5.2 Stochastic Process

We have already stated that in a random experiment, if a real variable X is associated with every out come then it is called a random variable or stochastic variable. This is equivalent to, having a function on the sample space S and this function is called a random function or a stochastic function. In this article we discuss a stochastic process called the *Markov process* which is such that the generation of the probability distributions depend only on the present state. Before we take up the actual discussion of this Markov process we present some basic definitions and concepts relating to stochastic process.

• Classification of Stochastic Processes

Let S be the sample space of a random experiment and R be the set of all real numbers. A random variable X is a function f from S to R *i.e.,* $X = f(s)$, $s \in S$. We define an *index set* $T \subset R$ indexed by the parameter t such as time. Let us suppose that the value of a random variable defined on S depends on $s \in S$ and $t \in T$. In this context a ***Stochastic process*** is a set of random variables $\{X(t),\ t \in T\}$ defined on S with a parameter t. Here $X_0 = X(0)$ is called as the initial state of the system.

The values assumed by the random variable $X(t)$ are called *states* and the set of all possible values forms the *state space* of the process. If the state space of a stochastic process is discrete then it is called a ***discrete state process*** also called a *chain.*

On the other hand if the state space is continuous then the stochastic process is called a ***continuous state process.***

Similarly if the index set T is discrete then we have a ***discrete parameter process.*** Otherwise (*i.e., when T is a continuous set*) we have a ***continuous parameter process.*** A discrete parameter process is also called a stochastic sequence denoted by $\{X_n\}$, $n \in T$.

The classification of the four different type of stochastic processes are presented in the form of a table.

	Discrete Index Set - T	Continuous Index Set - T
Discrete State Space	Discrete parameter stochastic process (chain)	Continuous parameter stochastic process (chain)
Continuous State Space	Discrete parameter continuous state stochastic process	Continuous parameter continuous state stochastic process

5.21 Definitions

Probability Vector : By a vector we simply mean n tuple of numbers $(v_1, v_2, \dots v_n)$ where the quantities $v_1, v_2, \dots v_n$ are called components of the vector.

A vector $v = (v_1, v_2, \dots v_n)$ is called a *probability vector* if each one of its components are non negative and their sum is equal to unity.

Examples : $u = (1, 0)$; $v = \left(1/2, 1/2\right)$, $w = \left(1/4, 1/4, 1/2\right)$ are all probability vectors.

Note : *If v is not a probability vector but each one of the v_i $(i = 1$ to $n)$ are non negative then λv is a probability vector where $\lambda = 1 / \sum_{i=1}^{n} v_i$.*

For example if $v = (1, 2, 3)$ then $\lambda = 1/6$ and $(1/6, 2/6, 3/6)$ is a probability vector.

Stochastic Matrix : A square matrix $P = (p_{ij})$ having every row in the form of a probability vector is called a *stochastic matrix.*

Examples : (i) Identity matrix *(I)* of any order.

$$I_{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; I_{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

Regular Stochastic Matrix : A stochastic matrix P is said to be a *regular stochastic matrix* if all the entries of some power P^n are positive.

Example : $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Consider $$A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$\therefore$ A is a regular stochastic matrix $(n = 2)$

- **Properties of a Regular Stochastic Matrix**

The following properties are associated with a regular stochastic matrix P of order n.

1. (a) P has a *unique fixed point* $x = (x_1, x_2, \cdots x_n)$ such that $xP = x$

   (b) P has a unique fixed probability vector $v = (v_1, v_2, \cdots v_n)$ such that $vP = v$ where $v_i = \dfrac{x_i}{\sum_{i=1}^{n} x_i}$

2. $P^2, P^3, \cdots$ approaches the matrix V whose rows are each the fixed probability vector v.

3. If u is any probability vector then the sequence of vectors $uP, uP^2, \cdots$ approaches the unique fixed probability vector v.

WORKED PROBLEMS

47. *If* $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ *is a stochastic matrix and* $v = [v_1, v_2]$ *is a probability vector, show that* vA *is also a probability vector.*

>> By data $a_1 + a_2 = 1, b_1 + b_2 = 1, v_1 + v_2 = 1$.

$$\therefore \quad vA = \begin{bmatrix} v_1, & v_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} v_1 a_1 + v_2 b_1, & v_1 a_2 + v_2 b_2 \end{bmatrix}$$

We have to prove that $(v_1 a_1 + v_2 b_1) + (v_1 a_2 + v_2 b_2) = 1$

LHS $= v_1(a_1 + a_2) + v_2(b_1 + b_2) = v_1 \cdot 1 + v_2 \cdot 1 = v_1 + v_2 = 1$

Thus **vA is also a probability vector.**

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48. *Prove with reference to two second order stochastic matrices that their product is also a stochastic matrix.*

>> Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be two stochastic matrices. Hence we have,

$$\left.\begin{aligned} a_{11} + a_{12} = 1 \;;\; b_{11} + b_{12} = 1 \\ a_{21} + a_{22} = 1 \;;\; b_{21} + b_{22} = 1 \end{aligned}\right\} \quad \ldots \text{(i)}$$

$$\therefore \quad AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\, b_{11} + a_{12}\, b_{21}, & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21}, & a_{21}\, b_{12} + a_{22}\, b_{22} \end{bmatrix}$$

We have to show that,

$$a_{11}\, b_{11} + a_{12}\, b_{21} + a_{11}\, b_{12} + a_{12}\, b_{22} = 1 \qquad \ldots \text{(ii)}$$

and $$a_{21}\, b_{11} + a_{22}\, b_{21} + a_{21}\, b_{12} + a_{22}\, b_{22} = 1 \qquad \ldots \text{(iii)}$$

LHS of (ii) can be written as,

$$a_{11}\,(b_{11} + b_{12}) + a_{12}\,(b_{21} + b_{22})$$

ie., $= a_{11} \cdot 1 + a_{12} \cdot 1 = a_{11} + a_{12} = 1$, by using (i).

