Measures of Expressivity for Quantum Channels and their Operational Meaning

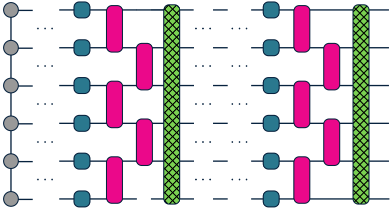
Matt Duschenes*, Diego García-Martín, Martín Larocca, Zoë Holmes, Marco Cerezo

Los Alamos National Laboratory

Winter, 2024

arXiv:2403.XXXXX

IQC Seminar

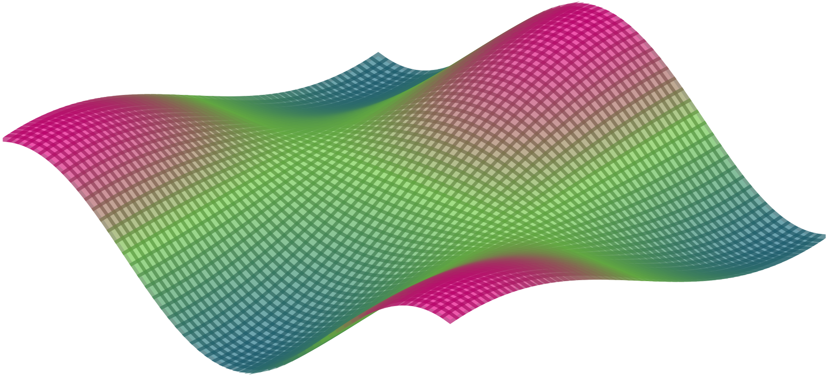


UNIVERSITY OF WATERLOO | IQC Institute for Quantum Computing

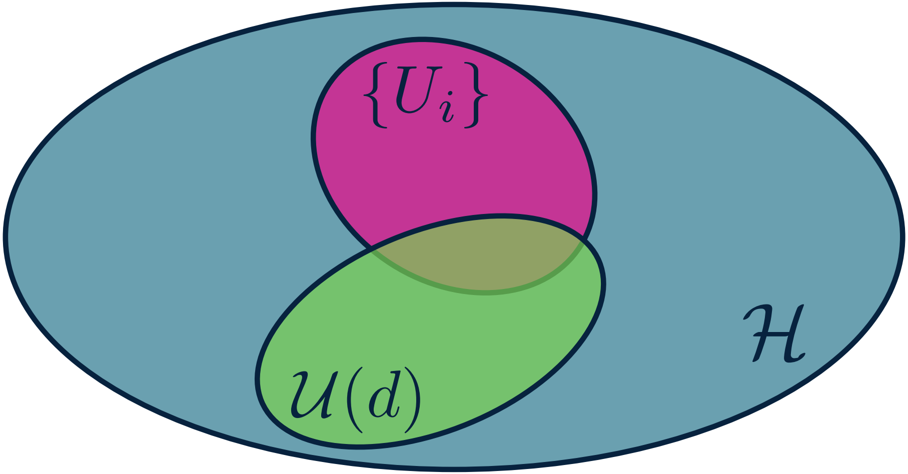
PERIMETER INSTITUTE

Los Alamos NATIONAL LABORATORY EST. 1943

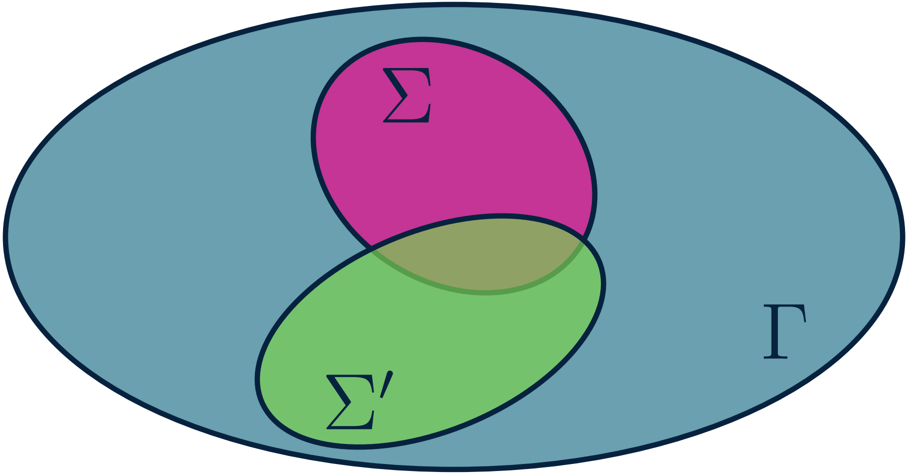
How Expressive are our Ansatze?



