that the slopes are $\pm\sqrt{3}$. But now we need to find an equation for our parabola. This is where plotting stuff on the coordinate grid comes into play. For the sake of our sanity, let the "bottom" of the triangle be at $(0, 0)$, and let the other two vertexes be at $(\frac{1}{2}, 1), (-\frac{1}{2}, 1)$. These two vertexes must also be on our parabola. We will use this information as follows. We write our parabola as

$$y = ax^2 + bx + c$$

In order to differentiate it easier. The derivative is

$$y' = 2ax + b$$

We know that the derivative at $x = 1/2$ is $\sqrt{3}$, and the derivative at $x = -1/2$ is $-\sqrt{3}$. Thus

$$\sqrt{3} = a + b$$

$$-\sqrt{3} = -a + b$$

By adding these two equations, we can see that b = 0 and a = $\sqrt{3}$. Next, we solve for c. We know that

$$y = \sqrt{3}x^2 + c$$

We can plug in $(\frac{1}{2}, 1)$ to solve for c.

$$1 = \frac{\sqrt{3}}{4} + c \implies c = 1 - \frac{\sqrt{3}}{4}$$

Now, the question asks us for the area between the line segment AB and the parabola. Line segment AB is the same as $y = 1$, so we can solve the following integral:

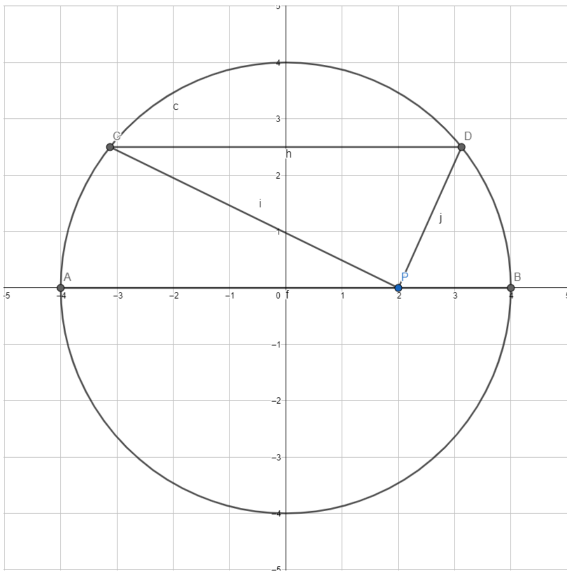
$$2\int_0^{\frac{1}{2}} 1 - (\sqrt{3}x^2 + 1 - \frac{\sqrt{3}}{4}) = \boxed{\frac{\sqrt{3}}{6}}$$

And we are done! Note that I took advantage of the symmetry of the parabola when taking the integral to make the bounds easier to deal with.

**Exercise 3.8.4.** $P$ is a fixed point on the diameter AB of a circle. Prove that for any chord CD of the circle that is parallel to AB, we have $PC^2 + PD^2 = PA^2 + PB^2$

Let's sketch this to get a better idea of whats going on.

While this seems like a geometry problem first, we will prevail by taking advantage of coordinates. WLOG, let $B = (r, 0), A = (-r, 0), P = (p, 0)$. We know that $PA^2 + PB^2 =$




$(r+p)^2+(r-p)^2=2r^2+2p^2$ by inspecting our graph. Now onto the LHS of our equation. We know that the y values of point C and D will be the same, and we also know that their x values will be opposites. Thus, let $D=(x_1,y_1), C=(-x_1,y_1)$. Using distance formula, the LHS of the equation is equivalent to

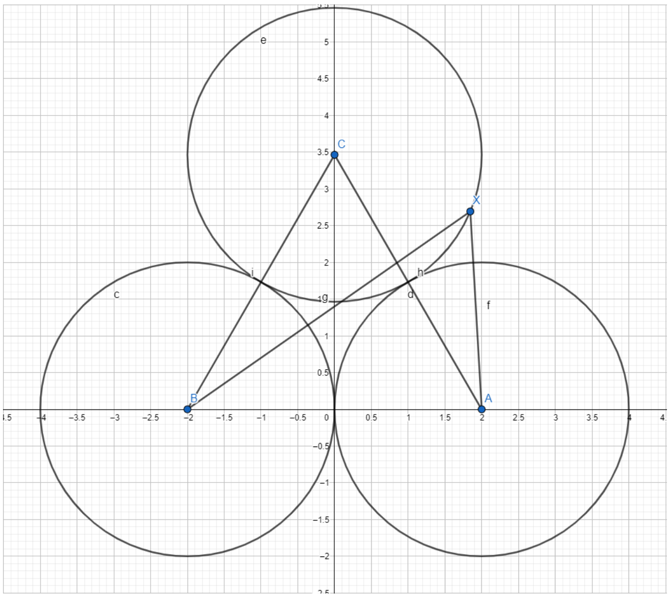
$$(x_1-p)^2+y_1^2+(-x_1-p)^2+y_1^2$$

$$=2(x_1^2+y_1^2)+p^2$$

However, we know that for any corresponding x and y value on our circle with center at (0,0), $x^2+y^2=r^2$. Thus LHS=RHS, and we are done.

**Exercise 3.8.5 (USAMTS).** Three circles with radius 2 are drawn in a plane such that each circle is tangent to the other two. Let the centers of the circles be A, B, and C. Point X is on the circle with center C such that $AX + XB = AC + CB$. Find the area of triangle AXB.

This is a really fun problem to graph, so I added a figure below.



The question gave us a really weird restriction for point x... at least we know how to find $AC + CB$. All of the radii are 2, so $AC = CB = 4$. We now must find a point X such that $AX + XB = 8$. Looking back at our figure, we did something similar when working with ellipses... we know that the sum of the distances from any point on an ellipse to its foci is always the length of the major axis, so why don't we let A and B be the foci of an ellipse with a major axis of length 8? Referring back to our diagram, let the center of the ellipse be at (0,0). Additionally, let $c = 2, a = 4$. Since $c^2 = a^2 - b^2$, we find that $b = 2\sqrt{3}$. If you solved for the side lengths of triangle ABC, you would have found that this is the distance from the origin to C! We can write this ellipse as

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Our problem now boils down to finding a point on this ellipse that is also on circle C. The equation of circle C is

$$x^2 + (y - 2\sqrt{3})^2 = 4$$

We can isolate $x^2$ and substitute it into our equation of the ellipse to find the y value that satisfies our desired condition as follows:

$$\frac{4-(y-2\sqrt{3})^2}{16}+\frac{y^2}{12}=1$$

Which simplifies to

$$y^2+12\sqrt{3}y-72=0$$

We can then use the quadratic formula to find that

$$y=\frac{-12\sqrt{3}\pm\sqrt{720}}{2}=-6\sqrt{3}\pm 6\sqrt{5}$$

Clearly, our value of y must be positive, so we choose $-6\sqrt{3}+6\sqrt{5}$ as our y. Then the area of triangle AXB is just $2(6\sqrt{5}-6\sqrt{3})=\boxed{12\sqrt{5}-12\sqrt{3}}$. For context, most contest problems will not require this amount of computation, but the USAMTS test is an online test that allows students to work on problems for a full month. These problems are usually around late AIME to early USAMO level.

# 4 Inequalities

## 4.1 Introduction

This handout is not intended to be long nor rigorous. It is simply intended to show you tools you may need to resort to when derivatives fail. This is one of the only handouts where I will say that it is completely OK if you don't understand where some of these inequalities are derived from, or how they work. However, these may show up on practice tests. The problem with inequalities in general is that most inequality problems consist of proving that inequalities are true, which you obviously can't do on a multiple choice test, therefore we will omit proving inequalities by "working backwards" in this handout.

## 4.2 Basic Inequalities

We brushed up on some neat inequality properties when we looked at the Binomial Theorem handout. So, let's go through some basic problems real quick.

**Fact 4.2.1.** Multiplying or dividing an inequality by a negative will flip the signs of the inequality. For example:

$$a < b < c \implies -a > -b > -c$$

This is especially important when tackling inequalities that include squared terms.

**Exercise 4.2.2.** Find all x such that

$$x^2 < 25$$

Our first thought may be to simply square the entire inequality. Thus, we obtain $x < 5$, thus our answer seems to be $x \in (-\infty, 5)$, however we immediately see that something is wrong with our answer if we plug in $x = -6$. The reason we got the wrong answer is that we didn't consider the case when x is negative. Doing so gives us two inequalities:

$$x < 5$$

$$-x < 5$$

The first inequality looks fine, but for the second one we will multiply by $-1$ to obtain

$$x > -5$$

Our values of x must satisfy all inequalities simultaneously, thus our answer is $x \in (-5, 5)$. Here is one more useful inequality property:

**Fact 4.2.3.** If we have 3 numbers such that $0 \leq a \leq b \leq c$, then

$$\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$$

There are a few things to note here. The first thing is our restriction that the numbers must be positive. If we try this with $-2 < 1$ and take the reciprocals, our property would suggest $-\frac{1}{2} > 1$, which is obviously not true. Thus we must restrict all of our numbers to be positive (or do we?).

## 4.3 Trivial Inequality

Our first important inequality may seem trivial, but it is indeed very useful!

**Fact 4.3.1 (Trivial Inequality).**

$$x^2 \geq 0$$

Let's see this super confusing and intricate inequality in action.

**Exercise 4.3.2.** Prove that if x and y are positive, then

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$$

Since x, y, and x+y are all non zero, we can remove the denominators to simplify the inequality:

$$(x)(y)(x+y)\left(\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}\right) \implies x^2 + 2xy + y^2 \geq 4xy$$

From here, we move everything to one side of the inequality, to obtain

$$x^2 - 2xy + y^2 \geq 0 \implies (x-y)^2 \geq 0$$

If you aren't familiar with common factorizations by now, please review!!! From here, it is clear that we can apply the trivial inequality to show that this inequality holds true.

Not much to this topic, let's move on.

## 4.4 QAGH!

We will first define two terms.

**Fact 4.4.1 (Arithmetic Mean).** The arithmetic mean of a set of n terms $a_1, a_2, \ldots a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

**Fact 4.4.2 (Geometric Mean).** The geometric mean of a set of n terms $a_1, a_2, \ldots a_n$ is

$$\sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

We will now look at one of the single most useful inequalities in all of high school math: AM-GM Inequality.

**Fact 4.4.3.** The arithmetic mean of any $n$ non-negative numbers is greater than or equal to the geometric mean of those numbers. We write this as:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

While this is fairly difficult to prove generally, we can get a better idea of what is going on if we analyze this for 2 numbers.

**Exercise 4.4.4.** Show that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and find when equality occurs.

We can simplify this inequality by squaring both sides, after which we obtain

$$\frac{a^2+2ab+b^2}{4} \geq ab \implies a^2+2ab+b^2 \geq 4ab$$

Much like our example with trivial inequality, we will move all the terms to the same side, to obtain

$$a^2-2ab+b^2 \geq 0 \implies (a-b)^2 \geq 0$$

When does equality occur? We guess that it is when $a=b$ based off of our final inequality, so let's see what happens when this is the case.

$$\frac{2a}{a} \geq \sqrt{a^2}$$

Equality does indeed occur! This is important to know, because it can show us how to obtain maximums and minimums of inequalities. Using the same logic, we can apply this logic generally as well.

> *Remark 4.4.5.* Equality in AM-GM occurs when all the numbers $a_1, a_2, \ldots a_n$ are equal.

Let's see AM-GM in action:

**Exercise 4.4.6.** Let x, y, and z be positive real numbers. Show that

$$(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9$$

We think about analyzing the AM-GM inequality of each factor on the LHS individually, since both factors are sums. We see that

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3} \geq \sqrt[3]{\frac{1}{xyz}}$$

Multiplying by 3 in both inequalities, we see that we can actually multiply both of the inequalities together to achieve our intended result of

$$(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9\sqrt[3]{\frac{xyz}{xyz}} = 1$$

**Exercise 4.4.7.** Prove that $\left(\frac{n+1}{2}\right)^n > n!$ for all positive integers $n \geq 2$.

In order to apply AM-GM, we want our sums to be free of powers. Thus, we take the nth root of the inequality to obtain

$$\frac{n+1}{2} > \sqrt[n]{n!} = \sqrt[n]{n(n-1)(n-2)\dots(2)(1)}$$

The LHS of our inequality doesn't seem to correlate to the RHS through AM-GM, so we focus on the right side and try to write it's AM-GM, which is

$$\frac{1+2+3+\cdots+n}{n} > \sqrt[n]{n!}$$

Note that our inequality is strict, since inequality will be impossible since all of our numbers are different. We convert the sum of the first n integers into closed form to obtain

$$\frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2} > \sqrt[n]{n!}$$

And we are back where we started!

**Exercise 4.4.8 (FAMAT Mu).** Find the maximum volume of a rectangular box whose diagonal length is 12

Let the side lengths of our box be x, y, and z. We see that $x^2+y^2+z^2 = 144$ by Pythagorean theorem. AM-GM seems like a natural next step, since we want a xyz term. We find that

$$\frac{x^2+y^2+z^2}{3} \geq \sqrt[3]{x^2y^2z^2}$$

$$48 \geq (xyz)^{2/3} \implies xyz \leq 48^{3/2}$$

We have equality when all the side lengths are the same, thus the maximum volume is $\boxed{48^{3/2}}$.

We will end this section by briefly covering Quadratic and Harmonic Means.

**Fact 4.4.9 (Quadratic Mean).** The quadratic mean of the real numbers $a_1, a_2, \ldots a_n$ is

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

**Fact 4.4.10 (Harmonic Mean).** The harmonic mean of the nonzero numbers $a_1, a_2, \ldots a_n$ is

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

**Fact 4.4.11 (Mean Inequality Chain).** For a set of real, positive numbers, it is true that

$$QM \geq AM \geq GM \geq HM$$

These are mostly non-important in FAMAT, but there's no point in not including it.

## 4.5 Cauchy Schwarz

Cauchy Schwarz may seem complicated at first, but it will become more natural as we work through some examples.

**Fact 4.5.1.** For any two sets of real numbers $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ it is true that

$$(b_1^2 + b_2^2 + \cdots + b_n^2)(a_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots a_n b_n)^2$$

**Exercise 4.5.2.** Prove that $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$

If we expand both sides of the inequality, we get

$$a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \geq a^2c^2 + 2abcd + b^2d^2$$

After simplifying:

$$a^2d^2 + b^2c^2 \geq 2abcd$$

Now we move all the terms to one side, in hopes of creating a trivial inequality situation. We get

$$a^2d^2 - 2abcd + b^2c^2 = (ad - bc)^2 \geq 0$$

Which is true by Trivial Inequality. Note that equality holds when $(ad - bc)^2 = 0$, so when $ad = bc$.

The general proof for Cauchy is somewhat advanced, so I will leave it as an exercise.

**Fact 4.5.3.** Equality holds in Cauchy-Schwarz if and only if either $a_i = 0$ for all $i$, or there exists a constant t such that $b_i = ta_i$ for all $i$.

**Exercise 4.5.4.** Show that $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$

We see a sum of squares on the LHS, so we think of using Cauchy. Multiplying the inequality by 3, we have

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2$$

Hmm... in our RHS, each of our a, b, and c is multiplied by one. So, if we had $(1^2 + 1^2 + 1^2$ on the LHS, this would be true by Cauchy, and it turns out we do! We can rewrite this as

$$(1 + 1 + 1)(a^2 + b^2 + c^2) \geq (a + b + c)^2$$

And thus the inequality is true by Cauchy.

**Exercise 4.5.5 (Hungary).** Let a and b be positive real numbers with $a + b = 1$. Show that

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \geq \frac{1}{3}$$

A useful tactic with Cauchy Schwarz comes from the fact that we multiply $a_i$ and $b_i$ together on the lesser side of the inequality. Imagine if $a_i$ and $b_i$ were reciprocals, that means on the lesser side they would cancel out to 1! Let's use that thinking here. We seek to cancel out the denominators of the RHS, so we introduce them into the inequalty:

$$[(a+1)+(b+1)]\left(\frac{a^2}{a+1}+\frac{b^2}{b+1}\right) \geq \left(\sqrt{a+1}\cdot\sqrt{\frac{a^2}{a+1}}+\sqrt{b+1}\cdot\sqrt{\frac{b^2}{b+1}}\right)$$

$$\implies [(a+1)+(b+1)]\left(\frac{a^2}{a+1}+\frac{b^2}{b+1}\right) \geq (a+b)$$

Now we can isolate $\frac{a^2}{a+1}+\frac{b^2}{b+1}$ to obtain

$$\frac{a^2}{a+1}+\frac{b^2}{b+1} \geq \frac{a+b}{a+b+2}$$

However, we are given that $a+b=1$, thus

$$\frac{a^2}{a+1}+\frac{b^2}{b+1} \geq \frac{1}{3}$$

And we are done!

> *Remark 4.5.6.* When you feel stuck on a problem, look back to see what information the problem gives you. You should always try to use every piece of information at least once!

**Exercise 4.5.7 (AOPS).** Jane has just drawn 6 rectangles, and each rectangle has a different length and width. The sum of the squares of the lengths is 40 and the sum of the squares of the widths is 20. What is the largest possible value of the sum of the areas of the rectangles?

Let the lengths be denoted as $l_n$ and the widths denoted as $w_n$. We are given the sum of $l_n^2$ series and the $w_n^2$ series, and we seek to relate this to the $l_n w_n$ series... this sounds like a job for Cauchy. We have:

$$(l_1^2+l_2^2+\cdots+l_6^2)(w_1^2+w_2^2+\cdots+w_6^2) \geq (l_1w_1+l_2w_2+\ldots l_6w_6)^2$$

However, we are given the values of the two series on the LHS! Thus we can plug in the given values to obtain

$$800 \geq (l_1w_1+l_2w_2+\cdots+l_6w_6)^2$$

Thus by square rooting both sides, we have

$$l_1w_1+l_2w_2+\cdots+l_6w_6 \leq 20\sqrt{2}$$

To show that equality is achievable, let $l_i = w_i\sqrt{2}$.

## 4.6 Triangle Inequality

This last section will be short and simple. Many of you may remember this inequality from geometry, which basically says that the sum of any two sides of a triangle must be greater than or equal to the remaining side. However, we can write this in terms of absolute values, which is more meaningful to us.

**Fact 4.6.1 (Triangle Inequality).**

$$|x + y| \leq |x| + |y|$$

What this means intuitively is that the straight line distance from one point to another is always the shortest path to take. We also have another form of this:

**Fact 4.6.2 (Reverse Triangle Inequality).**

$$|x - y| \geq ||x| - |y||$$

While these formulas may prove useful, the intuitive understanding of the inequality is more important.

# 5 Induction and Binomial Theorem

## 5.1 Introduction

The content covered in this handout is fairly advanced, but is still fairly important for FAMAT. Technically, all you need is to memorize the binomial theorem formula, and just apply it to whatever problem requires it, but blindly memorizing formulas is a bad habit. Instead, this handout will focus on how to prove the binomial theorem, and some advanced applications of it.

## 5.2 Important Factoids about Ratios

Since we will be dealing with n choose k notation and factorials, it's best to be comfortable with different properties of ratios, and why these properties exist. Let's go over two of the most fundamental properties.

**Exercise 5.2.1.** If it is true that $\frac{a}{b} = \frac{c}{d}$, show that

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$$

The approach we will use for this problem (and many other problems involving ratios) is to assign the key ratios to a variable. In this case, let $r = \frac{a}{b} = \frac{c}{d}$. We then have that $br = a, dr = c$. Substituting these values into the ratio we are trying to show holds true, we have

$$\frac{br + dr}{b+d} = \frac{r(b+d)}{b+d} = r = \frac{a}{b}$$

And we are done. It turns out that this tactic of letting a ratio be equal to a single variable is a very strong problem solving strategy, and you can apply it to problems that you wouldn't think of applying it to, such as this problem.

**Exercise 5.2.2 (AMC).** The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let a and b be the maximum and minimum values of $\frac{y}{x}$ over all points (x,y) on the ellipse. What is the value of a+b?

(I know this question is a little off topic, but it relates to the conics handout :b). Our initial thought is that we want to complete the square of this ellipse, but there is an annoying xy term. Let's think about what we need to solve for. We want to solve for the values of $\frac{y}{x}$, so let's try our strategy of assigning this to a variable and substituting it in. Let $r = \frac{y}{x} \implies rx = y$. After plugging this into the given equation, we have

$$(2 + r + 3r^2)x^2 - (11 + 20r)x + 40 = 0$$

Now, we need to find some way to have an equation only in terms of r, so we can find its maximum and minimum. We know that we can use a graphs discriminant to find information to find useful information, so we think how we can apply it to this problem. The question tells us that the ellipse is in the first quadrant. All the x's we are interested in are positive and real, thus the discriminant of this equation must be non-negative. Thus we set up the inequality

$$B^2 - 4AC = -80r^2 + 280r - 199 \geq 0$$

This is the graph of an upside down parabola, so the solution to this inequality will be the values of x in between this graphs two roots. More specifically, $x \in [r_1, r_2]$, where $r_1, r_2$ are the two roots of the parabola. Since this range is the only values that r can take, we know that $r_1$ will be it's minimum, and $r_2$ will be it's maximum. To solve the problem, we need to find $r_1 + r_2$, which we can easily find using vieta's formula, and thus the answer is $\frac{280}{80} = \boxed{\frac{7}{2}}$.

Next is another useful ratio identity.

**Exercise 5.2.3.** If $\frac{a}{b} = \frac{c}{d}$, show that

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

We will use the same strategy as before. Let $r = \frac{a}{b} = \frac{c}{d} \implies rb = a, rd = c$. Substituting these values in, we have

$$\frac{rb+b}{rb-b} = \frac{rd+d}{rd-d} \implies \frac{r+1}{r-1} = \frac{r+1}{r-1}$$

And we are done.

## 5.3 Induction

Induction is a very useful tool used to prove statements, mainly involving positive integers. The key idea of induction is to show that a statement is true for the very first case, and that if it is true for some case k, then it holds true for case k+1. This essentially creates a domino effect for all the possible cases, and sufficiently proves the statement. Let's go over a simple example here.

**Exercise 5.3.1.** Prove that

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Induction requires 2 main steps: The base case and the inductive step. We will go over the proof now.

*Base Case*: Our base case is $n = 1$. Observe that

$$1 = \frac{1(1+1)}{2} = 1$$

Thus our base case is satisfied.

*Inductive Step*: Assume that

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}$$

We want to show that

$$1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

To do this, we will add $k+1$ to both sides of our inductive hypothesis such that

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

Focusing on the RHS, we wish to simplify this into one single fraction, and hope that it simplifies into the form that we want. We have

$$\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Thus we have proved this statement.

You may be wondering when you would ever use this in high school, and the answer is pretty much never. FAMAT will never ask you to prove a mathematical statement. So, why are we learning this? FAMAT has asked about the steps of induction before in Theta/Alpha tests, so it's not a stretch to assume they may ask about it again. But more importantly, it is the tool that you use to solve problems that include patterns, you just never realized it because once you find the pattern, you assume it's true. Let's go over another example.

**Exercise 5.3.2 (AOPS).** The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 2$ and $x_{k+1} = x_k^2 - x_k + 1$ for all $k \geq 1$. Find $\sum_{k=1}^{\infty} \frac{1}{x_k}$

When a problem gives a formula for a pattern to follow for a sequence of numbers, the best approach is always to write down the first few terms and go from there. Let's find the first couple of terms in the sequence.

$$x_1 = 2$$

$$x_2 = 2^2 - 2 + 1 = 3$$

$$x_3 = 3^2 - 3 + 1 = 7$$

$$x_4 = 7^2 - 7 + 1 = 43$$

Hmm... there doesn't seem to be a clear pattern here. However, we notice that the question is asking for the value of an infinity summation. When dealing with problems like this that ask for the value of a summation, we want to observe the values of the *partial sums* of the summation. Basically, we want to evaluate the summation only up to a certain number of terms, and see if there is a pattern. Observe that

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$S_3 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$$

We see that every value we get is in the form $\frac{n-1}{n}$, and as we take the partial sums greater and greater, n will tend to infinity. This is basically equivalent to evaluating $\lim_{x\to\infty} \frac{x-1}{x} = 1 + \frac{1}{\infty} = \boxed{1}$, and thus the infinite summation must be equal to 1. If you are in algebra, don't worry about the limit we just evaluated. Intuitively, you can see that the numbers will tend very close to 1 without needing to find the limit.