LHS of (iii) can be written as,

$$a_{21}\,(b_{11} + b_{12}) + a_{22}\,(b_{21} + b_{22}) = a_{21} \cdot 1 + a_{22} \cdot 1 = 1$$

Thus AB is a stochastic matrix.

Remark : *In particular we can say that A^n ($n = 1, 2, 3, \ldots$) are all stochastic matrices.*

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49. *If A is a square matrix of order n whose rows are each the same vector $a = (a_1, a_2, \cdots a_n)$ and if $v = (v_1, v_2, \cdots v_n)$ is a probability vector, prove that $vA = a$*

>> By data we have, $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$

and $\quad v_1 + v_2 + \cdots v_n = 1$

Consider vA as a matrix product.

$$vA = \begin{bmatrix} v_1, & v_2, & \cdots v_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$= \left[v_1 a_1 + v_2 a_1 + \cdots + v_n a_1 ,\ v_1 a_2 + v_2 a_2 + \cdots + v_n a_2 ,\ \cdots v_1 a_n + v_2 a_n + \cdots v_n a_n \right]$$

$$= \left[a_1 (v_1 + v_2 \cdots + v_n),\ a_2 (v_1 + v_2 + \cdots + v_n), \cdots a_n (v_1 + v_2 + \cdots + v_n) \right]$$

$$= \left[a_1 ,\ a_2 ,\ \cdots a_n \right] = a \text{ since } v_1 + v_2 + \cdots + v_n = 1$$

Thus $vA = a$ as required.

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50. *Find the unique fixed probability vector of the regular stochastic matrix* $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

>> We have to find $v = (x, y)$ where $x + y = 1$ such that $vA = v$

$$\Rightarrow \begin{bmatrix} x, & y \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} x, & y \end{bmatrix}$$

i.e., $$\left[\frac{3}{4}x + \frac{1}{2}y,\ \frac{1}{4}x + \frac{1}{2}y \right] = \begin{bmatrix} x, & y \end{bmatrix}$$

$$\Rightarrow \quad \frac{3}{4}x + \frac{1}{2}y = x \qquad \ldots \text{(i)}$$

$$\frac{1}{4}x + \frac{1}{2}y = y \qquad \ldots \text{(ii)}$$

We can solve either of the two equations by using $y = 1 - x$.

Using $y = 1 - x$ in (i) we have, $\frac{3}{4}x + \frac{(1-x)}{2} = x$

or $\quad 3x + 2 - 2x = 4x \quad \therefore \quad x = 2/3$

Hence $y = 1 - x = 1/3$ and $v = (x, y) = (2/3,\ 1/3)$

Thus $(2/3,\ 1/3)$ is the unique fixed probability vector.

---

51. *Find the unique fixed probability vector for the regular stochastic matrix*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

>> We have to find $v = (x, y, z)$ where $x + y + z = 1$ such that $vA = v$

$$\Rightarrow \quad \begin{bmatrix} x, & y, & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} x, & y, & z \end{bmatrix}$$

i.e., $\left[\frac{y}{6},\ x+\frac{y}{2}+\frac{2z}{3},\ \frac{y}{3}+\frac{z}{3}\right]=\left[x,\ y,\ z\right]$

$\Rightarrow \quad \frac{y}{6}=x,\ x+\frac{y}{2}+\frac{2z}{3}=y,\ \frac{y}{3}+\frac{z}{3}=z$

i.e., $\quad y=6x,\ 6x+3y+4z=6y,\ y+z=3z$

i.e., $\quad y=6x,\ 6x-3y+4z=0,\ y-2z=0$

Using $y=6x$ and $z=1-x-y=1-x-6x=1-7x$ in $6x-3y+4z=0$ we have,

$$6x-18x+4-28x=0 \quad \therefore \quad x=1/10$$

Hence $y=6/10,\ z=3/10$

Thus the required unique fixed probability vector v is given by

$$v=\left(\mathbf{1/10},\ \mathbf{6/10},\ \mathbf{3/10}\right)$$

---

52. *With reference to the stochastic matrix A in Example-24, verify the property that the sequence $A^2,\ A^3,\ A^4$ approaches the matrix whose rows are each the fixed probability vector.*

>> We have $A=\begin{bmatrix}3/4 & 1/4\\ 1/2 & 1/2\end{bmatrix}$ and we have obtained in Problem-50 the fixed probability vector $v=(2/3,\ 1/3)$.

Let B be the matrix whose each row is v.

i.e., $\quad B=\begin{bmatrix}2/3 & 1/3\\ 2/3 & 1/3\end{bmatrix}$

Consider $A=\frac{1}{4}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}$

Now $A^2=\frac{1}{16}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}=\frac{1}{16}\begin{bmatrix}11 & 5\\ 10 & 6\end{bmatrix}=\begin{bmatrix}0.6875 & 0.3125\\ 0.625 & 0.375\end{bmatrix}$

$$A^3=\frac{1}{64}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}11 & 5\\ 10 & 6\end{bmatrix}=\frac{1}{64}\begin{bmatrix}43 & 21\\ 42 & 22\end{bmatrix}=\begin{bmatrix}0.671875 & 0.328125\\ 0.65625 & 0.34375\end{bmatrix}$$

$$A^4=\frac{1}{256}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}43 & 21\\ 42 & 22\end{bmatrix}=\frac{1}{256}\begin{bmatrix}171 & 85\\ 170 & 86\end{bmatrix}\approx\begin{bmatrix}0.67 & 0.33\\ 0.66 & 0.34\end{bmatrix}$$

