1 Coordinate Transformations

1.1 Curvilinear Coordinates

We will investigate general coordinate basis transformations

$$(x, y, z, \dots) \to (x', y', z', \dots) \tag{1}$$

between old tuples of coordinates $(\mu) = (x, y, z, \dots)$, and new tuples of coordinates $(\mu') = (x', y', z', \dots)$ in d dimensions. This description is very general, and not necessary to be understood, but describes general steps for performing coordinate transformations, of identifying coordinate functions to represent the relationships between individual coordinates, determining consistent basis unit vectors for the transformed coordinates in terms of the distance vector, and determining components of vectors and gradients in transformed coordinates. Examples for specific coordinate systems are below.

Here, arbitrary individual coordinates from the d-length tuples (μ) or (μ'), are denoted with letters $\mu, \nu, \dots$ or $\mu', \nu', \dots$. Basis vectors from the set of d coordinate basis vectors $\{\vec{\mu}\}$ or $\{\vec{\mu}'\}$, are denoted as $\vec{\mu}, \vec{\nu}, \dots$ or $\vec{\mu}', \vec{\nu}', \dots$. Unless denoted explicitly, arguments to functions $f(\mu)$ are assumed to be functions of all coordinates (old (μ) or new (μ'), depending on whether the variables are primed). Partial derivatives with respect to specific coordinates are denoted by $\partial_\mu = \partial/\partial\mu$. Finally sums over all d coordinates in the tuple are denoted as $\sum_\mu = \sum_{\mu \in \{x,y,z,\dots\}}$.

When considering the basis unit vectors $\{\hat{\mu}\}$ or $\{\hat{\mu}'\}$, it is important to know whether these vectors depend on position, and therefore derivates of any such vectors with respect to the coordinates, must include *derivatives of the unit vectors themselves*. We will assume the old coordinates are *standard*, or *Cartesian* coordinates, such that their unit vectors are *position-independent*, and the new coordinates are generally assumed to be *position-dependent*, such that

$$\partial_\nu \hat{\mu} = 0 \quad , \quad \partial_{\nu'} \hat{\mu}' \neq 0 \ . \tag{2}$$

We may define the coordinate transformations $\mu \to \mu'$ in terms of functions for each coordinate. Each old coordinate, is a function of the new coordinates

$$\mu = f_\mu(\nu') \ , \tag{3}$$

or inversely, each new coordinate, is a function of the old coordinates

$$\mu' = f_{\mu'}(\nu) \ . \tag{4}$$

Let the distance vector $\vec{l}$ be written in the old (standard) basis as

$$\vec{l} = \sum_\mu f_\mu \ \hat{\mu} \tag{5}$$

and therefore the differential distance vector may be written in terms of the new basis differentials as

$$d\vec{l} = \sum_{\mu'} \partial_{\mu'} \vec{l} \, d\mu' \ . \tag{6}$$

The new basis vectors may then be defined as the derivates of the distance vector with respect to each new coordinate

$$\vec{\mu}' = \partial_{\mu'} \vec{l} = \sum_{\nu} \partial_{\mu'} f_{\nu} \ \hat{\nu} \tag{7}$$

with magnitude

$$\varphi_{\mu'} = |\vec{\mu}'| = \sqrt{\sum_{\nu} |\partial_{\mu'} f_{\nu}|^2} \ . \tag{8}$$

From these unit vectors in the new coordinate system, the overlap components of the unit vectors in each coordinate system are

$$\hat{\nu}' \cdot \hat{\mu} = \frac{1}{\varphi_{\nu'}} \partial_{\nu'} f_{\mu} \ , \tag{9}$$

and therefore the basis unit vectors in the old or new coordinates may be expressed in terms of the other coordinates as

$$\hat{\mu}' = \sum_{\nu} \frac{1}{\varphi_{\mu'}} \partial_{\mu'} f_{\nu} \ \hat{\nu} \tag{10}$$

$$\hat{\mu} = \sum_{\nu'} \frac{1}{\varphi_{\nu'}} \partial_{\nu'} f_{\mu} \ \hat{\nu}' \ . \tag{11}$$

After performing coordinate transformations, we generally want to express other vectors, and gradients in terms of the new coordinates. Old components a_{μ} of vectors $\vec{a}$ may be expressed in terms of new components $a_{\mu'}$ with the change of variables

$$a_{\mu} = \sum_{\nu'} \frac{1}{\varphi_{\nu'}} \partial_{\nu'} f_{\mu} a_{\nu'} \ . \tag{12}$$

