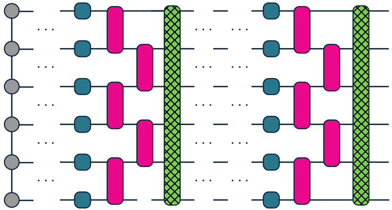
Simulation of Noisy Quantum Systems with POVM-MPS Tensor Networks

Matthew Duschenes

December 2, 2024

Carrasquilla Group Meeting



UNIVERSITY OF WATERLOO | IQC Institute for Quantum Computing

PERIMETER INSTITUTE

ETH zürich

How to Simulate Quantum Channels

Simulating Unitaries versus Channels

- Simulation of channels vs. unitaries is inherently a *higher dimensional* problem

$$U|\psi\rangle \in \mathbb{C}^{D^N} \quad \rightarrow \quad \Lambda(\rho) \in \mathbb{C}^{D^{2N}} \tag{1}$$

Simulating Unitaries versus Channels

- Simulation of channels vs. unitaries is inherently a *higher dimensional* problem

$$U|\psi\rangle \in \mathbb{C}^{D^N} \quad \rightarrow \quad \Lambda(\rho) \in \mathbb{C}^{D^{2N}} \tag{1}$$

- How do we differentiate between "quantum" correlations in *pure* states, versus "classical" correlations in *mixed* states?

Simulating Unitaries versus Channels

- Simulation of channels vs. unitaries is inherently a *higher dimensional* problem

$$U|\psi\rangle \in \mathbb{C}^{D^N} \quad \rightarrow \quad \Lambda(\rho) \in \mathbb{C}^{D^{2N}} \tag{1}$$

- How do we differentiate between "quantum" correlations in *pure* states, versus "classical" correlations in *mixed* states?

- Does *noise* make simulations *easier* [1]?

[1] Mele, A. *et al.*, *arXiv:2403.13927* (2024)

Simulating Unitaries versus Channels

- Simulation of channels vs. unitaries is inherently a *higher dimensional* problem

$$U|\psi\rangle \in \mathbb{C}^{D^N} \quad \rightarrow \quad \Lambda(\rho) \in \mathbb{C}^{D^{2N}} \tag{1}$$

- How do we differentiate between "quantum" correlations in *pure* states, versus "classical" correlations in *mixed* states?

- Does *noise* make simulations *easier* ?

- Is there an *area law* for noise/classical correlations?
  i.e) How does simulation complexity depend on system size, Kraus rank [2]?

[2] Noh, K. *et al.*, *Quantum* **4** (2020)

Simulating Unitaries versus Channels

- Simulation of channels vs. unitaries is inherently a *higher dimensional* problem

$$U|\psi\rangle \in \mathbb{C}^{D^N} \quad \rightarrow \quad \Lambda(\rho) \in \mathbb{C}^{D^{2N}} \tag{1}$$

- How do we differentiate between "quantum" correlations in *pure* states, versus "classical" correlations in *mixed* states?

- Does *noise* make simulations *easier* ?

- Is there an *area law* for noise/classical correlations?
  i.e) How does simulation complexity depend on system size, Kraus rank ?

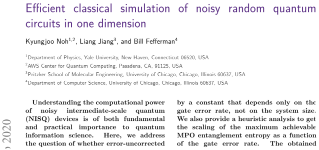
- Can we *efficiently* perform variational algorithms in non-unitary settings?

Numerical and Analytical Analyses of Noisy Simulations

- Various numerical tensor networks methods

Numerical and Analytical Analyses of Noisy Simulations

- Various numerical tensor networks methods
  - Matrix Product Operators (MPO) - Intuitive, but no guaranteed positivity [2]



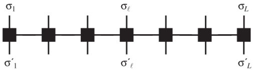
Efficient classical simulation of noisy random quantum circuits in one dimension

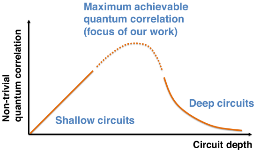
Kyungjoo Noh[1,2], Liang Jiang[3], and Bill Fefferman[4]

[1]Department of Physics, Yale University, New Haven, Connecticut 06520, USA
[2]AWS Center for Quantum Computing, Pasadena, CA, 91125, USA
[3]Pritzker School of Molecular Engineering, University of Chicago, Chicago, Illinois 60637, USA
[4]Department of Computer Science, University of Chicago, Chicago, Illinois 60637, USA

Understanding the computational power of noisy intermediate-scale quantum (NISQ) devices is of both fundamental and practical importance to quantum information science. Here, we address the question of whether error-uncorrected

by a constant that depends only on the gate error rate, not on the system size. We also provide a heuristic analysis to get the scaling of the maximum achievable MPO entanglement entropy as a function of the gate error rate. The obtained





[2] Noh, K. *et al.*, *Quantum* **4** (2020)

Numerical and Analytical Analyses of Noisy Simulations

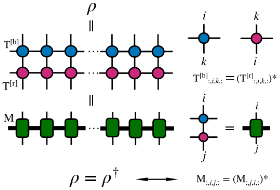
- Various numerical tensor networks methods
  - Matrix Product Operators (MPO) - Intuitive, but no guaranteed positivity
  - Locally Purified Density Operator (LPDO) - Positivity, but complicated [3]

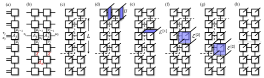

PHYSICAL REVIEW RESEARCH **3**, 023005 (2021)

Simulating noisy quantum circuits with matrix product density operators

Song Cheng,[1,2] Chenfeng Cao,[3] Chao Zhang,[2] Yongxiang Liu,[2] Shi-Yao Hou,[4,2] Pengxiang Xu,[2] and Bei Zeng[3]
[1] *Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 100407, China*
[2] *Center for Quantum Computing, Peng Cheng Laboratory, Shenzhen 518055, China*
[3] *Department of Physics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China*
[4] *College of Physics and Electronic Engineering, Center for Computational Sciences, Sichuan Normal University, Chengdu 610068, Ch*

(Received 7 July 2020; accepted 1 March 2021; published 1 April 2021)

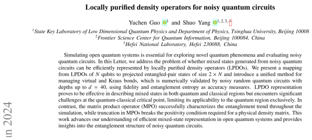




[3] Cheng, S. *et al.*, *Phys. Rev. Res.* **3**, 023005 (2021)

Numerical and Analytical Analyses of Noisy Simulations

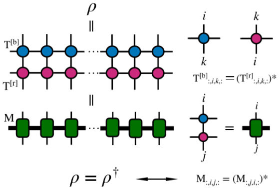
- Various numerical tensor networks methods
  - Matrix Product Operators (MPO) - Intuitive, but no guaranteed positivity
  - Locally Purified Density Operator (LPDO) - Positivity, but complicated [4]

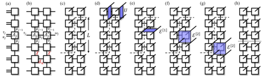


Locally purified density operators for noisy quantum circuits

Yuchen Guo [1] and Shuo Yang [1,2,3,*]
[1] State Key Laboratory of Low Dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 10008
[2] Frontier Science Center for Quantum Information, Beijing 100084, China
[3] Hefei National Laboratory, Hefei 230088, China

Simulating open quantum systems is essential for exploring novel quantum phenomena and evaluating noisy quantum circuits. In this Letter, we address the problem of whether mixed states generated from noisy quantum circuits can be efficiently represented by locally purified density operators (LPDOs). We present a mapping from LPDOs of N qubits to projected entangled-pair states of size $2 \times N$ and introduce a unified method for managing virtual and Kraus bonds, which is numerically validated by noisy random quantum circuits with depths up to $d = 40$, using fidelity and entanglement entropy as accuracy measures. LPDO representation proves to be effective in describing mixed states in both quantum and classical regions but encounters significant challenges at the quantum-classical critical point, limiting its applicability to the quantum region exclusively. In contrast, the matrix product operator (MPO) successfully characterizes the entanglement trend throughout the simulation, while truncation in MPOs breaks the positivity condition required for a physical density matrix. This work advances our understanding of efficient mixed-state representation in open quantum systems and provides insights into the entanglement structure of noisy quantum circuits.