Each row of A^4 is approaching $v=(2/3, 1/3)\approx(0.67, 0.33)$

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53. *Find the unique fixed probability vector of the regular stochastic matrix*

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

>> We have to find $v = (a, b, c, d)$ where $a+b+c+d = 1$ such that $vP = v$

$$\Rightarrow \quad [a, b, c, d] \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a, b, c, d]$$

i.e., $$\left[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}, \frac{a}{2}+\frac{c}{2}+\frac{d}{2}, \frac{a}{4}+\frac{b}{4}, \frac{a}{4}+\frac{b}{4}\right] = [a, b, c, d]$$

$$\Rightarrow \quad \frac{1}{2}(b+c+d) = a \text{ or } b+c+d = 2a$$

$$\frac{1}{2}(a+c+d) = b \text{ or } a+c+d = 2b$$

$$\frac{1}{4}(a+b) = c \text{ or } a+b = 4c$$

$$\frac{1}{4}(a+b) = d \text{ or } a+b = 4d$$

By using $b+c+d = 1-a$ and $a+c+d = 1-b$

(i) and (ii) respectively becomes $1-a = 2a$ and $1-b = 2b$

$\therefore \quad a = 1/3$ and $b = 1/3$

Hence we have from (iii) and (iv),

$4c = 2/3$ and $4d = 2/3$

$\therefore \quad c = 1/6$ and $d = 1/6$

Thus $v = (1/3, 1/3, 1/6, 1/6)$ **is the required unique fixed probability vector.**

---

54. *Show that* (a, b) *is a fixed point of the stochastic matrix* $P = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$ *What is the associated fixed probability vector ?*

Hence write down the fixed probability vector of each of the following matrices.

$$P_1 = \begin{bmatrix} 1/3 & 2/3 \\ 1 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}, \; P_3 = \begin{bmatrix} 7/10 & 3/10 \\ 8/10 & 2/10 \end{bmatrix}$$

>> Let $x = (a, b)$ and consider the matrix product

$$xP = [a, b] \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

$$= \left[a(1-b)+ba,\ ab+b(1-a)\right] = [a, b]$$

Thus $xp = x$ $\therefore$ $\boldsymbol{x = (a, b)}$ **is a fixed point of** $\boldsymbol{P}$.

Also $v = (a/a+b,\ b/a+b)$ **is the required fixed probability vector of** $\boldsymbol{P}$.

Comparing P_1, P_2, P_3 with P we have respectively

$$a = 1,\ b = 2/3\ ;\ a = 2/3,\ b = 1/2\ ;\ a = 8/10,\ b = 3/10$$

$$a+b = 5/3\ ;\ a+b = 7/6\ ;\ a+b = 11/10$$

The corresponding fixed probability vectors of P_1, P_2, P_3 be respectively denoted by v_1, v_2, v_3 where we have in general

$v = (a/a+b,\ b/a+b)$

Thus $v_1 = (3/5, 2/5)$; $v_2 = (4/7, 3/7)$; $v_3 = (8/11, 3/11)$

are the required fixed probability vectors of $\boldsymbol{P_1, P_2, P_3}$ in the respective order.

---

55. *If* $P_1 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$

show that P_1, P_2 *and* $P_1 P_2$ *are stochastic matrices.*

>> In P_1 we have $(1-a)+a = 1$ and $b+(1-b) = 1$

In P_2 we have $b+(1-b) = 1$ and $a+(1-a) = 1$

Thus $\boldsymbol{P_1}$ **and** $\boldsymbol{P_2}$ **are stochastic matrices.**

Now $P_1 P_2 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$

$$= \begin{bmatrix} (1-a)(1-b)+a^2, & (1-a)b+a(1-a) \\ b(1-b)+a(1-b), & b^2+(1-b)(1-a) \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \text{ (say)}$$

We shall show that $a_1+b_1 = 1$ and $a_2+b_2 = 1$

Now $a_1+b_1 = (1-a)(1-b) + (1-a)b+a^2+a(1-a)$

$$= (1-a)\{1-b+b\} + a\{a+1-a\}$$

$$= 1-a+a = 1. \quad \therefore \quad a_1+b_1 = 1$$

Also $a_2 + b_2 = b(1-b) + b^2 + a(1-b) + (1-b)(1-a)$

$= b\{(1-b)+b\} + (1-b)\{a+(1-a)\}$

$= b + 1 - b = 1.$ Thus $a_2 + b_2 = 1$

Thus $P_1 P_2$ is a stochastic matrix.

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56. *Show that* $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ *is a regular stochastic matrix. Also find the associated unique fixed probability vector.*

>> Consider $P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$$P \cdot P^2 = P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P \cdot P^3 = P^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P \cdot P^4 = P^5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

We observe that in P^5 all the entries are positive.

Thus P is a regular stochastic matrix.

Next we have to find $v = (a, b, c)$ where $a + b + c = 1$ such that $vP = v$

$$\Rightarrow \quad [a, b, c]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [a, b, c]$$

i.e., $\left[\frac{c}{2}, a + \frac{c}{2}, b\right] = [a, b, c]$

$$\Rightarrow \quad \frac{c}{2} = a, \; a + \frac{c}{2} = b, \; b = c$$

Using $c = 2a$ and $b = c = 2a$ in $a + b + c = 1$ we get

$5a = 1$ or $a = 1/5$ Hence $b = c = 2a = 2/5$

Thus $(1/5, 2/5, 2/5)$ is the required unique fixed probability vector of P.

