Channel Expressivity Measures

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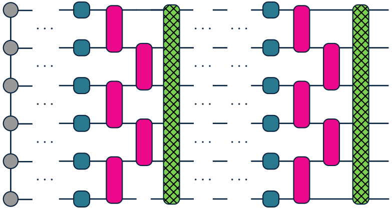
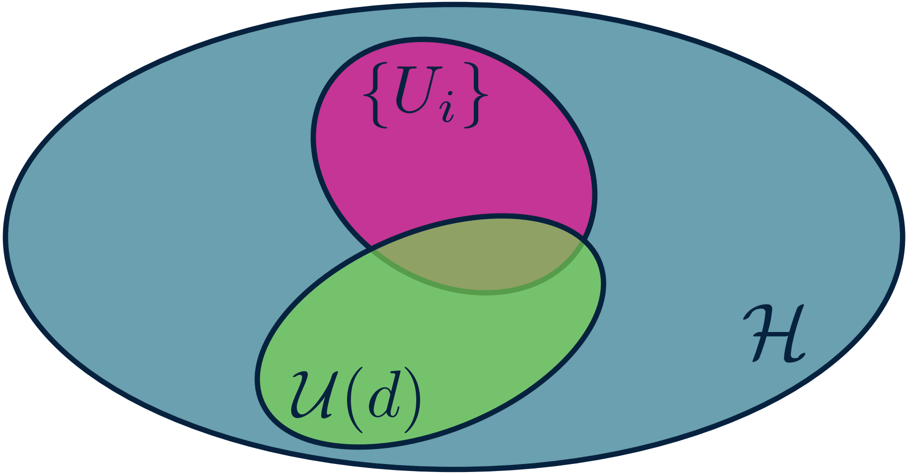


How Expressive are our Ansatze?



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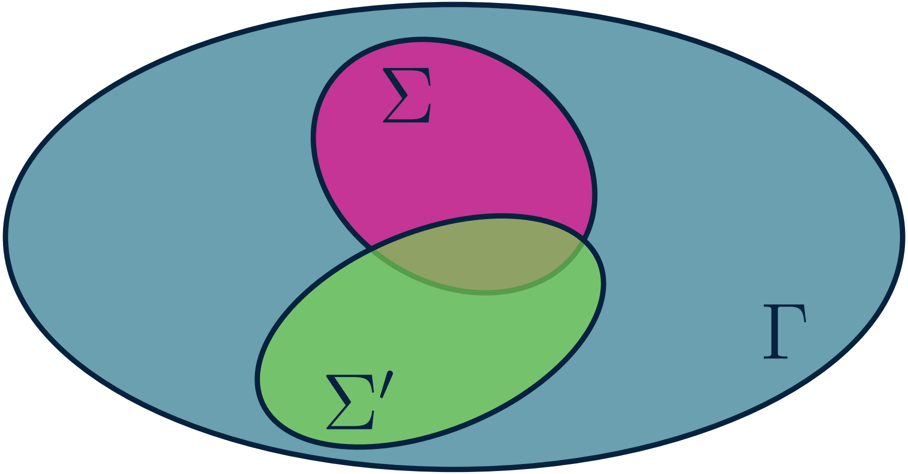

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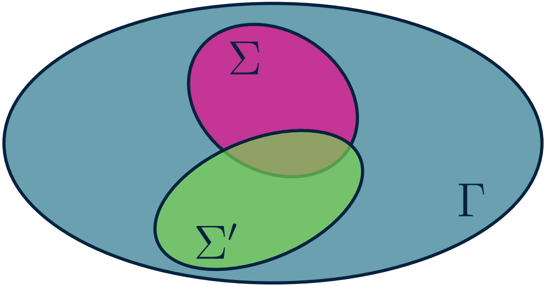

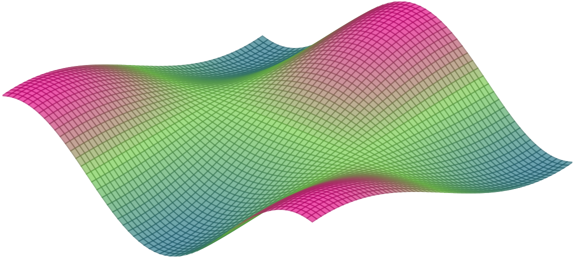

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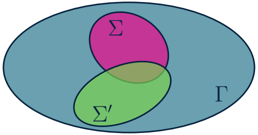
- Expressivity and trainability of *unitary ansatze* are well understood (Holmes, *et al.* , PRX Quantum, 2021)



