Overparameterization and Expressivity of Realistic Quantum Systems

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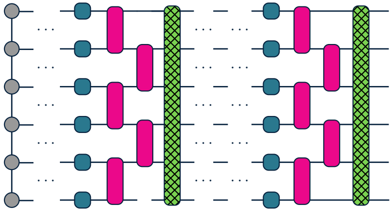
University of Waterloo, Institute for Quantum Computing, ETH Zurich, & Los Alamos National Laboratory

Seminar



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Fall, 2024

Quantum Tasks Of Interest

Ultimately, we want to do something *useful* with our quantum devices

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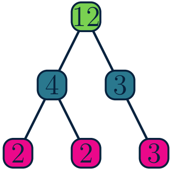

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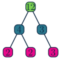
- *Quantum algorithms* i.e) Factoring numbers 

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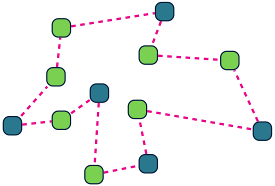

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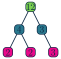
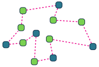
- *Quantum algorithms* i.e) Factoring numbers 

- *Optimization* problems i.e) Travelling Salesman Problem 
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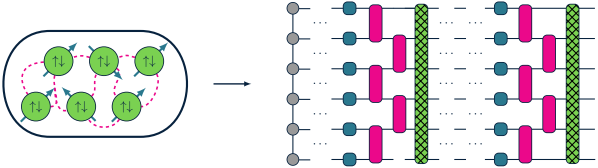

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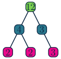
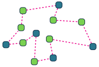
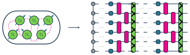
- *Quantum algorithms* i.e) Factoring numbers 

- *Optimization* problems i.e) Travelling Salesman Problem 
- *Compilation* tasks i.e) Form operators U given native gates $\{V\}$ 
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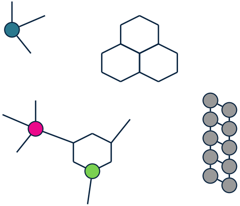

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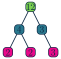

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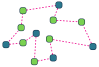

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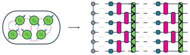

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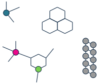

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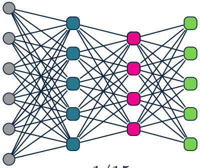

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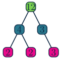

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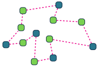

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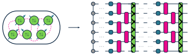

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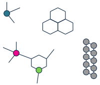

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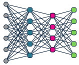
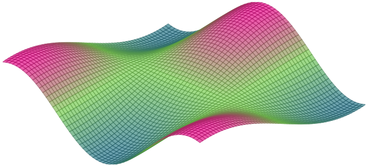
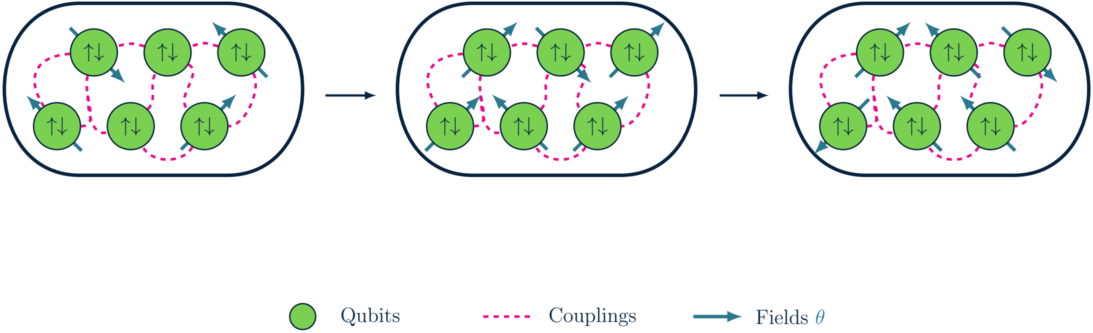




What Are Parameterized Quantum Systems?



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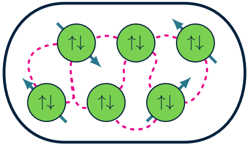
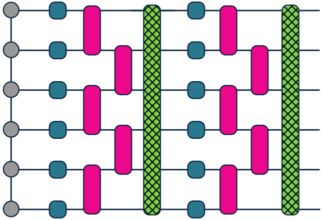


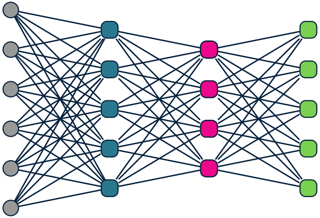
Quantum System

$H_\theta = \sum_\mu \theta_\mu G_\mu$



Quantum Circuit

$U_\theta = \prod_\mu U_\theta^\mu$

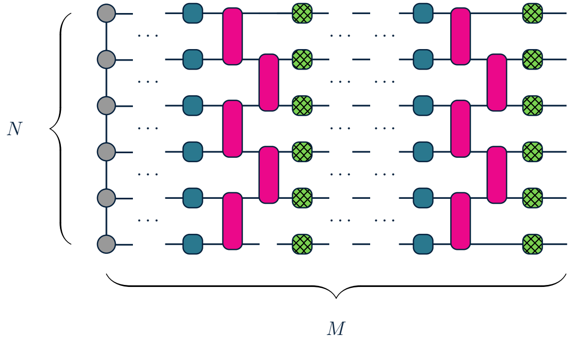


Classical Algorithm

$f_\theta = \circ_\mu f_\theta^\mu$

Tasks of Interest: Unitary Compilation, State Preparation

Learning Phenomena of Quantum Systems



Let our parameterized ansatz consist of unitary $\mathcal{U}_\theta$ and noise $\mathcal{N}_\gamma$ components

$$\Lambda_{\theta\gamma} = \bigcirc_{\mu}^{M} \mathcal{N}_{\gamma}^{(\mu)} \circ \mathcal{U}_{\theta}^{(\mu)} \tag{1}$$

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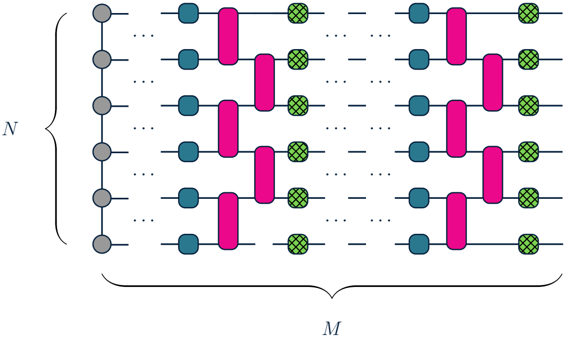

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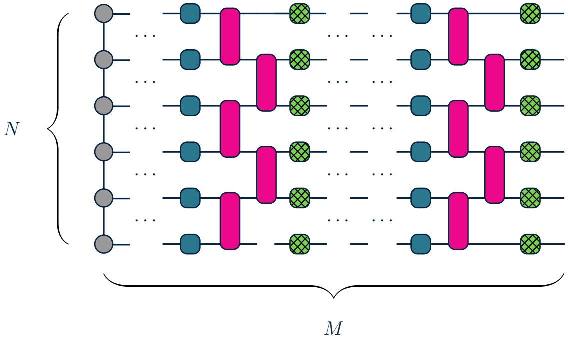
$$U_\theta^{(\mu)} = e^{-i\delta H_\theta^{(\mu)}} \;:\; H_\theta^{(\mu)} = \sum_\nu \theta_\nu^{(\mu)} G_\nu \tag{2}$$

$$\mathcal{N}_\gamma^{(\mu)} = \otimes_i^N \mathcal{N}_{\gamma_i} \;:\; \mathcal{N}_{\gamma_i} = (1-\gamma)\mathcal{I}_i + \gamma \mathcal{K}_i \tag{3}$$



