UNIT-5: MARKOV CHAIN & QUEUEING THEORY

Markov chain:

$$P(X_{k+1} = a_{k+1} \mid X_k = a_k)$$

→ Stochastic ([Matrix])

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & - & - & & \\ p_{31} & - & - & & \\ p_{n1} & & - & & p_{nn} \end{pmatrix}$$ ← probability vector

→ Prob vector i $V = (v_1, v_2, \dots v_n)$

$v_1, v_2, v_3, v_4, \dots, v_n \geq 0$

$v_1 + v_2 + v_3 + \dots + v_n = 1$

→ fixed prob vector $VP = V$

→ Regular Stochastic Matrix

$P^2, P^3, \dots, P^n$

SHOW THAT THE FOLLOWING MATRICES IS A REG STOCHIASTIC MATRIX.

(1)

$$P = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

Since every element in P^2 is $> 0 \Rightarrow P$ is a regular stochastic matrix.

(2)

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

$\frac{1}{16} + \frac{1}{16}$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1/8 & 5/16 & 9/16 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

$\frac{9}{16} + \frac{1}{16}$

$$P^3 = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.1562 & 0.6406 & 0.2031 \\ 0.125 & 0.3125 & 0.3625 \end{pmatrix}$$

(3)

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$\frac{2}{6} + \frac{1}{6}$

$$P^2 = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 1/4 & 5/12 & 1/6 \\ 1/4 & 1/6 & 5/12 \end{pmatrix}$$

$\frac{1}{6} + \frac{1}{4}$ $\frac{10}{24}$ $\frac{5}{12}$

$\frac{1}{6} + \frac{1}{4}$

FIND THE UNIQUE FIXED PROBILITY VECTOR FOR THE FOLLOWING

(1)

$$P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

Let V be the fixed probability vector

$V = (v_1, v_2)$, $v_1, v_2 \geq 0$ $v_1 + v_2 = 1$

$VP = V$

$$(v_1 \ v_2)_{1\times 2} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}_{2\times 2} = (v_1 \quad v_2)$$

$$\left(\frac{3}{4}v_1 + \frac{1}{2}v_2 \quad \frac{v_1}{4} + \frac{v_2}{2} \right) = (v_1 \quad v_2)$$

$$\frac{3}{4}v_1 + \frac{1}{2}v_2 = v_1$$

$$\frac{1}{4}v_1 + \frac{1}{2}v_2 = v_2$$

$$\Rightarrow -\frac{1}{4}v_1 + \frac{1}{2}v_2 = 0$$

$$\frac{1}{4}v_1 - \frac{1}{2}v_2 = 0$$

$$v_1 + v_2 = 1$$

$$-v_1 + 2v_2 = 0$$

$$+ \quad v_1 + v_2 = 1$$

$$3v_2 = 1$$

$$v_2 = 1/3$$

$$v_1 = 2/3$$

② $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/6 \\ 0 & 2/3 & 1/3 \end{pmatrix}$

$v = (v_1 \quad v_2 \quad v_3)$ $\quad v_1, v_2, v_3 \geq 0$ $\quad v_1 + v_2 + v_3 = 1$

$$(v_1 \ v_2 \ v_3) \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (v_1 \quad v_2 \quad v_3)$$

$$\left(\frac{v_2}{6}, \ v_1 + \frac{v_2}{2} + \frac{2v_3}{3}, \ \frac{v_2 + v_3}{3} \right) = (v_1 \quad v_2 \quad v_3)$$

$$\frac{v_2}{6} = v_1$$

$$v_1 + \frac{v_2}{2} + \frac{2v_3}{3} = v_2$$

$$\frac{v_2}{3} + \frac{v_3}{3} = v_3$$

$$\boxed{\begin{aligned} & v_1 + v_2 + v_3 = 1 \\ & v_1 - \frac{v_2}{6} + 0v_3 = 0 \\ & v_1 - \frac{v_2}{2} + \frac{2v_3}{3} = 0 \end{aligned}}$$

$v_1 = 1/10$

$v_2 = 6/10$

$v_3 = 3/10$

③ $P = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0 & 0.2 & 0.8 \end{pmatrix}$

$$(v_1 \ v_2 \ v_3) \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0 & 0.2 & 0.8 \end{pmatrix} = (v_1 \quad v_2 \quad v_3)$$