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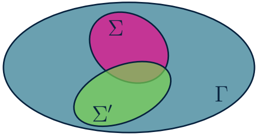

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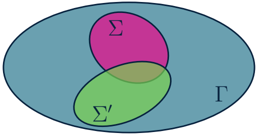
- How does an ansatz compare to a *maximally expressive* reference ansatz?
- How do generalized expressivity measures for channels depend on:
  - Noise induced phenomena
  - Underlying (parameterized) unitary evolution
  - Coupling with the environment



Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ have an *average* behaviour defined by the t-order *twirl* over t-copies of a state ρ

$$\mathcal{T}_{\Sigma}^{(t)}(\rho) = \int_{\Sigma} d\Lambda \; \Lambda^{\otimes t}(\rho^{\otimes t}) \tag{1}$$

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- *Depolarizing* $\sim I$ (all trace preserving operations contain the identity)

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}(\rho)} = \frac{\operatorname{tr}(\rho)^t}{d_{\mathcal{H}}^t} \; I^{\otimes t} \tag{7}$$

Behaviour of Random Quantum Channels

The t-order Haar, cHaar, Depolarize ensembles are related by

$$\lim_{d_{\mathcal{E}} \to 1} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}} \qquad \lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}} \to \infty} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}} \tag{8}$$

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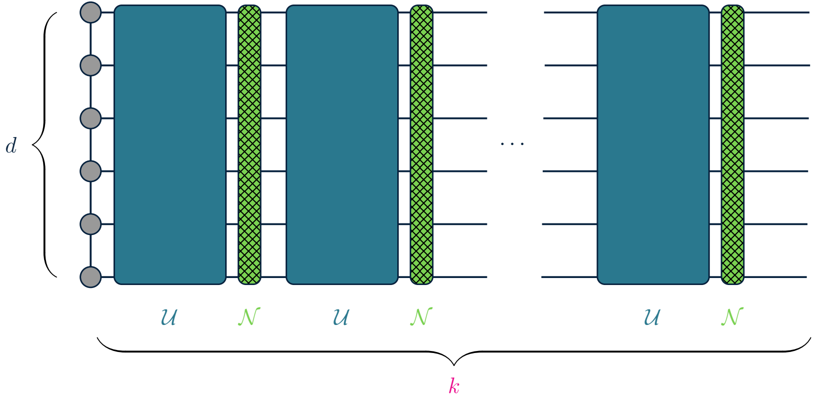
The k-concatenated, t-order cHaar ensemble is depolarizing and non-unital

$$\lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}} \to \infty} \mathcal{T}^{(t)k}_{\Sigma_{\text{cHaar}}}(\rho) = \boxed{\frac{\text{tr}(\rho)^t}{d_{\mathcal{H}}^t} I^{\otimes t}} + \boxed{O\left(\frac{1}{d_{\mathcal{H}}^2 d_{\mathcal{E}}}\right) \sum_{P \neq I^{\otimes t}} P} \tag{9}$$

Relationships between Noise and Expressivity

Analytical *expressivities* for k layers of specific channel *ansatze*

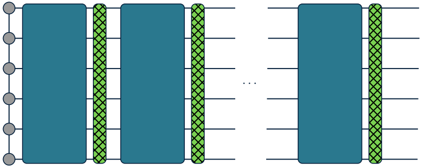
$$\Lambda^{(k)}_{\mathcal{U}_\gamma}(\rho) = (\mathcal{N}_\gamma \circ \mathcal{U})^k(\rho) = \frac{\mathrm{tr}(\rho)}{d} I + \Delta^{(k)}_\gamma(\rho) \tag{10}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + Fixed *Unital* Pauli Noise: *Increases* Expressivity

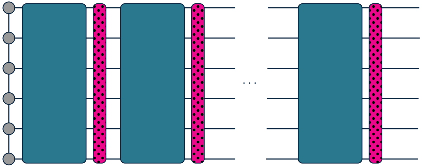
$$\mathcal{E}^{(t,k)}_{\mathcal{U}_\gamma} = O\left((1-\gamma)^{2k}\right) \tag{11}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + Fixed *Non-Unital* Pauli Noise: *Decreases* Expressivity

