Moments of Quantum Channels

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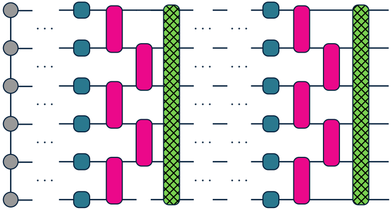
Institute for Quantum Computing, Perimeter Institute & Los Alamos National Laboratory

IQC Seminar

April 23, 2025



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Quantum Tasks Of Interest

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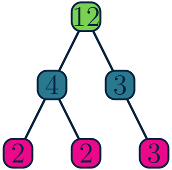

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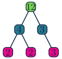

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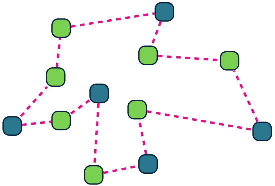

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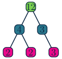
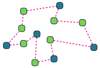
- *Quantum algorithms* i.e) Factoring numbers 

- *Optimization* problems i.e) Travelling Salesman Problem 
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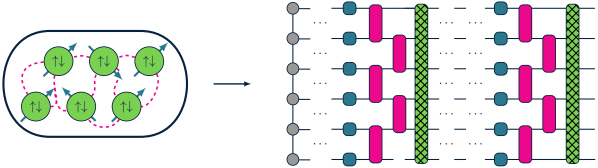

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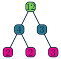
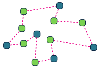
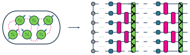
- *Quantum algorithms* i.e) Factoring numbers 

- *Optimization* problems i.e) Travelling Salesman Problem 
- *Compilation* tasks i.e) Form operators U given native gates $\{V\}$ 
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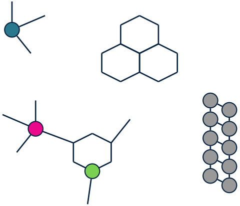

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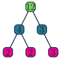
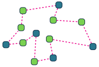
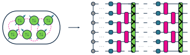
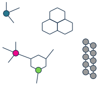
- *Quantum algorithms* i.e) Factoring numbers 

- *Optimization* problems i.e) Travelling Salesman Problem 
- *Compilation* tasks i.e) Form operators U given native gates $\{V\}$ 
- *Simulate* quantum systems i.e) Complicated molecules and chemical reactions 
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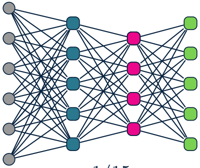

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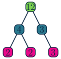

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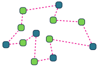

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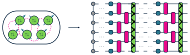

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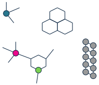

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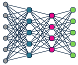
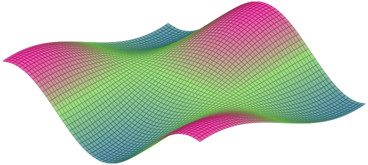
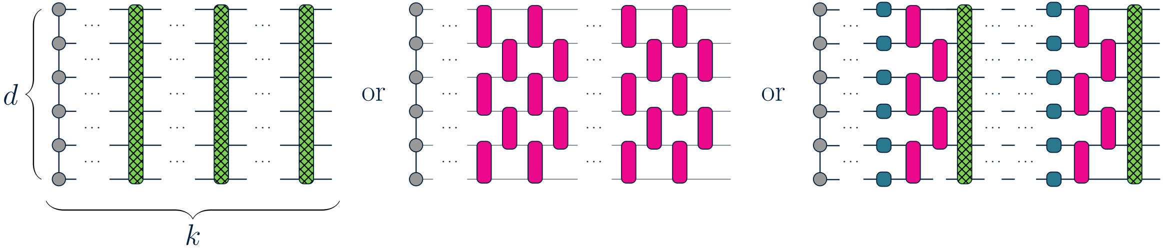




What Are (Random) Quantum Systems?



What *criteria* should we use when selecting an ansatz from an *ensemble*?

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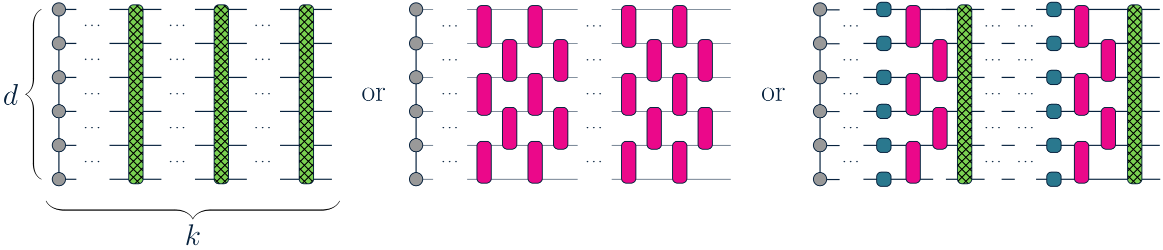

What if our ansatze are *noisy* or our dynamics are described by *quantum channels*?

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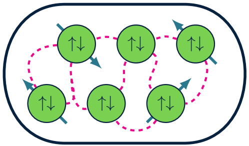
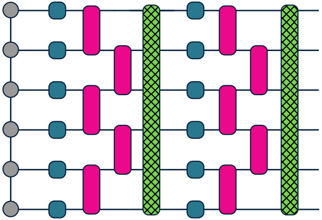


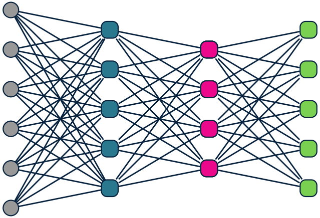
Quantum System

$H = \sum_\mu \theta_\mu G_\mu$



Quantum Circuit

$$\boxed{\Lambda = \prod_\mu \Lambda^\mu}$$



Classical Algorithm

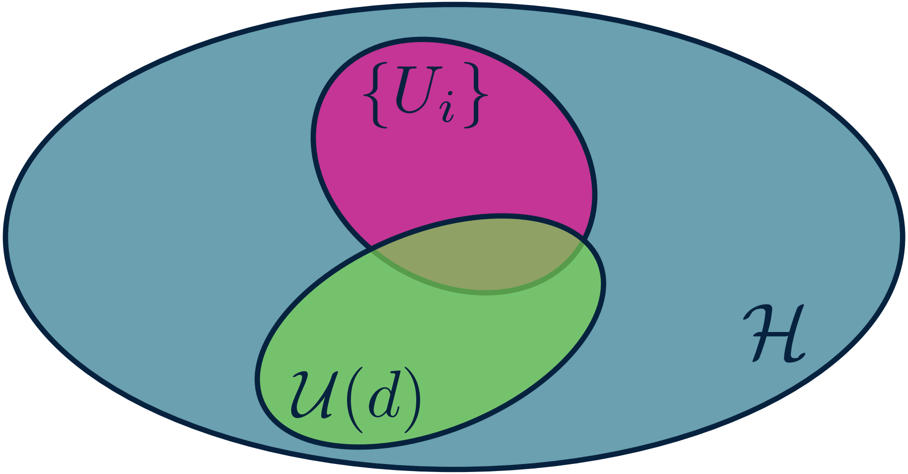
$f = \circ_\mu f^\mu$

What are Statistical Moments of Ensembles?

Suppose our channels $\Lambda \sim \Sigma$ are *randomly distributed*

What are the *statistical moments* of *random instances* of quantum systems?

What are Statistical Moments of Ensembles?