Old derivatives ∂_{μ} of gradients ∇ may be expressed in terms of new derivatives $\partial_{\mu'}$ with the chain rule

$$\partial_{\mu} = \sum_{\nu'} \partial_{\mu} f_{\nu'} \partial_{\nu'} \ . \tag{13}$$

Vectors $\vec{a}$ and gradients ∇ may be expressed as sums of components in any coordinate system

$$\vec{a} = \sum_{\mu} a_{\mu} \hat{\mu} = \sum_{\mu'} a_{\mu'} \hat{\mu}' = \sum_{\mu,\nu'} \frac{1}{\varphi_{\nu'}} \partial_{\nu'} f_{\mu} a_{\nu'} \hat{\mu} \tag{14}$$

$$\nabla = \sum_{\mu} \hat{\mu} \partial_{\mu} = \sum_{\mu'} \hat{\mu}' \frac{1}{\varphi_{\mu'}} \partial_{\mu'} = \sum_{\mu,\nu'} \hat{\mu} \partial_{\mu} f_{\nu'} \partial_{\nu'} \ . \tag{15}$$

Notice how the vectors and gradients transform inversely to each other, with vectors being transformed in terms of the old coordinate functions f_μ, and the gradients being transformed in terms of the new coordinate functions $f_{\mu'}$. In general, quantities, that appear in the old coordinate system as simple dot products between gradients and vectors, due to standard basis vectors being position independent, now have additional terms due to the new basis vectors being position dependent.

For example, the divergence of a vector $\vec{a}$ in the new coordinate system is

$$\nabla \cdot \vec{a} = \sum_\mu \partial_\mu a_\mu = \sum_\mu \left[\sum_{\nu'} \partial_\mu f_{\nu'} \partial_{\nu'}\right] \left[\sum_{\eta'} \frac{1}{\varphi_{\eta'}} \partial_{\eta'} f_\mu a_{\eta'}\right] . \tag{16}$$

1.2 Polar Coordinates

Let us consider $d = 2$ transformation between old Cartesian and new Polar coordinates

$$(x, y) \to (s, \phi) . \tag{17}$$

Our general procedure is as follows to derive quantities in terms of new coordinates, including unit vectors, vector components, and gradient partial derivatives:

1. Derive the distance vector $\vec{l} = x\hat{x} + y\hat{y}$ in terms of the new coordinates, given we know $x = f_x(s, \phi)$, $y = f_y(s, \phi)$, or equivalently $s = f_s(x, y)$, $\phi = f_\phi(x, y)$ for some functions for each coordinate, depending on the specific transformation.

2. Derive the new coordinate unit basis vectors $\hat{s}, \hat{\phi}$ in terms of the old coordinate unit basis vectors $\hat{x}, \hat{y}$, given (non-unit) basis vectors are defined as $\vec{s} = \frac{\partial \vec{l}}{\partial s}$, $\vec{\phi} = \frac{\partial \vec{l}}{\partial \phi}$.

3. Derive vector $\vec{a}$ old coordinate components a_x, a_y in terms of new coordinate components a_s, a_ϕ, given all vectors can be expressed as sums of orthogonal components in any coordinates $\vec{a} = a_x\hat{x} + a_y\hat{y} = a_s\hat{s} + a_\phi\hat{\phi}$.

4. Derive gradient partial derivatives with respect to old coordinates in terms of partial derivatives with respect to new coordinate, given the chain rule, and that the new coordinates depend the old coordinates,
$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \frac{\partial f_s(x,y)}{\partial x}\frac{\partial}{\partial s} + \frac{\partial f_\phi(x,y)}{\partial x}\frac{\partial}{\partial \phi}$, $\frac{\partial}{\partial y} = \frac{\partial s}{\partial y}\frac{\partial}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} = \frac{\partial f_s(x,y)}{\partial y}\frac{\partial}{\partial s} + \frac{\partial f_\phi(x,y)}{\partial y}\frac{\partial}{\partial \phi}$.

5. Derive quantities of interest in new coordinates, given known equations involving old coordinates, (such as the divergence $\nabla \cdot \vec{a} = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y$), given we know now the partial derivatives and the vector components, written in the old coordinates, in terms of the new coordinates.