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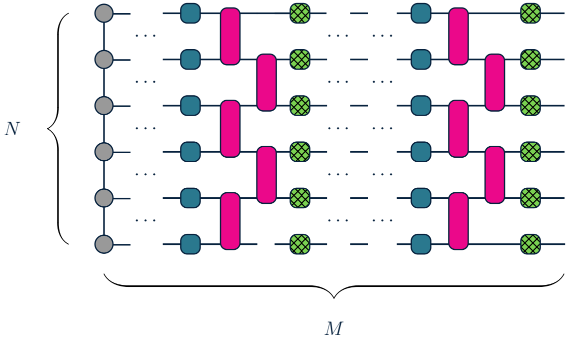
$$\text{NMR: } H_\theta^{(\mu)} = \sum_i^N \theta_i^{x(\mu)} X_i + \sum_i^N \theta_i^{y(\mu)} Y_i + \sum_i^N h_i Z_i + \sum_{i<j}^N J_{ij} Z_i Z_j \tag{4}$$



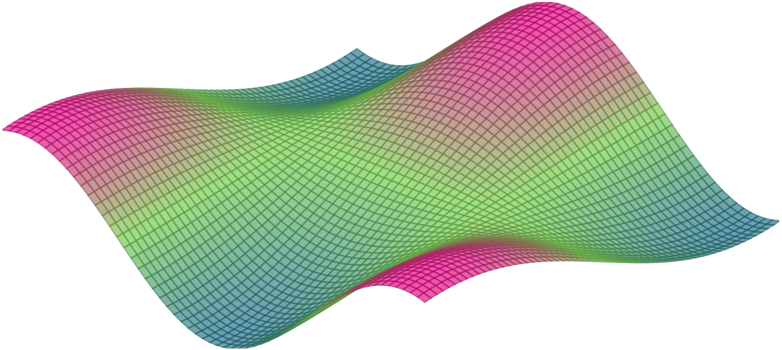
How does the amount of *noise* γ and the *evolution depth* M

of a *constrained* system

affect its classical simulation and optimization, and resulting infidelities

$$\mathcal{L}_{\theta^*_{\gamma}} \; : \; U_{\theta_\gamma} \approx U \;,\; \rho_{\theta_\gamma} \approx \rho \;?$$

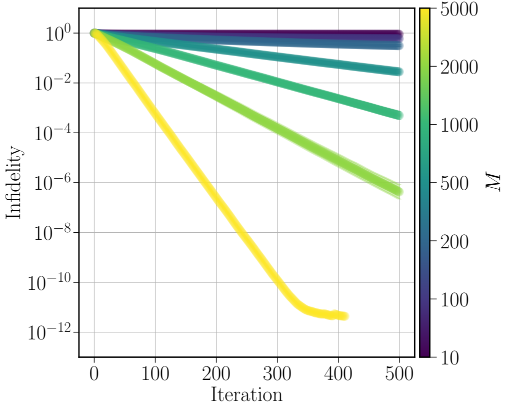
Learning Phenomena of Quantum Systems



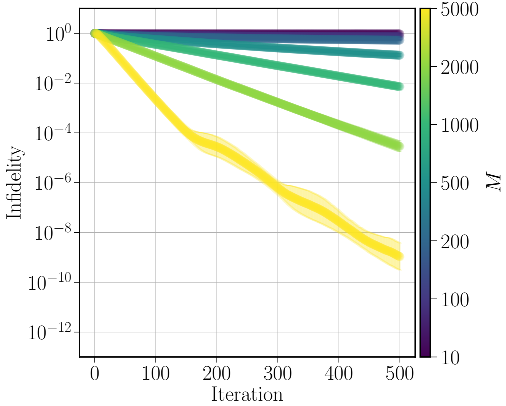
How can we leverage approaches from
quantum optimal control and *learning theory* to describe these relationships?

$$\textit{Infidelity}\text{: } 1 - \operatorname{tr}\left(\rho\rho_{\theta\gamma}\right)\ ,\ \textit{Impurity}\text{: } 1 - \operatorname{tr}\left(\rho_{\theta\gamma}^2\right)\ ,\ \textit{Entropy}\text{: } -\operatorname{tr}\left(\rho_{\theta\gamma}\log\rho_{\theta\gamma}\right)$$