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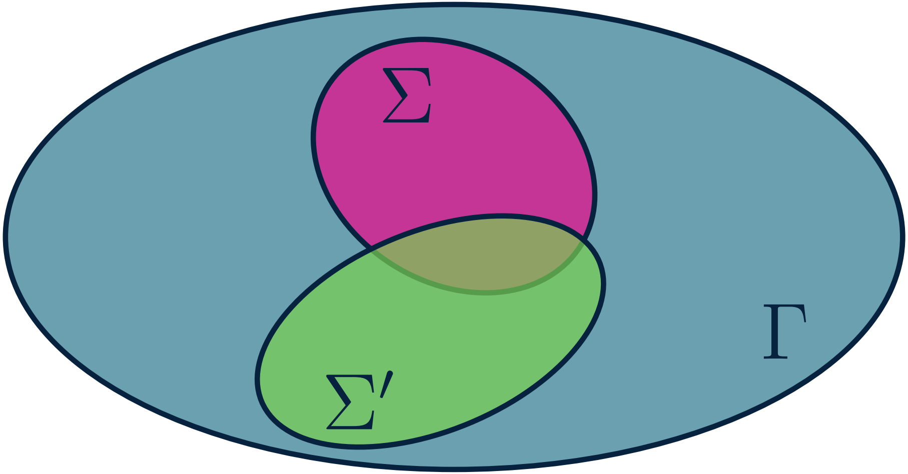

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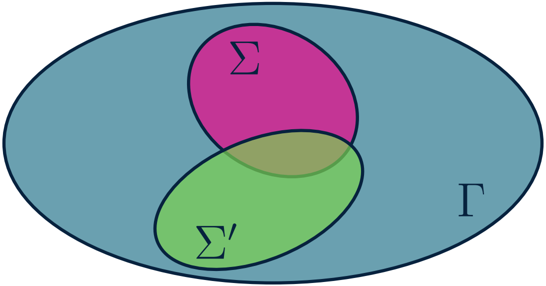

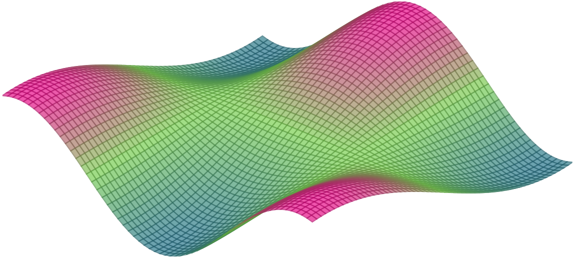

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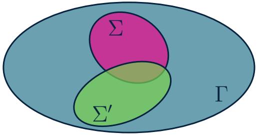
- How may we *compare* ansatz via statistical *moments*?



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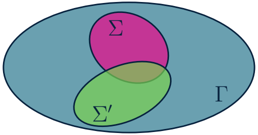

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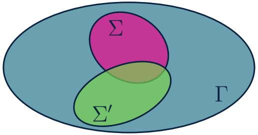
- Expressivity and trainability of *unitary ensembles* are well understood [1]



[1] Holmes, Z. *et al.*, *PRX Quantum* **3**, 010313 (2022)

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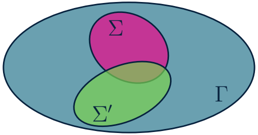
- Expressivity and trainability of *unitary ensembles* are well understood [1]
- How do moments of *quantum channel ensembles* depend on:
  - Noise induced phenomena
  - Underlying (parameterized) unitary evolution
  - Coupling with the environment



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Moment Operators

We are interested in computing t-order *moments* with respect to ensembles Σ,

$$\left\langle \left[\mathrm{tr}(\Lambda(\rho)O)\right]^t \right\rangle_{\Lambda\sim\Sigma} = \mathrm{tr}\left(\mathcal{T}_{\Sigma}^{(t)}(\rho^{\otimes t})O^{\otimes t}\right) = \langle\!\langle O^{\otimes t}|\widehat{\mathcal{T}}_{\Sigma}^{(t)}|\rho^{\otimes t}\rangle\!\rangle \tag{1}$$

via *twirls* or *moment operators*, with *vectorizations*,

$$\mathcal{T}_{\Sigma}^{(t)}(X) = \int_{\Sigma} d\Lambda\ \Lambda^{\otimes t}(X) \quad \rightarrow \quad \boxed{\mathcal{T}_{\Sigma}^{(t)}(X) \ \leftrightarrow\ \widehat{\mathcal{T}}_{\Sigma}^{(t)}|X\rangle\!\rangle} \tag{2}$$

Moment Operators

For example, channels Λ with Kraus operators $\{K_\Lambda\}$ may be represented as: Super-operators $\widehat{\Lambda}$, or Choi states Φ_Λ

$$\Lambda(X) = \sum_{K_\Lambda} K_\Lambda \; X \; K_\Lambda^\dagger \tag{3}$$

$$\leftrightarrow$$

$$\boxed{\widehat{\Lambda} = \sum_{K_\Lambda} K_\Lambda \otimes K_\Lambda^*} \quad , \quad \boxed{\Phi_\Lambda = (\Lambda \otimes I)(\Omega) = \sum_{K_\Lambda} |K_\Lambda\rangle\!\rangle\langle\!\langle K_\Lambda|} \tag{4}$$

$$X \to |X\rangle\!\rangle = X \otimes I \; |\Omega\rangle \; , \tag{5}$$

for un-normalized maximally entangled states Ω.

Moment Operators

For example, ensembles of *unitaries* $\mathcal{U}$ with a measure dU

$$\mathcal{T}_{\mathcal{U}}^{(t)}(X) = \int_{\mathcal{U}} dU \; U^{\otimes t} \; X \; U^{\otimes t\,\dagger} \qquad \rightarrow \qquad \widehat{\mathcal{T}}_{\mathcal{U}}^{(t)} = \int_{\mathcal{U}} dU \; U^{\otimes t} \otimes U^{\otimes t\,*} \tag{6}$$

Moment Operators

For example, ensembles of *channels* $\mathcal{C}$ with a measure $d\Lambda$

$$\mathcal{T}_{\mathcal{C}}^{(t)}(X) = \int_{\mathcal{C}} d\Lambda \; \Lambda^{\otimes t}(X) \qquad \rightarrow \qquad \widehat{\mathcal{T}}_{\mathcal{C}}^{(t)} = \int_{\mathcal{C}} d\Lambda \; \widehat{\Lambda^{\otimes t}} \tag{7}$$

Moment Operators

How do ensembles of *quantum channels* $\mathcal{C}$ differ from ensembles of *unitaries* $\mathcal{U}$?

t-Designs and Frame Potentials

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- Ensembles Σ form *ϵ-t-designs* with respect to a *reference* ensemble Σ' if their t-order moment operators correspond with respect to a (i.e) Frobenius) *norm*

$$\|\widehat{\mathcal{T}}_{\Sigma}^{(t)} - \widehat{\mathcal{T}}_{\Sigma'}^{(t)}\|^2 \;=\; \epsilon \quad \rightarrow \quad 0 \tag{8}$$

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$$\|\widehat{\mathcal{T}}_{\Sigma}^{(t)} - \widehat{\mathcal{T}}_{\Sigma'}^{(t)}\|^2 = \boxed{\|\widehat{\mathcal{T}}_{\Sigma}^{(t)}\|^2} - \boxed{\|\widehat{\mathcal{T}}_{\Sigma'}^{(t)}\|^2} + \cdots \tag{9}$$

- How are t-designs related to norms of moment operators, or *frame potentials*?

