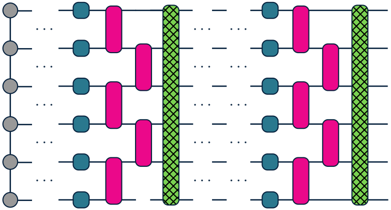
Simulation of Noisy Quantum Systems with POVM-MPS

Matthew Duschenes*, Roger Melko, Raymond Laflamme, Juan Carrasquilla

Institute for Quantum Computing, Perimeter Institute & ETH Zurich

IQC Grad Student Conference

July 23, 2025



Noisy Quantum Simulations

- Simulations of channels vs. unitaries are *higher dimensional* and *constrained*

$$\underbrace{U|\psi\rangle \;\in\; \mathbb{C}^{D^N}}_{\text{Arbitrary Normalized Vector}} \quad \to \quad \underbrace{\Lambda(\rho) \;\in\; \mathbb{C}^{D^N \times D^N}}_{\text{Positive-Semidefinite, Hermitian Matrix}} \tag{1}$$

Noisy Quantum Simulations

- Simulations of channels vs. unitaries are *higher dimensional* and *constrained*

$$\underbrace{U|\psi\rangle \;\in\; \mathbb{C}^{D^N}}_{\text{Arbitrary Normalized Vector}} \qquad \rightarrow \qquad \underbrace{\Lambda(\rho) \;\in\; \mathbb{C}^{D^N \times D^N}}_{\text{Positive-Semidefinite, Hermitian Matrix}} \tag{1}$$

- How to differentiate between "quantum" correlations in *pure* states versus "classical" correlations in *mixed* states?

Noisy Quantum Simulations

- Simulations of channels vs. unitaries are *higher dimensional* and *constrained*

$$\underbrace{U|\psi\rangle \;\in\; \mathbb{C}^{D^N}}_{\text{Arbitrary Normalized Vector}} \qquad \rightarrow \qquad \underbrace{\Lambda(\rho) \;\in\; \mathbb{C}^{D^N \times D^N}}_{\text{Positive-Semidefinite, Hermitian Matrix}} \tag{1}$$

- How to differentiate between "quantum" correlations in *pure* states versus "classical" correlations in *mixed* states?

- Does *noise* make simulations *easier* [1] ?

[1] Mele, A. *et al.*, *arXiv:2403.13927* (2024)

Noisy Quantum Simulations

- Simulations of channels vs. unitaries are *higher dimensional* and *constrained*

$$\underbrace{U|\psi\rangle \in \mathbb{C}^{D^N}}_{\text{Arbitrary Normalized Vector}} \quad \to \quad \underbrace{\Lambda(\rho) \in \mathbb{C}^{D^N \times D^N}}_{\text{Positive-Semidefinite, Hermitian Matrix}} \tag{1}$$

- How to differentiate between “quantum” correlations in *pure* states versus “classical” correlations in *mixed* states?

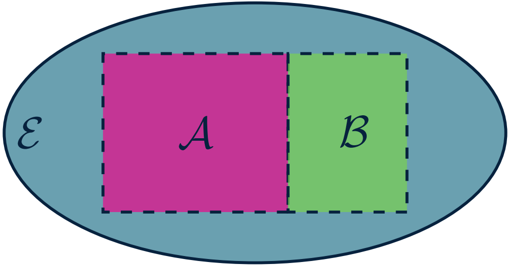
- Does *noise* make simulations *easier* ?

- Is there an *area law* for noise/classical correlations?
  i.e) How does simulation complexity depend on system size, Kraus rank [2] ?

[2] Noh, K. *et al.*, *Quantum* **4** (2020)

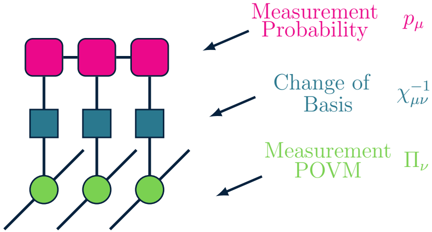
Noisy Quantum Simulations

What if we want to assess *capabilities* of capturing correlations in systems *strictly* between $\mathcal{A}, \mathcal{B}$, excluding correlations with environments $\mathcal{E}$?



POVM-MPS for Quantum Simulations

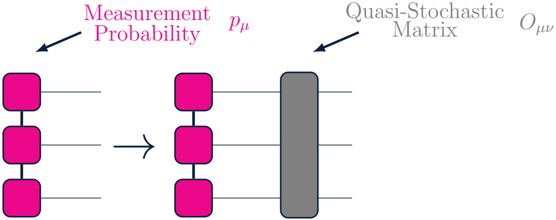
Suppose we model states in terms of measurement *probabilities* $\rho \cong p$



$$\rho_{\sigma\sigma'} = \sum_{\mu\nu} p_\mu \, \chi^{-1}_{\mu\nu} \, \Pi_{\nu\sigma\sigma'}$$

POVM-MPS for Quantum Simulations

Suppose we model dynamics in terms of *(quasi-)stochastic matrices* $\Lambda \cong O$



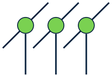
$$p_\mu \to \sum_\nu O_{\mu\nu} p_\nu$$

POVM-MPS for Quantum Simulations

To ensure *efficient* classical simulations:

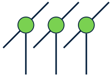
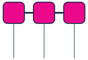
POVM-MPS for Quantum Simulations

To ensure *efficient* classical simulations:

- Measurements $\Pi = \otimes_i^N \Pi_i$ are *local*, *informationally complete* (POVM) [3] 

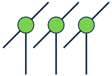
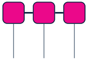
POVM-MPS for Quantum Simulations

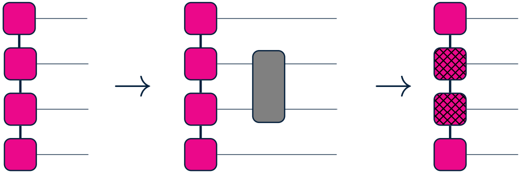
To ensure *efficient* classical simulations:

- Measurements $\Pi = \otimes_i^N \Pi_i$ are *local, informationally complete* (POVM) 
- Probabilities $p \approx p_\chi = \prod_i^N A_i$ are *tensor trains* (pMPS) 

POVM-MPS for Quantum Simulations

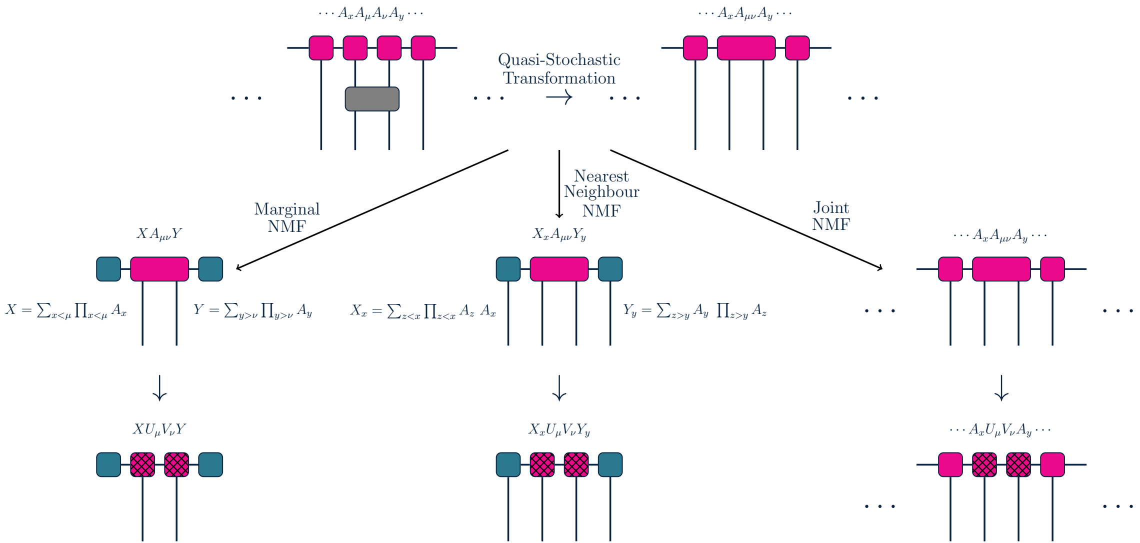
To ensure *efficient* classical simulations:

- Measurements $\Pi = \otimes_i^N \Pi_i$ are *local, informationally complete* (POVM) 
- Probabilities $p \approx p_\chi = \prod_i^N A_i$ are *tensor trains* (pMPS) [4] 
- How may we *normalize and truncate* updates $p \to Op \to p'_\chi$?