Unconstrained vs. Constrained Optimization

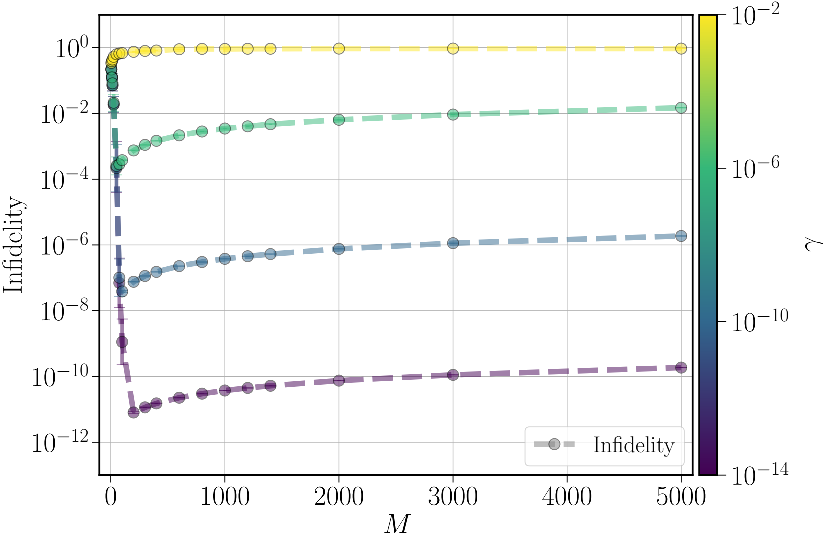


(a) Unconstrained Unitary Compilation



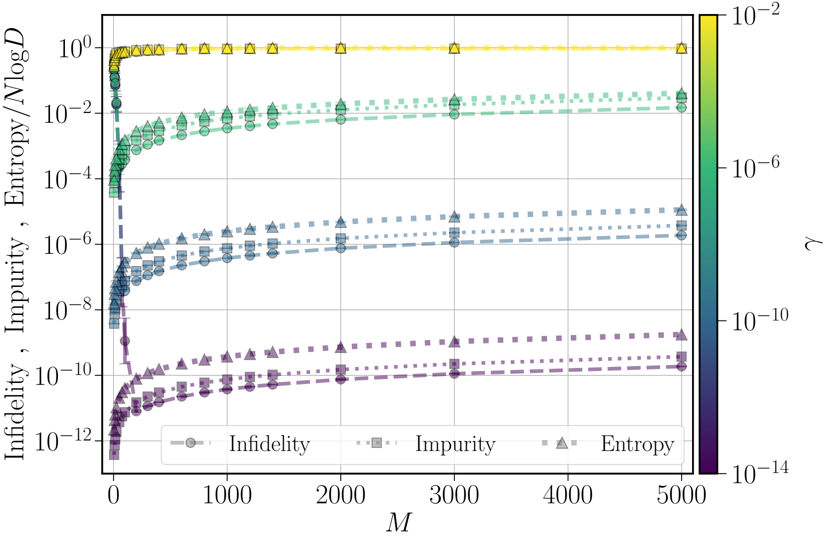
(b) Constrained Unitary Compilation

Regimes of Noisy Optimization



(c) Unital Dephasing for State Preparation

Regimes of Noisy Optimization



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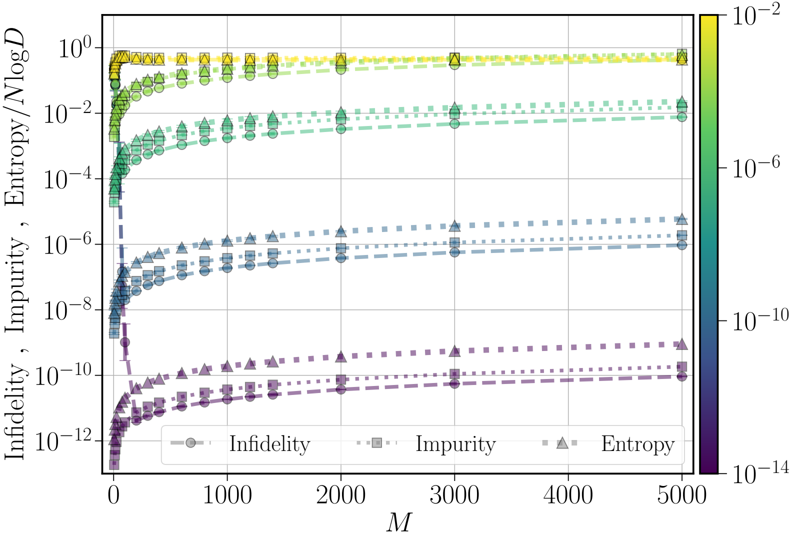

(e) Non-Unital Amplitude Damping for State Preparation

Noise Induced Critical Depth

Noise induces a critical depth (Fontana *et al.* PRA **104** (2021))

$$M_\gamma \sim \log 1/\gamma \ , \tag{5}$$

meaning the minimum infidelity is *linear-quadratic* $(1 \leq \alpha \leq 2)$ in noise

$$\mathcal{L}_{\theta^*\gamma|M_\gamma} \sim \gamma^\alpha \ , \tag{6}$$

and parameterized noise channels are therefore *robust* tof approximately

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Is it possible to derive the M, γ scaling of the optimal $\mathcal{L}_{\theta^{*}\gamma}$ *analytically*?

$$|\mathcal{L}_{\theta_{\gamma}} - \mathcal{L}_{\theta}| \leq 2|(1-\gamma)^{NM} - 1| \tag{8}$$

Representations of Channels and States

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- States can be represented as *Bloch* coefficients $\rho_{\theta\gamma} \approx \rho \;\longleftrightarrow\; \lambda_{\theta\gamma} \approx \lambda$

$$\rho_{\theta\gamma} \;=\; \frac{I + \lambda_{\theta\gamma} \cdot \omega}{d} \;=\; (1-\gamma)^K \rho + (1 - (1-\gamma)^K)\epsilon_{\theta\gamma} + \Delta_{\theta\gamma} \tag{10}$$

Representations of Channels and States

- Quantities of interest at *noiseless* optimality scale remarkably similarly

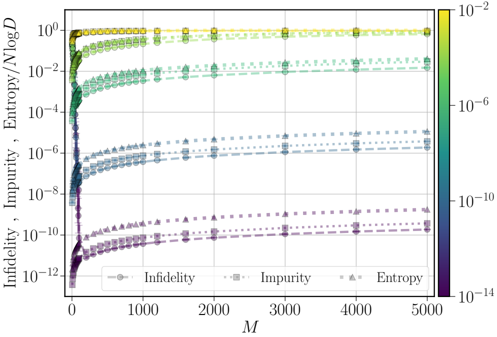
$$\mathcal{L}^{\rho}_{\theta\gamma} \sim \frac{1}{2}\mathcal{I}_{\theta\gamma} \sim \boxed{K\gamma\frac{d-1}{d}\left(1-\frac{\lambda\cdot\varepsilon_{\theta\gamma}}{\lambda^2}\right)} + O(\tbinom{K}{2}\gamma^2) \tag{11}$$

$$\mathcal{S}_{\theta\gamma} \sim \mathcal{D}^{\rho}_{\theta\gamma} \sim O(K\gamma) \tag{12}$$

Representations of Channels and States

Noise phenomena dominates at $M \geq M_{\gamma}$:
The scale of optimization and entropic driven infidelities *intersect*

$$\mathcal{L}^{\rho}_{\theta^*_{\gamma}\gamma} \sim e^{-\alpha M}|_{M_{\gamma}} \approx \mathcal{L}^{\rho}_{\theta^*{\gamma}} \sim NM\gamma|_{M_{\gamma}} \ , \tag{13}$$



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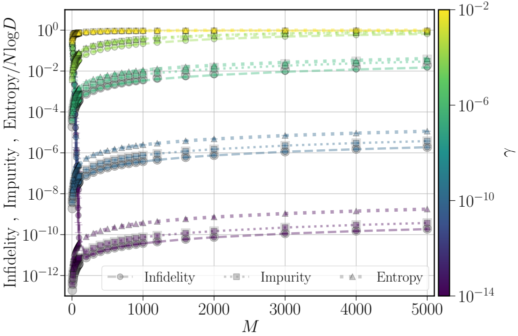
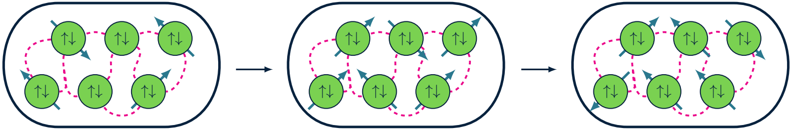
and we recover our numerically predicted noise-induced critical depth!

$$M_\gamma \sim \log 1/\gamma \tag{14}$$

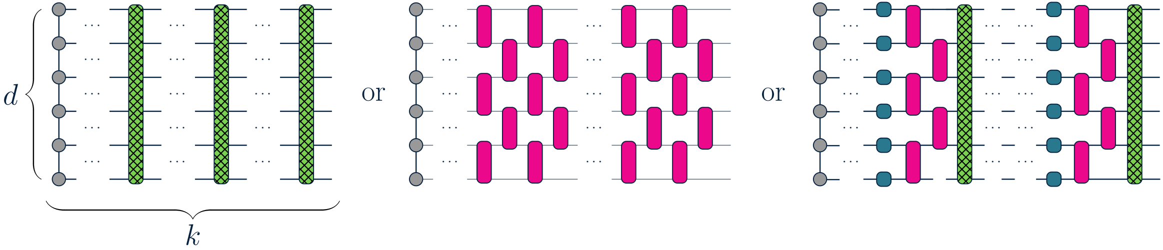


What Have We Learned About Noisy Overparameterization?

- Overparameterization is *robust* to constraints
- Accumulation of noise induces a *critical* depth M_γ that prevents convergence
- Fidelities, purities, entropies highly *correlated* in $\gamma \ll 1, M \gg 1$ regime
- What are other *noise-induced* effects on *trainability* versus *expressivity*?



How Expressive are our Ansatze?