t-Designs and Frame Potentials

- Super-operator norms and traces have *operational meaning*

i.e) Super-Operator Inner Products $\leftrightarrow$ Choi State Inner Products

For channels Λ, Γ:

$$\boxed{\left\langle \widehat{\Lambda}, \widehat{\Gamma} \right\rangle} = \mathrm{tr}\left(\widehat{\Lambda}^\dagger \widehat{\Gamma}\right) = \mathrm{tr}\left(\Phi_\Lambda \Phi_\Gamma\right) = \boxed{\left\langle \Phi_\Lambda, \Phi_\Gamma \right\rangle} \tag{10}$$

t-Designs and Frame Potentials

- Super-operator norms and traces have *operational meaning*

i.e) Norm: Choi state purity , Trace: Entanglement fidelity

$$\boxed{\|\widehat{\Lambda}\|^2} = \mathrm{tr}\left(\Phi_\Lambda^2\right) \qquad , \qquad \boxed{\mathrm{tr}\left(\widehat{\Lambda}\right)} = \mathrm{tr}(\Omega\ \Phi_\Lambda) \tag{11}$$

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- Unitary ensembles $\mathcal{U}$ consisting of strictly *unitaries* $\{U\}$

$$\boxed{\|\widehat{\mathcal{T}}_{\mathcal{U}}^{(t)}\|^2} = \int_{\mathcal{U}\times\mathcal{U}} dU\ dV\ |\langle\langle U|V\rangle\rangle|^{2t} \quad , \quad \boxed{\mathrm{tr}\left(\widehat{\mathcal{T}}_{\mathcal{U}}^{(t)}\right)} = \int_{\mathcal{U}} dU\ |\mathrm{tr}(U)|^{2t} \tag{12}$$

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- Channel ensembles $\mathcal{C}$ consisting of *channels* with Kraus operators $\{\Lambda \leftrightarrow \{K_\Lambda\}\}$

$$\boxed{\|\widehat{\mathcal{T}}_{\mathcal{C}}^{(t)}\|^2} = \int_{\mathcal{C}\times\mathcal{C}} d\Lambda\ d\Gamma \sum_{K_\Lambda, K_\Gamma} |\langle\langle K_\Lambda|K_\Gamma\rangle\rangle|^{2t} \quad , \quad \boxed{\operatorname{tr}\left(\widehat{\mathcal{T}}_{\mathcal{C}}^{(t)}\right)} = \int_{\mathcal{C}} d\Lambda \sum_{K_\Lambda} |\operatorname{tr}(K_\Lambda)|^{2t} \tag{13}$$

Reference Ensembles

Channels present several choices for reference ensembles

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- *Haar* ∼ Unitary Haar measure (uniformly random unitaries U) [1]

$$\boxed{\mathcal{T}^{(t)}_{\mathcal{U}(d)}(\rho)} = \int_{\mathcal{U}(d)} dU \; U^{\otimes t} \, \rho^{\otimes t} \, U^{\otimes t \, \dagger} \tag{14}$$

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- *cHaar* $\sim$ Stinespring Unitary Haar measure (random channels Λ) [2]

$$\boxed{\mathcal{T}^{(t)}_{\mathcal{C}(d,d_{\mathcal{E}})}(\rho)} = \mathrm{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(dd_{\mathcal{E}})} dU \; U^{\otimes t} \; \rho^{\otimes t} \otimes \; \nu_{\mathcal{E}}^{\otimes t} \; U^{\otimes t\,\dagger}\right) \tag{15}$$

[2] Kukulski, R. *et al.*, *J. Math. Phys.* **62**, 062201 (2021)

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- *Depolarizing* $\sim$ Maximally Depolarizing (single depolarizing channel $\mathcal{D}$) [3]

$$\boxed{\mathcal{T}^{(t)}_{\mathcal{D}(d)}(\rho)} = \frac{\mathrm{tr}\left(\rho^{\otimes t}\right)}{d^t} \; I^{\otimes t} \tag{16}$$

[3] Bai, J. *et al.*, *Quantum Information Processing* **23**, 1–18 (2024)

Computations with Invariant Ensembles

- *Invariant* ensembles (i.e) those corresponding with *group* structure $\mathcal{G}$) with *invariant measures* (i.e) Haar measures) have *projector* moment operators

$$\int_{\mathcal{G}} dU \; f(U \circ V) \;=\; \int_{\mathcal{G}} dU \; f(V \circ U) \;=\; \int_{\mathcal{G}} dU \; f(U) \qquad \forall \; V \in \mathcal{G} \tag{17}$$

$$\leftrightarrow$$

$$\boxed{\widehat{\mathcal{T}}_{\mathcal{G}}^{(t)} \; \widehat{\mathcal{T}}_{\mathcal{G}}^{(t)} \;=\; \widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}} \tag{18}$$

Computations with Invariant Ensembles

- Invariant ensemble moment operators *project* onto a set $\mathcal{S}_{\mathcal{G}}^{(t)}$ (the *commutant*)

$$\mathcal{T}_{\mathcal{G}}^{(t)}(X) = \frac{1}{d^t} \sum_{P \in \mathcal{S}_{\mathcal{G}}^{(t)}} \tau_{\mathcal{G}}^{(t)}(P, X)\ P \tag{19}$$

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$$U^{\otimes t} \;\; \mathcal{T}_{\mathcal{G}}^{(t)}(X) \;\; U^{\otimes t\;-1} \;=\; \mathcal{T}_{\mathcal{G}}^{(t)}(X) \qquad\qquad \forall\, U \in \mathcal{G}$$

$$\leftrightarrow$$

$$\boxed{\left[\mathcal{T}_{\mathcal{G}}^{(t)}(X)\;,\; U^{\otimes t}\right] = 0 \qquad\qquad \forall\, U \in \mathcal{G}} \tag{19}$$

$$\leftrightarrow$$

$$[P\;,\;U^{\otimes t}] = 0 \qquad\qquad \forall\, U \in \mathcal{G}\;\;,\;\; P \in \mathcal{S}_{\mathcal{G}}^{(t)}$$

Computations with Invariant Ensembles

- Subset-ensemble frame potentials are *lower bounded* by invariant frame potentials

$$\|\widehat{\mathcal{T}}_{\mathcal{H}}^{(t)} - \widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}\|^2 = \|\widehat{\mathcal{T}}_{\mathcal{H}}^{(t)}\|^2 - \|\widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}\|^2 \qquad \forall\, \mathcal{H} \subseteq \mathcal{G}$$

$$\leftrightarrow$$

$$\boxed{\|\widehat{\mathcal{T}}_{\mathcal{H}}^{(t)}\|^2 \geq \|\widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}\|^2 \qquad \forall\, \mathcal{H} \subseteq \mathcal{G}} \tag{20}$$

Computations with Invariant Ensembles

- Subset-ensemble frame potentials are *lower bounded* by invariant frame potentials

$$\|\widehat{\mathcal{T}}_{\mathcal{H}}^{(t)} - \widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}\|^2 = \|\widehat{\mathcal{T}}_{\mathcal{H}}^{(t)}\|^2 - \|\widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}\|^2 \qquad \forall\, \mathcal{H} \subseteq \mathcal{G}$$

$$\leftrightarrow$$

$$\boxed{\|\widehat{\mathcal{T}}_{\mathcal{H}}^{(t)}\|^2 \geq \|\widehat{\mathcal{T}}_{\mathcal{G}}^{(t)}\|^2 \qquad \forall\, \mathcal{H} \subseteq \mathcal{G}} \tag{20}$$

For example, subgroups of Haar distributed *unitaries* $\mathcal{U} \subseteq \mathcal{U}(d)$

$$\boxed{\|\widehat{\mathcal{T}}_{\mathcal{U}}^{(t)}\|^2 \geq \|\widehat{\mathcal{T}}_{\mathcal{U}(d)}^{(t)}\|^2 \quad \leftrightarrow \quad \mathcal{S}_{\mathcal{U}}^{(t)} \supseteq \mathcal{S}_{\mathcal{U}(d)}^{(t)}} \tag{21}$$

Computations with Invariant Ensembles

Do such *relationships* between moment operators and norms hold for ensembles of quantum channels $\mathcal{C} \supseteq \mathcal{U}$?