The reason I included this in the induction section is that normally, we would have to prove such a statement using induction. However, it is very time consuming, and normally we would never have time to prove this during a contest setting. Thus, the normal way of solving this is just to assume that a pattern is true, and use it.

## 5.4 Binomial Theorem in an Algebraic Setting

Before covering binomial theorem, it is important to understand the notation we are about to use.

**Fact 5.4.1.**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let's start by defining what the binomial theorem states.

**Fact 5.4.2.** The binomial theorem states that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{n-i} x^{n-i} y^i$$

When expanded, it looks like

$$(x+y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + \binom{n}{0}y^n$$

Before we prove binomial theorem, let's prove an identity that we will use in our proof of binomial theorem.

**Exercise 5.4.3.** Show that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

We have that

$$\frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}$$
$$\frac{(r+1)n!}{r!(n-r)!} + \frac{(n-r)n!}{(r+1)!(n-r)(n-r-1)!}$$
$$\frac{(r+1)n! + (n-r)n!}{(r+1)!(n-r)!} = \frac{(n+1)!}{(r+1)!(n-r)!} = \binom{n+1}{r+1}$$

This is called Pascal's Identity. Next, let's prove Binomial Theorem.

**Exercise 5.4.4.** Show that

$$(x+y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + \binom{n}{0}y^n$$

for all positive integers n.

We will prove this through induction. Our base case will be n=1, so that

$$(x+1)^1 = \binom{1}{1}x + \binom{1}{0}y$$

Which is true. Next, we assume that

$$(x+y)^k = \binom{k}{k}x^k + \binom{k}{k-1}x^{k-1}y + \binom{k}{k-2}x^{k-2}y^2 + \cdots + \binom{k}{1}xy^{k-1} + \binom{k}{0}y^k$$

We want to make the k on the LHS be k+1, and we can do this by multiplying both sides of the equation by $x+y$

$$(x+y)^{k+1} = x\left(\binom{k}{k}x^k + \cdots + \binom{k}{1}xy^{k-1} + \binom{k}{0}y^k\right) + y\left(\binom{k}{k}x^k + \cdots + \binom{k}{1}xy^{k-1} + \binom{k}{0}y^k\right)$$

$$= \left(\binom{k}{k}x^{k+1} + \cdots + \binom{k}{1}x^2y^{k-1} + \binom{k}{0}xy^k\right) + \left(\binom{k}{k}x^ky + \cdots + \binom{k}{1}xy^k + \binom{k}{0}y^{k+1}\right)$$
$$= \binom{k}{k}x^{k+1} + \left(\binom{k}{k-1} + \binom{k}{k}\right)x^ky + \cdots + \left(\binom{k}{0} + \binom{k}{1}\right)xy^k + \binom{k}{0}y^{k+1}$$

From here, we know that $\binom{k}{k} = \binom{k+1}{k+1}$ and $\binom{k}{0} = \binom{k+1}{0}$, so we can substitute these values into our equation. In addition, we can apply Pascal's Identity to each of the terms coeffecients, and this satisfies our inductive hypothesis.

Was this worth writing two chapters for to set up to this? I'm not sure! But let's do some problems.

**Exercise 5.4.5.** Find the coefficient of the $x^2$ term in the expansion of $(3x - \frac{2}{x^2})^5$

This is a perfect example of the types of questions that are asked in FAMAT, especially in theta. First, we want to find the term that has $x^2$, so we write out binomial theorem as follows

$$\binom{5}{n}(3x)^n(-\frac{2}{x^2})^{5-n}$$

Notice that it doesn't really matter which term we give which exponent, because they will add up to 5 anyways. This is due to binomial theorem's symmetric nature. Also, note that we have to be wary of negatives in our expansion. Any time that $5-n$ is odd, the term will be negative. To actually find which term is the $x^2$ term, we know that

$$n - (10 - 2n) = 2$$

We get this equation by noticing what happens when we multiply $x^n$ and $\frac{1}{x^{2(5-n)}}$. We subtract their exponents, and we want their subtraction to be equal to 2. Solving this equation, we get that $n = 4$. Thus to find our coefficient, we must evaluate

$$\binom{5}{4}(3x)^4(-\frac{2}{x^2}) = -810x^2$$

Thus our answer is $\boxed{-810}$. Whether a question asks you for the value of the constant, or a specific term involving two variables instead of just x's, you solve it exactly the same way.

While it's not specifically an application of Binomial Theorem, it is useful to be able to recognize when a certain expansion is present in a problem, as we will see here

**Exercise 5.4.6 (Mandelbrot).** Find the real solution to the equation $x^3+3x^2+3x=3$

This should look familiar. We have seen $x^3+3x^2+3x+1$ show up when we expand $(x+1)^3$. Thus we can rewrite the problem as

$$(x+1)^3-1=3 \implies (x+1)^3=4$$

From here, we can simply cube root both sides and subtract one to find that the real root of this equation is $\boxed{\sqrt[3]{4}-1}$. Next we will do a neat AIME problem

**Exercise 5.4.7 (AIME).** Find the sum of all the roots, real and nonreal, of the equation

$$x^{2001}+\left(\frac{1}{2}-x\right)^{2001}=0$$

given that there are no multiple roots.

The fact that we are looking for the sum of all roots should be a dead give away that we should use Vieta's formulas. Additionally, we know that we only care about the leading coefficient and the coefficient after the leading coefficient, so as long as we properly apply binomial theorem, this shouldn't be too hard. However, we have to be careful. We may think that $x^{2001}$ will be the leading term, and that the leading coefficient will be one, but this term actually cancels. We first expand the second term in the equation:

$$\left(\frac{1}{2}-x\right)^{2001}=\left(\frac{1}{2}\right)^{2001}+\cdots-\binom{2001}{2}\left(\frac{1}{2}\right)^2(x)^{1999}+\binom{2001}{1}\left(\frac{1}{2}\right)(x)^{2000}-x^{2001}$$

We see that the $x^{2001}$ terms cancel, and that our leading coefficient is now $\binom{2001}{1}\left(\frac{1}{2}\right)=\frac{2001}{2}$, and our second term's coefficient is $\binom{2001}{2}\left(\frac{1}{2}\right)^2=\frac{(2001)(2000)}{(2)(4)}$. Applying vieta's, our sum is now

$$\frac{\frac{2001*2000}{2*4}}{\frac{2001}{2}}=\boxed{500}$$

And we are done.

## 5.5 Binomial Theorem in a Number Theory Setting

Binomial Theorem can also be applied to solve a variety of NT problems. Let's go over a few.

**Exercise 5.5.1 (AOPS).** Find the remainder when $65^{20}$ is divided by 512.

When doing NT problems, prime factorizing important numbers is always a good first step. We see that $512 = 2^9$, so we want to find 2's somewhere. The fact that this problem is in a binomial theorem section suggests that we can split the 65 apart somehow and expand it, and this is exactly what we can do! We know that $65 = 64 + 1 = 2^6 + 1$, and when we expand $(64 + 1)^{20}$, we have

$$\binom{20}{20}64^{20} + \binom{20}{19}64^{19} + \cdots + \binom{20}{2}64^2 + \binom{20}{1}64 + \binom{20}{0}$$

Notice that $512|64^2$, so we only care about the last two terms in this expansion, since every other term will be evenly divided by 512. Evaluating the last two terms,

$$\binom{20}{1}64 + \binom{20}{0} = (20)(64) + 1 = 1281$$

The remainder of $1281/512 = \boxed{257}$, and we are done.

**Exercise 5.5.2 (AHSME).** What is the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$?

We think about expanding this expression using binomial theorem, but this won't lead us anywhere, since we will have square roots on every odd exponent, and square roots are not integers. The key to this problem is to observe the fact that the expansion of $(\sqrt{3} - \sqrt{2})^6$ will have negatives for all of the odd exponents. This will be more obvious when we expand each term:

$$(\sqrt{3}+\sqrt{2})^6 = \binom{6}{6}(\sqrt{3})^6 + \binom{6}{5}(\sqrt{3})^5(\sqrt{2}) + \binom{6}{4}(\sqrt{3})^4(\sqrt{2})^2 + \binom{6}{3}(\sqrt{3})^3(\sqrt{2})^3$$
$$+\binom{6}{2}(\sqrt{3})^2(\sqrt{2})^4 + \binom{6}{1}(\sqrt{3})(\sqrt{2})^5 + \binom{6}{0}(\sqrt{2})^6$$
$$(\sqrt{3}-\sqrt{2})^6 = \binom{6}{6}(\sqrt{3})^6 - \binom{6}{5}(\sqrt{3})^5(\sqrt{2}) + \binom{6}{4}(\sqrt{3})^4(\sqrt{2})^2 - \binom{6}{3}(\sqrt{3})^3(\sqrt{2})^3$$

$$+\binom{6}{2}(\sqrt{3})^2(\sqrt{2})^4 - \binom{6}{1}(\sqrt{3})(\sqrt{2})^5 + \binom{6}{0}(\sqrt{2})^6$$

But how does this help us solve the problem? Notice that $\sqrt{3}-\sqrt{2} < 1$, which means that we can add $(\sqrt{3}+\sqrt{2})^6 + (\sqrt{3}-\sqrt{2})^6)$, because this value is close to $(\sqrt{3}+\sqrt{2})^6$. Adding these, we get

$$2(3^3) + 30(3^2)(2) + 30(3)(2^2) + 2(2^3) = 970$$

Thus, $(\sqrt{3}+\sqrt{2})^6 + (\sqrt{3}-\sqrt{2})^6 = 970$, but since $0 < (\sqrt{3}-\sqrt{2})^6 < 1$, the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$ is 970.

This problem showcases a good tactic for dealing with some binomial theorem problems:

**Theorem 5.5.3**

Sometimes the odd exponents in binomial expansions can be annoying. Say our original binomial is

$$(a+b)^n$$

We can eliminate all of the terms with odd degrees by adding

$$(a+b)^n + (a-b)^n$$

This works because the -b is only positive when raised to an even power.

## 5.6 Some Extras

**Exercise 5.6.1.** Evaluate

$$\binom{100}{100} + \binom{100}{99} + \binom{100}{98} + \cdots + \binom{100}{1} + \binom{100}{0}$$

After writing out a few terms, we can see that there must be some kind of shortcut, there's no way we're going to evaluate all 100 terms, and they get very big, very fast. We think about where we have seen this type of notation before, and it should remind us of binomial expansion. Specifically, we know that any binomial we expand to the power of 100 will have these factors in the coefficients. So, we will chose a binomial with nice terms that will allow us to evaluate this, observe that

$$(1+x)^{100} = \binom{100}{100} + \binom{100}{99}x + \binom{100}{98}x^2 + \cdots + \binom{100}{1}x^{99} + \binom{100}{0}x^{100}$$

We take advantage of the very useful fact that the "1" term in our binomial does not affect the value of any term in our binomial expansion. From here, we see that this is identical to the expression we want to evaluate, besides all the x terms. We can easily eliminate this problem by letting $x = 1$, and thus the expression is equal to $\boxed{2^{100}}$.

Don't be locked into thinking that this is the only circumstance you can apply this strategy! There will be some problems in the exercises that you can cleverly use this strategy on that aren't so straight forward.

Next we will look at how we can extend Binomial Theorem to non-integer powers.

**Fact 5.6.2.** For all real numbers x,y, and r with $|x| > |y|$, we have

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \ldots$$

Note that this is an infinite series, and all of the x and y exponents always add up to r.

**Exercise 5.6.3 (Theta Logs and Exponents).** The value of $\sqrt{4.01}$ is approximated using the first three terms of the expansion of $(4+.01)^{\frac{1}{2}}$. What is the value of this approximation?

Applying our formula, the expansion of this binomial is equal to

$$4^{\frac{1}{2}} + \frac{1}{2}4^{-\frac{1}{2}}(.01) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}4^{-\frac{3}{2}}(.01)^2 + \ldots$$

$$= 2 + \frac{1}{400} - \frac{1}{640000}$$

Thus that is our answer. As you can see, most FAMAT problems aren't very involved, they usually only require one key insight/formula and then directly apply that to find the answer. There will be a more advanced application of this concept in the exercise problems.

To end this handout, here is trinomial theorem

**Fact 5.6.4.**

$$(a+b+c)^n = \sum_{\substack{i,j,k \\ i+j+k=n}} \frac{n!}{i!j!k!} a^i b^j c^k$$

I haven't seen any questions ask this, but here it is.

# 6 Sequences, Recursions, Summations

## 6.1 Introduction

Summations are a very important topic in Calculus, especially in BC Calculus. We use summations to describe Riemann Sums and Taylor Series, along with just analyzing summations in general. This handout will go over summations in an algebraic setting, so we won't be getting into Riemann Sums or Taylor Series. However, some problems in this handout may use derivatives/limits. Let's start with some definitions:

**Fact 6.1.1 (Sequence).** A sequence is a list of things, such as

$$1, 2, 3, 4, 5, \ldots$$

This is the infinite sequence of all positive integers.

**Fact 6.1.2 (Series).** A series is a sum of things, such as

$$1 + 2 + 3 + 4 + 5 + \ldots$$

This is the infinite series of all positive integers.

Note that by our definition of sequence and series, the "things" in our list of "things" don't have to be numbers.

## 6.2 Arithmetic Series

From here on out in this handout, I will call every sequence/series as simply a series, because we are mainly interested in evaluating it's sum. However, keep in mind that most of these series can also be written as sequences.

The most basic series that we can analyze are arithmetic and geometric series. Let's start with arithmetic series.

**Fact 6.2.1 (Arithmetic Series).** An arithmetic series is defined as a series where every term differs by a common difference from the previous term. For example:

$$1+2+3+4+5$$

Is an arithmetic series with a common difference of 1. Additionally, there are 5 terms in this series.

In working with series such as this, we often refer to the first term of the series as $a_1$, although it is not uncommon for some problems to start at $a_0$. Additionally, we can let the common difference be any variable we want, so we will let the common difference be $d$. Let's see what we can find out about this series.

**Exercise 6.2.2.** What is the nth term of an arithmetic series?

By our definition of arithmetic series, we know that the first term is $a_1$, and that the second term is $a_1 + d$. Additionally, the third term is $a_1 + d + d = a_1 + 2d$. We see that the nth term can be written as $a_1$ plus $n - 1$ common differences. Here is our formula

**Fact 6.2.3.** The nth term of an arithmetic series is

$$a_1 + (n-1)d$$

The definition of arithmetic series is simple enough that we don't need to get more rigorous than this.

The next question we may ask ourselves is how do we evaluate an arithmetic series? Let's find out

**Exercise 6.2.4.** What is the sum of the arithmetic series

$$a_1 + a_2 + \cdots + a_n$$

We can rewrite this series as

$$a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)$$

We can see that we have n $a_1$'s, and the coefficient of our d term seems to be the sum of the first $n - 1$ integers ... which is in itself an arithmetic series. We don't have the tools to

evaluate this yet (We will very soon!), so let's find another strategy.
Let's say that $a_n = b$, and $a_1 = a$ in our arithmetic series. We can then rewrite the series as

$$S = a + (a+d) + (a+2d) + \cdots + (b-2d) + (b-d) + b$$

We are able to do this because by our definition of an arithmetic sequence, $a_n - d = a_{n-1}$. Now we will set up the series as so.

$$S = a + (a+d) + (a+2d) + \cdots + (b-2d) + (b-d) + b$$

$$S = b + (b-d) + (b-2d) + \cdots + (a+2d) + (a+d) + a$$

Notice that these series are exactly the same, we just wrote them backwards. From this, we can see some nice symmetry going on, and we are inspired to add the two series, so we get

$$2S = \underbrace{(a+b) + (a+b) + \cdots + (a+b)}_{\text{n terms}}$$

We know there are n terms because $S$ has n terms, and none of our terms cancelled out. Thus our sum is

$$S = \frac{n(a+b)}{2}$$

We see that this is equal to the average of the first and last term in the series multiplied by the number of terms in the series, which is an easy way to memorize this.

**Fact 6.2.5.** An arithmetic series is equal to

$$\frac{n(a_1 + a_n)}{2}$$

Let's quickly use this new fact

**Exercise 6.2.6.** Evaluate $5 + 11 + 17 + 23 + 29 + \cdots + 107$

We see that $a_1 = 5$, $a_n = 107$, and $d = 6$. Now we just need to find how many terms are in the series... While there are many ways to do this, here is how I usually do it. I start by subtracting every term in the series by $a_1$. This leaves us with

$$0 + 6 + 12 + 18 + \cdots + 102$$

Next, we can divide every term in the series by 6 to get

$$0 + 1 + 2 + 3 + \cdots + 17$$

If you are doing this, remember the 0! It was still a term in our original series, so we must count it. Including 0, this series has 17 + 1 = 18 terms, so our original series has 18 terms. Thus the sum of our original series is

$$\frac{18(112)}{2} = 1008$$

There are faster ways to find the number of terms in the series, but this way feels more natural to me personally.
Now that we know how to evaluate an arithmetic series, let's find a definition for a very important series that comes up often.

**Exercise 6.2.7.** Evaluate the sum of the first n positive integers.

Our series looks like this

$$S = 1 + 2 + 3 + 4 + \cdots + n$$

This is an arithmetic series with a common difference of 1. However, we don't know what our $a_n$ is. If we want to create a general formula, we must rewrite the formula for an arithmetic series as

$$\frac{n(a_1 + (a_1 + (n-1)d)}{2}$$

Plugging in our $a_1$ and $d$, we have

$$S = \frac{n(1 + 1 + (n-1))}{2} = \frac{n(n+1)}{2}$$

And we are done!

**Fact 6.2.8.** The sum of the first n positive integers is

$$\frac{n(n+1)}{2}$$

There are many hard AIME level problems involving arithmetic series, but they are not very prevalent in Calculus, so I will leave some (much) harder problems as exercises.

## 6.3 Geometric Series

A geometric series is much like an arithmetic series, except instead of adding by a common difference for each consecutive term, we multiply by a common ratio.

**Fact 6.3.1.** A geometric series is a series for which there exists a constant r such that each term is r times the previous term. This constant r is called the common ratio. For example:

$$1 + 2 + 4 + 8 + 16 + \dots$$

Is an infinite geometric series where the common ratio is 2 and the first term is 1.

Much like we did for arithmetic series, let's find the nth term of the series.

**Exercise 6.3.2.** What is the nth term of a geometric series?

Letting the first term be $a_1$ and the common ratio be $r$, we write the first few terms out:

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \dots$$

We can see that, much like the arithmetic series, the nth term of this series is $a_1 r^{(n-1)}$.

**Fact 6.3.3.** The nth term of a geometric series is

$$a_1 r^{(n-1)}$$

Now we will find the sum by using a similar yet very different technique that we used for arithmetic series.

**Exercise 6.3.4.** What is the sum of the geometric series

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Let the common ratio be $r$ and let the first term be $a$, for simplicity. We write the series as

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{(n-1)}$$

We will now multiply the series by r to get

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{(n-1)}$$

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{(n-1)} + ar^n$$

From this, we see that

$$rS - S = ar^n - a$$

Now we can factor and simplify to get

$$(r-1)S = a(r^n - 1) \implies S = \frac{a(r^n - 1)}{r - 1}$$

This method is not only useful for geometric series, as we will see in the next section.

**Fact 6.3.5.** The sum of a geometric series is equal to

$$\frac{a(r^n - 1)}{r - 1}$$

You may be wondering why I have included infinite series in some of the definitions, but I have not touched them yet. Arithmetic series cannot be evaluated if they are infinite, because no matter what the common difference is, if there are an infinite number of terms in an arithmetic series, it will equal either $\infty$ or $-\infty$. However, geometric series are a different story. For example, let's analyze this series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

With the first term being 1 and the common ratio being $\frac{1}{2}$. We can see that as n gets very large, $a_n$ approaches 0, so there might be hope for us to evaluate this. Let's try using the method we used to find the sum of a finite geometric series.

$$S = 1 + 1/2 + 1/4 + 1/8 + 1/16 + \ldots$$

$$\frac{1}{2}S = 1/2 + 1/4 + 1/8 + \ldots$$

$$S - \frac{1}{2} = 1$$

$$S = 2$$

Wait, what did we just do? It looks like we made a lot of assumptions to get this answer, especially when we subtracted two infinite series. We saw in the finite series that the last

term of one of our series didn't cancel out, so why don't we have another term other than 1 when we subtract the series? We know that the last term of S is $\frac{1}{2^{(n-1)}}$, thus when we multiply the series by $1/2$, the last term will be $\frac{1}{2^n}$. This term does not cancel out with anything, so it is indeed present when we subtract the series. However, $\lim_{n\to\infty} \frac{1}{2^n} = 0$, or in other words, as n gets very large, our last term approaches 0, so it is irrelevant. Thus we can say that our series is equal to 2.
Let's now try to find a general formula for the sum of an infinite geometric series.

**Exercise 6.3.6.** Find the sum of

$$a + ar + ar^2 + ar^3 + \dots$$

If |r| < 1

First, let's discuss our restriction on r. In both finite and infinite geometric series, r cannot be equal to 1, since it makes the denominator of our formula for sum of a geometric series equal 0. Additionally, it is basically an arithmetic series with common difference 0. Can r be equal to -1? Let's see what happens if this is the case

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

This is impossible to evaluate if there are an infinite amount of terms, because the value changes from 1 to -1 after every term. Now let's examine why $|r| > 1$ is not possible. If this is the case, each term in the series is increasing, so as n goes to infinity, $a_n$ goes to either $\infty$ or $-\infty$, so we cannot properly evaluate the series. Now that we understand our restrictions, let's find the formula.

$$S = a + ar + ar^2 + ar^3 + \dots$$

$$rS = ar + ar^2 + ar^3 + \dots$$

$$S - rS = a$$

$$S = \frac{a}{1-r}$$

And we are done!

**Fact 6.3.7.** The value of an infinite geometric series with $|r| < 1$ is

$$\frac{a}{1-r}$$

Again, there are many hard AIME level problems involving geometric series that I will not put in this handout, but unlike arithmetic series, geometric series have more applications than just integers, so we will come back to these formulas in a later section.

## 6.4 Arithmetico-Geometric Series

Arithmetic and Geometric series should be review for most of you. However, we can actually combine properties of arithmetic and geometric series together to make an Arithmetico-Geometric series.

**Exercise 6.4.1.** Evaluate the series

$$1(2^0) + 2(2^1) + 3(2^2) + 4(2^3) + \cdots + n(2^{(n-1)}) + \cdots + 11(2^{10})$$

We will attempt to approach this like we did with our geometric series, as follows

$$S = 1(2^0) + 2(2^1) + 3(2^2) + 4(2^3) + \cdots + 11(2^{10})$$

$$2S = 1(2^1) + 2(2^2) + 3(2^3) + \cdots + 10(2^{10}) + 11(2^{11})$$

$$S - 2S = (2^0) + (2^1) + (2^2) + \cdots + (2^{10}) - 11(2^{11})$$

Miraculously, our series has turned into a geometric series with an extra term! We can now evaluate the series

$$-S = \frac{1(2^{11}-1)}{2-1} - 11(2^{11}) = 2^{11} - 1 - 11(2^{11}) = -10(2^{11}) - 1 = -20481$$

Thus $S = 20481$.

Note that the reason we chose to multiply the series by 2 is that 2 is the common ratio of the geometric part of the series. This is the same logic you want to use for any arithmetico-geometric series. It turns out, evaluating finite series is actually a bit more annoying than evaluating infinite series. Luckily, we can also solve infinite Arithmetico-Geometric series!