5.22 Markov Chains

A stochastic process which is such that the generation of the probability distribution depend only on the present state is called a *Markov process.*
If this state space is discrete (*finite or countably infinite*) we say that the process is a discrete state process or chain. Then the Markov process is known as a ***Markov chain.***

Further if the state space is continuous, the process is called a continuous state process.

We explicitly **define a Markov chain** as follows.

Let the outcomes $X_1, X_2, \ldots$ of a sequence of trials satisfy the following properties.

(i) Each outcome belong to the finite set (*state space*) of the outcomes $\{a_1, a_2, \ldots a_m\}$

(ii) The outcome of any trial depend at most upon the outcome of the immediate preceeding trial.

Probability p_{ij} is associated with every pair of states (a_i, a_j) that a_j occurs immediately after a_i occurs. Such a stochastic process is called a

finite Markov chain These probabilities (p_{ij}) which are non zero real numbers are called *transition probabilities* and they form a square matrix of order m called the *transition probability matrix* (*t.p.m*) denoted by P.

$$\text{i.e.,} \quad P = \left[p_{ij}\right] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

With each state a_i there corresponds the i^{th} row of transition probabilities $p_{i1}, p_{i2}, \ldots p_{im}$. It is evident that the elements of P have the following properties.

$$\text{(i)} \quad 0 \le p_{ij} \le 1 \quad \text{(ii)} \quad \sum_{j=1}^{m} p_{ij} = 1 \quad (i = 1, 2, 3, \ldots m)$$

The above two properties satisfy the requirement of a stochastic matrix and hence we conclude that the *transition matrix of a Markov chain is a stochastic matrix.*

Illustrative Examples for writing t.p.m of a Markov chain

1. *A person commutes the distance to his office everyday either by train or by bus. Suppose he does not go by train for two consecutive days, but if he goes by bus the next day he is just as likely to go by bus again as he is to travel by train.*

The state space of the system is $\{\text{train } (t), \text{ bus } (b)\}$
The stochastic process is a Markov chain since the outcome of any day depends only on the happening of the previous day.The $t \cdot p \cdot m$ is as follows.

$$P = \begin{array}{c c} & \begin{array}{cc} t & b \end{array} \\ \begin{array}{c} t \\ b \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

The first row of the matrix is related to the fact that the person does not commute two consecutive days by train and is sure to go by bus if he had travelled by train. The second row of the matrix is related to the fact that if the person had commuted in bus on a particular day he is likely to go by bus again or by train. Thus the probabilities are equal to $1/2$.

2. *Three boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A.*

State space $= \{A, B, C\}$ and the $t \cdot p \cdot m$ P is as follows.

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

- ## Higher transition probabilities

The entry p_{ij} in the transition probability matrix P of the Markov chain is the probability that the system changes from the state a_i to a_j in a single step. That is $a_i \to a_j$

The probability that the system changes from the state a_i to the state a_j in exactly n steps is denoted by $p_{ij}^{(n)}$

That is $a_i \to a_{r_1} \to a_{r_2} \to \cdots \to a_{r_{n-1}} \to a_j$

The matrix formed by the probabilities $p_{ij}^{(n)}$ is called the n - step transition matrix denoted by $P^{(n)}$

$[P^{(n)}] = [p_{ij}^{(n)}]$ is obviously a stochastic matrix.

It can be proved that the n step transition matrix is equal to the n^{th} power of P.

That is $P^{(n)} = P^n$

Let P be the t.p.m of the Markov chain and let $p = (p_i) = (p_1, p_2, \cdots p_m)$ be the probability distribution at some arbitrary time. Then $p\,P$, $p\,P^2 \cdots p\,P^n$ respectively are the probabilities of the system after one step, two steps,... n steps.

Let $p^{(0)} = [p_1^{(0)}, p_2^{(0)}, \cdots p_m^{(0)}]$ denote the initial probability distribution at the start of the process and let $p^{(n)} = [p_1^{(n)}, p_2^{(n)}, \cdots p_m^{(n)}]$ denote the n^{th} step probability distribution at the end of n steps. Thus we have

$$p^{(1)} = p^{(0)} P,\ p^{(2)} = p^{(1)} P = p^{(0)} P^2,\ \cdots p^{(n)} = p^{(0)} P^n$$

Illustrations

1. Let us consider the t.p.m of the earlier illustrated Example-1

$$P = \begin{array}{c c} & \begin{array}{cc} t & b \end{array} \\ \begin{array}{c} t \\ b \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array} = \begin{bmatrix} p_{tt} & p_{tb} \\ p_{bt} & p_{bb} \end{bmatrix}$$

We shall find P^2 and P^3

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} p_{tt}^{(2)} & p_{tb}^{(2)} \\ p_{bt}^{(2)} & p_{bb}^{(2)} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} p_{tt}^{(3)} & p_{tb}^{(3)} \\ p_{bt}^{(3)} & p_{bb}^{(3)} \end{bmatrix}$$

$p_{tb}^{(2)} = 1/2$ means that the probability that the system changes from the state t to b in exactly two steps is $1/2$

$p_{bt}^{(3)} = 3/8$ means that the probability that the system changes from the state b to t in exactly 3 steps is $3/8$.

Next let us create an initial probability distribution for the start of the process. Let us suppose that the person rolled a 'die' and decided that he will go by bus if the number appeared on the face is divisible by 3.

$\therefore \quad p(b) = 2/6 = 1/3$ and $p(t) = 2/3$

That is $p^{(0)} = (2/3, 1/3)$ is the initial probability distribution.

