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A proof of the Vose–Liepins conjecture

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A key result sufficient for the asymptotic stability of the mixing operator in infinite population genetic algorithms depended upon a conjecture. While empirical results supported the conjecture, no proof has been heretofore obtained. In this paper we supply a proof of the conjecture. In so doing, we obtain properties useful for the study of genetic algorithms.

1. Introduction

Vose and Liepins [5] presented a simplified and powerful model for infinite population genetic algorithms (GAs). This was extended by Nix and Vose [3] to finite populations. Both models were later unified and further extended by Vose [4].

In the Vose and Liepins [5] paper, a key result revolved around the asymptotic stability of a mapping. This, in turn, depended on the magnitude of the second largest eigenvalue of a matrix M_* (defined below) being less than one-half. The authors reported empirical confirmation that the second largest eigenvalue of the matrix M_* was less than one-half, but were unable to provide a proof (see their conjecture 1).

In this paper we provide a direct way to compute the spectrum of M_* and are able to confirm the Vose–Liepins conjecture. In the course of deriving our results, we provide several useful identities for studying GAs.

We start with a review of relevant material in section 2. In section 3 we derive the complete spectrum of M_*. These results are used to prove the Vose–Liepins conjecture in section 4.

2. Background

Throughout, we will consider population members as binary strings of $\gamma > 1$ bits. Each such member can also be represented by an integer from 0 to $2^\gamma - 1$. The following notation will be used:

$i \oplus j$ is the bitwise EXCLUSIVE OR of i and j;

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$i \otimes j$	is the bitwise AND of i and j;
$\mid j \mid$	is the number of non-zero bits of j;
$\Delta(i, j, h)$	is $\mid (2^h - 1) \otimes i \mid - \mid (2^h - 1) \otimes j \mid$;
$\lfloor y \rfloor$	is the largest integer less than or equal to the real number y;
$\delta(i)$	is one if i is zero and is zero otherwise;
$\text{rev}\,(i)$	is the bitwise reversal of i where i is treated as having γ bits (with leading zeros if necessary);
$\text{wid}\,(i)$	is the difference between the position of the highest non-zero bit and the lowest non-zero bit of i (for $i > 0$) and $\text{wid}\,(0) \equiv 0$;
$P(x, y)$	is the number of combinations of x objects taken y at a time;
e	is a column vector of ones;
v'	signifies the transpose of v;
e_i	is a column vector of zeros having a one in row $i+1$, $i = 0, \dots, 2^\gamma - 1$; and
I	is an identity matrix of appropriate size.

Vose and Liepins [5] derived an expression for the probability that parents i and j result in an offspring of 0 under a one point crossover and mutation. The crossover rate is $\chi > 0$ and the mutation rate is $0 < \mu < 1$. Using $\eta \equiv \mu/(1-\mu)$, then the probability that parents i and j result in 0 is

$$M_{i,j} = (1-\mu)^\gamma \left[\eta^{|i|} \left((1-\chi) + \chi \sum_{h=1}^{\gamma-1} \eta^{-\Delta(i,j,h)} \Big/ (\gamma - 1) \right) \right.$$

$$\left. + \eta^{|j|} \left((1-\chi) + \chi \sum_{h=1}^{\gamma-1} \eta^{+\Delta(i,j,h)} \Big/ (\gamma - 1) \right) \right] /2.$$

Let M be the 2^γ by 2^γ matrix of such values.

PROPOSITION 1 (PROPERTIES OF M [5])

M has the following properties:

(1) $\sum_{k=0}^{2^\gamma-1} M_{i\oplus k, j\oplus k} = 1$;

(2) $e'Me = 2^\gamma$;

(3) $M > 0$.

Vose and Liepins [5] use the twist of M, denoted M_*, throughout their analysis. M_* is defined below with several of its properties:

PROPOSITION 2 (DEFINITION AND PROPERTIES OF M_* [5])

The twist of M is denoted by M_* and defined by

$$(M_*)_{i,j} = M_{i\oplus j,i}.$$

Two properties of M_* are

(1) $e'M_* = e'$;

(2) 1 is the largest eigenvalue of M_*.

