GRAPH THEORY

Presented by

Dr. B. MALLIKARJUNA

Assistant Professor

Department of Mathematics

Topics

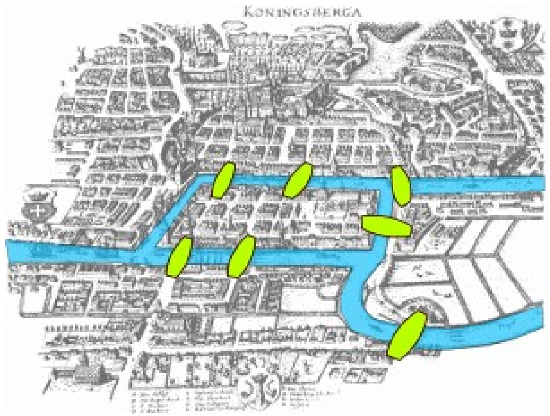
Basic Concepts:

- Types of Graphs,
- Order and Size of Graphs, In-degree and Out-degree
- Connected and Disconnected Graphs
- Eulerian Graph, Hamiltonian Graphs
- Sub-Graphs,
- Isomorphic Graphs
- Matrix Representation of Graphs: Adjacency and Incidence matrix
- Trees: Spanning and minimal spanning tree: Kruskal’s and Prim’s Algorithms
- Shortest Path: Dijkstra’s algorithms

History

The paper written by Leonhard Euler on the Seven Bridges of Königsberg and published in 1736 is regarded as the first paper in the history of graph theory.

Konigsberg Problem

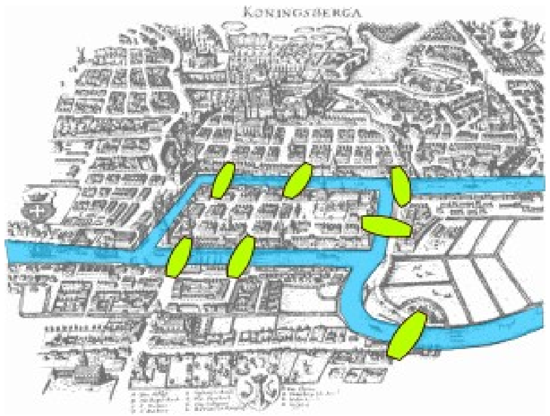


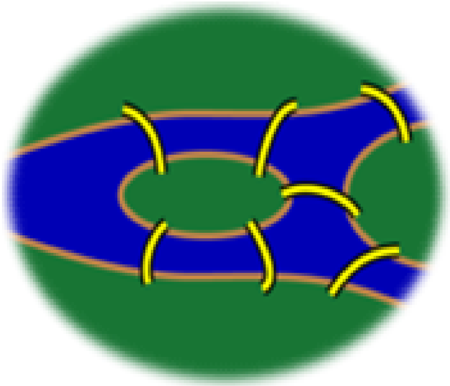
Graph theory started from a mathematical curiosity.

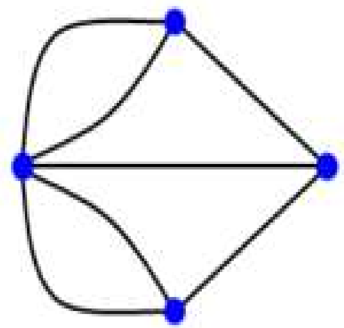
"The Seven Bridges of Königsberg is a problem inspired by an actual place and situation. The city of Kaliningrad, Russia (at the time, Königsberg, Germany) is set on the Pregolya River, and included two large islands which were connected to each other and the mainland by seven bridges. The question is whether it is possible to walk with a route that crosses each bridge exactly once, and return to the starting point. In 1736, Leonhard Euler proved that it was not possible."

Seven Bridges of Königsberg 2

"In proving the result, Euler formulated the problem in terms of graph theory, by abstracting the case of Königsberg -- first, by eliminating all features except the landmasses and the bridges connecting them; second, by replacing each landmass with a dot, called a vertex or node, and each bridge with a line, called an edge or link. The resulting mathematical structure is called a graph."







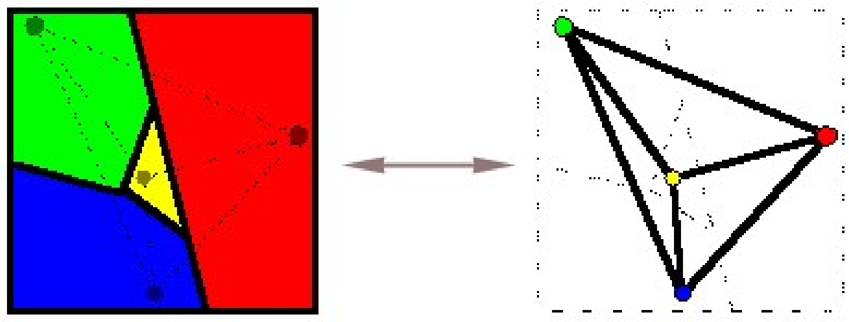
Four Color Problem

- One of the most famous and stimulating problems in graph theory is the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?" This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed.
- The four color problem remained unsolved for more than a century. In 1969 Heinrich Heesch published a method for solving the problem using computers.

Graph theory presentation of the theorem

"To formally state the theorem, it is easiest to rephrase it in graph theory. It then states that the vertices of every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color. Or "every planar graph is four-colorable" for short. Here, every region of the map is replaced by a vertex of the graph, and two vertices are connected by an edge if and only if the two regions share a border segment (not just a corner)"

Source: http://en.wikipedia.org/wiki/Four-color_theorem

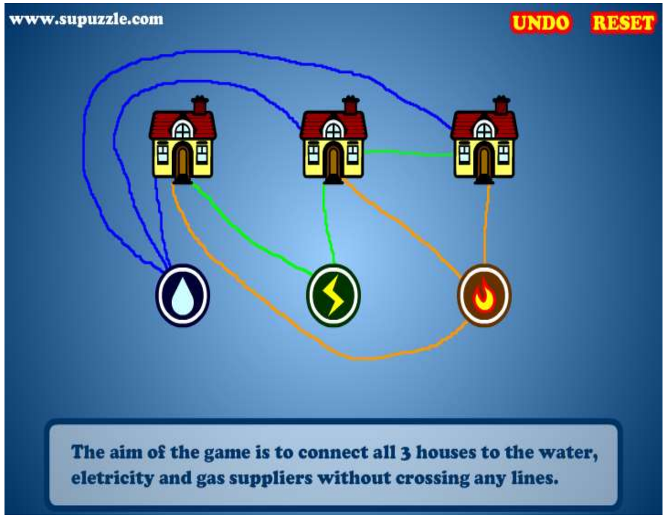


Three cottage problem

Source: http://en.wikipedia.org/wiki/Three_cottage_problem

"The three cottage problem is a well-known mathematical puzzle. It can be stated like this:

Suppose there are three cottages on a plane (or sphere) and each needs to be connected to the gas, water, and electric companies. Is there a way to do so without any of the lines crossing each other?"



Applications

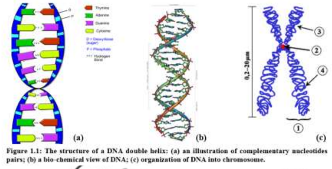
In computer science, graphs are used to represent

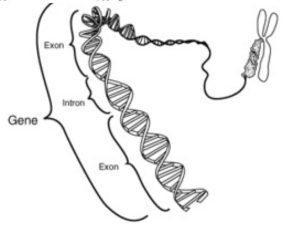
- networks of communication, data organization, computational devices, the flow of computation, etc. For instance, the link structure of a website can be represented by a directed graph, in which the vertices represent web pages and directed edges represent links from one page to another.
- A similar approach can be taken to problems in social media, travel, biology, computer chip design, mapping the progression of neuro-degenerative diseases, and many other fields.

