Graph theory

→ **Edges :-**

→ Simple edge

— multiple edges

. loop

→ Unlabelled graph

→ **Graph :-**

$G = (V, E)$

vertex $V = \{v_1, v_2, \ldots v_n\}$

Edge $E = \{(B, A), (C, B), (D, C)\}$
BA

directed　　undirected

B → A terminal vertex B

C ← D

↓ Initial vertex

→ $\{AB, BC, CD\}$

→ labelled graph

For directed edge $E = \{(A, B)\}$

Undirected ,, $E = \{AB\}$

,,

→ Simple graph :- No loops & multiple edges

→ Multi graph :- have multiple edges

→ loop free [Pseudo graph] :- Can have multiple edges but no loops

→ Null graph :-

→ Isolated vertex

Ex:- Null graph with 3 isolated vertices

→ General graph / graph :-

→ Order of a graph :-

cardinality of V $|V| = m$ is order of graph

→ Size of a graph :-

Cardinality of E $|E| = n$ is size of graph

$|E| = n$

$G = (V, E)$

(m, n) graph

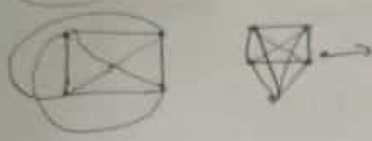
Complete graph :-

If $n \geq 2$, then it is complete graph


2 vertices


3 vertices


4 verti


Kuratowski's first graph

In degree & out degree of a vertex :-

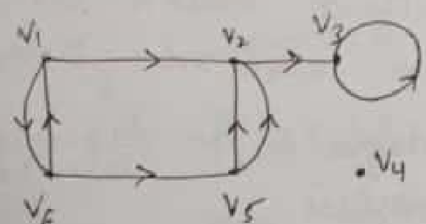
$d^+(v) =$ out $\qquad d^-(v) =$ in

First theorem of digraph theory

$$\Sigma d^+(v) = \Sigma d^-(v)$$

Verify the 1st theorem of digraph theory :-

1.



Sol. $d^+(v_1) = 2$ $\quad$ $d^-(v_1) = 1$

$d^+(v_2) = 1$ $\quad$ $d^-(v_2) = 3$

$d^+(v_3) = 1$ $\quad$ $d^-(v_3) = 2$

$d^+(v_4) = 0$ $\quad$ $d^-(v_4) = 0$

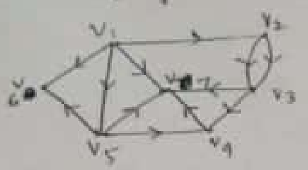
$d^+(v_5) = 2$ $\quad$ $d^-(v_5) = 1$

$d^+(v_6) = 2$ $\quad$ $d^-(v_6) = 1$

$\overline{8}$ $\quad$ $\overline{8}$

Hence, 1st theorem of digraph theorem is ~~verifi~~ verified.

2.

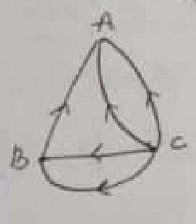


	v_1	v_2	v_3	v_4	v_5	v_6	v_7	Sum
Out degree	4	2	2	1	3	0	0	= 12
In degree	0	1	2	2	1	2	4	= 12

$\therefore \Sigma d^+(v) = \Sigma d^-(v)$

Hence verified.

3.



Write down vertex set & ~~determine~~ edge set. Determine out degree & In degree of all vertices.

Sol. $V = \{A, B, C\}$

$E = \{BA, CA, CB\}$

$d^+(A) = 0$, $d^-(A) = 3$

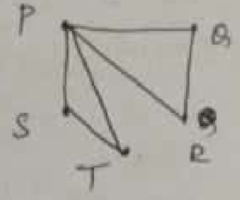
$d^+(B) = 1$, $d^-(B) = 2$

$d^+(C) = \ldots$ $\quad$ $d^-(C) = 0$

=0
)=2
=3
)=2
7

ing

have not played with eachother & no of games played by eachother.

