Skew symmetric :- $A = -A^T$ & diagonal should be zero

Orthogonal matrix :- $A \cdot A^T = I$

Upper triangular :
$$\begin{bmatrix} 2 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

Lower triangular :
$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 4 & 0 \\ 6 & 1 & 2 \end{bmatrix}$$

Idempotent matrix : $A^2 = A$

Involutory matrix :- $A^2 = I$

Complex matrix :- $A^H = (\bar{A})^T$

$$A = \begin{bmatrix} 2+8i & 5-3i & 4-7i \\ 6i & 1-4i & 3+2i \end{bmatrix}$$

$$A^H = \begin{bmatrix} 2-8i & -6i \\ 5+3i & 1+4i \\ 4+7i & 3-2i \end{bmatrix}$$

$\therefore (x_1, x_2, x_3)$

8

$$2x_1 - 3x_2 = -2$$
$$2x_1 + x_2 = 1$$
$$3x_1 + 2x_2 = 1$$

→ The augmented matrix is,

$$\left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right]$$

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$$r_3 \to 2r_3 - 3r_2 \to \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$$r_2 \to r_2 - r_1 \to \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 0 & 4 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$$r_3 \to 4r_3 - r_2 \to \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & -7 \end{array}\right]$$

$$\begin{array}{l} r_1 \to \frac{1}{2} r_1 \\ r_2 \to \frac{1}{4} r_2 \end{array} \to \left[\begin{array}{cc|c} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 0 & -7 \end{array}\right]$$

$$r_3 \to \frac{r_3}{-7} \left[\begin{array}{cc|c} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{array}\right]$$

The reduced system is,

$$x_1 - 3/2\, x_2 = -1$$

$$x_2 = 3/4$$

$$0 = 1$$

Here, $0 = 1$, which is not possible. So the given system is inconsistent and it has no soln.

Sol:

(8)

$$2x - 3y + 5z = 1$$
$$3x + y - z = 2$$
$$x + 4y - 6z = 1$$

$\longrightarrow$ The augmented matrix is,

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right]$$

$$r_3 \to 2r_3 - r_1 \to \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 0 & 11 & -17 & 1 \end{array}\right]$$

$$r_2 \to 2r_2 - 3r_1 \to \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 11 & -17 & 1 \end{array}\right]$$

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$$r_3 \to r_3 - r_2 \to \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_1 \to \frac{1}{2} r_1 \to \left[\begin{array}{ccc|c} 1 & -3/2 & 5/2 & 1/2 \\ 0 & 11 & -17 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_2 \to \frac{1}{11} r_2 \to \left[\begin{array}{ccc|c} 1 & -3/2 & 5/2 & 1/2 \\ 0 & 1 & -17/11 & 1/11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the reduced matrix is,

$$x - \frac{3}{2} y + \frac{5}{2} z = 1/2 \quad \text{--- (i)}$$

$$y - \frac{17}{11} z = \frac{1}{11} \quad \text{--- (ii)}$$

Here, no of equations = 2

no of variables = 3

$\therefore$ no of free variables $= 3 - 2 = 1$

Here, z is the free variable

let, $z = t$

(2) =>

$$y - \frac{17}{11}t = \frac{1}{11}$$

$$\Rightarrow y = \frac{1}{11} + \frac{17}{11}t$$

$$= \frac{1+17t}{11}$$

(1) =>

$$x = \frac{1}{2} + \frac{3}{2}y - \frac{5}{2}z$$

$$= \frac{1}{2} + \frac{3}{2} \times \frac{1+17t}{11} - \frac{5}{2}t$$

$$= \frac{11 + 3(1+17t) - 55t}{22}$$

$$= \frac{11 + 3 + 51t - 55t}{22}$$

$$= \frac{14 - 4t}{22}$$

$$= \frac{7 - 2t}{11}$$

(8) Determine the value of λ such that the following system has (i) no soln (ii) more than 1 (iii) unique soln.

$$x+y-z=1$$
$$2x+3y+\lambda z=3$$
$$x+\lambda y+3z=2$$

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the augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right]$$

$$r_3 \to r_3 - r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right]$$

$$r_2 \to r_2 - 2r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right]$$

$$= -(\lambda-1)(\lambda+2)+4$$
$$= -\lambda^2-\lambda+2+4$$
$$= 6-\lambda-\lambda^2$$
$$= (3+\lambda)(2-\lambda)$$

$$r_3 \to -(\lambda-1)r_2 + r_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & (2-\lambda)(3+\lambda) & 2-\lambda \end{array}\right]$$

The reduced system is,

$$x+y-z=1 \quad \text{--- (i)}$$
$$y+(2+\lambda)z=1 \quad \text{--- (ii)}$$
$$(3+\lambda)(2-\lambda)z = 2-\lambda \quad \text{--- (iii)}$$

(1) Now if we put $\lambda+3=0 \Rightarrow \lambda=-3$.

then eqn (iii) becomes $0=5$ which is contradiction. Therefore the system is inconsistent & it has no solution for $\lambda=-3$

(2) From eqn (iii) if $\lambda=2$ the it becomes $0=0$ In that case the system has 3 variables within 2 eqns. So that the given system has more than 1 solution for $\lambda=2$.