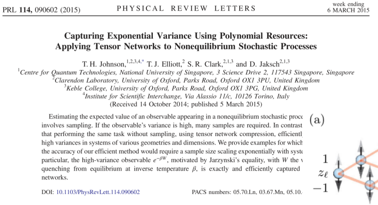




[4] Guo, Y. & Yang, S., *arXiv:2312.02854* (2023)

Numerical and Analytical Analyses of Noisy Simulations

- Various numerical tensor networks methods
  - Matrix Product Operators (MPO) - Intuitive, but no guaranteed positivity
  - Locally Purified Density Operator (LPDO) - Positivity, but complicated
  - Stochastic Matrix Product State (sMPS) - Classical processes, but ... [5]



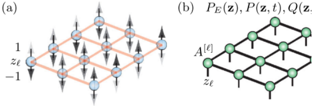
PRL **114**, 090602 (2015) PHYSICAL REVIEW LETTERS week ending 6 MARCH 2015

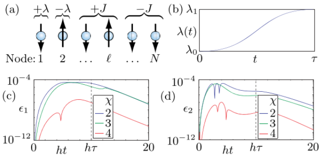
Capturing Exponential Variance Using Polynomial Resources: Applying Tensor Networks to Nonequilibrium Stochastic Processes

T. H. Johnson,[1,2,3,4,*] T. J. Elliott,[2] S. R. Clark,[2,1,3] and D. Jaksch[2,1,3]
[1]Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Singapore
[2]Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom
[3]Keble College, University of Oxford, Parks Road, Oxford OX1 3PG, United Kingdom
[4]Institute for Scientific Interchange, Via Alassio 11/c, 10126 Torino, Italy
(Received 14 October 2014; published 5 March 2015)

Estimating the expected value of an observable appearing in a nonequilibrium stochastic proc
involves sampling. If the observable's variance is high, many samples are required. In contrast
that performing the same task without sampling, using tensor network compression, efficientl
high variances in systems of various geometries and dimensions. We provide examples for which
the accuracy of our efficient method would require a sample size scaling exponentially with syst
particular, the high-variance observable $e^{-\beta W}$, motivated by Jarzynski's equality, with W the
quenching from equilibrium at inverse temperature β, is exactly and efficiently captured
networks.

DOI: 10.1103/PhysRevLett.114.090602 PACS numbers: 05.70.Ln, 03.67.Mn, 05.10.

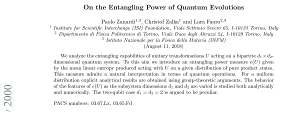




[5] Johnson, T. H. *et al.*, *PRE* **82**, 036702 (2010)

Numerical and Analytical Analyses of Noisy Simulations

- Various analytical random matrix theoretic methods
  - Operator Entanglement Measures - ”Entangling power” of operators [6]

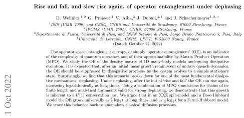


On the Entangling Power of Quantum Evolutions

Paolo Zanardi[1,3], Christof Zalka[1] and Lara Faoro[2,3]
[1] Institute for Scientific Interchange (ISI) Foundation, Viale Settimio Severo 65, I-10133 Torino, Italy
[2] Dipartimento di Fisica Politecnico di Torino, Viale Duca degli Abruzzi 24, I-10129 Torino, Italy
[3] Istituto Nazionale per la Fisica della Materia (INFM)
(August 11, 2018)

We analyze the entangling capabilities of unitary transformations U acting on a bipartite $d_1 \times d_2$-dimensional quantum system. To this aim we introduce an entangling power measure $e(U)$ given by the mean linear entropy produced acting with U on a given distribution of pure product states. This measure admits a natural interpretation in terms of quantum operations. For a uniform distribution explicit analytical results are obtained using group-theoretic arguments. The behavior of the features of $e(U)$ as the subsystem dimensions d_1 and d_2 are varied is studied both analytically and numerically. The two-qubit case $d_1 = d_2 = 2$ is argued to be peculiar.

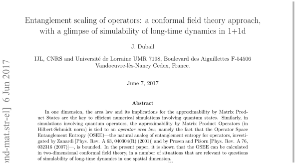
PACS numbers: 03.67.Lx, 03.65.Fd



Rise and fall, and slow rise again, of operator entanglement under dephasing

D. Wellnitz,[1,2] G. Preisser,[1] V. Alba,[3] J. Dubail,[4,1] and J. Schachenmayer[1,2,*]
[1] ISIS (UMR 7006) and CESQ, CNRS and Université de Strasbourg, 67000 Strasbourg, France
[2] IPCMS (UMR 7504), CNRS, 67000 Strasbourg, France
[3] Dipartimento di Fisica, Università di Pisa, and INFN Sezione di Pisa, Largo Bruno Pontecorvo 3, Pisa, Italy
[4] Université de Lorraine, CNRS, LPCT, F-54000 Nancy, France
(Dated: October 24, 2022)

The operator space entanglement entropy, or simply 'operator entanglement' (OE), is an indicator of the complexity of quantum operators and of their approximability by Matrix Product Operators (MPO). We study the OE of the density matrix of 1D many-body models undergoing dissipative evolution. It is expected that, after an initial linear growth reminiscent of unitary quench dynamics, the OE should be suppressed by dissipative processes as the system evolves to a simple stationary state. Surprisingly, we find that this scenario breaks down for one of the most fundamental dissipative mechanisms: dephasing. Under dephasing, after the initial 'rise and fall' the OE can rise again, increasing logarithmically at long times. Using a combination of MPO simulations for chains of infinite length and analytical arguments valid for strong dephasing, we demonstrate that this growth is inherent to a $U(1)$ conservation law. We argue that in an XXZ spin-model and a Bose-Hubbard model the OE grows universally as $\frac{1}{4}\log_2 t$ at long times, and as $\frac{1}{2}\log_2 t$ for a Fermi-Hubbard model. We trace this behavior back to anomalous classical diffusion processes.



Entanglement scaling of operators: a conformal field theory approach, with a glimpse of simulability of long-time dynamics in 1+1d

J. Dubail

IJL, CNRS and Université de Lorraine UMR 7198, Boulevard des Aiguillettes F-54506 Vandoeuvre-lès-Nancy Cedex, France.

June 7, 2017

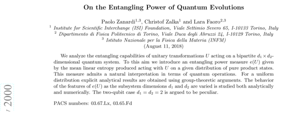
Abstract

In one dimension, the area law and its implications for the approximability by Matrix Product States are the key to efficient numerical simulations involving quantum states. Similarly, in simulations involving quantum operators, the approximability by Matrix Product Operators (in Hilbert-Schmidt norm) is tied to an *operator area law*, namely the fact that the Operator Space Entanglement Entropy (OSEE)—the natural analog of entanglement entropy for operators, investigated by Zanardi [Phys. Rev. A 63, 040304(R) (2001)] and by Prosen and Pižorn [Phys. Rev. A 76, 032316 (2007)]—, is bounded. In the present paper, it is shown that the OSEE can be calculated in two-dimensional conformal field theory, in a number of situations that are relevant to questions of simulability of long-time dynamics in one spatial dimension.

[6] Zanardi, P. *et al.*, *PRA* **62**, 4 (2000)

Numerical and Analytical Analyses of Noisy Simulations

- Various analytical random matrix theoretic methods
  - Operator Entanglement Measures - "Entangling power" of operators
  - Analysis of Dissipated Quantum Chaos - Entanglement dynamics [7]

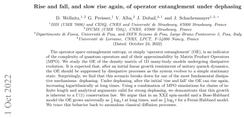


On the Entangling Power of Quantum Evolutions

Paolo Zanardi[1,3], Christof Zalka[1] and Lara Faoro[2,3]
[1] *Institute for Scientific Interchange (ISI) Foundation, Viale Settimio Severo 65, I-10133 Torino, Italy*
[2] *Dipartimento di Fisica Politecnico di Torino, Viale Duca degli Abruzzi 24, I-10129 Torino, Italy*
[3] *Istituto Nazionale per la Fisica della Materia (INFM)*
(August 11, 2018)

We analyze the entangling capabilities of unitary transformations U acting on a bipartite $d_1 \times d_2$-dimensional quantum system. To this aim we introduce an entangling power measure $e(U)$ given by the mean linear entropy produced acting with U on a given distribution of pure product states. This measure admits a natural interpretation in terms of quantum operations. For a uniform distribution explicit analytical results are obtained using group-theoretic arguments. The behavior of the features of $e(U)$ as the subsystem dimensions d_1 and d_2 are varied is studied both analytically and numerically. The two-qubit case $d_1 = d_2 = 2$ is argued to be peculiar.