Now $\quad p^{(2)} = p^{(0)} P^2 = \begin{bmatrix} 2/3, & 1/3 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 5/12, & 7/12 \end{bmatrix}$

$$p^{(3)} = p^{(0)} P^3 = \left[2/3,\ 1/3\right] \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix} = \left[7/24,\ 17/24\right]$$

$$p^{(3)} = \left[7/24,\ 17/24\right] = \left[p_t^{(3)},\ p_b^{(3)}\right]$$

This is the probability distribution after 3 days.

That is, probability of travelling by train after 3 days $= 7/24$

probability of travelling by bus after 3 days $= 17/24$

2. Let us consider the t.p.m of the earlier illustrated Example - 2.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Referring to Problem - 56, we have $P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$

Supposing that C was the person having the ball first then $p^{(0)} = (0,\ 0,\ 1)$

Consider $p^{(5)} = p^{(0)} P^5 = [0,\ 0,\ 1] \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$

$$p^{(5)} = \left[1/8,\ 3/8,\ 1/2\right] = \left[p_A^{(5)},\ p_B^{(5)},\ p_C^{(5)}\right]$$

This implies that after 5 throws the probability that the ball is with A is $1/8$, the ball with B is $3/8$, the ball with C is $1/2$.

• Stationary distribution of regular Markov chains

A Markov chain is said to be regular if the associated transition probability matrix P is regular.

If P is a regular stochastic matrix of the Markov chain, then the sequence of n step transition matrices $P^2,\ P^3,\ \cdots P^n$ approaches the matrix V whose rows are each the unique fixed probability vector v of P.

We have $p^{(n)} = p^{(0)} P^n$ where, $p^{(n)} = \left[p_1^{(n)},\ p_2^{(n)} \cdots p_m^{(n)}\right]$

Further as $n \to \infty$, $p_i^{(n)} = v_i$ where $i = 1,\ 2,\ 3,\ \cdots m$.

This is called the *stationary distribution* of the markov chain and $v = (v_1, v_2, \cdots v_m)$ is called the stationary (*fixed*) probability vector of the Markov chain.

Referring to the Illustrative Example - 2, the t.p.m of the Markov chain is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

and by Worked Problem - 56 the unique fixed probability vector of P is $(1/5, 2/5, 2/5)$.

Hence we conclude that in the long run $(n \to \infty)$ A will have thrown the ball 20% of the time, while B and C will have thrown the ball 40% of the time.

Note : A Markov chain is said to be ***irreducible*** if every state can be reached from every other state in a finite number of steps. That is to say that $p_{ij}^{(n)} > 0$ for some $n \geq 1$. This is equivalent to saying that a ***Markov chain is irreducible if the associated transition probability matrix is regular.***

• Absorbing state of a Markov chain

In a Markov chain the process reaches to a certain state after which it continues to remain in the same state. Such a state is called an *absorbing state* of the Markov chain. In an absorbing state the transition probabilities p_{ij} are such that

$$p_{ij} = 1 \text{ for } i = j \text{ and } p_{ij} = 0 \text{ otherwise.}$$

Thus a state a_i of the Markov chain is absorbing if the i^{th} row of the t.p.m has 1 on the principal diagonal and zeroes elsewhere.

Examples :

1. $P = \begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \end{array}$ The state a_2 is absorbing.

2. $P = \begin{array}{c c} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{bmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$ The states a_3 and a_4 are absorbing.

WORKED EXAMPLES

57. *The transition matrix P of a Markov chain is given by* $\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ *with the initial probability distribution* $p^{(0)} = (1/4, 3/4)$. *Define and find the following.*

(i) $p_{21}^{(2)}$ *(ii)* $p_{12}^{(2)}$ *(iii)* $p^{(2)}$ *(iv)* $p_1^{(2)}$

(v) the vector $p^{(0)}\, p^n$ *approches.*

(vi) the matrix P^n *approaches.*

>> **(i)** $p_{21}^{(2)}$ is the probability of moving from state a_2 to state a_1 in 2 steps. This can be obtained from the 2 - step transition matrix P^2

$$P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} \end{bmatrix}$$

$\therefore \quad p_{21}^{(2)} = 9/16$

(ii) $p_{12}^{(2)}$ is the probability of moving from state a_1 to a_2 in two steps $\therefore p_{12}^{(2)} = 3/8$

(iii) $p^{(2)}$ is the probability distribution of the system after 2 steps.

$$p^{(2)} = p^{(0)}P^2 = \begin{bmatrix} 1/4, & 3/4 \end{bmatrix}\begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} \frac{37}{64}, & \frac{27}{64} \end{bmatrix}$$

That is $p^{(2)} = [37/64, 27/64] = [p_1^{(2)}, p_2^{(2)}]$

(iv) $p_1^{(2)}$ is the probability that the process is in the state a_1 after 2 steps. Hence $p_1^{(2)} = 37/64$.

(v) The vector $p^{(0)}\, P^n$ approaches the unique fixed probability vector of P and we shall find the same.