$v_1 = 0$

$v_2 = 1/4$

$v_3 = 3/4$

$$0.1v_1 = v_1$$

$$0.3v_1 + 0.4v_2 + 0.2v_3 = v_2$$

$$0.6v_1 + 0.6v_2 + 0.8v_3 = v_3$$

④

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

$\frac{v_3}{2} = v_1$ $\qquad v_1 = 2/9$

$v_1 + \frac{v_3}{4} = v_2$ $\qquad v_2 = 3/9$

$v_2 + \frac{v_3}{4} = v_3$ $\qquad v_3 = 4/9$

$v = (2/9 \quad 3/9 \quad 4/9)$

⑤ A habitual gambler is a member of two clubs A & B, He visits either of clubs everyday for playing cards. He never visits club A on two consecutive days but if he visits club B on a particular day then the next day he's as likely to visit club A as club B

(i) Find the transition matrix of the Markov chain

(ii) Show that the matrix is irreducible

(iii) In the long run how often does he visit the clubs

(iv) If the gambler had visited club B on monday. ~~Find~~ find the prob that he visits club A on ~~thu~~ thursday

$$\begin{array}{ccc} & 1 & 2 \\ P = \begin{array}{c} 1 \\ 2 \end{array} & \multicolumn{2}{c}{\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}} \end{array}$$

irregular means regular stochastic matrix

(ii) $P^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \Rightarrow$ Irreducible

(iii) long run $(n \to \infty)$

Steady state distr $\Rightarrow$ fixed prob vector

$vP = v$

$v = (v_1, v_2)$ where $v_1, v_2 \geq 0$

$v_1 + v_2 = 1$

$v = (1/3 \quad 2/3)$

(iv) $p^{(n)} = p^0 p^n$

$p^0 \Rightarrow$ Initial state

$= (0 \quad 1)$

club A Club B

As he visits Club B on monday = 1

$p^{(3)} = p^0 \, p^3$

$= (0 \quad 1) \begin{pmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{pmatrix}$

$= (3/8 \quad 5/8)$

⑥ A computer device can be either in busy (state 1) or in idle mode (state 2). Being in a busy mode it can finish a task and enter an idle mode any minute with the prob 0.2. Thus with the prob 0.8 it stays another min in a busy mode. Being in an idle mode, it recieves a new task any min w the prob 0.1 and enters a busy mode thus it stays another in idle w prob 0.9. The initial state is idle. Let X_n be the state of device after n mins. ① X_2? ② Steady state dist X_n

$$
\begin{array}{c c} & \begin{array}{cc} \text{Busy} & \text{Idle} \end{array} \\ \begin{array}{c} \text{Busy } 0 \\ \text{Idle } 1 \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \end{array} = P
$$

$p^{(2)} = p^0 p^2$

$p^0 \to (0 \quad 1)$

$$p^2 = \begin{pmatrix} 0.66 & 0.34 \\ 0.17 & 0.83 \end{pmatrix}$$

$p^{(2)} = p^0 p^2 = (0.17 \quad 0.83)$

$0.2v_1 + 0.9v_2 = v_2$

$0.2v_1 - 0.1v_2 = 0$

$2v_1 - v_2 = 0$

$vP = v$

$v_1, v_2 \geq 0$

$v_1 + v_2 = 1$

$$(v_1 \; v_2)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} = (v_1 \; v_2)$$

$0.8v_1 + 0.1v_2 = v_1$

$-0.2v_1 + 0.1v_2 = 0$

$-2v_1 + v_2 = 0$

$-v_1 + \cancel{v_2} = \phi$

$v_1 = 1/3 \quad v_2 = 2/3$

$v = (1/3 \quad 2/3)$

(7) Two boys B_1 & B_2 & 2 $G_1 G_1 G_2$, and throwing ball

$B \to B \; 1/2$

$\;\;\to G \; 1/4$

$G \to B \; 1/2$

$\;\;\to G \; 0$

In long run how often does each recieve the ball

$$
P = \begin{array}{c c} & \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ G_1 \\ G_2 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array}
$$

$v = (0.333 \quad 0.333 \quad 0.166 \quad 0.166)$

$v = (1/3 \quad 1/3 \quad 1/6 \quad 1/6)$

$vP = v$

$v_1, v_2, v_3, v_4 \geq 0$

$v_1 + v_2 + v_3 + v_4 = 1$

$\dfrac{v_2 + v_3 + v_4}{2} = v_1$

$\dfrac{v_1 + v_3 + v_4}{2} = v_2$

$\dfrac{v_1 + v_2}{4} = v_3$

$\dfrac{v_1 + v_2}{4} = v_4$

(8) Pattern of sunny & rainy days on the planet rainbow in a homogenous markov chain with two states.

