Moments of Quantum Channels

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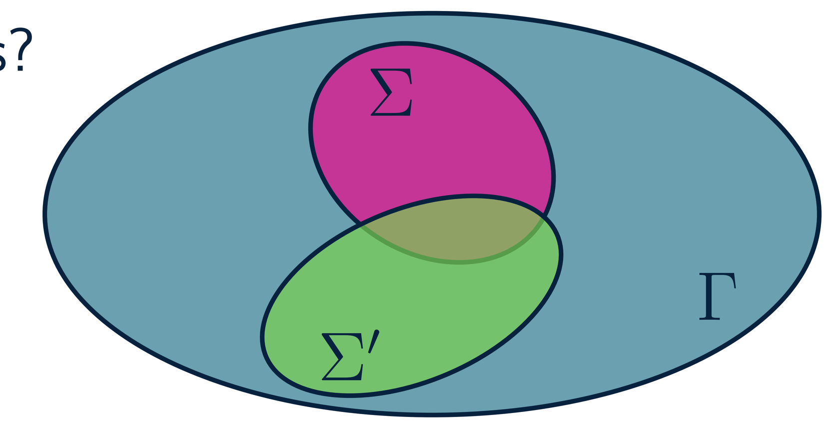
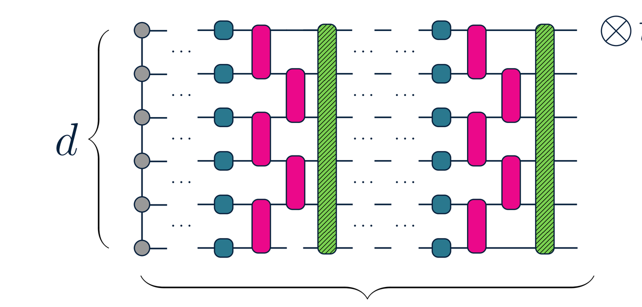
1. Statistics of Ensembles of Random Operators

- What are the *statistical moments* of ensembles of random quantum channels? [1]
- Can we develop *tools* that quantify moments of quantum channels and their *spectral* properties?
- How do statistics of ensembles, and comparisons to *reference ensembles* depend on:
  - *Noise* induced phenomena
  - Underlying (parameterized) *unitary* evolution
  - Coupling with the *environment*



2. Variables



Space Dimension	d, ϵ
Copies of Space	t
Layers	k
Ensembles	$\Sigma, \Sigma' \subseteq \Gamma$
Twirls	$\mathcal{T}^{(t)}_{\Sigma} = \int_{\Sigma} d\Lambda\ \Lambda^{\otimes t} = \mathcal{D}_d^{\otimes t} + \Delta^{(t)}_{\Sigma}$
Norms	$\mathcal{E}^{(t)}_{\Sigma\Sigma'} = \lVert \mathcal{T}^{(t)}_{\Sigma} - \mathcal{T}^{(t)}_{\Sigma'} \rVert \ ,\ \mathcal{E}^{(t)}_{\Sigma} = \lVert \Delta^{(t)}_{\Sigma} \rVert$

3. Reference Ensembles

- *Haar* $\sim$ Unitary Haar Measure (uniformly random unitaries)

$$\boxed{\mathcal{T}^{(t)}_{\mathcal{U}(d)}(\rho)} = \int_{\mathcal{U}(d)} dU\ U^{\otimes t}\ \rho^{\otimes t}\ U^{\otimes t\ \dagger} \tag{1}$$

- *cHaar* $\sim$ Stinespring Haar Measure (random channels) [2]

$$\boxed{\mathcal{T}^{(t)}_{\mathcal{C}(d,\epsilon)}(\rho)} = \mathrm{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(d\epsilon)} dU\ U^{\otimes t}\ \rho^{\otimes t} \otimes \nu_{\mathcal{E}}^{\otimes t}\ U^{\otimes t\ \dagger}\right) \tag{2}$$

- *Depolarize* $\sim$ Maximally Depolarizing Channel (single channel)