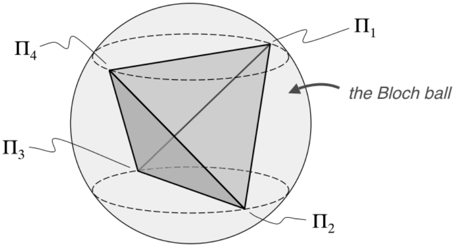
[4] Temme, K. & Verstraete, F., *PRL* **104**, 210502 (2010)

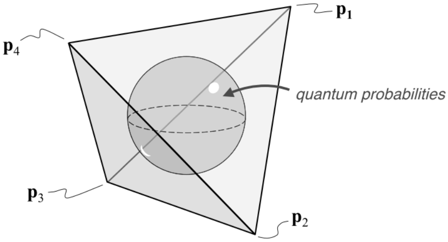
Tensor Factorizations and Truncations



Quantum Probability Simplexes

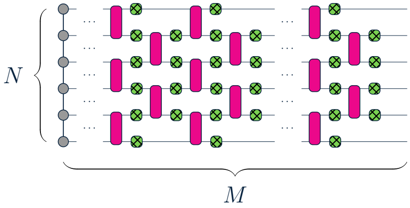
Not only do POVM-MPS updates $p \to p_\chi$ have to be *non-negative*, but they must obey non-trivial quantum probability simplex *constraints* [5]





[5] Appleby, D. M. *et al.*, *Found. Phys.* **41**, 564–579 (2009)

Random Noisy Quantum Circuits



M layers of Random *2-local* Unitaries (Pink) + Noise (Green) with *Product* Initial States

1. Convert states to probabilities and channels to quasi-stochastic matrices

$$\rho \to p \qquad , \qquad \Lambda \to O$$

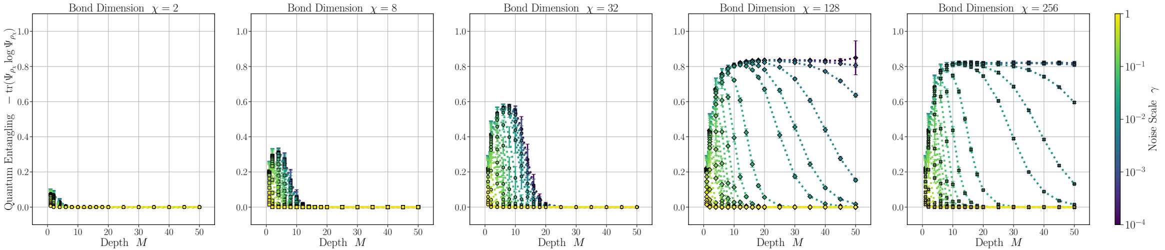
2. Apply channels and truncate bond-dimension to χ (SVD-based truncation)

$$p \to p' = Op \approx p'_\chi$$

3. Convert evolved probabilities to states ρ'_χ and compute quantities

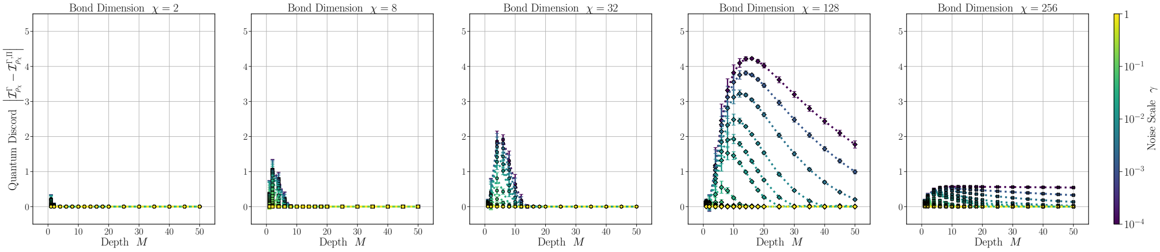
$$\mathcal{L}_{\rho_\chi \rho} \in \left\{ \mathrm{tr}\left(\sqrt{\rho\rho_\chi}\right) , \sqrt{p} \cdot \sqrt{p_\chi} , \mathrm{tr}\left(\rho\rho_\chi\right) \right\} \quad , \quad \mathcal{S}^{\mathcal{A}}_{\rho_\chi}$$

POVM-MPS Operator Entanglement



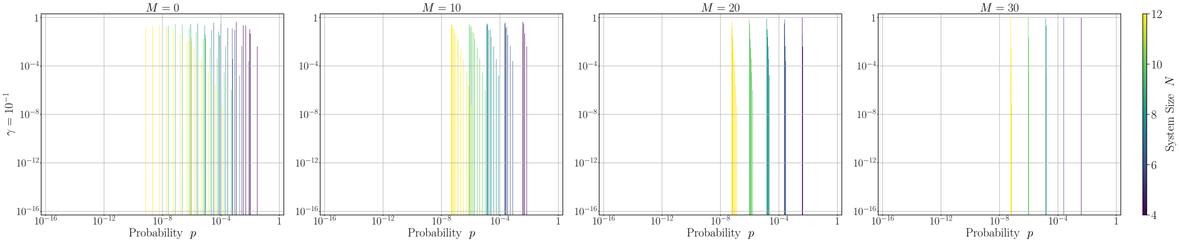
(a) Quantum Operator Entanglement of SVD-POVM-MPS ρ_χ for $N = 8$ qubits.

POVM-MPS Quantum Discord



(b) Quantum Discord of SVD-POVM-MPS ρ_χ for $N = 8$ qubits.

Distribution of POVM-MPS Probabilities



(c) POVM Probability Counts of Exact POVM-MPS ρ for N qubits.

Advantages and Disadvantages of POVM-MPS

- Pro: Dynamics are represented as single *matrix multiplications*
- Con: Channels must be converted to matrices at runtime
  i.e) Single qubit channels are "free" - No increase in bond dimension χ

Advantages and Disadvantages of POVM-MPS

- Pro: Dynamics are represented as single *matrix multiplications*
- Con: Channels must be converted to matrices at runtime
  i.e) Single qubit channels are "free" - No increase in bond dimension χ

- Pro: Single bond dimension χ captures quantum and classical *correlations*
- Con: Density matrix positivity constraints are *hard* and *slow in practice*
  i.e) NMF has *no closed forms*, *no convergence guarantees*, *gauge freedom*

Advantages and Disadvantages of POVM-MPS

- Pro: Dynamics are represented as single *matrix multiplications*
- Con: Channels must be converted to matrices at runtime
  i.e) Single qubit channels are "free" - No increase in bond dimension χ

- Pro: Single bond dimension χ captures quantum and classical *correlations*
- Con: Density matrix positivity constraints are *hard* and *slow in practice*
  i.e) NMF has *no closed forms*, *no convergence guarantees*, *gauge freedom*

Are physical mixed states *always* inherently harder to simulate *precisely*?

Appendix

Benchmarks for Noisy Quantum Simulations

Accuracy of simulations is characterized by *infidelity* and *entanglement*

Infidelity: Similarity of states $\rho \approx \sigma$ or distributions $p \approx s$

- Quantum

$$\mathcal{L}^{Q}_{\rho\sigma} = 1 - \mathrm{tr}\left(\sqrt{\rho\sigma}\right) \tag{1}$$

- Classical

$$\mathcal{L}^{C}_{\rho\sigma} = 1 - \sqrt{p} \cdot \sqrt{s} \tag{2}$$

- Pure

$$\mathcal{L}^{P}_{\rho\sigma} = 1 - \sqrt{\mathrm{tr}\left(\rho\sigma\right)} \tag{3}$$

Benchmarks for Noisy Quantum Simulations

Accuracy of simulations is characterized by *infidelity* and *entanglement*

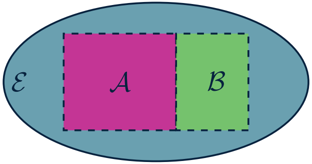
Entanglement: Entropy of states $\rho_{\mathcal{A}} = \mathrm{tr}_{\mathcal{B}}\left(\rho\right)$, $p_{\mathcal{A}} = \sum_{\mathcal{B}} p$, for partitions $\mathcal{A}$

- Quantum

$$\mathcal{S}_{\rho}^{Q\mathcal{A}} = -\,\mathrm{tr}\left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right) \tag{4}$$