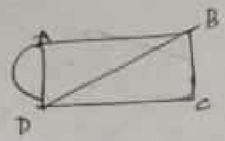


no of teams not played
~~TQ~~, TR, SQ, SR

no of games played
P - 4
Q - 2
R - 2
S - 2
T - 2

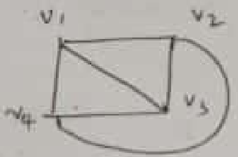
Identify the type of graph:

a)



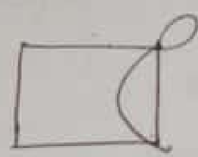
- Multi graph

b)



- complete graph

c)

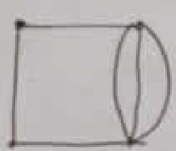


- general ~~simple~~ graph

d)

- simple graph

e)



- multi graph

f)

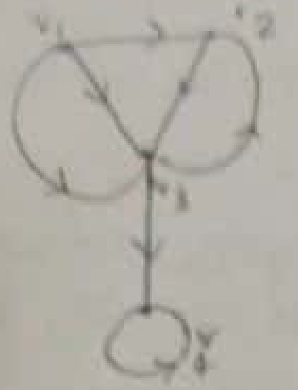
- multi graph

g)

- general graph

h)

- simple graph



$d^+(v_1) = 3$ $d^-(v_1) = 0$

$d^+(v_2) = 1$ $d^-(v_2) = 2$

$d^+(v_3) = 2$ $d^-(v_3) = 3$

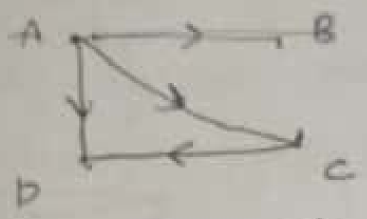
$d^+(v_4) = 1$ $d^-(v_4) = 2$

$\overline{\quad 7 \quad}$ $\overline{\quad 7 \quad}$

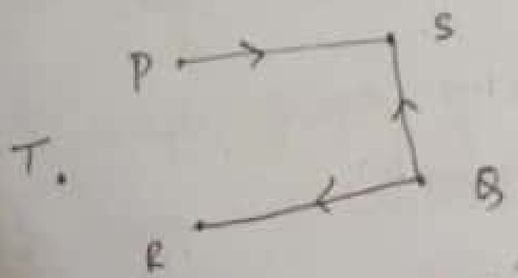
Hence verified.

5. Draw a graph $G = (V, E)$ in the following cases :-

1. $V = \{A, B, C, D\}$
   $E = \{AB, AC, AD, CD\}$



2. $G = (V, E)$
   $V = \{P, Q, R, S, T\}$
   $E = \{PS, QR, QS\}$



3. Let P, Q, R, S, T represent 5 cricket teams. suppose that teams P, Q, R have played one game with each ther and the teams P, S, T have played one game with eachother. Represent the situation in a graph, hence determine the teams that

graph is a pair (V, E) where V is non-empty set & E is set of unordered pairs of elements taken from set V.

→ For a graph G = (V, E) the elements of V are called vertices (points or nodes) and elements of E are called undirected edges or just edges.

→ The set V is called vertex set & E is called the edge set

→ A graph that does not contain loops & multiple edges is called simple graph.

→ A graph that doesn't contain a loop is called loop-free graph or Pseudo graph

→ A graph that contains multiple edges but no loops is called a multi graph

→ A graph which contains multiple edges or loops or both is called general graph

→ A simple graph of order greater than or equal to two in which there is an edge b/w every pair of vertices is called a complete or a full graph.

In other words, a complete graph is a simple graph of order ≥ 2 in which every pair of distinct vertices are adjacent.

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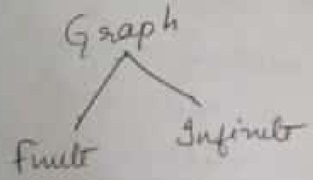
→ A complete graph with $n \geq 2$ vertices is denoted by K_n. The no. of vertices in a graph is called the order of graph & no of edges in it is called size.