(3) The system has unique soln if $(3+\lambda)(2-\lambda)$

i.e $\lambda \neq -3$ & $\lambda \neq 2$

(8) Determine the values of λ & μ such that the following system of linear eqn has (i) no soln (ii) more than 1 soln (iii) unique soln

$\longrightarrow$

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \tau & u \end{array}\right]$$

$$r_3 \to r_3 - r_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \tau-3 & u-10 \end{array}\right]$$

$$r_2 \to r_2 - r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \tau-3 & u-10 \end{array}\right]$$

Now the reduced system is,

$$x + y + z = 6 \quad \text{--- (i)}$$

$$y + 2z = 4 \quad \text{--- (ii)}$$

$$(\tau - 3)z = u - 10 \quad \text{--- (iii)}$$

1. if $\tau = 3$ & $u \neq 10$ then 3rd eqn will be $0 = a$ where $a = u - 10 \neq 0$ which is inconsistent. Thus the system has no soln for $\tau = 3$ & $u \neq 10$.

2. If $r = 3$ & $u = -10$ then eqn (3) vanishes. In that case the system has 3 variables within 2 eqns, so that the system has more than one solution for $r = 3$ & $u = -10$

3. For a unique soln the co-efficient of z in eqn (3) must be non-zero, i.e. $r \neq 3$ and u may have any value.

Linear Combination;

Q1 Express the following as linear combination of $\underline{u} = (2, 1, 4)$, $\underline{v} = (1, -1, 3)$ and $\underline{w} = (3, 2, 5)$

a. $(-9, -7, -15)$

let, $(-9, -7, -15) = k_1 \underline{u} + k_2 \underline{v} + k_3 \underline{w}$

$\Rightarrow (-9, -7, -15) = k_1 (2, 1, 4) + k_2 (1, -1, 3) + k_3 (3, 2, 5)$

Then we have,

$$2k_1 + k_2 + 3k_3 = -9$$
$$k_1 - k_2 + 2k_3 = -7$$
$$4k_1 + 3k_2 + 5k_3 = -15$$

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

$$r_3 \to r_3 - 2r_1 \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$r_2 \to 2r_2 - r_1 \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & -3 & 1 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$r_3 \to 3r_3 + r_2 \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

$$r_1 \to \frac{1}{2} r_1 \to \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

$r_2 \to r_2/-3$

$r_3 \to r_3/-2$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & -9/2 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Now the reduced system,

$$k_1 + \frac{k_2}{2} + \frac{3}{2}k_3 = -\frac{9}{2} \quad \text{--- (i)}$$

$$k_2 - \frac{1}{3}k_3 = \frac{5}{3} \quad \text{--- (ii)}$$

$$k_3 = -2 \quad \text{--- (iii)}$$

(ii) =>

$$k_2 = \frac{5}{3} + \frac{1}{3} \times -2$$

$$= \frac{5-2}{3} = 1$$

(i) =>

$$k_1 = -\frac{9}{2} - \frac{1}{2} + \frac{6}{2}$$

$$= -2$$

$$\therefore (-9, -7, -15) = -2\underline{u} + \underline{v} - 2\underline{w}$$

which is a linear combination

Q2 Let $\underline{u} = (1, 2, -1)$ and $\underline{v} = (6, 4, 2)$.

Show that $w = (9, 2, 7)$ is a linear combination of u & v

→ Let, $(9, 2, 7) = k_1\underline{u} + k_2\underline{v}$

$$= k_1(1, 2, -1) + k_2(6, 4, 2)$$

Then we have,

$$k_1 + 6k_2 = 9$$
$$2k_1 + 4k_2 = 2$$
$$-k_1 + 2k_2 = 7$$

So the augmented matrix is,

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right]$$

$$r_3 \to r_3 + r_1 \to \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 0 & 8 & 16 \end{array}\right]$$

$$r_2 \to r_2 - 2r_1 \to \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right]$$

$$r_3 \to r_3 - r_2 \to \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right]$$

So the reduced system,

$$k_1 + 6k_2 = 9$$
$$-8k_2 = -16$$
$$\therefore k_2 = 2$$
$$\therefore k_1 = -3$$
$$\therefore (9, 2, 7) = -3\underline{u} + 2\underline{v}$$

Linear dependent & independent :-

(B) Whether the following sets of vectors are linearly dependent or independent ?