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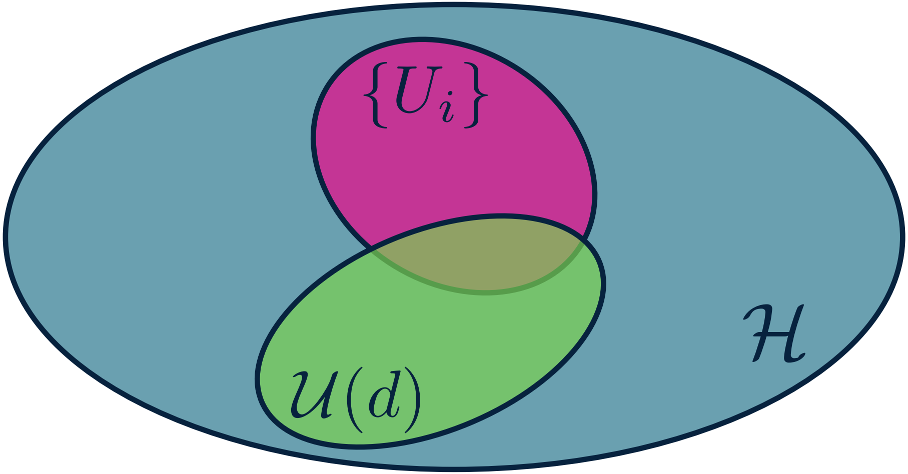

[1] Holmes, Z. *et al.*, *PRX Quantum* **3**, 010313 (2022)

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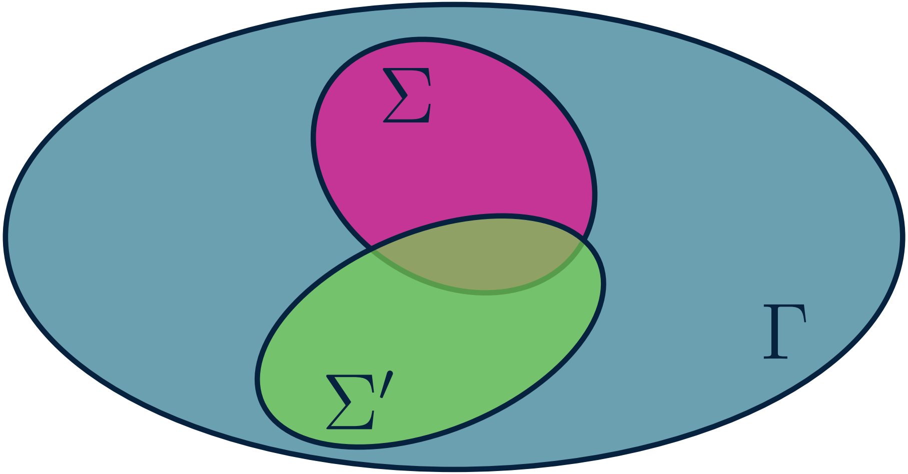

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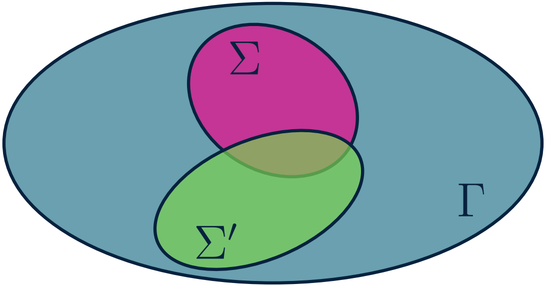
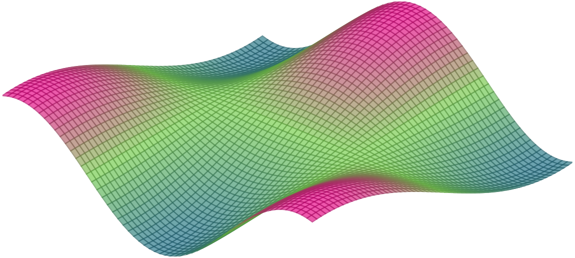
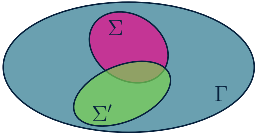




How Expressive are our Ansatze?

- Expressivity and trainability of *unitary ansatze* are well understood [1]



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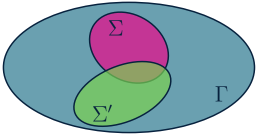

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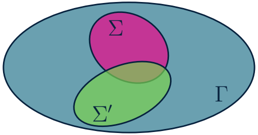
- How does an ansatz compare to a *maximally expressive* reference ansatz?
- How do generalized expressivity measures for channels depend on:
  - Noise induced phenomena
  - Underlying (parameterized) unitary evolution
  - Coupling with the environment



Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

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$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) \;=\; \boxed{\underbrace{\frac{\mathrm{tr}\,(\cdot)}{d^t} I}_{\text{Depolarizing}}} \;+\; \boxed{\underbrace{\Delta_{\Sigma}^{(t)}(\cdot)}_{\text{Deviations}}} \tag{17}$$

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- *Haar* ∼ Unitary Haar measure (uniformly random unitaries) [1]

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}(\rho)} = \int_{\mathcal{U}(d_{\mathcal{H}})} dU \; U^{\otimes t} \; \rho^{\otimes t} \; U^{\otimes t\,\dagger} \tag{19}$$

[1] Holmes, Z. *et al.*, *PRX Quantum* **3**, 010313 (2022)

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$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\mathrm{cHaar}}}(\rho)} = \mathrm{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(d_{\mathcal{H}} d_{\mathcal{E}})} dU \; U^{\otimes t} \; \rho^{\otimes t} \otimes \; \nu_{\mathcal{E}}^{\otimes t} \; U^{\otimes t\,\dagger}\right) \tag{20}$$

[2] Kukulski, R. *et al.*, *J. Math. Phys.* **62**, 062201 (2021)

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- *Depolarizing* $\sim$ Maximally Depolarizing (single channel)

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}(\rho)} = \frac{\operatorname{tr}\left(\rho^{\otimes t}\right)}{d_{\mathcal{H}}^{t}} \; I^{\otimes t} \tag{21}$$

Behaviour of Random Quantum Channels

The t-order Haar, cHaar, Depolarize ensembles are related by

$$\lim_{d_{\mathcal{E}} \to 1} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}} \qquad \lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}} \to \infty} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}} \tag{22}$$

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The k-concatenated, t-order cHaar ensemble is *depolarizing* and *non-unital* [3]

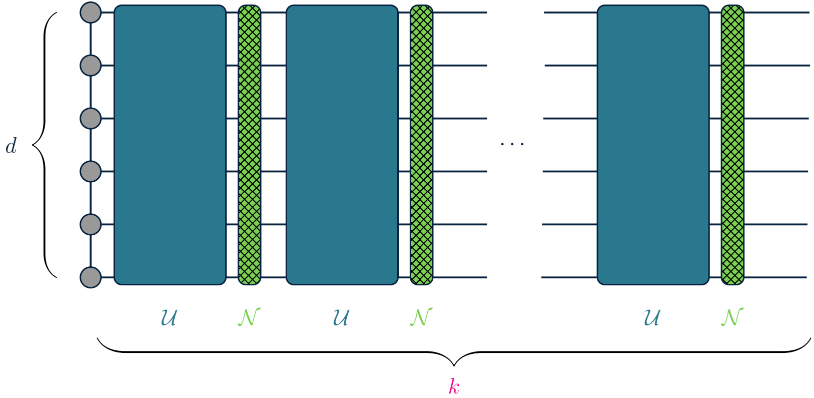
$$\lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}}\to\infty} \mathcal{T}^{(t)k}_{\Sigma_{\text{cHaar}}}(\rho) = \boxed{\underbrace{\frac{\operatorname{tr}\left(\rho^{\otimes t}\right)}{d^t_{\mathcal{H}}} I^{\otimes t}}_{\text{Depolarize}}} + \boxed{\underbrace{O\left(\frac{1}{d^2_{\mathcal{H}} d_{\mathcal{E}}}\right) \sum_{P\neq I^{\otimes t}} P}_{\text{Non-Unital}}} \tag{23}$$