Applications

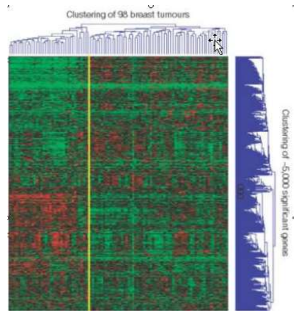
- Graph-theoretic methods, in various forms, have proven particularly useful in linguistics, since natural language often lends itself well to discrete structure.
- Graph theory is also used to study molecules in chemistry and physics.
- Graph theory is also widely used in sociology as a way, for example, to measure actors' prestige or to explore rumor spreading, notably through the use of social network analysis software.
- Biology and in various fields

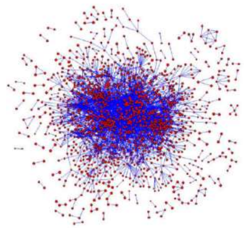
Applications





- Introduction to biology (cell, DNA, RNA, genes, proteins)
- Sequencing and genomics (sequencing technology, sequence alignment algorithms)
- Functional genomics and microarray analysis (array technology, statistics, clustering and classification)
- Introduction to biological networks
- Network properties
  - Network/node centralities
  - Network motifs
- Network models
- Software tools for network analysis





Typical Graph Applications 1

- Modelling a road network with vertices as towns and edge costs as distances.
- Modelling a water supply network. A cost might relate to current or a function of capacity and length. As water flows in only 1 direction, from higher to lower pressure connections or downhill, such a network is inherently an acyclic directed graph.
- Dynamically modelling the status of a set of routes by which traffic might be directed over the Internet.
- Minimising the cost and time taken for air travel when direct flights don't exist between starting and ending airports.
- Managing links between pages within a website and to analyse ease of navigation between different parts of the site.

graph theory: Definition

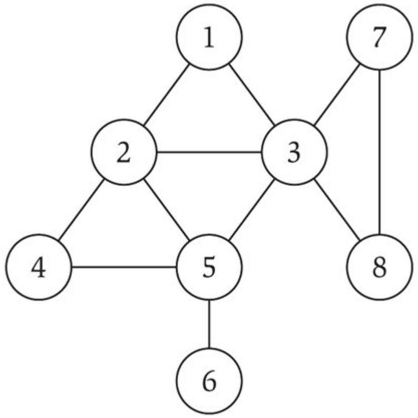
- **Graph** – mathematical object consisting of a set of:
  - *V* = **nodes** (vertices, points). Which is non empty set
  - *E* = **edges** (lines, curves) between pairs of nodes. Which may be empty set.
  - Edge **e** can be represented by {u, v} or simply uv.
  - Denoted by $G = (V, E)$.
  - The number of vertices in a graph is called the **order** of the graph and the number of edges in it is called its **size**
  - **Graph order and size** are denoted by: $n = |V|, m = |E|$.
  - Captures pairwise relationship between objects.

V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { {1,2}, {1,3}, {2,3}, {2,4}, {2,5}, {3,5}, {3,7}, {3,8}, {4,5}, {5,6} }

n = 8

m = 11



- How many diagrams can be drawn for a **graph**

Definitions

- A graph with only a finite number of vertices as well as finite number of edges is called **finite graph**,
- otherwise it is called **infinite graph**.
- A graph containing no edges is called a **null graph**.
- A null graph with only one vertex is called a **trivial graph**

Problems

Draw a diagram of the graph *G(V,E)* in each of the following cases:

(i) V={A, B, C, D}, E={AB, AC, AD, CD}

(ii) $V=\{v_1, v_2, v_3, v_4, v_5\}$, $E=\{v_1v_2, v_1v_3, v_2v_3, v_4v_5\}$

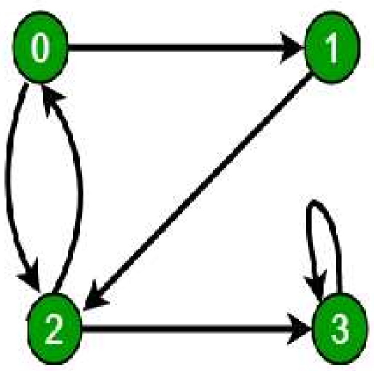
(iii) V={P, Q, R, S, T}, E={PQ, QR, QS}

(iv) V={A, B, C, D, E, F}, E={AD, AF, DF, CB, CE, BE}

Graph theory: Definitions

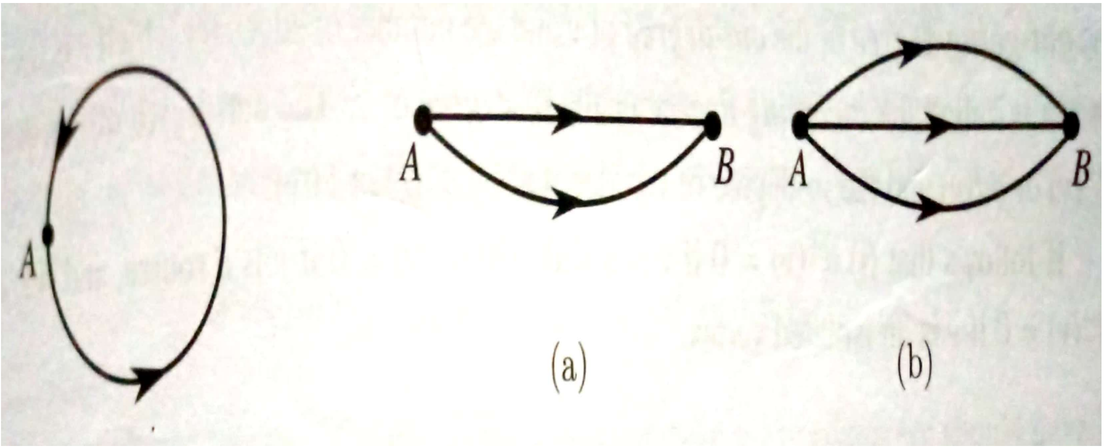
- Directed Graph:
  - A graph is said to be **directed** graph (or a digraph) if V is a nonempty set and E is a set of ordered pairs of elements taken from the set V. otherwise it is called as Undirected graph
  - E is called as directed edge set and elements in E are called as directed edges.

    Which can be represented by ordered pairs: (0,1), (1,2),(3,3), (0,2),(2,0), (2,3)
  - Initial Vertex and terminal vertex



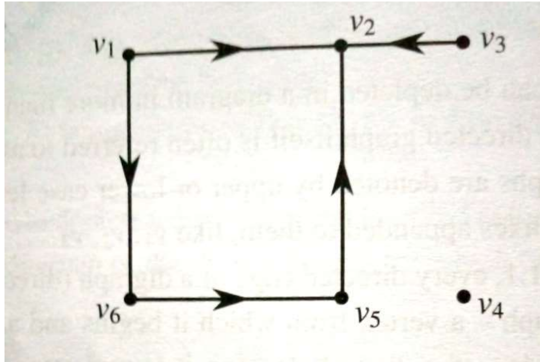
Graph theory: Definitions

- Directed Graph:
  - Directed loop
  - Parallel directed edges
  - Multiple directed edges



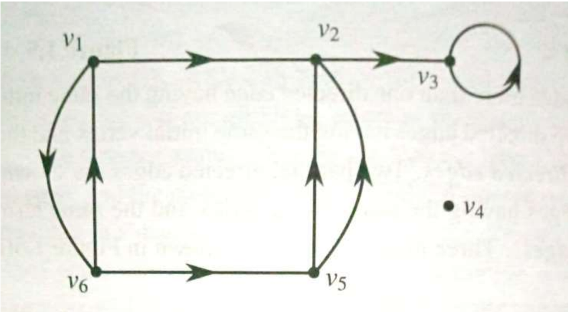
Graph theory: Definitions

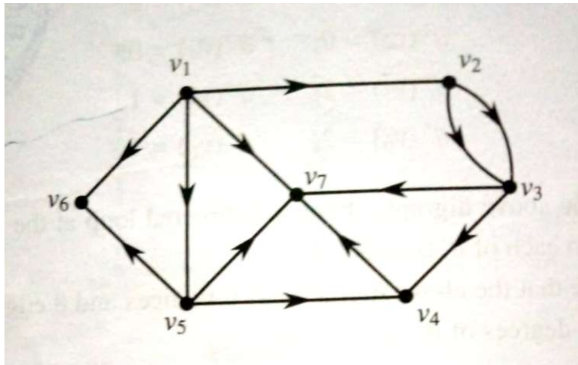
- Directed Graph:
  - Isolated Vertex: A vertex which is neither initial nor terminal
  - Source (head or origin): A non-isolated vertex which is not a terminal vertex
  - Sink (tail or target): A non-isolated vertex which is not an initial vertex