a. $(4, -1, 2), (-4, 10, 2)$

→ form the matrix,

$$\begin{bmatrix} 4 & -1 & 2 \\ -4 & 10 & 2 \end{bmatrix}$$

$$r_2 \to r_2 + r_1 \to \begin{bmatrix} 4 & -1 & 2 \\ 0 & 9 & 4 \end{bmatrix}$$

$$r_1 \to \tfrac{1}{4} r_1 \to \begin{bmatrix} 1 & -1/4 & 1/2 \\ 0 & 9 & 4 \end{bmatrix}$$

$$r_2 \to \frac{r_2}{9} \to \begin{bmatrix} 1 & -1/4 & 1/2 \\ 0 & 1 & 4/9 \end{bmatrix}$$

Since the no of non zero rows in echolon form $= 2 < 3$

so the ~~variables~~ vectors are linearly dependent.

b. $(1,-2,3), (5,6,-1), (3,2,1)$

$\longrightarrow$ form the matrix,

$$\begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$r_3 \to r_3 - 3r_1 \to \begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & 1 \\ 0 & 8 & -8 \end{bmatrix}$$

$$r_2 \to r_2 - 5r_1 \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 16 & -16 \\ 0 & 8 & -8 \end{bmatrix}$$

$$r_3 \to 2r_3 - r_2 \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 16 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_2 \to \frac{1}{16} r_2 \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the no. of non zero rows in echelon form $= 2 < 3$

so the vectors are linearly dependent.

c: $(-2,0,1), (3,2,5), (6,-1,1), (7,0,-2)$

→ form the matrix,

$$\begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & 5 \\ 6 & -1 & 1 \\ 7 & 0 & -2 \end{bmatrix}$$

$$r_3 \to r_3 - 2r_2 \to \begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & 5 \\ 0 & -5 & -9 \\ 7 & 0 & -2 \end{bmatrix}$$

$$r_1 \to \frac{1}{-2} r_1 \to \begin{bmatrix} 1 & 0 & -1/2 \\ 3 & 2 & 5 \\ 0 & -5 & -9 \\ 7 & 0 & -2 \end{bmatrix}$$

$-1/2 + 1/2$

$$r_4 \to r_4 - 7r_1 \to$$
$$r_3 \to 2r_3 + 5r_2 \to$$
$$r_2 \to r_2 - 3r_1 \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 2 & 13/4 \\ 0 & 0 & 7 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$y + \frac{3}{2} = \frac{1+3}{2}$

$$r_2 \to \frac{1}{2} r_2 \to$$
$$r_3 \to \frac{1}{7} r_3 \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 13/4 \\ 0 & 0 & 1 \\ 0 & 0 & 3/2 \end{bmatrix}$$

Since the no. of non-zero rows in echolon form = 3

So the vectors are linearly independent

<u>Rank:-</u> The number of non-zero rows of a matrix in its echolon form is known as rank.

<u>Nullity:-</u>

The number of free variables in the solution of $A\underline{x} = 0$ or dimension of null space is called nullity.

rank(A) + nullity (A) = no of columns of A.

1. Find the nullity

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & +4 \\ 7 & -6 & 2 \end{bmatrix}$$

$\rightarrow$ Here, $\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

$r_3 \rightarrow r_3 - 7r_1 \rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 0 & 1 & -19 \end{bmatrix}$

$r_2 \rightarrow r_2 - 5r_1 \rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$

$r_3 \rightarrow r_3 - r_2 \rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

$\cancel{r_3 \rightarrow \frac{1}{3} r_3 \rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 1 \end{bmatrix}}$

Here no. of non-zero row = 2

$\therefore R(A) = 2$

Now, Rank(A) + Nullity(A) = no of column of A

$\Rightarrow$ Nullity (A) $= 3 - 2$

$= 1$

2. $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow$ Given, $\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$r_2 \to r_2 - 2r_1 \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$r_1 \to \frac{1}{2} r_1 \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

no of non zero rows $= 2$

Rank (A) + Nullity (A) = no of columns of A

$\Rightarrow$ Nullity $(A) = 3 - 2$

$= 2$

3. Encode Decode

(a) Encode the message, "MATHEMATICS" with the key $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

→

M	A	T	H	E	M	A	T	I	C	S
13	1	20	8	5	13	1	20	9	3	19

forming matrices with the above numbers

$[13 \quad 1]$ $[20 \quad 8]$ $[5 \quad 13]$ $[1 \quad 20]$ $[9 \quad 3]$

$[19 \quad 0]$

to encrypt,

$$[13 \quad 1]\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = [13 \quad 28]$$

$$[20 \quad 8]\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = [20 \quad 56]$$

$$[5 \quad 13]\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = [5 \quad 36]$$

$$[1 \quad 20]\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = [1 \quad 42]$$

$$[9 \quad 3]\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = [9 \quad 24]$$

$$[19 \quad 0]\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = [19 \quad 38]$$