$S \to S \; 0.8$

$\;\;\to R \; 0.2$

$R \to R \; 0.6$

$\;\;\to S \; 0.4$

(i) Today is sunny on rainbow. chance of rain day after tomorrow

(ii) Compute the prob that April 1 next year is rainy on rainbow.

$$
\begin{array}{c c} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} S \\ R \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \end{array}
$$

$p^0 = (1 \quad 0)$ (S, R)

0 1 2

(i) $p^{(3)} = p^0 \; p^2$

$$= (1 \quad 0)\begin{pmatrix} \cancel{0.688} \; 0.72 & \cancel{0.312} \; 0.28 \\ \cancel{0.624} \; 0.56 & \cancel{0.376} \; 0.44 \end{pmatrix}$$

$= \cancel{(0.688 \quad 0.312)} \quad (0.72 \quad 0.28)$

(ii) $(2/3 \quad 1/3)$

Q A gambler's luck follows a pattern, if he wins a game $\xrightarrow{W} 0.6$, $\xrightarrow{L} 0.4$; lose a game $\xrightarrow{W} 0.3$, $\xrightarrow{L} 0.7$.

There is an even change of gambler winning the first game. If so

(i) Prob of winning second game 0.45

(ii) Third game 0.435

(iii) In long run: win? 0.4285

$$\begin{array}{cc} & \begin{array}{cc} W & L \\ 0 & 1 \end{array} \\ \begin{array}{c} W\ 0 \\ L\ 1 \end{array} & \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

$p^{(1)} = p^0 \cdot p^1$

$= (1/2 \quad 1/2)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$

$= \begin{array}{cc} 0 & 1 \end{array}$
$= (0.45 \quad 0.55)$

$p^0 = (1/2 \quad 1/2)$

$p^{(2)} = p^0 \cdot p^2$

$= (1/2 \quad 1/2)\begin{pmatrix} 0.48 & 0.52 \\ 0.39 & 0.61 \end{pmatrix}$

$= \begin{array}{cc} 0 & 1 \end{array}$
$= (0.435 \quad 0.565)$

$VP = P$

$V = (V_1, V_2)$

$(V_1 \quad V_2)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = (V_1 \quad V_2)$

$0.6V_1 + 0.3V_2 = V_1$

$0.4V_1 + 0.7V_2 = V_2$

$V_1 + V_2 = 1$

$-0.4V_1 + 0.3V_2 = 0$

$0.4V_1 + 0.4V_2 = 0.4$

$0.7V_2 = 0.4$

$V_2 = \frac{4}{7}$

$V_1 = \frac{3}{7}$

Q A salesman teritory consists of 3 cities

$A \to B$

B } twice as likely to sell in city A as
C } in the other city

Never on same city on succesive days.

(i) Long run, how often does he sell in each of the cities. $(V_1\ V_2\ V_3)$

(ii) Sopposing he sells in city B in week 1 find the prob of selling in city C in week 4. $P^{(3)}$ 0.2592

$B \to C$

$$\text{(i)}\quad \begin{array}{cc} & \begin{array}{ccc} A & B & C \\ 0 & 1 & 2 \end{array} \\ \begin{array}{c} A\ 0 \\ B\ 1 \\ C\ 2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} \end{array}$$

$(V_1 \quad V_2 \quad V_3)\begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} = (V_1 \quad V_2 \quad V_3)$

$\frac{2}{3}(V_2 + V_3) = V_1$

$V_1 + \frac{V_3}{3} = V_2$

$\frac{V_2}{3} = V_3$

$-V_1 + \frac{2}{3}V_2 + \frac{2}{3}V_3 = 0$

$V_1 + V_2 + V_3 = 1$

$V_1 - V_2 + \frac{V_3}{3} = 0$

$V_1 = \frac{2}{5}$

$V_2 = \frac{9}{20}$

$V_3 = \frac{3}{20}$

$A \to 2/5$

$B \to 9/20$

$C \to 3/20$

(ii) $p^0 = (0 \quad 1 \quad 0)$

$P^{(3)} = p^0 P^3 = \cancel{(0.6666 \quad 0 \quad 0.3333)}$

$p^{(3)} = (0.5185 \quad 0.2222 \quad 0.2592)$

Q Consider a game of ladder climbing
5 levels in the game. Level 1 is the lowest (bottom)
level 5 is the heighest (top)
Player starts at bottom. Each time a paired coin is tossed.

H → 1 rung
T → bottom.

Top → Bottom → Tails.
→ Top → Heads.

(i) Transmission prob matrix ✓
(ii) 2 step transition prob matrix ✓
(iii) Steady state dist.