The associated coordinate transformation functions are

$$x = s\cos(\phi) \quad , \quad y = s\sin(\phi) \tag{18}$$

$$s = \sqrt{x^2+y^2} \quad , \quad \phi = \tan^{-1}\frac{y}{x} \ . \tag{19}$$

The distance vector is

$$\vec{l} = s\cos(\phi)\hat{x} + s\sin(\phi)\hat{y} \tag{20}$$

$$d\vec{l} = \ (\cos(\phi)\hat{x} + \sin(\phi)\hat{y})\, ds \ + \ (-s\sin(\phi)\hat{x} + s\cos(\phi)\hat{y})\, d\phi \ . \tag{21}$$

The new basis vectors are

$$\vec{s} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y} \quad , \quad \vec{\phi} = -s\sin(\phi)\hat{x} + s\cos(\phi)\hat{y} \tag{22}$$

$$\varphi_s = 1 \quad , \quad \varphi_\phi = s \tag{23}$$

$$\hat{s} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y} \quad , \quad \vec{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y} \ . \tag{24}$$

Therefore the distance vector may be expressed as

$$\vec{l} = s\hat{s} \tag{25}$$

$$d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} \ . \tag{26}$$

The components between new and old basis unit vectors are

$$\hat{x}\cdot\hat{s} = \cos(\phi) \quad , \quad \hat{y}\cdot\hat{s} = \sin(\phi) \tag{27}$$

$$\hat{x}\cdot\hat{\phi} = -\sin(\phi) \quad , \quad \hat{y}\cdot\hat{\phi} = \cos(\phi) \ . \tag{28}$$

The old basis unit vectors are

$$\hat{x} = \cos(\phi)\hat{s} - \sin(\phi)\hat{\phi} \quad , \quad \hat{y} = \sin(\phi)\hat{s} + \cos(\phi)\hat{\phi} \ . \tag{29}$$

Vectors with old components expressed in the new components are

$$a_x = \cos(\phi)a_s - \sin(\phi)a_\phi \quad , \quad a_y = \sin(\phi)a_s + \cos(\phi)a_\phi \ . \tag{30}$$

Gradients with old derivatives expressed in the new derivatives are

$$\partial_x = \cos(\phi)\partial_s - \frac{1}{s}\sin(\phi)\partial_\phi \quad , \quad \partial_x = \sin(\phi)\partial_s + \frac{1}{s}\cos(\phi)\partial_\phi \ . \tag{31}$$

The divergence in new components is

$$\begin{aligned}\nabla\cdot\vec{a} = &\ \left(\cos(\phi)\partial_s - \frac{1}{s}\sin(\phi)\partial_\phi\right)\left(a_s\cos(\phi) - a_\phi\sin(\phi)\right) \ + \qquad (32)\\ &\ \left(\sin(\phi)\partial_s + \frac{1}{s}\cos(\phi)\partial_\phi\right)\left(a_s\sin(\phi) + a_\phi\cos(\phi)\right)\end{aligned}$$

$$\nabla\cdot\vec{a} = \frac{1}{s}\partial_s\left(sa_s\right) \ + \ \frac{1}{s}\partial_\phi\left(a_\phi\right) \ . \tag{33}$$

2 Integral Calculus

2.1 Dirac Delta Distributions

A useful object is the *Dirac-Delta* distribution $\delta(x)$, which is defined via integration for all (single variable) functions $f(x)$ as

$$\int_{-\infty}^{\infty} dy\ \delta(x-y)\ f(y) = f(x)\ , \tag{34}$$

which can be thought of as a *convolution* of δ with f, yielding the single point $f(x)$, where the argument $x-y$ of the $\delta(x-y)$ is zero. We can think of this as roughly

$$\delta(x) = \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases} \tag{35}$$

This definition can be extended to multi variable functions $f(\vec{l})$, $\vec{l}=(x,y,z,\dots)\in\mathbb{R}^d$ in d-dimensions via

$$\delta(\vec{l}) \equiv \delta^{(d)}(\vec{l}) = \delta(x)\delta(y)\delta(z)\cdots = \prod_{\mu\in\{x,y,z,\dots\}} \delta(\mu)\ . \tag{36}$$

When the argument of $\delta(g(x))$ is a function $g(x)$, a change of variables yields

$$\int_{-\infty}^{\infty} dx\ \delta(g(x))\ f(x) = \sum_{x\ :\ g(x)=0} \frac{1}{g'(x)} f(x)\ . \tag{37}$$

2.2 Triple Products

Vectors $\vec{a},\vec{b}\in\mathbb{R}^d$ may also be *vector-fields*, which are functions $\vec{a}(\vec{l})$ of spatial variables $\vec{l}$ in d-dimensions, with standard Cartesian coordinates $\vec{l}=(x,y,z,\dots)\in\mathbb{R}^d$. Products of such vectors may be defined in several ways.

