Lab 2

23rd September 2020

Exercise 2 c) (i)

Given a polynomial $p_m(x)$ of order m to a true function $f(x)$, which is assumed to be $m+1$ differentiable, the interpolation error can be found from the following analysis.

Assume that we have $n+1$ data points $\{(x_i, y_i)\}$ such that $y_i \approx f(x_i) \equiv f_i$, and $x_i \in \Omega$. We must find the polynomial $p_m(x)$ such that $p_m(x_i) = y_i$. Typically the system is over determined, meaning $n > m$, and there are more data points than terms in the polynomial. In this case, a least squares fitting of a polynomial, through the m-order Vandermonde matrix $X \in \mathbb{R}^{n+1 \times p+1}$ for the data, yields polynomial coefficients $\tilde{\gamma}$:

$$y \approx \tilde{y} = X\tilde{\gamma}. \tag{1}$$

This fit $\tilde{y}$ is imperfect for $n > m$, and does not perfectly fit to the $n+1$ data points, or the true function $f(x)$, but provides the least squares error for the residual of $r = y - \tilde{y}$. Typically this can be solved via a pseudo-inverse, or other iterative methods:

$$\tilde{\gamma} = (X^T X)^{-1} X^T y. \tag{2}$$

To determine the interpolation error, it is instead assumed that $n = m$, and there are $m+1$ data points for the m order polynomial. The error

$$e_m(x) = f(x) - p_m(x) \tag{3}$$

has the form given the data

$$e_m(x; X) = \frac{f^{(m+1)}(\xi)}{(m+1)!} w_X(x), \tag{4}$$

where the function over the data

$$w_X(x) = \prod_i (x - x_i), \tag{5}$$

and $\xi(x)$ is a root such that

$$e_m^{(m+1)}(\xi) - \frac{(m+1)!}{w_X(x)} e_m(x) = 0. \tag{6}$$

The maximum norm error $|e(x)|$ then corresponds to finding the maxima of w_X, given the data. The root ξ depends on the interval and the specific function used; and for the maximum norm, not any root, but

$$\xi = \text{argmax}_{x \in \Omega} |f^{(m+1)}(x)|.$$

We will look at the cases of $m = 1, 2$, given equally spaced data in one dimensional data:

$$\Omega =[a, b], \tag{7}$$
$$x_i =a + ih, \tag{8}$$
$$h =\frac{b-a}{m}. \tag{9}$$

We will also look at a function that is a k order polynomial with true coefficients β:

$$f = X\beta, \tag{10}$$

where X here is the k-order Vandermonde matrix. In this case, the polynomial function will have coefficients:

$$\beta = \{7.33 \times 10^{1}, 3.785 \times 10^{-1}, -1.229 \times 10^{-3}, 2.949 \times 10^{-6}, \tag{11}$$
$$-4.247 \times 10^{-9}, 3.12 \times 10^{-12}, -9.076 \times 10^{-16}\}. \tag{12}$$

a. m=1

For $m = 1$, it can be seen that

$$x_i = \{a, b\},$$

w_X is maximum at

$$x = \frac{b-a}{2} = m\frac{h}{2}.$$

ξ can be found through numerical approximation, and only considering the maximum of $f^{(2)}(x)$ over Ω, there is a bound of

$$f^{(2)}(\xi) < C_2.$$

Therefore the maximal error is:

$$|e(x)|_\infty \leq \frac{1}{2}C_2\frac{1}{4}m^2h^2 \tag{13}$$

b. m=2

For $m = 2$, it can be seen that

$$x_i = \{a, \frac{a+b}{2}, b\},$$

and the w_X is a maximum at

$$x = \frac{b-a}{2} = \frac{a+b}{2} - \frac{1}{\sqrt{3}}m\frac{h}{2}.$$

ξ, from $f^{(3)}(x)$ can be found through numerical approximation,, and only considering the maximum of $f^{(3)}(x)$ over Ω, there is a bound of

$$f^{(3)}(\xi) < C_3.$$

Therefore the maximal error is:

$$|e(x)|_\infty \leq \frac{1}{2}C_3\frac{1}{18}m^3h^3. \tag{14}$$

Exercise 2 c) (ii)

Given that the points are equally spaced, allows for the $w(x)$ to fluctuate, and doesn't necessarily yield minimal $|e(x)| \sim |w(x)|$, the data points should be chosen such that they minimize this $w(x)$ function. From the literature, it appears that switching to Chebyshev points, and rewriting the expressions in a Chebyshev basis, as opposed to this more Lagrange approach, will allow for better optimization of the interpolation error. The points should potentially be chosen based on the function $f(x)$ that is being fit as well, since there may be more complex regions of the function's domain that required more sampling for better fitting and to reduce the interpolation error.