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/2 & 1/4 & 0 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 0 & 1/4 \end{pmatrix}$$

$p^0 = (1\ 0\ 0\ 0\ 0)$

$p^{(2)} = p^0 P^2$

$= (1/2\ \ 1/4\ \ 1/4\ \ 0\ \ 0)$

(iii) $$(v_1\ v_2\ v_3\ v_4\ v_5) \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/2 & 1/4 & 0 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 0 & 1/4 \end{pmatrix} = (v_1, v_2, v_3, v_4, v_5)$$

$\frac{v_1+v_2+v_3+v_4+v_5}{2} = v_1$ $\quad -\frac{v_1}{2} + \frac{v_2}{2} + \frac{v_3}{2} + \frac{v_4}{}$

$\frac{v_1+v_2+v_3+v_4+v_5}{4} = v_2$

$\frac{v_1}{4} = v_3$ $\quad \frac{v_3+v_4+v_5}{4} = v_5$

$\frac{v_2}{4} = v_4$

$(1/2\ \ 1/4\ \ 1/8\ \ 1/16\ \ 1/16)$

Q Each year a man trades his car for a new car in 3 brands

A "standard" → "zen"
B "zen" → "Esteem"
C "Esteam" → Esteem 1/3
zenn 1/3
standard 1/3

2010
Esteem
prob that in 2012
2013 zen.
↓
0.2592

Esteem
↓
0.4444

10 0
11 1
12 2

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/3 & 1/3 & 1/3 \end{array}$$

$$p^0 = (0 \ 0 \ 1)$$

$$p^{(2)} = (0 \ 0 \ 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$p^{(2)} = (0.1111 \quad 0.4444 \quad 0.4444)$$

$$p^{(3)} = p^0 p^3$$

$$= (0.1481 \quad 0.2592 \quad 0.5925)$$

Q In a 24 hr service station vehicles arrive at a rate of 30/day on avg. The avg servicing time for a vehicle is 36 mins. Find
① Mean Queue Size
② Prob that queue size >9

Ans

λ - arrival rate
μ - service rate

$\mu = \frac{1}{36}$ mins   36 mins / service

$\lambda = 30/\text{day}$

$= \frac{30}{24 \times 60} = \frac{1}{48}$ per min

(i) Mean Queue Size.

$L_q = \frac{\rho^2}{1-\rho} = 2.25$

$\rho = \frac{\lambda}{\mu} = \frac{3}{4}$

(ii) $P(\text{Queue size} > 9)$

$= 1 - P(\text{Queue size} <= 9)$

$= 1 - [P_0 + P_1 + \ldots + P_9]$ $P_n = (1-\rho)\rho^n$

$= 1 - (1-\rho)[\rho^0 + \rho^1 + \rho^2 + \ldots + \rho^9]$

$= 1 - \frac{\cancel{(1-\rho)}(1-\rho^{10})}{\cancel{(1-\rho)}}$

$= \cancel{1} - \cancel{1} + \rho^{10}$

$= \rho^{10} = (0.75)^{10} = 0.0563$

Q The mean duration of telephone conversation is estimated to be 3 mins. If no more than a 3 min wait for the phone may be tolerated. Find the largest amount of incoming traffing that can be supported

$\mu = 1/3$ calls/min

$\lambda = ?$

$W_q = 3$ min

$3 = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$

$\lambda = \frac{3}{3}\left(\frac{1}{3} - \lambda\right)$

$2\lambda = \frac{1}{3}$

$\lambda = 1/6$

Q Customer arrive in a telephone booth at intervals of 10 mins on the average. The length of the phone call is 3 mins on the average

(i) Prob that a person arriving at the booth will have to wait

(ii) What is the average length of the queue that forms from time-to-time

(iii) The owner of the booth will install a second booth when convinced that an arrival would have to wait atleast 3 mins for the phone. By how much must the flow of arrivals be increased in order to justify second booth.

1

λ = Arrival rate

μ = Service rate

$\lambda = 1/10$ per min

$\mu = 1/3$ per min

$\rho = \frac{\lambda}{\mu} = \frac{3}{10} = 0.3$

(i) P(arrival has to wait) = P(busy)

= 1 − P(not busy/ idle)

$= 1 - P_0$

$= 1 - (1-\rho)$

$= \rho$

$= 0.3$

(ii) Queue size $= \frac{\rho^2}{1-\rho} = 0.128$

(iii) Let λ' be the new arrival rate such that

$W_q \geq 3$

$\frac{\lambda'}{\mu(\mu-\lambda')} \geq 3$

$\lambda' \geq 3\mu^2 - 3\mu\lambda'$

$\lambda'(1+3\mu) \geq 3\mu^2$

$\lambda' \geq \frac{3\mu^2}{1+3\mu}$

$\lambda' \geq \frac{1}{6}$

Q A TV repairman finds that average time spent on a job is 30 min. If he ~~~~ repairs TV sets in the order in which they come in & if the average arrival of sets is 10 per 8 hours per day what is the repairman's expected idle time each day.