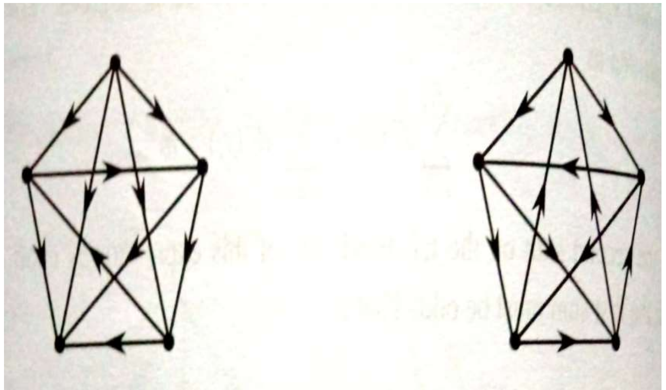


Graph theory: Definitions

- Directed Graph:
  - Examples

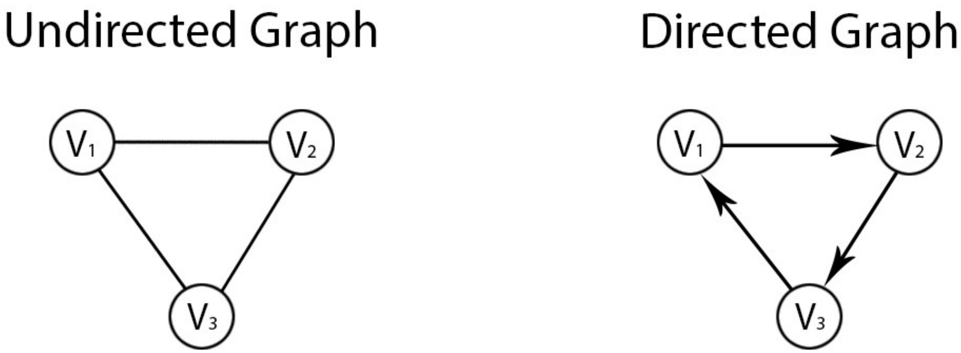






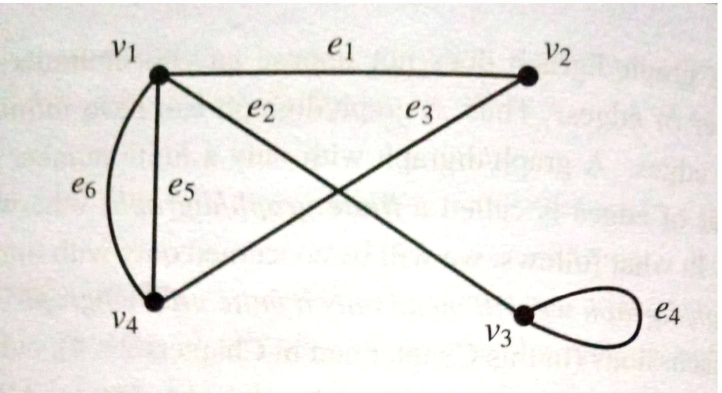
Graph theory: Definitions

- Edge types:
  - **Undirected**;
  - **Directed**; ordered pairs of nodes.



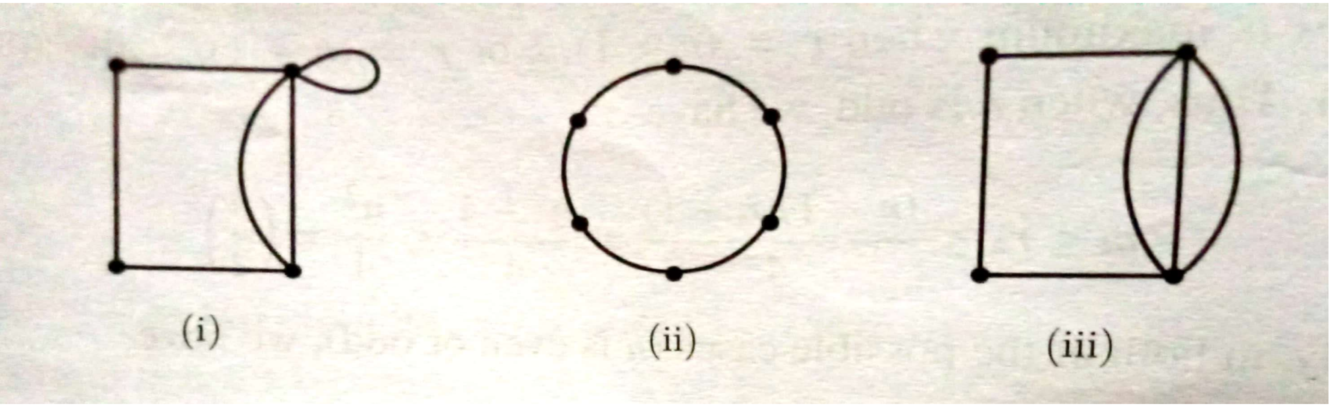
Graph theory: Definitions

- For undirected graph *G(V,E)*:
  - An edge where the two end vertices are the same is called a **loop**, or a **self-loop**
  - Two or more edges having same end vertices are called **multiple edges** and these are also called as **PARALLEL EDGES**
  - If edge e=*{u,v}* $\in E(G)$, we say that *u* and *v* are **adjacent** vertices or **neighbors** to the the edge "e" and "e" is called as incident on (or to) vertices u and v.
  - *u* and *v* are **incident** with *e*
  - *u* and *v* are **end-vertices** of *e*
  - Any two non-parallel edges are said to be **adjacent edges**, if they are incident on a common vertex (i.e., if they have a vertex in common).



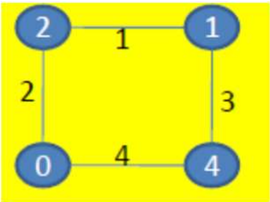
Definitions: Simple graph, Multigraph, General graph

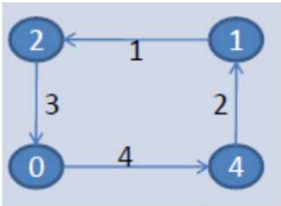
- **Simple graph:** A graph which does not contain loops and multiple edges is called a **simple graph**.
  - A graph which does not contain a loop is called a loop **free graph**
  - A graph which contains multiple edges but no loops is called a **multigraph.**
- A graph which contains multiple edges or loops (or both) is called a **general graph.**

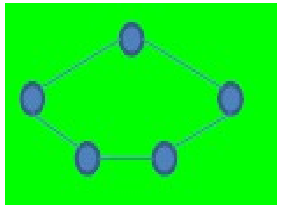
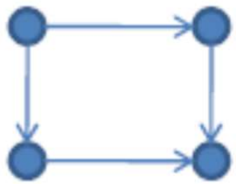


Definitions: Unlabeled and Labeled graphs

- If names are assigned to the vertices of a graph is called **labeled graph.** Otherwise it is called as **unlabeled graphs.**