- Classical

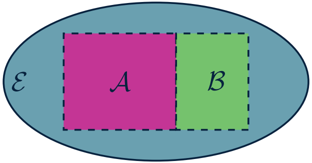
$$\mathcal{S}_{\rho}^{C\mathcal{A}} = -\,p_{\mathcal{A}} \cdot \log p_{\mathcal{A}} \tag{5}$$



- Renyi

$$\mathcal{S}_{\rho}^{R\mathcal{A}} = 1 - \mathrm{tr}\left(\rho_{\mathcal{A}}^{2}\right) \tag{6}$$

State versus Operator Entanglement



- *State Entanglement*

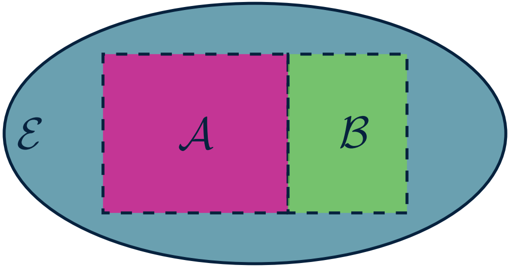
$$\rho \;\rightarrow\; \rho_{\mathcal{A}} \;\rightarrow\; \mathcal{S}_{\rho}^{\mathcal{A}} = -\mathrm{tr}\left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right) \tag{7}$$

- Captures entropy of reduced state within $\mathcal{A}$, which can be due to correlations with *any* other partition ($\mathcal{B}$, environment, ancilla etc.)

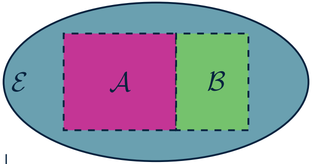
i.e) Maximum entanglement of $\mathcal{A}$ with environment
$\rightarrow$ Maximally mixed $\rho_{\mathcal{A}} = I$, even if no correlations with $\mathcal{B}$
$\rightarrow$ Maximum state entanglement $\mathcal{S}_{\rho}^{\mathcal{A}} = \log d_{\mathcal{A}}$, even though classical

State versus Operator Entanglement

- What if we just want to capture correlations *strictly* between $\mathcal{A}, \mathcal{B}$?



State versus Operator Entanglement



- *Operator Entanglement*

$$\rho \to |\rho\rangle\!\rangle = \rho \otimes I|\Omega\rangle \to \Psi_\rho = \frac{|\rho\rangle\!\rangle\langle\!\langle\rho|}{\langle\!\langle\rho|\rho\rangle\!\rangle} \to \Psi_{\rho_\mathcal{A}} \tag{8}$$

$$\mathcal{Q}_\rho^\mathcal{A} = \mathcal{S}_{\Psi_\rho}^\mathcal{A} = \mathcal{S}_{|\rho\rangle\!\rangle\langle\!\langle\rho|}^\mathcal{A} = -\mathrm{tr}\left(\Psi_{\rho_\mathcal{A}} \log \Psi_{\rho_\mathcal{A}}\right) \tag{9}$$

- Captures entropy of reduced vectorized states within $\mathcal{A}$

i.e) Pure States: $|\psi\rangle \to |\psi\rangle\langle\psi| \to |\psi\psi\rangle \to |\psi\psi\rangle\langle\psi\psi|$

i.e) Mixed States: $\sum_\lambda p_\lambda \rho_\lambda \to \sum_\lambda p_\lambda |\rho_\lambda\rangle\!\rangle \to \sum_{\lambda\kappa} p_\lambda p_\kappa |\rho_\lambda\rangle\!\rangle\langle\!\langle\rho_\kappa|$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\to\; |\rho\rangle\rangle = |\psi\psi\rangle \;\to\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\mathcal{A}}_{|\psi\rangle} = 2\mathcal{S}^{\mathcal{A}}_{|\psi\rangle} \tag{10}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\to\; |\rho\rangle\!\rangle = |\psi\psi\rangle \;\to\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\mathcal{A}}_{|\psi\rangle} = 2\mathcal{S}^{\mathcal{A}}_{|\psi\rangle} \tag{10}$$

- Product states: $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\to\; \Psi_\rho = \Psi_{\rho_{\mathcal{A}}} \otimes \Psi_{\rho_{\mathcal{B}}}$

$$\mathcal{Q}^{\mathcal{A}}_{\rho_{\mathcal{A}}\otimes\rho_{\mathcal{B}}} = \mathcal{Q}_{\rho_{\mathcal{A}}} = \mathcal{S}_{|\rho_{\mathcal{A}}\rangle\!\rangle\langle\!\langle\rho_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{\rho} = \mathcal{S}_{\rho_{\mathcal{A}}} \tag{11}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\!\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\mathcal{A}}_{|\psi\rangle} = 2\mathcal{S}^{\mathcal{A}}_{|\psi\rangle} \tag{10}$$

- Product states: $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_{\mathcal{A}}} \otimes \Psi_{\rho_{\mathcal{B}}}$

$$\mathcal{Q}^{\mathcal{A}}_{\rho_{\mathcal{A}}\otimes\rho_{\mathcal{B}}} = \mathcal{Q}_{\rho_{\mathcal{A}}} = \mathcal{S}_{|\rho_{\mathcal{A}}\rangle\!\rangle\langle\!\langle\rho_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{\rho} = \mathcal{S}_{\rho_{\mathcal{A}}} \tag{11}$$

- Maximum entanglement with environment:

  $\rho = I_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\rightarrow\; \Psi_\rho = |\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| \otimes \Psi_{\mathcal{B}}$

$$\mathcal{Q}^{\mathcal{A}}_{I} = \mathcal{S}_{\Omega_{\mathcal{A}}} = \mathcal{S}_{|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{I} = \log d_{\mathcal{A}} \tag{12}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\!\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\mathcal{A}}_{|\psi\rangle} = 2\mathcal{S}^{\mathcal{A}}_{|\psi\rangle} \tag{10}$$

- Product states: $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_{\mathcal{A}}} \otimes \Psi_{\rho_{\mathcal{B}}}$

$$\mathcal{Q}^{\mathcal{A}}_{\rho_{\mathcal{A}}\otimes\rho_{\mathcal{B}}} = \mathcal{Q}_{\rho_{\mathcal{A}}} = \mathcal{S}_{|\rho_{\mathcal{A}}\rangle\!\rangle\langle\!\langle\rho_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{\rho} = \mathcal{S}_{\rho_{\mathcal{A}}} \tag{11}$$

- Maximum entanglement with environment:
$\rho = I_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\rightarrow\; \Psi_\rho = |\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| \otimes \Psi_{\mathcal{B}}$

$$\mathcal{Q}^{\mathcal{A}}_{I} = \mathcal{S}_{\Omega_{\mathcal{A}}} = \mathcal{S}_{|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{I} = \log d_{\mathcal{A}} \tag{12}$$

- Maximum entanglement with $\mathcal{B}$: $\rho = \Omega_{\mathcal{AB}} \rightarrow \Psi_\rho = \Omega_{\mathcal{AB}} \otimes \Omega_{\mathcal{AB}}$

$$\mathcal{Q}^{\mathcal{A}}_{\Omega_{\mathcal{AB}}} = \mathcal{S}_{I_{\mathcal{A}}\otimes I_{\mathcal{A}}} = 2\log d_{\mathcal{A}} \tag{13}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\mathcal{A}}_{|\psi\rangle} = 2\mathcal{S}^{\mathcal{A}}_{|\psi\rangle} \tag{10}$$

- Product states: $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_{\mathcal{A}}} \otimes \Psi_{\rho_{\mathcal{B}}}$

$$\mathcal{Q}^{\mathcal{A}}_{\rho_{\mathcal{A}}\otimes\rho_{\mathcal{B}}} = \mathcal{Q}_{\rho_{\mathcal{A}}} = \mathcal{S}_{|\rho_{\mathcal{A}}\rangle\rangle\langle\langle\rho_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{\rho} = \mathcal{S}_{\rho_{\mathcal{A}}} \tag{11}$$

- Maximum entanglement with environment:
$\rho = I_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \;\rightarrow\; \Psi_\rho = |\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| \otimes \Psi_{\mathcal{B}}$

$$\mathcal{Q}^{\mathcal{A}}_{I} = \mathcal{S}_{\Omega_{\mathcal{A}}} = \mathcal{S}_{|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|} = 0 \;\leq\; \mathcal{S}^{\mathcal{A}}_{I} = \log d_{\mathcal{A}} \tag{12}$$

- Maximum entanglement with $\mathcal{B}$: $\rho = \Omega_{\mathcal{AB}} \rightarrow \Psi_\rho = \Omega_{\mathcal{AB}} \otimes \Omega_{\mathcal{AB}}$