Property two gives the largest eigenvalue of M_*. In this paper, we wish to determine the full spectrum of M_*.

Also instrumental in Vose–Liepin's results are Walsh matrices. We will denote a 2^γ by 2^γ Walsh matrix by W where

$$W_{i,j} = (-1)^{|\operatorname{rev}(i)\otimes j|} = (-1)^{|\operatorname{rev}(j)\otimes i|}.$$

Below are several well-known properties of Walsh matrices.

PROPOSITION 3 (PROPERTIES OF WALSH MATRICES [1, 2, 5])

Walsh matrices satisfy

(1) $W = W'$;

(2) $WW = 2^\gamma I$;

(3) $We = 2^\gamma e_0$ (so $We_0 = e$);

(4) $W_{i\oplus j,k} = W_{i,k}W_{j,k}$.

We wish to compute the spectrum of M_*. This job is made easier by Walsh matrices and the following nicely derived result of Vose and Liepins.

PROPOSITION 4 (M_* AND C [5])

Let $C = WM_*W$. Then C is lower triangular.

This immediately gives us a way to approach the determination of the spectrum of M_* as shown below.

COROLLARY 1 (EIGENVALUES OF M_*)

The eigenvalues of M_* are $C_{i,i}/2^\gamma$, $i = 0, \ldots, 2^\gamma - 1$.

Proof

Let λ be an eigenvalue of M_* and x the associated eigenvector. Then

$$M_* x = \lambda x.$$

Let $Wy = x$. Then

$$M_* Wy = \lambda Wy$$

so

$$Cy = WM_* Wy = \lambda W Wy = \lambda 2^\gamma y.$$

The rest follows from the fact that $C_{i,i}$ are the eigenvalues of the lower triangular matrix C, for $i = 0, \ldots, 2^\gamma - 1$. □

Each $C_{i,i}$ can be computed as shown below.

LEMMA 1

The $C_{i,i}$ values can be computed from

$$C_{i,i} = \sum_{j=0}^{2^\gamma - 1} W_{j,i} \sum_{k=0}^{2^\gamma - 1} M_{j,k} = (WMe)_i.$$

Proof

Clearly

$$C_{i,i} = \sum_{j=0}^{2^\gamma - 1} W_{i,j} \sum_{k=0}^{2^\gamma - 1} (M_*)_{j,k} W_{k,i}$$

$$= \sum_{j=0}^{2^\gamma - 1} \sum_{k=0}^{2^\gamma - 1} (M_*)_{j,k} W_{k,i} W_{i,j}.$$

So, by symmetry of W, the definition of M_*, and property (4) in proposition 3, we get

$$C_{i,i} = \sum_{j=0}^{2^\gamma - 1} \sum_{k=0}^{2^\gamma - 1} M_{k \oplus j, k} W_{k \oplus j, i}.$$

Letting $h = k \oplus j$ gives

$$C_{i,i} = \sum_{j=0}^{2^\gamma-1} \sum_{h=0}^{2^\gamma-1} M_{h,h\oplus j} W_{h,i},$$

$$C_{i,i} = \sum_{h=0}^{2^\gamma-1} W_{h,i} \sum_{j=0}^{2^\gamma-1} M_{h,h\oplus j}$$

which gives our desired result. □

Note that $C_{0,0}$ is easily determined from identities already presented. $W = W'$ by proposition 3, part (1); $We_0 = e$ by proposition 3, part (3); and $e'M_* e = 2^\gamma$ by proposition 2, part (1). Thus

$$e_0' C e_0 = C_{0,0} = e_0' W M_* W e_0 = 2^\gamma.$$

We now turn to a derivation of the spectrum of M_*.

3. Spectrum of M_*

A useful set of identities (given below in lemma 2) is needed before our main result. First, recall the following well-known combinatorial identities.

