Channel Expressivity Measures and their Operational Meaning

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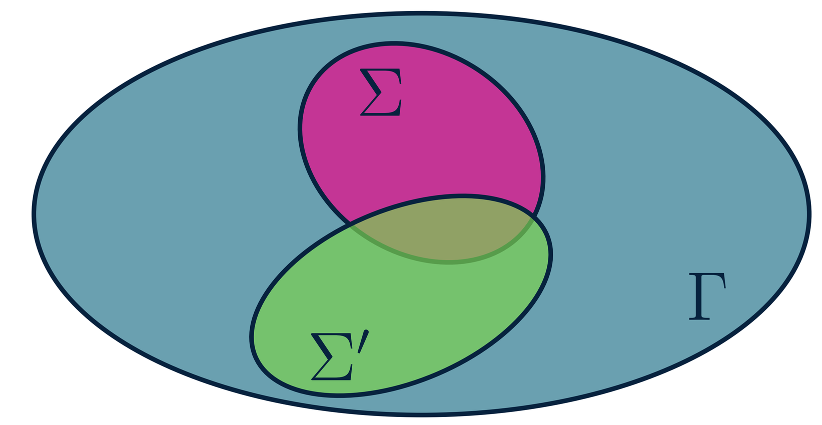
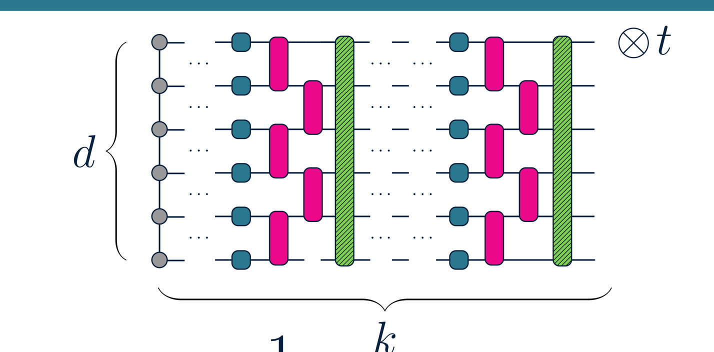
1. How Expressive are our Ansatze?

- How does an ensemble of quantum channels compare to a *maximally expressive* reference ensemble? [1]
- How may we *quantify* and *compute* expressivity measures for quantum channels?
- How do generalized expressivity measures and reference ensembles depend on:
  - *Noise* induced phenomena
  - Underlying (parameterized) *unitary* evolution
  - Coupling with the *environment*



2. Variables

Space Dimension	d
Copies of Space	t
Layers	k
Ensembles	$\Sigma, \Sigma' \subseteq \Gamma$
Twirls	$\mathcal{T}^{(t)}_{\Sigma} = \int_{\Sigma} d\Lambda\ \Lambda^{\otimes t} = \frac{1}{d^t}\lvert I\rangle\langle I\rvert \ + \ \Delta^{(t)}_{\Sigma}$
Expressivity	$\mathcal{E}^{(t)}_{\Sigma\Sigma'} = \lVert \mathcal{T}^{(t)}_{\Sigma} - \mathcal{T}^{(t)}_{\Sigma'} \rVert \ , \ \mathcal{E}^{(t)}_{\Sigma} = \lVert \Delta^{(t)}_{\Sigma} \rVert$



3. Reference Ensembles

- *Haar* $\sim$ Unitary Haar Measure (uniformly random unitaries)

$$\boxed{\mathcal{T}^{(t)}_{\mathbb{U}(d)}(\rho)} = \int_{\mathbb{U}(d)} dU\ U^{\otimes t}\ \rho^{\otimes t}\ U^{\otimes t\ \dagger} \qquad (1)$$

- *cHaar* $\sim$ Stinespring Haar Measure (random channels) [2]

$$\boxed{\mathcal{T}^{(t)}_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})}(\rho)} = \mathrm{tr}_{\mathcal{E}}\left(\int_{\mathbb{U}(d_{\mathcal{H}}d_{\mathcal{E}})} dU\ U^{\otimes t}\ \rho^{\otimes t} \otimes\ \nu_{\mathcal{E}}^{\otimes t}\ U^{\otimes t\ \dagger}\right) \qquad (2)$$

- *Depolarize* $\sim$ Maximally Depolarizing Channel (single channel)