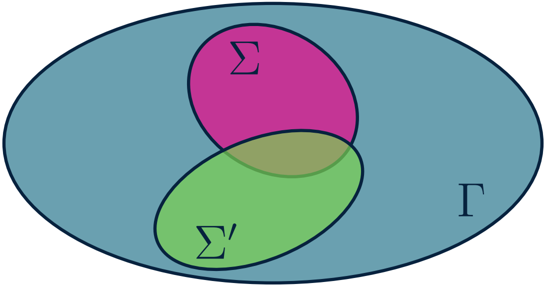
How Expressive are our Ansatze?

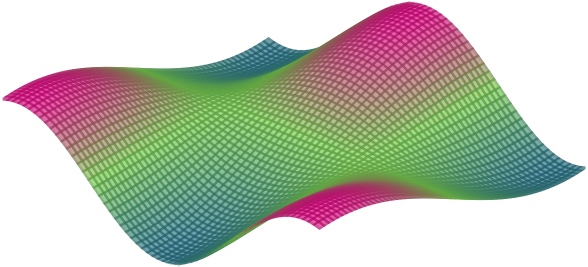


How Expressive are our Ansatze?



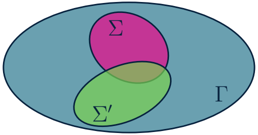
How Expressive are our Ansatze?





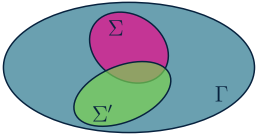
How Expressive are our Ansatze?

- Expressivity and trainability of *unitary ansatze* are well understood (Holmes, *et al.* , PRX Quantum, 2021)



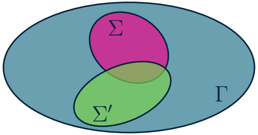
How Expressive are our Ansatze?

- Expressivity and trainability of *unitary ansatze* are well understood (Holmes, *et al.* , PRX Quantum, 2021)
- How does an ansatz compare to a *maximally expressive* reference ansatz?



How Expressive are our Ansatze?

- Expressivity and trainability of *unitary ansatze* are well understood (Holmes, *et al.* , PRX Quantum, 2021)
- How does an ansatz compare to a *maximally expressive* reference ansatz?
- How do generalized expressivity measures for channels depend on:
  - Noise induced phenomena
  - Underlying (parameterized) unitary evolution
  - Coupling with the environment



Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) = \int_{\Sigma} d\Lambda \; \Lambda^{\otimes t}(\cdot) \tag{1}$$

Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) = \int_{\Sigma} d\Lambda \ \Lambda^{\otimes t}(\cdot) \tag{1}$$

- This allows us to define an *expressivity* measure between ensembles

$$\mathcal{E}_{\Sigma\Sigma'}^{(t)}(\cdot) = \left\| \mathcal{T}_{\Sigma}^{(t)}(\cdot) - \mathcal{T}_{\Sigma'}^{(t)}(\cdot) \right\| \tag{2}$$

Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) = \int_{\Sigma} d\Lambda \; \Lambda^{\otimes t}(\cdot) \tag{1}$$

- This allows us to define an *expressivity* measure between ensembles

$$\mathcal{E}_{\Sigma\Sigma'}^{(t)}(\cdot) = \left\| \mathcal{T}_{\Sigma}^{(t)}(\cdot) - \mathcal{T}_{\Sigma'}^{(t)}(\cdot) \right\| \tag{2}$$

- Twirls over (quasi) *invariant* measures Σ are (quasi) *projections* onto its (quasi) *commutant* $\mathcal{S}_{\Sigma}^{(t)}$, allowing adapted *Weingarten* calculus approaches

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) \;=\; \boxed{\frac{\mathrm{tr}\,(\cdot)}{d^t} I} \;+\; \boxed{\Delta_{\Sigma}^{(t)}(\cdot)} \tag{3}$$

Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) = \int_{\Sigma} d\Lambda \; \Lambda^{\otimes t}(\cdot) \tag{1}$$

- This allows us to define an *expressivity* measure between ensembles

$$\mathcal{E}_{\Sigma\Sigma'}^{(t)}(\cdot) = \left\| \mathcal{T}_{\Sigma}^{(t)}(\cdot) - \mathcal{T}_{\Sigma'}^{(t)}(\cdot) \right\| \tag{2}$$