Let $v = (x, y)$ where $x + y = 1$ and we must have $vP = v$

That is $[x, y]\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = [x, y]$

$\therefore \quad x/2 + 3y/4 = x$ and $x/2 + y/4 = y$

Using $y = 1 - x$ the first equation becomes

$$\frac{x}{2} + \frac{3(1-x)}{4} = x \text{ or } 2x + 3(1-x) = 4x \;\therefore\; x = 3/5$$

since $x = 3/5$, $y = 2/5$

The vector $p^{(0)} P^n$ approaches the vector $(3/5, 2/5)$

(vi) P^n approaches the matrix V whose rows are each the fixed probability vector of P.

$$P^n \text{ approaches the matrix } \begin{bmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{bmatrix}$$

---

58. *The $t \cdot p \cdot m$ of a Markov chain is given by*

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

and the initial probability distribution is $p^{(0)} = (1/2, 1/2, 0)$

Find $p_{13}^{(2)}$, $p_{23}^{(2)}$, $p^{(2)}$ and $p_1^{(2)}$

>> First let us find the two step transition matrix P^2

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$\therefore \quad p_{13}^{(2)} = 3/8 \quad \text{and} \quad p_{23}^{(2)} = 1/2$$

$$p^{(2)} = p^{(0)} P^2 = \left[1/2, 1/2, 0\right] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$= \left[7/16, 1/8, 7/16\right]$$

$$\therefore \quad p^{(2)} = (7/16, 1/8, 7/16) \quad \text{and} \quad p_1^{(2)} = 7/16$$

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59. *Prove that the Markov chain whose $t \cdot p \cdot m$ is*

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \text{, is irreducible.}$$

Find the corresponding stationary probability vector.

>> We shall show that P is a regular stochastic matrix. For convenience we shall write the given matrix in the form

$$P = \frac{1}{6} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

Consider $P^2 = \frac{1}{36}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$

Since all the entries in P^2 are positive we conclude that the t.p.m P is regular.

Hence the Markov chain having t.p.m P is irreducible.

Next we shall find the fixed probability vector of P.

If $v = (x, y, z)$ we shall find v such that $vP = v$ where $x+y+z = 1$.

That is $[x, y, z] \cdot \frac{1}{6}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} = [x, y, z]$

$\Rightarrow \quad \frac{1}{6}[3y+3z, 4x+3z, 2x+3y] = [x, y, z]$

$\Rightarrow \quad 3y+3z = 6x \; ; \; 4x+3z = 6y \; ; \; 2x+3y = 6z$

Solving these by using $x+y+z = 1$ we obtain $x = 1/3, y = 10/27, z = 8/27$

Thus $v = (1/3, 10/27, 8/27)$ **is the required stationary probability vector.**

---

60. *A habitual gambler is a member of two clubs A and B. He visits either of the clubs everyday for playing cards. He never visits club A on two consecutive days. But, if he visits club B on a particular day, then the next day he is as likely to visit club B or club A. Find the transition matrix of this Markov chain. Also,*

(a) *show that the matrix is a regular stochastic matrix and find the unique fixed probability vector.*

(b) *if the person had visited club B on Monday, find the probability that he visits club A on Thursday.*

>> The transition matrix P of the Markov chain is formulated as follows.

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The first row corresponds to the fact that he never goes to club A on two consecutive days which implies that he is sure to visit club B. The second row corresponds to the fact that if he goes to B on a particular day he visits B or A on the following day. Probability of going to A is $1/2$ and probability of going to B is also $1/2$.

(a) Now consider $P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

Since all the entries of P^2 are positive **P is a regular stochastic matrix.**

We shall find the unique fixed probability vector. That is to find

$$v = (x, y) \text{ such that } vP = v \quad \text{where} \quad x + y = 1$$

i.e., $\begin{bmatrix} x, & y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} x, & y \end{bmatrix}$

or $\begin{bmatrix} \dfrac{y}{2}, & x + \dfrac{y}{2} \end{bmatrix} = \begin{bmatrix} x, & y \end{bmatrix}$

$\Rightarrow \quad \dfrac{y}{2} = x \; ; \; x + \dfrac{y}{2} = y$. But $y = 1 - x$

$\therefore \quad \dfrac{1-x}{2} = x$ or $x = \dfrac{1}{3} \quad \therefore \; y = \dfrac{2}{3}$

Thus $\quad v = $ **$(1/3, 2/3)$**

(b) Let us suppose Monday as day 1, then Thursday will be 3 days after Monday. Given that the person had visited club B on Monday the probability that he visits club A after 3 days is equivalent to finding $a_{21}^{(3)}$ from P^3.

Now $\quad P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$

$\therefore \quad a_{21}^{(3)} = 3/8$. **Thus the required probability is 3/8.**

---

61. *A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, he is 60% sure not to study the next night. In the long run how often does he study ?*

>> The state space of the system is $\{A, \; B\}$ where A : Studying B : Not studying.

The associated transition matrix P is as follows .

$$P = \begin{array}{c} \\ A \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

In order to find the happening in the long run we have to find the unique fixed probability vector v of P. That is to find

$$v = (x, y) \text{ such that } vP = v \quad \text{where} \quad x + y = 1$$

i.e., $\begin{bmatrix} x, & y \end{bmatrix}\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} x, & y \end{bmatrix}$

i.e., $\begin{bmatrix} 0.3x + 0.4y, & 0.7x + 0.6y \end{bmatrix} = \begin{bmatrix} x, & y \end{bmatrix}$

$\Rightarrow 0.3x + 0.4y = x \ ; \ 0.7x + 0.6\,y = y$

Using $y = 1 - x$ in the first of the equations we have
$0.3x + 0.4(1 - x) = x$ or $1.1x = 0.4 \ \therefore \ x = 4/11$

Since $x = 4/11, \ y = 7/11, \ v = (4/11 \text{ and } 7/11) = (p_A, p_B)$

Thus we conclude that in the long run the student will study 4/11 of the time or 36.36% of the time.

---

62. *A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2. On the other hand, if he smokes non filter cigarettes one week there is a probability of 0.7 that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes ?*

>> The state space of the system is $\{A, \ B\}$ where
A : Smoking filter cigarettes, B : Smoking non filter cigarettes

The associated transition matrix is as follows.