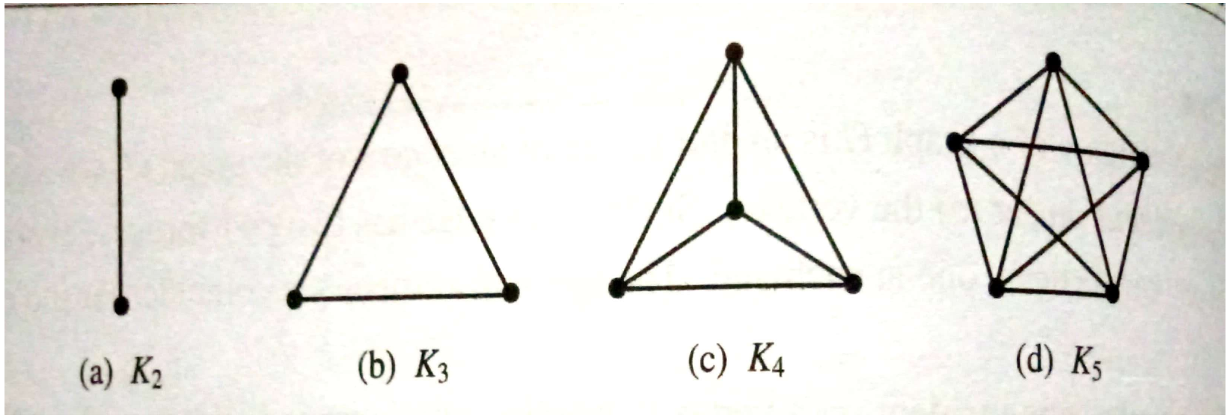




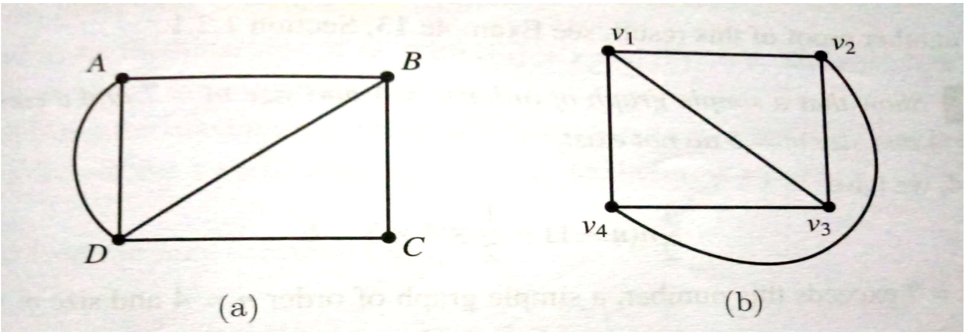
Definitions: Complete Graph

- A simple graph of order ≥ 2 in which there is an edge between every pair of vertices is called a **COMPLETE GRAPH (or a full graph)**
- In other words, a complete graph is a simple graph of order ≥ 2 in which every pair of distinct vertices are adjacent.
- A complete graph with n (≥ 2) vertices is **denoted by K_n.**
- Complete graph with five vertices is called the **KURATOWSKI'S FIRST GRAPH.**

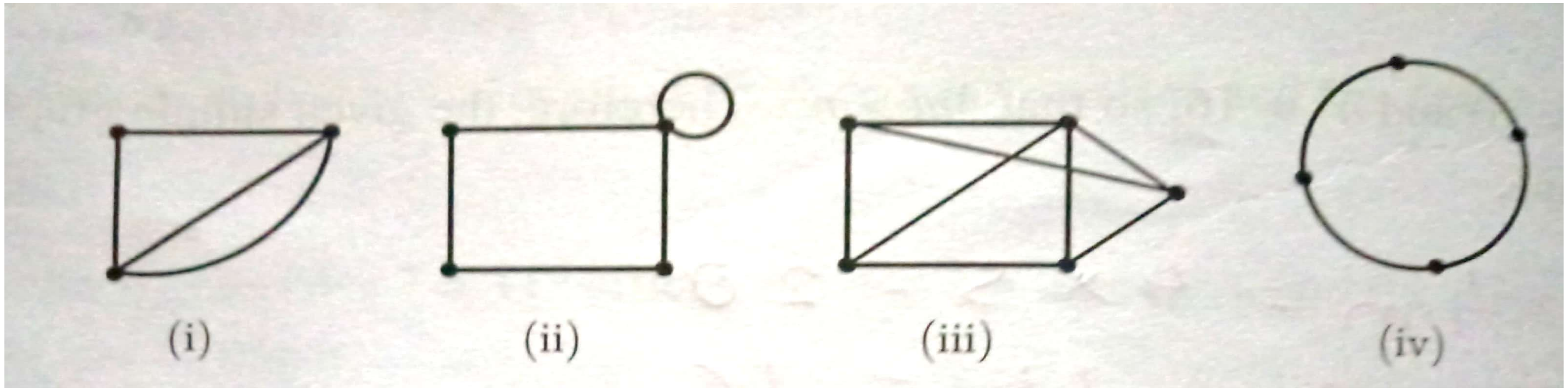


Definitions: Complete Graph

- Which of the following are complete graphs?



(a) (b)

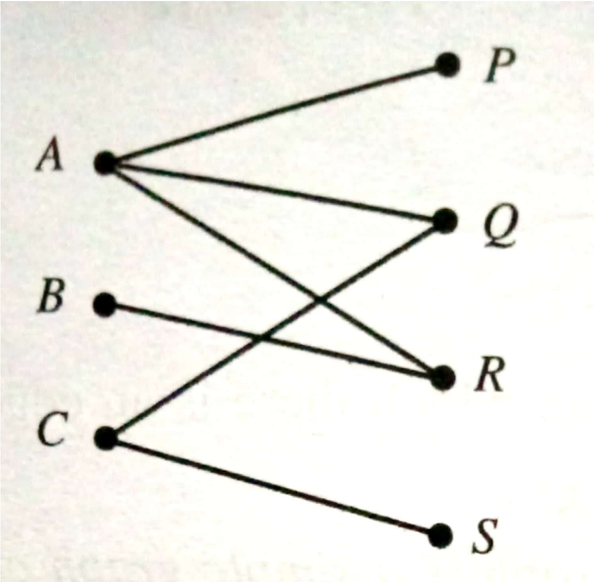

(i) (ii) (iii) (iv)

Definitions: Bipartite Graph

- A simple graph G=G(V,E) is said to be **BIPARTITE GRAPH,**

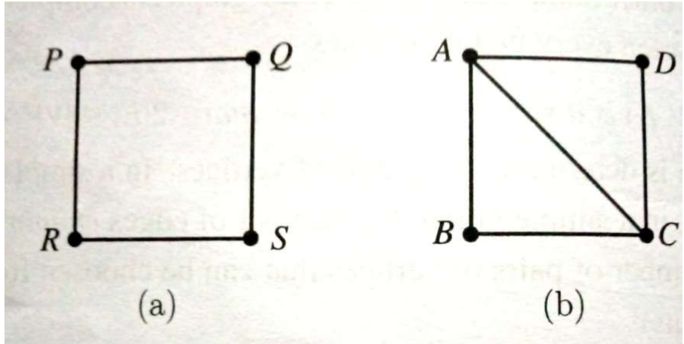
if V is the union of two of its mutually disjoint nonempty subsets V1 and V2 which are such that each edge in G joins a vertex in V1 and a vertex in V2.

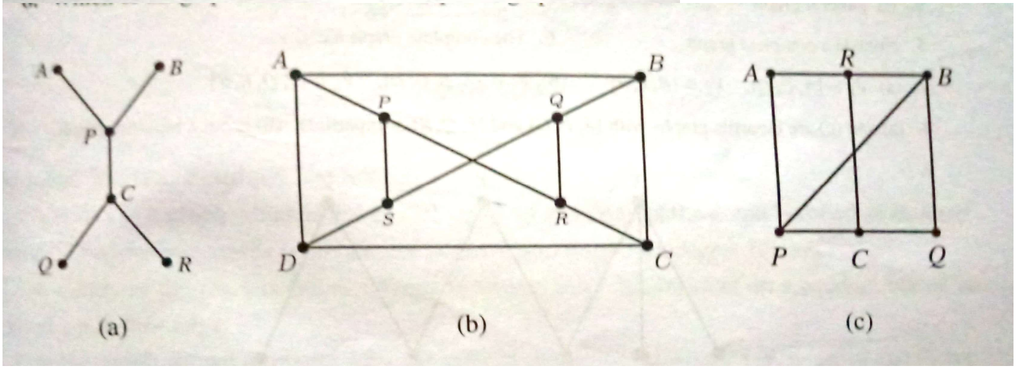
- V1 and V2 are called as **Bipartites**
- This is also represented by **G=G(V1, V2; E).**