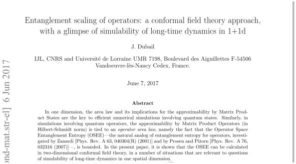
PACS numbers: 03.67.Lx, 03.65.Fd



Rise and fall, and slow rise again, of operator entanglement under dephasing

D. Wellnitz,[1,2] G. Preisser,[1] V. Alba,[3] J. Dubail,[4,1] and J. Schachenmayer[1,2,*]
[1] *ISIS (UMR 7006) and CESQ, CNRS and Université de Strasbourg, 67000 Strasbourg, France*
[2] *IPCMS (UMR 7504), CNRS, 67000 Strasbourg, France*
[3] *Dipartimento di Fisica, Università di Pisa, and INFN Sezione di Pisa, Largo Bruno Pontecorvo 3, Pisa, Italy*
[4] *Université de Lorraine, CNRS, LPCT, F-54000 Nancy, France*
(Dated: October 24, 2022)

The operator space entanglement entropy, or simply 'operator entanglement' (OE), is an indicator of the complexity of quantum operators and of their approximability by Matrix Product Operators (MPO). We study the OE of the density matrix of 1D many-body models undergoing dissipative evolution. It is expected that, after an initial linear growth reminiscent of unitary quench dynamics, the OE should be suppressed by dissipative processes as the system evolves to a simple stationary state. Surprisingly, we find that this scenario breaks down for one of the most fundamental dissipative mechanisms: dephasing. Under dephasing, after the initial 'rise and fall' the OE can rise again, increasing logarithmically at long times. Using a combination of MPO simulations for chains of infinite length and analytical arguments valid for strong dephasing, we demonstrate that this growth is inherent to a $U(1)$ conservation law. We argue that in an XXZ spin-model and a Bose-Hubbard model the OE grows universally as $\frac{1}{4}\log_2 t$ at long times, and as $\frac{1}{2}\log_2 t$ for a Fermi-Hubbard model. We trace this behavior back to anomalous classical diffusion processes.



Entanglement scaling of operators: a conformal field theory approach, with a glimpse of simulability of long-time dynamics in 1+1d

J. Dubail

IJL, CNRS and Université de Lorraine UMR 7198, Boulevard des Aiguillettes F-54506 Vandoeuvre-lès-Nancy Cedex, France.

June 7, 2017

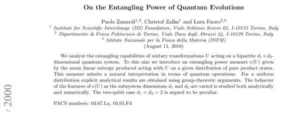
Abstract

In one dimension, the area law and its implications for the approximability by Matrix Product States are the key to efficient numerical simulations involving quantum states. Similarly, in simulations involving quantum operators, the approximability by Matrix Product Operators (in Hilbert-Schmidt norm) is tied to an *operator area law*, namely the fact that the Operator Space Entanglement Entropy (OSEE)—the natural analog of entanglement entropy for operators, investigated by Zanardi [Phys. Rev. A 63, 040304(R) (2001)] and by Prosen and Pižorn [Phys. Rev. A 76, 032316 (2007)]—, is bounded. In the present paper, it is shown that the OSEE can be calculated in two-dimensional conformal field theory, in a number of situations that are relevant to questions of simulability of long-time dynamics in one spatial dimension.

[7] Wellnitz, D. *et al.*, *PRL* **129** (2022)

Numerical and Analytical Analyses of Noisy Simulations

- Various analytical random matrix theoretic methods
  - Operator Entanglement Measures - ”Entangling power” of operators
  - Analysis of Dissipated Quantum Chaos - Entanglement dynamics
  - Conformal Field Theoretic Analysis - Operator entanglement scaling [8]

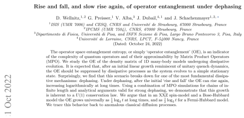


On the Entangling Power of Quantum Evolutions

Paolo Zanardi[1,3], Christof Zalka[1] and Lara Faoro[2,3]
[1] *Institute for Scientific Interchange (ISI) Foundation, Viale Settimio Severo 65, I-10133 Torino, Italy*
[2] *Dipartimento di Fisica Politecnico di Torino, Viale Duca degli Abruzzi 24, I-10129 Torino, Italy*
[3] *Istituto Nazionale per la Fisica della Materia (INFM)*
(August 11, 2018)

We analyze the entangling capabilities of unitary transformations U acting on a bipartite $d_1 \times d_2$-dimensional quantum system. To this aim we introduce an entangling power measure $e(U)$ given by the mean linear entropy produced acting with U on a given distribution of pure product states. This measure admits a natural interpretation in terms of quantum operations. For a uniform distribution explicit analytical results are obtained using group-theoretic arguments. The behavior of the features of $e(U)$ as the subsystem dimensions d_1 and d_2 are varied is studied both analytically and numerically. The two-qubit case $d_1 = d_2 = 2$ is argued to be peculiar.

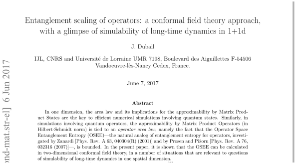
PACS numbers: 03.67.Lx, 03.65.Fd



Rise and fall, and slow rise again, of operator entanglement under dephasing

D. Wellnitz,[1,2] G. Preisser,[1] V. Alba,[3] J. Dubail,[4,1] and J. Schachenmayer[1,2,*]
[1] *ISIS (UMR 7006) and CESQ, CNRS and Université de Strasbourg, 67000 Strasbourg, France*
[2] *IPCMS (UMR 7504), CNRS, 67000 Strasbourg, France*
[3] *Dipartimento di Fisica, Università di Pisa, and INFN Sezione di Pisa, Largo Bruno Pontecorvo 3, Pisa, Italy*
[4] *Université de Lorraine, CNRS, LPCT, F-54000 Nancy, France*
(Dated: October 24, 2022)

The operator space entanglement entropy, or simply ‘operator entanglement’ (OE), is an indicator of the complexity of quantum operators and of their approximability by Matrix Product Operators (MPO). We study the OE of the density matrix of 1D many-body models undergoing dissipative evolution. It is expected that, after an initial linear growth reminiscent of unitary quench dynamics, the OE should be suppressed by dissipative processes as the system evolves to a simple stationary state. Surprisingly, we find that this scenario breaks down for one of the most fundamental dissipative mechanisms: dephasing. Under dephasing, after the initial ‘rise and fall’ the OE can rise again, increasing logarithmically at long times. Using a combination of MPO simulations for chains of infinite length and analytical arguments valid for strong dephasing, we demonstrate that this growth is inherent to a $U(1)$ conservation law. We argue that in an XXZ spin-model and a Bose-Hubbard model the OE grows universally as $\frac{1}{4}\log_2 t$ at long times, and as $\frac{1}{2}\log_2 t$ for a Fermi-Hubbard model. We trace this behavior back to anomalous classical diffusion processes.



Entanglement scaling of operators: a conformal field theory approach, with a glimpse of simulability of long-time dynamics in 1+1d

J. Dubail

IJL, CNRS and Université de Lorraine UMR 7198, Boulevard des Aiguillettes F-54506 Vandoeuvre-lès-Nancy Cedex, France.

June 7, 2017

Abstract

In one dimension, the area law and its implications for the approximability by Matrix Product States are the key to efficient numerical simulations involving quantum states. Similarly, in simulations involving quantum operators, the approximability by Matrix Product Operators (in Hilbert-Schmidt norm) is tied to an *operator area law*, namely the fact that the Operator Space Entanglement Entropy (OSEE)—the natural analog of entanglement entropy for operators, investigated by Zanardi [Phys. Rev. A 63, 040304(R) (2001)] and by Prosen and Pižorn [Phys. Rev. A 76, 032316 (2007)]—, is bounded. In the present paper, it is shown that the OSEE can be calculated in two-dimensional conformal field theory, in a number of situations that are relevant to questions of simulability of long-time dynamics in one spatial dimension.