- Twirls over (quasi) *invariant* measures Σ are (quasi) *projections* onto its (quasi) *commutant* $\mathcal{S}_{\Sigma}^{(t)}$, allowing adapted *Weingarten* calculus approaches

$$\boxed{\mathcal{E}_{\Sigma}^{(t)}(\cdot) = \left\| \Delta_{\Sigma}^{(t)}(\cdot) \right\|} \tag{4}$$

Reference Ensembles

Channels present several choices for reference ensembles

Reference Ensembles

Channels present several choices for reference ensembles

- *Haar* $\sim$ Unitary Haar measure (uniformly random unitaries)

$$\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}(\cdot) = \int_{\mathcal{U}(d_{\mathcal{H}})} dU \; U^{\otimes t} \;\cdot\; U^{\otimes t\,\dagger} \tag{5}$$

Reference Ensembles

Channels present several choices for reference ensembles

- *Haar* $\sim$ Unitary Haar measure (uniformly random unitaries)

$$\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}(\cdot) = \int_{\mathcal{U}(d_{\mathcal{H}})} dU \; U^{\otimes t} \; \cdot \; U^{\otimes t \; \dagger} \tag{5}$$

- *cHaar* $\sim$ Stinespring Unitary Haar measure (Kukulski, J. Math. Phys, 2020)

$$\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}(\cdot) = \operatorname{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(d_{\mathcal{H}\otimes\mathcal{E}})} dU \; U^{\otimes t} \; \cdot \otimes \nu \; U^{\otimes t \; \dagger} \right) \tag{6}$$

Reference Ensembles

Channels present several choices for reference ensembles

- *Haar* $\sim$ Unitary Haar measure (uniformly random unitaries)

$$\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}(\cdot) = \int_{\mathcal{U}(d_{\mathcal{H}})} dU \; U^{\otimes t} \; \cdot \; U^{\otimes t\,\dagger} \tag{5}$$

- *cHaar* $\sim$ Stinespring Unitary Haar measure (Kukulski, J. Math. Phys, 2020)

$$\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}(\cdot) = \mathrm{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(d_{\mathcal{H}\otimes\mathcal{E}})} dU \; U^{\otimes t} \; \cdot \otimes \nu \; U^{\otimes t\,\dagger}\right) \tag{6}$$

- *Depolarizing* $\sim I$ (all trace preserving operations contain the identity)

$$\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}(\cdot) = \frac{\mathrm{tr}\,(\cdot)}{d^t_{\mathcal{H}}} I \tag{7}$$

Hierarchy of Ensembles

The k-concatenated, t-order twirl super-operator is found to be

$$\mathcal{T}_{\Sigma}^{(t)k} = \boxed{\frac{1}{d_{\mathcal{H}}^{t}} \left| I \right\rangle \left\langle I \right|} + \Delta_{\Sigma}^{(t,k)} \tag{8}$$

where the cHaar ensemble

$$\mathcal{E}_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} = \left\| \Delta_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} \right\| = O\left(\binom{t}{2} \frac{1}{d_{\mathcal{E}}^{k}} \right) \tag{9}$$

Hierarchy of Ensembles

The k-concatenated, t-order twirl super-operator is found to be

$$\mathcal{T}_{\Sigma}^{(t)k} = \boxed{\frac{1}{d_{\mathcal{H}}^{t}} \left|I\right\rangle \left\langle I\right|} + \Delta_{\Sigma}^{(t,k)} \tag{8}$$

where the cHaar ensemble

$$\mathcal{E}_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} = \left\| \Delta_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} \right\| = O\left(\binom{t}{2} \frac{1}{d_{\mathcal{E}}^{k}} \right) \tag{9}$$

The ensembles have exact limits of

$$\lim_{d_{\mathcal{E}} \to 1} \boxed{\mathcal{T}_{\Sigma_{\mathrm{cHaar}}}^{(t)}} \to \boxed{\mathcal{T}_{\Sigma_{\mathrm{Haar}}}^{(t)}} \quad \lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}} \to \infty} \boxed{\mathcal{T}_{\Sigma_{\mathrm{cHaar}}}^{(t)}} \to \boxed{\mathcal{T}_{\Sigma_{\mathrm{Depolarize}}}^{(t)}} \tag{10}$$