Computations with Invariant Ensembles

Do such *relationships* between moment operators and norms hold for ensembles of quantum channels $\mathcal{C} \supseteq \mathcal{U}$?

Issue: Ensembles of channels generally have less *structure* and are thus *non-invariant* and *non-projective*

i.e) Depolarization $\widehat{\mathcal{D}} \circ \widehat{\mathcal{T}}_{\mathcal{C}}^{(t)} = \widehat{\mathcal{D}} \neq \widehat{\mathcal{T}}_{\mathcal{C}}^{(t)} \circ \widehat{\mathcal{D}}$

i.e) Non-invariance under k *concatenations* $\widehat{\mathcal{T}}_{\mathcal{C}}^{(t)k} \neq \widehat{\mathcal{T}}_{\mathcal{C}}^{(t)}$

Projections and Permutations

t-copies of Haar unitaries $\mathcal{U}(d)$ *project*, and cHaar channels $\mathcal{C}(d, d_{\mathcal{E}})$ *map* onto t-order permutations

$$\mathcal{S}_t = \{e \ , \ \tau \ , \ \sigma \ , \ \ldots\}$$

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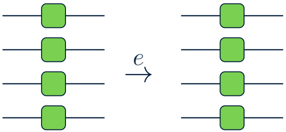
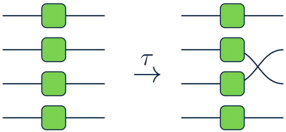
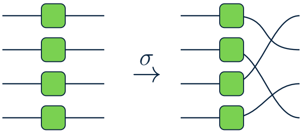


$e \cong I$

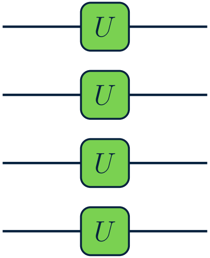


$\tau \cong \mathrm{SWAP}$



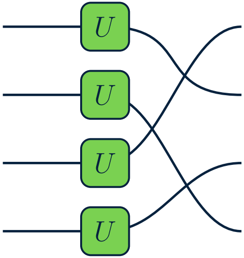
$\sigma \cong \mathrm{PERMUTE}$

Projections and Permutations



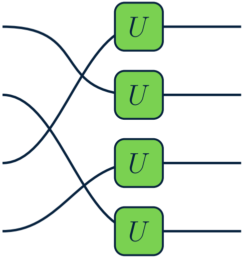
$U^{\otimes 4}$

$\xrightarrow{\sigma}$



$\sigma = (1\ 2\ 4\ 3)$

$=$



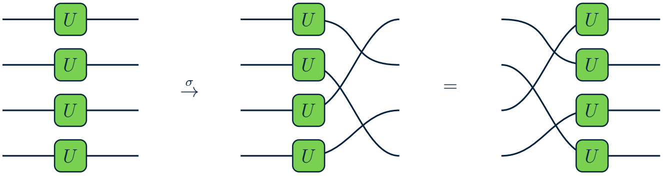
$U^{\otimes 4}\sigma = \sigma U^{\otimes 4}$

Projections and Permutations

- Haar unitaries *project*, and cHaar channels *map* onto permutations

$$\boxed{\widehat{\mathcal{T}}^{(t)}_{\mathcal{U}(d)}} = \int_{\mathcal{U}(d)} dU \; U^{\otimes t} \otimes U^{\otimes t\,*} = \frac{1}{d^t} \sum_{\sigma,\pi \in \mathcal{S}_t} \tau^{(t)}_{\mathcal{U}(d)}(\sigma,\pi) \; |\sigma\rangle\!\rangle\langle\!\langle\pi| \tag{22}$$

$$\boxed{\widehat{\mathcal{T}}^{(t)}_{\mathcal{C}(d,d_{\mathcal{E}})}} = \langle\!\langle I_{\mathcal{E}}^{\otimes t}| \; \widehat{\mathcal{T}}^{(t)}_{\mathcal{U}(dd_{\mathcal{E}})} \; |\nu_{\mathcal{E}}^{\otimes t}\rangle\!\rangle = \frac{1}{d^t} \sum_{\sigma,\pi \in \mathcal{S}_t} \tau^{(t)}_{\mathcal{C}(d,d_{\mathcal{E}})}(\sigma,\pi) \; |\sigma\rangle\!\rangle\langle\!\langle\pi| \tag{23}$$



$U^{\otimes 4}$ $\qquad$ $\sigma = (1\ 2\ 4\ 3)$ $\qquad$ $U^{\otimes 4}\sigma = \sigma U^{\otimes 4}$

Projections and Permutations

- *Permutations* $\sigma, \pi \in \mathcal{S}_t$ are *non-orthogonal*,

$$\boxed{\langle\langle \sigma | \pi \rangle\rangle \propto 1/d^{|\sigma^{-1}\pi|}} \,, \tag{24}$$

which complicates analysis of moment operators and their *concatenations*

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which complicates analysis of moment operators and their *concatenations*

- *Localized permutations* $[\sigma], [\pi] \in [\mathcal{S}_t]$ are *block-diagonal with respect to support*,

$$\boxed{\langle\!\langle [\sigma] | [\pi] \rangle\!\rangle \propto \delta_{\mathrm{supp}(\sigma) = \mathrm{supp}(\pi)}}\,, \tag{25}$$

which reveals important similarities and differences between moment operators

Localized Permutation Basis

Permutations σ have *definite* support $\text{supp}(\sigma) \subseteq [t]$ over a *subset* of t indices

$$\leftrightarrow$$

Suppose we expand permutations in t-copies of an orthogonal basis

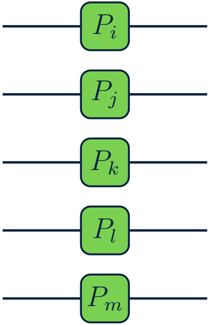
$$\mathcal{P}_d^{\otimes t} = \{P_1 \otimes P_2 \otimes \cdots \otimes P_t \quad : \quad P_1, P_2, \ldots, P_t \in \mathcal{P}_d\}$$

$$\leftrightarrow$$

Can we find a *basis* for permutations defined by their *support*?