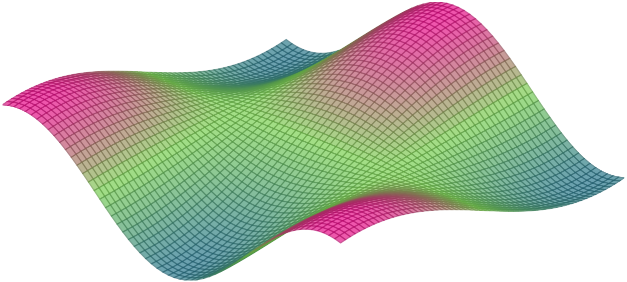
$$\mathcal{E}_{\mathcal{U}\gamma\eta}^{(t,k)} = O\left(\eta\right) \tag{12}$$



Relationships between Noise and Expressivity

Objective $\mathcal{L}$ and Gradient $\partial\mathcal{L}$ *Concentration*: $p\left(|\mathcal{L}-\mu_{\mathcal{L}}| \geq \epsilon\right) \leq \sigma_{\mathcal{L}}^2/\epsilon^2$

$$\mathcal{L}(\rho, O) = \mathrm{tr}(O\Lambda(\rho))$$



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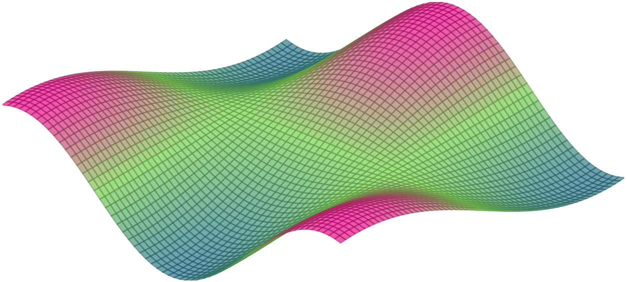
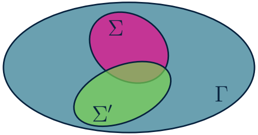
$$\sigma_{\mathcal{L}}^2 \sim O\left(\frac{1}{d_{\mathcal{H}} d_{\mathcal{E}}}\right) + \|O\|_2^2\, \mathcal{E}^{(2)}_{\Sigma\Sigma'}(\rho) \quad \text{(with } \textit{caveats} \text{ on } \Sigma', \rho, O \text{ locality)} \tag{14}$$



Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually channel *expressivity* phenomena!



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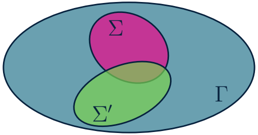
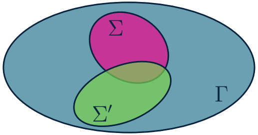


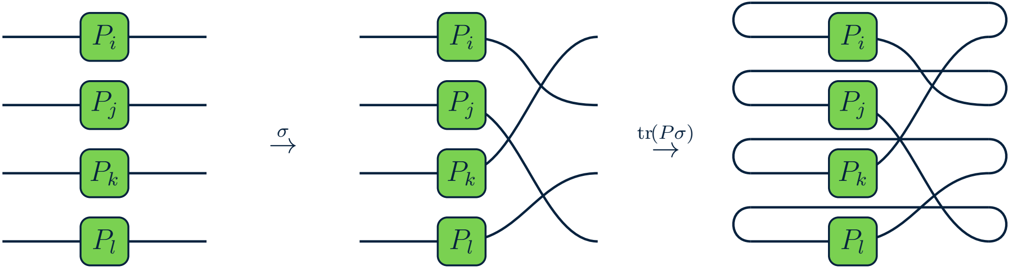
Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually channel *expressivity* phenomena!
- Channel expressivity is more subtly related to *usefulness* or *capability*
- Are there relationships between channel expressivity and their *simulability*?



Appendix

Diagrammatic Expansions of Permutations



$P = P_i \otimes P_j \otimes P_k \otimes P_l$ $\quad$ $\sigma = (ijlk)$ $\quad$ $\mathrm{tr}(P\sigma) = \mathrm{tr}\left(\prod_{\mu\in\sigma} P_\mu\right)$

$$\boxed{\tau = \frac{1}{d}\sum_{P\in\mathcal{P}_d} P\otimes P^\dagger} \quad \to \quad \boxed{\sigma = \frac{1}{d^{|\sigma|}}\sum_{P\in\mathcal{P}_d^{(\sigma)}} P} \tag{15}$$