So we have the following encrypted message,

13 28 20 56 5 36 1 42 9 24

38

(3) Decode 13 28 20 56 5 36 1 42 9 2

19 38 with the key = $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

→ Given, $k = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

then, $k^{-1} = \frac{1}{2+0} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}$

forming the matrices,

[13 28] [20 56] [5 36] [1 42] [9 24]

[19 38]

then,

$[13 \quad 28] \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = [13 \quad 1]$

$[20 \quad 56] \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = [20 \quad 8]$

$[5 \quad 36] \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = [5 \quad 36]$

$[1 \quad 42] \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = [1 \quad 42]$

$[9 \quad 24] \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = [9 \quad 3]$

$$[19 \quad 38] \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = [19 \quad 0]$$

then the decoded message,

13 1 20 8 5 36 1 42 19 3 19 6

M A T H E M A T I C S

4. Polynomial curve fitting

(3) Determine the polynomial $P(x) = a_0 + a_1 x + a_2 x^2$ whose graph passes through the points $(1,4)$, $(2,0)$ & $(3,12)$

$\longrightarrow$ substituting, $x = 1, 2, 3$

$$P(1) = a_0 + a_1 + a_2$$
$$P(2) = a_0 + 2a_1 + 4a_2$$
$$P(3) = a_0 + 3a_1 + 9a_2$$

$$\therefore \quad a_0 + a_1 + a_2 = 4$$
$$a_0 + 2a_1 + 4a_2 = 0$$
$$a_0 + 3a_1 + 9a_2 = 12$$

Augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 12 \end{array}\right]$$

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$$R_3 \to R_3 - R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 0 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 2 & 16 \end{array}\right]$$

$$R_3 \to \frac{1}{2} R_3 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

So the reduced system,

$$a_0 + a_1 + a_2 = 4 \quad \text{--- (i)}$$

$$a_1 + 3a_2 = -4 \quad \text{--- (ii)}$$

$$a_2 = 8 \quad \text{--- (iii)}$$

(ii) $\Rightarrow$ $a_1 = -4 - 3 \times 8 = -28$

(i) $\Rightarrow$ $a_0 = 4 - a_1 - a_2 = 4 + 28 - 8$

$= 24$

So, the polynomial function is,

$$p(x) = 24 - 28x + 8x^2$$

5. linear programming

A computer production company "ASUS" wishes to produce 3 types of electronic devices type x, y, z. To manufacture a type x device requires 2 minutes each on machine I & II and 3 minutes on machine III. A type of Y device requires 2 mins on machine I, 3 minutes on machine II and 4 mins on machine III. A type Z requires 3 mins on machine I, 4 mins on machine II. There are 3.5 hours available on machine I, 4.5 hours available on machine II & 5 hours available on machine III. How many devices of each type should company make in order to use all the available time?

Here,

3.5 hours = 210 minutes

4.5 hours = 270 minutes

5 hours = 300 minutes

Let x, y, z be the number of devices of type X, Y, Z. Then the following system of equations can be formed,

$$2x + 2y + 3z = 210$$
$$2x + 3y + 4z = 270$$
$$3x + 4y + 3z = 300$$

$3 - \frac{9}{2}$

$6 - \frac{9}{2}$

$-\frac{1}{2}$

The augmented matrix,

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 210 \\ 2 & 3 & 4 & 270 \\ 3 & 4 & 3 & 300 \end{array}\right]$$

$$r_1 \to \frac{1}{2} r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 105 \\ 2 & 3 & 4 & 270 \\ 3 & 4 & 3 & 300 \end{array}\right]$$

$$r_3 \to r_3 - 3r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 105 \\ 2 & 3 & 4 & 270 \\ 0 & 1 & -3/2 & -15 \end{array}\right]$$

$$r_2 \to r_2 - 2r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 105 \\ 0 & 1 & 1 & 60 \\ 0 & 1 & -3/2 & -15 \end{array}\right]$$

$$r_3 \to r_3 - r_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 105 \\ 0 & 1 & 1 & 60 \\ 0 & 0 & -5/2 & -75 \end{array}\right]$$

$$r_3 \to -\frac{2}{5} \times r_3 \to \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 105 \\ 0 & 1 & 1 & 60 \\ 0 & 0 & 1 & 30 \end{array}\right]$$