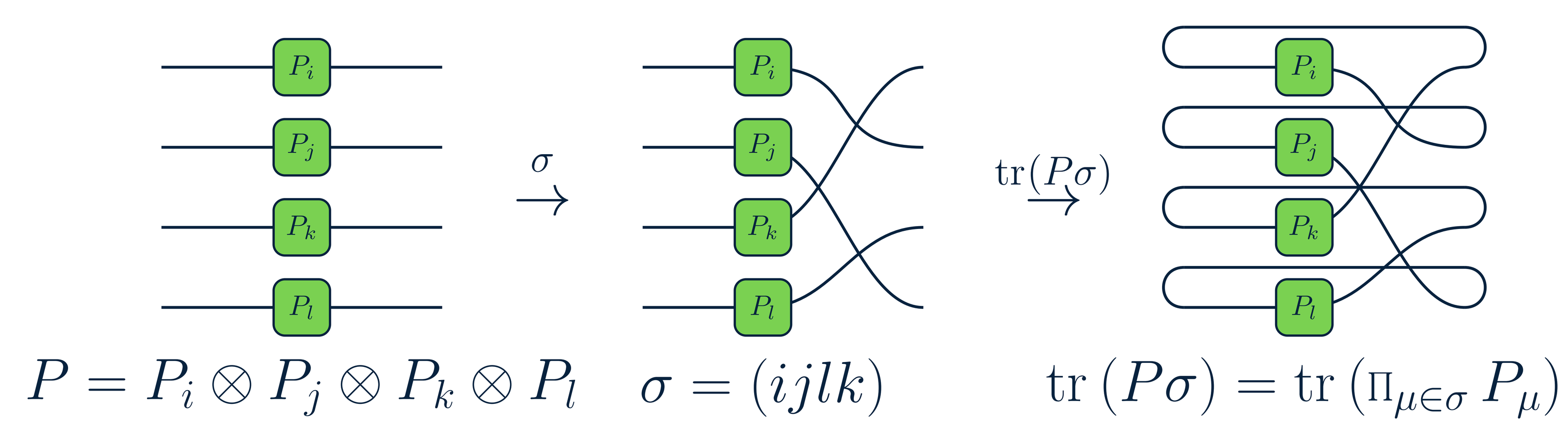
$$\boxed{\mathcal{T}^{(t)}_{\mathcal{D}(d)}(\rho)} = \mathcal{D}_d^{\otimes t}(\rho^{\otimes t}) = \frac{\mathrm{tr}\,(\rho^{\otimes t})}{d^t}\ I^{\otimes t} \tag{3}$$

$$\boxed{\lVert \mathcal{T}^{(t)}_{\mathcal{D}(d)} \rVert_2} \underset{d\epsilon\to\infty}{\leq} \boxed{\lVert \mathcal{T}^{(t)k}_{\mathcal{C}(d,\epsilon)} \rVert_2} \underset{d\epsilon\to d}{\leq} \boxed{\lVert \mathcal{T}^{(t)}_{\mathcal{U}(d)} \rVert_2} \tag{4}$$

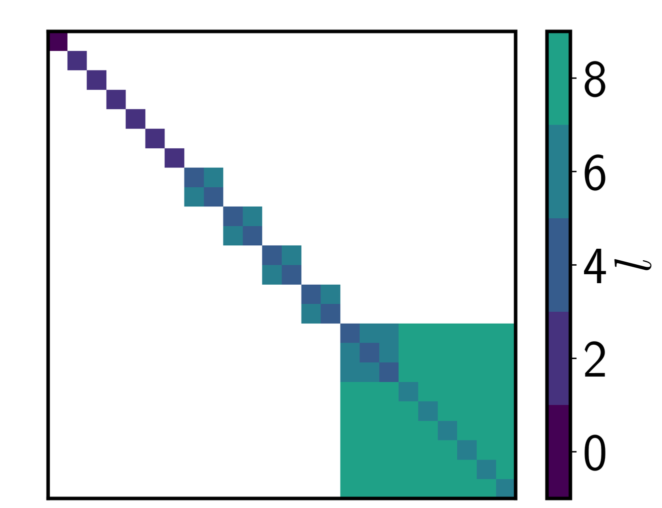
4. Super-Operators, Permutations, and Localized Bases