[8] Dubail, J., *JPA* **50** (2016)

Benchmarks for Noisy Simulations

Accuracy of simulations is characterized by *infidelity* and *entanglement*

Infidelity: Similarity of states $\rho \approx \sigma$ or distributions $p \approx s$

- Quantum

$$\mathcal{L}^{Q}_{\rho\sigma} = 1 - \operatorname{tr}\left(\sqrt{\rho\sigma}\right) \tag{2}$$

- Classical

$$\mathcal{L}^{C}_{\rho\sigma} = 1 - \sqrt{p} \cdot \sqrt{s} \tag{3}$$

- Pure

$$\mathcal{L}^{P}_{\rho\sigma} = 1 - \sqrt{\operatorname{tr}\left(\rho\sigma\right)} \tag{4}$$

Benchmarks for Noisy Simulations

Accuracy of simulations is characterized by *infidelity* and *entanglement*

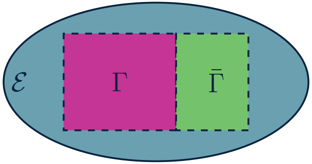
Entanglement: Entropy of states $\rho_\Gamma = \mathrm{tr}_{\bar{\Gamma}}(\rho)$, $p_\Gamma = \sum_{\bar{\Gamma}} p$, for partitions Γ

- Quantum

$$\mathcal{S}_\rho^{Q\Gamma} = -\,\mathrm{tr}\left(\rho_\Gamma \log \rho_\Gamma\right) \tag{5}$$

- Classical

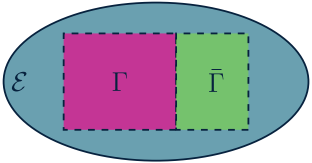
$$\mathcal{S}_\rho^{C\Gamma} = -\,p_\Gamma \cdot \log p_\Gamma \tag{6}$$



- Renyi

$$\mathcal{S}_\rho^{R\Gamma} = 1 - \mathrm{tr}\left(\rho_\Gamma^2\right) \tag{7}$$

State versus Operator Entanglement



- *State Entanglement*

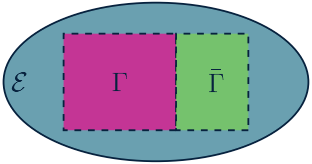
$$\rho \;\rightarrow\; \rho_\Gamma \;\rightarrow\; \mathcal{S}_\rho^\Gamma = -\mathrm{tr}\left(\rho_\Gamma \log \rho_\Gamma\right) \tag{8}$$

- Captures entropy of reduced state within Γ, which can be due to correlations with *any* other partition ($\bar{\Gamma}$, environment, ancilla etc.)

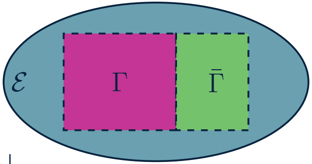
  i.e) Maximum entanglement of Γ with environment
  $\rightarrow$ Maximally mixed $\rho_\Gamma = I$, even if no correlations with $\bar{\Gamma}$
  $\rightarrow$ Maximum state entanglement $\mathcal{S}_\rho^\Gamma = \log d_\Gamma$, even though classical

State versus Operator Entanglement

- What if we just want to capture correlations *strictly* between $\Gamma, \bar{\Gamma}$?



State versus Operator Entanglement



- *Operator Entanglement*

$$\rho \to |\rho\rangle\rangle = \rho \otimes I|\Omega\rangle \to \Psi_\rho = \frac{|\rho\rangle\rangle\langle\langle\rho|}{\langle\langle\rho|\rho\rangle\rangle} \to \Psi_{\rho_\Gamma} \tag{9}$$

$$\mathcal{Q}^\Gamma_\rho = \mathcal{S}^\Gamma_{\Psi_\rho} = \mathcal{S}^\Gamma_{|\rho\rangle\rangle\langle\langle\rho|} = -\mathrm{tr}\left(\Psi_{\rho_\Gamma} \log \Psi_{\rho_\Gamma}\right) \tag{10}$$

- Captures entropy of reduced vectorized states within Γ

i.e) Pure States: $|\psi\rangle \to |\psi\rangle\langle\psi| \to |\psi\psi\rangle \to |\psi\psi\rangle\langle\psi\psi|$

i.e) Mixed States: $\sum_\lambda p_\lambda \rho_\lambda \to \sum_\lambda p_\lambda |\rho_\lambda\rangle\rangle \to \sum_{\lambda\kappa} p_\lambda p_\kappa |\rho_\lambda\rangle\rangle\langle\langle\rho_\kappa|$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\Gamma}_{|\psi\rangle} = 2\mathcal{S}^{\Gamma}_{|\psi\rangle} \tag{11}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\!\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\Gamma}_{|\psi\rangle} = 2\mathcal{S}^{\Gamma}_{|\psi\rangle} \tag{11}$$

- Product states: $\rho = \rho_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_\Gamma} \otimes \Psi_{\rho_{\bar{\Gamma}}}$

$$\mathcal{Q}^{\Gamma}_{\rho_\Gamma \otimes \rho_{\bar{\Gamma}}} = \mathcal{Q}_{\rho_\Gamma} = \mathcal{S}_{|\rho_\Gamma\rangle\!\rangle\langle\!\langle\rho_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{\rho} = \mathcal{S}_{\rho_\Gamma} \tag{12}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\!\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\Gamma}_{|\psi\rangle} = 2\mathcal{S}^{\Gamma}_{|\psi\rangle} \tag{11}$$

- Product states: $\rho = \rho_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_\Gamma} \otimes \Psi_{\rho_{\bar{\Gamma}}}$

$$\mathcal{Q}^{\Gamma}_{\rho_\Gamma \otimes \rho_{\bar{\Gamma}}} = \mathcal{Q}_{\rho_\Gamma} = \mathcal{S}_{|\rho_\Gamma\rangle\!\rangle\langle\!\langle\rho_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{\rho} = \mathcal{S}_{\rho_\Gamma} \tag{12}$$

- Maximum entanglement with environment: $\rho = I_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = |\Omega_\Gamma\rangle\langle\Omega_\Gamma| \otimes \Psi_{\bar{\Gamma}}$

$$\mathcal{Q}^{\Gamma}_{I} = \mathcal{S}_{\Omega_\Gamma} = \mathcal{S}_{|\Omega_\Gamma\rangle\langle\Omega_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{I} = \log d_\Gamma \tag{13}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$
$$\mathcal{Q}^{\Gamma}_{|\psi\rangle} = 2\mathcal{S}^{\Gamma}_{|\psi\rangle} \tag{11}$$
- Product states: $\rho = \rho_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_\Gamma} \otimes \Psi_{\rho_{\bar{\Gamma}}}$
$$\mathcal{Q}^{\Gamma}_{\rho_\Gamma \otimes \rho_{\bar{\Gamma}}} = \mathcal{Q}_{\rho_\Gamma} = \mathcal{S}_{|\rho_\Gamma\rangle\rangle\langle\langle\rho_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{\rho} = \mathcal{S}_{\rho_\Gamma} \tag{12}$$
- Maximum entanglement with environment: $\rho = I_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = |\Omega_\Gamma\rangle\langle\Omega_\Gamma| \otimes \Psi_{\bar{\Gamma}}$
$$\mathcal{Q}^{\Gamma}_{I} = \mathcal{S}_{\Omega_\Gamma} = \mathcal{S}_{|\Omega_\Gamma\rangle\langle\Omega_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{I} = \log d_\Gamma \tag{13}$$
- Maximum entanglement with $\bar{\Gamma}$: $\rho = \Omega_{\Gamma\bar{\Gamma}} \rightarrow \Psi_\rho = \Omega_{\Gamma\bar{\Gamma}} \otimes \Omega_{\Gamma\bar{\Gamma}}$
$$\mathcal{Q}^{\Gamma}_{\Omega_{\Gamma\bar{\Gamma}}} = \mathcal{S}_{I_\Gamma \otimes I_\Gamma} = 2\log d_\Gamma \tag{14}$$

State versus Operator Entanglement

- Pure states: $\rho = |\psi\rangle\langle\psi| \;\rightarrow\; |\rho\rangle\rangle = |\psi\psi\rangle \;\rightarrow\; \Psi_\rho = \rho \otimes \rho$

$$\mathcal{Q}^{\Gamma}_{|\psi\rangle} = 2\mathcal{S}^{\Gamma}_{|\psi\rangle} \tag{11}$$

- Product states: $\rho = \rho_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = \Psi_{\rho_\Gamma} \otimes \Psi_{\rho_{\bar{\Gamma}}}$

$$\mathcal{Q}^{\Gamma}_{\rho_\Gamma \otimes \rho_{\bar{\Gamma}}} = \mathcal{Q}_{\rho_\Gamma} = \mathcal{S}_{|\rho_\Gamma\rangle\rangle\langle\langle\rho_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{\rho} = \mathcal{S}_{\rho_\Gamma} \tag{12}$$