So the reduced system,

$$x + y + \frac{3}{2} z = 105 \quad \text{--- (i)}$$

$$y + z = 60 \quad \text{--- (ii)}$$

$$z = 30 \quad \text{--- (iii)}$$

(ii) $\Rightarrow$ $y = 30$

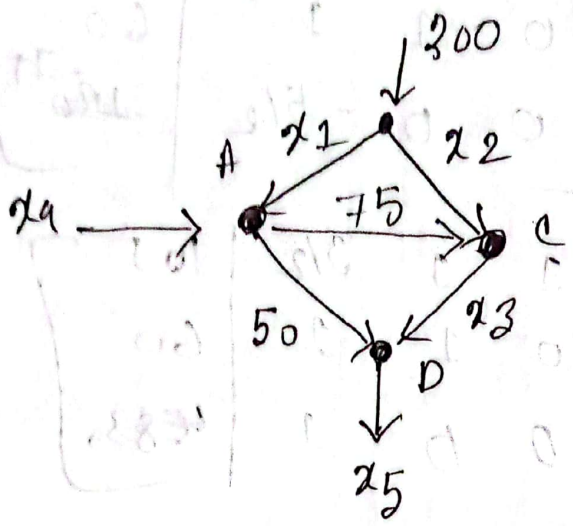
(i) $\Rightarrow$ $x = 105 - 30 - 45$

$= 30$

6. Graph & networking

for each node,

flow in = flow out



the equations are,

$$x_1 + x_4 = 75 + 50$$

$$200 = x_1 + x_2$$

$$x_2 + 75 = x_3$$

$$50 + x_3 = x_5$$

$$200 + x_4 = x_5$$

In standard form,

$$x_1 + x_4 = 125$$

$$x_1 + x_2 = 200$$

$$-x_2 + x_3 = 75$$

$$-x_3 + x_5 = 50$$

$$-x_4 + x_5 = 200$$

The augmented matrix is,

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 1 & 0 & 0 & 75 \\ 0 & 0 & -1 & 0 & 1 & 50 \\ 0 & 0 & 0 & -1 & 1 & 200 \end{array}\right]$$

$$r_4 \to r_4 + r_1 \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 1 & 0 & 0 & 75 \\ 1 & 0 & -1 & 1 & 1 & 175 \\ 0 & 0 & 0 & -1 & 1 & 200 \end{array}\right]$$

$$r_4 \to r_4 - r_2 \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 1 & 0 & 0 & 75 \\ 0 & -1 & -1 & 1 & 1 & -75 \\ 0 & 0 & 0 & -1 & 1 & 200 \end{array}\right]$$

$$r_4 \to r_4 + r_3 \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 1 & 0 & 0 & 75 \\ 0 & -2 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & -1 & 1 & 200 \end{array}\right]$$

$$r_4 \to r_4 + 2r_2 \to \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right.$$

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$$R_4 \leftrightarrow R_5 \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & -1 & 1 & 200 \\ 0 & 0 & -1 & 0 & 1 & 50 \end{array}\right]$$

$$R_4 \rightarrow -1 \times R_4 \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & -1 & 0 & 1 & 50 \end{array}\right]$$

$$R_3 \rightarrow R_2 + R_3 \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 1 & 0 & 1 & 0 & 0 & 275 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & -1 & 0 & 1 & 50 \end{array}\right]$$

$$R_3 \rightarrow R_3 - R_1 \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & -1 & 0 & 1 & 50 \end{array}\right]$$

$$R_2 \rightarrow R_2 - R_1 \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 0 & 1 & 0 & -1 & 0 & 75 \\ 0 & 0 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & -1 & 0 & 1 & 50 \end{array}\right]$$

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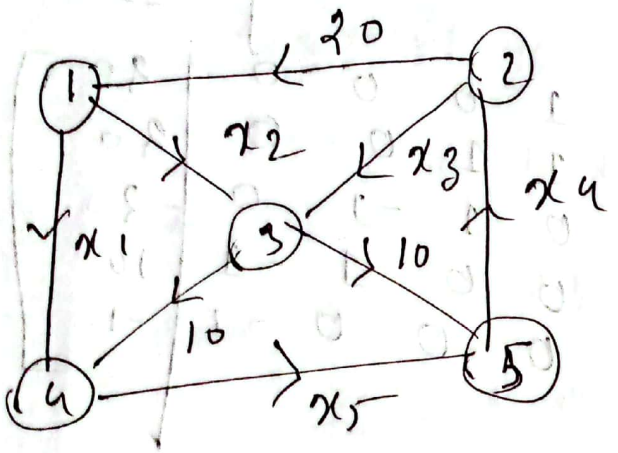
$$R_5 \to R_5 + R_3 \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 0 & 1 & 0 & -1 & 0 & 75 \\ 0 & 0 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & -1 & 1 & 200 \end{array}\right]$$

$$R_5 \to R_5 + R_4 \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 125 \\ 0 & 1 & 0 & -1 & 0 & 75 \\ 0 & 0 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(8)



→ Linear Equations are,

$$x_1 + x_2 = 20$$
$$-x_3 + x_4 = 20$$
$$x_2 + x_3 = 20$$
$$x_1 - x_5 = -10$$
$$-x_4 + x_5 = -10$$

The augmented matrix is,

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & -1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

$R_2 \leftrightarrow R_3$
$R_4 \leftrightarrow R_5$ $\Rightarrow$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & -1 & 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & -1 & 1 & -10 \\ 1 & 0 & 0 & 0 & -1 & -10 \end{array}\right]$$