Ans

1 — 8 — 10

1 — 1 — $\frac{10}{8}$

1 — 24 — $\frac{10}{8} \times 24$

1 — 24 — 30 ~~day~~

Arrival Rate $\lambda = \frac{30}{24 \times 60} = \frac{1}{48}$ TV/min.

Service Rate $\mu = 1/30$ TV/min

$\rho = \frac{\lambda}{\mu} = 0.625$

Idle time $= 1 - \rho$

$= 0.375$

Q A barber takes 25 mins to complete 1 haircut on the average. If the customer arrive at an average interval of 40 mins. How long on the avg must the customer wait for the service

Ans

service rate = 1/25 haircut/min

waiting rate = 1/40 haircut/min

$$\therefore W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/40}{1/25(1/25 - 1/40)}$$

$$= 41.667 \text{ min}$$

Q The rate of arrival of a plane at an international airport is 20/hr & the airport can land 30/hr on an avg, when there is congestion the planes are forced to fly over the field

(i) How many planes in da air

(ii) How long will be plane in air & process of lan

$\lambda = 20$

$\mu = 30$

Ans

(i) Queue length $= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{4}{3} = 1.333$

(ii) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{30-20} = 0.1 \text{ hr}$

$= 6 \text{ mins}$

Q Customers arrive at a first class ticket counter of a railway station at the rate of 12 per hour. There is a clerk serving the customers @ 30/hr

(i) Probability that there is no customer at the counter

(ii) Probability that there are more than two customers in the queue

(iii) Probability that customer is being served and no-one is waiting.

Ans

$\lambda = 12$

$\mu = 30$

(i) $P_0 = (1-\rho)\rho^0 = 1-\rho = 1-\frac{\lambda}{\mu} = 1-\frac{12}{30}$

$= \frac{18}{30} = \frac{3}{5}$

(ii) $P(\text{Queue size} > 2)$

$= P(\text{Queue size} \geq 3) = \rho^3 = \left(\frac{2}{5}\right)^3 = \frac{8}{125}$

(iii) $P_1 = (1-\rho)\rho^1 = \frac{9}{15} \times \frac{2}{5} = \frac{18}{75} = \frac{6}{25}$

Q In a departmental store, one cashier is there to serve the customers & customers pick up their needs by themselves. The arrival rate is 9 customers for every 10 minutes and the cashier can serve 10 customers in 5 minutes. Assuming Poisson arrival rate & exponential dist for service rate. Find (i) Avg no of customers in the system, (ii) Avg no of customers in the queue (iii) Avg time a customer spends in the system. (iv) Avg time a customer waits before getting served.

Ans $\lambda = 54$ customers/hr $\rightarrow$ 9 / 10 mins
$\mu = 120$ customers/hr $\rightarrow$ 20 / 10 mins

$M/M/1 : \infty / \infty$

(i) $\rho = \frac{9}{20}$

(i) $L_s = \frac{\rho}{1-\rho} = 0.8181$

(ii) $L_q = \frac{\rho^2}{1-\rho} = 0.3681$

(iii) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{\mu(1-\rho)} = 0.0909$ / 10 min

(iv) $W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)} = 0.0409$ / 10 min

Q. In a railway marshalling yard goods train arrive @ 30 trains / day, Assuming that
$\lambda \rightarrow$ ~~Poisson~~ exp dist
$\mu \rightarrow$ (time taken to hump a train) exp dist
with avg of 36 mins. (iv) Esto the fraction of a that the machine you will be busy
(i) Avg no of trains in the yard.
(ii) Expected waiting time in da queue.
(iii) Avg no of trains in the queue.

Ans
$\lambda = \left(\frac{30}{24 \times 60}\right)$ / min $\quad \mu = \frac{1}{36} \quad \rho = \frac{3}{4}$

$\lambda = \frac{1}{48}$ / min $\quad \mu = \frac{1}{36}$

(i) $L_s = \frac{\rho}{1-\rho} = 3$

(ii) $W_q = \frac{\rho}{\mu(1-\rho)} = 108$ minutes.

(iii) $L_q = 2.25$

(iv) $\rho = \underline{0.75}$ $\frac{3}{4}$