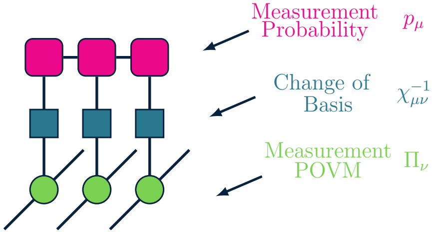
$$\mathcal{Q}^{\mathcal{A}}_{\Omega_{\mathcal{AB}}} = \mathcal{S}_{I_{\mathcal{A}}\otimes I_{\mathcal{A}}} = 2\log d_{\mathcal{A}} \tag{13}$$

- Local operations: $\rho = \sum_\lambda p_\lambda \rho_{\lambda_{\mathcal{A}}} \otimes \rho_{\lambda_{\mathcal{B}}} \;\rightarrow\; \Psi_\rho = \sum_{\lambda\kappa} p_\lambda p_\kappa |\rho_{\lambda_{\mathcal{A}}}\rangle\rangle\langle\langle\rho_{\kappa_{\mathcal{A}}}| \otimes |\rho_{\lambda_{\mathcal{B}}}\rangle\rangle\langle\langle\rho_{\kappa_{\mathcal{B}}}|$

$$\mathcal{Q}^{\mathcal{A}}_{\sum_\lambda p_\lambda \rho_{\lambda_{\mathcal{A}}}\otimes\rho_{\lambda_{\mathcal{B}}}} \geq 0 \tag{14}$$

POVM-MPS for Quantum Simulations

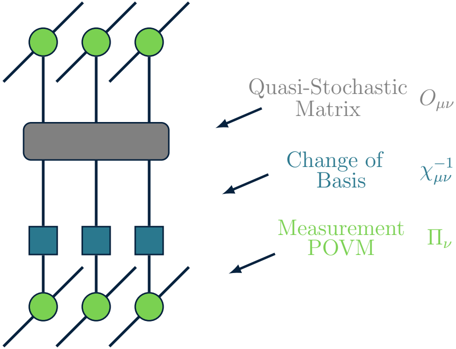
Suppose we model dynamics of states in terms of measurement *probabilities*



$$\rho_{\sigma\sigma'} = \sum_{\mu\nu} p_\mu \, \chi^{-1}_{\mu\nu} \, \Pi_{\nu\sigma\sigma'}$$

POVM-MPS for Quantum Simulations

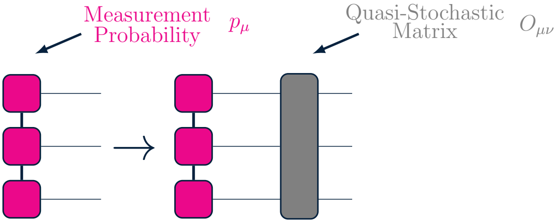
Suppose we model dynamics of states in terms of measurement *probabilities*



$$\Lambda_{\substack{\sigma\sigma' \\ \pi\pi'}} = \sum_{\substack{\mu\nu \\ \eta\kappa}} O_{\mu\nu}\, \chi^{-1}_{\mu\eta}\, \Pi_{\eta\sigma\sigma'}\, \Pi_{\nu\pi\pi'}$$

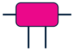
POVM-MPS for Quantum Simulations

Suppose we model dynamics of states in terms of measurement *probabilities*

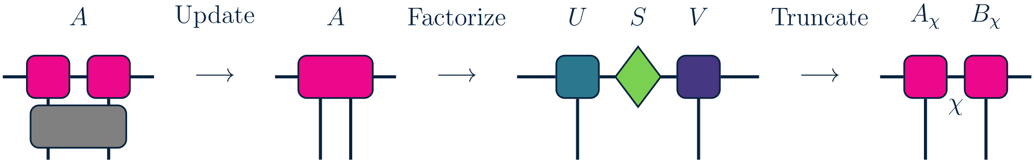


$$p_\mu \to \sum_\nu O_{\mu\nu} p_\nu$$

Tensor Factorizations and Truncations

Factorize tensors $A_{\sigma\pi\alpha\beta}$  , via decompositions/conditional-independence

Truncate virtual bonds to dimension $\chi \ll O(D^N)$



Retain largest *singular-values/probabilities* $S_\chi = \{\lambda_\alpha : \alpha < \chi\}$, with *error* [4]

$$\epsilon_\chi^2 = 1 - \sum_{\alpha<\chi} \lambda_\alpha^2 \quad , \quad \epsilon_\chi = 1 - \sum_{\alpha<\chi} \lambda_\alpha \; . \tag{15}$$

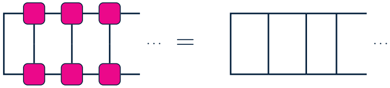
[4] Temme, K. & Verstraete, F., *PRL* **104**, 210502 (2010)

Tensor Factorizations and Truncations

- Tensor Network States

  i.e) MPS: Amplitudes - Singular Value Decomposition (SVD)

$$\psi_\sigma = \prod_i A_{\sigma_i} \quad : \quad |\psi|^2 = \sum_\sigma |\psi_\sigma|^2 = 1$$



(15)

- Tensor Network Probabilities

  i.e) pMPS: Probabilities - Non-Negative Matrix Factorization (NMF)

$$p_\mu = \prod_i A_{\mu_i} \quad : \quad |p| = \sum_\mu p_\mu = 1$$

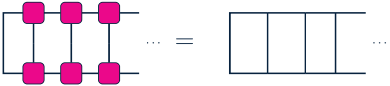
(16)

Tensor Factorizations and Truncations

- Tensor Network States

  i.e) MPS: Amplitudes - Singular Value Decomposition (SVD)

$$\psi_\sigma = \prod_i A_{\sigma_i} \quad : \quad |\psi|^2 = \sum_\sigma |\psi_\sigma|^2 = 1 \tag{15}$$



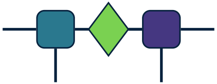
- Tensor Network Probabilities

  i.e) pMPS: Probabilities - Non-Negative Matrix Factorization (NMF)

$$p_\mu = \prod_i A_{\mu_i} \quad : \quad |p| = \sum_\mu p_\mu = 1 \tag{16}$$

Tensor SVD versus NMF Updates

- SVD algorithms $A \to USV \;:\; 0 \not\leq A, U, V \not\leq 1$



  i.e) *Boundaries* may be arbitrary

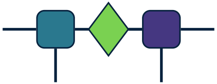
  i.e) *Closed forms*, *Convergence guarantees*, Local unitary *gauge freedom* [6]

  i.e) Power methods, or Arnoldi or Krylov iterations

[6] Boutsidis, C. *et al.*, *Patt. Recog.* **41**, 1350–1362 (2008)

Tensor SVD versus NMF Updates

- SVD algorithms $A \to USV \;\; : \; 0 \not\leq A, U, V \not\leq 1$

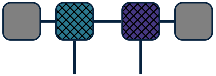


  i.e) *Boundaries* may be arbitrary

  i.e) *Closed forms*, *Convergence guarantees*, Local unitary *gauge freedom*

  i.e) Power methods, or Arnoldi or Krylov iterations

- NMF algorithms $A \to UV \;\; : \; 0 \leq A, U, V \leq 1$



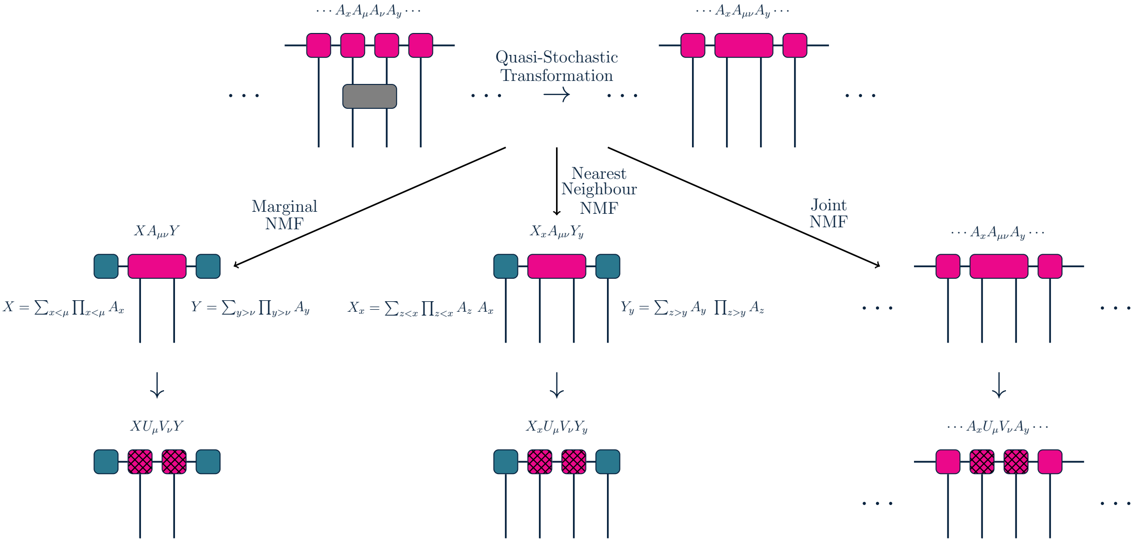
  i.e) *Boundaries* must enforce non-negativity

  i.e) *No closed forms*, *No convergence guarantees*, Inverse *gauge freedom* [7]

  i.e) Multiplicative updates, Alternating least squares, Gradient methods

[7] Cichocki, A. *et al.* (John Wiley and Sons, 2009)