PROPOSITION 5 (COMBINATORIAL IDENTITIES)

(1) Hypergeometric rule: Suppose we are given x objects of one type and $n - x$ objects of another type. Then, the number of groups of r objects with y of the first type and $r - y$ of the second type is

$$P(x, y)\, P(n - x, r - y).$$

(2) For any whole number n,

$$\sum_{i=0}^{n} P(n, i) = 2^n.$$

(3)

$$\sum_{i=0}^{n} P(x, i) y^i = \sum_{i=0}^{n} P(n - x, i)(1 + y)^{n-i}(-y)^i$$

and

$$\sum_{i=0}^{n} P(x, i)(-i)^i = (-1)^n P(x - 1, n).$$

The following identities have proven instrumental in our analysis.

LEMMA 2

For $h = 1, \ldots, \gamma$ and $i = 0, \ldots, 2^\gamma - 1$, the following identities are true:

(1)
$$\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|k|} = (1-2\mu)^{|i|}(1-\mu)^{-\gamma}.$$

(2)
$$\begin{aligned}&\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|k|-|(2^h-1)\otimes k|}\\ &= 2^h(1-2\mu)^{|i|}(1-\mu)^{h-\gamma} \quad \text{if } i = i \otimes (2^{\gamma-h}-1)\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

(3)
$$\begin{aligned}&\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|(2^h-1)\otimes k|}\\ &= 2^{\gamma-h}(1-2\mu)^{|i|}(1-\mu)^{-h} \quad \text{if } i \otimes (2^{\gamma-h}-1) = 0\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Proof

We will prove (3). (1) is a special case of (3), and (2) follows a similar line of reasoning as (3).

Clearly

$$\begin{aligned}\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|(2^h-1)\otimes k|} &= \sum_{s=0}^{2^{\gamma-h}-1}\sum_{r=0}^{2^h-1} W_{i,r+s2^h}\eta^{|(2^h-1)\otimes(r+s2^h)|}\\ &= \sum_{s=0}^{2^{\gamma-h}-1}\sum_{r=0}^{2^h-1} W_{i,r+s2^h}\eta^{|r|}.\end{aligned}$$

Now, $|i \otimes \mathrm{rev}(r+s2^h)| = |i \otimes \mathrm{rev}(r)| + |i \otimes \mathrm{rev}(s2^h)|$, so

$$W_{i,r+s2^h} = W_{i,r}W_{i,s2^h}.$$

Thus

$$\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|(2^h-1)\otimes k|} = \sum_{s=0}^{2^{\gamma-h}-1} W_{i,s2^h}\sum_{r=0}^{2^h-1} W_{i,r}\eta^{|r|}.$$

Consider the last sum. First note that, since $r = 0, \dots, 2^h - 1$, we have

$$| i \otimes \operatorname{rev}(r) | = | (i - i \otimes (2^{\gamma - h} - 1)) \otimes \operatorname{rev}(r) |.$$

For notational convenience, let $z \equiv (i - i \otimes (2^{\gamma - h} - 1))$. Then

$$\sum_{r=0}^{2^h-1} W_{i,r} \eta^{|r|} = \sum_{r=0}^{2^h-1} (-1)^{|z \otimes \operatorname{rev}(r)|} \eta^{|r|}.$$

For $t = 0, \dots, h$, we know from proposition 5, part (1), that

$$P(|z|, k)\, P(h - |z|, t - k)$$

terms will multiply η^t and have a value of $(-1)^k$ where $k = 0, \dots, \gamma$. Thus

$$\sum_{r=0}^{2^h-1} W_{i,r} \eta^{|r|} = \sum_{t=0}^{h} \sum_{k=0}^{\gamma} (-1)^k \eta^t P(|z|, k)\, P(h - |z|, t - k).$$

The right side can be rearranged to

$$\sum_{k=0}^{\gamma} (-1)^k P(|z|, k) \sum_{t=0}^{h} \eta^t P(h - |z|, t - k).$$

The non-zero terms of $P(h - |z|, t - k)$ have $t \geq k$ and $h - |z| \geq t - k$. Also, $h \geq h - |z| + k$. Thus, the last term yields

$$\begin{aligned} \sum_{t=0}^{h} \eta^t P(h - |z|, t - k) &= \sum_{t=k}^{h-|z|+k} \eta^t P(h - |z|, t - k) \\ &= \sum_{t=0}^{h-|z|} \eta^{t+k} P(h - |z|, t). \end{aligned}$$