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array} = \begin{bmatrix} 8/10 & 2/10 \\ 3/10 & 7/10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix}$$

We have to find the unique fixed probability vector, $v = (x, y)$ such that $vP = v$ where $x + y = 1$

i.e., $\begin{bmatrix} x, & y \end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} x, & y \end{bmatrix}$

i.e., $\begin{bmatrix} 8x + 3y, & 2x + 7y \end{bmatrix} = \begin{bmatrix} 10x, & 10y \end{bmatrix}$

$\Rightarrow$ $8x + 3y = 10\,x, \ 2\,x + 7y = 10\,y$

Using $y = 1 - x$ in the first equation, we get,

$8x + 3(1 - x) = 10x$

or $x = 3/5 \quad \therefore \quad y = 2/5$

Hence $v = (x, \ y) = (3/5, \ 2/5) = (p_A, p_B)$

In the long run, he will smoke filter cigarettes 3/5 or 60% of the time.

---

63. *Each year a man trades his car for a new car in 3 brands of the popular company Maruti Udyog limited. If he has a 'Standard' he trades it for 'Zen'. If he has a 'Zen' he trades it for a 'Esteem'. If he has a 'Esteem' he is just as likely to trade it for a new 'Esteem' or for a 'Zen' or a 'Standard' one. In 1996 he bought his first car which was Esteem.*

(i) Find the probability that he has

(a) 1998 Esteem (b) 1998 Standard

(c) 1999 Zen (d) 1999 Esteem.

(ii) In the long run, how often will he have a Esteem?

>> The state space of the system is $\{A, B, C\}$ where A : Standard B : Zen C : Esteem.

The associated transition matrix is as follows.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(i) With 1996 as the first year, 1998 is to be regarded as 2 years after and 1999 as 3 years after.

We need to compute P^2 and P^3

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 7/27 & 16/27 \end{bmatrix}$$

$$\left.\begin{array}{ll} (a)\ \text{1998 Esteem} & = a_{33}^{(2)} = \mathbf{4/9} \\ (b)\ \text{1998 Standard} & = a_{31}^{(2)} = \mathbf{1/9} \end{array}\right\} \text{with reference to } P^2$$

$$\left.\begin{array}{ll} (c)\ \text{1999 Zen} & = a_{32}^{(3)} = \mathbf{7/27} \\ (d)\ \text{1999 Esteem} & = a_{33}^{(3)} = \mathbf{16/27} \end{array}\right\} \text{with reference to } P^3$$

(ii) We have to find the unique fixed probability vector $v = (x, y, z)$ such that $vP = v$ where $x + y + z = 1$

i.e., $$\begin{bmatrix} x, & y, & z \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} x, & y, & z \end{bmatrix}$$

i.e., $$\begin{bmatrix} x, & y, & z \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x, & y, & z \end{bmatrix}$$

i.e., $$\begin{bmatrix} z, & 3x+z, & 3y+z \end{bmatrix} = \begin{bmatrix} 3x, & 3y, & 3z \end{bmatrix}$$

$$\Rightarrow \quad z = 3x, \quad 3x+z = 3y, \quad 3y+z = 3z$$

Consider $3x+z = 3y$; Using $z = 3x$ and $y = 1-x-z$ we get $6x = 3(1-x-z)$ or $6x = 3-3x-3z$ or $18x = 3$ $\therefore$ $x = 1/6$

Hence we obtain $y = 1/3$, $z = 1/2$

$$\therefore \quad v = \begin{bmatrix} x, & y, & z \end{bmatrix} = \begin{bmatrix} 1/6, & 1/3, & 1/2 \end{bmatrix} = \begin{bmatrix} p^A, & p^B, & p^C \end{bmatrix}$$

In the long run, probability of he having Esteem is $p^{(C)}$ **= 1/2**

Thus in the long run in 50% of the time he will have Esteem.

---

64. *Three boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that after three throws*

(i) *A has the ball* ***(ii)*** *B has the ball* ***(iii)*** *C has the ball*

>> State space $= \{A, B, C\}$ and the associated $t \cdot p \cdot m$ is as follows.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Intially if C has the ball, the associated initial probability vector is given by $p^{(0)} = (0, 0, 1)$

Since the probabilities are desired after three throws we have to find $p^{(3)} = p^{(0)} P^3$

Refering to the Problem - 56, $P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

$$\therefore \quad p^{(3)} = p^{(0)} P^3 = \begin{bmatrix} \frac{1}{4}, & \frac{1}{4}, & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} p_A^{(3)}, & p_B^{(3)}, & p_C^{(3)} \end{bmatrix}$$

Thus after three throws the probability that the ball is with A is 1/4, with B is 1/4 and with C is 1/2.

---

65. *Two boys B_1, B_2 and two girls G_1, G_2 are throwing ball from one to the other. Each boy throws the ball to the other boy with probability 1/2 and to each girl with probability 1/4. On the otherhand each girl throws the ball to each boy with probability 1/2 and never to the other girl. In the long run how often does each receive the ball.*

>> State space $= \{B_1, B_2, G_1, G_2\}$ and the associated $t \cdot p \cdot m$ P is as follows.

$$P = \begin{array}{c} \\ B_1 \\ B_2 \\ G_1 \\ G_2 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

We need to find the fixed probability vector $v = (a, b, c, d)$ such that $vP = v$

Refering to Problem - 53. We have $v = (1/3, 1/3, 1/6, 1/6)$

Thus we can say that in the long run **each boy receives the ball 1/3 of the time and each girl 1/6 of the time.**

---

66. *A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However if he loses a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so*

(a) *What is the probability of he winning the second game ?*

(b) *What is the probability of he winning the third game ?*

(c) *In the long run, how aften he will win ?*

>> State space {Win (W), Lose (L)} and the associated $t \cdot p \cdot m$ is as follows.

$$P = \begin{array}{c} \\ W \\ L \end{array} \begin{array}{c} \begin{array}{cc} W & L \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{array} = \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

Probability of winning the first game is $1/2$.