Hierarchy of Ensembles

The k-concatenated, t-order twirl super-operator is found to be

$$\mathcal{T}_{\Sigma}^{(t)k} = \boxed{\frac{1}{d_{\mathcal{H}}^{t}} \left| I \right\rangle \left\langle I \right|} + \Delta_{\Sigma}^{(t,k)} \tag{8}$$

where the cHaar ensemble

$$\mathcal{E}_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} = \left\| \Delta_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} \right\| = O\left(\binom{t}{2} \frac{1}{d_{\mathcal{E}}^{k}} \right) \tag{9}$$

The cHaar ensemble norm scales to leading order as

$$\boxed{\left\| \mathcal{T}_{\Sigma_{\mathrm{cHaar}}}^{(t)} \right\|^{2}} = \frac{1}{d_{\mathcal{E}}^{2t}} \binom{d_{\mathcal{E}}^{2} + t - 1}{t} \left| \mathcal{S}_{\Sigma}^{(t)} \right| \left[1 + O\left(\frac{1}{d_{\mathcal{H}} d_{\mathcal{E}}} \right) \right] \leq \left| \mathcal{S}_{\Sigma}^{(t)} \right| \tag{11}$$

Hierarchy of Ensembles

The k-concatenated, t-order twirl super-operator is found to be

$$\mathcal{T}_{\Sigma}^{(t)k} = \boxed{\frac{1}{d_{\mathcal{H}}^{t}} \left| I \right\rangle \left\langle I \right|} + \Delta_{\Sigma}^{(t,k)} \tag{8}$$

where the cHaar ensemble

$$\mathcal{E}_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} = \left\| \Delta_{\Sigma_{\mathrm{cHaar}}}^{(t,k)} \right\| = O\left(\binom{t}{2} \frac{1}{d_{\mathcal{E}}^{k}} \right) \tag{9}$$

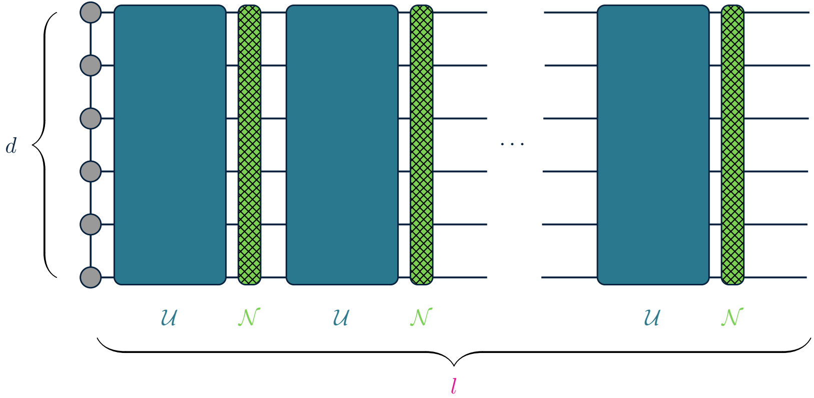
From $d_{\mathcal{H}\otimes\mathcal{E}} \to \infty$ limits, there is the conjectured hierarchy of norms

$$1 = \boxed{\left\| \mathcal{T}_{\Sigma_{\mathrm{Depolarize}}}^{(t)} \right\|} \leq \boxed{\left\| \mathcal{T}_{\Sigma_{\mathrm{cHaar}}}^{(t)} \right\|} \leq \boxed{\left\| \mathcal{T}_{\Sigma_{\mathrm{Haar}}}^{(t)} \right\|} = \sqrt{\left| \mathcal{S}_{\Sigma}^{(t)} \right|} \tag{12}$$

Relationships between Noise and Expressivity

Analytical *expressivities* for M layers of channels

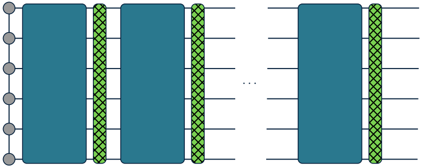
$$\Lambda_{\mathcal{U}_\gamma}^{(M)} = (\mathcal{N}_\gamma \circ \mathcal{U})^{M} = \frac{\mathrm{tr}\,(\cdot)}{d} I + \Delta_\gamma^{(M)}(\cdot) \tag{13}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + *Unital* Pauli Noise: *Increases* Expressivity