[3] Bai, J. *et al.*, *Quantum Information Processing* **23**, 1–18 (2024)

Relationships between Noise and Expressivity

Analytical *expressivities* for k layers of channels

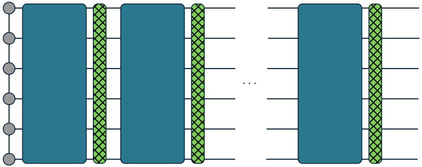
$$\Lambda_{\mathcal{U}_\gamma}^{(k)} = (\mathcal{N}_\gamma \circ \mathcal{U})^k = \frac{\operatorname{tr}(\cdot)}{d} I + \Delta_\gamma^{(k)}(\cdot) \tag{24}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + Fixed *Unital* Noise: *Increases* Expressivity

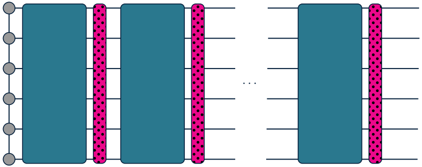
$$\mathcal{E}_{\mathcal{U}_{\gamma}}^{(t,k)2} = \binom{t}{2} (1-\gamma)^{4k} + O\left((1-\gamma)^{6k}\right) \tag{25}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + Fixed *Non-Unital* Noise: *Decreases* Expressivity

$$\mathcal{E}_{\mathcal{U}\gamma\eta}^{(t,k)2} = t\left(d^2 - 1\right)\ \eta^2\ +\ O\left(\eta^4\right) \tag{26}$$



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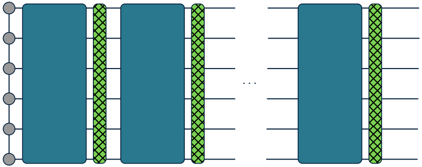
Parameterized Random Unitaries + Fixed *Unital* Noise: *Increases* Expressivity

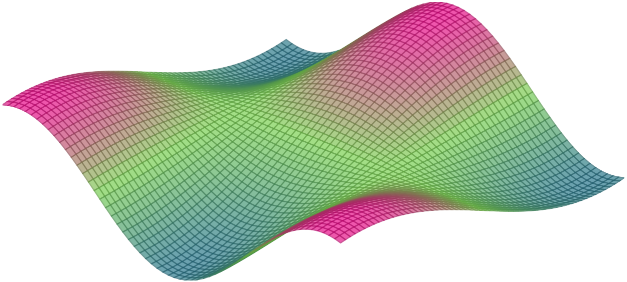
$$\mathcal{E}_{\mathcal{G}\gamma}^{(t,k)2} = t|\mathcal{S}_G \setminus \{I\}| \left(1-\gamma\right)^{2k} + O\left(\left(1-\gamma\right)^{2k+2}\right) \tag{27}$$



Relationships between Noise and Expressivity

Objective $\mathcal{L}$ and Gradient $\partial\mathcal{L}$ *Concentration*: $p\left(|\mathcal{L} - \mu_{\mathcal{L}}| \geq \epsilon\right) \leq \sigma_{\mathcal{L}}^2/\epsilon^2$

$$\mathcal{L}(\rho, O) = \mathrm{tr}\left(O\Lambda(\rho)\right)$$



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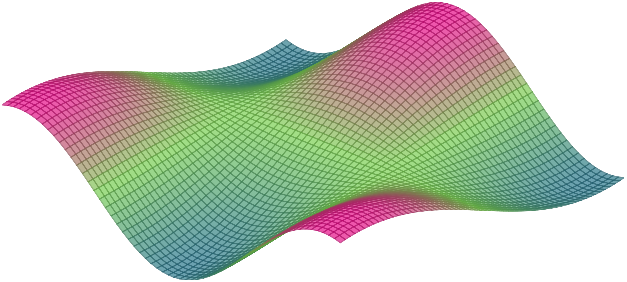
$$\sigma_{\mathcal{L}}^2\sim O\left(\frac{1}{d_{\mathcal{H}}d_{\mathcal{E}}}\right)\;+\;\|O\|_2^2\,\mathcal{E}_{\Sigma\Sigma'}^{(2)}(\rho)\quad(\text{with } \textit{caveats} \text{ on } \Sigma',\rho,O \text{ locality})\tag{28}$$



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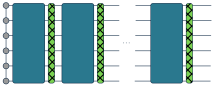

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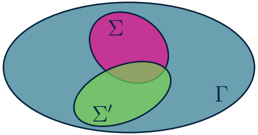
Channels versus Unitary Ensembles

Many *subtle* differences between ensembles of channels and unitaries



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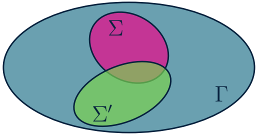

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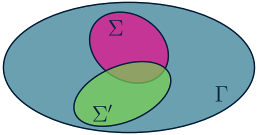

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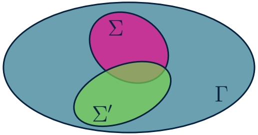

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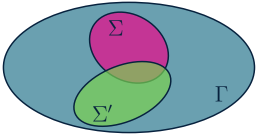
- Subtleties in realizing channel t-designs in *practice*



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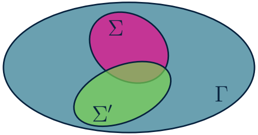
Ensembles of channels have inherently different *interpretations*



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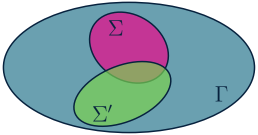

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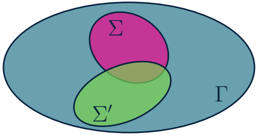

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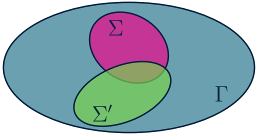

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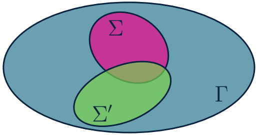
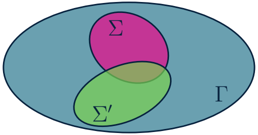
- *Tomography*: Depolarizing channels maximize *uncertainty* in measurements
- *Scrambling*: Depolarizing channels maximally *scramble* information



Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually channel *expressivity* phenomena!



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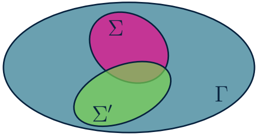

Operational Meaning of Channel Expressivity Measures

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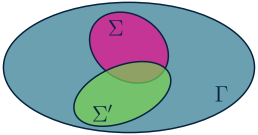
- Channel expressivity is more subtly related to *usefulness* or *capability*
- Are there relationships between channel expressivity and their *simulability*?