Tensor SVD versus NMF Updates



Tensor SVD versus NMF Updates

$$A = A_{x\mu\nu y} \;=\; X_{x\alpha} Z_{\alpha\mu\nu\beta} Y_{\beta z} \;\approx\; X_{x\alpha} U_{\alpha\mu\gamma} V_{\gamma\nu\beta} Y_{\beta z} \;+\; E_{x\mu\nu y} \;=\; XUVY \;+\; E$$

Frobenius Norm - Multiplicative Updates

$$\|A - XUVY\| \quad \rightarrow \quad \begin{cases} U \rightarrow \frac{A \cdot XVY}{XUVY \cdot XVY} \odot U \\ V \rightarrow \frac{A \cdot XUY}{XUVY \cdot XUY} \odot V \end{cases} \tag{17}$$

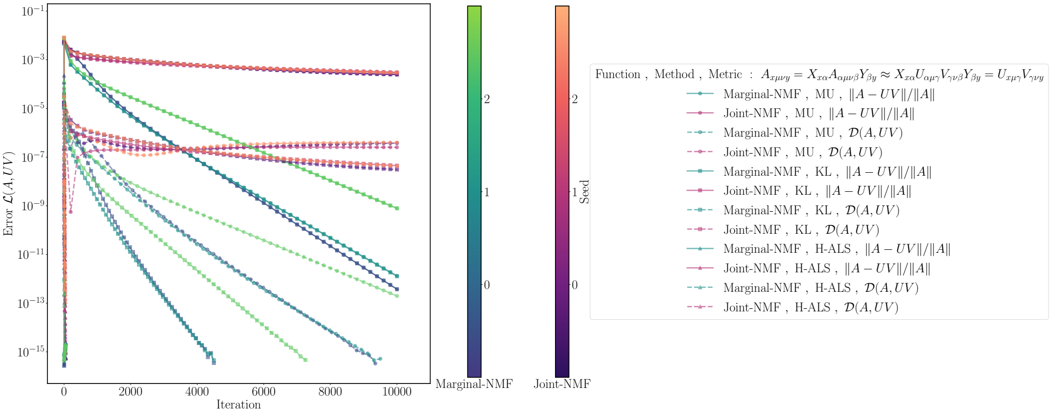
KL Divergence - Multiplicative Updates

$$-A \cdot \log \frac{XUVY}{A} \quad \rightarrow \quad \begin{cases} U \rightarrow \frac{\frac{A}{UV} \cdot V}{1 \cdot V} \odot U \\ V \rightarrow \frac{\frac{A}{UV} \cdot U}{1 \cdot U} \odot V \end{cases} \tag{18}$$

Least Squares - Local Updates

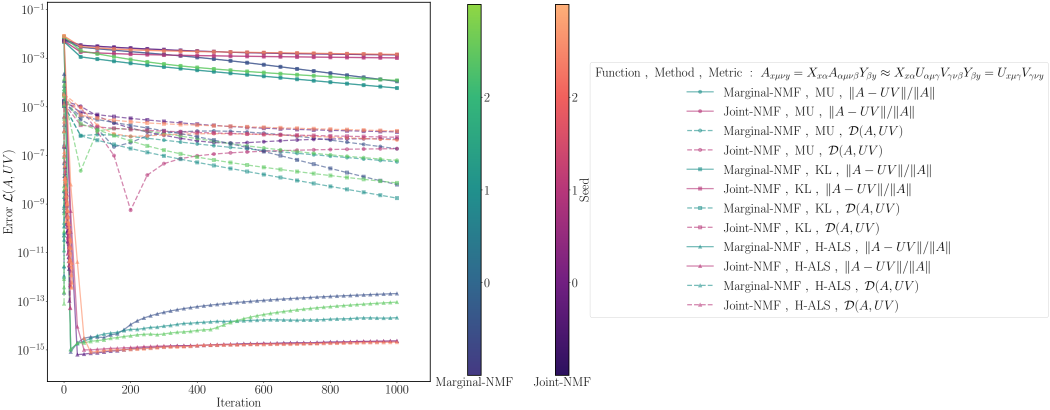
$$\|A - XUVY + X_i U_i V_i Y_i - X_i U_i V_i Y_i\| \quad \rightarrow \quad \begin{cases} U_i \rightarrow \left[\frac{E_i \cdot X_i V_i Y_i}{X_i V_i Y_i \cdot X_i V_i Y_i} \right]_{+\epsilon} \\ V_i \rightarrow \left[\frac{E_i \cdot X_i U_i Y_i}{X_i U_i Y_i \cdot X_i U_i Y_i} \right]_{+\epsilon} \end{cases} \tag{19}$$

Tensor SVD versus NMF Updates



(d) NMF Convergence for Multiplicative, and Hierarchical Least Squares Updates, for Random Tensor Trains of size $D^{N-2} \times D^2 \times D^N$, for $D = 2$, $N = 6$.

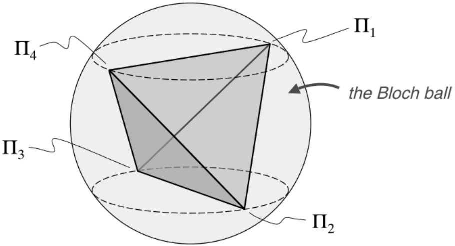
Tensor SVD versus NMF Updates

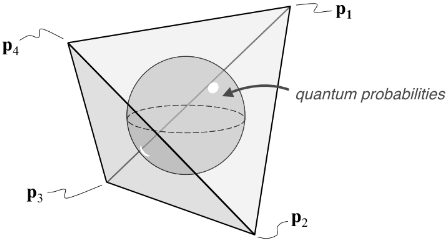


(e) NMF Convergence for Multiplicative, and Hierarchical Least Squares Updates, for Random Tensor Trains of size $D^{N-2} \times D^2 \times D^N$, for $D = 2$, $N = 6$.

Quantum Probability Simplexes

Not only do POVM-MPS updates $p \to p_\chi$ have to be *non-negative*, but they must obey non-trivial quantum probability simplex *constraints* [5]





[5] Appleby, D. M. *et al.*, *Found. Phys.* **41**, 564–579 (2009)

Quantum Probability Simplexes

Not only do POVM-MPS updates $p \to p_\chi$ have to be *non-negative*, but they must obey non-trivial quantum probability simplex *constraints*

i.e) Symmetric POVM's: $\chi^{-1}_{\mu\nu} = \xi \delta_{\mu\nu} \; - \; 1_\mu 1_\nu \quad : \quad \xi = D(D+1)$

$$\sum_\mu p_\mu = 1 \quad , \quad 1 \; \leq \; \xi \sum_\mu p_\mu^2 \; \leq \; 2 \quad , \quad 1 \; \leq \; \xi^2 \sum_{\mu\nu\upsilon} \chi_{\mu\nu\upsilon} \, p_\mu p_\nu p_\upsilon \; \leq \; 4 \quad (20)$$

i.e) Tensor Products of N SIC-POVM's: $\Pi_\mu \to \Pi_{\mu\sigma\omega\ldots} = \otimes_i^N \Pi_{\mu_i}$

i.e) Quasi-stochastic transformations $O_{\mu\nu}$ are *stochastic* for *unital* channels

Quantum Probability Simplexes

Tensor Products of N SIC-POVM's: $\Pi_\mu \rightarrow \Pi_{\mu\sigma\omega\ldots} = \otimes_i^N \Pi_{\mu_i}$