Definitions: Bipartite Graph

- Which of the following are Bipartite graphs





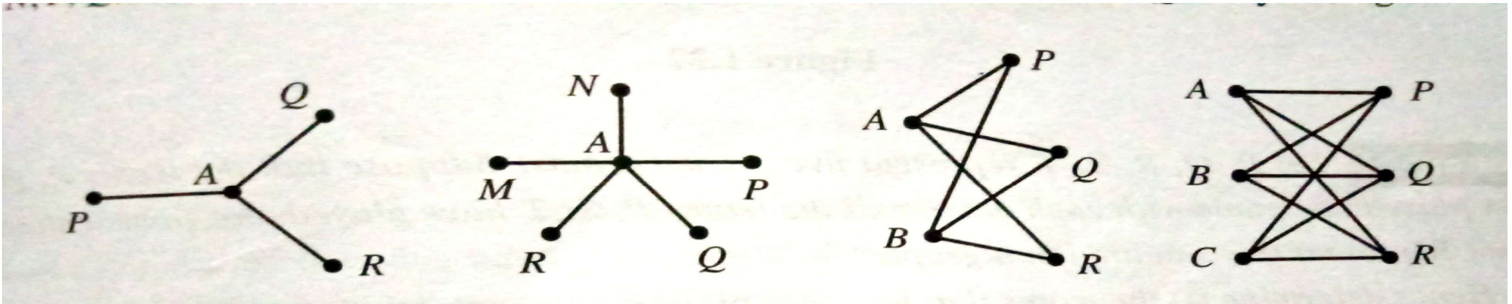
28

Definitions: Complete Bipartite Graph

- A bipartite graph G=G(V1, V2, E) is called a **COMPLE BIPARTITE GRAPH** if there is an edge between every vertex in V1 and every vertex in V2.
- A complete bipartite graph G=G(V1, V2, E) in which the bipartites V1 and V2 contain *r* and *s* vertices respectively, with $r \leq s$, is denoted **by $K_{r,s}$**
- In these graphs each of *r* vertices in V1 is joined to each of *s* vertices in V2. Therefor, $K_{r,s}$ has **r+s vertices** and **r.s edges**:
- Hence **$K_{r,s}$ is of order $r+s$ and size rs.**
- *A* graph $K_{3,3}$ is called as **Kuratowski's second graph.**

Note: Is every complete bipartite graph, a complete graph?

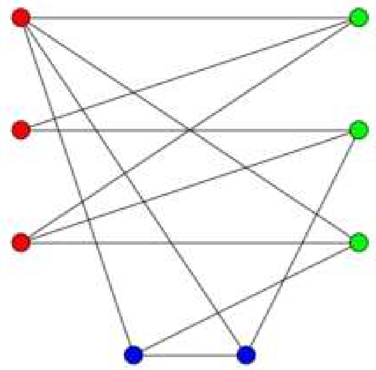
NO



(a) : $K_{1,3}$ (b) : $K_{1,5}$ (c) : $K_{2,3}$ (d) : $K_{3,3}$

Definitions: Complete Bipartite Graph

- Which of the following are Complete Bipartite Graphs



PROBLEMS

- Prove that a complete graph with *n* vertices, namely K_n, has n(n-1)/2 edges.

Proof: In a complete graph, there exists exactly one edge between every pair of vertices.

As such, the number of edges in a complete graph is equal to the number of pairs of vertices.

If the number of vertices is *n,* then the number of pairs of vertices is

$$\binom{n}{k} = \frac{n(n-1)}{2}$$

PROBLEMS

- Verify whether simple graph of order 4 and size 7 and a complete graph of order 4 and size 5 do exist?

Proof: For *n=4,* then the number of pairs of vertices is

$$\binom{n}{2} = \frac{n(n-1)}{2} = 6$$

Since, *m=7* exceeds this number, a simple graph of order 4 and size 7 does not exist.

Similarly since, *m=5* is not equal to 6, a complete graph of order 4 and size 5 does not exit.

PROBLEMS

- A) How many vertices and how many edges are there in the complete bipartite graphs $K_{4,7}$ and $K_{7,11}$?

  B) If the graph $K_{r,12}$ has 72 edges, what is *r*?

Proof: Complete bipartite graph $K_{r,s}$ has $r+s$ vertices and rs edges.

A) The graph $K_{4,7}$ has 4+7=11 vertices and 4*7=28 edges,

   and the graph $K_{7,11}$ has 18 vertices and 77 edges.

B) If the graph $K_{r,12}$ has 72 edges,

   we have 12r=72 so that r=6.

Definitions: Complete Bipartite Graph

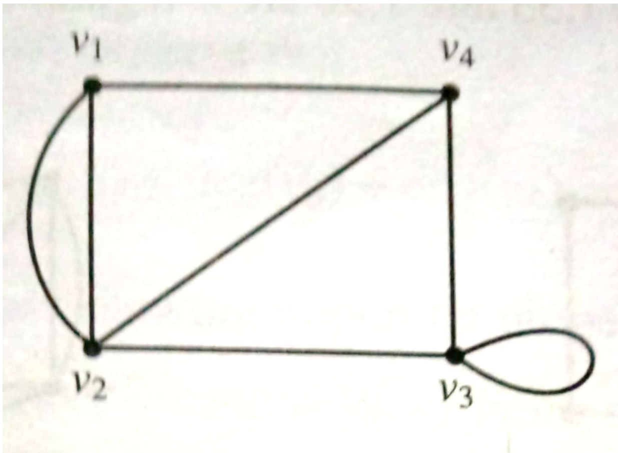
Property: Let $G=(V,E)$ be a simple graph of order $|V|=n$ and size $|E|=m$. If G is a bipartite graph then $4m \leq n^2$.

Prove that a simple graph of order 4 and size 5 cannot be a bipartite graph.

Vertex Degree

Let G=(V,E) be an undirected graph and *v* be a vertex of G. Then, the number of edges of G that are incident on *v* (i.e., the number of edges that join *v* to other vertices of G) with the loops counted twice is called the ***degree*** of the vertex *v* and is denoted by deg(v), or d(v).

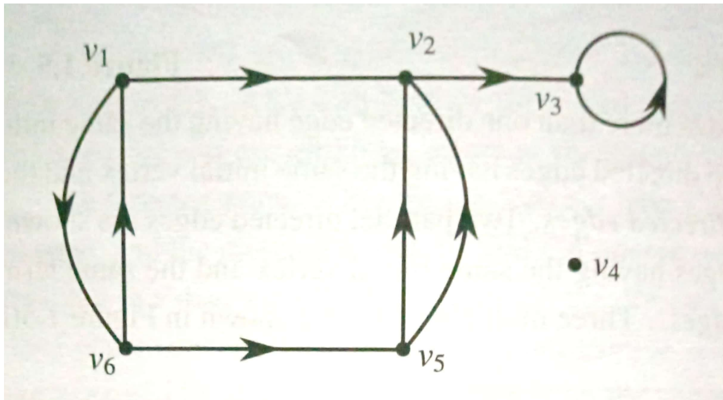
- The degrees of all vertices of a graph arranged in non-decreasing order is called the **degree sequence** of the graph.
- Minimum of the degrees of vertices of a graph is called the **degree of the graph.**



In Degree and Out Degree

Let G=(V,E) be a directed graph and *v* be a vertex of G.

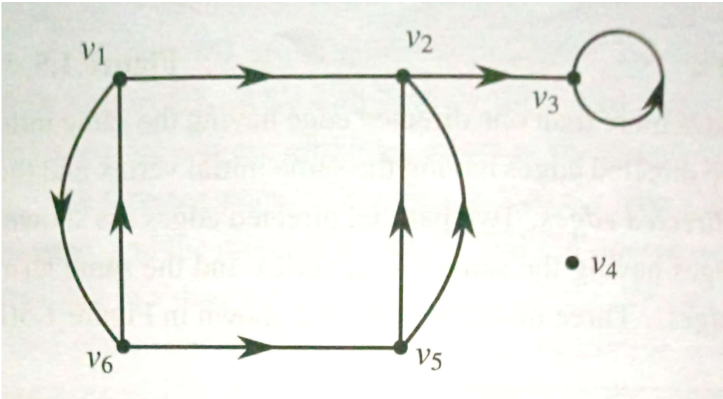
- The number of edges for which *v* is the initial vertex is called out-going degree or the out-degree of *v*
- The number of edges for which *v* is the terminal vertex is called in coming degree or the in-degree of v.
- The out-degree of *v* is denoted by $d^+(v)$ or od(v)
- The in-degree of *v* is denoted by $d^-(v)$ or id(v)
- If od(v) = 0 then v is a sink and id(v) = 0 then v is a source
- If od(v) = id(v) = 0 then v is called isolated vertex



In Degree and Out Degree

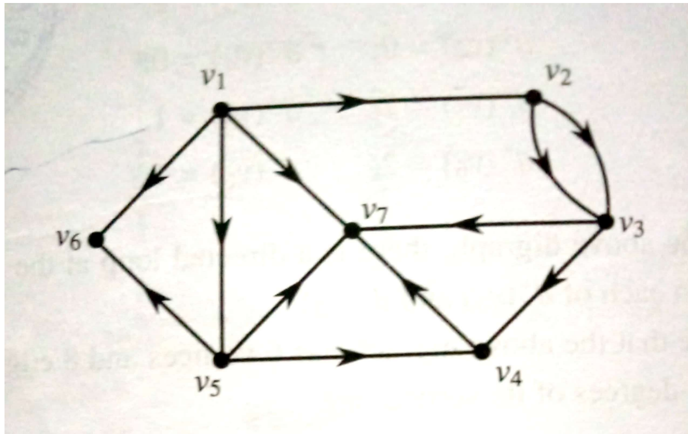
FIRST THEOREM OF DIGRAPHS

Property: In every digraph D, the sum of the out-degrees of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.