$\therefore$ initial probability vector $p^{(0)} = (1/2, 1/2)$

(a) Now $p^{(1)} = p^{(0)} P = \dfrac{1}{2}[1, 1] \cdot \dfrac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = \dfrac{1}{20}[9, 11]$

Hence $p^{(1)} = \left[9/20, 11/20\right] = \left[p^{(W)}, p^{(L)}\right]$

Thus the probability of he winning the second game is 9/20.

(b) $p^{(2)} = p^{(1)} P = \frac{1}{20}[9, 11] \cdot \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = \frac{1}{200}[87, 113]$

Hence $p^{(2)} = \left[87/200, 113/200\right] = \left[p^{(W)}, p^{(L)}\right]$

Thus **the probability of he winning the third game is 87/200.**

(c) We hall find the fixed probability vector

$$v = (x, y) \text{ such that } vP = v \text{ where } x + y = 1$$

That is $[x, y] \cdot \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = [x, y]$

$\Rightarrow \qquad 6x + 3y = 10x, \; 4x + 7y = 10y$

or $\qquad 3y = 4x$ and by using $y = 1 - x$ we get

$$3(1 - x) = 4x \quad \therefore \quad x = 3/7 \text{ and } y = 4/7$$

Hence $v = \left[3/7, 4/7\right] = \left[p^{(W)}, p^{(L)}\right]$

Thus in the long run he wins 3/7 of the time.

EXERCISES

1. Identify the probability vectors from the following.

   (a) $(2/5, 3/5)$ (b) $(0, -1/3, 4/3)$

   (c) $(1/3, 0, 1/6, 1/2, 1/3)$

   (d) $(1/3, 0, 1/6, 1/2)$ (e) $(0.1, 0.2, 0.3, 0.4)$

2. Find the associated probability vector to each of the following tuples.

   (a) $(1, 3, 5)$ (b) $(4, 0, 1, 2)$

   (c) $(1/2, 2/3, 0, 2, 5/6)$

3. $A = [a_{ij}]$ is a stochastic matrix of order $n \times n$ and $v = (v_1, v_2, \cdots v_n)$ is a probability vector, show that vA is also a probability vector.

4. If A and B are two stochastic matrices of order 3×3, prove that AB is also a stochastic matrix.

5. Show that $(cf + ce + de \, , \; af + bf + ae \, , \; ad + bd + bc)$ is a fixed point of the stochastic matrix

$$P = \begin{bmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{bmatrix}$$

6. Show that the following matrix P is a regular stochastic matrix and also find its unique fixed probability vector.

$$P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

7. Given the $t \cdot p \cdot m$ $P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$ with initial probability distribution $p^{(0)} = (1/3 \, , \; 2/3)$, find the following.

(a) $p_{21}^{(3)}$ (b) $p^{(3)}$ (c) $p_2^{(3)}$

8. A software engineer goes to his office everyday by motorbike or by car. He never goes by bike on two consecutive days, but if he goes by car on a day then he is equally likely to go by car or by bike the next day. Find the $t \cdot p \cdot m$ of the Markov chain. If car is used on the first day of the week find the probability that after 4 days (a) bike is used (b) car is used.

9. A salesman's territory consists of 3 cities A, B, C. He never sells in the same city for 2 consecutive days. If he sells in city A, then the next day he sells in city B. However if he sells in either B or C, then the next day he is twice as likely to sell in city A as in the other city. In the long run how often does he sell in each of the cities ?

10. Show that the Markov chain with $t \cdot p \cdot m$ given by

$$P = \frac{1}{10} \begin{bmatrix} 6 & 2 & 2 \\ 1 & 8 & 1 \\ 6 & 0 & 4 \end{bmatrix}$$ is irreducible. Find the corresponding stationary probability vector.

ANSWERS

1. (a) , (d) , (e) are probability vectors.
2. (a) (1/9 , 3/9 , 5/9) (b) (4/7 , 0 , 1/7 , 2/7)
   (c) (1/8 , 1/6 , 0 , 1/2 , 5/24)
6. (4/11 , 4/11 , 3/11)
7. (a) 7/8 (b) (11/12 , 1/12) (c) 1/12
8. $\begin{array}{cc} & \begin{array}{cc} B & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$ (a) 5/16 (b) 11/16
9. v = (0.4 , 0.45 , 0.15). In the long run he sells 40% of the time in city A, 45% of the time in B, 15% of the time in C
10. v = (4/10 , 4/10 , 2/10)

BEATING THE MEMORY

[Formulae, Properties and Results to be remembered from all the modules at a glance]

Module - 1

Numerical Methods

Formulae for solving the initial value problem :

$$\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0. \text{ To compute } y(x_0+h).$$

➤ *Taylor's series formula*

$$y(x) = y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\cdots$$

➤ *Modified Euler's formula* [M.E.F]

Taking $x_1 = x_0+h$ and $y_1 = f(x_1)$

$$y_1^{(0)} = y_0+hf(x_0, y_0) \text{ ... [Initial approx. / Euler's formula]}$$

$$y_1^{(1)} = y_0+\frac{h}{2}\left[f(x_0, y_0)+f(x_1, y_1^{(0)})\right] \text{... [First approx./ M.E.F]}$$

$$y_1^{(2)} = y_0+\frac{h}{2}[f(x_0, y_0)+f(x_1, y_1^{(1)})] \text{... [Second approx. / M.E.F]}$$

➤ *Runge - Kutta formula*

$$y(x_0+h) = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4] \text{ where}$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$k_4 = hf(x_0+h, y_0+k_3)$$