1 Trapped Ions - Sideband Pulse Definitions

Let a real angle be θ, a complex phase be $e^{i\phi}$, for real angles θ, ϕ, and for θ, ϕ-dependent Hamiltonians $H(\theta, \phi)$, let associated θ, ϕ-dependent unitaries be

$$R(\theta, \phi) = e^{i\frac{t}{\hbar}H(\theta,\phi)} \ . \tag{1}$$

Carrier:

$$\theta = \Omega t \tag{2}$$

$$H_{Carrier}(\theta, \phi) = \frac{\hbar\Omega}{2}\left(e^{i\phi}|e\rangle\langle g| \ + \ e^{-i\phi}|g\rangle\langle e|\right) \tag{3}$$

$$\begin{aligned} R_{Carrier}(\theta, \phi) = \ &\cos\left(\frac{\theta}{2}\right)|g\rangle\langle g| \ + \ \cos\left(\frac{\theta}{2}\right)|e\rangle\langle e| \\ &+ \ e^{i\phi}\sin\left(\frac{\theta}{2}\right)|e\rangle\langle g| \ + \ e^{-i\phi}\sin\left(\frac{\theta}{2}\right)|g\rangle\langle e| \end{aligned} \tag{4}$$

Blue:

$$\theta = \eta\Omega t \tag{5}$$

$$H_{Blue}(\theta, \phi) = \frac{\hbar\eta\Omega}{2}\left(ie^{i\phi}\hat{a}^\dagger|e\rangle\langle g| - ie^{-i\phi}\hat{a}|g\rangle\langle e|\right) \tag{6}$$

$$\begin{aligned} R_{Blue}(\theta, \phi) = \ &\cos\left(\frac{\theta}{2}\sqrt{\hat{n}+1}\right)|g\rangle\langle g| \ + \ \cos\left(\frac{\theta}{2}\sqrt{\hat{n}}\right)|e\rangle\langle e| \\ &- \ e^{i\phi}\hat{a}^\dagger\frac{1}{\sqrt{\hat{n}+1}}\sin\left(\frac{\theta}{2}\sqrt{\hat{n}+1}\right)|e\rangle\langle g| \ + \ e^{-i\phi}\hat{a}\frac{1}{\sqrt{\hat{n}}}\sin\left(\frac{\theta}{2}\sqrt{\hat{n}}\right)|g\rangle\langle e| \end{aligned} \tag{7}$$

Red:

$$\theta = \eta\Omega t \tag{8}$$

$$H_{Red}(\theta, \phi) = \frac{\hbar\eta\Omega}{2}\left(ie^{i\phi}\hat{a}|e\rangle\langle g| \ - \ ie^{-i\phi}\hat{a}^\dagger|g\rangle\langle e|\right) \tag{9}$$

$$\begin{aligned} R_{Red}(\theta, \phi) = \ &\cos\left(\frac{\theta}{2}\sqrt{\hat{n}}\right)|g\rangle\langle g| \ + \ \cos\left(\frac{\theta}{2}\sqrt{\hat{n}+1}\right)|e\rangle\langle e| \\ &- \ e^{i\phi}\hat{a}\frac{1}{\sqrt{\hat{n}}}\sin\left(\frac{\theta}{2}\sqrt{\hat{n}}\right)|e\rangle\langle g| \ + \ e^{-i\phi}\hat{a}^\dagger\frac{1}{\sqrt{\hat{n}+1}}\sin\left(\frac{\theta}{2}\sqrt{\hat{n}+1}\right)|g\rangle\langle e| \end{aligned} \tag{10}$$

Note: Here, we use phase conventions which differ from the original Cirac-Zoller definitions, where they absorb the i-factors on each term of H_{Blue}, H_{Red} into the phase factors $e^{i\phi}$.

To derive the form of $R(\theta,\phi)$, given the corresponding Hamiltonian $H(\theta,\phi)$, we expand the exponential of the operator

$$R(\theta,\phi) = e^{i\frac{t}{\hbar}H(\theta,\phi)} = \sum_{k=0}^{\infty}\frac{(i\frac{\theta}{2})^k}{k!}A^k \ , \tag{11}$$

where $A(\theta,\phi) = 2tH(\theta,\phi)/\theta\hbar$, and even and odd powers of A may be considered.

For example, for the Red case, $A_{Red}(\theta,\phi) = ie^{i\phi}\hat{a}|e\rangle\langle g| \ - \ ie^{-i\phi}\hat{a}^\dagger|g\rangle\langle e|$, with powers,

$$\begin{aligned}
A^{2k}_{Red}(\theta,\phi) &= (\hat{n})^k|g\rangle\langle g| \ + \ (\hat{n}+1)^k|e\rangle\langle e| && (12)\\
&= \sqrt{\hat{n}}^k|g\rangle\langle g| \ + \ \sqrt{\hat{n}+1}^k|e\rangle\langle e| && (13)\\
A^{2k+1}_{Red}(\theta,\phi) &= ie^{i\phi}\hat{a}(\hat{n})^k|e\rangle\langle g| \ - \ ie^{-i\phi}\hat{a}^\dagger(\hat{n}+1)^k|g\rangle\langle e| && (14)\\
&= ie^{i\phi}\hat{a}\frac{1}{\sqrt{\hat{n}}}\sqrt{\hat{n}}^{2k+1}|e\rangle\langle g| \ - \ ie^{-i\phi}\hat{a}^\dagger\frac{1}{\sqrt{\hat{n}+1}}\sqrt{\hat{n}+1}^k|g\rangle\langle e| \ , && (15)
\end{aligned}$$