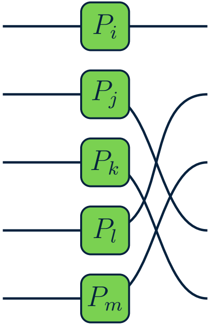
$$\sigma \quad \sim \quad \sum_{\substack{P \in \mathcal{P}_d^{\otimes t} \\ \text{supp}(P) \subseteq \text{supp}(\sigma)}} \text{tr}(\sigma P) \;\; P$$

Localized Permutation Basis



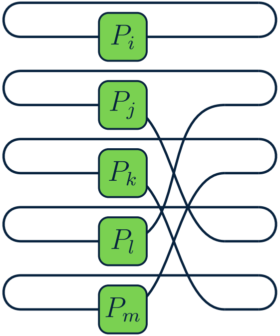
$$P = P_i \otimes P_j \otimes P_k \otimes P_l \otimes P_m$$

$\xrightarrow{\sigma}$

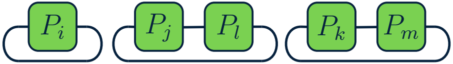


$$\sigma = (jl)(km)$$

$\xrightarrow{\mathrm{tr}(P\sigma)}$

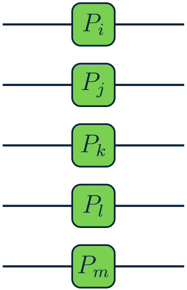


$=$

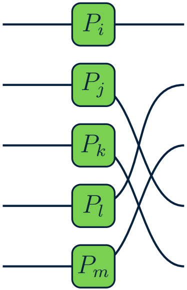


$$\mathrm{tr}(P\sigma) = \mathrm{tr}\left(\prod_{\mu\in\sigma} P_\mu\right)$$

Localized Permutation Basis



$\xrightarrow{\sigma}$



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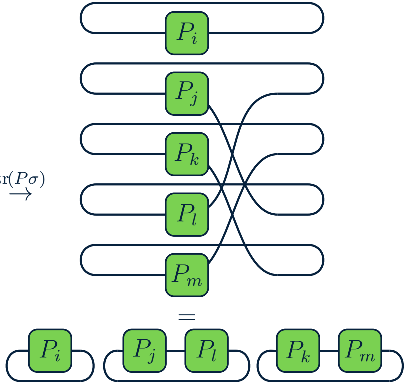


$P = P_i \otimes P_j \otimes P_k \otimes P_l \otimes P_m$

$\sigma = (jl)(km)$

$\mathrm{tr}(P\sigma) = \mathrm{tr}\left(\prod_{\mu\in\sigma} P_\mu\right)$

$$\boxed{\tau = \frac{1}{d}\sum_{P\in\mathcal{P}_d} P \otimes P^\dagger} \quad \rightarrow \quad \boxed{\sigma = \prod_{\tau} \tau = \prod_{\tau=(ij)} \left[\frac{1}{d}\sum_{P\in\mathcal{P}_d} P_i \otimes P_j^\dagger\right]} \tag{26}$$

Localized Permutation Basis

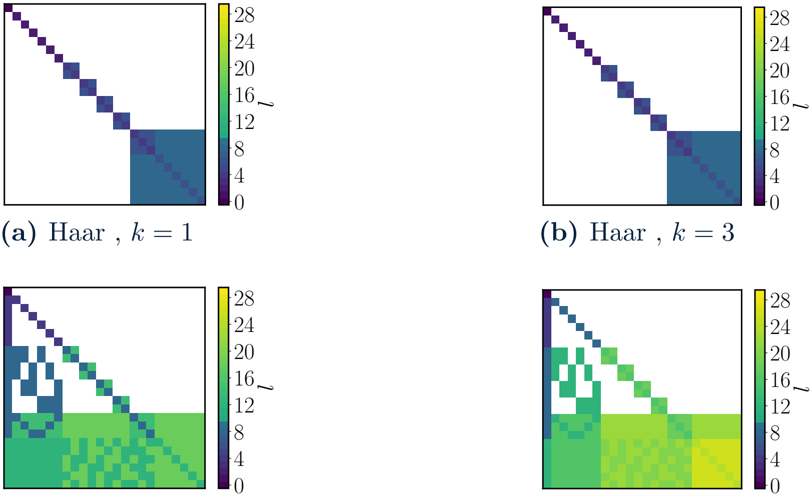
By *grouping* products of tensor-product basis operators comprising permutations by their support, we may define *localized permutations*

$$[\mathcal{S}_t] = \{[\sigma] \quad : \quad \sigma \in \mathcal{S}_t \quad , \quad \langle [\sigma], [\pi] \rangle \propto \delta_{\mathrm{supp}(\sigma)=\mathrm{supp}(\pi)}\}$$

$$[\sigma] \quad \sim \sum_{\substack{P \in \mathcal{P}_d^{\otimes t} \\ \mathrm{supp}(P)=\mathrm{supp}(\sigma)}} P$$

Permutations σ may then be expanded in terms of localized permutations (corresponding to their *sub-permutations* $|\pi^{-1}\sigma| = |\sigma| - |\pi|$)

$$\sigma \quad = \quad \frac{1}{d^{|\sigma|}} \sum_{\pi \subseteq \sigma} [\pi] \tag{27}$$



(a) Haar , $k = 1$

(b) Haar , $k = 3$

(c) cHaar , $k = 1$

(d) cHaar , $k = 3$

Figure 1: Transfer matrices for $t = 4$, k-concatenations, from the smallest identity (top-left), to the largest cycles (bottom-right), scaling as $\tau_{\mathcal{U},\mathcal{C}}^{(t,k)} \sim \mathcal{O}(1/d^l)$.

Hierarchy of Ensembles

The t-order Haar, cHaar, Depolarize ensembles are related by

$$\lim_{d_{\mathcal{E}}\to 1} \boxed{\widehat{\mathcal{T}}^{(t)}_{\mathcal{C}(d,d_{\mathcal{E}})}} \to \boxed{\widehat{\mathcal{T}}^{(t)}_{\mathcal{U}(d)}} \qquad \lim_{\frac{d}{d_{\mathcal{E}}}\to\infty} \boxed{\widehat{\mathcal{T}}^{(t)}_{\mathcal{C}(d,d_{\mathcal{E}})}} \to \boxed{\widehat{\mathcal{T}}^{(t)}_{\mathcal{D}(d)}} \tag{28}$$

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The k-concatenated, t-order cHaar ensemble is *depolarizing* and *non-unital* [3]

$$\lim_{\frac{d}{d_{\mathcal{E}}}\to\infty} \widehat{\mathcal{T}}^{(t)k}_{\mathcal{C}(d,d_{\mathcal{E}})} = \boxed{\underbrace{\widehat{\mathcal{D}}^{\otimes t}}_{\text{Depolarize}}} + \boxed{\underbrace{\mathcal{O}\left(\frac{1}{d^2 d_{\mathcal{E}}}\right) \widehat{\Delta}^{(t,k)}_{\mathcal{C}(d,d_{\mathcal{E}})}}_{\text{Non-Unital}}} \tag{29}$$

[3] Bai, J. *et al.*, *Quantum Information Processing* **23**, 1–18 (2024)

Hierarchy of Ensembles

The k-concatenated t-th cHaar moment operator has a single leading eigenvalue $\lambda = 1$, with other sub-leading eigenvalues of,

$$0 < |\lambda| \leq \mathcal{O}\left(\frac{1}{d_{\mathcal{E}}}\right) < 1 \;, \tag{30}$$

and in the fixed t and large $dd_{\mathcal{E}}$ limit, has norms and traces of,

$$\|\widehat{\mathcal{T}}^{(t)k}_{\mathcal{C}(d,d_{\mathcal{E}})}\|^2 \leq 1 \; + \; \mathcal{O}\left(\frac{1}{d_{\mathcal{E}}^2}\right) \tag{31}$$

$$\mathrm{tr}\left(\widehat{\mathcal{T}}^{(t)k}_{\mathcal{C}(d,d_{\mathcal{E}})}\right) \leq 1 \; + \; \mathcal{O}\left(\frac{1}{d_{\mathcal{E}}^k}\right) \;. \tag{32}$$