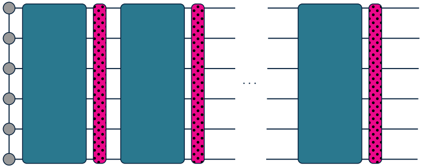
$$\mathcal{E}_{\mathcal{U}\gamma}^{(t,M)} = \left[\binom{t}{2}\left(1-\frac{1}{d^2}\right)\left(1-\gamma\right)^2\right]^M + O\left(\frac{1}{d}(1-\gamma)^{2M}\right) \tag{14}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + *Non-Unital* Pauli Noise: *Decreases* Expressivity

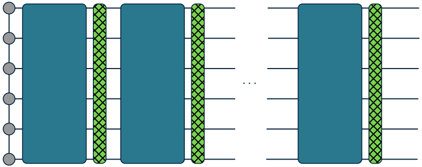
$$\mathcal{E}^{(t,M)}_{\mathcal{U}\gamma\eta} = \sqrt{t\left(d^2-1\right)}\,\eta \;+\; O\left(\eta^2\right) \tag{15}$$



Relationships between Noise and Expressivity

Pauli Parameterized Unitaries + *Unital* Pauli Noise: *Local* terms in Pauli commutant dominate

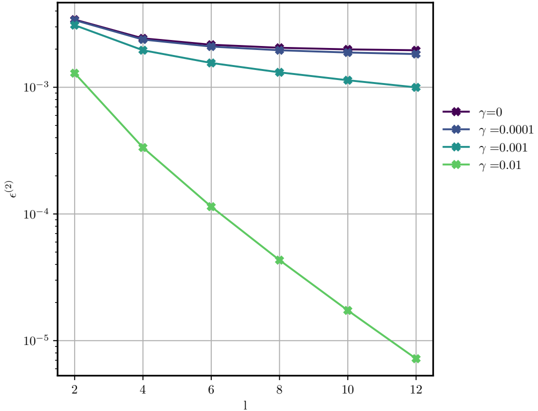
$$\mathcal{E}_{\mathcal{G}\gamma}^{(t,M)} = \sqrt{t\left(|\mathcal{S}_G| - 1\right)}\,(1-\gamma)^M + O\left((1-\gamma)^{M+1}\right) \tag{16}$$



Relationships between Noise and Expressivity

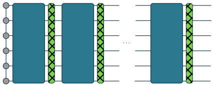
Objective $\mathcal{L}$ and Gradient $\partial\mathcal{L}$ *Concentration:* $p\left(|\mathcal{L}-\mu_{\mathcal{L}}|\geq\epsilon\right)\leq\sigma_{\mathcal{L}}^2/\epsilon^2$

$$\sigma_{\mathcal{L}}^2 \sim O\left(\frac{1}{d_{\mathcal{H}}d_{\mathcal{E}}}\right) + \|O\|_2^2\,\mathcal{E}^{(2)}_{\Sigma\Sigma'}[\rho] \quad \text{(with \textit{caveats} on } \Sigma', \rho, O \text{ locality)} \qquad (17)$$



TFIM vs. Depolarize $t=2$ expressivity as a function of depth $l=M$, for $N=6$ qubits in the initial $|+\rangle$ state, and local depolarizing noise γ

Relationships between Noise and Expressivity



- *Haar* Random Unitaries + *Unital* Pauli Noise:

$$\mathcal{E}_{\mathcal{U}\gamma}^{(t,M)} = \left[\binom{t}{2}\left(1-\frac{1}{d^2}\right)\left(1-\gamma\right)^2\right]^M + O\left(\frac{1}{d}(1-\gamma)^{2M}\right) \tag{14}$$

- *Haar* Random Unitaries + *Non-Unital* Pauli Noise:

$$\mathcal{E}_{\mathcal{U}\gamma\eta}^{(t,M)} = \sqrt{t\left(d^2-1\right)}\,\eta + O\left(\eta^2\right) \tag{15}$$

- *Pauli* Parameterized Unitaries + *Unital* Pauli Noise:

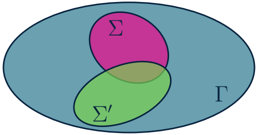
$$\mathcal{E}_{\mathcal{G}\gamma}^{(t,M)} = \sqrt{t\left(|\mathcal{S}_G|-1\right)}\,(1-\gamma)^M + O\left((1-\gamma)^{M+1}\right) \tag{16}$$

- Objective $\mathcal{L}$ and Gradient $\partial\mathcal{L}$ *Concentration*: $p\left(|\mathcal{L}-\mu_{\mathcal{L}}|\geq\epsilon\right)\leq\sigma_{\mathcal{L}}^2/\epsilon^2$

$$\sigma_{\mathcal{L}}^2 \sim O\left(\frac{1}{d_{\mathcal{H}}d_{\mathcal{E}}}\right) + \|O\|_2^2\,\mathcal{E}_{\Sigma\Sigma'}^{(2)}[\rho] \tag{17}$$