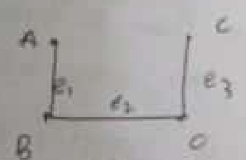
In otherwords, for the graph $G = (V, E)$ the cardinality for the set V namely $|V|$ is called the order of G & the cardinality of set E namely $|E|$ is called size of G.

A graph of order m and size n is called (m, n) graph.

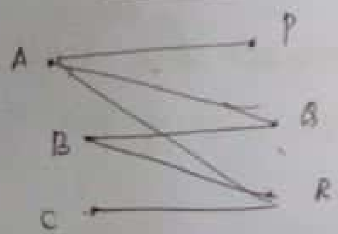
11 Oct'19

→ 

→ Incidence



→ **Bipartite Graph**



$V = \{A, B, C, P, Q, R\}$

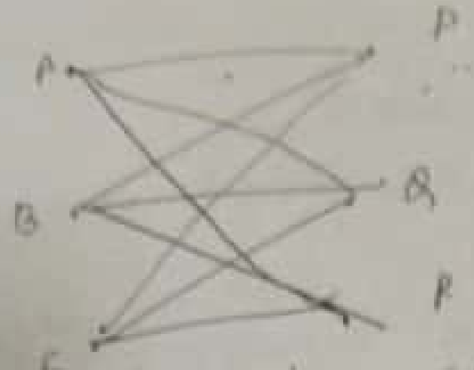
$V_1 \cup V_2 = V$

$V_1 \cap V_2 = \phi$

$V_1 = \{A, B, C\}$

$V_2 = \{P, Q, R\}$

complete bipartite graph



→ Kuratowski's second graph

$K_{r,s}$ $\quad$ $|V| = r+s$

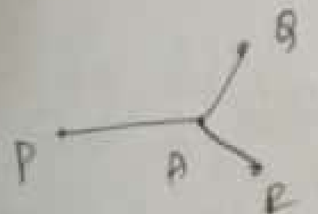
$|V_1| = r$ $\quad$, $r \leq s$ $\quad$ $|E| = rs$

$|V_2| = s$

this holds good for complete bipartite graph

→ Identify bipartite graph :-

1



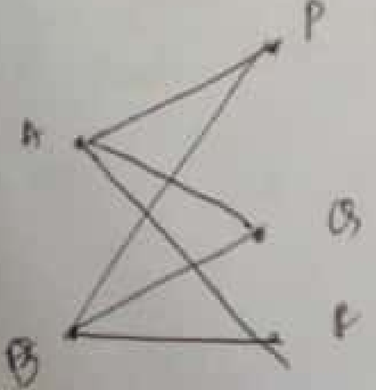
$V_1 = \{A\}$

$V_2 = \{P, Q, R\}$

$E = \{AP, AQ, AR\}$

complete bipartite graph

2.



$V_1 = \{A, B\}$

$V_2 = \{P, Q, R\}$

$E = \{AP, AQ, AR, BP, BQ, BR\}$

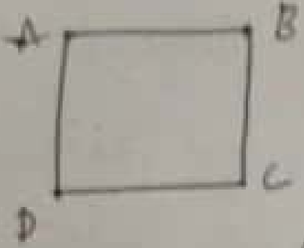
complete bipartite graph

4

5

6

7

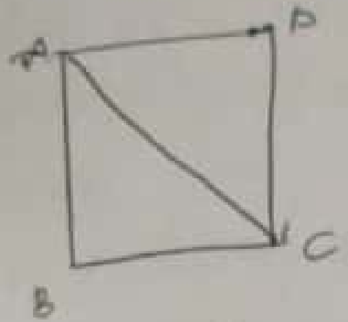
3. 

$V_1 = \{A, C\}$

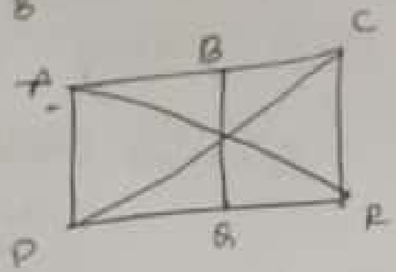
$V_2 = \{B, D\}$

Complete bipartite

$E = \{AB, AD, BC, CD\}$

4. 

