Let $X_1, X_2 \dots X_n$: sample from $N(\mu, \sigma^2)$

sample variance : $S^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (X_i - \underbrace{\bar{X}}_{\text{mean}})^2$

then the test stat: $\frac{(n-1)\cdot S^2}{\sigma^2} \sim \chi^2(n-1)$

1/ Standardization

$$X_i \sim N(\mu, \sigma^2) \Rightarrow Z_i = \underbrace{\frac{X_i - \mu}{\sigma}}_{N(0,1)}$$

then $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$

2/ $\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)$

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right) = \underbrace{\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2}_{\chi^2(n)} + \underbrace{n\left(\frac{\bar{X} - \mu}{\sigma}\right)}_{\chi^2(1)}$$

3/

because $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

4/ so the remaining :

$$\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 = \underbrace{\frac{(n-1)\cdot S^2}{\sigma^2}}_{\chi^2(n-1)}$$

Chi-squared with df = n-1