$N = 1$:

$$\begin{gathered}
\rho = \xi \; p_\mu \Pi_\mu \;\; - \;\; I \\
O_{\mu\nu} = \xi \; \langle\!\langle \Pi_\mu | \Lambda(\Pi_\nu) \rangle\!\rangle \; - \; \langle\!\langle \Pi_\mu | \Lambda(I) \rangle\!\rangle 1_\nu \\
1_\mu p_\mu = 1 \\
1 \;\leq\; \xi \; p_\mu p_\mu \;\leq\; 2 \\
1 \;\leq\; 2\xi \; p_\mu p_\mu - 1 \;\leq\; \xi^2 \; \chi_{\mu\nu\upsilon} p_\mu p_\nu p_\upsilon \;\leq\; 3\xi \; p_\mu p_\mu - 2 \;\leq\; 4
\end{gathered} \tag{21}$$

Quantum Probability Simplexes

Tensor Products of N SIC-POVM's: $\Pi_\mu \rightarrow \Pi_{\mu\sigma\omega\ldots} = \otimes_i^N \Pi_{\mu_i}$

$N = 2$:

$$\rho = \xi^2 \; p_{\mu\sigma}\Pi_{\mu\sigma} \;-\; \xi \; p_\mu \Pi_\mu \;-\; \xi \; p_\sigma \Pi_\sigma \;+\; I$$

$$O_{\mu\sigma\nu\pi} = \xi^2 \langle\!\langle \Pi_{\mu\sigma}|\Lambda(\Pi_{\nu\pi})\rangle\!\rangle \;-\; \xi \langle\!\langle \Pi_{\mu\sigma}|\Lambda(\Pi_\nu)\rangle\!\rangle 1_\pi \;-\; \xi \langle\!\langle \Pi_{\mu\sigma}|\Lambda(\Pi_\pi)\rangle\!\rangle 1_\nu \;+\; \langle\!\langle \Pi_{\mu\sigma}|\Lambda(I)\rangle\!\rangle 1_{\nu\pi}$$

$$1_{\mu\sigma} p_{\mu\sigma} = 1 \tag{22}$$

$$1 \;\le\; \xi\,(p_\mu p_\mu + p_\sigma p_\sigma) - 1 \;\le\; \xi^2 \; p_{\mu\sigma}p_{\mu\sigma} \;\le\; \xi\,(p_\mu p_\mu + p_\sigma p_\sigma) \;\le\; 4$$

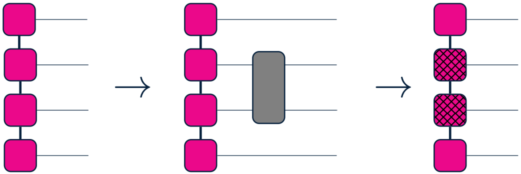
$$2 - 5\xi\,(p_\mu p_\mu + p_\sigma p_\sigma) \;\le\; \xi^4 \chi_{\mu\nu\upsilon}\chi_{\sigma\pi\tau} p_{\mu\sigma} p_{\nu\pi} p_{\upsilon\tau} \;-\; 2\xi^3\,(\chi_{\mu\nu\upsilon} p_{\mu\sigma} p_{\nu\sigma} p_\upsilon + \chi_{\sigma\pi\tau} p_{\mu\sigma} p_{\mu\pi} p_\tau)$$

$$5 - 6\xi\,(p_\mu p_\mu + p_\sigma p_\sigma) \;\ge$$

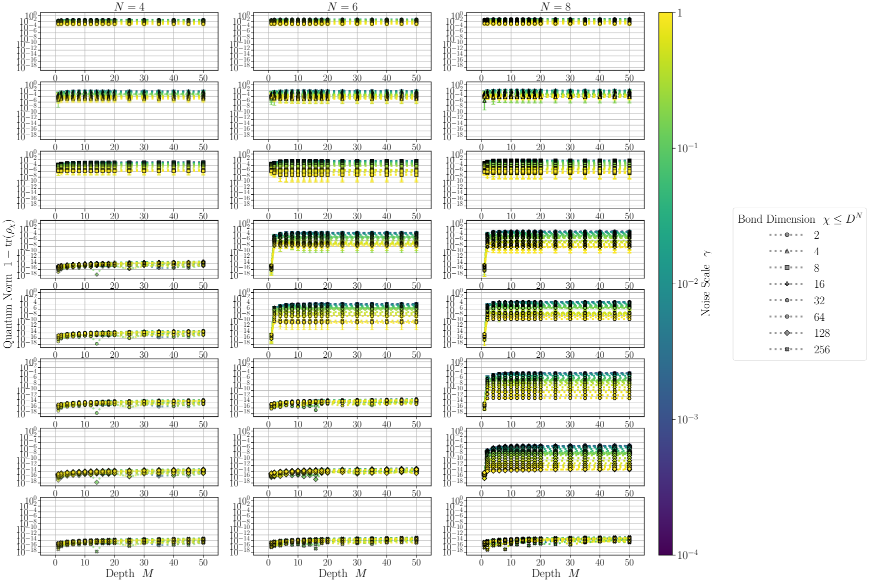
Quantum Probability Simplexes

Not only do POVM-MPS updates $p \rightarrow p_\chi$ have to be *non-negative*, but they must obey non-trivial quantum probability simplex *constraints*

What other structure does $O_{\mu\nu}$ have i.e) for 2-local unitaries that allows for *efficient*, *convergent* updates?

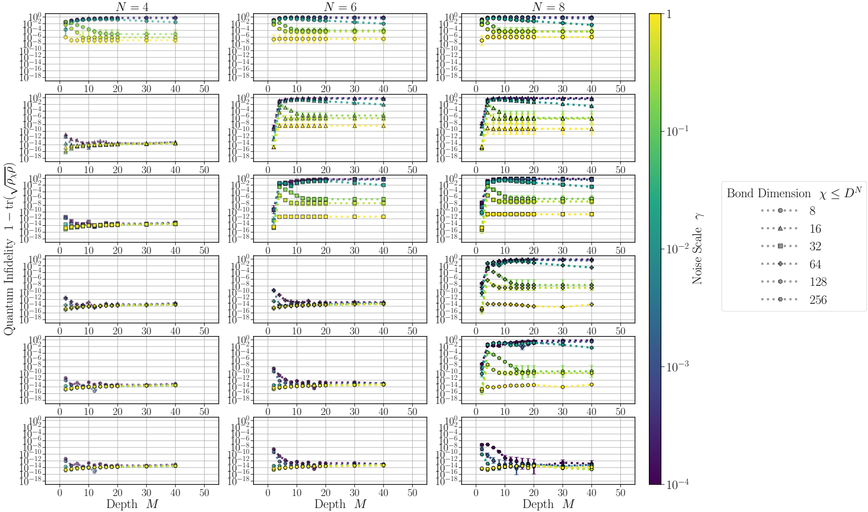


POVM-MPS Norm



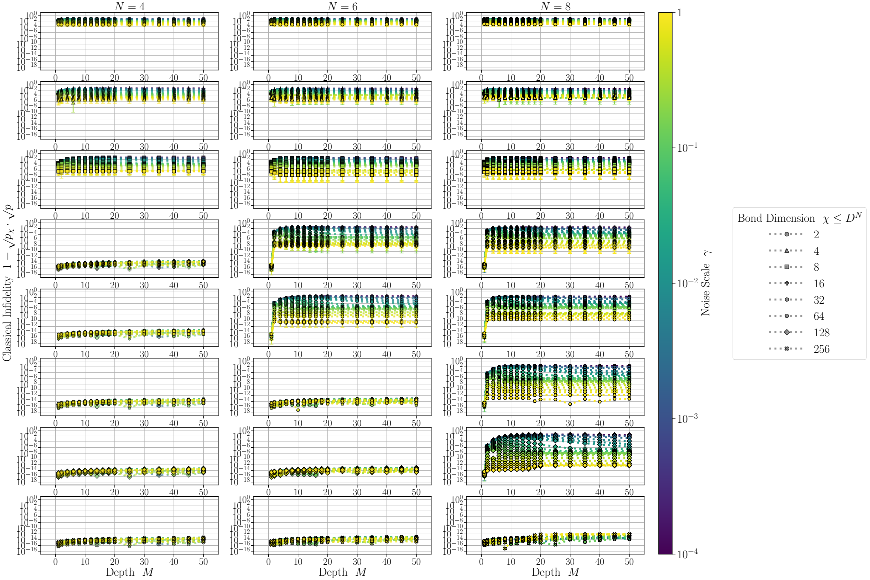
(f) Trace of SVD-POVM-MPS ρ_χ for N qubits.