- Maximum entanglement with environment: $\rho = I_\Gamma \otimes \rho_{\bar{\Gamma}} \;\rightarrow\; \Psi_\rho = |\Omega_\Gamma\rangle\langle\Omega_\Gamma| \otimes \Psi_{\bar{\Gamma}}$

$$\mathcal{Q}^{\Gamma}_{I} = \mathcal{S}_{\Omega_\Gamma} = \mathcal{S}_{|\Omega_\Gamma\rangle\langle\Omega_\Gamma|} = 0 \;\leq\; \mathcal{S}^{\Gamma}_{I} = \log d_\Gamma \tag{13}$$

- Maximum entanglement with $\bar{\Gamma}$: $\rho = \Omega_{\Gamma\bar{\Gamma}} \rightarrow \Psi_\rho = \Omega_{\Gamma\bar{\Gamma}} \otimes \Omega_{\Gamma\bar{\Gamma}}$

$$\mathcal{Q}^{\Gamma}_{\Omega_{\Gamma\bar{\Gamma}}} = \mathcal{S}_{I_\Gamma \otimes I_\Gamma} = 2\log d_\Gamma \tag{14}$$

- Local operations: $\rho = \sum_\lambda p_\lambda \rho_{\lambda_\Gamma} \otimes \rho_{\lambda_{\bar{\Gamma}}} \;\rightarrow\; \Psi_\rho = \sum_{\lambda\kappa} p_\lambda p_\kappa |\rho_{\lambda_\Gamma}\rangle\rangle\langle\langle\rho_{\kappa_\Gamma}| \otimes |\rho_{\lambda_{\bar{\Gamma}}}\rangle\rangle\langle\langle\rho_{\kappa_{\bar{\Gamma}}}|$

$$\mathcal{Q}^{\Gamma}_{\sum_\lambda p_\lambda \rho_{\lambda_\Gamma} \otimes \rho_{\lambda_{\bar{\Gamma}}}} \geq 0 \tag{15}$$

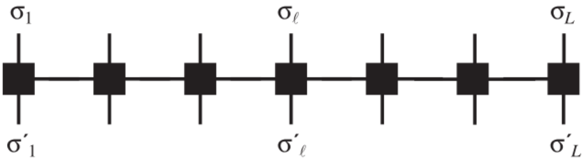
State of the Art Noisy Simulations

Tensor networks: Limited infidelity analysis, good entanglement correspondence

State of the Art Noisy Simulations

Tensor networks: Limited infidelity analysis, good entanglement correspondence

- Matrix Product Operators (MPO) [2] :
  Unclear fidelity and expected entanglement with adequate bond dimension

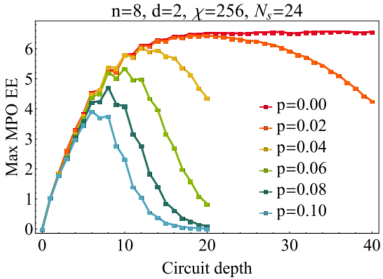


(a) MPO Tensor Network for Density Matrices, with bond dimension χ.

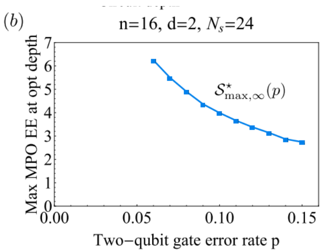
State of the Art Noisy Simulations

Tensor networks: Limited infidelity analysis, good entanglement correspondence

- Matrix Product Operators (MPO) [2] :
  Unclear fidelity and expected entanglement with adequate bond dimension



(a) MPO Operator Entanglement for various noise scales p as a function of depth, for $n = 8$ qubits.

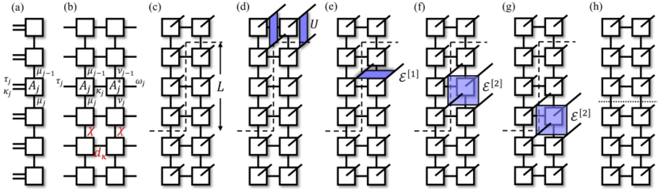


(b) MPO Maximum Operator Entanglement as a function of noise scale p, for $n = 16$ qubits, and bond dimension $\chi = O(100)$ such that $\mathrm{tr}\,(\rho) \geq 0.99$.

State of the Art Noisy Simulations

Tensor networks: Limited infidelity analysis, good entanglement correspondence

- Locally Purified Density Operator (LPDO) [4]:
  Poor infidelity and unexpected entanglement with small bond dimension



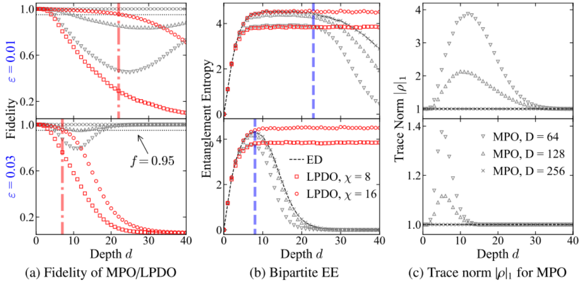
(c) LPDO Tensor Network for Density Matrices, with "unitary" χ and "noise" d_κ bond dimensions.

[4] Guo, Y. & Yang, S., *arXiv:2312.02854* (2023)

State of the Art Noisy Simulations

Tensor networks: Limited infidelity analysis, good entanglement correspondence

- Locally Purified Density Operator (LPDO) [4]:
  Poor infidelity and unexpected entanglement with small bond dimension



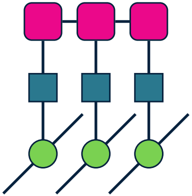
(d) LPDO Pure Fidelity and Operator Entanglement for various noise scales ϵ as a function of depth d, for $N = 8$ qubits.

[4] Guo, Y. & Yang, S., *arXiv:2312.02854* (2023)

POVM-MPS Simulation

POVM-MPS for Non-Unitary Dynamics

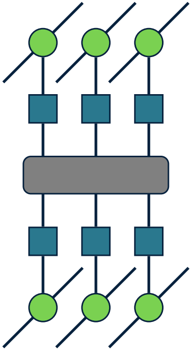
- Suppose we model dynamics of states in terms of measurement *probabilities*



$$\rho_{\sigma\sigma'} = \sum_{\mu\nu} p_\mu \, \chi^{-1}_{\mu\nu} \, \Pi_{\nu\sigma\sigma'}$$

POVM-MPS for Non-Unitary Dynamics

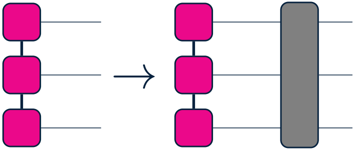
- Suppose we model dynamics of states in terms of measurement *probabilities*



$$\Lambda_{\substack{\sigma\sigma' \\ \pi\pi'}} = \sum_{\substack{\mu\nu \\ \eta\kappa}} O_{\mu\nu}\ \chi^{-1}_{\mu\eta}\ \chi^{-1}_{\nu\kappa}\ \Pi_{\eta\sigma\sigma'}\ \Pi_{\kappa\pi\pi'}$$

POVM-MPS for Non-Unitary Dynamics

- Suppose we model dynamics of states in terms of measurement *probabilities*



$$p_\mu \quad \rightarrow \sum_\nu O_{\mu\nu} p_\nu$$

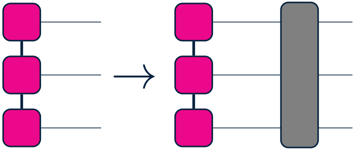
- POVMs are *local* Informationally-Complete measurements [9]

$$\Pi_{\mu\sigma\sigma'} = \otimes_i^N \Pi_{\mu_i \sigma_i \sigma_i'} \quad \leftrightarrow \quad \chi_{\mu\nu} = \langle \Pi_\mu | \Pi_\nu \rangle \; \rightarrow \; \chi_{\mu\nu}^{-1} = \otimes_i^N \chi_{\mu_i \nu_i}^{-1} \; , \tag{16}$$

[9] Carrasquilla, J. *et al.*, *PRA* **104**, 032610 (2019)