Appendix

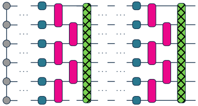
How May We Control Quantum Systems?

- Represented as *channels* $\Lambda_{\theta_\gamma} = \mathcal{N}_\gamma \circ \mathcal{U}_\theta$ with unitary evolution $\mathcal{U}_\theta$, and noise $\mathcal{N}_\gamma$
- Evolution generated by Hamiltonians with localized generators $\{G_\mu\}$

$$H_\theta^{(\mu)} = \sum_\nu \theta_\nu^{(\mu)} G_\nu \;\rightarrow\; U_\theta \approx \prod_{\mu,\nu}^{M} U_\theta^{(\mu,\nu)} \;:\; U_\theta^{(\mu,\nu)} = e^{-i\delta H_\theta^{(\mu,\nu)}} \approx e^{-i\delta\theta_\nu^{(\mu)} G_\nu} \tag{29}$$

i.e) *NMR* with variable transverse fields and constant longitudinal fields (Peterson *et al.* , PRA **13** (2020)) (Coloured in circuit ↘)

$$H_\theta^{(\mu)} = \sum_i \theta_i^{x(\mu)} X_i + \sum_i \theta_i^{y(\mu)} Y_i + \sum_i h_i Z_i + \sum_{i<j} J_{ij} Z_i Z_j \tag{30}$$

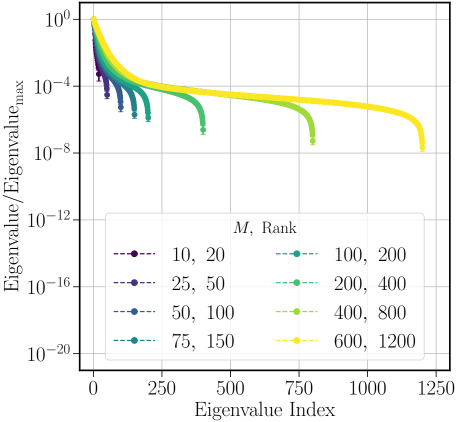


- Noise generated by constant *Kraus* operators $\{\mathcal{K}_{\gamma_\alpha}\}$
  i.e) Dephasing $\{\sqrt{1-\gamma}\; I,\; \sqrt{\gamma}\; Z\}$

$$\rho \;\rightarrow\; \rho_{\Lambda_{\theta_\gamma}} = \circ_\mu^M \mathcal{N}_\gamma \circ \mathcal{U}_\theta^{(\mu)}(\rho) = \circ_\mu^M \left[\sum_\alpha \mathcal{K}_{\gamma_\alpha}\, U_\theta^{(\mu)}\, \rho\, {U_\theta^{(\mu)}}^\dagger\, {\mathcal{K}_{\gamma_\alpha}}^\dagger\right] \tag{31}$$

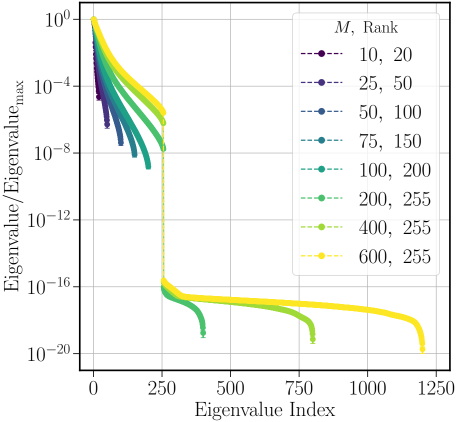
Overparameterization Phenomena

- *Overparameterized* regime is reached with constraints for sufficient depth $M > O(G)$ (Dynamical Lie Algebra $\mathcal{G}_{\mathrm{NMR}}$, with dimension $G = 2^{2N} - 1$)



(h) Hessian Rank Saturation

$$\mathcal{H}_{\mu\nu} = \partial_{\mu\nu}\mathcal{L}_\theta$$

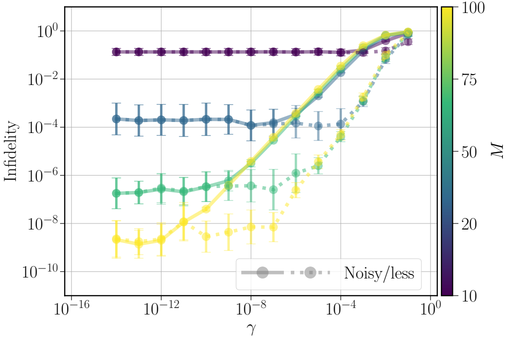


(i) Fisher Information Rank Saturation

$$\mathcal{F}_{\mu\nu} = \frac{1}{d}\mathrm{tr}\left(\partial_\mu U_\theta^\dagger \partial_\nu U_\theta\right) - \frac{1}{d^2}\mathrm{tr}\left(\partial_\mu U_\theta^\dagger U_\theta\right)\mathrm{tr}\left(U_\theta^\dagger \partial_\nu U_\theta\right)$$

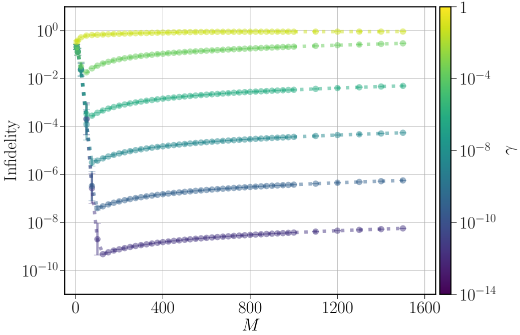
Noisy Optimization

- Haar random state preparation for $N = 4$ qubits, with independent dephasing



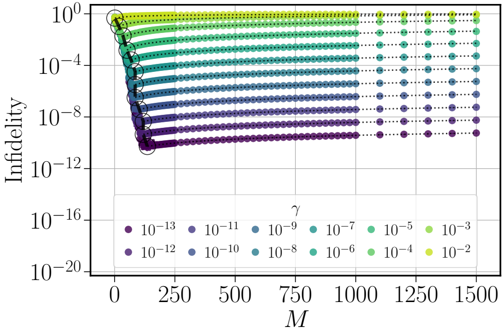
(j) Trained Noisy Infidelity, and Tested Infidelity of Noisy Parameters in Noiseless Ansatz

$\partial\mathcal{L}_{\theta\gamma} \sim \sum_\eta \alpha_\eta \, \mathcal{L}_{\theta+\eta\ \gamma}$

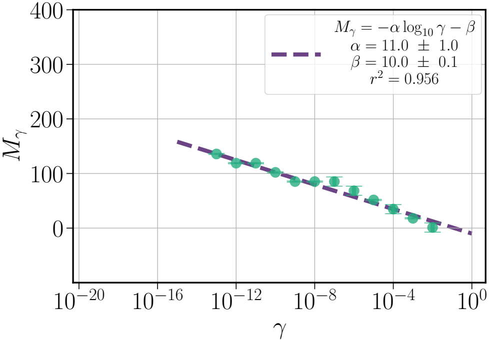


(k) Critical Depth for Noisy Infidelity

Noise Induced Critical Depth



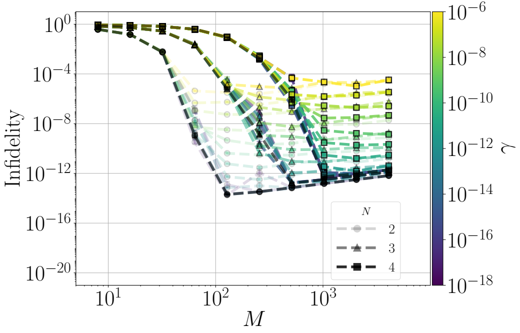
(l) Piecewise Fit of Noisy Infidelity



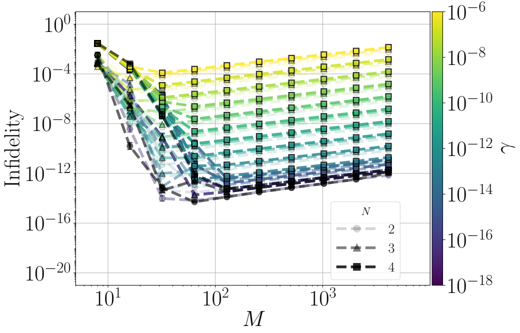
(m) Linear-Log Fit of Critical Depth

Universal Effects of Noise

- Effects of infidelities on noise for Haar random targets in $d = D^N$ dimensions