**Exercise 6.4.2 (Mandelbrot).** Determine the value of the infinite product $2^{1/3} \cdot 4^{1/9} \cdot 8^{1/27} \cdot 16^{1/81} \ldots$

We see repeated powers of 2, so we think it will be helpful to rewrite all the powers of 2 so that they have a base of 2. The result of this is

$$P = (2^{1/3})(2^{2/9})(2^{3/27})(2^{4/81})\ldots$$

We achieve these new exponents by use of exponent rules. This manipulation is superbly helpful, because now we can rewrite the product as

$$P = 2^{(\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\ldots)}$$

The exponent itself is an Arithemtico-Geometric series! So let's evaluate it. We have

$$S = 1/3 + 2/9 + 3/27 + 4/81 + \ldots$$

$$3S = 1 + 2/3 + 3/9 + 4/27 + \ldots$$

$$3S - S = 1 + 1/3 + 1/9 + 1/27 + \ldots$$

$$2S = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$S = \frac{3}{4}$$

Thus our final answer is $2^{3/4}$.

Let's look at a few more examples.

**Exercise 6.4.3 (MAO Nats 2021 Alpha Seq/Series).** If $1 + 2x + 3x^2 + 4x^3 + \cdots = 25$, what is x?

We see that the coefficients form an arithmetic progression, and the x terms form a geometric progression. This inspires us to approach this the same way we approached the last few problems:

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots = 25$$

$$xS = x + 2x^2 + 3x^3 + 4x^4 + \cdots = 25x$$

$$S - xS = 1 + x + x^2 + x^3 + \cdots = 25 - 25x$$

$$S(1-x) = \frac{1}{1-x} \implies S = \frac{1}{(1-x)^2}$$

We are able to write the infinite series of x's as that fraction because it is simply an infinite geometric series with common ratio x. Since we know that S = 25, we must have

$$25 = \frac{1}{(1-x)^2}$$

$$(1-x)^2 = \frac{1}{25}$$

$$(1-x) = \frac{1}{5}$$

$$x = \frac{4}{5}$$

Wait, shouldn't we have two solutions, since $(1-x)$ can be positive or negative? Let's see what happens when we try this:

$$(1-x)^2 = \frac{1}{25}$$

$$\pm(1-x) = \frac{1}{5}$$

When we take the negative version, we get

$$(x-1) = \frac{1}{5} \implies x = \frac{6}{5}$$

However, this would mean that $|r| > 1$, and since our series equals a finite value, this solution is extraneous.

> *Remark 6.4.4.* Don't think that you are restricted to using these strategies on series that consist of numbers. You can extend these strategies to functions with variables too!

Here is the last problem we will look at in this section:

**Exercise 6.4.5 (Mandelbrot).** Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Find the value of the infinite sum

$$\frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \cdots + \frac{F_n}{3^n} + \dots$$

The denominators follow a geometric progression but... what is the pattern of the numerators? The numerators follow a *recursive pattern*, this one specifically is called the fibonacci sequence. We should first take a look at what the pattern means so we have a better understanding of the problem.

The recursive formula tells us that each term is the sum of the two previous terms. Writing a few out, we get

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

And so forth. We note that the first numerator in our series correlates to $F_1$ in this sequence, and the numerators do indeed follow this pattern. We don't have to guess this either, since the problem tells us that the nth term of this series is indeed $\frac{F_n}{3^n}$. But how do we evaluate this sum? Let's go back to our tried and true method:

$$S = \frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \frac{3}{81} + \ldots$$

$$3S = 1 + \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{5}{81} + \ldots$$

Let's see what happens when we subtract them:

$$3S - S = 1 + \frac{1}{9} + \frac{1}{27} + \frac{2}{81} + \ldots$$

Aha! We can rewrite this as

$$2S = 1 + \frac{1}{3}S$$

$$6S = 3 + S \implies S = \frac{3}{5}$$

This problem shows that not only can we evaluate Artihmetico-Geometric series, but we can also evaluate series with a recursive pattern in the numerator and a geometric pattern in the denominator! In the next section, we will learn how to concisely write these different summations, which will help us see these patterns faster.

## 6.5 Summations

**Fact 6.5.1.** A summation is a type of mathematical notation that we use to concisely write series. For example:

$$\sum_{k=1}^{4} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

Is a geometric series with common ratio 2, with 4 terms.

**Fact 6.5.2.** We also have a specific notation for repeated multiplication. We will call it product notation. For example:

$$\prod_{k=1}^{4} \frac{1}{2^k} = (\frac{1}{2})(\frac{1}{2^2})(\frac{1}{2^3})(\frac{1}{2^4})$$

In each of these notations, the variable we use is called the *dummy variable*, or *index*. Rewriting series into summation notation is a very important skill to have, so we will do one for practice.

**Exercise 6.5.3 (HMMT).** Express, as concisely as possible, the value of the product

$$(0^3 - 350)(1^3 - 349)(2^3 - 348)(3^3 - 347)\dots(349^3 - 1)(350^3 - 0)$$

To rewrite this in product notation, let's first focus on the $n^3$ terms. It starts at $0^3$, and ends at $350^3$. Thus, our starting index should be 0, and are last index should be 350. Now we have

$$\prod_{n=0}^{350} (n^3 - (350 - n))$$

Thus, each term in our product is equivalent to $n^3 + n - 350$. Our next thought should be to attempt to factor this quadratic, just because doing so usually gives us good information about a problem. After trying different values, we find that 7 is a root of the quadratic. But wait! Our product ranges from $n \in [0, 350]$, which means that $n = 7$ for some term in our product. This is all we need to solve the problem, since we would be multiplying by 0, thus the answer is $\boxed{0}$.

Another important skill is understanding how to *shift the index* of a summation. Let's do a quick one as an example

**Exercise 6.5.4.** Express the sum $\sum_{i=4}^{12} \frac{2i}{i-3}$ as a summation in which the dummy variable ranges from 7 to 15.

In order to accomplish this, we want to define a new dummy variable, say k. This k must satisfy $k = i + 3$, since we are adding 3 to the limits of the summation. However, we need to rewrite the argument of the summation (the thing inside the summation) in terms of k.

We know that $i = k - 3$, so our new summation is now

$$\sum_{k=7}^{15} \frac{2k-6}{k-7}$$

You can write out the first few terms of each summation to confirm that they are equivalent. Now that we are more comfortable with how summations work, let's go over a few important properties of summations.

**Fact 6.5.5.** If c is a constant, then

$$\sum_n ca_n = c\sum_n a_n$$

We can see this is true if we write out a few terms of the series:

$$ca_1 + ca_2 + ca_3 + \dots$$

We can clearly factor c out of every term, thus this rule is true.

**Fact 6.5.6.**

$$\sum_n a_n + b_n = \sum_n a_n + \sum_n b_n$$

This one is a little less obvious, but still easy to see once we write out a few terms. We have

$$a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \cdots = (a_1 + a_2 + a_3 \dots) + (b_1 + b_2 + b_3 + \dots)$$

Thus, we can split the single summation into two summations. Let's see these rules in action

**Exercise 6.5.7.** Compute

$$\sum_{k=0}^{\infty} \frac{1+4^k}{5^k}$$

We can split the argument up into $\frac{1}{5^k} + (\frac{4}{5})^k$, thus we can rewrite the summation into two simpler summations:

$$\sum_{k=0}^{\infty} \frac{1}{5^k} + \sum_{k=0}^{\infty} (\frac{4}{5})^k$$

These are both infinite geometric series, so we can evaluate them as follows

$$\frac{1}{1-1/5}+\frac{1}{1-4/5}=\frac{25}{4}$$

And we are done.

## 6.6 Nested Sums

Nested sums are what we call having a summation inside of a summation. While the following problems may seem boring/not interesting, we will do these so that you become comfortable with this notation.

**Exercise 6.6.1.** Evaluate

$$\sum_{i=1}^{4}\sum_{j=1}^{6}(i+j)$$

We will work from the inside out. We can first split the inside summation into

$$\sum_{i=1}^{4}(\sum_{j=1}^{6}i+\sum_{j=1}^{6}j)$$

From this, the importance of the dummy variable becomes apparent. In the summation $\sum_{j=1}^{6} i$, the dummy variable is j, not i, so i acts as a constant, thus we can write it as $i\sum_{j=1}^{6} 1$. The second summation on the inside, $\sum_{j=1}^{6} j$, is just the sum of the first 6 positive integers, which we have a formula for. Thus our summation is now

$$\sum_{i=1}^{4}(6i+21)$$

We can now split this summation into

$$6\sum_{i=1}^{4}i+\sum_{i=1}^{4}21=144$$

Splitting the summation is a very useful tool for simplifying complicated summations. Let's look at a few more examples.

**Exercise 6.6.2.** Evaluate

$$\sum_{i=1}^{5}\sum_{j=1}^{5} ij$$

Again, we work from the inside out. The i in the inside summation is not the dummy variable, so we can factor it out as a constant, giving us

$$\sum_{i=1}^{5}\left(i\sum_{j=1}^{5} j\right)$$

We can now evaluate the inside sum and rewrite as

$$\sum_{i=1}^{5} i(1+2+3+4+5)$$

This is equal to

$$1(1+2+3+4+5)+2(1+2+3+4+5)+\cdots+5(1+2+3+4+5)$$

$$=(1+2+3+4+5)(1+2+3+4+5)=225$$

Hmm, it seems that we could have rewritten the original sum as $\left(\sum_{i=1}^{5} i\right)\left(\sum_{j=1}^{5} j\right)$! In general, it turns out that if we can factor out a variable that is not the dummy variable out of our inside summation, and it is the dummy variable of the outside summation, we can split them apart! Let's try this in our next problem.

**Exercise 6.6.3.** Evaluate

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{1}{2^{i+j}}\right)$$

We first separate the dummy variables in the inside summation as

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{1}{2^i}\cdot\frac{1}{2^j}\right)$$

$$=\sum_{i=1}^{\infty}\left(\frac{1}{2^i}\cdot\sum_{j=1}^{\infty}\frac{1}{2^j}\right)$$

$$= \left(\sum_{i=1}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=1}^{\infty} \frac{1}{2^j}\right)$$

From here, we can easily evaluate both summations as geometric series, and we find that the answer is $\boxed{1}$.

Let's look at one more example

**Exercise 6.6.4.** Evaluate

$$\sum_{i=0}^{\infty}\left(\sum_{j=i}^{\infty} \frac{1}{3^j}\right)$$

This is different than our previous example, because only one dummy variable is present in the summation, but it relates to the dummy variable of the outside summation. To tackle this, we will seek the *closed form* of the inside summation. This just means that we want to find an expression that will be equal to the value of the summation for any value of i we want. Focusing on the inside summation, we see that it is a geometric series, with initial term $\frac{1}{3^i}$, thus we can express the inside summation as

$$\sum_{j=i}^{\infty} \frac{1}{3^j} = \frac{\frac{1}{3^i}}{1-\frac{1}{3}} = \frac{3}{2}\cdot\frac{1}{3^i}$$

Now all we have to do is evaluate

$$\sum_{i=0}^{\infty} \frac{3}{2}\cdot\frac{1}{3^i}$$

We can easily evaluate this by factoring out the $3/2$ and solving the geometric series, and thus the answer is $\boxed{\frac{9}{4}}$. For our last question in this section (for real this time!) we will look at an actual competition problem.

**Exercise 6.6.5 (Edit of AMC).** Suppose Michael, Divij, and Lucas are playing a game, where Michael thinks of a number that is equal to $1/2$ raised to some positive integer power, Divij thinks of a number that is equal to $1/3$ raised to some positive integer power, and Lucas thinks of a number that is equal to $1/5$ raised to some positive integer power. What is the sum of all the possible products of their numbers?

To solve this problem, we want to express the problem in terms of a summation, and solve such summation. Michael's number is $(\frac{1}{2})^a$, where $a \in [1, \infty)$, Divij's number is $(\frac{1}{3})^b$, where $b \in [1, \infty)$, and Lucas's number is $(\frac{1}{2})^c$, where $c \in [1, \infty)$. Note that we must use different variables for each numbers exponent, because each person's exponent does not need to be the same (although they can be!). We can express this as

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{1}{2^a}\frac{1}{3^b}\frac{1}{5^c}$$

Make sure you understand why this makes sense! We are choosing all possible values of c, then all possible values of b, then finally all possible values of a. From our previous problem, we can factor out the dummy variables we don't care about from the first summation, and then again from the second one. Now our summation becomes

$$\sum_{a=1}^{\infty}\frac{1}{2^a}\cdot\sum_{b=1}^{\infty}\frac{1}{3^b}\cdot\sum_{c=1}^{\infty}\frac{1}{5^c}$$

This is just 3 geometric series, so the answer is $1 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. In fact, we can shorten the notation we used at first into a single summation!

$$\sum_{a,b,c\geq 1}\frac{1}{2^a}\frac{1}{3^b}\frac{1}{5^c}$$

This is basically saying that we are taking all possible combinations of a, b, and c where $a, b, c \geq 1$ and summing them together. Additionally, we are allowed to split them apart like we did originally as well. Let's see why. Let's say we have a summation such that

$$\sum_{1\leq n,k\leq 20} a_n b_k$$

This is fundamentally the same as

$$(a_1 + a_2 + a_3 + \cdots + a_{20})(b_1 + b_2 + b_3 + \ldots b_{20})$$

Because when we multiply these together using distributive property, the terms will be all possible combinations of n and k.

## 6.7 Intermission: Factorizations

I know I covered a few in the first algebra handout, but there are a few that we want to be familiar with for the upcoming section.

**Exercise 6.7.1.** How can we rewrite $1 + x + x^2 + x^3 + x^4$?

We see that this is a geometric series with common ratio x, so we can rewrite this using our formula for the sum of a finite geometric series to get

$$\frac{x^5 - 1}{x - 1}$$

This tactic will come in handy for many summation problems that involve simplification of polynomials. Here is one more that you may or may not need.

**Exercise 6.7.2.** Factor $x^4 + 2x^3 + 3x^2 + 2x + 1$

Let's try factoring this into two quadratics. Because of the symmetric nature of this polynomial, we suspect that it is a perfect square. Let's experiment! Since the leading coefficient is 1, the leading coefficient of our quadratic is 1. Similarly, since every term is positive and the constant is 1, the constant of our polynomial is most likely 1. Thus we suspect this is equal to

$$(x^2 + ax + 1)^2 = x^4 + 2ax^3 + (2 + a^2)x^2 + 2ax + 1$$

If we let a = 1, we see that $(x^2 + x + 1)^2$ is indeed equal to our original polynomial. If you see a long polynomial that follows the same pattern as our fourth degree polynomial, you can factor it in a similar fashion.

## 6.8 Telescoping Sums

Telescoping sums are essentially summations where a majority of the terms cancel out with each other, leaving only a few terms. Let's see it in action.

**Exercise 6.8.1.** Evaluate

$$\sum_{n=1}^{10} \frac{1}{n} - \frac{1}{n+1}$$

Writing out a few terms, we see this summation is equivalent to

$$(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{10} - \frac{1}{11})$$

All the terms cancel out except 1 and $-\frac{1}{11}$, so our answer is $\frac{10}{11}$.
Of course, a telescoping problem will never be this easy. We must turn a summation into a form such as this that allows us to telescope. One of the best tools we have to do this is called *Partial Fraction Decomposition*. Let's learn how to do it.

**Exercise 6.8.2.** If

$$\frac{1}{n(n+1)} = \frac{A}{n} - \frac{B}{n+1}$$

Find A and B

We will multiply both sides by $n(n+1)$ to obtain

$$1 = A(n+1) + Bn$$

To isolate A and B, we can choose values of n so that it gets rid of one of the variables and leaves the other one. First, if we let n=-1, we obtain $-B = 1 \implies B = -1$. Similarly, if we let n=0, we obtain $1 = A$. Substituting these values into the original equation, we have

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

And this is the exact same expression we had in the argument of our telescoping series! You may be thinking that we just did a whole lot of work to split this up, and that there has to be an easier way to do partial fraction decomposition since it is so important in both series and integration, and there is! It is called the cover up method, and I will go over it in practice, but if you can't make it in person I believe there is an article on brilliant.org about it.

Keep in mind that the numerator of a rational expression does not have to be 1 in order to do partial fraction decomposition, it can be any number, or even a polynomial. Also,

it is possible to perform PFD on denominators that have linear factors of a multiplicity greater than 1, and even irreducible quadratics, but I believe that they are mostly irrelevant when it comes to telescoping sums, although I encourage you to peruse PFD on brilliant.org to learn the different techniques.
Remember back when we found the sum of the first n positive integers? Let's explore another way to find this result using telescoping sums.

**Exercise 6.8.3.** Find a closed form for the sum of the first n positive integers using a telescoping sum.

First, let's simplify $(k+1)^2 - k^2$. We have

$$(k+1)^2 - k^2 = 2k+1$$

Thus it is true that

$$\sum_{k=1}^{n} 2k+1 = \sum_{k=1}^{n} (k+1)^2 - k^2 = (2^2 - 1^2) + (3^2 - 2^2) + \cdots + ((n+1)^2 - n^2)$$

$$= (n+1)^2 - 1^2 = n^2 + 2n$$

This let's us set up the equation

$$\sum_{k=1}^{n} 2k+1 = n^2 + 2n$$

We can now split up our summation and solve the parts that we know how to solve.

$$2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = n^2 + 2n$$

$$2\sum_{k=1}^{n} k = n^2 + n \implies \sum_{k=1}^{n} n = \frac{n(n+1)}{2}$$

Let's try to apply the same logic to find the sum of the first n positive squares!

**Exercise 6.8.4.** Find a closed form of $\sum_{k=1}^{n} k^2$

Inspired by our last solution, let's try to use $(k+1)^3 - k^3$. Simplifying this, we get $3k^2 + 3k + 1$. We now set up the summation

$$\sum_{k=1}^{n} 3k^2 + 3k + 1 = \sum_{k=1}^{n} (k+1)^3 - k^3 = (2^3 - 1^3) + (3^3 - 2^3) + \cdots + ((n+1)^3 - n^3)$$

Thus we have

$$\sum_{k=1}^{n} 3k^2 + 3k + 1 = (n+1)^3 - 1 = n^3 + 3n^2 + 3n$$

We now split up our summation as

$$3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3n^2 + 3n$$

Simplifying, we end up with the result

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

*Remark 6.8.5.* It is very easy to read through some of these problems that require a bit of algebra, and just assume that you know how to simplify the equation, and that going through the work is a waste of time. However, this is a very bad mindset to have, because you will lose the "muscle memory" of going through the proof, and getting a better understanding of where the formulas come from, thus being less prone to forgetting important formulas when competition day comes!

We can apply this same logic to derive the closed form of the first n positive cubes, and these closed forms combine to form the ever-so important polynomial summations:

**Fact 6.8.6.**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Note that it is technically possible to find the closed form of any sum in the form of $\sum_{k=1}^{n} k^a$, where a is a positive integer, using this technique. Let's do some problems!

**Exercise 6.8.7 (ARML).** Compute the value of

$$\frac{1}{2}+\frac{1}{6}(1^2+2^2)+\frac{1}{12}(1^2+2^2+3^2)+\frac{1}{20}(1^2+2^2+3^2+4^2)+\cdots+\frac{1}{3660}(1^2+\cdots+60^2)$$

When tackling problems like this, the best strategy is to just turn it into a summation. However, that seems to be the hard part of this problem. Let's first figure out the range that our sum needs to take. We see that the first term in the series is $\frac{1}{2}(1^2)$, and the last term has a $60^2$ term in it's series, thus the limit of our summation should go from 1 to 60, however we may have to use two summations to express this series, so let's not get too ahead of ourselves. The pattern that each term's own series follows is pretty clear, so let's try to find a pattern in the fractional coefficients of each term. We see that

$$2 = 2 \cdot 1$$

$$6 = 3 \cdot 2$$

$$12 = 4 \cdot 3$$

$$\vdots$$

$$3660 = 61 \cdot 60$$

Thus we can express each coefficient as $\frac{1}{n(n+1)}$. Here is how we can express this as a summation(s):

$$\sum_{k=1}^{60}\left(\frac{1}{k(k+1)}\sum_{n=1}^{k} n^2\right)$$

If you don't understand how we got to this sum, we just had to think cautiously and carefully about the expression. Since the nth term of this series has a factor that is a series with n terms, we know that we have to express each terms series in terms of a summation with an upper limit of a dummy variable, since it is changing for each term. Again, this takes a lot of practice, which you will have a lot of if you work through the exercise problems. From here, we can easily evaluate the inner summation, and rewrite the outer summation as

$$\sum_{k=1}^{60} \frac{1}{k(k+1)} \cdot \frac{k(k+1)(2k+1)}{6}$$

$$= \sum_{k=1}^{60} \frac{1}{3}k + \frac{1}{6}$$

Evaluating this sum, we arrive at a final answer of 620.

We will wrap up this section by going through a select few nice telescoping problems.

**Exercise 6.8.8 (USAMTS + Many MAO tests).** Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$$

This problem first showed up in USAMTS, and for some reason has showed up around 3 other times in MAO tests, so I guess it's a well known problem. Firstly, we want to rewrite this as a summation. We see reciprocals of squares, so we think that this will telescope with some hard work and PFD. It seems that each term is

$$\sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}}$$

Let's ignore the square root right now so we can just focus on simplifying the inside. We combine the two fractions to get

$$1 + \frac{2n^2 + 2n + 1}{n^2(n+1)^2}$$

This doesn't seem to be anything too exciting, so we combine the fraction with the 1, to get

$$\frac{n^4 + 2n^3 + 3n^2 + 2n + 1}{n^2(n+1)^2}$$

It would be awesome if we could somehow write this numerator into a perfect square, since we are trying to square root this... wait a minute, I think I saw this factorization earlier in this handout! We can rewrite this as

$$\frac{(n^2+n+1)^2}{n^2(n+1)^2}$$

This let's us finally get rid of the square root, so each term in the summation is equal to

$$\frac{n^2+n+1}{n(n+1)}$$

From here, we would really like to perform PFD, but we run into a bit of an issue if we try to. The degree of the numerator is greater than that of the denominator! This happens often in integration, and we can work around this by splitting up the fraction. Specifically, we can rewrite it as

$$\frac{n^2+n+1}{n(n+1)} = \frac{n(n+1)+1}{n(n+1)} = 1 + \frac{1}{n(n+1)}$$

This isn't some manipulation that we pulled out of thin air! We wanted to lower the degree of the numerator, so we did exactly that. From here it's clear how we can rewrite this.

$$1 + \frac{1}{n(n+1)} = 1 + \frac{1}{n} - \frac{1}{n+1}$$

Now we just need to write this as a summation and solve it! Note that there are 1999 terms, so our summation is

$$\sum_{n=1}^{1999} 1 + \frac{1}{n} - \frac{1}{n+1} = 1999 + (1 - \frac{1}{2000})$$

Thus our final answer is $\boxed{2000 - \frac{1}{2000}}$.

**Exercise 6.8.9 (Putnam).** Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$$

The first obvious step is to factor the numerator and denominator as a sum and difference of cubes. We then obtain

$$\prod_{n=2}^{\infty} \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$$

And... now what? It seems that we can't really do much else, so we try writing out the first few terms in the product:

$$\frac{1\cdot 7}{3\cdot 3}\cdot\frac{2\cdot 13}{4\cdot 7}\cdot\frac{3\cdot 21}{5\cdot 13}\cdots$$

Aha! It seems that this product miraculously telescopes! The first factor in each numerator is all of the positive integers, and the denominators contain each of the positive integers starting at 3, so we have cancellation of all of those coinciding factors. Next, we see that all the terms that resulted from the quadratic factors seem to cancel out as well. Thus our answer seems to be $\boxed{\frac{2}{3}}$, which is true, but why do the quadratic terms cancel?