$R_2 \to -1 \times R_2$
$R_4 \to -1 \times R_4$ $\Rightarrow$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 1 & 0 & 0 & 0 & -1 & -10 \end{array}\right]$$

$R_3 \to R_3 + R_4 \to$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 1 & 0 & 0 & 0 & -1 & -10 \end{array}\right]$$

$R_5 \to R_5 - R_1 \to$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & -1 & 0 & 0 & 0 & -30 \end{array}\right]$$

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$$R_5 \to R_5 + R_2 \to \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & 0 & -10 \end{array}\right]$$

$$R_5 \to R_5 - R_3 \to \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$x_1 + x_2 = 20$$

$$x_2 + x_3 = 20$$

$$x_3 - x_4 = -20$$

$$x_4 - x_5 = 10$$

No of free variable $5 - 4 = 1$

x_5 is free variable

Let, $x_5 = t$

$\therefore x_4 = 10 + t$

$$x_3 = -20 + 10 + t = t - 10$$

$$x_2 = 20 - t + 10 = 30 - t$$

$$x_1 = 20 - 30 + t = t - 10$$

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{array}\right]$$

20 + 6

~~8~~0 - 14

$$r_2 \to r_2 - 3r_1 \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 2 & 4 & 8 \end{array}\right]$$

$\cancel{r_3 \to r_3 +}$

$$r_3 \to 5r_3 - 2r_2 \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & 26 & 26 \end{array}\right]$$

$$r_2 \to \frac{1}{5} r_2 \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -3/5 & 7/5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$r_3 \to \frac{r_3}{26}$$

so the reduced system,

$$I_1 - I_2 + I_3 = 0$$

$$I_2 - \frac{3}{5} I_3 = 7/5$$

$$I_3 = 1$$

$$(ii) \Rightarrow \quad I_2 = 7/5 + 3/5 = 2$$

$$(i) \Rightarrow \quad I_1 = I_2 - I_3 = 2 - 1 = 1$$

Final suggestion

Cayley-Hamilton theorem;

Every square matrix satisfies its own characteristic equation.

Eigen Values & Eigen vectors

(8) $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$M_{11} = -3 - 8 = -11$

$M_{22} = 8 + 6 = 14$

$M_{33} = -24 + 32 = 8$

$\rightarrow \quad |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - [\text{sum of diag element}]\lambda^2 + [\text{sum of diag minors}]\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \lambda = 1, 2, 3 \rightarrow$ Eigenvalues

for $\lambda = 1$, $(A - \lambda I)X = 0$

Eigenvectors

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using cramer's rule,

$$7x - 8y - 2z = 0 \quad \text{—(i)}$$

$$4x - 4y - 2z = 0 \quad \text{—(ii)}$$

$$\frac{x}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{8} = \frac{-y}{-6} = \frac{z}{4}$$

$$\Rightarrow \frac{x}{8} = \frac{y}{6} = \frac{z}{4}$$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$$

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for $\lambda = 2$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using crammers rule,

$$6x - 8y - 2z = 0$$

$$4x - 5y - 2z = 0$$

crammers rule,

$$\frac{x}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{6} = \frac{-y}{-4} = \frac{z}{2}$$

$$\Rightarrow \frac{x}{6} = \frac{y}{4} = \frac{z}{2}$$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Dimension:-

Number of vectors in a basis called dimension.

(Q) Determine the basis and dimension of the system.

a) $x_1 + x_2 + x_3 = 0$

$-2x_1 - x_2 + 2x_3 = 0$

$-x_1 \quad + x_3 = 0$

→ The augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$$

$$r_3' \to r_3 + r_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$r_2' \to 2r_1 + r_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

so the system becomes,

$$x_1 + x_3 - x_3 = 0$$

$$x_2 = 0$$

Here, no of equation = 2

no of variable = 3

$\therefore$ no of free variable = 3 − 2 = 1

Here x_3 is the free variable

let, $x_3 = t$

$\Rightarrow x_1 - t = 0$

$\Rightarrow t = x_1$

so the soln of the given system,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

let, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \{v_1\}$ is the basis of the solution space

$\therefore$ Dimension = 1

(b) $x_1 - 3x_2 + x_3 = 0$

$2x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 9x_2 + 3x_3 = 0$

$\rightarrow$ The augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right]$$

$$r_3' \rightarrow r_1 + r_2 - r_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_2' \rightarrow 2r_1 - r_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the system becomes,

$x_1 - 3x_2 + x_3 = 0$

no of equation $= 1$

no of variables $= 3$

$\therefore$ no of free variables $= 3 - 1 = 2$

Here the free variables are x_2 & x_3

let, $x_2 = s$ & $x_3 = t$ then we get,

$$x_1 - 3s + t = 0$$

$$\Rightarrow x_1 = 3s - t$$

so the solution of the becomes,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3s \\ s \\ t \end{bmatrix} - \begin{bmatrix} t \\ s \\ t \end{bmatrix}$$

$$= s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

let, $v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore \{v_1, v_2\}$ is the basis of the solution