Channels versus Unitary Ensembles

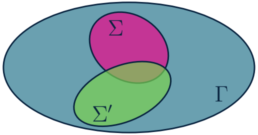
Many *subtle* differences between ensembles of channels and unitaries



Channels versus Unitary Ensembles

Many *subtle* differences between ensembles of channels and unitaries

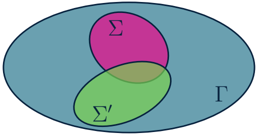
- Twirls are quasi-projections (quasi-commutant may defined via *dilation*)



Channels versus Unitary Ensembles

Many *subtle* differences between ensembles of channels and unitaries

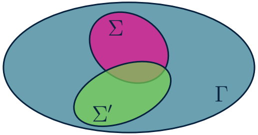
- Twirls are quasi-projections (quasi-commutant may defined via *dilation*)
- Open systems have degrees of freedom (environment *size*, *coupling* strength)



Channels versus Unitary Ensembles

Many *subtle* differences between ensembles of channels and unitaries

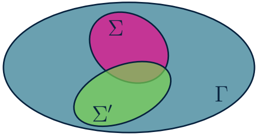
- Twirls are quasi-projections (quasi-commutant may defined via *dilation*)
- Open systems have degrees of freedom (environment *size*, *coupling* strength)
- Adjoint channels are not strictly *physical* channels (concentration *caveats*)



Channels versus Unitary Ensembles

Many *subtle* differences between ensembles of channels and unitaries

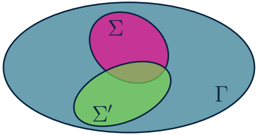
- Twirls are quasi-projections (quasi-commutant may defined via *dilation*)
- Open systems have degrees of freedom (environment *size*, *coupling* strength)
- Adjoint channels are not strictly *physical* channels (concentration *caveats*)
- Subtleties in realizing channel t-designs in *practice*



Channels versus Unitary Ensembles

Ensembles of channels have inherently different *interpretations*

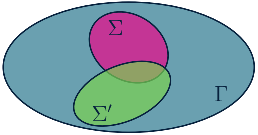
- *Uniformly Random*: cHaar channels are a *uniform* random ensemble



Channels versus Unitary Ensembles

Ensembles of channels have inherently different *interpretations*

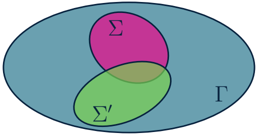
- *Uniformly Random*: cHaar channels are a *uniform* random ensemble
- *Capacities*: Depolarizing channels maximize environment *exchange entropy*



Channels versus Unitary Ensembles

Ensembles of channels have inherently different *interpretations*

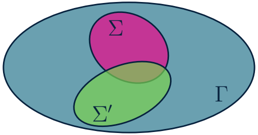
- *Uniformly Random*: cHaar channels are a *uniform* random ensemble
- *Capacities*: Depolarizing channels maximize environment *exchange entropy*
- *Tomography*: Depolarizing channels maximize *uncertainty* in measurements



Channels versus Unitary Ensembles

Ensembles of channels have inherently different *interpretations*

- *Uniformly Random*: cHaar channels are a *uniform* random ensemble
- *Capacities*: Depolarizing channels maximize environment *exchange entropy*
- *Tomography*: Depolarizing channels maximize *uncertainty* in measurements
- *Scrambling*: Depolarizing channels maximally *scramble* information



Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually *channel* expressivity phenomena!

Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually *channel* expressivity phenomena!
- Many *subtle* differences between ensembles of unitary and non-unitary channels

Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually *channel* expressivity phenomena!
- Many *subtle* differences between ensembles of unitary and non-unitary channels
- Choice of reference ensemble reflects retained *coherent information*

