Scientific Computing Homework 2

1 Spectral Radius of Fixed Point Methods

Fixed Point iterative methods involve finding the solutions to

$$Ax = b \tag{1}$$

by decomposing the matrix into a form such that iterations are

$$x_{n+1} = Bx_n + c, \tag{2}$$

where B is matrix that is typically a function of A, and c is a function of A and b. These iteration matrices are generally most easily expressed when A is decomposed as its diagonal, lower, and upper triangular components:

$$A = D + L + U, \tag{3}$$

since the forms of these matrices make them easily invertible for use in the iterative updates to x.

Given an initial guess x_0 the error $e_n = x_n - x$ is

$$e_n = B^n e_0, \tag{4}$$

and the relative norm of the error is:

$$\frac{\|e_n\|}{\|e_0\|} \leq \|B^n\|. \tag{5}$$

This norm of $\|B^n\|$ can be approximated by upper bounds, and in the case of the 2-norm, in terms of the spectral radius $\rho(B)$:

$$\|B^n\| \leq \|B\|^n \leq \rho(B)^n. \tag{6}$$

Therefore the spectral radius of B can be found from either directly calculating (through some other iterative method using a numerical library) for the eigenvalues of B, or from measuring the norms of the residual over the iterations, with the last iteration likely giving the best estimate for the spectral radius:

$$\rho(B) = \max_{\lambda} |\lambda(B)| = \left(\frac{\|e_n\|}{\|e_0\|}\right)^{1/n}. \tag{7}$$

1.1 Jacobi Method

For the Jacobi method, the iterations proceed such that

$$B = -D^{-1}(L + U) \tag{8}$$

$$c = D^{-1}b. \tag{9}$$

1.2 Gauss-Seidel Method

For the Gauss-Seidel method, the iterations proceed such that

$$B = -(D+L)^{-1}U \tag{10}$$
$$c = (D+L)^{-1}b. \tag{11}$$