$\therefore$ Dimension $= 2$

Linear Transformation

$T : R^n \to R^m$

L.T $\iff$ $\bar{a}, \bar{b} \in \mathbb{R}^n$

$T(\bar{a}+\bar{b}) = T(\bar{a}) + T(\bar{b})$

$T(\alpha \bar{a}) = \alpha T(\bar{a})$

(8) $T : R^2 \to \mathbb{R}^2$

$T(x_1, x_2) = (x_1 + x_2, 3x_1)$

or, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}$

$\to$ let $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\bar{a} + \bar{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$

$T(\bar{a}+\bar{b}) = T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$

$T(\bar{a}) = T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ 3a_1 \end{bmatrix}$

$T(\bar{b}) = T\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} b_1 + b_2 \\ 3b_2 \end{bmatrix}$

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$$T(\bar{a}) + T(\bar{b}) = \begin{bmatrix} a_1 + a_2 \\ 3a_1 \end{bmatrix} + \begin{bmatrix} b_1 + b_2 \\ 3b_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$$

let α be a scalar,

$$T(\alpha \bar{a}) = T\left(\begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha a_1 + \alpha a_2 \\ 3\alpha a_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} a_1 + a_2 \\ 3a_1 \end{bmatrix}$$

$$= \alpha T(\bar{a}_1)$$

$\therefore T(\bar{a} + \bar{b}) = T(\bar{a}) + T(\bar{b})$ &

$T(\alpha \bar{a}) = \alpha T(\bar{a})$

we can say, T is a linear transformation.

(3) $T: R^3 \to R^2$; $T(x, y, z) = (2x + y + z, 3x + 2y + 4z)$

$\to$ let, $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $\bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\bar{a} + \bar{b} + \bar{c} = \begin{bmatrix} a_1 + b_1 + c_1 \\ a_2 + b_2 + c_2 \\ a_3 + b_3 + c_3 \end{bmatrix}$$

$$T(\bar{a} + \bar{b} + \bar{c}) = T\left(\begin{bmatrix} a_1 + b_1 + c_1 \\ a_2 + b_2 + c_2 \\ a_3 + b_3 + c_3 \end{bmatrix}\right) =$$

$$\begin{bmatrix} 2a_1 + 2b_1 + 2c_1 + a_2 + b_2 + c_2 + a_3 + b_3 + c_3 \\ 3a_1 + 3b_1 + 3c_1 + 2a_2 + 2b_2 + 2c_2 + 4a_3 + 4b_3 + 4c_3 \end{bmatrix}$$

$$T(\bar{a}) = T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 + a_2 + a_3 \\ 3a_1 + 2a_2 + 4a_3 \end{bmatrix}$$

$$T(\bar{b}) = T\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) = \begin{bmatrix} 2b_1 + b_2 + b_3 \\ 3b_1 + 2b_2 + 4b_3 \end{bmatrix}$$

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$$T(\bar{c}) = T\left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right) = \begin{bmatrix} 2c_1 + c_2 + c_3 \\ 3c_1 + 2c_2 + 4c_3 \end{bmatrix}$$

$$\therefore T(\bar{a}) + T(\bar{b}) + T(\bar{c}) = T(\bar{a} + \bar{b} + \bar{c})$$

Again, let α be the scalar

$$T(\alpha\bar{a}) = T\left(\begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{bmatrix}\right) = \begin{bmatrix} 2\alpha a_1 + \alpha a_2 + \alpha a_3 \\ 3\alpha a_1 + 2\alpha a_2 + 4\alpha a_3 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2a_1 + a_2 + a_3 \\ 3a_1 + 2a_2 + 4a_3 \end{bmatrix}$$

$$= \alpha\, T(\bar{a})$$

$\therefore T$ is a linear transformation

(8) $T: R^2 \to R$, $T(x, y) = xy$

$\to$ let, $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\bar{a} + \bar{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

$$T(\bar{a}+\bar{b}) = T\left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}\right) = (a_1+b_1)(a_2+b_2)$$

$$T(\bar{a}) = T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = a_1 a_2$$

$$T(\bar{b}) = T\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = b_1 b_2$$

Here, $T(\bar{a}+\bar{b}) \neq T(\bar{a}) + T(\bar{b})$

Hence, T is not a linear transformation.