To investigate, let's first think about how things actually telescope in the first place. It is true that for any polynomial P(x), the summation

$$\sum_{x=1}^{n} \frac{1}{P(x)} - \frac{1}{P(x+1)}$$

Will telescope (make sure you understand why!). Thus, we suspect that if $P(n) = n^2 - n + 1$, then $P(n+1) = n^2 + n + 1$. Plugging in this value into our polynomial, we see that this is indeed true! Neat!

To wrap up this section, I will present you a summation which we will not prove in this handout, but is well known enough that it may be asked in FAMAT

**Fact 6.8.10.**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

## 6.9 Recursions

We have already seen recursions previously in this handout, when we tackled a problem dealing with the Fibonacci sequence. In that case, the *recurrence relationship* was that

$F_n = F_{n-1} + F_{n-2}$. Just like we did with summations, we can also index shift recurrence relationships. Using the Fibonacci sequence as an example, we can write $F_{n+1} = F_n + F_{n-1}$. Let's try some problems.

**Exercise 6.9.1 (AMC 10).** Let $a_k$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for all positive integers m and n. Find $a_{12}$.

We have a relationship involving two variables that aren't necessarily restricted, which means we can choose any positive integer we want for each one. (This is a common technique used to solve functional equations). For example, if we let n=1, we simplify the relationship as

$$a_{m+1} = a_m + a_1 + m = a_m + 1 + m$$

Writing out the first few terms, we have

$$a_2 = a_1 + 2$$

$$a_3 = a_2 + 3$$

$$\vdots$$

$$a_{12} = a_{11} + 12$$

From here, we can "substitute backwards" all of the $a_n$ terms on the RHS of this equation, and we find that

$$a_{12} = 1 + 2 + 3 + \cdots + 12 = \frac{12(13)}{2} = 72$$

And we are done.

**Exercise 6.9.2 (2021 Gemini).** Given that $a_1 = a_2 = 1$, and $a_n = a_{n-1} + 2a_{n-2}$, find $a_{2021} + a_{2020}$

We wish to introduce the sum of two consecutive terms into our recurrence relationship, so we are inspired to add $a_{n-1}$ to both sides of the recurrence. Doing so gives us

$$a_n + a_{n-1} = 2(a_{n-1} + a_{n-2})$$

Often in recursion problems, if you can get the recurrence down to a form that repeats a simple pattern that is countable, you can solve the problem. This is a perfect show case of that. We can start writing a few terms from the inside out as such:

$$a_{2021} + a_{2020} = 2(a_{2020} + a_{2019})$$

$$a_{2021} + a_{2020} = 2(2(a_{2019} + a_{2019})$$

$$\vdots$$

$$a_{2021} + a_{2020} = 2^{2019}(a_2 + a_1) = \boxed{2^{2020}}$$

Sometimes, a question may ask to find the nth term of a recursion, where n is very large. Whenever this occurs, it is often a *periodic* recursion, meaning that the recursion repeats. You will see this in action in the next exercise:

**Exercise 6.9.3 (ARML).** If $a_1 = a_2 = 1$ and $a_{n+2} = \frac{a_{n+1}+1}{a_n}$ for $n \geq 1$, compute $a_t$, where $t = 1998^5$.

In recursions that we suspect are periodic, we can often solve it by just writing out a few terms and seeing if there is a pattern. We have

$$a_1 = 1$$

$$a_2 = 1$$

$$a_3 = 2$$

$$a_4 = 3$$

$$a_5 = 2$$

$$a_6 = 1$$

$$a_7 = 1$$

Since each term is dependent on the two previous terms, we know this is periodic since we have another pair of 1's. Thus, we see that $a_1 = a_6, a_2 = a_7$, etc. We say that the period of this recursion is 5, since $a_n = a_{n+5}$. To find the $1998^5$th term, we need to find the remainder when we divide this by 5. You can achieve this by using binomial theorem, or just by using modular arithmetic.

$$1998^5 \equiv 3^5 \equiv 243 \equiv 3 \mod 5$$

What this tells us is that the $1998^5$th term will be equivalent to a term that leaves a remainder of 3 when divided by 5. Thus, it is the same as $a_3 = \boxed{2}$.

The last technique I will touch on is finding the nth term of a recursion as n approaches infinity. Let's take a look:

**Exercise 6.9.4 (Mu Jan Regional).** If $a_n$ is defined as a sequence with $a_1 = 2015$, and $a_{k+1} = \frac{2a_k}{5} + 1$ for any natural number, k, what is $\lim_{k\to\infty} a_k$?

The trick to this question is to think about what should happen as n approaches infinity. We suspect that since we are being asked to evaluate a term near infinity, then the recurrence should converge. What this means is that all the terms start to approach a single number as we write out many, many terms. So, we can think of this as $\lim_{k\to\infty} a_k = \lim_{k\to\infty} a_{k+1}$. Thus, it suffices to solve $a_k = \frac{2a_k}{5} + 1$, because at infinity $a_{k+1} = a_k$. We thus have

$$\frac{3}{5}a_k = 1 \implies a_k = \frac{5}{3}$$

And we are done.

Sorry this handout is so long! Although I believe there is a lot of good information in this handout, if you want to become very comfortable with series and sequences I highly suggest working through the exercise problems (or at least those that seem compelling to you!).

# 7 Advanced Trigonometry

## 7.1 Introduction

This handout will focus purely on algebraic trigonometry instead of trigonometric applications to geometry. Thus, things such as law of sines and law of cosines will not show up on this handout, but will in a future handout, as these skills are very important for certain related rates and optimization problems. Additionally, this handout will assume that you know how to graph these functions and what their graphs/translations look like.

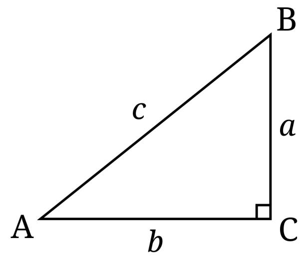
As you go through this handout, the formulas will become more and more complicated, and thus harder to memorize. One of the biggest keys to memorizing trig formulas is actually to not memorize them, but to memorize how they are derived from simpler formulas. Therefore, you shouldn't stare at the formulas in an attempt to memorize them for a week and forget them, but to instead learn how to derive them by hand without help from a solution.

Finally, this handout will include some late AMC/ early AIME level problems. While they may be very intimidating, you should at least learn the solutions for the ones in the handout, as I believe that they are all beneficial and something can be learned from all of them.

## 7.2 Defining our Functions

I know that you guys already know about SOH CAH TOA and how to use trig functions to solve triangles, so this section will be very brief. Suppose we have a right triangle as shown

Let $x = \angle BAC$. Thus we define our functions as follows:




**Fact 7.2.1.**

$$\sin x = \frac{a}{c}$$
$$\cos x = \frac{b}{c}$$
$$\tan x = \frac{a}{b}$$
$$\cot x = \frac{b}{a}$$
$$\csc x = \frac{c}{a}$$
$$\sec x = \frac{c}{b}$$

Note that $\cot x = \frac{1}{\tan x}$, $\csc x = \frac{1}{\sin x}$, and $\sec x = \frac{1}{\cos x}$. Let's use these definitions to "prove" the Pythagorean Identities.

**Exercise 7.2.2.** Show that $\sin^2 x + \cos^2 x = 1$

*Solution:* We have that $\sin x = \frac{a}{c}$ and $\cos x = \frac{b}{c}$. We then have $\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2}$ However, we also know $a^2 + b^2 = c^2$ because of Pythagorean's theorem. Thus $\frac{a^2+b^2}{c^2} = \frac{c^2}{c^2}$ and we are done.

We can derive the other Pythagorean identities from the first by dividing the entire formula by $\sin^2 x$ and $\cos^2 x$, respectively. We have thus derived three formulas:

**Theorem 7.2.3 (Pythagorean Identities)**

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Additionally, the first formula is in fact the graph of the unit circle, where the radius is 1, $\sin x$ is the y-coordinate and $\cos x$ is the x-coordinate. This is important because it also shows us that the range of both $\sin x$ and $\cos x$ is $[-1, 1]$. Additionally, the period of all trig functions besides tangent and cotangent is $2\pi$. Tangent and cotangent have a period of $\pi$.

Another thing to note is that while the range of both $\sin x$ and $\cos x$ is $[-1, 1]$, the range of $\tan x$ is $[-\infty, \infty]$. This means that if we have an equation involving a variable $a$, then we can say that for any value $a$, $a = \tan B$, where $B$ is some angle. This will come in handy later in this handout.

Furthermore, it is very important to note that sin is an odd function, while cos is an even function.

To finish off this section, I will briefly touch on the graphs of these functions.

**Fact 7.2.4.** Trig functions can be graphed in the form:

$$y = a\sin(b(x - c)) + d$$

Where a = amplitude, period = $\frac{2\pi}{b}$ ($\frac{\pi}{b}$ for tangent and cotangent), c = phase shift, and d = vertical shift.

## 7.3 Inverses

The inverse of trig functions are often referred to as "arc" functions, such as arcsin, arccos, etc. They work just like regular inverse functions do. For example, if $\sin a = b$, then $\arcsin b = a$. Well, sort of. All inverse trig functions are one to one, meaning that for every input, there is only one output. This is in contrast to how our normal trig functions have an infinite number of inputs that can lead to the same output. To accomplish this, we restrict the ranges of our inverse functions.

Normally when dealing with inverse functions, the domain and ranges of the two functions flip. This also occurs with inverse trig functions. For example, since the range of cosine is $[-1, 1]$, the domain of arccos is $[-1, 1]$. We just have to be careful about our ranges. Arcsin, arctan, and arccsc all have a range of $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and arccos, arccot, and arcsec all have a range of $[0, \pi]$. Let's try a few problems.

**Exercise 7.3.1.** Evaluate $\arcsin(\sin(\frac{5\pi}{6}))$

*Solution:* It seems easy to say that the answer is $\frac{5\pi}{6}$, and we're done, but remember that we have to mind our functions range. $\sin(\frac{5\pi}{6}) = \frac{1}{2}$, so we then must evaluate $\arcsin(\frac{1}{2})$. The only angle in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that accomplishes this is $\boxed{\dfrac{\pi}{6}}$, so we are done.

**Exercise 7.3.2.** Simplify $\sin(\arccos(a))$

*Solution:* Whenever you see a problem that features two different trig functions that you need to simplify in some way, always think back to your known formulas to find relationships between these functions that may be helpful. In particular, we know that $\sin^2 x + \cos^2 x = 1$. But how can we use this to solve the problem? In general, it is true that $F(F^{-1}(x)) = x$, so let's have $x = \arccos(a)$ in our Pythagorean identity and see if it goes anywhere. We then have

$$\sin^2(\arccos(a)) + \cos^2(\arccos(a)) = 1$$

Using the inverse function relationship shown earlier, we then have

$$\sin^2(\arccos(a)) + a^2 = 1$$

Rearranging and taking the square root, we have

$$\sin(\arccos(a)) = \boxed{\sqrt{1 - a^2}}$$

And we are done.

**Exercise 7.3.3 (AOPS).** Evaluate $\sin(\arctan(2))$

*Solution:* There are many ways to approach this problem, but I will walk through a solution very similar to the previous exercise. Let

$$\sin(\arctan(2)) = y$$

By taking the arcsin of both sides, we have

$$\arctan(2) = \arcsin(y)$$

Then taking the tangent of both sides, we have

$$2 = \tan(\arcsin(y))$$

$$2 = \frac{\sin(\arcsin(y))}{\cos(\arcsin(y))}$$

$$2 = \frac{y}{\sqrt{1-y^2}}$$

Rearranging, we have

$$5y^2 = 4$$

$$y^2 = \frac{4}{5}$$

$$y = \pm\frac{2\sqrt{5}}{5}$$

Is our solution positive, negative, or both? We know that the range of arctan is in only in the 4th and 1st quadrants of the unit circle, and since $\tan = 2$, we know that the angle is in the first quadrant since tangent is positive. Thus, the sin of that angle must be positive, so our only solution is $\boxed{\frac{2\sqrt{5}}{5}}$ Again, the beauty of problems involving trig functions is that there is often many ways of approaching a problem. This may not have been the fastest solution, but it is a solution nonetheless.

## 7.4 Sum/Difference of Angles

These formulas are very easy to prove using geometry, but I will leave that as an exercise. For now, just know these formulas:

**Theorem 7.4.1 (Sum/Difference of Angles)**

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$
$$\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

You can derive the (a-b) forms of these equations by replacing b with -b, and taking advantage of the functions evenness and oddness. Deriving $\tan(a+b)$ just requires you to simplify $\frac{\sin(a+b)}{\cos(a+b)}$.

**Theorem 7.4.2**

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

From these, it is easy to show the following VERY important identities

**Theorem 7.4.3**

$$\sin(90^\circ - x) = \cos(x)$$
$$\cos(90^\circ - x) = \sin(x)$$
$$\sin(180^\circ - x) = \sin(x)$$
$$\cos(180^\circ - x) = -\cos(x)$$

There are similar identities for every trig function, but they are so simple that it is usually easier to just derive it during a test.

From these, we can derive double-angle formula by letting the functions argument be (a+a).

> *Remark 7.4.4.* If you are reading this before the school year starts, don't worry, we will be deriving these together during practice.

**Theorem 7.4.5**

$$\sin(2a) = 2\sin a\cos a$$
$$\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$
$$\tan(2a) = \frac{2\tan a}{1-\tan^2 a}$$

From here, we can derive our half-angle formulas by using different forms of the double angle for cosine, and then using these formulas to derive the tangent half angle formula.

**Theorem 7.4.6**

$$\cos(\frac{x}{2}) = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos x}{2}}$$
$$\tan(\frac{x}{2}) = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

Now we can solve some problems.

**Exercise 7.4.7.** Evaluate $\cos 15^\circ$

The most obvious approach is to use the half-angle formula, so we will do it this way first. We have that

$$\cos 15^\circ = \sqrt{\frac{1+\cos 30}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Let's see what happens when we evaluate this using the sum-difference formulas. We have

$$\cos 45^\circ - 30^\circ = \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Why did we get two different values? It turns out that these two values are actually equivalent. It is important to note this, because sometimes a question will only have one of these two forms as an answer choice, and you have to be able to identify if they are equivalent. (To see why it is equivalent, square $\frac{\sqrt{6}+\sqrt{2}}{4}$, simplify, then square root it.)

**Exercise 7.4.8 (ARML).** Compute x if $\arctan x + \arctan 1 = 2(\arctan x - \arctan \frac{1}{3})$

*Solution:* First note that $\arctan 1 = \frac{\pi}{4}$, so we can instantly make that substitution. Doing so and rearranging, we get

$$\arctan x = 2\arctan\frac{1}{3} + \frac{\pi}{4}$$

We then take the tangent of both sides and apply tangent sum of angles formula to get

$$x = \frac{\tan(2\arctan\frac{1}{3}) + 1}{1 - \tan(2\arctan\frac{1}{3})}$$

We then must evaluate $\tan(2\arctan\frac{1}{3}$, which we can do by applying tangent double angle formula as follows

$$\tan(2\arctan\frac{1}{3}) = \frac{\tan(\arctan\frac{1}{3}) + \tan(\arctan\frac{1}{3})}{1 - \tan^2(\arctan\frac{1}{3})} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$$

Using this result, we find that

$$x = \frac{\frac{3}{4} + 1}{1 - \frac{3}{4}} = \boxed{7}$$

## 7.5 Product-to-Sum

**Exercise 7.5.1.** Find a formula for $\cos a \cos b$ using sum-difference formulas

We first think about formulas we know that have a $\cos a \cos b$ term. We see this in $\cos(a+b)$ and $\cos(a-b)$. Let's next make a system of equations:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

We can now add both formulas to get

$$\cos(a+b)+\cos(a-b) = 2\cos a\cos b$$

$$\frac{1}{2}(\cos(a+b)+\cos(a-b)) = \cos a\cos b$$

**Theorem 7.5.2**

$$\frac{1}{2}(\cos(a+b)+\cos(a-b)) = \cos a\cos b$$

**Exercise 7.5.3.** Find a formula for $\sin a\sin b$ using sum-difference formulas

Using the same thought process as before, using $\cos(a+b)$ and $\cos(a-b)$ to make a system of equations we have

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

But instead of adding these equations, we will instead subtract them to get

$$\cos(a+b)-\cos(a-b) = -2\sin a\sin b$$

$$-\frac{1}{2}(\cos(a+b)-\cos(a-b)) = \sin a\sin b$$

Note that if you did $\cos(a-b)-\cos(a+b)$, you wouldn't have a negative.

**Theorem 7.5.4**

$$-\frac{1}{2}(\cos(a+b)-\cos(a-b)) = \sin a\sin b$$

We use this same logic for the other formulas, except we use $\sin(a+b)$ and $\sin(a-b)$. I will leave these as an exercise to derive.

**Theorem 7.5.5**

$$\frac{1}{2}(\sin(a+b)+\sin(a-b)) = \sin a\cos b$$

Before we do any problems, let's look at one last formula that ties into these formulas.

## 7.6 Sum-To-Product

**Exercise 7.6.1.** Derive a formula for $\cos a + \cos b$ using Product-to-Sum formulas

We know that $\frac{1}{2}(\cos(a+b)+\cos(a-b)) = \cos a \cos b$, so let's try to use this. However, we have a problem. We want a formula for $\cos a + \cos b$, not $\cos(a+b)+\cos(a-b)$. To work around this, let $(a+b) = y$ and $(a-b) = x$. Using systems of equations, we find that $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$. Thus by rearranging the equation we have our first Sum-to-Product formula

**Theorem 7.6.2**

$$\cos a + \cos b = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$$

Using similar logic, we can derive the other formulas

**Theorem 7.6.3**

$$\cos a - \cos b = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$$
$$\sin a + \sin b = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$$
$$\sin a - \sin b = 2\sin(\frac{a-b}{2})\cos(\frac{a+b}{2})$$

Now we can do some problems

**Exercise 7.6.4 (ARML).** Compute

$$\frac{\sin 13^\circ + \sin 47^\circ + \sin 73^\circ + \sin 107^\circ}{\cos 17^\circ}$$

> *Remark 7.6.5.* Whenever you are faced with a problem containing multiple trig functions with not-nice degrees, try to find symmetry, or pairs that add up to a common number.

*Solution:* We obviously want to utilize Sum-to-Product on the numerator, but which functions do we choose to group together? We notice that 13+107 = 47+73, so we will group them as such. After applying Sum-to-Product, we have

$$\frac{2\sin 60^\circ \cos 47^\circ + 2\sin 60^\circ \cos 13^\circ}{\cos 17^\circ} = \frac{\sqrt{3}(\cos 47^\circ + \cos 13^\circ)}{\cos 17^\circ}$$

The next step seems to be another application of Sum-to-Product, which after applying we get

$$\frac{3\cos 17^\circ}{\cos 17^\circ} = \boxed{3}$$

To round off this section, let's solve an AIME problem!

**Exercise 7.6.6.** Find the smallest positive integer solution to

$$\tan 19x^\circ = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$$

It seems that we would want to do something using Sum-to-Product identity, but we don't have a formula for $\cos + \sin$. However, we know that $\sin(a) = \cos(90^\circ - a)$. By applying this, we have

$$\frac{\cos(96^\circ) + \cos(-6^\circ)}{\cos(96^\circ) - \cos(-6^\circ)} = \frac{\cos(96^\circ) + \cos(6^\circ)}{\cos(96^\circ) - \cos(6^\circ)}$$

Now, after applying Sum-to-Product, we get

$$\frac{2\cos(51^\circ)\cos(45^\circ)}{-2\sin(51^\circ)\cos(45^\circ)} = -\cot(51^\circ) = \tan(19x^\circ)$$

Now, there are several ways to proceed from here, but here's how I originally did it (you may find an easier way to reach a similar result)

$$\frac{-\cos(51^\circ)}{\sin(51^\circ)} = \frac{\sin(19x^\circ)}{\cos(19x^\circ)}$$

And by cross multiplying we get

$$-\cos(51^\circ)\cos(19x^\circ) = \sin(19x^\circ)\sin(51^\circ)$$

$$\cos(51^\circ)\cos(19x^\circ) + \sin(19x^\circ)\sin(51^\circ) = 0$$

$$\cos(19x^\circ - 51^\circ) = 0$$

Now, we know that cos is 0 at $90^\circ + 180^\circ k$, thus

$$19x - 51 = 90 + 180k$$

$$19x = 141 + 180k$$

Now here comes the bs AIME stuff. We seek to find x such that it is an integer, and we want it to be the smallest possible integer. For x to be an integer, $141 + 180k$ must be divisible by 19. By dividing both sides by 19, we get

$$x = 7 + \frac{8}{19} + 9k + \frac{9k}{19} = 7 + 9k + \frac{8 + 9k}{19}$$

Now for x to be an integer, 8 + 9k must be divisible by 19. We can find this by the following

$$8 + 9k = 19n$$

Where n is some positive integer. Next we will solve for k:

$$k = \frac{19n - 8}{9} = 2n + \frac{n - 8}{9}$$

The smallest integer n that will make k an integer is n = 8. Thus, k = 16. Now we can solve for x.

$$x = 7 + 9(16) + \frac{8 + 9(16)}{19} = \boxed{159^\circ}$$

The main point of including this problem in the handout was the steps leading to $\tan 19x = -\cot 51$. Number theory usually isn't tested in FAMAT calculus. (But it's still a cool problem imo).

## 7.7 Treating Functions as variables

Let's cover a very common problem solving strategy involving substitution. Note that nothing changes, and all we are doing is just substituting something for a variable to make our lives easier.

**Exercise 7.7.1.** Find all x given that $0 \leq x \leq 2\pi$ such that

$$\sin^3 x - \sin^2 x + \sin x - 1 = 0$$

We wish to solve this as we would a normal polynomial. Let's let $U = \sin x$. We then have

$$U^3 - U^2 + U - 1 = 0$$

To factor this, we see (either by inspection or rational root theorem) that $(U - 1)$ is a factor of this polynomial. We then factor as

$$(U - 1)(U^2 + 1) = 0$$

For this to equal 0, either $U - 1 = 0$ or $U^2 + 1 = 0$. We know that the latter isn't possible with real numbers, so we shift our focus to the first factor. After substituting back in $\sin x$, we are left with

$$\sin x = 1$$

The only x that satisfies this condition within the restricted interval is $\boxed{\frac{\pi}{2}}$, and we are done.

I don't know why I made this it's own section tbh, it just shows up often enough in FAMAT that it's worth knowing.