Not a bipartite

5. 

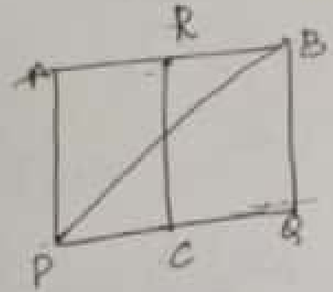
$V_1 = \{A, C, Q\}$

$V_2 = \{B, P, R\}$

$E = \{AB, AP, AR, CB, CR, CP, QB, QP, QR\}$

5

Complete bipartite

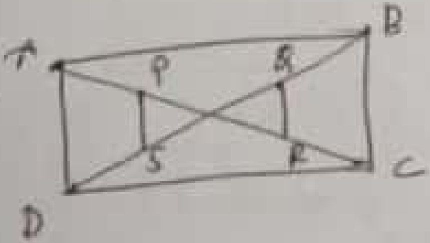
6. 

$V_1 = \{B, C, A\}$

$V_2 = \{P, Q, R\}$

$E = \{AP, AR, BP, BQ, BR, CP, CR, CQ\}$

Bipartite

7. 

$V_1 = \{ A$

$V_2 = \{.$

P_1

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How many vertices & edges are there in bipartite graph $K_{4,7}$ & $K_{7,11}$

vertices = 11
edges = 28

vertices = 18
edges = 77

② If the graph $K_{R,12}$ has 72 edges what is value of R

$R = 6$

$12 \times 6 = 72$ edges.

③ If $G = (V, E)$ is a simple graph. P.T two times $2|E| \le |V|^2 - |V|$

(n, m)
$n = |V|$
$m = |E|$

$^nC_2$ ways

$m \le {}^nC_2$

$m \le \frac{n(n-1)}{2}$

$2m \le n^2 - n$

$2|E| \le |V|^2 - |V|$

Each edge of graph is determined by pair of vertices in a simple graph there occur no multiple edges as such in a simple graph, the no. of edges cannot exceed no. of pairs of vertices. The no. of pairs of vertices that can be chosen from n vertices is $^nC_2$ (continued above)

④ S.T a complete graph with n vertices namely K_n has ~~n×n-1~~ $\frac{n(n-1)}{2}$ edges.

Sol. In a complete graph there exists exactly 1 edge b/w every pair of vertices as such the no. of edges in a complete is equal to no. of pairs of vertices. If the no. of vertices is 'n' then the no. of pairs of vertices is $^nC_2$

Hence, No. of edges in a complete graph with 'n' vertices $= \frac{n(n-1)}{2}$

⑤ S.T a simple graph of ordered 'n' $= 4$ & size $m=7$ & complete graph of order $n=4$ & size $m=5$ do not exists.

i) $n=4$

no. of edges $= \frac{4 \times 3}{2} = 6$ $\therefore 7>6$

$\Rightarrow$ Hence not possible

ii) $n=4$

Since for complete graph $m=6$.

$5<6$

$\therefore$ Hence not possible.

⑥ State whether the following graph can or can't exist : $\rightarrow$ order $\rightarrow$ size

① Simple graph of $n=3$, $m=2$

$n=5$, $m=12$

" $n=5$, $m=10$

② complete graph

Bipartite graph n=4, m=3

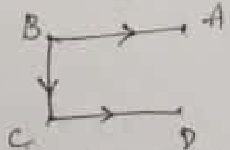
(5) " " n=3, m=4

(6) complete bipartite graph n=4, m=4

→ A digraph is a pair D(V, E) ~~where~~

where V is non-empty set and E is a set of ordered pairs of elements taken from set D. For a directed graph (V, E) the elements of V are called vertices (points or node) and elements of E are called directed edges. The set V is a vertex set and set E is directed ~~edeg~~ edge set

eg:-



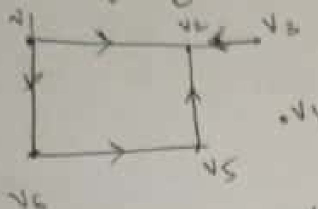
$V = \{A, B, C, D\}$

$E = \{(B, A), (B, C), (C, D)\}$

Here (B, A) is directed edge which begins at the vertex B & ends at A.