Hierarchy of Ensembles

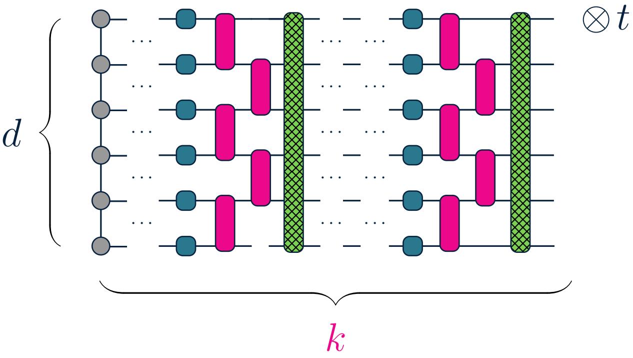
Given it holds analyticity in the fixed t and large $dd_{\mathcal{E}}$ limit, and numerically for $t \leq 4$, we *conjecture* the following hierarchy holds absolutely,

$$1 = \boxed{\left\|\widehat{\mathcal{T}}^{(t)}_{\mathcal{D}(d)}\right\|^2} \underset{dd_{\mathcal{E}}\to\infty}{\leq} \boxed{\left\|\widehat{\mathcal{T}}^{(t)k}_{\mathcal{C}(d,d_{\mathcal{E}})}\right\|^2} \underset{dd_{\mathcal{E}}\to d}{\leq} \boxed{\left\|\widehat{\mathcal{T}}^{(t)}_{\mathcal{U}(d)}\right\|^2} = t! \qquad (33)$$

Relationships between Noise and Channel t-designs

Analytical *moment operator norms* for k layers of specific channel *ansatze*

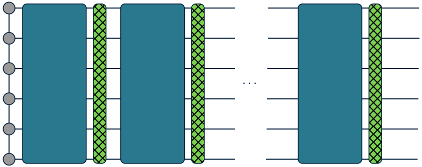
$$\Lambda^{k}_{\mathcal{U}_\gamma}(\rho) = (\mathcal{N}_\gamma \circ \mathcal{U})^k(\rho) = \frac{\mathrm{tr}(\rho)}{d} I + \Delta^{(k)}_\gamma(\rho) \tag{34}$$



Relationships between Noise and Channel t-designs

Haar Random Unitaries + Fixed *Unital* Pauli Noise: *Decreases* Norms

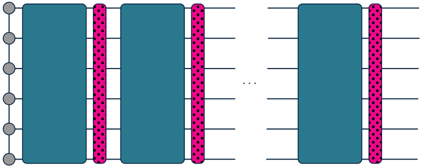
$$\|\widehat{\mathcal{T}}^{(t,k)}_{\mathcal{U}\gamma} - \widehat{\mathcal{T}}^{(t)}_{\mathcal{D}}\|^2 = \mathcal{O}\big((1-\gamma)^{2k}\big) \tag{35}$$



Relationships between Noise and Channel t-designs

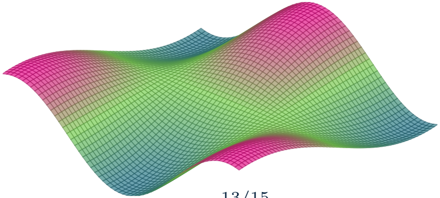
Haar Random Unitaries + Fixed *Non-Unital* Pauli Noise: *Increases* Norms

$$\|\widehat{\mathcal{T}}_{\mathcal{U}\gamma\eta}^{(t,k)} - \widehat{\mathcal{T}}_{\mathcal{D}}^{(t)}\|^2 = \mathcal{O}(\eta) \tag{36}$$



Designs versus Trainability

- Ensemble-dependent functions $\mathcal{F}$ may *concentrate* $\boxed{p\left(|\mathcal{F}-\mu_{\mathcal{F}}| \geq \epsilon\right) \leq \sigma_{\mathcal{F}}^2/\epsilon^2}$ (with *caveats* on ensembles, locality, norms, ...)



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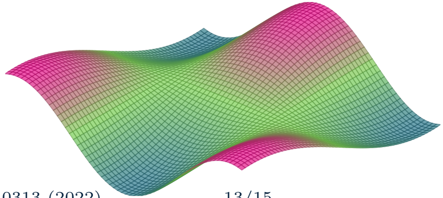
- *Objectives* and *gradients* $\boxed{\mathcal{L} = \mathrm{tr}(\Lambda(\rho)O) \to \partial\mathcal{L}}$ variances may decay [1]



[1] Holmes, Z. *et al.*, *PRX Quantum* **3**, 010313 (2022)

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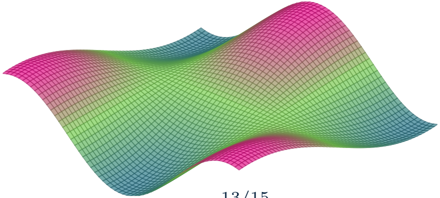
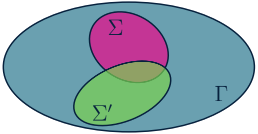
$$\sigma_{\mathcal{L}}^2\ ,\ \sigma_{\partial\mathcal{L}}^2 \ \sim\ \underbrace{\frac{\|\rho\|_2^2\ \|O\|_2^2}{\mathcal{O}(\mathrm{poly}(d, d_{\mathcal{E}}))}}_{\text{Inherent}} + \underbrace{\frac{\|\rho\|_p^2\ \|O\|_q^2}{\mathcal{O}(\mathrm{poly}(d, d_{\mathcal{E}}))}\ \|\mathcal{T}_{\Sigma}^{(2)} - \mathcal{T}_{\Sigma'}^{(2)}\|_p}_{\text{Designs}} \tag{37}$$



Interpretations of Ensemble Statistics

Many *subtle* differences between ensembles of channels and unitaries



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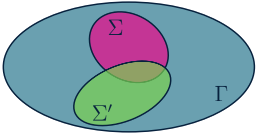

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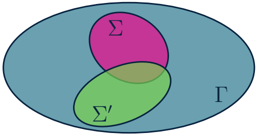

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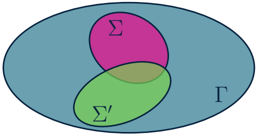

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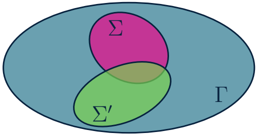

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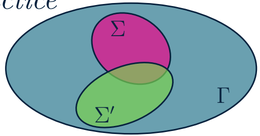
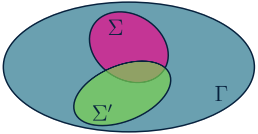
- Open systems have extra degrees of freedom (environment *size*, *coupling* strength)
- Adjoint channels are not necessarily *physical* channels (concentration *caveats*)
- Difficulties in deriving channel t-designs and their structure in *practice*



Interpretations of Ensemble Statistics

Ensembles and depolarization arise in many areas of *quantum information*



Interpretations of Ensemble Statistics

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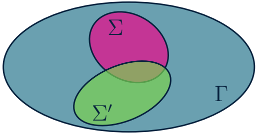
- *Capacity*: Depolarizing channels minimize *channel capacity* [4]



[4] King, C., *IEEE Trans. Inf. Theory* **49**, 221–229 (2002)