Hence, from proposition 5, part (3), we get that

$$\sum_{t=0}^{h-|z|} \eta^{t+k} P(h - |z|, t) = \eta^k (1 + \eta)^{h-|z|}.$$

Thus,

$$\sum_{r=0}^{2^h-1} W_{i,r} \eta^{|r|} = \sum_{k=0}^{\gamma} (-1)^k P(|z|, k) \eta^k (1 + \eta)^{h-|z|}$$

which gives, again using proposition 5, part (3):

$$(1+\eta)^{h-|z|}(1+\eta)^{|z|}.$$

So, using this result, we get

$$\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|(2^h-1)\otimes k|} = \sum_{s=0}^{2^{\gamma-h}-1} W_{i,s2^h}(1+\eta)^{h-|z|}(1-\eta)^{|z|}.$$

Now consider

$$\sum_{s=0}^{2^{\gamma-h}-1} W_{i,s2^h}.$$

Note that

$$\begin{aligned}|\operatorname{rev}(i)\otimes(s2^h)| &= |\operatorname{rev}(i)\otimes\operatorname{rev}(2^{\gamma-h}-1)\otimes(s2^h)|\\ &= |\operatorname{rev}(i\otimes(2^{\gamma-h}-1))\otimes(s2^h)|.\end{aligned}$$

Let $v\equiv i\otimes(2^{\gamma-h}-1)$. Then, since s ranges from 0 to $2^{\gamma-h}-1$, the following are equivalent:

$$\sum_{s=0}^{2^{\gamma-h}-1} W_{i,s2^h} = \sum_{s=0}^{2^{\gamma-h}-1} W_{v,s2^h}.$$

Clearly, if $|v|=0$, each term of the last sum is $+1$. For $|v|>0$, there will be as many terms equal to $+1$ as -1. Thus

$$\begin{aligned}\sum_{s=0}^{2^{\gamma-h}-1} W_{i,s2^h} &= 2^{\gamma-h} \quad \text{if } |v|=0\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=0}^{2^\gamma-1} W_{i,k}\eta^{|(2^h-1)\otimes k|} &= 2^{\gamma-h}(1+\eta)^{h-|z|}(1-\eta)^{|z|} \quad \text{if } |v|=0\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

But, $0 = |v| \equiv |i \otimes (2^{\gamma-h} - 1)|$ implies $|z| \equiv |(i - i \otimes (2^{\gamma-h} - 1))| = |i|$. Thus

$$\sum_{k=0}^{2^\gamma-1} W_{i,k} \eta^{|(2^h-1)\otimes k|} = 2^{\gamma-h}(1+\eta)^{h-|i|}(1-\eta)^{|i|} \quad \text{if } |v| = 0$$

$$= 0 \qquad \text{otherwise.}$$

Substituting $\mu/(1-\mu)$ for η gives the desired result. □

A useful implication of lemma 2 (where $i = 0$) is given in the following corollary.

COROLLARY 2

(1)

$$\sum_{k=0}^{2^\gamma-1} \eta^{|k|} = (1-\mu)^{-\gamma}.$$

(2)

$$\sum_{k=0}^{2^\gamma-1} \eta^{|k|-|(2^h-1)\otimes k|} = 2^h(1-\mu)^{h-\gamma}.$$

(3)

$$\sum_{k=0}^{2^\gamma-1} \eta^{|(2^h-1)\otimes k|} = 2^{\gamma-h}(1-\mu)^{-h}.$$

To compute the eigenvalues of M_*, lemma 1 shows that we need the row sums of M. The following lemma provides a way to compute these sums.