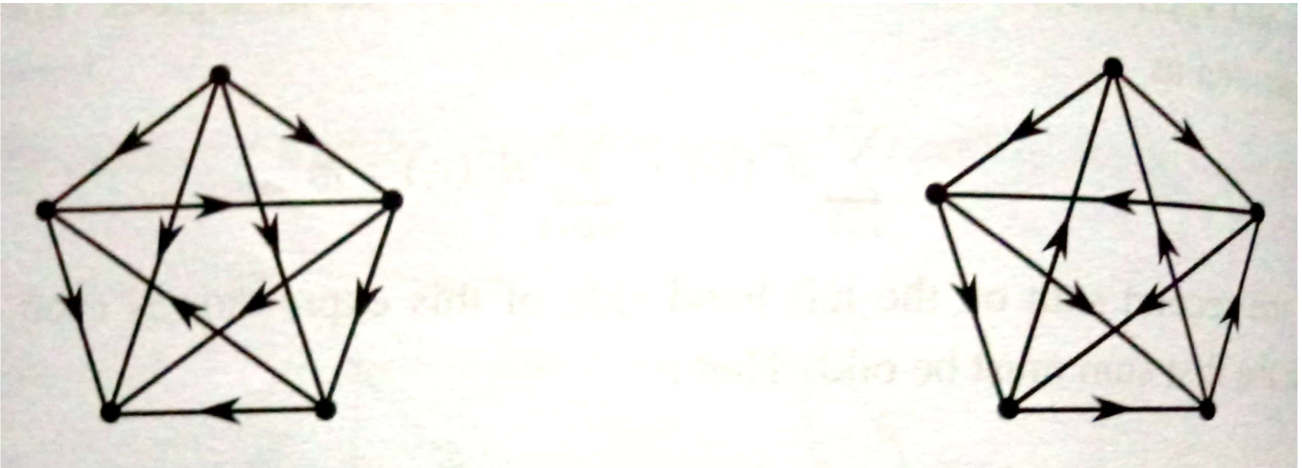
In Degree and Out Degree

Problem: Find the in-degrees and out-degrees of the following graph



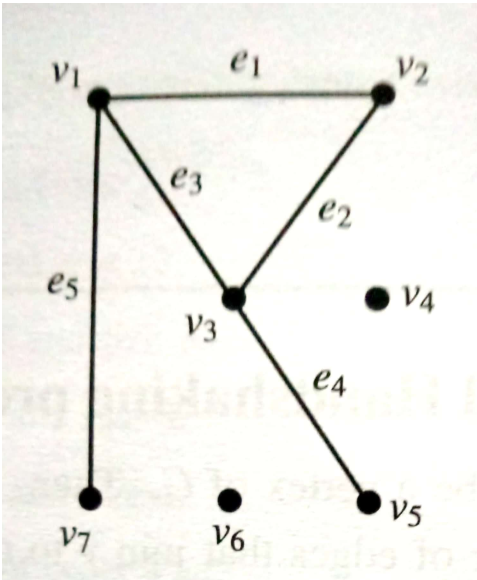
In Degree and Out Degree

Verify the First theorem of Digraph for the following graphs



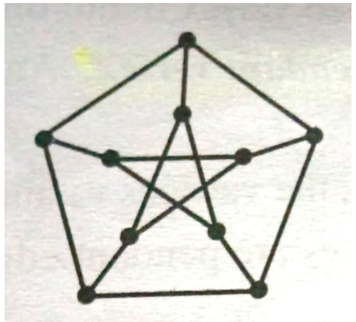
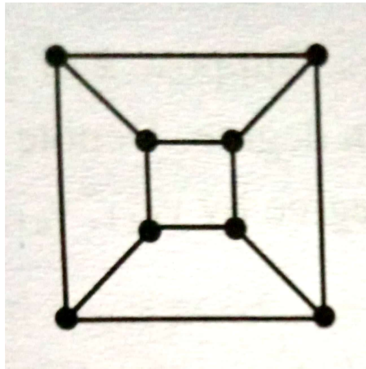
Isolated Vertex, Pendent Vertex

- A vertex in an undirected graph which is not an end vertex of any edge of the graph is called an **isolated vertex.**
- Therefore, degree of an isolated vertex is **<u>zero</u>**
- A vertex of degree "1" is called a pendant vertex. An edge incident on a pendant vertex is called a pendant edge.



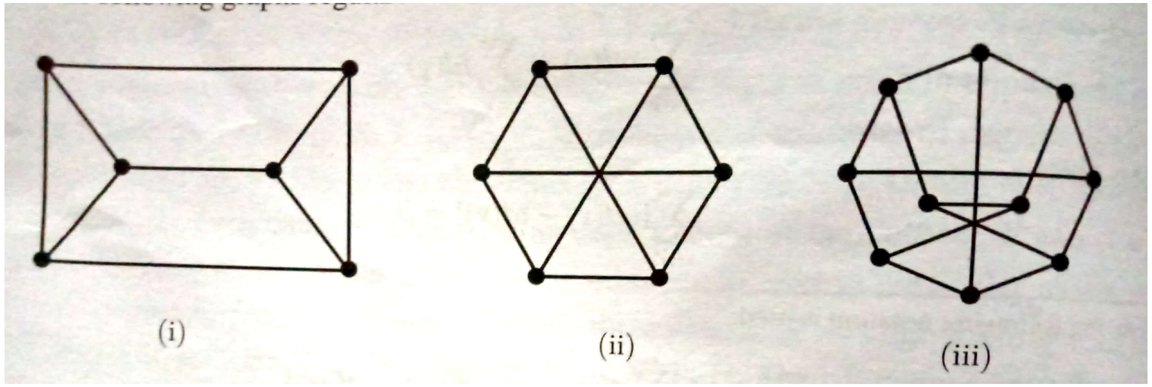
Regular Graph

- A graph in which all the vertices are of the same degree k is called **a regular graph of degree k, or a k-regular graph.**
- In particular, a 3-regular graph is called a ***cubic graph***.
- A cubic graph with 10 vertices and 15 edges is called the **Peterson graph.**
- For any positive integer k, a loop-free k-regular graph with 2^k vertices is called the k-dimensional hypercube (or k-cube) and is denoted by Q_k.

Regular Graph

- Are the following graphs regular



Handshaking Property

- The sum of the degrees of all the vertices in a graph is an even number: and this number is equal to twice the number of edges in the graph. This property is called ***HANDSHAKING PROPERTY.***

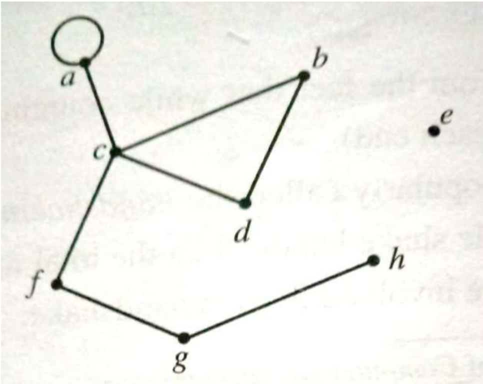
That is, for a graph G=(V,E).

$$\sum_{v \in V} \deg(v) = 2|E|$$

Property: In every graph, the number of vertices of odd degrees is even.