POVM-MPS for Non-Unitary Dynamics

- Suppose we model dynamics of states in terms of measurement *probabilities*



$$p_\mu \quad \to \sum_\nu O_{\mu\nu} p_\nu$$

- POVMs are *local* Informationally-Complete measurements

$$\Pi_{\mu\sigma\sigma'} = \otimes_i^N \Pi_{\mu_i \sigma_i \sigma'_i} \quad \leftrightarrow \quad \chi_{\mu\nu} = \langle \Pi_\mu | \Pi_\nu \rangle \ \to \ \chi^{-1}_{\mu\nu} = \otimes_i^N \chi^{-1}_{\mu_i \nu_i} \ , \qquad (16)$$

- Represent POVM probability $p \approx p_\chi$ as *Matrix Product State* (MPS)

Tensor Network Normalization

- Tensor Network States i.e) MPS are *canonically normalized* according to normalizing *amplitudes* i.e) SVD factorizations

$$\psi_\sigma = \prod_i A^{[i]}_{\sigma_i} \quad : \quad |\psi|^2 = \sum_\sigma |\psi_\sigma|^2 = 1 \tag{17}$$

Tensor Network Normalization

- Tensor Network States i.e) MPS are *canonically normalized* according to normalizing *amplitudes* i.e) SVD factorizations

$$\psi_\sigma = \prod_i A^{[i]}_{\sigma_i} \quad : \quad |\psi|^2 = \sum_\sigma |\psi_\sigma|^2 = 1 \tag{17}$$

- Tensor Network Probabilities i.e) pMPS are *normalized* according to normalizing *probabilities* i.e) Non-negative matrix factorizations

$$p_\mu = \prod_i A^{[i]}_{\mu_i} \quad : \quad |p| = \sum_\mu p_\mu = 1 \tag{18}$$

Tensor Network Normalization

- Tensor Network States i.e) MPS are *canonically normalized* according to normalizing *amplitudes* i.e) SVD factorizations

$$\psi_\sigma = \prod_i A^{[i]}_{\sigma_i} \quad : \quad |\psi|^2 = \sum_\sigma |\psi_\sigma|^2 = 1 \tag{17}$$

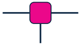
- Tensor Network Probabilities i.e) pMPS are *normalized* according to normalizing *probabilities* i.e) Non-negative matrix factorizations

$$p_\mu = \prod_i A^{[i]}_{\mu_i} \quad : \quad |p| = \sum_\mu p_\mu = 1 \tag{18}$$

- How may we *normalize and truncate* POVM-MPS representations of p_μ? [10]

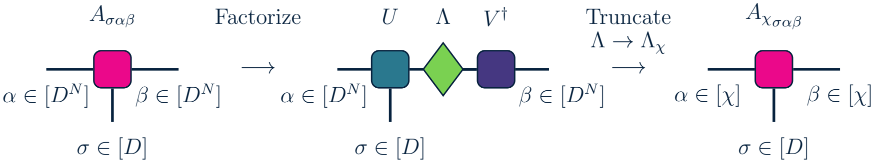
[10] Temme, K. & Verstraete, F., *PRL* **104**, 210502 (2010)

Factorization and Truncation

Factorize tensors $A_{\sigma\alpha\beta}$  , via decompositions/conditional-independence

i.e) bi-partite entangled/correlated states $|\psi_{AB}\rangle = \sum_i^D \psi_i |i_A\rangle|i_B\rangle$

Truncate virtual bonds to dimension $\chi \ll O(D^N)$



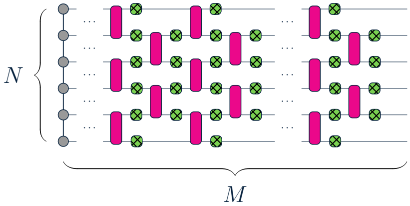
Retain largest *singular-values/probabilities* $\Lambda_\chi = \{\lambda_\alpha : \alpha < \chi\}$, with *error* [10]

$$\epsilon_\chi^2 = 1 - \sum_{\alpha<\chi} \lambda_\alpha^2 \quad , \quad \epsilon_\chi = 1 - \sum_{\alpha<\chi} \lambda_\alpha \ . \tag{19}$$

[10] Temme, K. & Verstraete, F., *PRL* **104**, 210502 (2010)

POVM-MPS Infidelity and Operator Entanglement Scaling

Random Noisy Quantum Circuits



M layers of Random *2-local* Unitaries (Pink) + Noise (Green) γ with *Product* Initial States

1. Convert states to probabilities and channels to quasi-stochastic matrices

$$\rho \to p \qquad , \qquad \Lambda \to O$$

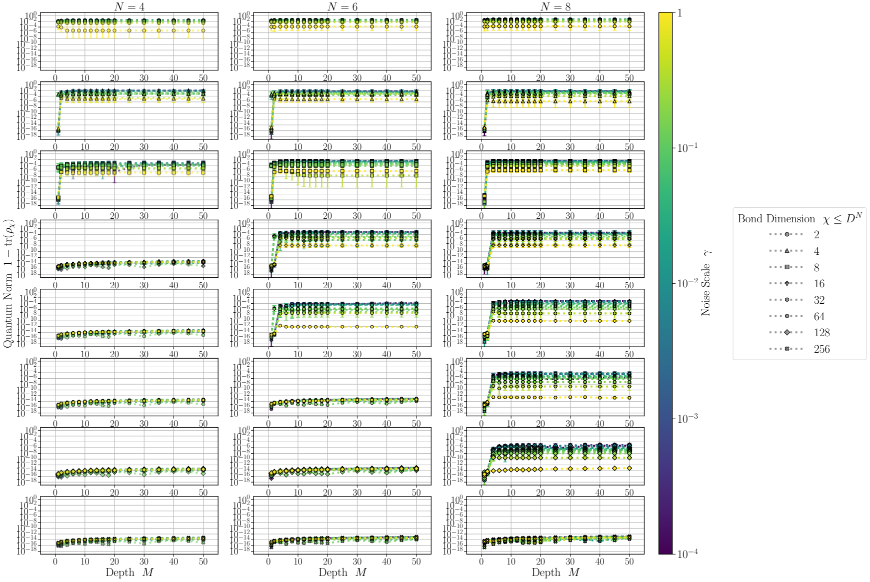
2. Apply channels and truncate bond-dimension to χ (SVD-based truncation)

$$p \to p' = Op \approx p'_\chi$$

3. Convert evolved probabilities to states ρ'_χ and compute quantities

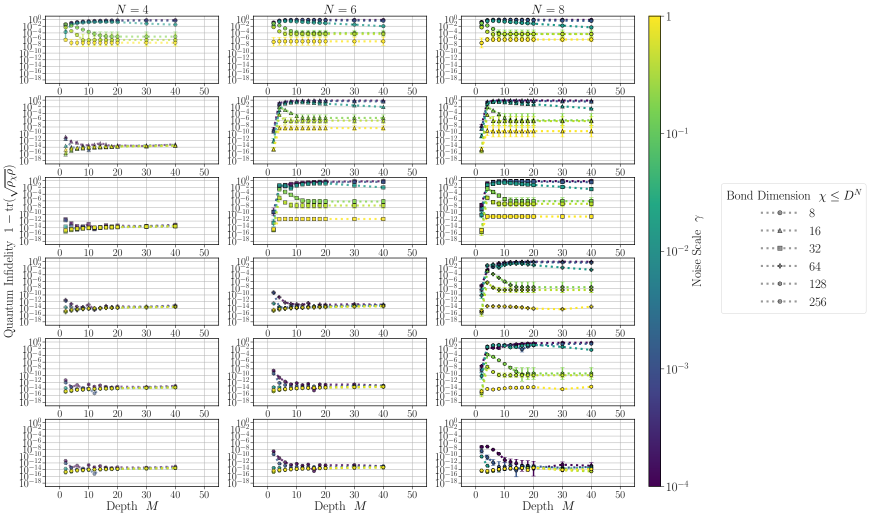
$$\mathcal{L}_{\rho_\chi \rho} \in \{\mathrm{tr}\left(\sqrt{\rho\rho_\chi}\right) , \sqrt{p} \cdot \sqrt{p_\chi} , \mathrm{tr}\left(\rho\rho_\chi\right)\} \quad , \quad \mathcal{S}^\Gamma_{\rho_\chi} \quad , \quad \mathcal{Q}^\Gamma_{\rho_\chi}$$