Specifically in $d=3$-dimensions with standard Cartesian coordinates (x,y,z), there exists two operations of

Dot product (yields a scalar that is the projection of $\vec{a}$ on $\vec{b}$)

$$\begin{aligned} \vec{a}\cdot\vec{b} &= a_x b_x\ +\ a_y b_y\ +\ a_z b_z && (38)\\ &= \sum_{\mu\in\{x,y,z\}} a_\mu b_\mu && (39) \end{aligned}$$

Cross product (yields a vector that is orthogonal to $\vec{a}$ and $\vec{b}$, via the *right-hand-rule* of the permutation of the cycle $x\to y\to z$)

$$\begin{aligned} \vec{a}\times\vec{b} &= (a_y b_z\ -\ a_z b_y)\hat{x}\ +\ (a_z b_x\ -\ a_x b_z)\hat{y}\ +\ (a_x b_y\ -\ a_y b_x)\hat{z} && (40)\\ &= \sum_{\substack{\mu,\nu,\eta\ \in\ \{x,y,z\} \\ \mu\neq\nu<\eta}} (-1)^{\mu=y}(a_\nu b_\eta\ -\ a_\eta b_\nu)\hat{\mu}\ . && (41) \end{aligned}$$

Useful *triple-product* identities for vectors $\vec{a}, \vec{b}, \vec{c}$ can be derived from the definitions of dot and cross products of vectors, and often involve nice, symmetrical shifts, or permutations of the order of the vectors and their components:

Scalar triple product (can shift the order of vectors, all equivalent to parallelogram formed by vectors)

$$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{c} \cdot \vec{a} \times \vec{b} = \vec{a} \cdot \vec{b} \times \vec{c} \tag{42}$$

Vector triple product (can relate the components along the directions of the vectors in the second cross product)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \ . \tag{43}$$

Jacobi identity (anti-symmetry of cross products means permutations of vectors cancel)

$$\vec{a} \times (\vec{b} \times \vec{c}) \ + \ \vec{c} \times (\vec{a} \times \vec{b}) \ + \ \vec{b} \times (\vec{c} \times \vec{a}) = 0 \ . \tag{44}$$

3 Vector Calculus

3.1 Line Integrals

We will investigate integrals of functions $f(\vec{l})$ of variables in d-dimensions with standard Cartesian coordinates $\vec{l} = (x, y, z, \dots) \in \mathbb{R}^d$, along a line described by $\vec{l}$. Such *line-elements* along the integral can be defined using the chain rule,

$$df = (\frac{\partial f}{\partial x}\hat{x} \ + \ \frac{\partial f}{\partial y}\hat{y} \ + \ \frac{\partial f}{\partial z}\hat{z} \ + \ \cdots) \cdot (dx \ \hat{x} \ + \ dy \ \hat{y} \ + \ dz \ \hat{z} \ + \ \cdots) = \nabla f \cdot d\vec{l} \tag{45}$$

where the *gradient* of a function is defined as the vector

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} \ + \ \frac{\partial f}{\partial y}\hat{y} \ + \ \frac{\partial f}{\partial z}\hat{z} \ + \ \cdots \ = \sum_{\mu \in \{x,y,z,\dots\}} \hat{\mu}\frac{\partial f}{\partial \mu} \tag{46}$$

for a given line element

$$d\vec{l} = dx \ \hat{x} \ + \ dy \ \hat{y} \ + \ dz \ \hat{z} \ + \ \cdots \ = \sum_{\mu \in \{x,y,z,\dots\}} d\mu \ \hat{\mu} \ . \tag{47}$$

3.2 Grad and Curl Operators

Derivatives of vectors can take several forms, based on defining the gradient operator

$$\nabla = \hat{x}\frac{\partial}{\partial x} \ + \ \hat{y}\frac{\partial}{\partial y} \ + \ \hat{z}\frac{\partial}{\partial z} \ + \ \cdots = \sum_{\mu \in \{x,y,z,\dots\}} \hat{\mu}\frac{\partial}{\partial \mu} \tag{48}$$

as roughly behaving like a vector, excluding the fact that derivatives obey the product rule.