Handshaking Property

- Verify the Handshaking property to the following graphs



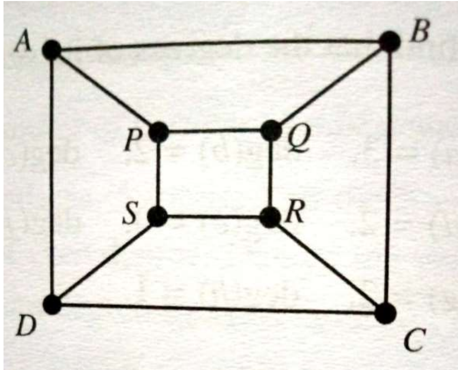
Handshaking Property

Problems

1) Can there be a graph consisting of the vertices A, B, C, D with deg(A)=2, deg(B)=3, deg(C)=2 and deg(D)=2?
2) Can there be a graph with 12 vertices such that two of the vertices have degree 3 each and the remaining 10 vertices have degree 4 each?
3) For a graph G=(V,E), what is the largest possible value for |V|, if |E|=19 and deg(v) ≥4 for all *v in V?*

Handshaking Property

Prove that the hypercube Q_3 is a bipartite graph which is not a complete bipartite graph.



Proof:

Handshaking Property

Prove that k-dimensional hypercube Q_k has $k2^{k-1}$ edges. Determine the number of edges in $Q_{8.}$

Proof:

Handshaking Property

1) What is the dimension of the hypercube with 524288 edges?
2) How many vertices are there in a hypercube with 4980736 edges?

Solution:

Handshaking Property

Determine the order |V| of the graph in the following cases

1) G is a cubic graph with 9 edges.
2) G is regular with 15 edges.
3) G has 10 edges with 2 vertices of degree 4 and all other vertices of degree 3.

Handshaking Property

Let G be a graph of order 9 such that each vertex has degree 5 or 6. Prove that at least 5 vertices have degree 6 or atleast 6 vertices have degree 5.

Handshaking Property

Prove that there is no graph with 28 edges and 12 vertices in the following cases

1) The degree of a vertex is either 3 or 4.
2) The degree of a vertex is either 3 or 6.

Handshaking Property

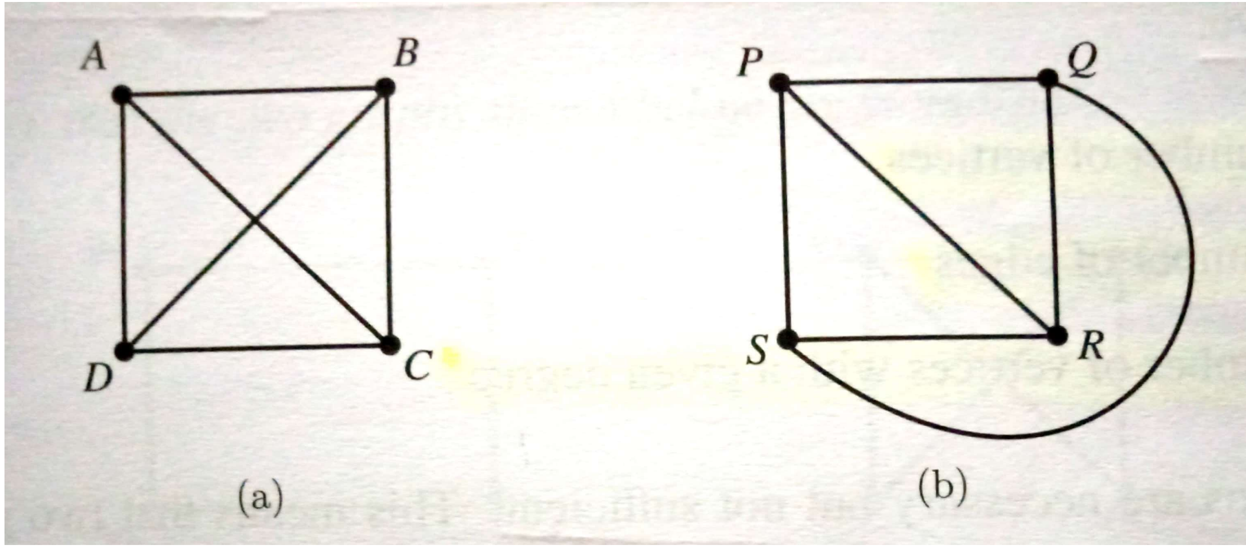
For a graph with n vertices and m edges, if δ is the minimum and Δ is the maximum of the degrees of vertices, Prove that

$$\delta \leq \frac{2m}{n} \leq \Delta$$

ISOMORPHISM

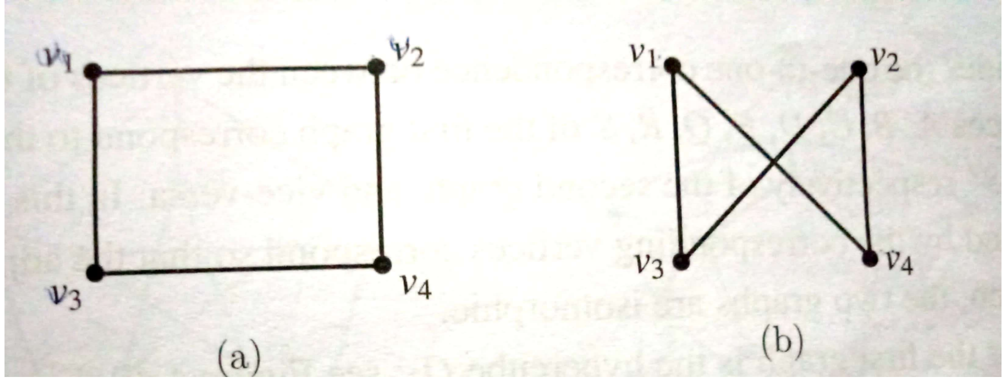
Definition: Consider two graphs G=(V,E) and G' = (V',E'). Suppose there exists a function $f: V \to V'$ such that (i) f is one to one correspondence and (ii) for all vertices A,B of G, {A,B} is an edge of G if and only if {f(A, f(B)} is an edge of G'. Then f is called an ***isomorphism*** between G and G', and we say that G and G' are ***isomorphic graphs.***

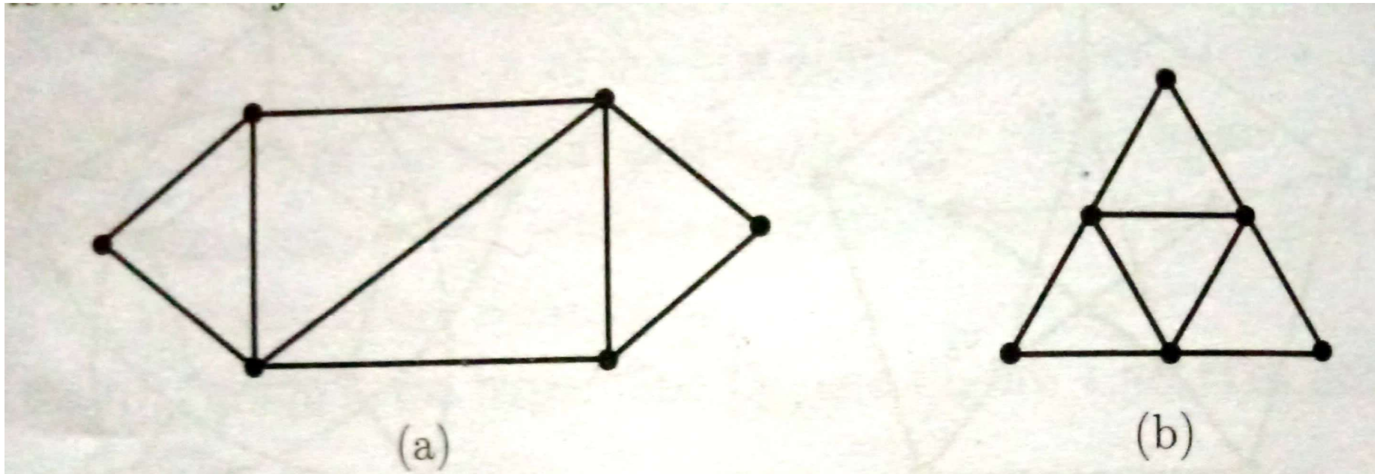
In other words, two graphs are said to be isomorphic if there is one to one correspondence between their vertices and between their edges such that adjacency of vertices is preserved.