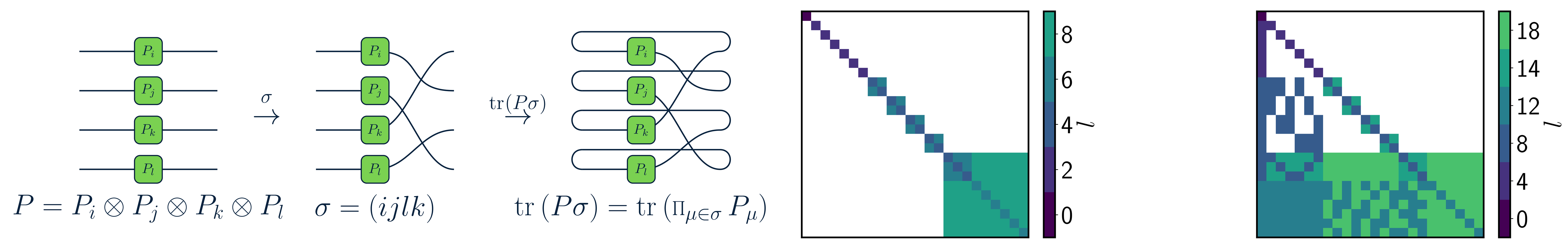
$$\boxed{\mathcal{T}^{(t)}_{\mathbb{D}(d)}(\rho)} = \frac{\mathrm{tr}\left(\rho^{\otimes t}\right)}{d^t}\ I^{\otimes t} \qquad (3)$$

$$\boxed{\lVert\mathcal{T}^{(t)}_{\mathbb{D}(d_{\mathcal{H}})}\rVert_2} \underset{d_{\mathcal{H}}d_{\mathcal{E}}\to\infty}{\leq} \boxed{\lVert\mathcal{T}^{(t)k}_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})}\rVert_2} \underset{d_{\mathcal{H}}d_{\mathcal{E}}\to d_{\mathcal{H}}}{\leq} \boxed{\lVert\mathcal{T}^{(t)}_{\mathbb{U}(d_{\mathcal{H}})}\rVert_2} \qquad (4)$$

4. Super-Operators, Permutations, and Cycle Operators

- Twirls over ensembles Σ may be expressed in a super-operator *basis* $\boxed{\mathcal{T}^{(t)}_{\Sigma} = \frac{1}{d^t}\sum_{P,S\in\mathcal{S}^{(t)}_{\Sigma}} \tau^{(t)}_{\Sigma}(P,S)\lvert P\rangle\langle S\rvert}$ Is $\mathcal{S}^{(t)}_{\Sigma} \subseteq \langle \mathcal{P}^t_d \rangle$ *orthogonal*?
- Haar random unitary $\mathbb{U}(d)$ twirls project onto *permutations* $\mathcal{S}_t$ [3], which may be expanded in orthogonal *cycle operators* $\mathcal{P}^{(\mathcal{S}_t)}_d$,

$$\tau = \frac{1}{d}\sum_{P\in\mathcal{P}_d} P\otimes P^{\dagger} \ \rightarrow\ \sigma = \frac{1}{d^{|\sigma|}}\sum_{P\in\mathcal{P}^{(\sigma)}_d} P \ \rightarrow\ \mathcal{T}^{(t)}_{\mathbb{U}(d)} = \frac{1}{d^t}\sum_{P,S\in\mathcal{P}^{(\mathcal{S}_t)}_d} \tau^{(t)}_{\mathbb{U}(d)}(P,S)\lvert P\rangle\langle S\rvert \ \rightarrow\ \mathcal{T}^{(t)}_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})} = \frac{1}{d^t}\sum_{P,S\in\mathcal{P}^{(\mathcal{S}_t)}_{d_{\mathcal{H}}}} \tau^{(t)}_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})}(P,S)\lvert P\rangle\langle S\rvert \qquad (5)$$



(a) Cycle Operator Structure for $P\in\mathcal{P}^{(\mathcal{S}_t)}_d$, for $t=4$ (b) Haar Twirl $\tau^{(t)}_{\mathbb{U}(d)}(P,S)\sim O(1/d^l)$ (c) cHaar Twirl $\tau^{(t)}_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})}(P,S)\sim O(1/d^l)$