- Twirls over ensembles Σ may be expressed in a super-operator *basis* $\boxed{\mathcal{T}^{(t)}_{\Sigma} = \sum_{\sigma,\pi\in\mathcal{S}^{(t)}_{\Sigma}} \tau^{(t)}_{\Sigma}(\sigma,\pi)\ |\sigma\rangle\langle\pi|}$ $\rightarrow$ Is $\mathcal{S}^{(t)}_{\Sigma}$ *orthogonal*?
- Haar random unitary $\mathcal{U}(d)$ twirls project [3] onto non-orthogonal *permutations* $\mathcal{S}_t$ $\rightarrow$ one-to-one with *localized permutations* $[\mathcal{S}_t]$

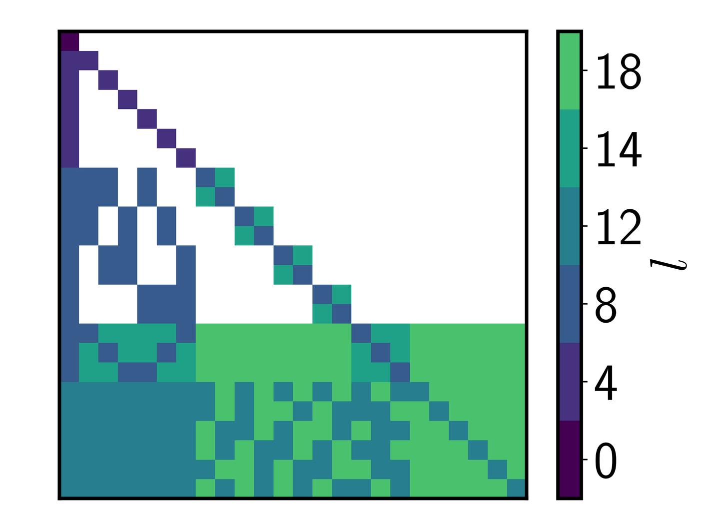
$$\sigma = \sum_{\pi\subseteq\sigma} [\pi] \qquad [\sigma] = \sum_{\mathrm{supp}(P)=\mathrm{supp}(\sigma)} \chi^{(t)}_d([\sigma],P)\ P \quad \rightarrow \quad \mathcal{T}^{(t)}_{\mathcal{U}(d)} = \sum_{\substack{\sigma,\pi\in\mathcal{S}_t \\ \mathrm{supp}(\sigma)=\mathrm{supp}(\pi)}} \tau^{(t)}_{\mathcal{U}(d)}(\sigma,\pi)\ |[\sigma]\rangle\langle[\pi]| \quad \rightarrow \quad \mathcal{T}^{(t)}_{\mathcal{C}(d,\epsilon)} = \sum_{\substack{\sigma,\pi\in\mathcal{S}_t \\ \mathrm{supp}(\sigma)\supseteq\mathrm{supp}(\pi)}} \tau^{(t)}_{\mathcal{C}(d,\epsilon)}(\sigma,\pi)\ |[\sigma]\rangle\langle[\pi]| \tag{5}$$