POVM-MPS Infidelity



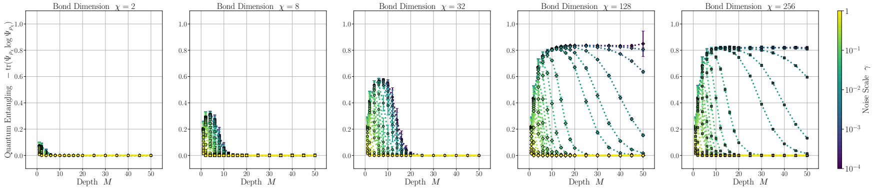
(g) Quantum Infidelity of POVM-MPS $\rho_\chi \approx \rho$ for N qubits.

POVM-MPS Infidelity



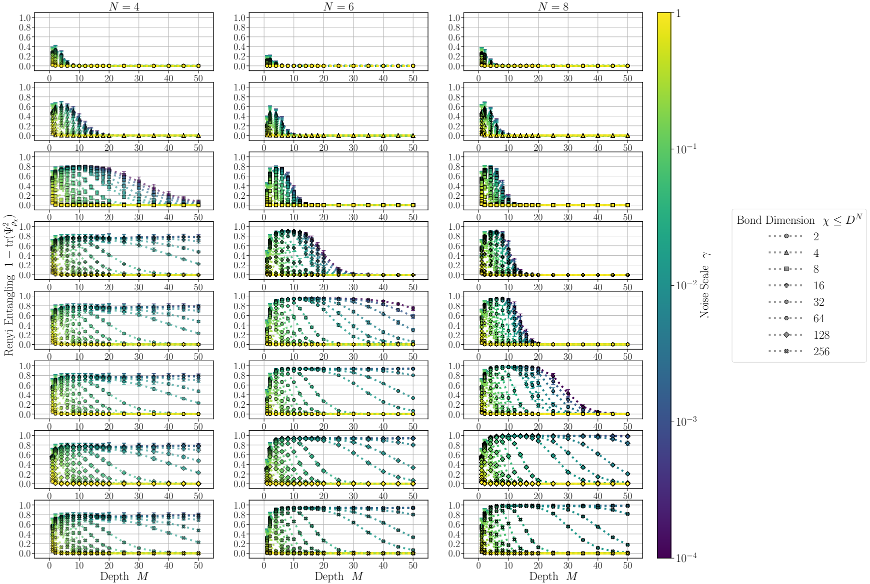
(h) Classical Infidelity of POVM-MPS $p_\chi \approx p$ for N qubits.

POVM-MPS Operator Entanglement



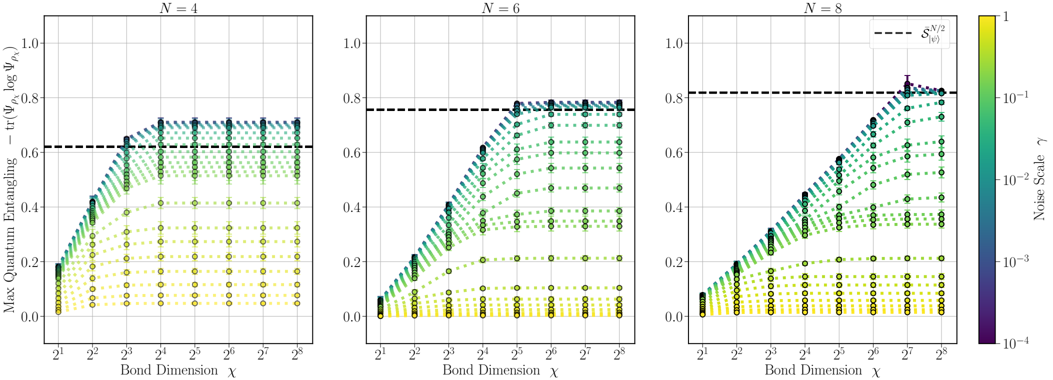
(i) Quantum Operator Entanglement of SVD-POVM-MPS ρ_χ for N qubits.

POVM-MPS Operator Entanglement



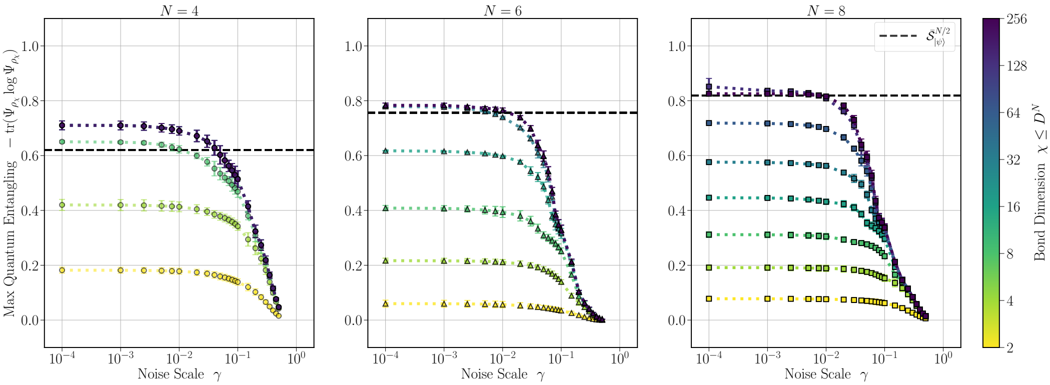
(j) Renyi Operator Entanglement of SVD-POVM-MPS ρ_χ for N qubits.

POVM-MPS Operator Entanglement Scaling



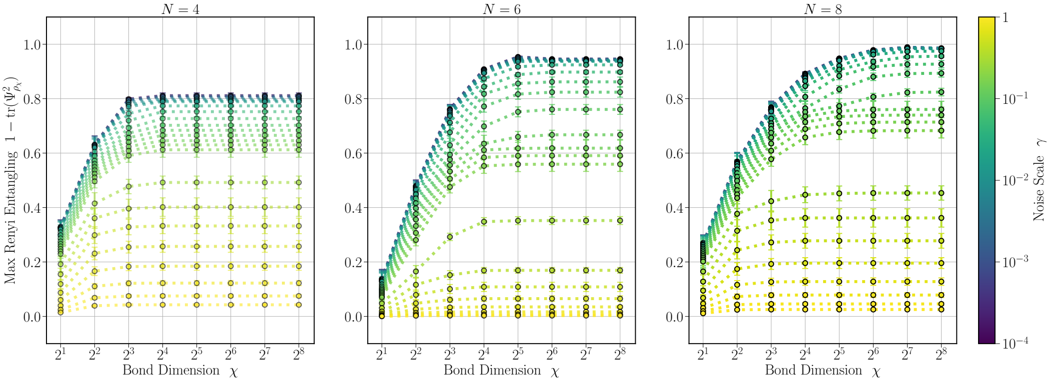
(k) Quantum Operator Entanglement of SVD-POVM-MPS ρ_χ with bond dimension χ, for N qubits.

POVM-MPS Operator Entanglement Scaling



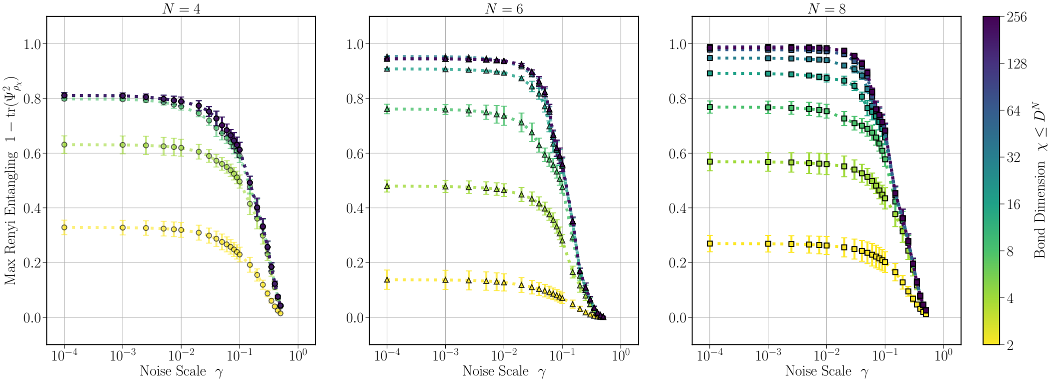
(1) Quantum Operator Entanglement of SVD-POVM-MPS ρ_χ with noise scale γ, for N qubits.