## 7.8 When Trig shows up in unexpected places

The key take away from this section is that if you are stuck on a problem and the variables seem to resemble a trig identity, try using trig anyways. Let's look at some examples

**Exercise 7.8.1 (Mandelbrot).** Let a, b, c, d be positive real numbers such that

$$a^2 + b^2 = 1$$

$$c^2 + d^2 = 1$$

$$ac - bd = \frac{1}{2}$$

Calculate $ad + bc$

I'm sure there are many ways to solve this problem without using trigonometry, but it is definitely an easier and more interesting method. Firstly, we know that the first and second equations resemble our Pythagorean identities, and the third equation seems to resemble an angle-addition identity.Let's start by assigning these variables to trig functions, as such

$$a = \sin x$$

$$b = \cos x$$

$$c = \sin y$$

$$d = \cos y$$

Such that x and y are both first quadrant angles. Now let's substitute these values into the third equation

$$\sin x \sin y - \cos x \cos y = \frac{1}{2}$$

By multiplying both sides by -1 and applying angle addition identity we have

$$\cos(x + y) = -\frac{1}{2}$$

Since x and y are both first quadrant angles, we know that $x + y$ must be in either the first or second quadrant, so $x + y = \frac{2\pi}{3}$. Now let's look at the equation we need to solve for

$$\sin x \cos y + \cos x \sin y$$

This is the same as $\sin(x + y)$, and since $x + y = \frac{2\pi}{3}$, we know that the answer is $\boxed{\frac{\sqrt{3}}{2}}$. Let's look at one more:

**Exercise 7.8.2 (AOPS).** Solve the system

$$2x + yx^2 = y$$

$$2y + zy^2 = z$$

$$2z + xz^2 = x$$

Since this question was put in a trigonometry handout, the first thing we should look for is possible trig identities we can take advantage of. My gut is leaning towards some sort of tangent identity, but let's first try manipulating one of the equations. Rearranging the first one we have

$$2x = y - yx^2$$

$$2x = y(1 - x^2)$$

$$\frac{2x}{1 - x^2} = y$$

Aha! This is the same as tangent double angle formula! Rearranging the remaining equations, we have

$$\frac{2y}{1 - y^2} = z$$

$$\frac{2z}{1 - z^2} = x$$

But... how do we apply the double angle identity here? Remember in the beginning of the handout when I mentioned that the range of $\tan x = (-\infty, \infty)$? We will utilize this fact here by making some substitutions

$$x = \tan A$$

$$y = \tan B$$

$$z = \tan C$$

Thus we now know that

$$y = \tan B = \tan 2A$$

$$z = \tan C = \tan 2B$$

$$x = \tan A = \tan 2C$$

Thus

$$\tan(2A) = \tan(B)$$

$$\tan(4A) = \tan(2B) = \tan(C)$$

$$\tan(8A) = \tan(4B) = \tan(2C) = \tan(A)$$

But this means that $\tan(8A) = \tan(A)$, which is OK, because we know that the period of tangent is $\pi$. Therefore,

$$8A = A + \pi k$$

Where k is some integer. Thus

$$7A = \pi k$$
$$A = \frac{\pi k}{7}$$

From our initial equations, we can then deduce that

$$\tan B = \tan(\frac{2\pi k}{7})$$

$$\tan C = \tan(\frac{4\pi k}{7})$$

Thus our solutions are

$$\boxed{(x, y, z) = \left(\tan(\frac{\pi k}{7}), \tan(\frac{2\pi k}{7}), \tan(\frac{4\pi k}{7})\right)}$$

## 7.9 Some advanced problems

These following problems will be fairly hard, but don't be intimidated! You already know everything you need to learn to solve these problems, so just try to take it slow and understand every step of the solutions.

**Exercise 7.9.1 (AOPS).** Find $(\cos 20^\circ)(\cos 40^\circ)(\cos 80^\circ)$

We first start by evaluating our options. We see that we are multiplying cosines together, so we can try solving using Product-to-Sum (Try it on your own!) but we also see that each angle is double the previous, which leads us to consider using double angle identities. However, using cosine double angle identity only makes this problem uglier. If only we could use sine double angle instead... but in fact we can! We just need to put a sine term in there ourselves.

$$y = \cos 20^\circ \cos 40^\circ \cos 80^\circ$$

$$y \sin 20^\circ = \sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ$$

$$y\sin 20^\circ = \frac{1}{2}\sin 40^\circ \cos 40^\circ \cos 80^\circ$$
$$y\sin 20^\circ = \frac{1}{4}\sin 80^\circ \cos 80^\circ$$
$$y\sin 20^\circ = \frac{1}{8}\sin 160^\circ$$
$$y = \frac{\sin 160^\circ}{8\sin 20^\circ}$$

Well... what now? That pattern of double-sines sure seemed intended, but now we are left with two sine terms that we can't evaluate by hand! Let's think a bit harder. We see that 20 + 160 = 180... Aha! We know that $\sin(\pi - x) = \sin(x)$, therefore $\sin 20^\circ = \sin 160^\circ$, and our answer is $\boxed{\frac{1}{8}}$.

> *Remark 7.9.2.* If you are stuck on a question, think about what would make the problem easier to deal with, and see if you can find a way to make it easier yourself

> *Remark 7.9.3.* Always look at relationships between angles, and always look for sines/cosines that can cancel with each other

**Exercise 7.9.4 (AIME).** Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2(\cos a \sin a + \cos 4a \sin 2a + \cos 9a \sin 3a + \cdots + \cos(n^2 a)\sin(na))$$

is an integer.

Product-to-Sum seems like the most obvious candidate for this problem, so let's apply it a few times and see if we find a pattern.

$$\sin a \cos a = \frac{1}{2}(\sin(2a) + \sin(0))$$
$$\sin 2a \cos 4a = \frac{1}{2}(\sin(6a) + \sin(-2a)) = \frac{1}{2}(\sin(6a) - \sin(2a))$$
$$\sin 3a \cos 9a = \frac{1}{2}(\sin(12a) + \sin(-6a)) = \frac{1}{2}(\sin(12a) - \sin(6a))$$
$$\vdots$$

$$\sin na \cos n^2 a = \frac{1}{2}(\sin((n+n^2)a) + \sin((na - n^2)a)$$

We see that many of the terms cancel out in a telescoping series fashion, and by following this pattern we know that the initial expression is equal to

$$(\sin((n+n^2)a) + \sin(0) = \sin((n+n^2)a)$$

Now we must find out when $(\sin((n+n^2)a)$ is an integer. The only time that a trig function is an integer is when it is equal to either -1 or 1, thus we can say that

$$((n+n^2)a) = \frac{\pi}{2}k$$

Where k is some integer. Next, the problem tells us that $a = \frac{\pi}{2008}$. Plugging this in, we end up with

$$(n+n^2) = n(n+1) = 1004k = (2*2*251*k)$$

For this to be an integer, the RHS of the equation must be in the form $n(n+1)$, and we want to minimize this number. After careful inspection, we see that 252 is divisible by 4, thus when $k = 63$ we meet our condition. This means that n = $\boxed{251}$ and we are done. Let's end this handout with one last AIME problem

**Exercise 7.9.5 (AIME).** Find the value of $10\cot(\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21)$

We see that we are adding multiple angles in our cotangent argument, so the only feasible identity we can use seems to be Sums of Angles, however we don't have an identity for cotangent yet, so let's derive one. We have

$$\cot(a+b) = \frac{\cos(a+b)}{\sin(a+b)} = \frac{\cos a \cos b - \sin a \sin b}{\sin a \cos b + \sin b \cos a}$$

We then divide both the numerator and the denominator by $\sin a \sin b$ to get

$$\cot(a+b) = \frac{\cot a \cot b - 1}{\cot b + \cot a}$$

To be able to apply this identity, we group our inverse cotangents together in two groups. But how do we choose which terms to group together? There doesn't seem to be an apparent pattern with their arguments, so let's assume it doesn't matter. We then have

$$\frac{\cot(\cot^{-1} 3 + \cot^{-1} 7)\cot(\cot^{-1} 13 + \cot^{-1} 21) - 1}{\cot(\cot^{-1} 13 + \cot^{-1} 21) + \cot(\cot^{-1} 3 + \cot^{-1} 7)}$$

We then apply our cotangent angle addition identity again to each of the two unique cotangent terms. We find that

$$\cot(\cot^{-1} 3 + \cot^{-1} 7) = 2$$

$$\cot(\cot^{-1} 13 + \cot^{-1} 21) = 8$$

After plugging these values back into our previous expression, we get

$$\frac{(8)(2) - 1}{8 + 2} = \frac{15}{10}$$

But remember that the question asked for 10 times this number, so our final answer is $\boxed{15}$

While this handout may have gone over all the relevant trigonometric identities you need to solve any algebraic trig problem, there are many more problem solving strategies not showcased in this handout that are present in the exercise problems that go along with this. In all honesty, I would only do a majority of the exercise problems if you are taking Trig at states or something, or if you find math fun (haha imagine). The next trig handout will cover geometric applications of trigonometry, including law of sines and law of cosines. Thank you for reading! =D

# 8 Geometric Trigonometry

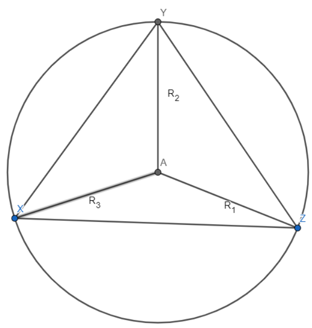
## 8.1 Introduction

This handout will assume that you know how to solve basic right triangles using trigonometry, and not much else. Before we begin, there are a few important geometric terms that we must define.

**Fact 8.1.1.** The *semiperimeter* of $\triangle ABC$ is half the sum of the sides, or more concisely:

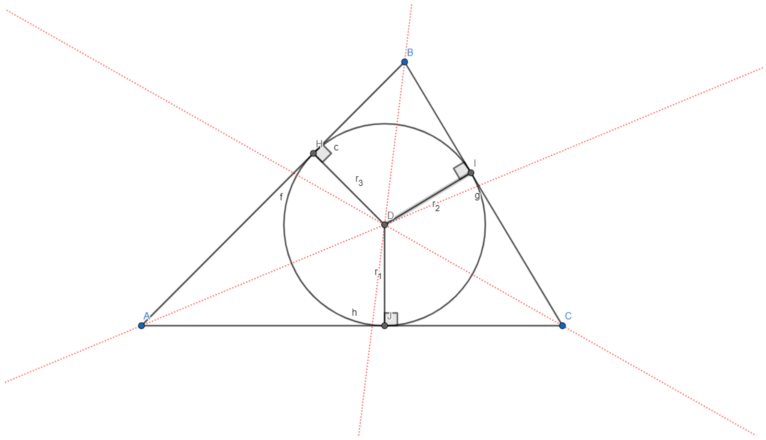
$$s = \frac{a+b+c}{2}$$

**Fact 8.1.2.** The *circumcenter* of a triangle is the center of the circle that circumscribes the triangle. The circumcircle's radius is labelled as $R$. This is shown in the figure below:




**Fact 8.1.3.** The *incenter* of a triangle is the point where the triangle's angle bisectors intersect. This also happens to be the center of the circle that is inscribe within the triangle. The radius of the incircle is labelled as $r$. This is shown in the figure below:

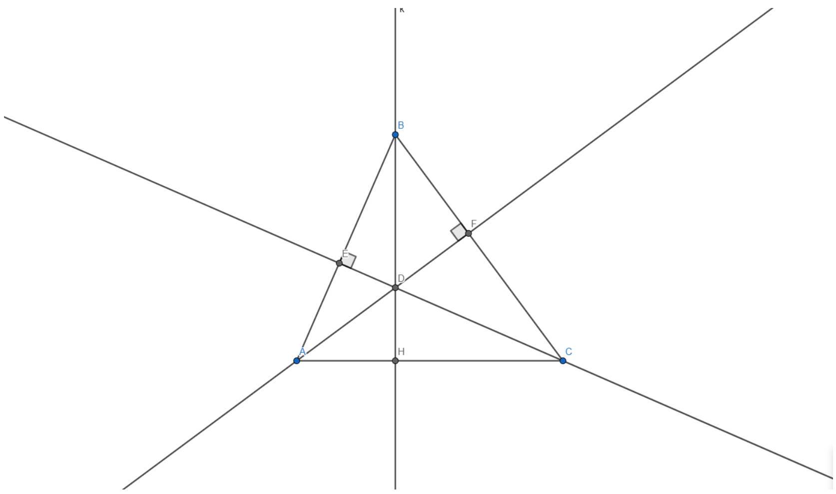
In the above figure, the red lines are angle bisectors, and we can see that their intersection is the center of the inscribed circle.



## 8.2 Triangle Centers

Before we get to trigonometry, let's go over 3 very important centers of a triangle, and tie them all together in an unexpected way. We already talked about the incenter enough in the introduction, so let's start with our second center.

**Fact 8.2.1.** The *orthocenter* of a triangle is where the altitudes intersect. Note that this can be outside the triangle. This is shown in the figure below.




Let's exercise our angle chasing skills with this next problem.

**Exercise 8.2.2.** Suppose triangle $\triangle ABC$ has orthocenter $H$. Find angle $\angle AHC$ in terms of $\angle B$. Notice any other cool angles?

I'm not going to include figures for every example in this handout, because I don't have the time, but the solutions should be sufficient. Firstly, since we are looking for $\angle AHC$, we shouldn't draw in the altitude down from $B$. Let the point where the altitude from $A$ intersects the triangle to be $D$, and let the point where the altitude from $C$ intersects the triangle be $E$. We will first focus on the quadrilateral formed by $BEHD$. Note that angles $\angle BEH$ and $\angle BDH$ are right angles. Also, note that the angles of a quadrilateral must add up to $360°$. (If you don't know why this is true, try breaking a quadrilateral into two triangles, and add up the angles. For an additional challenge, try generalizing this to any n sided convex polygon!). Thus,

$$\angle B + \angle EHD = 180^\circ \implies \angle EHD = 180^\circ - \angle B$$

From this result, we can see that $\angle AHC = \angle EHD = 180^\circ - \angle B$. However, there are other neat angles in this triangle that we can find. For example, look at $\angle EHA$. This is supplementary to angle $\angle EHD$, thus $\angle EHA = \angle B$. Pretty neat! From here it should be clear how to solve for the rest of the angles in this triangle.

Next up is the circumcenter. Through these next results, we will prove Inscribed Angle Theorem.

**Exercise 8.2.3.** Suppose triangle $\triangle ABC$ has circumcenter $O$. Analyze some of its angles and prove Inscribed Angle Theorem.

Much like the previous example we did, we will not draw one of the circumradii. To follow along, draw in all the circumradii in the triangle. We will focus on $\angle A$ and $\angle BOC$. Note that the circumradius $AO$ splits $\angle A$ into two angles. We will label the two angles $x$ and $y$, respectively. Now note that $\triangle AOB$ and $\triangle AOC$ are isosceles. This tells us that $\angle BAO = \angle OBA$ and $\angle OAC = \angle OCA$. If we let $\angle BAO = x$ and $\angle OAC = y$, we can do the following:

$$\angle ABO = x, \angle OCA = y \implies \angle AOB = 180^\circ - 2x, \angle AOC = 180^\circ - 2y$$

From here, we note that $\angle AOB + \angle AOC + \angle BOC = 360^\circ$. Thus

$$360^\circ - 2(x+y) + \angle BOC = 360^\circ$$

$$\angle BOC = 2(x+y)$$

Finally, note that $x+y = \angle A$, thus $\angle BOC = 2\angle A$! This is the basis of Inscribed Angle Theorem.

However, consider the case where $\triangle ABC$ is obtuse (to draw this, just draw a triangle in a circle such that the center of the circle is not in the triangle). We cannot apply this same rule to this type of triangle, so let's develop one. Assume that vertex $A$ is the vertex that is farthest from the circumcenter $O$. Draw circumradii to all of the vertices of the triangle. Again, we see that $\angle A$ is split into two angles. Label $\angle BAO = x, \angle CAO = y$. Again, we see that we have isosceles triangles $\triangle ABO$ and $\triangle ACO$. We can progress much the same way we did with the last example.

$$\angle ABO = x, \angle ACO = y \implies \angle AOB = 180^\circ - 2x, \angle AOC = 180^\circ - 2y$$

We thus find that the smaller angle formed by $\angle BOC = 360^\circ - 2\angle A$, but surprisingly the larger angle is $2\angle A$.

You may be wondering why we are doing these examples when this handout is supposed to be based in trigonometry. I believe that these topics can be turned into pretty neat calculus problems (which you might see in a practice competition!) but also since trigonometry has such a close relationship with angles, having some angle chasing practice won't hurt. I don't want to drag this section out too long, so I will leave a very neat exercise that ties all these centers together as an exercise problem.

## 8.3 Law of Sines/Cosines

First I will state law of sines explicitly, and then we will prove that it is true.

**Fact 8.3.1.** In $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We will prove this for acute triangles, but the proofs for obtuse and right triangles are mostly the same. Construct $\triangle ABC$, and then draw altitude $AT$. Note that $a$ denotes the side opposite to $\angle A$, etc. Now note that $\sin C = \frac{AT}{b}, \sin B = \frac{AT}{c}$. From this, we see that

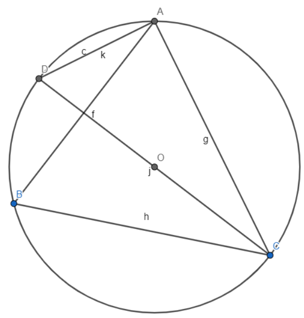
$b \sin C = c \sin B \implies \frac{b}{\sin B} = \frac{c}{\sin C}$. After this, we can drop another altitude, say $BJ$, and repeat the steps with angles $A$ and $C$ to complete the proof.

However, this is not all there is to the law of sines. We can extend the law of sines as follows:

**Fact 8.3.2.** In $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

This proof is fairly straight forward, and will give some insight into properties of cyclic quadrilaterals! Refer to the figure below: First, note that $\angle ADC = \angle B$ since they inscribe



the same arc. Now let's examine $\triangle ACD$. We see that

$$\frac{b}{\sin \angle ADC} = \frac{2R}{\sin 90} = 2R$$

However, since $\angle ADC = \angle B$, we know that

$$\frac{b}{\sin B} = 2R$$

And this basically proves the extended law of sines, since we can just slap 2R into our law of sines relationship on $\triangle ABC$. Of course, we still haven't covered the proofs for right triangles nor obtuse triangles, but extended law of sines applies to both cases, and this handout would be too long with those proofs. However, when using law of sines, tread with caution! Angles of a triangle can be any angle $0 < x < 180$ degrees. However, the sine function will give the same value for two different angles, which is obvious by looking at its behavior on the unit circle. Therefore, sometimes it may be possible for two different triangles to meet a problems criteria. This is usually only a problem met in the Alpha division, but still keep it in mind!

To wrap up this section, we will prove law of cosines.

**Fact 8.3.3.** In $\triangle ABC$,
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Just by looking at the formula, it is clear that this is a very powerful tool for solving for 1 side of a triangle. As before, we will prove this for the case of acute triangles, but this formula holds true for any triangle. Construct $\triangle ABC$, and drop an altitude down from $B$ to $AC$, and let the point of intersection be $P$. We will now attempt to solve for $c$. Note that

$$\sin C = \frac{BP}{a} \implies BP = a\sin C$$

$$\cos C = \frac{PC}{a} \implies PC = a\cos C$$

We know that $AC = b$, so $AP = b - a\cos C$. From here, we can simply compute Pythagorean's Theorem on $\triangle ABP$ as follows:

$$a^2\sin^2 C + (b^2 - 2ab\cos C + a^2\cos^2 C = a^2 + b^2 - 2ab\cos C = c^2$$

And we are done. There will be more practice problems focusing on just these formulas in the exercise problems, but for now we will just use these results to help prove some other results.

## 8.4 Areas, Areas, Areas!

Area formulas for triangles are one of the most important geometry tools to have in your tool belt for calculus. We will denote the area of $\triangle ABC$ as $[ABC]$ for the rest of the handout. Most everyone knows the basic area formula for a triangle:

**Fact 8.4.1.**
$$[ABC] = \frac{1}{2}bh$$

This isn't a very descriptive definition, but most of you know this formula by heart already. However, what if we wanted to find the rate of change of the area of a triangle as the angle is changing? We obviously can't use this formula, because it doesn't involve any angles, so let's make it involve some angles.

To accomplish this, construct $\triangle ABC$, and drop an altitude from $\angle A$ to $BC$. Label the point of intersection $x$. From our previous definition of the area of a triangle, it is clear that the area of this triangle is
$$[ABC] = \frac{1}{2}(AP)(a)$$
Let's try rewriting $AP$. If we focus on $\angle C$, we see that

$$\sin C = \frac{AP}{b} \implies AP = b\sin C$$

Therefore,
$$[ABC] = \frac{1}{2}ab\sin C$$
This is one of the most important triangle area formulas in calculus, so make sure you memorize this! Let's develop one more area formula from our results. We know that

$$\frac{c}{\sin C} = 2R \implies \sin C = \frac{c}{2R}$$

Therefore it is also true that
$$[ABC] = \frac{abc}{4R}$$
These three formulas are the most essential triangle area formulas to know for calculus, although we will show you a few more throughout this handout.

**Fact 8.4.2.** For $\triangle ABC$,

$$[ABC] = \frac{1}{2}(\textbf{base})(\textbf{height}) = \frac{1}{2}ab\sin C = \frac{abc}{4R}$$

## 8.5 Some neat derivations

We will now use what we learned in the past few sections to derive some neat formulas that may or may not be useful. You don't have to memorize all of these, instead, remember how you derived them.

**Exercise 8.5.1.** Construct $\triangle ABC$. Let the point of tangency of the incircle of $\triangle ABC$ and $AB$ be $F$. Show that $AF = s - a$, and $BF = s - b$.

First, label the point of tangency with $AC$ as E, and label the remaining point of tangency as $D$. The key insight of this problem is to notice that $AE = AF$, $BF = DB$, and $CE = CD$ by a circle formula we will not prove in this handout. Now, label $AF = x, BF = y, CD = z$. Now note the following:

$$x + y = c$$

$$x + z = b$$

$$y + z = a$$

Next, we seek $s - a$, which is equivalent to $\frac{b+c-a}{2}$. This inspires us to add the first two equations, and subtract the third, giving us

$$(x + y) + (x + z) - (y + z) = b + c - a$$

$$2x = b + c - a$$

$$x = \frac{b + c - a}{2} = s - a$$

Note that since $x = AF$, we have shown that $AF = s - a$ is true. We can use similar logic for y and z.

**Exercise 8.5.2.** Show that $\tan \frac{A}{2} = \frac{r}{s-a}$

We can reuse the same triangle from the last exercise for this problem. Remember that the center of the incircle is formed by finding the point of intersection of the angle bisectors. Thus, if we let the incenter be $O$, we can create right triangle $\triangle AOF$. This basically finishes the proof, since the opposite side from $\angle\frac{A}{2}$ is $r$ and the adjacent is $s - a$.

**Exercise 8.5.3.** Show that $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Sadly, it will be pretty hard to reuse our construction from the last two examples, since we don't have a very obvious way of finding the hypotenuse of $\triangle AOF$ without having an r in the result, so we will solve this using a different approach. Remember that $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$. This seems pretty useful. Additionally, in order to get this solely in terms of sides, we can solve for $\cos A$ using law of cosines as follows.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$-\frac{a^2-b^2-c^2}{2bc} = \frac{b^2+c^2-a^2}{2bc} = \cos A$$

Substituting this in, we get

$$\cos\frac{A}{2} = \sqrt{\frac{1+\frac{b^2+c^2-a^2}{2bc}}{2}}$$

$$= \sqrt{\frac{2bc+b^2+c^2-a^2}{4bc}} = \sqrt{\frac{(b+c)^2-a^2}{4bc}}$$

$$= \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}} = \sqrt{\frac{(2s)(2s-2a)}{4bc}} = \sqrt{\frac{s(s-a)}{bc}}$$

We will use this result to prove another very important triangle area formula.

**Exercise 8.5.4.** Show that for $\triangle ABC$,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

To prove this, we must first use the fact that $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$. This results pretty naturally from a combination of our previous result and the pythagorean identity, so I won't spend time showing why it is true here. Next, we want to use the fact that

$$[ABC] = \frac{1}{2}bc\sin A$$

However, this is in terms of $\angle A$, and we want it in terms of $\angle\frac{A}{2}$. Luckily, we have a way to transform this into our liking, the double angle formula! We can now rewrite this as

$$[ABC] = bc\cos\frac{A}{2}\sin\frac{A}{2}$$

$$= bc\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sqrt{s(s-a)(s-b)(s-c)}$$

And we are done! This result is known as **Heron's Formula**.

We will now show our final triangle area formula for this handout.

**Exercise 8.5.5.** Show that for $\triangle ABC$,

$$[ABC] = rs$$

Reuse the same construction we used in our first exercise in this section. Now, draw inradii $OF, OD, OE$. Examine $\triangle AOB$. We see that $[AOB] = \frac{1}{2}rc$. Similarly, $[AOC] = \frac{1}{2}b, [BOC] = \frac{1}{2}a$. Thus,

$$[ABC] = [AOB] + [AOC] + [BOC] = \frac{1}{2}r(a+b+c) = rs$$

**Exercise 8.5.6.** Prove **Angle Bisector Theorem**, which states that an angle bisector of a triangle will divide the opposite side into two segments that are proportional to the other two sides of the triangle.

