COLLEGE OF ENGINEERING
NUCLEAR ENGINEERING & RADIOLOGICAL SCIENCES
UNIVERSITY OF MICHIGAN

Lecture 6
Algorithms for Linear Algebra

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Outline

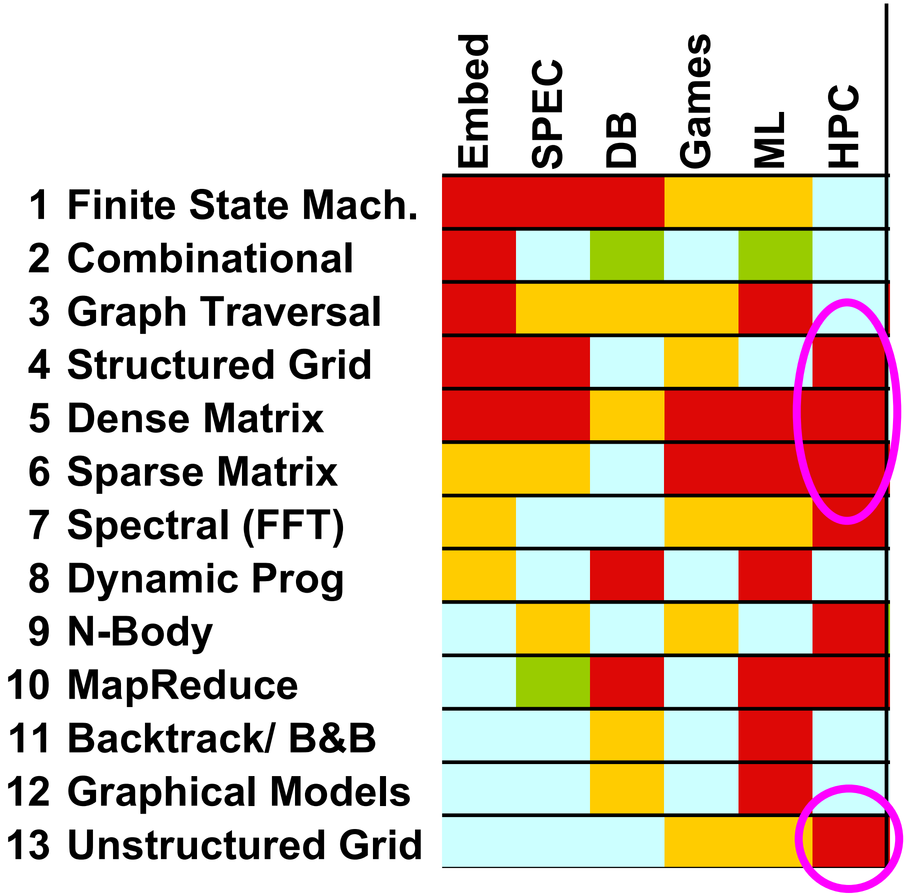
- Overview
- Elementary Operations
- Matrix Decompositions
- Solving Linear Systems

Learning Objectives

- Refresh our memory about basic operations that go into algorithms for linear algebra
- Understand common matrix factorizations and how they can be used
- Understand "10,000 foot view" of solving linear systems

Overview

- What types of problems are solved with Linear Algebra?
  - Linear systems of equations
  - Eigenvalue problems
  - matrix factorization
  - Overdetermined system of equations
    - (Data fitting)



Linear System Fundamentals

Basics of Linear Systems

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$a_{1,1} x_1 + a_{2,1} x_2 + \cdots + a_{n,1} x_n = b_1$$

$$a_{1,2} x_1 + a_{2,2} x_2 + \cdots + a_{n,2} x_n = b_2$$

$$\vdots$$

$$a_{n,1} x_1 + a_{2,n} x_2 + \cdots + a_{n,n} x_n = b_n$$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Norms of Vectors (1)

Like a measuring stick

$\underline{r} \equiv \underline{\underline{A}}\,\underline{\tilde{x}} - \underline{b}$ $\quad \underline{r}$ is residual

$\|r\|$

$\|r\|_1 = 1\text{-norm}$

$\|r\|_2 = 2\text{-norm}$

$\|r\|_\infty = \infty\text{-norm}$

p-norms $= \|r\|_p$

Norms of Vectors (2)