Operational Meaning of Channel Expressivity Measures

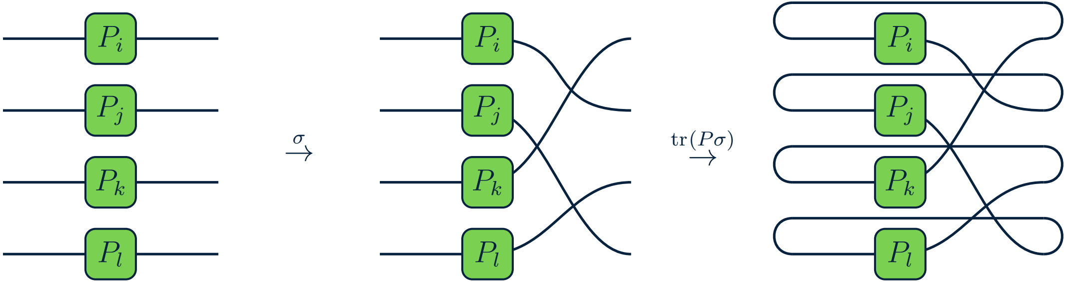
- *Noise induced* phenomena are actually *channel* expressivity phenomena!
- Many *subtle* differences between ensembles of unitary and non-unitary channels
- Choice of reference ensemble reflects retained *coherent information*
- Are there *practical* ensembles of channels that approach t-designs?

Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually *channel* expressivity phenomena!
- Many *subtle* differences between ensembles of unitary and non-unitary channels
- Choice of reference ensemble reflects retained *coherent information*
- Are there *practical* ensembles of channels that approach t-designs?
- Are there relationships between channel expressivity and their *simulability*?

Appendix

Diagrammatic Expansions of Permutations



$P = P_i \otimes P_j \otimes P_k \otimes P_l$ $\qquad$ $\sigma = (ijlk)$ $\qquad$ $\mathrm{tr}\left(P\sigma\right) = \mathrm{tr}\left(\prod_{\mu\in\sigma} P_\mu\right)$

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) = \frac{\mathrm{tr}\left(\cdot\right)}{d^t} I \; + \; \frac{1}{d^t} \sum_{P \in \mathcal{P}_d^{(\mathcal{S}_{\Sigma}^{(t)})} \setminus \{I\}} \tau_d^{(t)}(P, \cdot)\; P \tag{18}$$

Haar, cHaar, and Depolarizing Ensembles

$\Sigma \backslash t$	1	2
Haar	$\frac{\mathrm{tr}(\cdot)}{d_{\mathcal{H}}} I$	$\frac{\mathrm{tr}(\cdot)}{d_{\mathcal{H}}^2} I + \frac{1}{d_{\mathcal{H}}^2} \sum_{P \in \mathcal{P}_{d_{\mathcal{H}}}^{(\mathcal{S}_{\Sigma}^{(2)})} \setminus \{I\}} \tau_{d_{\mathcal{H}}}^{(2)}(P, \cdot) \; P \otimes P^{-1}$
cHaar	$\frac{\mathrm{tr}(\cdot)}{d_{\mathcal{H}}} I$	$\frac{\mathrm{tr}(\cdot)}{d_{\mathcal{H}}^2} I + \frac{1}{d_{\mathcal{H}}^2} \sum_{P \in \mathcal{P}_{d_{\mathcal{H}}}^{(\mathcal{S}_{\Sigma}^{(2)})} \setminus \{I\}} \tau_{d_{\mathcal{H}}, d_{\mathcal{E}}}^{(2)}(P, \cdot) \; P \otimes P^{-1}$
Depolarizing	$\frac{\mathrm{tr}(\cdot)}{d_{\mathcal{H}}^t} I$	

Table 1: Twirls $\mathcal{T}_{\Sigma}^{(t)}$ for various ensembles and moments

Subtleties with Concentration of Channel Ansatz

Given $\Lambda \sim \mathcal{U}_\theta$ with involutory generators $U_\theta = e^{-i\theta G}$, and pure inputs ρ:

Objective $\mathcal{L}_\Lambda^O$ variance concentrates as

$$\sigma^2_{\mathcal{L}_\Lambda^O|\Sigma}[\rho] \leq \begin{cases} O\left(\frac{1}{d_\mathcal{H} d_\mathcal{E}}\right) + \min\left\{\|O\|_2^2 \mathcal{E}^{(2)}_{\Sigma\Sigma'}[\rho] \ , \ \|O\|_\infty^2 \mathcal{E}^{(2|\diamond)}_{\Sigma\Sigma'}\right\} & \left\{O_{\text{Pauli}} \ , \ \Sigma'_{\text{cHaar}}\right\} \\ O\left(\frac{1}{d_\mathcal{H}^2}\right) + \min\left\{\|O\|_2^2 \mathcal{E}^{(2)}_{\Sigma\Sigma'}[\rho] \ , \ \|O\|_\infty^2 \mathcal{E}^{(2|\diamond)}_{\Sigma\Sigma'}\right\} & \left\{O_{\text{Projector}} \ , \ \Sigma'_{\text{cHaar}}\right\} \\ 0 + \min\left\{\|O\|_2^2 \mathcal{E}^{(2)}_{\Sigma\Sigma'}[\rho] \ , \ \|O\|_\infty^2 \mathcal{E}^{(2|\diamond)}_{\Sigma\Sigma'}\right\} & \left\{O_{\text{Pauli}} \ , \ \Sigma'_{\text{Depolarize}}\right\} \\ O\left(\frac{1}{d_\mathcal{H}^2}\right) + \min\left\{\|O\|_2^2 \mathcal{E}^{(2)}_{\Sigma\Sigma'}[\rho] \ , \ \|O\|_\infty^2 \mathcal{E}^{(2|\diamond)}_{\Sigma\Sigma'}\right\} & \left\{O_{\text{Projector}} \ , \ \Sigma'_{\text{Depolarize}}\right\} \end{cases} \tag{19}$$

Subtleties with Concentration of Channel Ansatz

Given $\Lambda \sim \mathcal{U}_\theta$ with involutory generators $U_\theta = e^{-i\theta G}$, and pure inputs ρ:

Objective gradient $\partial_\mu \mathcal{L}^O_\Lambda$ variance concentrates as

$$\sigma^2_{\partial_\mu \mathcal{L}^O_\Lambda | \Sigma\Sigma'_{RL}}[\rho] \leq \sigma^{2\,RL}_{\partial_\mu \mathcal{L}^O_\Lambda | \Sigma'_{\mu\,RL}}[\rho] + \begin{cases} \min_{\frac{1}{p}+\frac{1}{q}=1} O\left(\frac{1}{d_{\mathcal{E}}^2 d_{\mathcal{H}}^2}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[X^{\otimes 2}] + & \{O_{\text{Orthogonal}}, G_{\text{Involutory}}, \Sigma'_{\text{cHaar}}\} \\ \qquad O\left(\frac{1}{d_{\mathcal{E}} d_{\mathcal{H}}^2}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[O^{\otimes 2}] + 4\,\mathcal{E}^{(2|p^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[X^{\otimes 2}]\,\mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu\,L}\Sigma'_{\mu\,L}}[O^{\otimes 2}] & \\ \min_{\frac{1}{p}+\frac{1}{q}=1} O\left(\frac{1}{d_{\mathcal{E}}^2 d_{\mathcal{H}}^3}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[X^{\otimes 2}] + & \{O_{\text{Projector}}, G_{\text{Involutory}}, \Sigma'_{\text{cHaar}}\} \\ \qquad O\left(\frac{1}{d_{\mathcal{E}} d_{\mathcal{H}}^2}\|S\|_p\right) \mathcal{E}^{(2|q^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[O^{\otimes 2}] + 4\,\mathcal{E}^{(2|p^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[X^{\otimes 2}]\,\mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu\,L}\Sigma'_{\mu\,L}}[O^{\otimes 2}] & \\ \min_{\frac{1}{p}+\frac{1}{q}=1} 4\,\mathcal{E}^{(2|p^*)}_{\Sigma_{\mu\,R}\Sigma'_{\mu\,R}}[X^{\otimes 2}]\,\mathcal{E}^{(2|\dagger q^*)}_{\Sigma_{\mu\,L}\Sigma'_{\mu\,L}}[O^{\otimes 2}] & \left\{\Sigma'_{\text{Depolarize}}\right\} \end{cases} \tag{20}$$

where the *left* (L) and *right* (R) 2-design gradient variance is

$$\sigma^{2\,RL}_{\partial_\mu \mathcal{L}^O_\Lambda | \Sigma'_{\mu\,RL}}[\rho] = \begin{cases} O\left(\frac{1}{d_{\mathcal{E}}^2 d_{\mathcal{H}}}\right) & \{X_{\text{Pure}}, O_{\text{Orthogonal}}, G_{\text{Involutory}}, \Sigma'_{\text{cHaar}}\} \\ O\left(\frac{1}{d_{\mathcal{E}}^2 d_{\mathcal{H}}^2}\right) & \{X_{\text{Pure}}, O_{\text{Projector}}, G_{\text{Involutory}}, \Sigma'_{\text{cHaar}}\} \\ 0 & \left\{\Sigma'_{\text{Depolarize}}\right\} \end{cases} \tag{21}$$