LEMMA 3 (ROW SUMS OF M)

The jth row sum of M is given by

$$S_j \equiv \sum_{k=0}^{2^\gamma-1} M_{j,k} = (1-\chi)/2 + 2^\gamma(1-\mu)^\gamma(1-\chi)\eta^{|j|}/2$$

$$+ \chi/(2\gamma-2)\sum_{h=1}^{\gamma-1}(2-2\mu)^h G(h,j),$$

where

$$G(h,j) = \eta^{|(2^h-1)\otimes j|} + \eta^{|j|-|(2^{\gamma-h}-1)\otimes j|}.$$

Proof

First consider

$$\sum_{k=0}^{2^\gamma-1}(1-\mu)^\gamma(\eta^{|j|}(1-\chi)+\eta^{|k|}(1-\chi))/2.$$

Using corollary 2, part (1) gives that this is equal to

$$(1-\chi)/2+2^\gamma(1-\mu)^\gamma(1-\chi)\eta^{|j|}/2$$

which gives the first two terms of our result. Now consider

$$\sum_{k=0}^{2^\gamma-1}(1-\mu)^\gamma\left[\eta^{|j|}\left(\chi\sum_{h=1}^{\gamma-1}\eta^{-\Delta(j,k,h)}/(\gamma-1)\right)\right.$$
$$\left.+\eta^{|k|}\left(\chi\sum_{h=1}^{\gamma-1}\eta^{+\Delta(j,k,h)}/(\gamma-1)\right)\right]/2.$$

$|j|-\Delta(j,k,h)=|j|-|(2^h-1)\otimes j|+|(2^h-1)\otimes k|$, so applying corollary 2, part (3) gives

$$\sum_{k=0}^{2^\gamma-1}\eta^{|j|-\Delta(j,k,h)}=2^{\gamma-h}(1-\mu)^{-h}\eta^{|j|-|(2^h-1)\otimes j|}.$$

Similarly, applying corollary 2, part (2) gives

$$\sum_{k=0}^{2^\gamma-1}\eta^{|k|+\Delta(j,k,h)}=2^h(1-\mu)^{h-\gamma}\eta^{|(2^h-1)\otimes j|}.$$

Before substituting these into the last term of our row sum, note that

$$\sum_{h=1}^{\gamma-1}\eta^{|j|-|(2^h-1)\otimes j|}=\sum_{g=1}^{\gamma-1}\eta^{|j|-|(2^{\gamma-g}-1)\otimes j|}.$$

Collecting the above gives the final term of our row sum:

$$\chi/(2\gamma-2)\sum_{h=1}^{\gamma-1}(2-2\mu)^hG(h,j). \qquad \square$$

We now give our main result.

THEOREM 1 (SPECTRUM OF M_*)

The spectrum of M_* is

$$(1-2\mu)^{|i|}(1-\chi\,\mathrm{wid}\,(i)/(\gamma-1))/2 \quad i=0,\ldots,2^\gamma-1.$$

Proof

Pulling together corollary 1, lemma 1, and lemma 3 gives that 2^γ times the ith eigenvalue of M_* is

$$\sum_{j=0}^{2^\gamma-1} W_{i,j}S_j = \sum_{j=0}^{2^\gamma-1} W_{i,j}\left[(1-\chi)/2+2^\gamma(1-\mu)^\gamma(1-\chi)\eta^{|j|}/2 \right.$$
$$\left. +\chi(2\gamma-2)\sum_{h=1}^{\gamma-1}(2-2\mu)^h G(h,j)\right].$$

Note from proposition 3, part (3), that

$$\sum_{j=0}^{2^\gamma-1} W_{i,j} = 2^\gamma\delta(i).$$

Deriving a general expression for the eigenvalues of M_* would involve carrying along details that relate only to $i=0$. However, we have already shown that $C_{0,0}=2^\gamma$. Hence, we will restrict our attention to the cases where $i>0$.