ISOMORPHISM

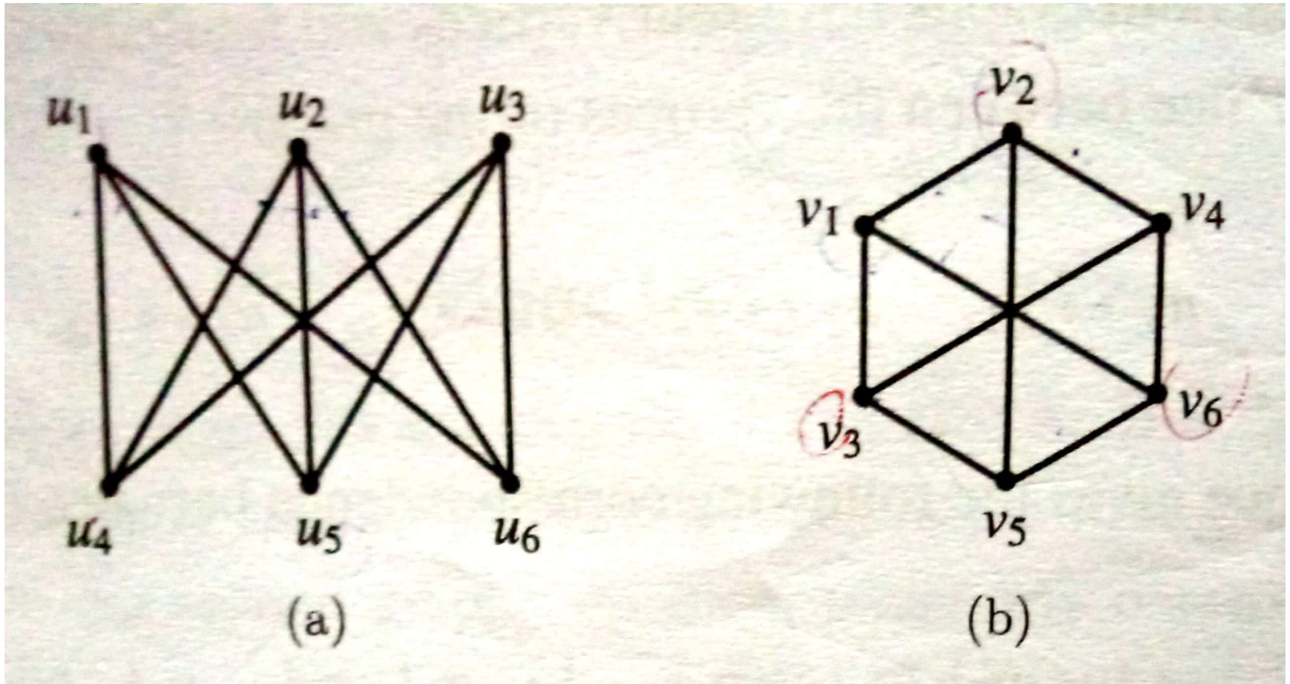
Verify the following graphs are isomorphic or not





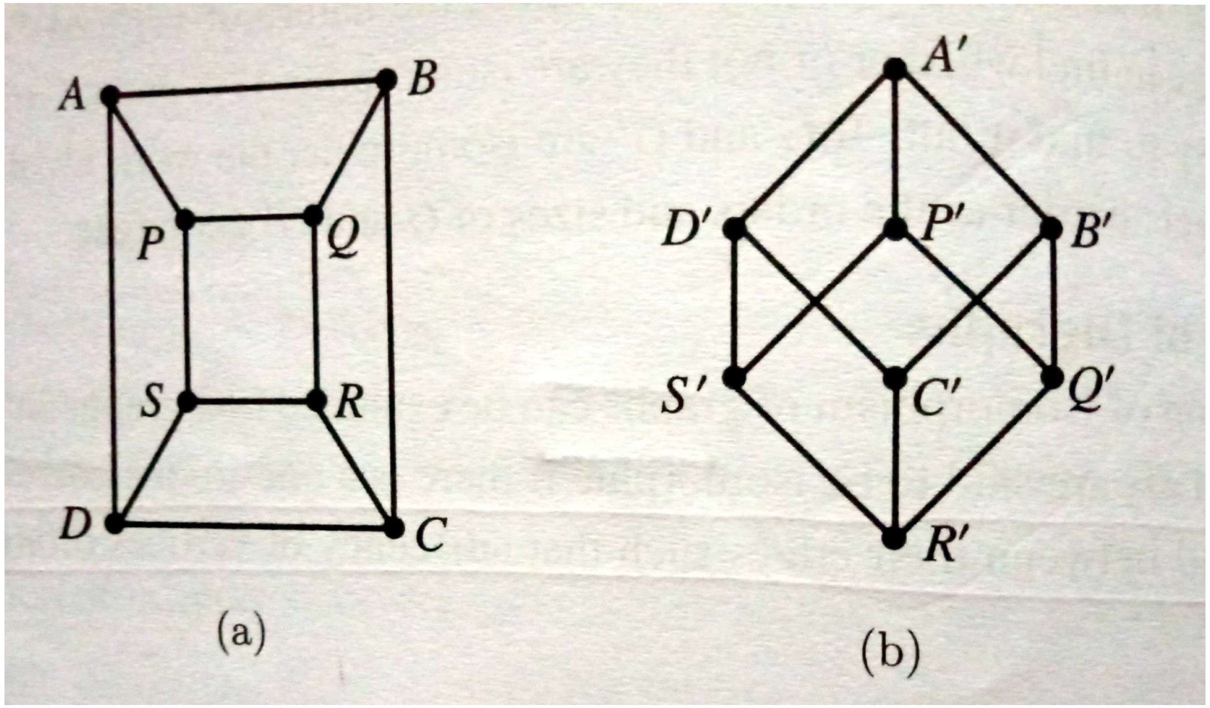
ISOMORPHISM

Verify the following graphs are isomorphic or not



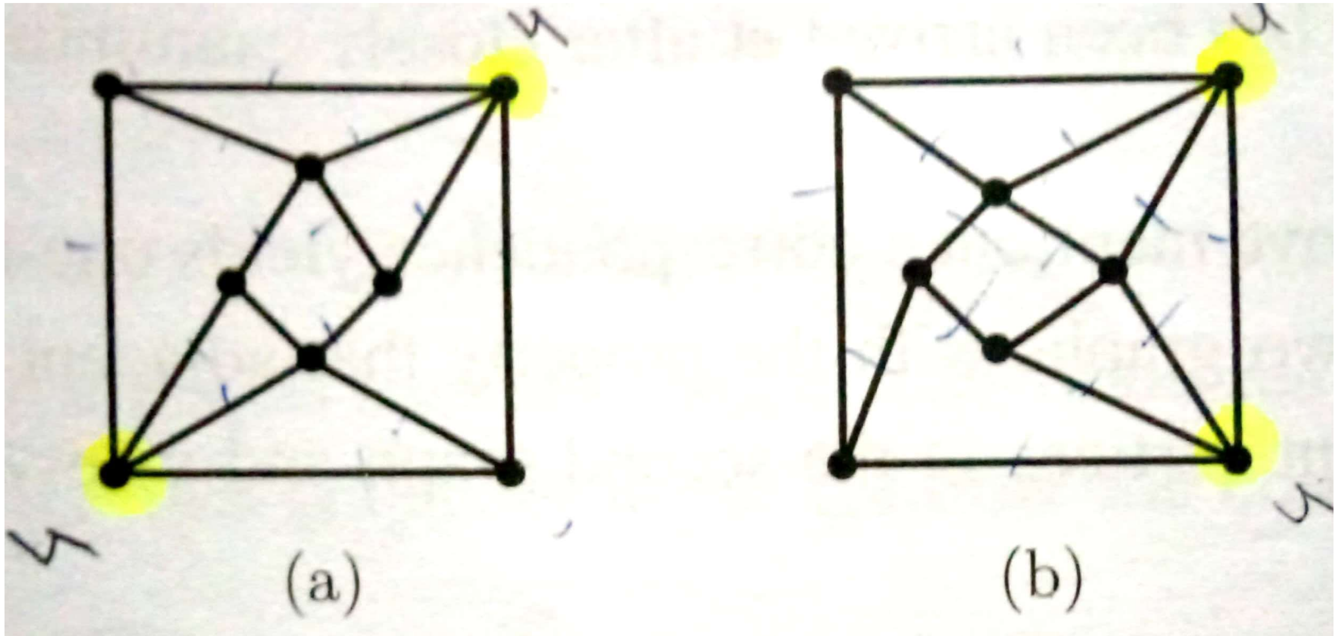
ISOMORPHISM

Verify the following graphs are isomorphic or not



ISOMORPHISM

Verify the following graphs are isomorphic or not



Thank you