→ A vertex of digraph which is neither an initial vertex nor a terminal vertex of any directed edge is called an isolated vertex of digraph.

→ A non-isolated vertex which is not a terminal for any directed edge is called a source and an non-isolated vertex which is not at initial vertex for any directed edge is called a sink.



→ If V is a vertex of a digraph the no. of edges for which V → initial vertex is called the out going degree or ~~out~~ out degree of V & the no. of edges for which V is terminal vertex is called incoming degree or in degree of V. The out degree of V is denoted by $d^+(v)$ or od(v), the in degree is denoted by $d^-(v)$ or id(v).

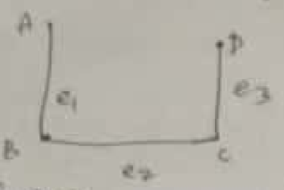
→ In every digraph D the sum of the out degrees of all vertices is equal to sum of in degrees of all vertices each sum being equal to the no. of edges in D.

Incidence :-

When a vertex V of graph G is an n-vertex of an edge E of the graph G we say that edge E is incident on vertex V. When a edge E is incidence on vertex V, we say the V is incident with E.

Two non-parallel edges are said to be adjacent edges if they are incident on a common vertex.

vertices are said to be adjacent vertices if there is an edge joining them



Adjacent vertices A & B, B & C, C & D

Adjacent edges e_1 & e_2, e_2 & e_3

Suppose,

→ A simple graph G is such that, its vertex V is union of two mutually disjoint non-empty subsets V_1, V_2 which are such that each edge in G joins a vertex in V_1 & a vertex in V_2. G is called bipartite graph & denoted by $G = (V_1, V_2 ; E)$

V_1 & V_2 are called the bipartites of vertex set V

→ A bipartite graph G is called a complete bipartite graph if there is an edge b/w every vertex in V_1 & every vertex in V_2.

NOTE :-

Let G = (V;E) be a simple graph of order |V| = n & size of |E| = m. If G is a bipartite graph then $4m <= n^2$

* S.T a simple graph of order $n=4$ & size cannot be a Bipartite graph.

$$4m \le n^2$$

$$4 \times 5 = 20$$

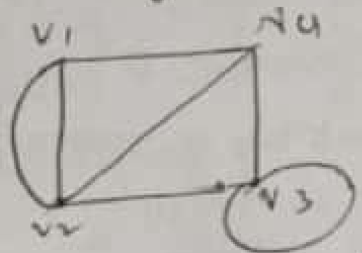
$$n^2 = 16$$

$$20 > 16 \Rightarrow \text{not possible.}$$

<u>Degree of the graph :-</u>

→ degree sequence of a graph

→ min deg - degree of a graph.



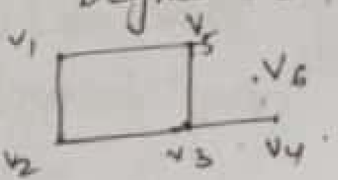
$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 4$

$d(v_4) = 3$

Degree $= 3, 3, 4, 4$



$v_4 = d(v_4) = 1$

degree $= 1$, it is called pend[ant]

→ Regular graph :-

k-regular graph (each vertex is of degree k)

(as 2-regular graph mean 2 degree $2^4 = 4$)

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For any +ve integer k, loop free k regular graph with 2^k vertices is called the kth dimensional hyper cube or k-cube & denoted by Q_k

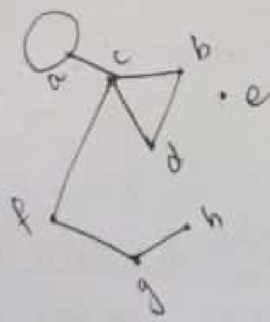
Handshaking Property :-

The sum of the degrees of all the vertices in a graph is an even number & this number is equal to twice the no. of edges in the graph.