Specifically in $d = 3$-dimensions, derivatives of vectors have the form

Gradient of a scalar (yields a vector of the changes in a function along its input directions)

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} \ + \ \frac{\partial f}{\partial y}\hat{y} \ + \ \frac{\partial f}{\partial z}\hat{z} \tag{49}$$

Divergence of a vector (yields a scalar of the total change with respect to its inputs, along the input vector component directions)

$$\nabla \cdot \vec{a} = \frac{\partial a_x}{\partial x} \ + \ \frac{\partial a_y}{\partial y} \ + \ \frac{\partial a_z}{\partial z} \tag{50}$$

Curl of a vector (yields a vector of the change in vector components, orthogonal to the vector component directions)

$$\nabla \times \vec{a} = (\frac{\partial a_z}{\partial y} \ - \ \frac{\partial a_y}{\partial z})\hat{x} \ + \ (\frac{\partial a_x}{\partial z} \ - \ \frac{\partial a_z}{\partial x})\hat{y} \ + \ (\frac{\partial a_y}{\partial x} \ - \ \frac{\partial a_x}{\partial y})\hat{z} \ . \tag{51}$$

There are then similar triple product identities (but *not* identical to in the case of vectors due to the product rule of derivatives)

Gradient of a dot product

$$\nabla(\vec{a} \cdot \vec{b}) = \vec{a} \times (\nabla \times \vec{b}) \ + \ \vec{b} \times (\nabla \times \vec{a}) \ + \ (\vec{b} \cdot \nabla)\vec{a} \ + \ (\vec{a} \cdot \nabla)\vec{b} \ . \tag{52}$$

Divergence of a cross product

$$\nabla \cdot (\vec{a} \times \vec{b}) = (\nabla \times \vec{a}) \cdot \vec{b} \ - \ \vec{a} \cdot (\nabla \times \vec{b}) \tag{53}$$

Curl of a cross product

$$\nabla \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla)\vec{a} \ - \ (\vec{a} \cdot \nabla)\vec{b} \ + \ (\nabla \cdot \vec{b})\vec{a} \ - \ (\nabla \cdot \vec{a})\vec{b} \ . \tag{54}$$

There are also identities related to the *commutativity* of partial derivatives (in Cartesian coordinates), meaning

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}, \tag{55}$$

which hold for all functions and vectors, therefore

Curl of gradient (a gradient is maximally in the direction of its components)

$$\nabla \times \nabla f = 0 \tag{56}$$

Divergence of curl (a curl is always orthogonal to the direction of the gradient components)

$$\nabla \cdot \nabla \times \vec{a} = 0 \ . \tag{57}$$

4 Divergence Theorem

There exist several theorems relating integrals of vectors $\vec{a}$ and their derivatives, over volumes V, surfaces S, with normal directions $\hat{n}$, and curves L with tangent directions $\hat{t}$:

Divergence Theorem (sum of sources within a volume equals the total flux at its surface)

$$\int_V dV\ \nabla \cdot \vec{a} = \int_S dS\ \hat{n} \cdot \vec{a} \tag{58}$$

Stoke's Theorem (total flux of curl at a surface equals the sum of circulation around its boundary)

$$\int_S dS\ \hat{n} \cdot \nabla \times \vec{a} = \int_L dl\ \hat{t} \cdot \vec{a}\ . \tag{59}$$

Using vector calculus identities and the previous integral theorems (through generally defining $\vec{a} = \vec{b} \times \vec{c}$, with a variable part $\vec{b}$ and a constant part $\vec{c}$, or $\vec{a} = f\hat{c}$, with a variable magnitude function f and a constant direction $\hat{c}$), we can also derive similar divergence like theorems

Divergence Theorem of Gradients

$$\int_V dV\ \nabla f = \int_S dS\ \hat{n} f \tag{60}$$

Divergence Theorem of Curl

$$\int_V dV\ \nabla \times \vec{a} = \int_S dS\ \hat{n} \times \vec{a} \tag{61}$$

Stoke's Theorem of Gradients

$$\int_S dS\ \hat{n} \times \nabla f = \int_L dl\ \hat{t} f\ . \tag{62}$$

These theorems allow us to interpret geometrically the meaning of curl and divergence operations, by taking the volume and surface area to zero:

Divergence (Radial flux of vector through surface)

$$\nabla \cdot \vec{a} = \lim_{V \to 0} \frac{1}{V} \int_S dS\ \hat{n} \cdot \vec{a} \tag{63}$$

Curl (Component of vector changing along tangent of curve)

$$\nabla \times \vec{a} = \lim_{V \to 0} \frac{1}{V} \int_S dS\ \hat{n} \times \vec{a} \tag{64}$$

$$\hat{n} \cdot \nabla \times \vec{a} = \lim_{S \to 0} \frac{1}{S} \int_L dl\ \hat{t} \cdot \vec{a}\ . \tag{65}$$