POVM-MPS Norm



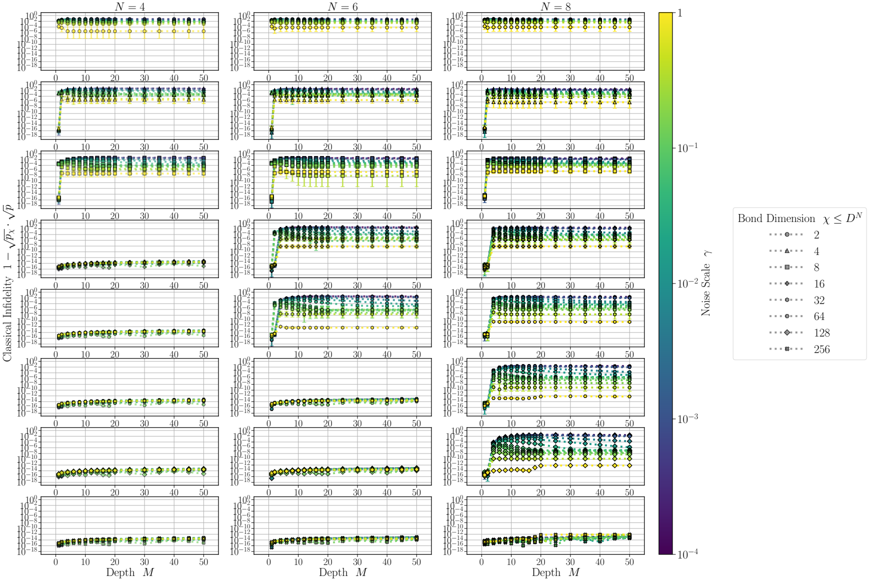
(e) Trace of POVM-MPS ρ_χ for N qubits.

POVM-MPS Infidelity



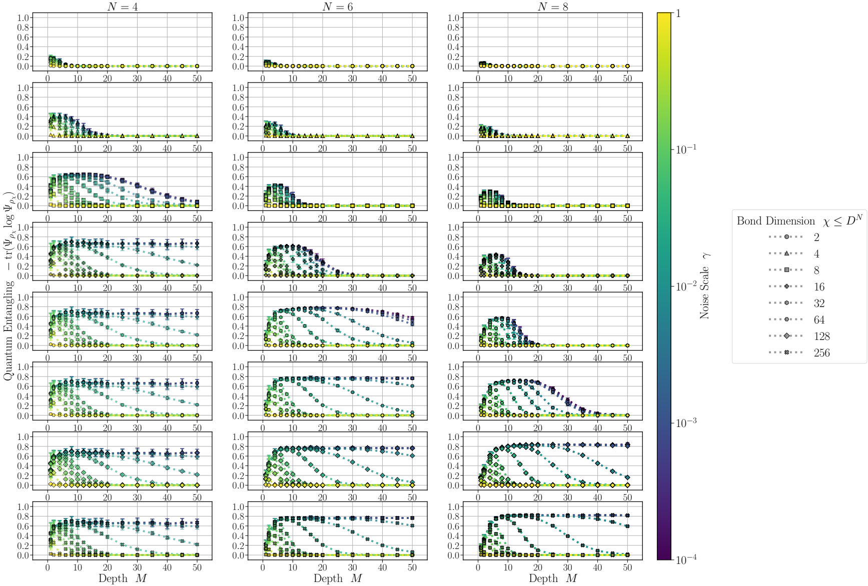
(f) Quantum Infidelity of POVM-MPS $\rho_\chi \approx \rho$ for N qubits.

POVM-MPS Infidelity



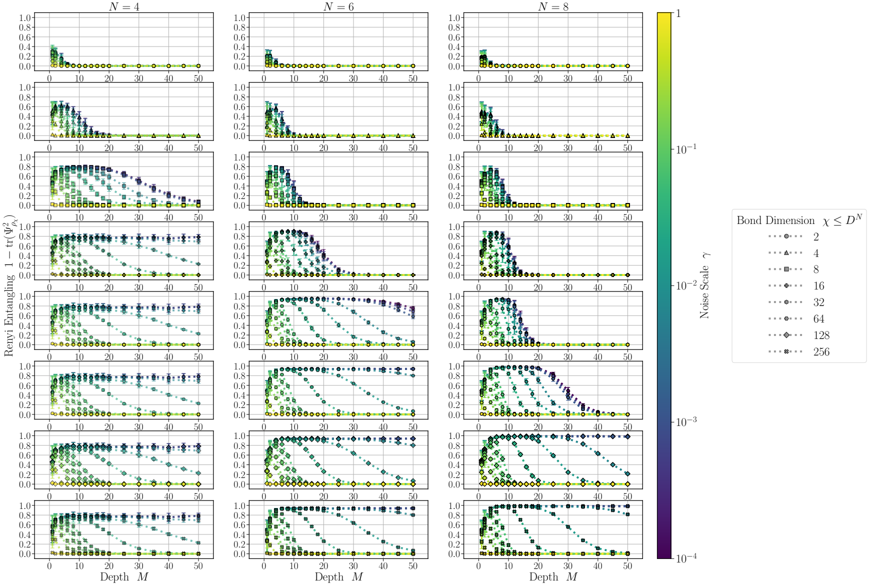
(g) Classical Infidelity of POVM-MPS $p_\chi \approx p$ for N qubits.

POVM-MPS Operator Entanglement



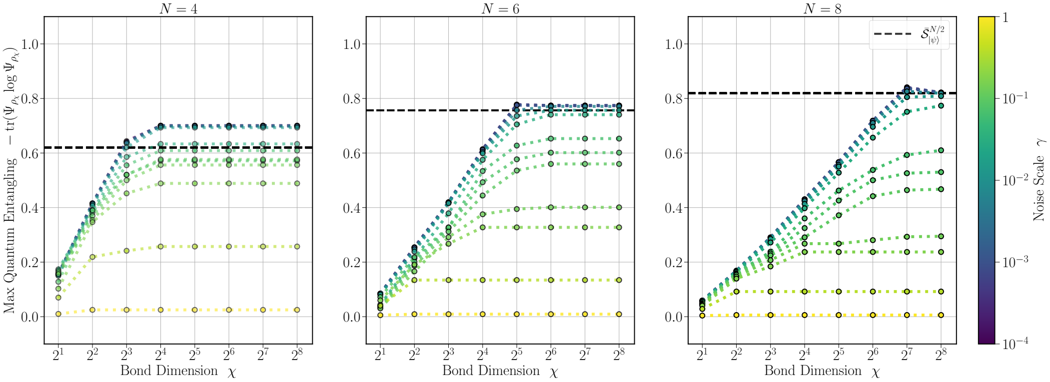
(h) Quantum Operator Entanglement of POVM-MPS ρ_χ for N qubits.

POVM-MPS Operator Entanglement



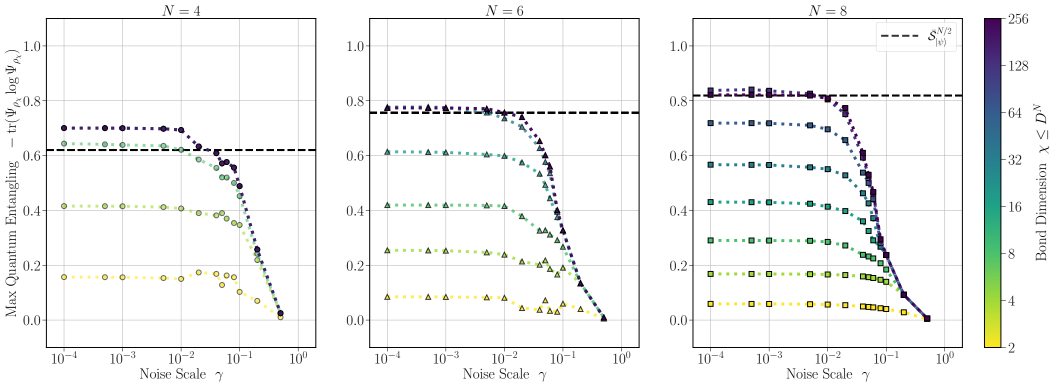
(i) Renyi Operator Entanglement of POVM-MPS ρ_χ for N qubits.