To prove this, construct $\triangle ABC$, and construct the angle bisector of $\angle B$. Label the point of intersection of the angle bisector and $AC$ as $D$. Now, let's do some labeling. Label $AB = a, AD = b, BC = d, CD = c$. First, notice that $\angle ABD = \angle DBC/$ Second, notice that $\angle ADB = \pi - \angle BDC$. This will prove useful when comparing sine values. For simplicity, let $\angle ABD = \theta, \angle ADB = \alpha$. Now note that

$$\frac{b}{\sin\theta} = \frac{a}{\sin\alpha} \implies \frac{b}{a} = \frac{\sin\alpha}{\sin\theta}$$

$$\frac{c}{\sin\theta} = \frac{d}{\sin(\pi-\alpha)} \implies \frac{c}{d} = \frac{\sin\alpha}{\sin\theta}$$

Note that $\sin(\pi-\alpha) = \sin\alpha$! We can thus conclude that $\frac{b}{a} = \frac{c}{d}$, which is Angle Bisector Theorem. Finally, we will prove **Stewart's Theorem**.

**Exercise 8.5.7.** Construct $\triangle ABC$. Let $D$ be a point on $BC$. Let $BD = m, DC = n, AB = c, AC = b, AD = d$. Show that $b^2m + c^2n = amn + d^2a$.

Remember that clever trick we did by using supplementary angles with sine? Let's do the same thing here, only we will instead want to use it with the law of cosines. Let $\angle BDA = \theta$. Therefore, $\angle ADC = \pi - \theta$. We can now set up the following equations

$$c^2 = m^2 + d^2 - 2dm\cos\theta$$

$$b^2 = n^2 + d^2 + 2dn\cos\theta$$

Note that $\cos(\pi - \theta) = -\cos\theta$. From here, we wish to cancel out the cosines, so we will multiply the first equation by n and the second equation by m and add them together.

$$nc^2 = nm^2 + nd^2 - 2dmn\cos\theta$$

$$mb^2 = mn^2 + md^2 + 2dmn\cos\theta$$
$$nc^2 + mb^2 = nm^2 + nd^2 + mn^2 + md^2$$

To simplify this, we will factor the left side as follows.

$$nc^2 + mb^2 = d^2(m+n) + mn(m+n)$$

Note that $m + n = a$, so we can simplify as follows.

$$nc^2 + mb^2 = ad^2 + amn$$

An easier way to write this in order to memorize it easier is

$$dad + man = cnc + bmb$$

And we are done!

# 9 Limit Techniques

## 9.1 Introduction

This aim of this handout is to give a complete overview of all the ways we can evaluate limits. If you are taking Limits and Derivatives at states, I strongly recommend going over Delta-Epsilon definition, however I feel it is unnecessary for this handout. Enjoy!

## 9.2 Domination

Some of the simplest limits to analyze are those of rational polynomial functions. For example:

$$\lim_{x\to\infty} \frac{3x^2+6x+10}{x^2+1}$$

You may see immediately that the answer is 3. What does this mean? This is the horizontal asymptote of the function. As x goes towards infinity, this function approaches the line y=3. However, how do we *actually* evaluate this? This brings up the discussion of **dominating functions**. In other words, functions that increase faster than other functions. In this example, $x^2$ dominates $x$ and obviously dominates the constant. But how can we actually show this? Observe:

$$\frac{3x^2+6x+10}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{3+6/x+10/x^2}{1+1/x^2}$$

Now if we try to evaluate the limit, it becomes much clearer. Every term with an x as the denominator will go to zero as x goes to infinity.

On the topic of dominating functions, it should be apparent that exponential functions grow much faster than polynomial functions. For example, it is true that

$$\lim_{x\to\infty} \frac{x^{100}}{1.5^x} = 0$$

Because the exponential dominates polynomial terms. This becomes very apparent if you analyze this using L'Hopital: as you take the derivative of the top, it will eventually go to

0, however the derivative of an exponential will never get to 0. However, be careful! In the case of $a^x$, if $|a| < 1$ the function will approach zero as x approaches infinity (why is this true?). Let's look at a more advanced and important example of dominating functions:

**Exercise 9.2.1.** Let p and q be real numbers. Evaluate the limit:

$$\lim_{n\to\infty} \sqrt[n]{p^n + q^n}$$

We will analyze this in two cases, the first case being if one number is greater than the other, and the second case being if they are equal. We will start with the former. Let $p > q$. We can then factor out a $p^n$ to obtain

$$\lim_{n\to\infty} p\sqrt[n]{1 + \frac{q}{p}^n} = p\lim_{n\to\infty} \sqrt[n]{1} = p$$

If we let $q > p$, we similarly find that the limit will evaluate to q. Let's look at our final case where $p = q$. We have

$$\lim_{n\to\infty} \sqrt[n]{2p^n} = p\lim_{n\to\infty} \sqrt[n]{2} = p$$

Thus we see that this limit evaluates to $\boxed{\max(p,q)}$. Keep this in mind, it shows up on famat.

## 9.3 Conjugates

Many limits can be evaluated simply by analyzing conjugates. This is such an important concept I am dedicating an entire section to it. Let's work through some problems.

**Exercise 9.3.1 (FAMAT).** Evaluate

$$\lim_{x\to\infty} \sqrt{16x^2 - 24x + 23} - 4x$$

We have a rational, so multiplying by the conjugate seems like a good place to start. We have

$$(\sqrt{16x^2 - 24x + 23} - 4x) \cdot \frac{\sqrt{16x^2 - 24x + 23} + 4x}{\sqrt{16x^2 - 24x + 23} + 4x} = \frac{-24x + 23}{\sqrt{16x^2 - 24x + 23} + 4x}$$

Now we are almost done. We do the following in order to determine the dominating functions:

$$\frac{-24x+23}{\sqrt{16x^2-24x+23}+4x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\frac{-24+23/x}{\sqrt{16-24/x+23/x^2}+4}$$

The trick is that $\frac{1}{x}=\frac{1}{\sqrt{x^2}}$, so it applies inside the rational as well. From here we evaluate the limit in pieces:

$$\lim_{x\to\infty}\frac{-24+23/x}{\sqrt{16-24/x+23/x^2}+4}=\lim_{x\to\infty}\frac{-24}{\sqrt{16}+4}=\boxed{-3}$$

The key takeaway from this problem is that when conjugates don't seem to simplify, remember to try finding dominating functions.

**Exercise 9.3.2 (Stanford Math Tournament).** Evaluate

$$\lim_{x\to 0}\frac{(1-\cos x)^2}{x^2-x^2\cos^2 x}$$

The first thing we see to do is factor the $x^2$ in the denominator, and then simplify using pythagorean identity. We now have

$$\lim_{x\to 0}\frac{(1-\cos x)^2}{x^2\sin^2 x}$$

Even if we wanted to use L'Hopital for this, all the squared terms make taking the derivative hard. However, we can take the square out of the limit:

$$\left(\lim_{x\to 0}\frac{1-\cos x}{x\sin x}\right)^2$$

We can do this because, as long as $f(x)$ is continuous, we can say that

$$\lim_{x\to a}f(g(x))=f(\lim_{x\to a}g(x))$$

(We will see this later when we deal with exponentials). From here, we still have an indeterminant form, and L'Hopital isn't helping at all. Thus we resort to multiplying by the conjugate of the numerator. We think of doing this because it will create a $\sin^2 x$ on the top and the bottom will be a sum.

$$\frac{1-\cos x}{x\sin x}\cdot\frac{1+\cos x}{1+\cos x}=\frac{\sin^2 x}{x\sin x(1+\cos x)}=\frac{\sin x}{x(1+\cos x)}$$

We can now split up the limit as follows

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}=1\cdot\frac{1}{2}$$

But remember we need to square this limit, so the answer is $\boxed{\frac{1}{4}}$.
However, most conjugate problems will just consist of rational radical functions, and when you do the conjugate everything simplifies immediately. These are just extreme examples.

## 9.4 L'Hopital

You should already know how to do L'Hopital, so I'm not going to dwell on the basics, but here is a very basic definition.

**Fact 9.4.1 (L'Hopital).** If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

If we have time at practice I will show a very simple derivative of this theorem, however it is not very important. Instead, I will show you some very cool problems involving L'Hopital that aren't boring.

**Exercise 9.4.2 (Stanford Math Tournament).** Evaluate

$$\lim_{n\to\infty} n^2\int_0^{\frac{1}{n}} x^{2018x+1}dx$$

The key to this problem is to note that the upper bound of the integral approaches 0, which means the entire integral approaches 0 as n approaches 0. We can thus force this into indeterminate form as follows:

$$\lim_{n\to\infty}\frac{\int_0^{\frac{1}{n}} x^{2018x+1}dx}{\frac{1}{n^2}}$$

Note that both the top and bottom approach 0, so we can now use L'Hopital. However, BE VERY CAREFUL! When we differentiate, we do so with respect to the variable that corresponds to the limit. It's not really an issue for this problem, but it can show up and be an issue (2021 States L+D Q26). After differentiating the top and bottom, we get

$$\lim_{n\to\infty} \frac{(\frac{1}{n})(\frac{1}{n})^{\frac{2018}{n}}(-\frac{1}{n^2})}{-\frac{2}{n^3}} = \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{n}^{\frac{1}{n}}\right)^{2018}$$

We will be able to show very soon that the limit inside in fact evaluates to 1. Thus the answer is $\boxed{\frac{1}{2}}$.

NGL L'Hopital is kind of boring, just don't rely on it too heavily as test makers will try to trick you on purpose, also just know you can do it repeatedly and know when something is indeterminant and how to make something indeterminant.

## 9.5 Exponentials

You may have seen this definition of the number e before:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

But how do we actually evaluate this limit? Notice that the annoying part about this limit is the exponent. Thus we wish to get rid of the exponent somehow. We can do this by taking the natural log of the limit. Let the limit we wish to find be equal to y. Thus

$$\ln y = \lim_{x\to\infty} x\ln(1+\frac{1}{x})$$

From here, we can force the RHS to be indeterminant by moving the x to the denominator, and from there we can easily apply L'Hopital. We obtain

$$\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}$$

From here, we find that this limit evaluates to 1. HOWEVER, remember the LHS of our initial equation. We have

$$\ln y = 1 \implies y = e$$

Thus the limit is indeed e. A few more important facts:

**Fact 9.5.1.**

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$$

The above can be shown by a substitution $u = \frac{1}{x}$. Additionally,

$$\lim_{x\to\infty}\left(1+\frac{m}{x}\right)^{nx} = e^{mn}$$

You should try to prove this fact yourself as it show up CONSTANTLY.

Let's look at one more example of this technique.

**Exercise 9.5.2 (HMMT).** Evaluate

$$\lim_{x\to 1} x^{\left(\frac{x}{\sin(1-x)}\right)}$$

Again, set the limit equal to y and take the natural log of both sides:

$$\ln y = \lim_{x\to 1}\frac{x\ln x}{\sin(1-x)}$$

Our limit is already in indeterminant form, so we can use L'Hopital:

$$\lim_{x\to 1}\frac{\ln x + 1}{-\cos(1-x)} = -1$$

Thus

$$\ln y = -1 \implies y = \frac{1}{e}$$

And we are done.

# 10 Derivative Techniques

## 10.1 Product Rule

By now, you should be very familiar with the product rule for derivatives:

**Fact 10.1.1.**

$$(fg)' = f'g + fg'$$

But how can we apply product rule to derivatives with more than 2 functions multiplied together? Turns out that it is just as easy!

**Fact 10.1.2 (Generalized Product Rule).**

$$(f_1 f_2 f_3 \dots f_n)' = \sum_{k=1}^{n} \frac{f_1 f_2 f_3 \dots f_n}{f_k} \cdot (f_k)'$$

This looks confusing, but if you play around with it a bit you will see that it's basically the same thing we have been doing. For example:

$$(fgh)' = f'gh + fg'h + fgh'$$

We go function by function basically. Let's apply this to a simple example.

**Exercise 10.1.3.** If $f(x) = (x-2)(x-1)^2(x+1)$, find $f'(0)$

**Solution:** We take advantage of generalized product rule. The derivative is thus

$$f'(x) = (x-1)^2(x+1) + 2(x-2)(x-1)(x+1) + (x-2)(x-1)^2$$

Plugging in 0, we find $f'(0) = 1 + 4 - 2 = \boxed{3}$. We will go over some much harder examples when we do some problem solving workshops.

## 10.2 Chain Rule

The way that the Chain Rule is taught in class... leaves something to be desired. Here is a more useful "definition"

**Fact 10.2.1 (Chain Rule).**
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Note that this isn't *technically* a true definition, since we are treating the derivative notation as a fraction when it really isn't, but it tends to work out as such. Let's look at a prime example.

**Exercise 10.2.2 (January Regional).** If $y = \sin(3x^2 - 12x + 1)$, then what is $\frac{dy}{d(x^2-2x+3)}$ at $x = 3$?

Using our definition of the chain rule, we know that

$$\frac{dy}{dx} = \frac{dy}{d(x^2-2x+3)}\frac{d(x^2-2x+3)}{dx}$$

$$\frac{dy}{dx} = (6x-12)\cos(3x^2-12x+1)$$

$$\frac{d(x^2-2x+3)}{dx} = 2x - 2$$

Now we evaluate at $x = 3$:

$$6\cos(-8) = \frac{dy}{d(x^2-2x+3)}(4)$$

$$\frac{3}{2}\cos(8) = \frac{dy}{d(x^2-2x+3)}$$

Notice that we turned the 8 inside the cosine positive since cosine is an even function.

## 10.3 Exponential

While this isn't super common, it is a neat way to tidy up a long derivative. Here is the most useful example that may show up.

**Exercise 10.3.1.** Find the derivative of **Heron's Formula**,

$$\sqrt{(s)(s-a)(s-b)(s-c)}$$

Given that all sides of the triangle are changing with respect to time.

We will use a method that is common among exponential limits, and that is to take the ln of this equation. Here is how:

$$\ln([ABC]) = \frac{1}{2}\ln((s)(s-a)(s-b)(s-c)) = \frac{1}{2}[\ln(s) + \ln(s-a) + \ln(s-b) + \ln(s-c)]$$

Note that this is very useful because we have effectively gotten rid of any use of the product rule. Be careful when taking the derivative however:

$$\frac{[ABC]'}{[ABC]} = \frac{1}{2}[\frac{s'}{s} + \frac{(s-a)'}{s-a} + \frac{(s-b)'}{s-b} + \frac{(s-c)'}{s-c}]$$

Note that in a problem such as this, the area at a given time is usually easy to find, and the semiperimeters simplify with the side lengths, so this derivative is not as messy as it looks!

## 10.4 Leibniz

This is probably the most complicated derivative formula that shows up in famat. However, if you understand binomial theorem, you will understand this easily! It is essentially binomial theorem for a product of two functions. Here is the statement:

**Fact 10.4.1 (Leibniz Rule).** If two functions $f$ and $g$ are both n-times differentiable, then their product is also n-times differentiable, and this derivative can be found as such:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Note that in this notation, $f^{(n)}$ refers to the n-th derivative of that function.

Let's look at an example:

**Exercise 10.4.2 (January Team Round).** Let $f(x) = x^3e^x$. Compute $f^{(100)}(3)$

The answer is not 0! Although the product is not 100 times differentiable, it is still 4 times differentiable, so we will have 4 terms that are not 0. Using Leibniz, we have the expansion as

$$\binom{100}{100}x^3e^x + \binom{100}{99}3x^2e^x + \binom{100}{98}6xe^x + \binom{100}{97}6e^x$$
$$= e^x(x^3 + 300x^2 + \frac{100 \cdot 99}{2}6x + \frac{100 \cdot 99 \cdot 98}{6}(6))$$

Evaluating at x=3, we get

$$e^3(27 + 2700 + 89100 + 970200) = e^3(1062027)$$

## 10.5 Some derivations

Here are some derivations so that you won't have to just memorize some of these formulas.

**Exercise 10.5.1.** Find a formula for $f^{-1}(x)$

Just use chain rule!

$$f(f^{-1}(x)) = x$$
$$f'(f^{-1}(x))(f^{-1}(x))' = 1$$
$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

And done.

**Exercise 10.5.2.** Find rules for the derivative of $a^x$ and $\log_a x$

For the first case, it is true that $e^{\ln a} = a$. Thus

$$e^{x \ln a} = a^x$$

$$e^{x \ln a} \cdot \ln a = (a^x)'$$

$$a^x \cdot \ln a = (a^x)'$$

The second case is even easier. By change of base, $\log_a x = \frac{\ln x}{\ln a}$. Thus

$$(\log_a x)' = \frac{1}{x \ln a}$$

In both cases, note that $\ln a$ is a constant.

## 10.6 Dirichlet Function

Here is a question to ponder: Is it possible for a function to be nowhere continuous? What would it even look like? It turns out that there are such functions, and a very common example is that of the Dirichlet Function:

$$f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$$

Or in other words, the function is equal to 1 if x is a rational number, and 0 if x is not equal to a rational number (an irrational number). This function seems to imply that between any real number, there is an irrational number. I'm sure the proof of this is simple but I haven't learned it yet. But it sounds true!

## 10.7 To infinity and beyond!

For the last section of this handout we will look at functions which repeat onto infinity. First I will show how to deal with such functions in a general since.

**Example 10.7.1**
Evaluate

$$\sqrt{25 + \sqrt{25 + \ldots}}$$

Set this expression equal to y. Now note that

$$y = \sqrt{25 + y}$$

Make sure you see why this is true. From here the steps are very straight forward.

$$y^2 = 25 + y \implies y^2 - y - 25 = 0$$

$$\frac{1 \pm \sqrt{101}}{2}$$

We want the positive answer since square roots can not be negative in the real number system. Thus

$$y = \frac{1 + \sqrt{101}}{2}$$

Let's apply this idea to derivatives.

**Exercise 10.7.2.** Find the derivative of

$$y = x^{x^{x^{\cdots}}}$$

Note that this is equal to

$$y = x^y$$

We can thus use our idea of taking the log of both sides again (starting to notice a trend?)

$$\ln y = y \ln x$$

$$\frac{y'}{y} = y' \ln x + \frac{y}{x}$$

$$y'(\frac{1}{y} - \ln x) = \frac{y}{x}$$

$$y' = \frac{y}{x(\frac{1}{y} - \ln x)}$$

What you do after this depends on what the question asks for but it should be straight forward after this. Try to come up with infinitely repeating functions on your own!

# 11 Analyzing Functions

## 11.1 Introduction

I want to keep these calculus handouts short and full of fresh content, therefore this handout will assume you know how to find minimums/maximums or points of inflection.

## 11.2 Tips and Tricks

Let's go over some neat tricks to help with max/min and concavity problems.

**Exercise 11.2.1.** If $f''(x) = (x-1)^3(x-2)^3(x-3)$, identify the points of inflection of f(x).

**Solution:** Remember that the definition of a point of inflection is whenever the second derivative crosses the x-axis. Putting the second derivative in factored form helps us with this greatly. Observe that any root with an even multiplicity will only touch the x axis, while any root with an odd multiplicity will cross it (convince yourself this is true). Thus the points of inflection are $\boxed{x = 1, 3}$. Using this technique saves a lot of time during tests.

**Exercise 11.2.2 (HMMT).** Let p be a monic, cubic polynomial such that $p(0) = 1$ and such that all the zeros of $p'(x)$ are also zeros of p(x). Find $p$.

**Solution:** Firstly, the term monic means that the leading coefficient is 1. Keep an eye out for this term: it often gives crucial information in solving a problem. Next, we need to find a cubic polynomial such that itself and it's derivative share the same zeros. We really have three cases: p has 3 distinct zeros, p has 2 distinct zeroes, or p has 1 zero. Checking the first case, it clearly won't work since the derivative won't share the same zeros. For the second case, one of the zero's is shared, but the zero with a multiplicity of 1 is not shared. Thus, the polynomial must have one zero with a multiplicity of 3. To figure out this root, we know that $p(0) = 1$, so our polynomial thus must be $\boxed{(x+1)^3}$.

**Exercise 11.2.3 (HMMT).** A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

**Solution:** When all else fails, remember that all polynomials can be written as $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. Writing this out, we have

$$a_nx^n + \cdots + a_1x + a_0 = (na_nx^{n-1} + \cdots + a_1)(n(n-1)a_nx^{n-2} + \cdots + 2a_2)$$

Note that we only care about the leading coefficient, $a_n$, so every other term that doesn't include this coefficient are essentially useless. The other key to solving this problem is that since both sides are equal, the greatest degree term on the LHS needs to be the same as the highest degree term on the RHS. This is essentially using the principle of matching coefficients. Observing this, we see that

$$a_nx^n = n^2(n-1)a_n^2x^{2n-3}$$

Again, observe that these two terms must be identical, so the degree of the x's need to be the same. Thus $n = 2n - 3 \implies n = 3$. Using this information, we can easily find that $\boxed{a_n = \frac{1}{18}}$.

**Exercise 11.2.4 (FAMAT Jan Regional).** Find the product of the maximum and minimum values of $e^{x^2-\pi x+1}$ on the interval $[-1, 2]$.

**Solution:** A key observation that makes this problem much easier is that $e^x$ is *monotonic*, or increasing everywhere. What this means for us is we only have to find the minimum and maximum of the exponent of e, and then those will be our minimum and maximum values (convince yourself that this is true). Taking the derivative of the exponent, we obtain $2x - \pi$. Thus the minimum of the exponent occurs at $x = \frac{\pi}{2}$, and the maximum occurs at $x = -1$. Now we need to plug these values into the function and multiply, so our answer is $\boxed{e^{3+\pi-\frac{\pi^2}{4}}}$.

## 11.3 Theorems

**Fact 11.3.1 (Rolle's Theorem).** If $f(x)$ is continuous and differentiable on the interval $[a, b]$ and $f(a) = f(b)$, then $f'(x) = 0$ somewhere when $a \leq x \leq b$.

**Fact 11.3.2 (Fermat's Theorem).** If a function has a local extremum at some point and is differentiable at that point, then the functions derivative must be 0 at that point.

# 12 Approximation Techniques

## 12.1 Linear Approximation

This is the easiest and most common method of approximation seen in FAMAT. Questions may refer to this as using a Linear Approximation, a Tangent Line Approximation, or Differentials. It all means the same thing. Tangent Line approximation essentially takes the tangent line at some point of a function, and uses that line to approximate points very close to the point of tangency. It can be written as:

$$f(x) = f'(x_0)(x - x_0) + f(x_0)$$

Where $x_0$ is the point of tangency, and $x$ is the the x-coordinate of the point you want to approximate. This formula is derived from the point-slope definition of a line, or

$$y - y_0 = m(x - x_0)$$

Which is a **very useful** tool for FAMAT calculus and is not restricted to approximations.

**Exercise 12.1.1.** Let $E(x)$ be the error in the approximation of $f(x)$ using a tangent line approximation at x=a. Prove that

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0$$

**Solution:** Observe that $E(x) = f(x) - (f'(a)(x - a) + f(a))$. From here it is simple algebra:

$$\lim_{x \to a} \frac{f(x) - (f'(a)(x - a) + f(a))}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a)$$

$$= \lim_{x \to a} f'(a) - f'(a) = 0$$

What this result means is that the error of the approximation approaches 0 much faster than $x - a$ approaches zero, which means that it gives an accurate approximation when x is "near a".

We can also view the equation for linear approximation as

$$\Delta y = f'(x_0)\Delta x$$

We can apply this to find how much our approximations might be off by. Let's do a few problems to practice.