Note :- In every graph the no. of vertices of odd degrees is even.

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$ (vertices with odd degrees shld be even so that sum is even)

1) For the given graph, indicate degree of each vertex & verify the hand shaking property.



d(a) = 3 d(e) = 0

d(b) = 2 d(f) = 2

d(c) = 4 d(g) = 2

d(d) = 2 d(h) = 1

sum = 16

No. of edges = 8

2) Can there be graph consisting of vertices A, B, C, D
$d(A) = 2, d(B) = 3, d(C) = 2, d(D) = 2$

Ans Here sum of degree is odd number which is not true.
Hence, such graph cannot exist

3) Can there be a graph with 12 vertices such that two of the vertices have degree 3 each & remaining 10 have degree 4 each.

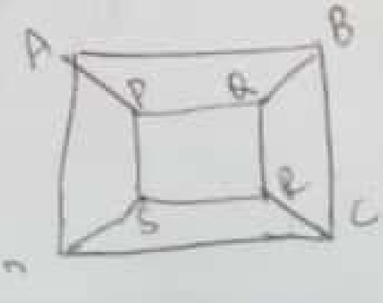
$$(2\times 3) + (10 \times 4) = 46$$
$$= 2(23) \text{ Hence possible}$$

4) For a graph $G = (V; E)$ what is largest possible value for $|V|$ if $|E| = 19$ & $\deg(v) \geq 4$ for all $v \in V$

Ans. $G = (V; E)$
$\deg(v) \geq 4$ for all $v \in V$
Let $|V| = n$
$$\sum_{i=0}^{n} \deg(v_i) \geq 4n$$
$$2 \times 19 \geq 4n$$
$$n = 9$$

5) S.T the hyper cube Q_3 is a bipartite graph which is not complete bipartite graph.

Sol. $2^K = 2^3$ vertices



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$V = \{A, C, R, E\}$ (1)

$B = \{B, D, P, F\}$ (2)

$E = \{AB, AD, AP, CB, CD, CE, RB, RP, RQ, SB, SD, SE\}$ (3)

(4)

6) P.T. that n dimensional hypercube has $k \times 2^{k-1}$ edges. Determine the no. of edges in Q_8.

Sol. hypercube Q_k

$\Rightarrow 2^k =$ no. vertices

Each vertex is of degree k

Sum of the degree of vertices $= k\,2^k$ — (1)

By handshaking property $= 2$(no. of edges) (2)

$$k\,2^k = 2(\text{no. of edges})$$

$$k\,2^{k-1} = \text{no. of edges} = |E|$$

Q_8

$k = 8$

No. of edges $= 8 \times 2^8$

7) Determine the order of $|V|$ of the graph $G = (V, E)$ in the following cases:—

① G is cubic graph with 9 edges.
② G has 10 edges with 2 vertices of degree 4 & all other vertices of degree 3
③ G is regular with 15 edges

Sol.
① no. of edges $|E| = 9$
Let n be no. of vertices
→ cubic graph = each vertex's of degree 3
Sum of degrees of vertices $= 3n$
By HSP, $\Rightarrow 3n = 2 \times 9$
Let n be no. of vertices $n = 6$

② $(2 \times 4) + (n-2)3 = 2 \times 10$
$n = 6$

③ Let n be the no. of vertices
Each vertex is of degree k (Q_k)
∴ Sum of degrees of vertices $= nk$
By HSK $\Rightarrow nk = 2(15) = 30$
$k = 30/n$
k has to be a +ve integer
$\Rightarrow n$ shld be a factor of 30
→ possible values of n are
$= 1, 2, 3, 5, 6, 10, 15, 30$

8) Let G be a graph of order 9 such that each vertex has degree 5 or 6. PT atleast 5 vertices have degree 6 or atleast 6 vertices have degree 5

let 'p' be the number of vertices of degree 5

$(9-p)$ be " " 6

By HSP, $5p + 6(9-p) = 2m$ $\quad 0 \le p \le 9$

$54 - p$ = even number $\quad 0 \le q \le 9$

$p = 0 \rightarrow q = 9$

$p = 2 \rightarrow q = 7$

$p = 4 \rightarrow q = 5$

$p = 6 \rightarrow q = 3$

$p = 8 \rightarrow q = 1$

we observe that in all the above cases either q is ≥ 5 or $p \ge 6$ this means atleast 5 vertices have degree 6 or atleast 6 vertices have degree 5.