Inner products $\sqrt{V^T \cdot V}$

$$\|r\|_1 = |r_1| + |r_2| + \cdots + |r_n|$$

Outer product

$V \cdot V^T$

$$\|r\|_2 = \sqrt{r_1^2 + r_2^2 + \cdots + r_n^2} \longrightarrow \text{Euclidean Norm}$$

$$\|r\|_\infty = \max(|r_1|, |r_2|, \ldots |r_n|)$$

$$\|r\|_p = \left(|r_1|^p + |r_2|^p + \cdots + |r_n|^p\right)^{1/p}$$

Types of operations in linear algebra

BLAS

Vector - vector → 1 loop
scalar - vector → 1 loop
Level 1

Matrix - vector → 2 loops
Scalar - Matrix → 2 loops
Level 2

Matrix - Matrix → 3 loops
Level 3

Going from Chalkboard to Terminal

$$\|r\|_2 = \left(\sum_{i=1}^{n} r_i\right)^{1/2}$$

```
sum = 0 ←
Do i = 1, n
  sum = sum + r(i)*r(i)
ENDDO
nrm = SQRT(sum)
```

Matrix Factorizations

Types of Matrices

Identity – Diagonal
Invertable
Triangular
Orthogonal
Symmetric
Singular

Anti-Symmetric
Asymmetric
Tridiagonal → Striped
Banded
Circulant
Sparse
Dense

Factorizing Matrices (1)

$A = LU$

Factorizing

$\underline{A = L + D + U}$

Decomposing

$Ax = b \Rightarrow LUx = b$

$x = A^{-1}b \Rightarrow L^{-1}LUx = \underbrace{L^{-1}b = y}$

$Ux = y$

$U = \begin{bmatrix} \diagdown & \\ 0 & \diagdown \end{bmatrix} \quad L = \begin{bmatrix} \diagdown & 0 \\ & \diagdown \end{bmatrix}$

1) $Ly = b$

2) $Ux = y$

$l_{1,1} y_2 = b_1 \Rightarrow y_1 = \frac{b_1}{l_{1,1}}$

$l_{2,1} y_1 + l_{2,2} y_2 = b_2 \quad y_2 = \frac{b_2 - l_{2,1} y_1}{l_{2,2}}$

$y_i = \left(b_i - \sum_{j=1}^{i-1} l_{i,j} y_j\right) / l_{i,i}$

Factorizing Matrices (2)

$Ux = y$

$U = \begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix}$

$x_n = \dfrac{y_n}{u_{n,n}}$

$$x_i = \left(y_i - \sum_{j=n,i+1,-1} u_{i,j} x_j\right) \Big/ u_{i,i}$$

$Ly = b$ Forward Elimination

$Ux = y$ Backward Substitution

Factorizing Matrices (3)

$A = QR$ $\quad R = [\,0\,\nabla\,]$

$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$

$q_i^T \cdot q_j = 0 \quad i \neq j \rightarrow$ orthogonal

$q_i^T \cdot q_j = 1 \quad i = j \Rightarrow$ normal

orthonormal

Gram-Schmidt (modified)

Householder

$\underline{q_1} = \dfrac{\underline{a_1}}{\|a_1\|_2}$ $\quad \text{proj}_u V = \dfrac{u^T \cdot v}{u^T \cdot u} u$

Krylov $\Rightarrow$ GMRES $\rightarrow$ each step involves orthogonalizing

Factorizing Matrices (4)

Singular Value Decomposition

$$A = U \Sigma V^T$$

U: Left singular values (unitary)

Σ: Diagonal

V^T: right singular values

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_n \end{bmatrix}$$

$$U^* U = I \qquad U^T = U^{-1}$$

$$U^* = U^{-1}$$

↳ If U is real

$$|\sigma_1| > |\sigma_2| > |\sigma_3| \dots$$

Factorizing Matrices (5)

Eigenvalue Decomposition

$$A = Q \Lambda Q^{-1}$$

orthonormal

columns are eigenvectors

eigenvalues (diagonal)