Interpretations of Ensemble Statistics

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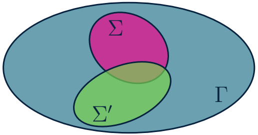
- *Capacity*: Depolarizing channels minimize *channel capacity*
- *Scrambling*: Depolarizing channels maximally *scramble* information [5]



[5] Nielsen, M. A. & Chuang, I. L. (Cambridge Univ. Press, 2000)

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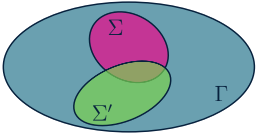

[6] Stratton, B. *et al.*, *Phys. Rev. A* **110**, L050601 (2024)

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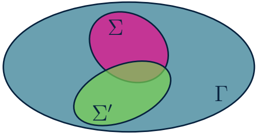
- *Capacity*: Depolarizing channels minimize *channel capacity*
- *Scrambling*: Depolarizing channels maximally *scramble* information
- *Discrimination*: State distinguishability depends on *Choi state rank*
- *Error Mitigation*: Sample complexity depends on *Choi state purity* [7]



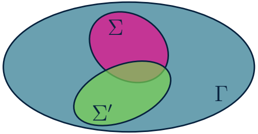
[7] Quek, Y. *et al.*, *arXiv:2210.11505* (2022)

Interpretations of Ensemble Statistics

Spectral properties of moment operators

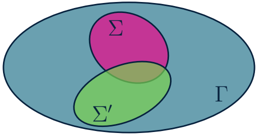
$$\leftrightarrow$$

Average ability of ensembles to transmit quantum information?



Operational Meaning of Moments of Quantum Channels

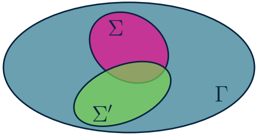
- How does *structure* of ensembles relate to *design* properties?



Operational Meaning of Moments of Quantum Channels

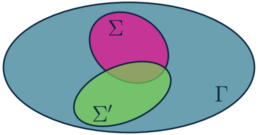
- How does *structure* of ensembles relate to *design* properties?

- Channel designs are more subtly related to *usefulness* or *capability*



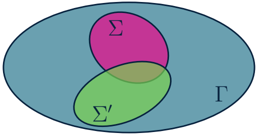
Operational Meaning of Moments of Quantum Channels

- How does *structure* of ensembles relate to *design* properties?
- Channel designs are more subtly related to *usefulness* or *capability*
- *Noise induced* phenomena are actually channel *design* phenomena



Operational Meaning of Moments of Quantum Channels

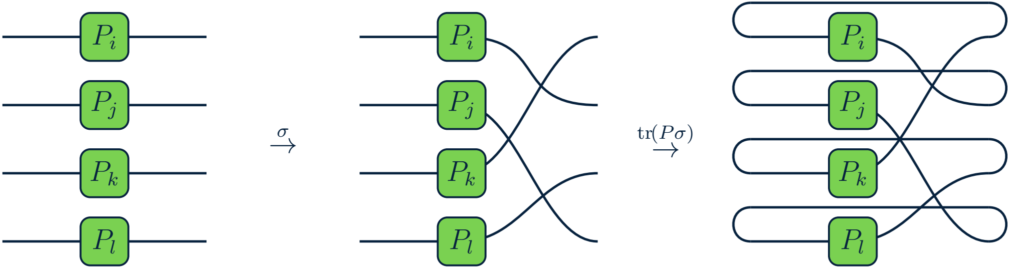
- How does *structure* of ensembles relate to *design* properties?
- Channel designs are more subtly related to *usefulness* or *capability*
- *Noise induced* phenomena are actually channel *design* phenomena
- Are there relationships between channel designs and their *simulability*? [8]



[8] Mele, A. *et al.*, *arXiv:2403.13927* (2024)

Appendix

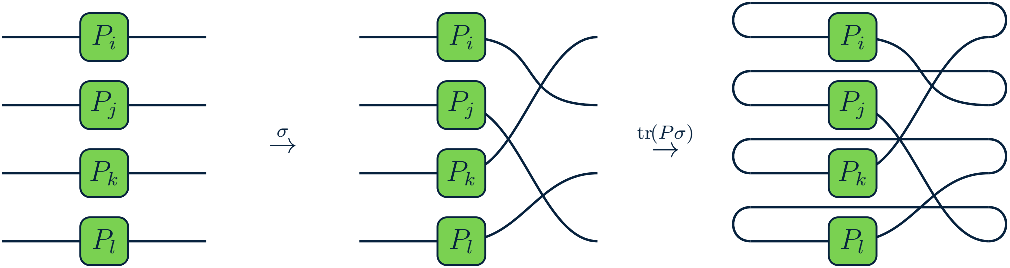
Diagrammatic Expansions of Permutations



$P = P_i \otimes P_j \otimes P_k \otimes P_l$ $\qquad$ $\sigma = (ijlk)$ $\qquad$ $\mathrm{tr}(P\sigma) = \mathrm{tr}\left(\prod_{\mu\in\sigma} P_\mu\right)$

$$\boxed{\tau = \frac{1}{d}\sum_{P\in\mathcal{P}_d} P\otimes P^\dagger} \quad \to \quad \boxed{\sigma = \frac{1}{d^{|\sigma|}}\sum_{P\in\mathcal{P}_d^{(\sigma)}} P} \tag{38}$$

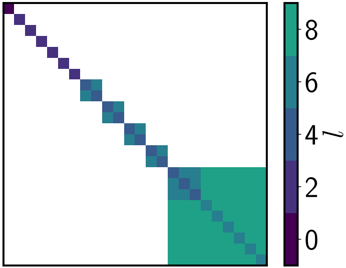
Diagrammatic Expansions of Permutations



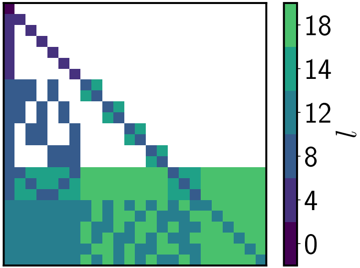
$P = P_i \otimes P_j \otimes P_k \otimes P_l$ $\qquad$ $\sigma = (ijlk)$ $\qquad$ $\mathrm{tr}(P\sigma) = \mathrm{tr}\left(\prod_{\mu \in \sigma} P_\mu\right)$

$$\widehat{\mathcal{T}}_{\Sigma}^{(t)} = \boxed{\frac{1}{d^t} \sum_{\sigma,\pi \in \mathcal{S}_{\Sigma}^{(t)}} \tau_d^{(t)}(\sigma,\pi) |\sigma\rangle\langle\pi|} = \boxed{\frac{1}{d^t} |I\rangle\langle I| + \frac{1}{d^t} \sum_{\substack{P,S \in \mathcal{P}_d^{(\mathcal{S}_{\Sigma}^{(t)})} \\ P \notin \{I\}}} \tau_d^{(t)}(P,S)\ |P\rangle\langle S|} \tag{39}$$

Transfer Matrices



(a) Haar Transfer Matrix Elements
$\tau^{(t)}_{\mathcal{U}(d)}(P,S) \sim O(1/d^l)$ for $t=4$



(a) cHaar Transfer Matrix Elements
$\tau_{\mathcal{C}(d,d_{\mathcal{E}})}(P,S) \sim O(1/d^l)$ for $t=4$

$$\widehat{\mathcal{T}}^{(t)}_{\Sigma} = \frac{1}{d^t} \sum_{P,S \in \mathcal{S}^{(t)}_{\Sigma}} \tau^{(t)}_{\Sigma}(P,S) |P\rangle\langle S| \tag{40}$$