**Exercise 12.1.2 (AOPS).** Paul is trying to measure the height of a building using a sextant. He stands 100 meters away and determines that the angle from the ground where he is standing to the top of the building is $30°$, so he computes that the height of the building is $\frac{100\sqrt{3}}{3}$ meters. But, if his sextant is only accurate to within 1 degree, then (approximately) by how much might his measurement of the height of the building be off?

**Solution:** After setting up the problem, we see that our function for height is $f(x) = 100\tan x$. But wait! The problem gives our angles in terms of degrees, and we need to eventually take the derivative of this function. **If we don't convert our x's to radians, we will get the wrong derivative.** To accomplish this, note that $x° = \frac{\pi}{180°}x$, thus $f(x) = 100\tan(\frac{\pi}{180}x)$. Now observe that

$$f'(x) = \frac{5\pi}{9}\sec^2(\frac{\pi}{180}x)$$

$$f'(30) = \frac{20\pi}{27}$$

$$\Delta y \approx f'(30)\Delta x$$

Observe that our error in our height correlates to $\Delta y$, and the error in our degrees correlates to $\Delta x$. Thus:

$$|\Delta y| \approx |f'(30)\Delta x| \leq f'(30) = \frac{20\pi}{27}$$

Thus our height may be off by as much as $\frac{20\pi}{27}$ meters.

**Exercise 12.1.3.** A sphere is being measured by a sphere measuring device thingy. The sphere is measured to have a diameter of 4 cm, however the SMDT is known to introduce a measurement error of withing $\pm 1\%$. What is the estimated percentage error in the computed volume of the sphere?

**Solution:** First, we can write the volume as

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(\frac{d}{2})^3 = \frac{\pi}{6}d^3$$

$$V' = \frac{\pi}{2}d^2$$

We want to use the relationship:

$$\Delta V \approx V'(4)\Delta d$$

Additionally,

$$|\Delta d| \leq .01d = .04, V'(4) = 8\pi$$

Thus

$$|\Delta V| \approx |8\pi\Delta d| \leq .32$$

However, we want the percentage error of V, which we can find by computing

$$\frac{|\Delta V|}{V} = \frac{.32\pi}{\frac{32\pi}{3}} = \boxed{3\%}$$

However, an even more useful observation is that we could have evaluated $\frac{|\delta V|}{V}$ directly. Observe:

$$\frac{|\Delta V|}{V} \approx \frac{|V'\Delta d|}{V} = \frac{\frac{\pi}{2}d^2|\Delta d|}{\frac{\pi}{6}d^3} = 3\frac{|\Delta d|}{d}$$

However, the problem tells us that $\frac{|\Delta d|}{d} \leq .01$, thus the percentage error of the volume is $\leq 3\%$. This method saved us a lot of time!

## 12.2 Newton's Method

Newton's method is a recursion used to find roots of functions. The way it works is that we choose an initial value of x, of which we will find the tangent line at that x. Then, we find the x intercept of that tangent line, and we use that x value as our next value, and we repeat this recursion until we find a root. The formal way of writing this is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Ponder the following exercises on your own.

**Exercise 12.2.1.** Derive the formula for Newton's Method (it is essentially just repeated tangent line approximations!)

**Exercise 12.2.2.** Try using Newton's Method on $f(x) = \sqrt[3]{x}$ with an initial x value of 2. Notice a problem? Why doesn't our method work?

Let's work out an example together.

**Exercise 12.2.3 (FAMAT Jan Regional).** Approximate $\sqrt{\pi}$ by using Newton's Method on $f(x) = \sin(x^2)$ to find $x_1$, given that $x_0 = 1$.

**Solution:** We know that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, thus our answer is simply

$$1 - \frac{\sin 1}{2\cos 1} = 1 - \frac{1}{2}\tan 1$$

There are a few nuances with Newton's Method. For starters, you may have seen already if you did the previous exercises that it can fail at times, so watch out for this. Secondly, you use Newton's Method to find roots, so you need to set your $f(x)$ to 0 in order to use it. For example, a famous question to solve using Newton's Method is to approximate when $\cos x = x$. However, in order to do this you must rearrange the equation to equal 0 before you can use Newton's Method. Finally, you can use Newton's Method to approximate the value of some numbers. Let's look at an example.

**Exercise 12.2.4.** Using two iterations of Newton's Method, approximate the value of $\sqrt{3}$.

**Solution:** The glaringly obvious problem with this question is that we need some function equal to 0 so we can find roots! However, notice that $\sqrt{3}$ is a root of the equation $x^2 - 3 = 0$, so thus we only need to use Newton's Method on this polynomial! We will use an initial approximation of 2, since $\sqrt{4}$ is relatively close to $\sqrt{3}$. We thus get

$$x_1 = 2 - \frac{1}{4} = \frac{7}{4}$$

$$x_2 = \frac{7}{4} - \frac{1}{56} = \frac{97}{56}$$

Using a calculator to check our answer, we see that our approximation is accurate to 3 decimal places!
Finally, I will showcase a problem that I have personally pioneered, and if this shows up on FAMAT they stole the idea from me and I am suing.

**Exercise 12.2.5.** Define a recurrence relation by $x_0 = 4$, and

$$x_n = x_{n-1} - \frac{x_{n-1}^2 + 7x_{n-1} - 30}{2x_{n-1} + 7}$$

for $n \geq 1$. Find

$$\lim_{n\to\infty} x_n$$

**Solution:** Hmm... This looks suspiciously like Newton's Method. In fact, by looking at the fraction part of the relationship, it seems that $f(x) = x^2 + 7x - 30$ and the denominator is it's derivative. Factoring, we see that $f(x) = (x+10)(x-3)$. Since we start at x=4, Newton's Method will approximate whichever root is closest to our initial x, which was 4. Thus, since Newton's Method becomes more accurate as we take more iterations, it seems that the limit should evaluate to 3, which it indeed does by checking with a calculator! (The result from wolframalpha is actually pretty neat).

## 12.3 Euler's Method

This is really a technique that should be saved toward's the end of the year when we touch on Differential Equations, but it showed up on a January Regional so we should probably learn it now. Euler's Method is basically a way of approximating values of functions given differential equations using tangent line approximations. Let's start with an easy example. Suppose we are given that $y' = -\frac{x}{y}$, and we want to estimate $y(0.3)$, given that $y(0) = 1$. Our first step is to break up the interval $[0, 0.3]$ into a set amount of partitions. Luckily, problems usually do this for us, but for this case we will do it with 3 partitions, $[0, .1] \cup [.1, .2] \cup [.2, .3]$. Next, we will approximate $y(0.1), y(0.2)$, and then finally $y(0.3)$ using tangent line approximations as follows:

$$y(0) = 1 \implies y'(0) = 0$$

$$y(0.1) \approx y'(0)(0.1 - 0) + y(0)$$

$$y(0.1) \approx 1$$

$$y(0.1) \approx 1 \implies y'(0.1) \approx -0.1$$

$$y(0.2) \approx y'(0.1)(0.2 - 0.1) + y(0.1)$$

$$y(0.2) \approx .99 \implies y'(0.2) \approx -0.202$$

$$y(0.3) \approx y'(0.2)(0.3 - 0.2) + y(0.2)$$

$$y(0.3) \approx 0.970$$

Phew, that took a while. There exists a tabular strategy to make this process more formulaic, but I will not include it in this handout. Note that generally speaking, more partitions = more accurate. Let's see how this is asked in FAMAT:

**Exercise 12.3.1 (FAMAT Jan Regional).** Approximate $\sqrt{\pi}$ using Euler's Method, given that $y' = \frac{1}{2\sqrt{x}}$ with initial point $(1, 1)$. If the interval $[1, \pi]$ is broken up into $[1, 2] \cup [2, 3] \cup [3, \pi]$, what is the approximation?

**Solution:** The only difference between this problem and our first example we did is that we need to see that $y(1) = 1$ from the initial point given. Now observe how we handle the unequal partitions:

$$y(2) \approx y'(1)(2 - 1) + y(1) = \frac{3}{2}$$

$$y'(2) = \frac{1}{2\sqrt{2}}$$

$$y(3) \approx y'(2)(3 - 2) + y(2) = \frac{1}{2\sqrt{2}} + \frac{3}{2}$$

$$y'(3) = \frac{1}{2\sqrt{3}}$$

$$y(\pi) \approx y'(3)(\pi - 3) + y(3) = \frac{\pi - 3}{2\sqrt{3}} + \frac{1}{2\sqrt{2}} + \frac{3}{2}$$

From here you had to combine the terms using common denominators and then rationalize the result, which you should know how to do by now. Notice that the partitions did not have to be of equal size!

I'm tired and this is about the extent of my approximation knowledge, so I'll end the handout here. Hope you enjoyed finding tangent lines!

# 13 Area Approximation

## 13.1 Introduction

This chapter (aha not calling them handouts anymore) will be quite short, as I will only go over approximating area under curves. I will not write a chapter on integration because Conner's handout is infinitely better than anything I can write.

## 13.2 Mr. Riemann

The reader should already be very familiar with these techniques. If not I recommend learning them and coming back. One tip I would give is to be able to comfortably do these approximations without needing to draw a picture, it saves a lot of time. I will give one problem as practice and move onto a more interesting topic.

**Exercise 13.2.1.** Approximate the total area bound by $a(x) = -x^3 + 4x^2 + 2$ and the x-axis using a Right-Hand Riemann Sum and 6 subdivisions of equal length on the interval [0,6].

**Solution:** The immediate reaction I had to this problem was "wait, it wants total area and the curve most likely goes under the x axis at some time during this interval, how do I deal with this using riemann sums?" The answer is that as long as you take the absolute value of each rectangle, it's fine. The answer is just $|f(1)| + |f(2)| + |f(3)| + |f(4)| + |f(5)| + |f(6)| = 121$.

## 13.3 Simpson's Rule

**Theorem 13.3.1 (Simpson's Rule)**

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\Delta x}{3}(f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots + f(x_k))$$

You may be wondering what the hell I just made you read. That is a completely valid response and I will explain it now. Simpson's rule is a way to approximate the area under a curve using parabolas. Specifically, we pick a parabola that is equal to our function at the endpoints and the midpoint of the interval we are looking at. The formula shown above is for the composite Simpson's rule, where we have to use sub-intervals, much like using riemann sums. The summation of f(x)'s alternates between a coefficient of 4 and 2, until we reach the end of our interval, when it is just a coefficient of 1.Besides the formula itself, there are two very important things to understand about Simpson's Rule.

**1.** For polynomials with a degree of 3 or less, Simpson's Rule gives the **exact value of the area under the curve**. This has to do with it being a parabola or whatever. So basically if a problem asks you to use Simpson's rule on a polynomial with degree of 3 just integrate normally lol.

**2.** $S = \frac{2M+T}{3}$, where S is Simpson's Rule, M is midpoint approximation and T is trapezoidal approximation. Note that these aren't approximation's done with subintervals, or composite approximations. The reason this works is because the errors cancel out or something IDK! Important to know for invitationals or AV perhaps.

## 13.4 Scary Summation

A common question in FAMAT is where a problem will ask you to evaluate a summation that is a hidden riemann sum. If you do not know how riemann sums are written in summation form, learn it and come back. I will give one practice problem and call it quits for this chapter.

**Exercise 13.4.1.** Evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}$$

**Solution:** We want to get our i's on top of some n's, preferably with equal degrees so we can find some dx's. This motivates us to factor out $n^2$ from the denominator, leaving us with

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{1+\frac{i^2}{n^2}}}\cdot\frac{1}{n}$$

And this essentially solves the problem. If we let a = 0, we get b = 1 and that the summation is equivalent to

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\mathrm{d}x$$

I will leave solving this (very difficult) integral as an exercise to the reader. Onto the next chapter!

# 14 An Exploration of Volumes

## 14.1 Introduction

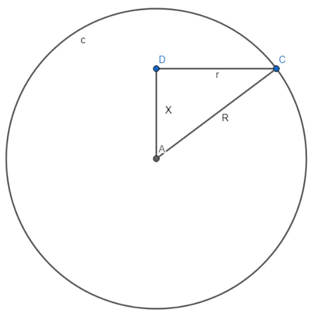
This chapter will assume prior knowledge of the basic methods of finding volumes of revolution like Disk and Cylindrical method. What you are about to witness in the following sections is some reality defying quantum meta-physical volume determination methods

## 14.2 Slice Slice Slice

One way of determining the volume of solids is by placing them on a coordinate space, and finding a function for the cross sectional area, and then integrating that function along the height of the solid. Here is an example.

**Exercise 14.2.1.** Derive the volume of a sphere.

**Solution:** Wow, talk about an over complicated question am I right /s. The first thing we will do is place the sphere on a coordinate space such that it's height lies along the x-axis. Now we will draw a picture of the cross-sectional area.




Notice what we are doing here: As we integrate along the x axis, we are taking the area of "slices" of the sphere, or in other words, we need to find the radius of the cross sectional area for each x. Our picture gives us a clear idea of how to go about doing this. We see that $r = \sqrt{R^2 - x^2}$, where r is our cross sectional radius and R is the radius of the sphere. Since our height is along the x-axis, we are integrating from -r to r, so our integral is

$$\pi \int_{-R}^{R} R^2 - x^2 \mathrm{d}x$$

Since the area of each slice is $\pi r^2$. It should be no surprise that this integral evaluates to $\frac{4}{3}\pi R^3$. Let's try another example.

**Exercise 14.2.2.** Make a reasonable assumption of the volume of a 4 dimensional sphere

**Solution:** I know what you're thinking, but hear me out. We can make a reasonable definition of dimensions using cross sections. The cross section of 2 dimensional shapes are lines, which are 1 dimensional. The cross section of 3 dimensional solids are 2 dimensional shapes. Therefore it only seems reasonable that the cross section of a 4 dimensional object would be a 3 dimensional solid, and in this case it would make sense if they were spheres. Even though our object is 4 dimensional, it still has the same shape as a sphere, so we can apply the same function we used in the last example for the radius of the cross sections, only now our integral looks like this:

$$\frac{4\pi}{3} \int_{-R}^{R} (R^2 - x^2)^{\frac{3}{2}} \mathrm{d}x$$

This integral is actually quite a bit more difficult than the previous example. First, we make the substitution $x = R\sin\theta \implies \mathrm{d}x = R\cos\theta \mathrm{d}\theta$. This simplifies the integral to:

$$\frac{4\pi}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^4 \cos^4\theta \mathrm{d}\theta$$

From here just apply the cosine double angle formula to reduce the fourth power, and you should end up with $\frac{1}{2}\pi^2 R^4$.

I suggest practicing this technique by just deriving volume formulas for different solids, such as cubes, cones, pyramids, etc.

## 14.3 Pappu's Theorem

In your spare time studying calculus, you may have come across the question of finding the volume of a torus, or essentially a donut. Using the conventional methods, this is a fairly difficult problem, however it is made trivial by Pappu's Theorem.

**Theorem 14.3.1 (Pappu's Theorem)**
The volume of a solid of revolution can be found by

$$V = A_c D$$

Where $A_c$ is the cross sectional area, and $D$ is the distance that it travels, or the circumference of it's circular path

Now let's try the aforementioned problem:

**Exercise 14.3.2.** Derive the volume of a torus

**Solution:** Let the radius of our cross section be $r$, and the radius of the torus (distance from center of cross section to the center of the torus) be $R$. It follows that $A_c = \pi r^2$, and $D = 2\pi R$, so the volume is simply $2\pi^2 R r^2$.

With using Pappu's Theorem, there are 2 very important formulas you should know. The first formula is the distance from a point to a line:

**Fact 14.3.3 (Distance from Point to Line).** The distance from point $(x_1, y_1)$ and line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

When defining Pappu's Theorem, I left out a very important detail. To calculate the circular path a solid of revolution takes, the distance from the line of rotation to the center of mass of the shape is the radius you use to calculate the circumference of the path. For circles this is easy, it's just the regular center of the circle. For other shapes such as triangles, it is not as easy. Let's look at some example problems.

**Exercise 14.3.4.** Find the volume of the solid obtaining by revolving the circle $(x-5)^2+(y-1)^2=1$ about the line $y=x$.

**Solution:** Let's start by finding $A_c$. The radius of our circle is 1, so the area is just $\pi$. Next, let's find the distance of the path. The radius of our circular path is the distance from the center of our circle $(5,1)$ to the line $x-y=0$. Using our formula, the distance is

$$\frac{|5-1|}{\sqrt{1+1}}=2\sqrt{2}$$

Now we see that the circumference of our path is just $2\cdot\pi\cdot 2\sqrt{2}=4\sqrt{2}\pi$, and our volume is $4\sqrt{2}\pi^2$. (Note that this solid is also a torus!).

Here is another fact that may come in handy:

**Fact 14.3.5.** The center of mass of a triangle is the intersection of a triangle's medians. If the triangle has vertices at $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ then the coordinates of the center of mass is

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

.

What if we had a region bound by a downwards-facing parabola and the x-axis, and we wanted to find the volume of this region revolved around a line, such as y=x? Pappu's theorem still applies, but we need to find the center of mass of this region. This is when the following formulas come in handy:

**Fact 14.3.6 (Center of Mass).** For a region bound by two functions, you can find the coordinates of the center of mass as follows:

$$x_c=\frac{1}{A}\int_a^b x(f(x)-g(x))\mathrm{d}x$$

$$y_c = \frac{1}{A}\int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)\mathrm{d}x$$

Where $A$ is the area of the region

With this information, you want to find the distance between the center of mass and the line of revolution, and use that as the radius of the circular path the region takes.

I will end this section with an idea of sorts, I guess. Say you want to find the volume of half a torus, where the circle only goes through half the path of revolution. It's pretty clear that you would just do Pappu's for a torus and then divide by 2, since it is literally half the torus. However, this seems to imply that you can find the volume of such a solid by finding the area of the cross sectional area and the length of the line it "moves" through. In fact, this is exactly how we are able to find the volume of a cylinder of any base. We just multiply the cross sectional area of the solid by the height, where in that case the solid just moves through a straight line. These results seem to imply that, for example, if I were to create a solid where the cross section area was constant throughout, and the cross sections "moved" through a curve, we can find the volume by multiplying the length of the curve it "moves" through and the cross sectional area. Some clear cases where this doesn't work is any curve with a cusp, such as |x|, since the cross sectional areas would overlap. But whenever your curve is smooth, this sounds like it would work. Just a thought.

# 15 Algebra Practice Problems

## 15.1 Problems

These problems are in no particular order, and while that is mainly because I am lazy, this is how you will encounter problems on a test. They won't be in order of topic covered to give you hints, you must be able to reach into your toolbox of mathematical knowledge and pull out the write idea to solve any given problem. Enjoy

> "In real life, I assure you, there is no such thing as algebra."
>
> *Fran Lebowitz*

**Problem 1 (AMC 12).** Let $f(x^2+1) = x^4 + 5x^2 + 3$. What is $f(x^2-1)$?

**Problem 2 (AOPS).** Let $f(x) = x^{10} - 2x^5 + 3$. Find the remainder when $f(x)$ is divided by $2x-4$.

**Problem 3 (AOPS).** Suppose that $f(x)$ is a polynomial with integer coefficients such that $f(2) = 3$ and $f(7) = -7$. Show that $f(x)$ has no integer roots. (Hint: 0 is not odd).

**Problem 4 (Mandelbrot).** There is a unique polynomial $P(x)$ of the form

$$P(x) = 7x^7 + c_1x^6 + c_2x^5 + \cdots + c_6x + c_7$$

such that $P(1) = 1, P(2) = 2, \ldots, P(7) = 7$. Find $P(0)$.

**Problem 5 (Mandelbrot).** Consider the system of equations

$$a_1 + 8a_2 + 27a_3 + 64a_4 = 1$$

$$8a_1 + 27a_2 + 64a_3 + 125a_4 = 27$$

$$27a_1 + 64a_2 + 125a_3 + 216a_4 = 125$$

$$64a_1 + 125a_2 + 216a_3 + 323a_4 = 343$$

These four equations determine $a_1, a_2, a_3, a_4$. Find $a_1 + a_2 + a_3 + a_4$ and $64a_1 + 27a_2 + 8a_3 +$

$a_4$.

**Problem 6 (AOPS).** One of the roots of $x^3 + ax + b = 0$ is $1 + 2i$, where $a$ and $b$ are real numbers. Find $a$ and $b$.

**Problem 7 (ARML).** Find the two values of $k$ for which $2x^3 - 9x^2 + 12x - k$ has a double root.

**Problem 8 (AIME).** For certain real values of $a, b, c, d$, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four nonreal roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$. Find $b$. (The coefficients are real numbers, how can this help you solve the problem?)

**Problem 9 (HMMT).** Find all the values of $m$ for which the zeros of $2x^2 - mx - 8$ differ by $m - 1$.

**Problem 10 (HMMT).** Find all real solutions $(x, y)$ of the system $x^2 + y = 12 = y^2 + x$.

**Problem 11 (AOPS).** Find the roots of $x^2 + (a - \frac{1}{a})x - 1 = 0$ in terms of a.

**Problem 12 (ARML).** If $P(x)$ is a polynomial in x, and $x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9) \cdot P(x)$ for all values of x, compute the sum of the coefficients of $P(x)$.

**Problem 13 (AOPS).** Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 + x$.

**Problem 14 (ARML).** Let $P(x)$ be a polynomial whose degree is 1996. If $P(n) = \frac{1}{n}$ for $n = 1, 2, 3, \ldots, 1997$, compute the value of $P(1998)$.

**Problem 15 (Mandelbrot).** Determine $(r+s)(s+t)(t+r)$ if $r, s, t$ are the three real roots of the polynomial $x^3 + 9x^2 - 9x - 8$.

## 15.2 Answers

1. $x^4 + x^2 - 3$
2. 963
3. Prove that $f(x)$ is odd for all even and odd integers x, thus $f(0)$ is also odd, and thus there are no integer roots since 0 is not odd. 4. -35280
5. 8, 729
6. $x^3 + x + 10$
7. 4, 5
8. 51
9. 6, -10/3
10. $(3,3), (-4,-4), (\frac{1+3\sqrt{5}}{2}, \frac{1-3\sqrt{5}}{2}), (\frac{1-3\sqrt{5}}{2}, \frac{1+3\sqrt{5}}{2})$
11. $-a, \frac{1}{a}$
12. 18
13. $5x$
14. $\frac{1}{999}$
15. 73

# 16 Conic Exercise Problems

## 16.1 Problems

These problems are (roughly) in order of difficulty. Many of these problems will deal with specific strategies that we did not have the chance to cover in practice, so please attempt these! If you need help with any of the problems, feel free to ask in the discord.

> "Along a parabola life like a rocket flies,Mainly in darkness, now and then on a rainbow."
>
> *Andrey Voznesensky*

**Problem 16.** Parabola $P$ has a vertical axis of symmetry and passes through the points $(4, 5)$, $(-2, 11)$, and $(-4, 21)$. Find an equation whose graph is $P$.

**Problem 17.** The graph of $y = x^2 + 2x - 2$ is reflected over the line $y = x$. How many points of intersections does the original graph have with its reflection?

**Problem 18 (CEMC).** Find the length of the common chord of the two circles whose equations are $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x + 2 = 0$.

**Problem 19 (AMC).** Square ABCD has sides of length 4, and $M$ is the mid point of $\overline{CD}$. A circle with radius 2 and center $M$ intersects a circle with radius 4 and center $A$ at points $P$ and $D$. Waht is the distance from $P$ to $\overline{AD}$?

**Problem 20.** An ellipse has foci at $(0, 0)$ and $(14, 0)$ and passes through the vertex of the parabola with equation $y = x^2 - 10x + 37$. Find the length of the major axis of this ellipse.

**Problem 21.** Find the foci of the graph of $xy = 1$.

**Problem 22 (FAMAT Analytical Geometry).** The latus rectum is not only for parabola. The latus rectum is most generally defined as the line segment that is perpendicular to the graphs axis of symmetry, and crosses through the foci of the graph, such that the focus is the midpoint of the latus rectum. Now, find the length of a latus rectum of the graph of $xy = 1$.