where the operators $\hat{a}^\dagger/\sqrt{\hat{n}+1}, \hat{a}/\sqrt{\hat{n}}$ are canonically normalized raising and lowering operators with unit, n-independent eigenvalues, yielding a unitary of,

$$\begin{aligned}
R_{Red}(\theta,\phi) = \ &\cos\left(\frac{\theta}{2}\sqrt{\hat{n}}\right)|g\rangle\langle g| \ + \ \cos\left(\frac{\theta}{2}\sqrt{\hat{n}+1}\right)|e\rangle\langle e| \qquad (16)\\
&- \ e^{i\phi}\hat{a}\frac{1}{\sqrt{\hat{n}}}\sin\left(\frac{\theta}{2}\sqrt{\hat{n}}\right)|e\rangle\langle g| \ + \ e^{-i\phi}\hat{a}^\dagger\frac{1}{\sqrt{\hat{n}+1}}\sin\left(\frac{\theta}{2}\sqrt{\hat{n}+1}\right)|g\rangle\langle e| \ .
\end{aligned}$$

Given these phase conventions, general θ,ϕ Blue and Red sideband pulses act on states,

$$R_{Blue}(\theta,\phi) \ : \ \begin{cases} |e\ n\rangle & \to \ \cos\left(\frac{\theta}{2}\sqrt{n}\right)|e\ n\rangle + e^{-i\phi}\sin\left(\frac{\theta}{2}\sqrt{n}\right)|g\ n-1\rangle \\ |g\ n-1\rangle & \to \ \cos\left(\frac{\theta}{2}\sqrt{n}\right)|g\ n-1\rangle - e^{i\phi}\sin\left(\frac{\theta}{2}\sqrt{n}\right)|e\ n\rangle \end{cases} , \tag{17}$$

$$R_{Red}(\theta,\phi) \ : \ \begin{cases} |e\ n\rangle & \to \ \cos\left(\frac{\theta}{2}\sqrt{n+1}\right)|e\ n\rangle + e^{-i\phi}\sin\left(\frac{\theta}{2}\sqrt{n+1}\right)|g\ n+1\rangle \\ |g\ n+1\rangle & \to \ \cos\left(\frac{\theta}{2}\sqrt{n+1}\right)|g\ n+1\rangle - e^{i\phi}\sin\left(\frac{\theta}{2}\sqrt{n+1}\right)|e\ n\rangle \end{cases} , \tag{18}$$

and for example, $\theta = \pi/2, \phi = 0$, Blue sideband pulses act on states,

$$R_{Blue}(\pi/2,0) \ : \ \begin{cases} |g0\rangle & \to \ \frac{1}{\sqrt{2}}\left(|g0\rangle - |e1\rangle\right) \\ |e0\rangle & \to \ |e0\rangle \\ |g1\rangle & \to \ \cos\left(\frac{\pi}{2\sqrt{2}}\right)|g1\rangle - \sin\left(\frac{\pi}{2\sqrt{2}}\right)|e2\rangle \\ |e1\rangle & \to \ \frac{1}{\sqrt{2}}\left(|g0\rangle + |e1\rangle\right) \end{cases} , \tag{19}$$

and for example, $\theta = \pi, \phi = 0$, Red sideband pulses act on states as

$$R_{Red}(\pi,0) \ : \ \begin{cases} |g0\rangle & \to \ |g0\rangle \\ |e0\rangle & \to \ |g1\rangle \\ |g1\rangle & \to \ -|e0\rangle \\ |e1\rangle & \to \ \cos\left(\frac{\pi}{\sqrt{2}}\right)|e1\rangle + \sin\left(\frac{\pi}{\sqrt{2}}\right)|g2\rangle \end{cases} . \tag{20}$$

2 Trapped Ions - Cirac-Zoller Gate Definitions

Given these phase conventions, Controlled phase gates, via Cirac-Zoller gates, may be implemented by the sequence of sideband pulses

$$U_{CZ} = R_{Red_1}(\pi,0)R_{Rabi_2}(2\pi,0)R_{Red_1}(\pi,0)\ . \tag{21}$$

The state $|\psi\rangle_{12M} = (a|g\rangle_1 + b|e\rangle_1)(c|g\rangle_2 + d|e\rangle_2)|0\rangle_M$ is transformed under U_{CZ} as,

$$\begin{aligned}
|\psi\rangle_{12M} &\overset{R_{Red_1}(\pi,0)}{\rightarrow} |g\rangle_1(c|g\rangle_2 + d|e\rangle_2)(a|0\rangle_M + b|1\rangle_M) && (22)\\
&\overset{R_{Rabi_2}(2\pi,0)}{\rightarrow} |g\rangle_1(ac|g0\rangle_{2M} - bc|g1\rangle_{2M} + ad|e0\rangle_{2M} + bd|e1\rangle_{2M}) && (23)\\
&\overset{R_{Red_1}(\pi,0)}{\rightarrow} (ac|gg\rangle_{12} + bc|eg\rangle_{12} + ad|ge\rangle_{12} - bd|ee\rangle_{12})|0\rangle_M\ , && (24)
\end{aligned}$$

and therefore states are transformed as

$$U_{CZ}\ :\ \begin{cases} |gg\rangle & \rightarrow & |gg\rangle \\ |eg\rangle & \rightarrow & |eg\rangle \\ |ge\rangle & \rightarrow & |ge\rangle \\ |ee\rangle & \rightarrow & -|ee\rangle \end{cases}\ . \tag{25}$$