The sum over the first two terms is found using proposition 3, part (3) and lemma 2, part (1). This yields

$$2^{\gamma-1}(1-\chi)(1-2\mu)^{|i|}.$$

Now consider

$$\sum_{j=0}^{2^\gamma-1} W_{i,j}G(h,j).$$

First, applying lemma 2, parts (2) and (3), gives

$$\sum_{j=0}^{2^\gamma-1} W_{i,j}\eta^{|(2^h-1)\otimes j|} = 2^{\gamma-h}(1-2\mu)^{|i|}(1-\mu)^{-h}\delta(i\otimes(2^{\gamma-h}-1))$$

and

$$\sum_{j=0}^{2^\gamma-1} W_{i,j}\eta^{|j|-|(2^{\gamma-h}-1)\otimes j|} = 2^{\gamma-h}(1-2\mu)^{|i|}(1-\mu)^{-h}\delta(i-i\otimes(2^h-1)).$$

Thus

$$\begin{aligned}&\sum_{h=1}^{\gamma-1}(2-2\mu)^h\sum_{j=0}^{2^\gamma-1} W_{i,j}G(h,j)\\ &= 2^\gamma(1-2\mu)^{|i|}\sum_{h=1}^{\gamma-1}[\delta(i\otimes(2^{\gamma-h}-1))+\delta(i-i\otimes(2^h-1))].\end{aligned}$$

The first term of the sum is equal to f where f is the position of the first non-zero bit of i (counting from zero). Similarly, the second sum is equal to $\gamma-g-1$ where g is the position of the last non-zero bit of i. Thus, we get

$$2^\gamma(1-2\mu)^{|i|}(\gamma-1-\mathrm{wid}\,(i)).$$

Pulling together the various parts yields the following:

$$\sum_{k=0}^{2^\gamma-1} W_{i,j}S_j = 2^{\gamma-1}(1-\chi)(1-2\mu)^{|i|}+2^{\gamma-1}(1-2\mu)^{|i|}(\gamma-1-\mathrm{wid}\,(i))\chi/(\gamma-1).$$

Dividing by 2^γ and simplifying gives the desired result for $i>0$. Finally, as noted above, proposition 2, part (2) and proposition 3, part (3), gives an eigenvalue of one for the case $i=0$. □

4. Vose–Liepins conjecture

In [5], Vose and Liepins provide the following conjecture.

CONJECTURE 1 (VOSE–LIEPINS CONJECTURE [5])

If $0<\mu<0.5$, then

(1) The second largest eigenvalue of M_* is $0.5-\mu$.

(2) The third largest eigenvalue of M_* is

$$2(0.5-\mu)^2(1-\chi/(\gamma-1)).$$

No proof was supplied, but the authors reported empirical justification. To prove that the conjecture is true, we start with a readily-apparent, direct-consequence of theorem 1.

COROLLARY 3 (RANKING OF THE EIGENVALUES OF M_*)

If $0 < \mu < 0.5$, then the eigenvalues of M_*, denoted

$$\lambda_0 \equiv 1,$$
$$\lambda_i \equiv (1-2\mu)^{|i|}(1-\chi \operatorname{wid}(i)/(\gamma-1))/2, \quad i = 1,\dots,2^\gamma - 1,$$

are decreasing in both $|i|$ and $\operatorname{wid}(i)$. Furthermore, if $|i| = |j|$, then $\lambda_i > \lambda_j$ if $\operatorname{wid}(i) < \operatorname{wid}(j)$.

Hence, as already known (proposition 2, part (2)), the largest eigenvalue is $\lambda_0 = 1$. The second largest eigenvalue is any one of

$$\lambda_2 s = (1-2\mu)/2, \quad s = 0,\dots,\gamma-1.$$

The third largest eigenvalue is any eigenvalue λ_j where $|j| = 2$ and $\operatorname{wid}(j) = 1$. Hence

$$\lambda_j = (1-2\mu)^2(1-\chi/(\gamma-1))/2.$$

Thus, the Vose–Liepins conjecture is true.

5. Conclusion

Corollary 3, and its application, shows that the Vose–Liepins conjecture is true. Theorem 1 provides a direct way to compute the entire spectrum of M_*.

Both lemma 2 and its corollary provide useful identities for studying GAs. Indeed, they were instrumental in our analysis.

Dedication and acknowledgements

This paper is dedicated to the memory of Gunar E. Liepins.

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