5. Expressivity Measures

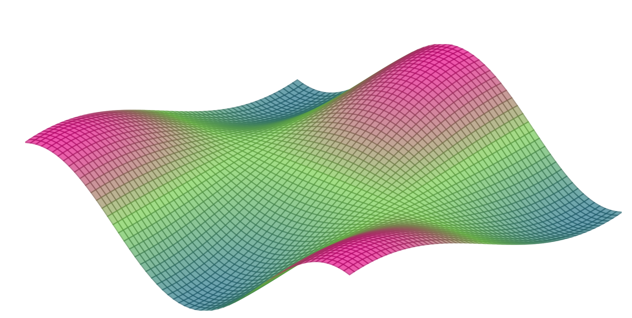
Unitary + Noise $(\mathcal{U}\circ\mathcal{N})^k$	Expressivity $\mathcal{E}^{(t,k)}_{\mathcal{U}\mathcal{N}}$
Haar $\mathbb{U}(d)$ $\circ$ Unital $\mathcal{N}_\gamma$	*Increases* $O((1-\gamma)^{2k})$
Haar $\mathbb{U}(d)$ $\circ$ Non-Unital $\mathcal{N}_\eta$	*Decreases* $O(\eta)$
Parameterized $\mathcal{G}_\theta$ $\circ$ Unital $\mathcal{N}_\gamma$	*Increases* $O((1-\gamma)^{k})$

(6)

$$\boxed{\mathcal{T}^{(t)k}_{\mathbb{E}(d_{\mathcal{H}},d_{\mathcal{E}})} = \underbrace{\frac{1}{d^t_{\mathcal{H}}}\lvert I\rangle\langle I\rvert}_{\text{Depolarize}} + \underbrace{O\left(\frac{1}{d^2_{\mathcal{H}}d_{\mathcal{E}}}\right)\frac{1}{d^t_{\mathcal{H}}}\lvert P\rangle\langle I\rvert}_{\text{Non-Unital}}} \qquad (7)$$

6. Expressivity versus Trainability

- Ensemble-dependent functions $\mathcal{F}$ may *concentrate* $\boxed{p\left(\lvert\mathcal{F}-\mu_{\mathcal{F}}\rvert \geq \epsilon\right) \leq \sigma^2_{\mathcal{F}}/\epsilon^2}$ (with *caveats* on ensembles, locality, norms, ...)
- Parameterized *objectives* and *gradients* $\boxed{\mathcal{L} = \mathrm{tr}\left(O\Lambda(\rho)\right) \rightarrow \partial\mathcal{L}}$ variances decay due to *inherent* and *expressivity* terms [4]



$$\sigma^2_{\mathcal{L}}\ ,\ \sigma^2_{\partial\mathcal{L}} \ \sim\ \underbrace{O\left(\frac{1}{\mathrm{poly}(d_{\mathcal{H}},d_{\mathcal{E}})}\right)\lVert\rho\rVert^2_2\ \lVert O\rVert^2_2}_{\text{Inherent}} \ +\ \underbrace{\min_{\frac{1}{p}+\frac{1}{q}=1} O\left(\frac{1}{\mathrm{poly}(d_{\mathcal{H}},d_{\mathcal{E}})}\right)\lVert\rho\rVert^2_p\ \lVert O\rVert^2_q\ \mathcal{E}^{(2|p)}_{\Sigma\Sigma'}}_{\text{Expressivity}} \qquad (8)$$

7. Conclusions

- *Noise induced* phenomena are actually channel *expressivity* phenomena!
- Depolarization arises often in quantum information, what else may expressivity *quantify*?
- Channel expressivity is more subtly related to *usefulness* or *capability*
- Are there relationships between channel expressivity and their *simulability*? [5]

8. References

[1] M. Duschenes, D. Garcia-Martin, Z. Holmes, M. Cerezo. arXiv:arXiv:2408.XXXXX, Report: LA-UR-24-20854 , (2024).
[2] R. Kukulski, I. Nechita, L. Pawela, Z. Puchala, K. Zyczkowski. Journal of Mathematical Physics **62**, 062201 (2021).
[3] J. Bai, J. Wang, Z. Yin. Quantum Information Processing **23**, 1–18 (2024).
[4] Z. Holmes, K. Sharma, M. Cerezo, P. J. Coles. PRX Quantum **3**, 010313 (2022).
[5] A. A. Mele, A. Angrisani, S. Ghosh, S. Khatri, J. Eisert, D. S. França, Y. Quek. arXiv:2403.13927 , (2024).