**Problem 23.** Find $r$ if $r$ is positive and the line whose equation is $x + y = r$ is tangent to the circle whose equation is $x^2 + y^2 = r$.

**Problem 24.** Find an equation whose graph is the circumcircle of a triangle with vertices $(-2, 5), (-4, -3), (0, -3)$.

**Problem 25 (AMC).** Points A and B are on the parabola $y = 4x^2 + 7x - 1$, and the origin of the Cartesian plane is the midpoint of $\overline{AB}$. What is the length of $\overline{AB}$?

**Problem 26 (AMC).** Let $f(x) = x^2 + 6x + 1$, and let $R$ denote the set of points $(x, y)$ in the coordinate plane such that $f(x) + f(y) \leq 0$ and $f(x) - f(y) \leq 0$. Find the area of $R$.

**Problem 27 (NYSML).** Two parabolas have the same focus, namely the point $(3, -28)$. Their directrixes are the x-axis and y-axis, respectively. Compute the slope of their common chord.

**Problem 28.** $P$ is a fixed point on the diameter $\overline{AB}$ of a circle. Prove that for any chord $\overline{CD}$ of the circle that is parallel to $\overline{AB}$, we have $PC^2 + PD^2 = PA^2 + PB^2$.

**Problem 29 (AIME).** An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, and one altitude is contained in the y-axis. Find the length of each side of the triangle.

## 16.2 Answers

1. $y = \frac{1}{2}x^2 - 2x + 5$
2. 4
3. $2\sqrt{3}$
4. $16/5$
5. 28
6. $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$
7. $2\sqrt{2}$
8. 2
9. $(x+2)^2 + (y - \frac{3}{4})^2 = \frac{289}{16}$
10. $5\sqrt{2}$
11. $8\pi$
12. -1
13. Ask in discord.
14. $\frac{16\sqrt{3}}{13}$

# 17 Advanced Problems and Their Solutions

## 17.1 Introduction

This is just a compilation of problems I have solved throughout this year preparing for the FAMAT season that are non-trivial/unique/advanced. These are just in order of when I solved them. Feel free to CTRL+F search certain topics to find specific problems pertaining to those topics.

## 17.2 Problems

**[2021 Jan Reg Indiv Q8]** Given that

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find $f'(0)$.

**Solution:** First we must check if the function is continuous at x=0. We can do so by evaluating $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$. Since the limit equals zero, and there is no way for this function to make a sharp turn in it's graph, the piece wise function is continuous. Next, to evaluate the derivative, it is clear that directly differentiating $x^2 \sin(\frac{1}{x})$ and plugging in 0 will not work, so we will cleverly use the limit definition of the derivative to help us. $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \frac{h^2 \sin(\frac{1}{h})}{h} = \lim_{h\to 0} h \sin(\frac{1}{h}) = 0$. Thus the derivative does exist and it is equal to 0. Important things to note about this problem: If it were not a piece wise function, this derivative would not exist. The only reason it does is because f(x) is continuous and we have a value of f(0) to plug into the limit definition of a derivative.

**[2021 Jan Reg Indiv Q11]** If $f_n(x) = x^{x^{x^{\cdots}}}$ where there are n x's, what is $\lim_{x\to 0^+} f_n(x)$ in terms of n.

**Solution:** A simple power tower problem. Be familiar with the power tower function: it shows up frequently. The tricky thing with this problem is that power towers are constructed *top to bottom*, not bottom to top as the problem suggests. For example, $f_3(x) = x^{(x^x)}$. From here, we simply find a pattern. $f_1(0) = 0, f_2(0) = 1, f_3(0) = 0\ldots$ etc. The function oscillates between 1 and 0. Testing the answer choices, the only one that works is $\frac{(-1)^n+1}{2}$.

**[2021 Jan Reg Indiv Q12]** A camera can currently take 100 photos with a quality of 3. For every 0.5 increase in quality, the camera can take 8 less photos. How many photos can the camera take if you wish to maximize number of photos multiplied by the quantity?

**Solution:** These maximization questions show up frequently but are straight forward. We start by constructing a function that models the camera: $f(x) = (3 + 0.5x)(100 - 8x)$. Now we simply expand and maximize. $f(x) = -4x^2 + 26x + 300$, $f'(x) = -8x + 26$, thus $x = \frac{26}{8}$, and when we plug into the factor of our function that deals with number of photos, we find that the number of photos will be 74.

**[2021 Jan Reg Indiv Q22]** Find the average value of the derivative of $f(x) = \sin x \tan x \sec x$ over the interval $[0, \frac{\pi}{4}]$.

**Solution:** The formula for average value is $\frac{1}{b-a}\int_a^b f(x)$. Before we find the derivative, we notice that we can simplify $f(x)$ as $\tan^2 x$. Thus $f'(x) = 2\tan x \sec^2 x$. However note that we will just be integrating this again to find average value, thus the problem is just $\frac{1}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} f'(x) = \frac{4}{\pi}(\tan^2 \frac{\pi}{4} - \tan^2 0) = \frac{4}{\pi}$.

**[2021 Jan Reg Indiv Q23]** Given that $f(x) = \int_0^{x^2}(x + t^2)^3 dt$, what is $f'(1)$?

**Solution:** While this seems like a direct application of Fundamental Theorem of Calculus, be careful! Since the integrand has an x term in it, you are not allowed to directly use FTC here. Thus we must integrate this by hand. Using binomial theorem saves us some work in expanding the integrand, and we get that the inside of the integral is $x^3 + 3x^2t^2 + 3xt^4 + t^6$. Integrating with respect to t, we have $x^3t + x^2t^3 + \frac{3}{5}xt^5 + \frac{1}{7}t^7$.

Plugging in the bounds and simplifying, we have $f(x) = x^5 + x^8 + \frac{3}{5}x^{11} + \frac{1}{7}x^{14}$. Thus $f'(1) = \frac{108}{5}$.

**[2021 Jan Reg Indiv Q25]** Find the sum of the x coordinates of the inflection points of $f(x) = x^4 + 3x^3 + 2x^2 + x - 15$.

**Solution:** We find that $f'(x) = 4x^3 + 9x^2 + 4x + 1$ and $f''(x) = 12x^2 + 18x + 4$. Since $f''(x)$ clearly doesn't have any double roots, we can find the sum of the roots using vieta's and thus the sum of the x coordinates of the inflection points is $-\frac{18}{12} = -\frac{3}{2}$. There were several slick things to make this problem faster. First, notice that the inflection points are simply the roots of the second derivative that have odd multiplicity (pass through the x axis). Thus if there is no double roots or imaginary roots you can simply use vieta's to quickly sum the roots.

**[2019 Stuart Sidney Competition Q1]** Let $f(x)$ be a function that is odd and differentiable everywhere. Prove that its derivative is even, and is the converse of the statement true.

**Solution:** The definition of an odd function is that $f(-x) = -f(x)$. Taking the derivative of $f(-x)$, we get $[-f(x)]' = -f'(x)$, $[f(-x)]' = -f'(-x)$. Since $f(x)$ is odd, $[-f(x)]' = [f(-x)]'$, thus $-f'(x) = -f'(-x)$, which implies the functions derivative is even. The converse is not true, because for a function to be even it must be symmetric about the origin, which means that $f(0) = 0$. However, just knowing that $f'(x)$ is even will not give us that information. For example: $(x^3 + x)' = 3x^2 + 1$, which is even, but the functions that give $3x^2 + 1$ as its derivative are in the form of $x^3 + x + k$, where k is some constant. This function is only odd if k = 0.

**[2019 Stuart Sidney Competition Q2]** Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Find a closed form for the power series

$$1 \cdot 2 + (2 \cdot 3)x + (3 \cdot 4)x^2 + (4 \cdot 5)x^3 + \dots$$

Given that $|x| < 1$.

**Solution:** When seeing incomplete factorials, we usually think of either derivatives or combinatorics (at the time of writing I am noob at combinatorial summations but give me a week its coming up in my textbook ;-;). Analyzing the derivatives of the given geometric series, we have

$$-\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$
$$\frac{1}{(1-x)^3} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + \dots$$

This is exactly what the answer was looking for, so this is the closed form. Also the solution said the first derivative was positive??? Pretty sure that's wrong but lmk.

**[SMT 2020 Q5]** Evaluate

$$(2020)^2 + \frac{(2021)^2}{1!} + \frac{(2022)^2}{2!} + \frac{(2023)^3}{3!} + \dots$$

**Solution:** Really good taylor series problem. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Our goal here is to manipulate this series to be in the form of $\frac{(n+2020)^2}{n!}$. The steps to do so are below.

$$x^{2020}e^x = \sum_{n=0}^{\infty} \frac{x^{n+2020}}{n!}$$

Now we take the derivative in order to get the exponent in front as a factor.

$$2020x^{2019}e^x + x^{2020}e^x = \sum_{n=0}^{\infty} \frac{(n+2020)x^{n+2019}}{n!}$$

We are getting closer. We want (n+2020) to be squared, so we multiply by x again.

$$2020x^{2020}e^x + x^{2021}e^x = \sum_{n=0}^{\infty} \frac{(n+2020)x^{n+2020}}{n!}$$

We differentiate once more:

$$2020^2x^{2019}e^x + 2020x^{2020}e^x + 2021x^{2020}e^x + x^{2021}e^x = \sum_{n=0}^{\infty} \frac{(n+2020)^2 x^{n+2019}}{n!}$$

From here, letting x=1 gives us the form we want, and the answer is $2020^2e + 4042e = 4084442e$.

# 18 The Vault

## 18.1 Algebra/Trig

Let r, s, and t be the three roots of the equation $F(x) = x^3 + 5x - 10$. Find $r^2 + s^2 + t^2$. = $\boxed{-10}\star$

Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are a+b, b+c, and c+a. Find t. $\boxed{23}$ (AIME)

Find all solutions $0 \leq x \leq \pi$ to $(\tan x \sec x + \tan x - 2\sin x) - (\frac{1}{2}\sec^2 x + \frac{1}{2}\sec x - 1) = 0 \star \boxed{0, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{2\pi}{3}}$

The sum of the zeroes, the product of the zeroes, and the sum of the coefficients of the function $f(x) = 21x^2 + bx + c$ are equal. What do they equal? $\boxed{21}$ (AMC 12)

Suppose that we have two quadratics such that $ax^2 + bx + c = dx^2 + ex + f$. Must their coefficients be equivalent? $(a = d, b = e, c = f)$. If yes, then show why.

Derive the quadratic formula. (Hint: Complete the square of $ax^2 + bx + c$).

Let $f(x) = \sqrt[3]{x^2} + 4\sqrt[3]{x} + 4$ and $g(x) = \sqrt{x}$. Evaluate $g(g(f(x)))$.

Suppose $f(x) = x + \sqrt{x}$ and $g(x) = x + 1/4$. Evaluate $g(f(g(f(g(f(7)))))) = \boxed{\dfrac{37}{4} + 3\sqrt{7}}$(AOPS)

Describe all pairs of real numbers $(x, y)$ such that there is only one real number $t$ for which $6 = 3x^2t - 2yt^2 = \boxed{x^4 = \dfrac{16}{3}y}$

Find the roots of $x^2 + (269 - \frac{1}{269})x - 1 = 0$. (Hint: Try to find a general result for the roots of $x^2 + (a - \frac{1}{a})x - 1$. $\boxed{\dfrac{1}{269}, -269}\star$

Let a and b be real numbers such that $0 < a < b$ and $a^2 + b^2 = 6ab$. Find $\frac{a+b}{a-b} = \boxed{\sqrt{2}}$ (AOPS)

Find all x such that $\frac{x-a}{b} + \frac{x-b}{a} = \frac{b}{x-a} + \frac{a}{x-b}$, where a and b are constants = $\boxed{x = 0, a+b, (a+b)^2/(a+b)}$(A

Find the only prime number that is followed by a perfect seventh power = $\boxed{127}\star$.

Two parabolas have the same focus, namely the point $(3, -28)$. Their directrixes are the x-axis and the y-axis, respectively. Compute the slope of their common chords = $\boxed{-1}$. (NYSML)

Let $f(x) = x^2 + bx + 1$, and let R denote the set of points (x,y) in the coordinate plane such that $f(x) + f(y) \leq 0$ and $f(x) - f(y) \leq 0$. Find the area of R = $\boxed{8\pi}$(AMC)
P is a fixed point on the diameter AB of a circle. Prove that for any chord CD of the circle that is parallel to AB, we have $PC^2 + PD^2 = PA^2 + PB^2$. (AOPS)
An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is (0,1), and one altitude is contained in the y-axis. Find the length of each side of the triangle. $= \boxed{\frac{16\sqrt{3}}{13}}$(AIME)
Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have? $= \boxed{73}$(AHSME)
Three circles with radius 2 are drawn in a plane such that each circle is tangent to the other two. Let the centers of the circles be A, B, and C. Point X is on the circle with center C such that AX + XB = AC + CB. Find the area of triangle AXB $= \boxed{12(\sqrt{5} + \sqrt{3})}$(USAMTS)
Find the unique triple of real numbers (x,y,z) that satisfy the equation

$$(5x^2 - 10x + \frac{23}{4})(4y^2 + 4y + \frac{7}{2})(3z^2 + 9z + \frac{59}{4}) = 15$$

$= \boxed{(1, \frac{-1}{2}, \frac{-3}{2})}\star$
5 runners are running on a circular track, with a diameter of 10. It is true that for any possible orientation of the 5 runners on the track, at least two of them are less than D distance apart. What is D? $= \boxed{5\sqrt{2}}\star$
What is the maximum amount of times a polynomial with degree 7 can intersect a polynomial with degree 2? $= \boxed{7}\star$.
Suppose $f(x) = 3x^5 + 4x^4 - 2x^3 + 2x + 21$. Consider all lines that intersect $f(x)$ at 5 distinct points: $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$. Show whether or not the value of

$$\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}{5}$$

is constant for all such lines, and if so, please evaluate. $= \boxed{\frac{52}{9}}\star$
Two dolphins are having a diving contest, and each dolphin is scored based on the absolute value of the average of the x-coordinates that they dive into the water and break out of the water on the cartesian plane. Dolphins are quite imaginative, so these positions can also be nonreal. However, the water has a slight concave to it, such that it is modeled by

the function $f(x) = -\frac{1}{100}x^2 + 1$. The first dolphin's motion is modeled by the function $g(x) = 21x^{20} + 20x^{19} + 19x^{18} + \cdots + 2x + 1$. The second dolphin's motion is modeled by the function $g(g(g(x)))$. Which dolphin wins? [They both tie with a score of $\frac{20}{21}$]$\star$

Show that $44^2 + 44 + 41$ is not prime.

Find the value of x that satisfies the equation $1 + 2x + 3x^2 + 4x^3 \cdots = \frac{9}{4} = \boxed{\frac{1}{3}}\star$

Suppose that two circles of radius 2 are tangent to each other. Let their centers be A and B, respectively. For a point on either circle X, there are 4 such points that satisfy the condition $|XA - XB| = AB$. If all the points both on and not on the circles that satisfy this condition form a graph with perpendicular asymptotes, what is the area of the triangle formed by any 3 of such points on the circles? $= \boxed{2(12)^{\frac{1}{4}} + 2(108)^{\frac{1}{4}}}\star$

Let $P(x_1, x_2, x_3, \ldots x_m) = \prod_{n=1}^{m} \sum_{k=1}^{n} kx_k$. Find the sum of the coefficients of the m-variable polynomial P $= \boxed{\frac{(m!)^2(m+1)}{2}}\star$

Express $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots$ as f(x)/g(x), where f(x) and g(x) are polynomials with finite degree $= \boxed{\frac{1}{1-x}}$

If $60^2 + 61^2 + 3660^2 = k^2$ for some positive integer k, find k $= \boxed{3661}\star$

Evaluate $3(3!) + 4(4!) + 5(5!) + \cdots + 2021(2021!) = \boxed{2022! - 6}\star$

Find the minimum value of

$$\frac{16}{(a-b)(a+b)b} + 2a + b$$

Given that $a > b$. $\star$

Let $F(x)$ be a function with a domain over the real numbers. Let $S$ be the sum of the roots of $\frac{F(x)+F(-x)}{2}$, and let $M$ be the product of the roots of $\frac{F(x)-F(-x)}{2}$. Solve

$$3x^2 - Sx + M = 147$$

for x. $= \boxed{\pm 7}\star$

Let $F_n = \frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \cdots + \frac{n}{4^n}$, and let $U_n = \frac{1}{4^n F_n F_{n+1}}$. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n+1}{4} U_k = \boxed{4}\star$

Evaluate $\sum_{n=1}^{\infty} \frac{2x+4}{x^4+8x^3+23x^2+28x+12} = \boxed{\frac{1}{6}}\star$

Find the minimum value of $x^{22} + \frac{1}{x^{21}}$ Hint: Find a general formula for the minimum of $x^m + \frac{1}{x^n} = \boxed{\frac{43}{\sqrt[43]{22^{22}21^{21}}}}\star$

Let $B \in \mathbf{R}$, then find B if

$$B \arctan 1 = \pi + \frac{\pi}{2^2} + \frac{\pi}{3^2} + \frac{\pi}{4^2} + \dots$$

= $\boxed{\frac{2}{3}\pi^2}\star$.

Let $f(x) = \frac{1}{1+x}$. Let the notation $f^n(x)$ denote composing $f(x)$ onto itself n times. (For example, $f^3(x) = f(f(f(x)))$). Evaluate

$$\prod_{n=1}^{\infty} f^n(\pi)$$

= $\boxed{0}\star$

## 18.2 Limits

$\lim_{n\to\infty}(p^n + q^n)^{\frac{1}{n}} = \boxed{\text{max(p,q)}}$ (AOPS)

$\lim_{x\to 1} x^{\frac{x}{\sin(1-x)}} = \boxed{\frac{1}{e}}$ (AOPS)

$\lim_{n\to\infty}(\frac{n^{n+1}+(n+1)^n}{n^{n+1}})^n = \boxed{e^e}$ (FAMAT)

$\lim_{x\to\infty} \sqrt{9x^2+2x+5} - 3x = \boxed{\frac{1}{3}}$ (FAMAT)

## 18.3 Derivatives

Let $\alpha = \frac{x^{17}-y^{17}}{x^{16}+x^{15}y+x^{14}y^2 \cdots + xy^{15}+y^{16}}$. If $\alpha \in \mathbb{R}$ for all $x, y \in \mathbb{R}$, find $\frac{dy}{dx}$ = $\boxed{1}$

Let $\beta = \frac{x^{17}+y^{17}}{x^{16}-x^{15}y+x^{14}y^2 \cdots - xy^{15}+y^{16}}$. If $\beta \in \mathbb{R}$ for $x, y \in \mathbb{R}$, Find $\int_{69}^{420}[\beta - \alpha]dy$ = $\boxed{171639}\star$

Let $\left[\frac{x^4+4y^4}{x^2+2y^2+2xy}\right]^2 = 2021$. Find $\frac{dy}{dx}\star$

Let $f(x) = (x-2020)^2(x-2021)(x-2022)^2$. Evaluate $f'(2023) = \boxed{69}$

Let $f(x) = e^x ln(x) sin(x) cos(x)$. Evaluate $f''(1) = \boxed{2e\cos^2 1 + e\cos 1 \sin 1 - 2e\sin^2 1}\star$

Let $a$ be an angle such that $0 < a < 180$, and let $F(x) = \sum_{i=0}^{3} \sin(\frac{\pi}{2}i - a)x^i$. Evaluate the sum of the reciprocals of the roots of $F'(x) = \boxed{\tan a}\star$

A triangularly shaped racetrack is experiencing an unnatural phenomenon, such that each side of the track is expanding by 1 unit per second. The track does so in a way that it remains a triangle. Find the rate of change of the inradius of the track at the moment the triangle has a 90 degree angle with legs of length 3 and 4, respectively. $= \boxed{\frac{153}{72}} \star$

Let $|x-a|-|x-b|=0$. Find the rate of change of x if $a>b$ and $a'=1, b'=-1$. $= \boxed{0} \star$

Let the slope of line $L$ be $m_2$, and let the slope of line $K$ be $m_1$. Find the rate of change of the acute angle between $L$ and $K$ when $m_2=2, m_1=\frac{1}{2}$ if $m_2'=\frac{5}{2}$ and $m_1'=\frac{3}{2}$ = $\boxed{-\frac{7}{10}} \star$

Let $f(x)=x^5\sin x$. Evaluate the fifth derivative of f(x) at $x=\pi$ = $\boxed{-5\pi^5+200\pi^3-600\pi} \star$.

Suppose we have isosceles triangle $\triangle ABC$ such that $\angle B=\angle C$. Let this triangle's orthocenter be at point O. IF the angles of this triangle change such that the rate of change of $\angle A=2$, and such that the triangle stays an isosceles triangle, what is the rate of change of $\angle BOC$ ? = $\boxed{-2} \star$

Let $P(x)$ be an even quadratic with roots $r_1, r_2$, whose roots are changing such that $P(x)$ stays even, and the rate of change of the product of the roots is -5 units. What is the rate of change of $r_1^2+r_2^2$? $= \boxed{10} \star$.

## 18.4 Integration

$\int_0^1 \ln^{2020}(x)\mathrm{d}x = \boxed{2020!}$ (MIT)

$\int \sin(\cos(x))\cot(x)-\cos(\cos(x))\sin(x)\ln(\sin(x))\mathrm{d}x = \boxed{\sin(\cos(x))\ln(\sin(x))+C} \star$

$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} = \boxed{5-\ln 6}$ (MIT)

$\int_0^{x^2}(x+t^2)^3\mathrm{d}t$, $f'(1) = \boxed{\frac{108}{5}}$ (FAMAT)

$\int_0^1 x^{-x}\mathrm{d}x = \boxed{\sum_{n=1}^{\infty}\frac{1}{n^n}}$ (Youtube)

$\int_0^{\pi} \frac{x}{1+\sin x}\mathrm{d}x = \boxed{\pi}$ (Cambridge)

$\int \frac{\sin x+\sin 3x+\sin 5x}{\cos x+\cos 3x+\cos 5x}$

## 18.5 Series

Let $a_n$ be the nth term of a tribonacci series, where $a_0 = a_1 = a_2 = 1$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Evaluate $\sum_{n=0}^{\infty} a_n(1/5)^n = \boxed{60/47}\star$
Let $F(x) = x^2e^x$. Evaluate $F^{(100)}(x) = \boxed{9900}$ (Youtube)
Lucas has created a perpetual motion machine that creates rectangles for him. Each rectangle has a real-valued length and width. For any n number of rectangles he makes, the sum of the squares of their lengths can be expressed as

$$\sum_{i=0}^{n} \frac{4^n}{n!}$$

And the sum of the squares of their widths can be expressed as

$$\sum_{i=0}^{n} \frac{(-1)^n(\frac{\pi}{3})^{2n+1}}{(2n+1)!}$$

What is the maximum possible value of the sum of the areas of the rectangles as n tends to infinity? $\star$
Evaluate $2 + \frac{3}{1!} + \frac{4}{2!} + \frac{5}{3!} + \cdots = \boxed{3e}\star$

## 18.6 Parametrics/Polar

Write a parametric equation for a particle tracing a circle with center (0,0) and radius 1, starting at (1,0) at time t=0, moving counterclockwise with speed $\frac{\sqrt{1-t^2}}{1-t^2} = \boxed{(\sqrt{1-t^2}, t)}\star$
Find the area enclosed by the graph of the parametric equations $x = \frac{4}{1+t^2}, y = \frac{4t}{1+t^2} = \boxed{4\pi}\star$

## 18.7 Other

Shigetora High School has 5 students interested in competing in their Mu Alpha Theta program. Each student may only choose one of the following division to compete in: Algebra, Pre-Calc, and Calculus. How many ways can the students choose divisions, provided that no division has 0 students? $= \boxed{150}\star$