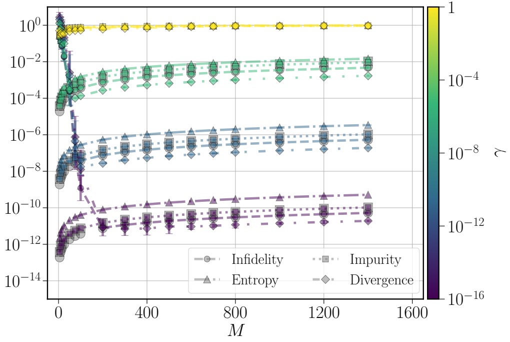
(n) Classical floating point noise for unitary compilation
$|\mathcal{L}_{\theta_\gamma} - \mathcal{L}_\theta| \leq |(1+\gamma)^{O(2NM)} - 1|$



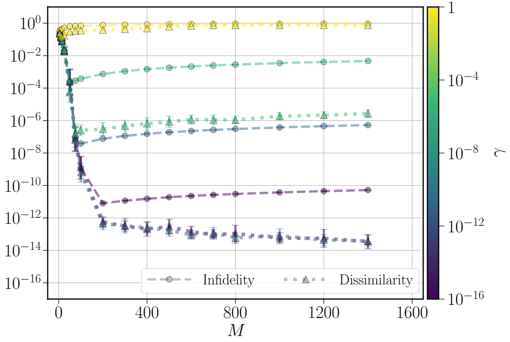
(o) Quantum dephasing noise for state preparation
$|\mathcal{L}_{\theta_\gamma} - \mathcal{L}_\theta| \leq 2|(1-\gamma)^{NM} - 1|$

Correlated Quantities

- Haar random state preparation for $N = 4$ qubits, with independent dephasing

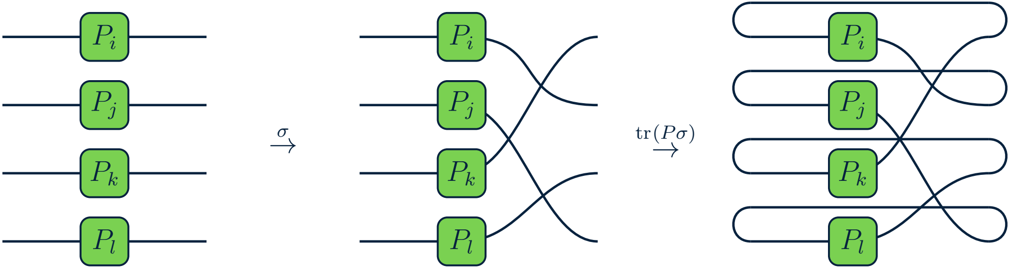


(p) Impurity, Entropy, Divergence



(q) Cosine Dissimilarity

Diagrammatic Expansions of Permutations



$$P = P_i \otimes P_j \otimes P_k \otimes P_l \qquad \sigma = (ijlk) \qquad \mathrm{tr}\left(P\sigma\right) = \mathrm{tr}\left(\prod_{\mu\in\sigma} P_\mu\right)$$

$$\boxed{\tau = \frac{1}{d}\sum_{P\in\mathcal{P}_d} P\otimes P^{-1}} \quad \rightarrow \quad \boxed{\sigma = \frac{1}{d^{|\sigma|}}\sum_{P\in\mathcal{P}_d^{(\sigma)}} P} \tag{32}$$

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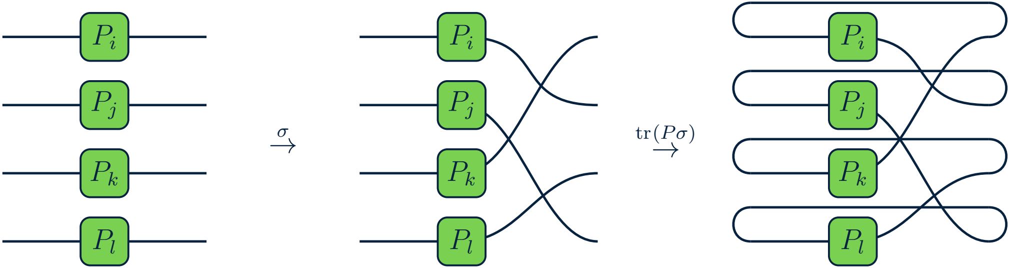
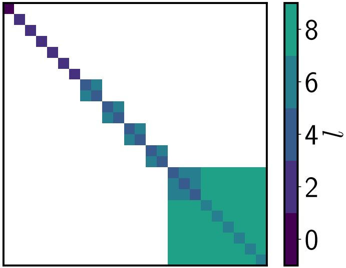


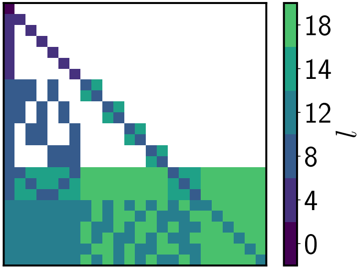
$$P = P_i \otimes P_j \otimes P_k \otimes P_l \qquad \sigma = (ijlk) \qquad \mathrm{tr}\,(P\sigma) = \mathrm{tr}\left(\prod\nolimits_{\mu\in\sigma} P_\mu\right)$$

$$\mathcal{T}_{\Sigma}^{(t)} = \boxed{\frac{1}{d^t}\sum_{\sigma,\pi\in\mathcal{S}_{\Sigma}^{(t)}} \tau_d^{(t)}(\sigma,\pi)|\sigma\rangle\langle\pi|} = \boxed{\frac{1}{d^t}|I\rangle\langle I| + \frac{1}{d^t}\sum_{P\in\mathcal{P}_d^{(\mathcal{S}_{\Sigma}^{(t)})}\setminus\{I\}} \tau_d^{(t)}(P,S)\,|P\rangle\langle S|} \tag{33}$$

Twirl Expansion Coefficients



(r) Haar Twirl Cycle Operator Coefficients $\tau^{(t)}_{\mathbb{U}(d)}(P,S) \sim O(1/d^l)$ for $t=4$



(r) cHaar Twirl Cycle Operator Coefficients $\tau_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})}(P,S) \sim O(1/d^l)$ for $t=4$

$$\mathcal{T}^{(t)}_{\Sigma} = \frac{1}{d^t} \sum_{P,S \in \mathcal{S}^{(t)}_{\Sigma}} \tau^{(t)}_{\Sigma}(P,S)|P\rangle\langle S| \tag{34}$$