9. S.T there is no graph with 28 edges & 12 vertices in the following cases

① The degree of a vertex is either 3 or 4

② The degree of vertex is either 3 or 6

sol ① let 'p' be number of vertices of degree 3

" " " degree 4

$(12-p)$ be "

$$\Sigma \deg(v) = 3p + 4(12-p)$$

$$48 - p = 2 \times 28$$

$$\Rightarrow p = -8$$

Not possible

Such graph doesn't exist.

ques 5 ② Let 'p' be ~~degree~~ no. of vertices of degree 3

6 (12-p) " " " 6

$\Sigma \deg(v) = 3p + 6(12-p) = 2(28)$

$\Rightarrow 72 - 3p = 2(28)$

$3p = 16$

$p = 16/3$

Not possible

Such ~~graph~~ graph doesn't exist.

10. ① If a graph with 'n' vertices & 'm' edges is k-regular, show that $m = \frac{k \times n}{2}$ ② Does that exists a cubic graph with 11 vertices

③ Does there exists a four regular graph

ⓐ 15 edges ⓑ 10 edges

sol ① n - no. of vertices

m - no. of edges

k-regular ⇒ Each vertex is of degree k

By HSP ⇒ $nk = 2m$

$m = \frac{k \times n}{2}$

② $n = 11$

cubic graph ⇒ $k = 3$

$m = \frac{3 \times 11}{2} = \frac{33}{2}$

Not possible

Graph doesn't exist.

12.

(a) $m = 15$, $k = 4$

$k \times n = 2m$

$n = \frac{2 \times 15}{4} = \frac{30}{4}$

∴ Not possible

graph doesn't exists.

(b) $m = 10$, $k = 4$

$k \times n = 2m$

$n = \frac{2 \times 10}{4} = 5$

∴ possible

graph exists

11. ① If k is odd. Show that no. of vertices in a k-regular graph is even. 13.

② S.T it is not possible to have a set of 7 persons ~~show~~ such that ~~each~~ each person in the set knows exactly 3 other persons in the set

sol ① ~~no. of vertices = 2(n)~~

k - odd

To prove n = even

$n = \frac{2m}{k}$

$n = 2\left(\frac{m}{k}\right)$ it is even.

②

12. S.T if a bipartite graph $G=(V_1,V_2;E)$ is regular then $|V_1|=|V_2|$

Sol. $G=(V_1,V_2;E)$

then $|V_1|=|V_2|$

Let G be a simple graph

$$\Sigma \deg(V_1)=\Sigma \deg(V_2)$$

Let $|V_1|=r$

$|V_2|=s$

& k-regular

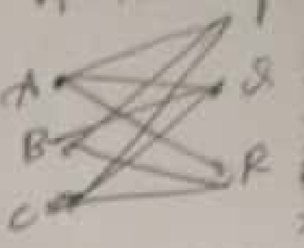
$rk=sk$

$r=s$

$\therefore |V_1|=|V_2|$

Eg: $V_1=\{A,B,C\}$

$V_2=\{P,Q,R\}$



13. S.T every simple graph of order ≥ 2 must have atleast two vertices of same degree.

Sol. Let n be the no. of vertices

Let all vertices be of different degrees

→ Let G be simple graph with 'n' vertices

suppose all vertices have degrees then since every vertex must have degree & since all such degrees must be b/w 0 & n-1. The degrees must be 0, 1, 2, ..., n-1

Let A be the vertex whose degree is 0 & B be the vertex whose degree is n-1 then n-1 edges are incident on B. This means B is joined to all other vertices by an edge & in particular to A also. Hence the degree of A is not zero.

This is a contradiction. Hence all vertices of G cannot have different degrees atleast two of them must have same degree.

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