POVM-MPS Operator Entanglement Scaling



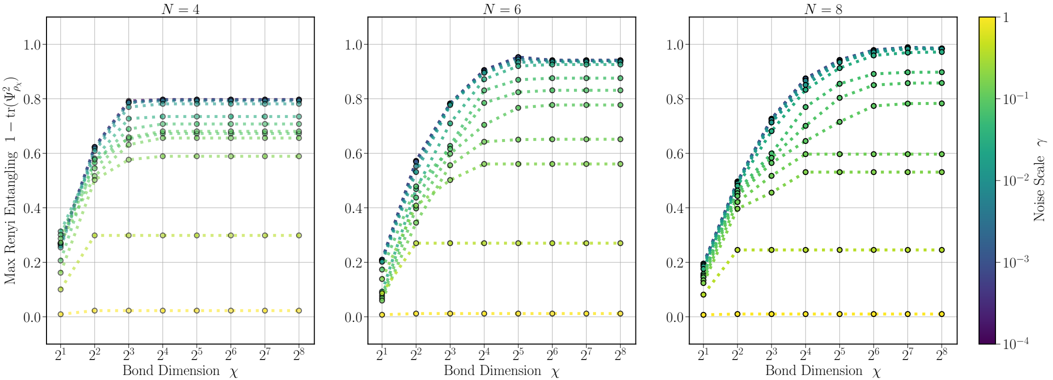
(j) Quantum Operator Entanglement of POVM-MPS ρ_χ with bond dimension χ, for N qubits.

POVM-MPS Operator Entanglement Scaling



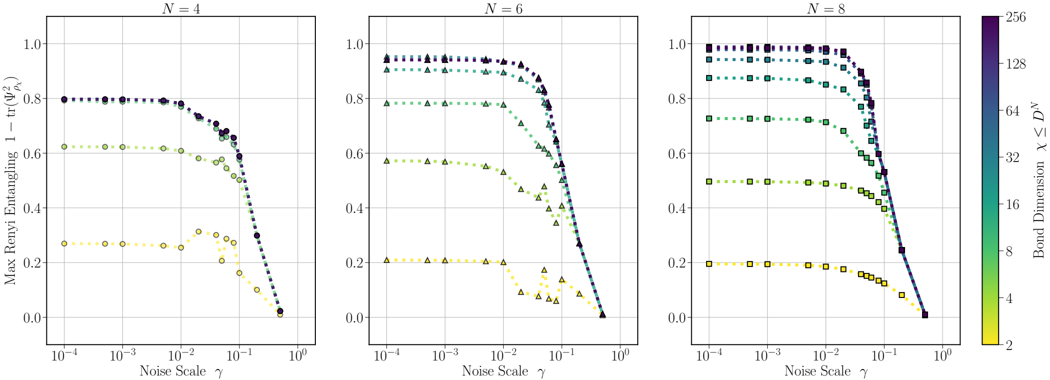
(k) Quantum Operator Entanglement of POVM-MPS ρ_χ with noise scale γ, for N qubits.

POVM-MPS Operator Entanglement Scaling



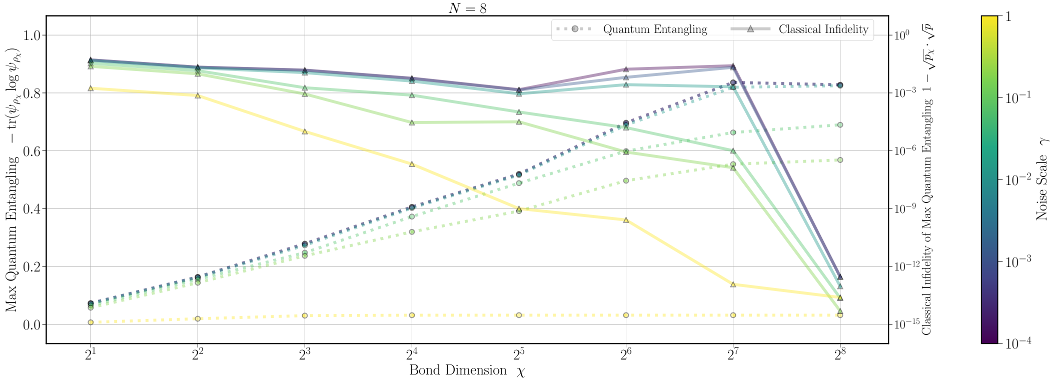
(1) Renyi Operator Entanglement of POVM-MPS ρ_χ with bond dimension χ, for N qubits.

POVM-MPS Operator Entanglement Scaling



(m) Renyi Operator Entanglement of POVM-MPS ρ_χ with noise scale γ, for N qubits.

POVM-MPS Operator Entanglement Scaling



(n) Quantum Operator Entanglement and Classical Infidelity of POVM-MPS ρ_χ with bond dimension χ, for N qubits.

Advantages and Disadvantages of POVM-MPS

- Pro: Easy to calculate traces of polynomial functions of states $\mathrm{tr}\left(\rho^2\right)$
- Con: Singular values of states λ are not automatically calculated during normalizations to calculate spectral functions of states $\mathrm{tr}\left(\rho \log \rho\right)$

Advantages and Disadvantages of POVM-MPS

- Pro: Easy to calculate traces of polynomial functions of states $\mathrm{tr}\left(\rho^2\right)$
- Con: Singular values of states λ are not automatically calculated during normalizations to calculate spectral functions of states $\mathrm{tr}\left(\rho \log \rho\right)$

- Pro: Dynamics are represented as single $(D^{2L} \times D^{2L})$ matrix multiplications
- Con: Channels must be converted to quasi-stochastic matrices at runtime

Advantages and Disadvantages of POVM-MPS

- Pro: Easy to calculate traces of polynomial functions of states $\mathrm{tr}\left(\rho^2\right)$
- Con: Singular values of states λ are not automatically calculated during normalizations to calculate spectral functions of states $\mathrm{tr}\left(\rho \log \rho\right)$

- Pro: Dynamics are represented as single $(D^{2L} \times D^{2L})$ matrix multiplications
- Con: Channels must be converted to quasi-stochastic matrices at runtime

- Pro: Single bond dimension χ to capture quantum and classical correlations
- Con: POVM probability bond dimension χ is less interpretable than MPO χ or LPDO $\chi_{\text{unitary,noise}}$ bond dimensions

What Have We Learned About Noisy Simulations?

- Infidelities, for small bond dimension POVM-MPS bond dimension χ, improve with increased noise γ, to $\mathcal{L} \sim O(10^{-6})$, is there an optimal γ?

What Have We Learned About Noisy Simulations?

- Infidelities, for small bond dimension POVM-MPS bond dimension χ, improve with increased noise γ, to $\mathcal{L} \sim O(10^{-6})$, is there an optimal γ?

- Interesting transition for different noise γ between "quantum" $\chi \sim O(e^{\mathcal{Q}})$ and "classical" $\chi \sim O(1)$, can this be verified?

What Have We Learned About Noisy Simulations?

- Infidelities, for small bond dimension POVM-MPS bond dimension χ, improve with increased noise γ, to $\mathcal{L} \sim O(10^{-6})$, is there an optimal γ?

- Interesting transition for different noise γ between "quantum" $\chi \sim O(e^{\mathcal{Q}})$ and "classical" $\chi \sim O(1)$, can this be verified?

- Physics Questions
  - Why is maximum depth before entering classical regime so small $M \sim O(10)$?
  - Why do all methods struggle with sub-maximal bond dimension $\chi < D^{2N}$?
  - How do POVM-MPS perform in higher $d > 1$ spatial dimensions [11]?

[11] Zhang, M. *et al.*, *PRA* **106** (2022)

What Have We Learned About Noisy Simulations?

- Infidelities, for small bond dimension POVM-MPS bond dimension χ, improve with increased noise γ, to $\mathcal{L} \sim O(10^{-6})$, is there an optimal γ?
- Interesting transition for different noise γ between "quantum" $\chi \sim O(e^{\mathcal{Q}})$ and "classical" $\chi \sim O(1)$, can this be verified?
- Physics Questions
  - Why is maximum depth before entering classical regime so small $M \sim O(10)$?
  - Why do all methods struggle with sub-maximal bond dimension $\chi < D^{2N}$?
  - How do POVM-MPS perform in higher $d > 1$ spatial dimensions ?
- Numerical Questions
  - Are SVD truncations adequate or are non-negative factorizations necessary?
  - Can the implementations be improved to solve variational problems?
  - How else should the methods be benchmarked?