Symmetric Positive Definite

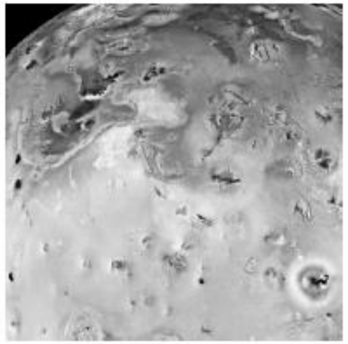
$$Q \Lambda Q^{-1} = U \Sigma V^{-1}$$

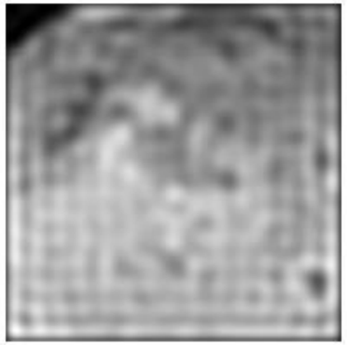
$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\lambda_1 > \lambda_2 > \lambda_3$$

Example of SVD Applied to Images

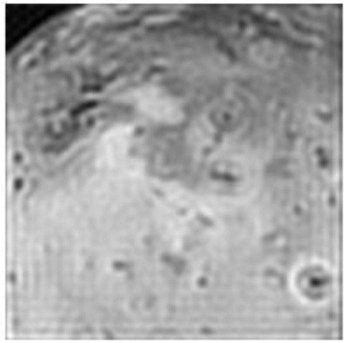
Original
(512x512)
~k=262144

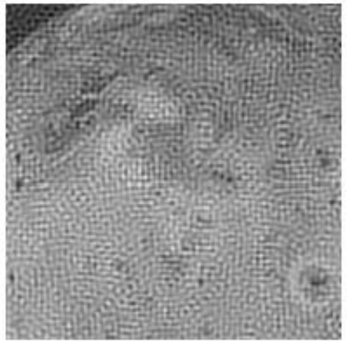




k=658 (too small, oversmoothed)

k=2813



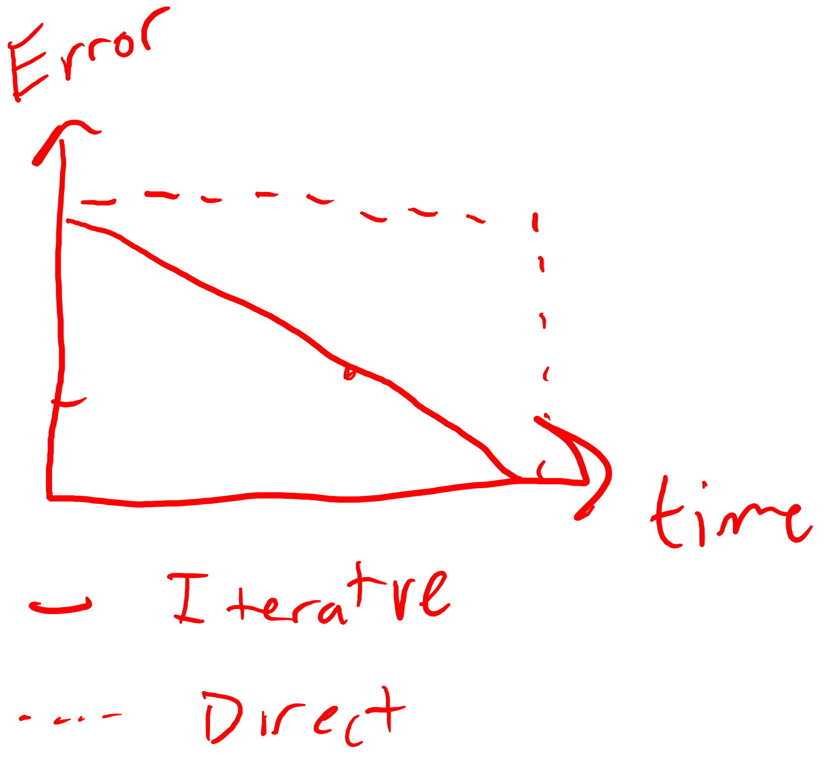


k=7243 (undersmoothed)

Fig 5.1 from P.C. Hansen, et al., *Deblurring Images: Matrices, Spectra, and Filtering*, SIAM, 2006.
https://doi.org/10.1137/1.9780898718874

Solving Linear Systems

Solving Linear Systems



Iterative & Direct

↓ repeated application of Mat-Vec

$O(?\, n^2)$

↓ matrix-matrix

$O(?\, n^3)$