Haar, cHaar, and Depolarizing Ensembles

Σ \ t	1	2
Haar	$\frac{1}{d}\|I\rangle\!\langle I\|$	$\frac{1}{d^2}\|I\rangle\!\langle I\| + \frac{1}{d^2}\frac{1}{d^2-1}\sum_{P,S\in\mathcal{P}_d^{(\tau)}\setminus\{I\}}\|P\rangle\!\langle S\|$
cHaar	$\frac{1}{d}\|I\rangle\!\langle I\|$	$\frac{1}{d^2}\|I\rangle\!\langle I\| + \frac{1}{d^2}\frac{d_{\mathcal{E}}-1}{d^2d_{\mathcal{E}}^2-1}\sum_{P\in\mathcal{P}_d^{(\tau)}\setminus\{I\}}\|P\rangle\!\langle I\| + \frac{1}{d^2}\frac{d_{\mathcal{E}}}{d^2d_{\mathcal{E}}^2-1}\sum_{P,S\in\mathcal{P}_d^{(\tau)}\setminus\{I\}}\|P\rangle\!\langle S\|$
Depolarize	$\frac{1}{d^t}\|I\rangle\!\langle I\|$	

Table 1: Moment operators $\widehat{\mathcal{T}}_{\Sigma}^{(t)}$ for various ensembles and moments

Monotonic Convergence and Hierarchy of Ensembles

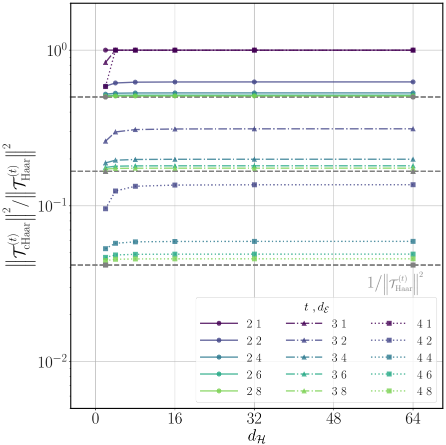


Figure 3: cHaar t-order moment operator norms convergence with $d, d_{\mathcal{E}}$.

$$1 \;=\; \|\widehat{\mathcal{T}}^{(t)}_{\mathcal{D}(d)}\|^2 \;\leq\; \|\widehat{\mathcal{T}}^{(t)k}_{\mathcal{C}(d,d_{\mathcal{E}})}\|^2 \;\leq\; \|\widehat{\mathcal{T}}^{(t)}_{\mathcal{U}(d)}\|^2 \;=\; t! \tag{41}$$

Subtleties with Concentration of Channel Ansatz

Given $\Lambda \sim \mathcal{U}_\theta$ for $U_\theta = e^{-i\theta G}$, with involutory generators G and pure inputs ρ: Objective $\mathcal{L}_\Lambda$ variance concentrates as

$$\sigma^2_{\mathcal{L}} \sim \mathcal{O}\left(\frac{1}{dd_{\mathcal{E}}}\right) + \|O\|_p^2 \, \mathcal{E}^{(2|q)}_{\Sigma\Sigma'}[\rho] \tag{42}$$

$$\sigma^2_{\mathcal{L}_\Lambda|\Sigma}[\rho, O] \leq \begin{cases} \mathcal{O}\left(\frac{d_{\mathcal{O}}}{d_{\mathcal{E}}}\frac{1}{d^2}\right) + \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \, \|\mathcal{T}^{(2)}_\Sigma - \mathcal{T}^{(2)}_{\Sigma'}\|_p & \{O_{\text{Pauli}} \,,\, \mathcal{C}(d, d_{\mathcal{E}})'\} \\ \mathcal{O}\left(\frac{1}{d^2}\right) + \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \, \|\mathcal{T}^{(2)}_\Sigma - \mathcal{T}^{(2)}_{\Sigma'}\|_p & \{O_{\text{Projector}} \,,\, \mathcal{C}(d, d_{\mathcal{E}})'\} \\ \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \, \|\mathcal{T}^{(2)}_\Sigma - \mathcal{T}^{(2)}_{\Sigma'}\|_p & \{O_{\text{Pauli}} \,,\, \mathcal{D}(d)'\} \\ \mathcal{O}\left(\frac{1}{d^2}\right) + \min_{\frac{1}{p}+\frac{1}{q}=1} \|O\|_q^2 \, \|\mathcal{T}^{(2)}_\Sigma - \mathcal{T}^{(2)}_{\Sigma'}\|_p & \{O_{\text{Projector}} \,,\, \mathcal{D}(d)'\} \end{cases} . \tag{43}$$

Subtleties with Concentration of Channel Ansatz

Given $\Lambda \sim \mathcal{U}_\theta$ for $U_\theta = e^{-i\theta G}$, with involutory generators G and pure inputs ρ: Objective gradient $\partial_\mu \mathcal{L}_\Lambda$ variance concentrates as

$$\sigma^2_{\partial_\mu \mathcal{L}} \sim \mathcal{O}\left(\frac{1}{d d_{\mathcal{E}}}\right) + \mathcal{O}\left(\mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\ \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O]\right) \tag{44}$$

$$\sigma^2_{\partial_\mu \mathcal{L}_\Lambda|\Sigma\Sigma'_{RL}}[\rho, O] \leq \sigma^{2RL}_{\partial_\mu \mathcal{L}_\Lambda|\Sigma'_{\mu_{RL}}}[\rho, O] + \begin{cases} \min_{\frac{1}{p}+\frac{1}{q}=1} \mathcal{O}\left(\frac{1}{d^2 d_{\mathcal{E}}^2}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho] + & \{O_{\mathrm{Orthogonal}},\ \Sigma'_{\mathrm{cHaar}}\} \\ \quad \mathcal{O}\left(\frac{1}{d^2 d_{\mathcal{E}}}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[O] + 4\, \mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\, \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O] & \\ \min_{\frac{1}{p}+\frac{1}{q}=1} \mathcal{O}\left(\frac{1}{d^3 d_{\mathcal{E}}^2}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho] + & \{O_{\mathrm{Projector}},\ \Sigma'_{\mathrm{cHaar}}\} \\ \quad \mathcal{O}\left(\frac{1}{d^2 d_{\mathcal{E}}}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[O] + 4\, \mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\, \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O] & \\ \min_{\frac{1}{p}+\frac{1}{q}=1} 4\, \mathcal{E}^{(2|p^*)}_{\Sigma_{\mu_R}\Sigma'_{\mu_R}}[\rho]\, \mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu_L}\Sigma'_{\mu_L}}[O] & \left\{\Sigma'_{\mathrm{Depolarize}}\right\} \end{cases} \tag{45}$$

where the *left* (L) and *right* (R) 2-design gradient variance is

$$\sigma^{2RL}_{\partial_\mu \mathcal{L}_\Lambda|\Sigma'_{\mu_{RL}}}[\rho, O] = \begin{cases} \mathcal{O}\left(\frac{1}{d\, d_{\mathcal{E}}^2}\right) & \{O_{\mathrm{Orthogonal}},\ \Sigma'_{\mathrm{cHaar}}\} \\ \mathcal{O}\left(\frac{1}{d^2 d_{\mathcal{E}}^2}\right) & \{O_{\mathrm{Projector}},\ \Sigma'_{\mathrm{cHaar}}\} \\ 0 & \left\{\Sigma'_{\mathrm{Depolarize}}\right\} \end{cases} \tag{46}$$