Haar, cHaar, and Depolarizing Ensembles

$\Sigma \backslash t$	1	2
Haar	$\frac{1}{d_{\mathcal{H}}}\|I\rangle\!\langle I\|$	$\frac{1}{d_{\mathcal{H}}^2}\|I\rangle\!\langle I\| + \frac{1}{d_{\mathcal{H}}^2}\frac{1}{d_{\mathcal{H}}^2-1}\sum_{P,S\in\mathcal{P}_{d_{\mathcal{H}}}^{(\tau)}\setminus\{I\}}\|P\rangle\!\langle S\|$
cHaar	$\frac{1}{d_{\mathcal{H}}}\|I\rangle\!\langle I\|$	$\frac{1}{d_{\mathcal{H}}^2}\|I\rangle\!\langle I\| + \frac{1}{d_{\mathcal{H}}^2}\frac{d_{\mathcal{E}}-1}{d_{\mathcal{H}}^2 d_{\mathcal{E}}^2-1}\sum_{P\in\mathcal{P}_{d_{\mathcal{H}}}^{(\tau)}\setminus\{I\}}\|P\rangle\!\langle I\| + \frac{1}{d_{\mathcal{H}}^2}\frac{d_{\mathcal{E}}}{d_{\mathcal{H}}^2 d_{\mathcal{E}}^2-1}\sum_{P,S\in\mathcal{P}_{d_{\mathcal{H}}}^{(\tau)}\setminus\{I\}}\|P\rangle\!\langle S\|$
Depolarize	$\frac{1}{d_{\mathcal{H}}^t}\|I\rangle\!\langle I\|$	

Table 1: Twirls $\mathcal{T}_{\Sigma}^{(t)}$ for various ensembles and moments

Monotonic Convergence and Hierarchy of cHaar Twirl Norms

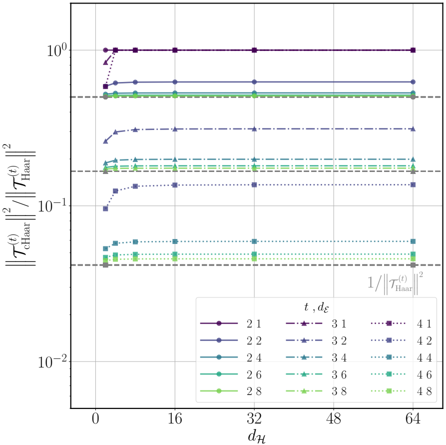


Figure 10: cHaar t-order twirl norms convergence with $d_{\mathcal{H}}, d_{\mathcal{E}}$ towards $1/\|\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}\|^2$.

$$\boxed{1 = \|\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Depolarize}}}\|^2 \leq \|\mathcal{T}^{(t)k}_{\Sigma_{\mathrm{cHaar}}}\|^2 \leq \|\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}\|^2 = |\mathcal{S}_t|} \tag{35}$$

Subtleties with Concentration of Channel Ansatz

Given $\Lambda \sim \mathcal{U}_\theta$ for $U_\theta = e^{-i\theta G}$, with involutory generators G and pure inputs ρ: Objective $\mathcal{L}_\Lambda$ variance concentrates as

$$\sigma^2_{\mathcal{L}} \sim O\left(\frac{1}{d_{\mathcal{H}} d_{\mathcal{E}}}\right) + \|O\|_p^2 \, \mathcal{E}^{(2|q)}_{\Sigma\Sigma'}[\rho] \tag{36}$$

$$\sigma^2_{\mathcal{L}_\Lambda|\Sigma}[\rho, O] \leq \begin{cases} O\left(\frac{d_{\mathcal{O}}}{d_{\mathcal{E}}} \frac{1}{d^2_{\mathcal{H}}}\right) \; + \; \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \; \mathcal{E}^{(2|p)}_{\Sigma\Sigma'} & \{O_{\text{Pauli}} \; , \; \Sigma'_{\text{cHaar}}\} \\ O\left(\frac{1}{d^2_{\mathcal{H}}}\right) \; + \; \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \; \mathcal{E}^{(2|p)}_{\Sigma\Sigma'} & \{O_{\text{Projector}} \; , \; \Sigma'_{\text{cHaar}}\} \\ \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \; \mathcal{E}^{(2|p)}_{\Sigma\Sigma'} & \left\{O_{\text{Pauli}} \; , \; \Sigma'_{\text{Depolarize}}\right\} \\ O\left(\frac{1}{d^2_{\mathcal{H}}}\right) \; + \; \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \; \mathcal{E}^{(2|p)}_{\Sigma\Sigma'} & \left\{O_{\text{Projector}} \; , \; \Sigma'_{\text{Depolarize}}\right\} \end{cases} . \tag{37}$$

Subtleties with Concentration of Channel Ansatz

Given $\Lambda \sim \mathcal{U}_\theta$ for $U_\theta = e^{-i\theta G}$, with involutory generators G and pure inputs ρ:
Objective gradient $\partial_\mu \mathcal{L}_\Lambda$ variance concentrates as

$$\sigma^2_{\partial_\mu \mathcal{L}} \sim O\left(\frac{1}{d_{\mathcal{H}} d_{\mathcal{E}}}\right) + O\left(\mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\ \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O]\right) \tag{38}$$

$$\sigma^2_{\partial_\mu \mathcal{L}_\Lambda|\Sigma\Sigma'_{RL}}[\rho, O] \leq \sigma^{2RL}_{\partial_\mu \mathcal{L}_\Lambda|\Sigma'_{\mu_{RL}}}[\rho, O] + \begin{cases} \min_{\frac{1}{p}+\frac{1}{q}=1} O\left(\frac{1}{d^2_{\mathcal{H}} d^2_{\mathcal{E}}}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho] + & \{O_{\text{Orthogonal}}\ ,\ \Sigma'_{\text{cHaar}}\} \\ O\left(\frac{1}{d^2_{\mathcal{H}} d_{\mathcal{E}}}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[O] + 4\ \mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\ \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O] & \\ \min_{\frac{1}{p}+\frac{1}{q}=1} O\left(\frac{1}{d^3_{\mathcal{H}} d^2_{\mathcal{E}}}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho] + & \{O_{\text{Projector}}\ ,\ \Sigma'_{\text{cHaar}}\} \\ O\left(\frac{1}{d^2_{\mathcal{H}} d_{\mathcal{E}}}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[O] + 4\ \mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\ \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O] & \\ \min_{\frac{1}{p}+\frac{1}{q}=1} 4\ \mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\ \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O] & \left\{\Sigma'_{\text{Depolarize}}\right\} \end{cases} \tag{39}$$

where the *left* (L) and *right* (R) 2-design gradient variance is

$$\sigma^{2RL}_{\partial_\mu \mathcal{L}_\Lambda|\Sigma'_{\mu_{RL}}}[\rho, O] = \begin{cases} O\left(\frac{1}{d_{\mathcal{H}} d^2_{\mathcal{E}}}\right) & \{O_{\text{Orthogonal}}\ ,\ \Sigma'_{\text{cHaar}}\} \\ O\left(\frac{1}{d^2_{\mathcal{H}} d^2_{\mathcal{E}}}\right) & \{O_{\text{Projector}}\ ,\ \Sigma'_{\text{cHaar}}\} \\ 0 & \left\{\Sigma'_{\text{Depolarize}}\right\} \end{cases} \tag{40}$$