POVM-MPS Operator Entanglement Scaling



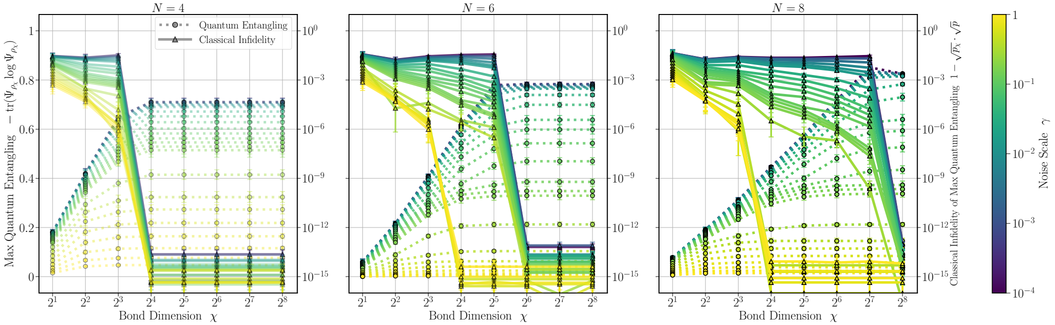
(m) Renyi Operator Entanglement of SVD-POVM-MPS ρ_χ with bond dimension χ, for N qubits.

POVM-MPS Operator Entanglement Scaling



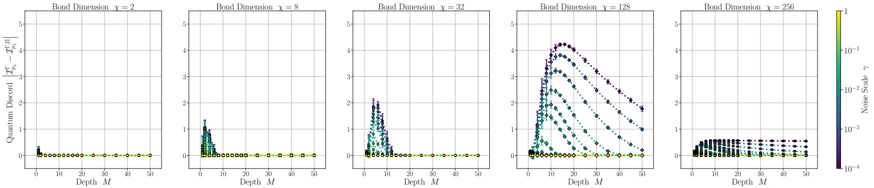
(n) Renyi Operator Entanglement of SVD-POVM-MPS ρ_χ with noise scale γ, for N qubits.

POVM-MPS Operator Entanglement Scaling



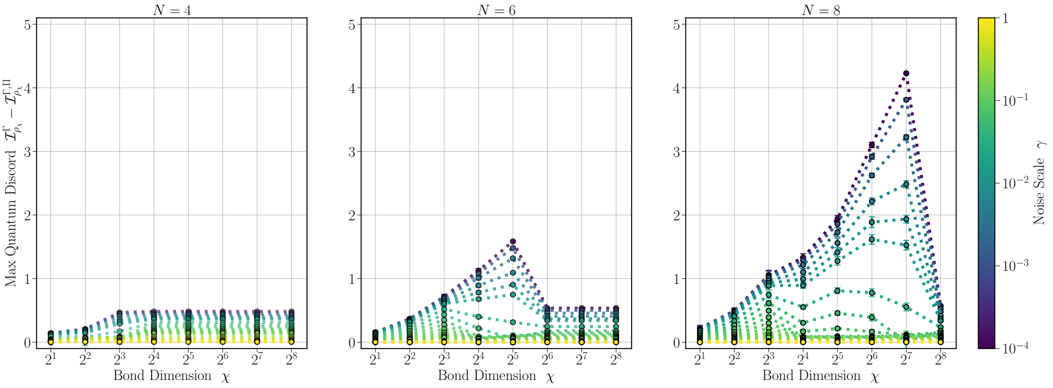
(o) Quantum Operator Entanglement and Classical Infidelity of SVD-POVM-MPS ρ_χ with bond dimension χ, for N qubits.

POVM-MPS Quantum Discord



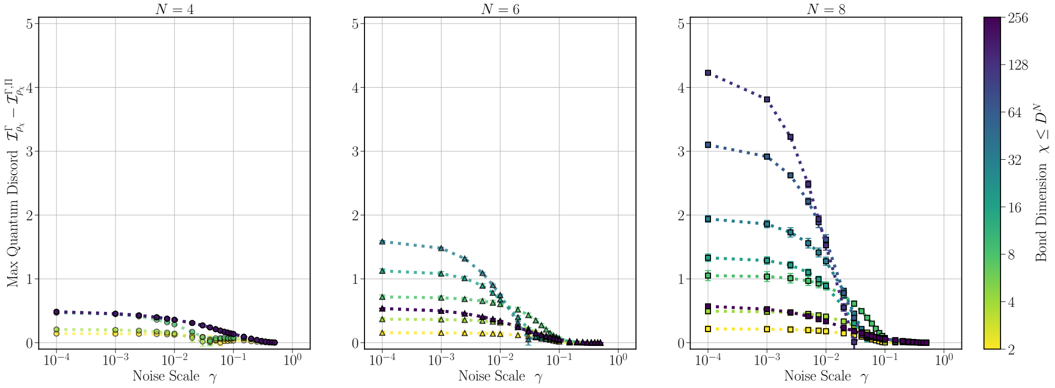
(p) Quantum Discord of SVD-POVM-MPS ρ_χ for N qubits.

POVM-MPS Quantum Discord Scaling



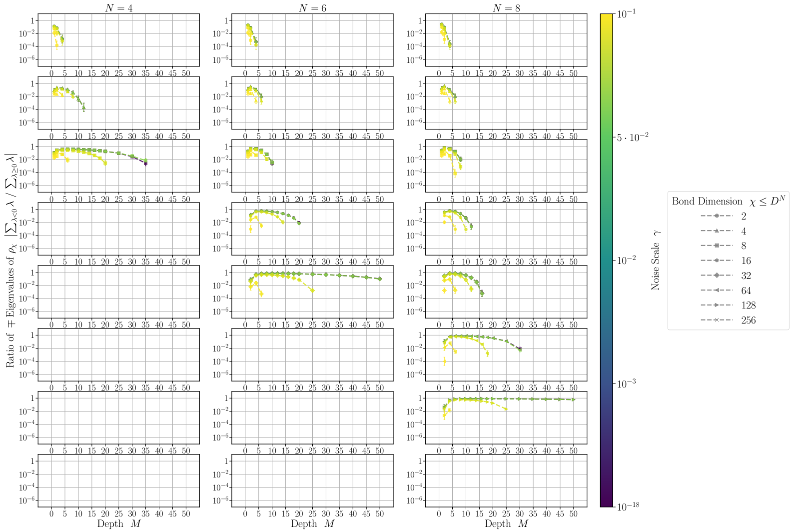
(q) Quantum Discord of SVD-POVM-MPS ρ_χ with bond dimension χ, for N qubits.

POVM-MPS Quantum Discord Scaling



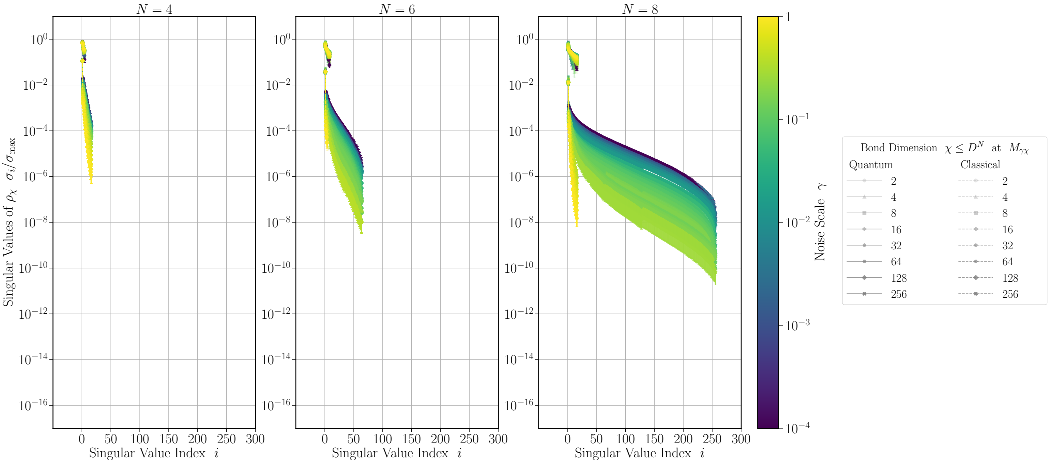
(r) Quantum Discord of SVD-POVM-MPS ρ_χ with noise scale γ, for N qubits.

POVM-MPS Quantum and Classical Spectrum Scaling



(s) Ratio of Negative/Positive Spectrum of SVD-POVM-MPS ρ_χ with noise scale γ, bond dimension χ, as a function of depth M, for N qubits.

POVM-MPS Quantum and Classical Spectrum Scaling



(t) Quantum and Classical Spectrum of SVD-POVM-MPS ρ_χ with noise scale γ, bond dimension χ, for N qubits.