(a) Operator String and Permutation Overlaps, for $t = 4$



(b) Haar Twirl $\tau^{(t)}_{\mathcal{U}(d)}(\sigma,\pi) \sim O(1/d^l)$



(c) cHaar Twirl $\tau^{(t)}_{\mathcal{C}(d,d^2)}(\sigma,\pi) \sim O(1/d^l)$

5. Twirl Norms

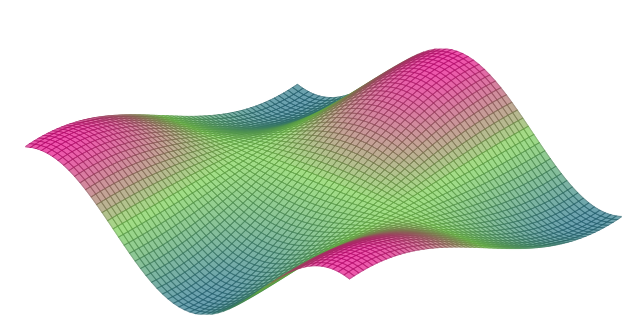
Unitary + Noise $(\mathcal{N}\circ\mathcal{U})^k$	Norm $\lVert(\mathcal{N}\circ\mathcal{T}^{(t)}_{\mathcal{U}})^k - \mathcal{D}_d^{\otimes t}\rVert^2$
Haar $\mathcal{U}(d)$, Unital $\mathcal{N}_\gamma$	*Decreases* $O((1-\gamma)^{2k})$
Haar $\mathcal{U}(d)$, Non-Unital $\mathcal{N}_\eta$	*Increases* $O(\eta)$
Parameterized $\mathcal{G}_\theta$, Unital $\mathcal{N}_\gamma$	*Decreases* $O((1-\gamma)^k)$

(6)

$$\mathcal{T}^{(t)k}_{\mathcal{C}(d,\epsilon)} = \underbrace{|I\rangle\langle I|}_{\text{Depolarize}} + \underbrace{O\left(\frac{1}{d^2\epsilon}\right)|T\rangle\langle I|}_{\text{Non-Unital}} + \underbrace{O\left(\frac{1}{d^2\epsilon^k}\right)|P\rangle\langle S|}_{\text{Unital}} \tag{7}$$

6. Expressivity versus Trainability

- Ensemble-dependent functions $\mathcal{F}$ may *concentrate* $\boxed{p\,(|\mathcal{F}-\mu_{\mathcal{F}}| \geq \delta) \leq \sigma^2_{\mathcal{F}}/\delta^2}$ (with *caveats* on ensembles, locality, norms, ...)
- Parameterized *objectives* and *gradients* $\boxed{\mathcal{L} = \mathrm{tr}\,(O\Lambda(\rho)) \rightarrow \partial\mathcal{L}}$ variances decay due to *inherent* and *expressivity* terms [4]



$$\sigma^2_{\mathcal{L}}\ ,\ \sigma^2_{\partial\mathcal{L}} \sim \underbrace{O\left(\frac{1}{\mathrm{poly}(d,\epsilon)}\right) \lVert\rho\rVert_2^2\ \lVert O\rVert_2^2}_{\text{Inherent}} + \underbrace{\min_{\frac{1}{p}+\frac{1}{q}=1} O\left(\frac{1}{\mathrm{poly}(d,\epsilon)}\right) \lVert\rho\rVert_p^2\ \lVert O\rVert_q^2\ \mathcal{E}^{(2|p)}_{\Sigma\Sigma'}}_{\text{Expressivity}} \tag{8}$$

7. Conclusions

- Spectral properties of twirls correspond to ability to *depolarize* or *transmit* quantum information
- Unlike unitaries, channel design properties are more subtly related to their *usefulness* or *capability*
- *Noise induced* phenomena are actually channel *design* phenomena!
- Are there relationships between channel statistics and their *simulability*? [5]

8. References

[1] M. Duschenes, D. Garcia-Martin, Z. Holmes, M. Cerezo. arXiv:arXiv:2510.XXXXX, Report: LA-UR-24-20854 , (2025).
[2] R. Kukulski, I. Nechita, L. Pawela, Z. Puchala, K. Zyczkowski. Journal of Mathematical Physics **62**, 062201 (2021).
[3] J. Bai, J. Wang, Z. Yin. Quantum Information Processing **23**, 1–18 (2024).
[4] Z. Holmes, K. Sharma, M. Cerezo, P. J. Coles. PRX Quantum **3**, 010313 (2022).
[5] A. A. Mele, A. Angrisani, S. Ghosh, S. Khatri, J. Eisert, D. S. França, Y. Quek. arXiv:2403.13927 , (2024).