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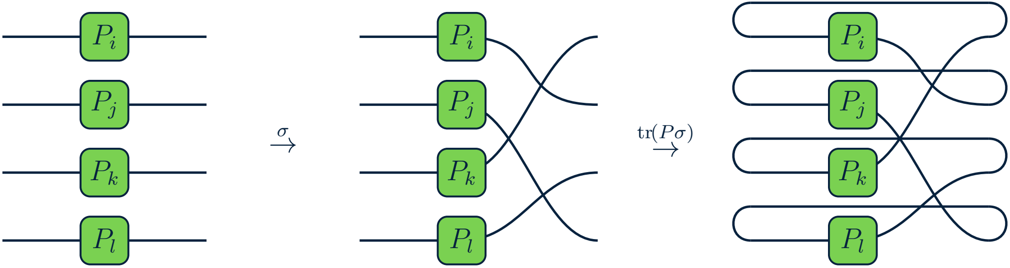

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$$\mathcal{T}_{\Sigma}^{(t)} = \boxed{\frac{1}{d^t}\sum_{\sigma,\pi\in\mathcal{S}_{\Sigma}^{(t)}} \tau_d^{(t)}(\sigma,\pi)|\sigma\rangle\langle\pi|} = \boxed{\frac{1}{d^t}|I\rangle\langle I| + \frac{1}{d^t}\sum_{\substack{P,S\in\mathcal{P}_d^{(\mathcal{S}_{\Sigma}^{(t)})}\\ P\notin\{I\}}} \tau_d^{(t)}(P,S)\ |P\rangle\langle S|} \tag{16}$$

Haar, cHaar, and Depolarizing Ensembles

Σ \ t	1	2
Haar	$\frac{1}{d_{\mathcal{H}}}\|I\rangle\!\rangle\langle\!\langle I\|$	$\frac{1}{d_{\mathcal{H}}^2}\|I\rangle\!\rangle\langle\!\langle I\| + \frac{1}{d_{\mathcal{H}}^2}\frac{1}{d_{\mathcal{H}}^2-1}\sum_{P,S\in\mathcal{P}_{d_{\mathcal{H}}}^{(\tau)}\setminus\{I\}}\|P\rangle\!\rangle\langle\!\langle S\|$
cHaar	$\frac{1}{d_{\mathcal{H}}}\|I\rangle\!\rangle\langle\!\langle I\|$	$\frac{1}{d_{\mathcal{H}}^2}\|I\rangle\!\rangle\langle\!\langle I\| + \frac{1}{d_{\mathcal{H}}^2}\frac{d_{\mathcal{E}}-1}{d_{\mathcal{H}}^2 d_{\mathcal{E}}^2-1}\sum_{P\in\mathcal{P}_{d_{\mathcal{H}}}^{(\tau)}\setminus\{I\}}\|P\rangle\!\rangle\langle\!\langle I\| + \frac{1}{d_{\mathcal{H}}^2}\frac{d_{\mathcal{E}}}{d_{\mathcal{H}}^2 d_{\mathcal{E}}^2-1}\sum_{P,S\in\mathcal{P}_{d_{\mathcal{H}}}^{(\tau)}\setminus\{I\}}\|P\rangle\!\rangle\langle\!\langle S\|$
Depolarize	$\frac{1}{d_{\mathcal{H}}^t}\|I\rangle\!\rangle\langle\!\langle I\|$	

Table 1: Twirls $\mathcal{T}_{\Sigma}^{(t)}$ for various ensembles and moments

Monotonic Convergence and Hierarchy of cHaar Twirl Norms

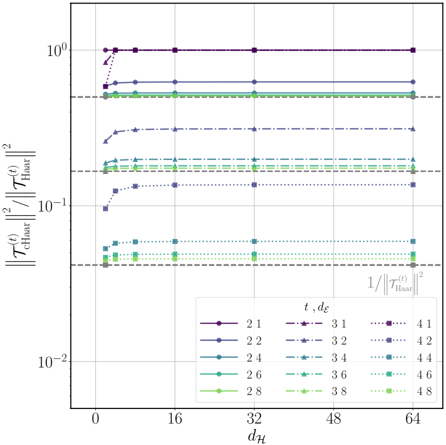

Figure 1: cHaar t-order twirl norms convergence with $d_{\mathcal{H}}, d_{\mathcal{E}}$ towards $1/\|\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}\|^2$.

$$\boxed{1 \;=\; \|\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Depolarize}}}\|^2 \;\leq\; \|\mathcal{T}^{(t)k}_{\Sigma_{\mathrm{cHaar}}}\|^2 \;\leq\; \|\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}\|^2 \;=\; |\mathcal{S}_t|} \tag{17}$$