Dot & Scalar product

Line Integral

(8) Evaluate $\int_C F\,dr$, $\bar{F} = x^2y^2\,\hat{i} + y\,\hat{j}$ &

C is curve $y^2 = 4x$ in xy plane from

$(0,0,0)$ to $(4,4)$

$$\longrightarrow \int_C F\,dr = \int_C (x^2y^2\,\hat{i} + y\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C x^2y^2\,dx + y\,dy$$

$$= \int_c x^2 4x\,dx + 2x\,dx$$

$$= \int_0^4 (4x^3 + 2)\,dx$$

$$= \left[\frac{4x^4}{4} + 2x\right]_0^4$$

$$= \left[x^4 + 2x\right]_0^4$$

$$= (4)^4 + 2(4)$$

$$= 264$$

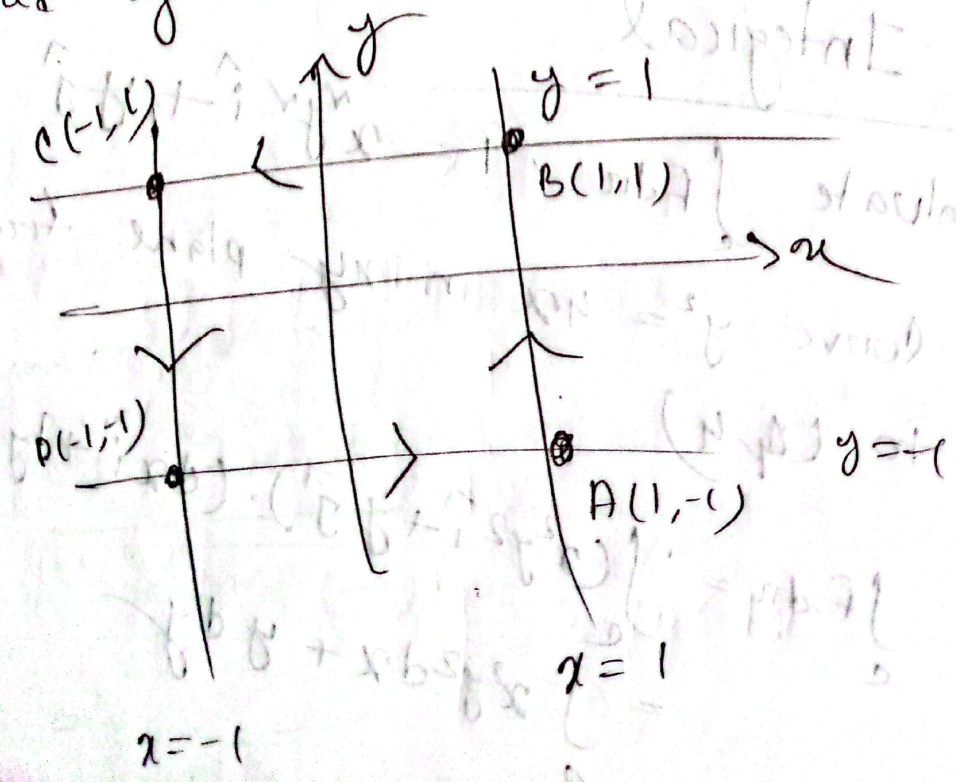
$y^2 = 4x$

$\Rightarrow 2y\,dy = 4\,dx$

$\Rightarrow y\,dy = 2\,dx$

Q) Evaluate the line integral $\int_c (x^2+xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by line $x = \pm 1$, $y = \pm 1$

$\rightarrow$



$\rightarrow$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} F\,dr + \int_{BC} F\,dr + \int_{CD} F\,dr + \int_{DA} F\,dr \quad (1)$$

$$F = (x^2+xy)\,\hat{i} + (x^2+y^2)\,\hat{j}$$

Along AB, $x = 1,\ dx = 0$

$y = -1$ to 1

$$\int_{AB} F\,dr = \int_{-1}^{1} 0 + \int_{-1}^{1} (1+y^2)\,dy$$

$$= 2\int_0^1 (1+y^2)\,dy$$

$$= 2\left(y + \frac{y^3}{3}\right)_0^1$$

$$= 2\left(1 + 1/3\right) = 8/3$$

Along BC, $y = 1,\ dy = 0$

$x = 1$ to -1

$$\int_{BC} F\,dr = \int_{1}^{-1} (x^2+x)\,dx + \int_{1}^{-1} 0\,\cancel{dy}$$

$$= -\left[\int_{-1}^{1} x^2\,dx + \int_{-1}^{1} x\,dx\right]$$

$$= -2\int_0^1 x^2\,dx$$

$$= -2\left(x^3/3\right)_0^1$$

$$= -2/3$$

Along CD, $x=-1$, $dx=0$

$y = 1$ to -1

$$\int_{CD} F\,dr = \int_1^{-1} (1+y^4)\,dy$$

$$= -2\int_0^1 (1+y^4)\,dy$$

$$= -2\left[y + y^3/3\right]_0^1$$

$$= -8/3$$

Along DA, $y=-1$, $dy=0$

$x = -1$ to 1

$$\int_{DA} F\,dr = \int_{-1}^{1} (x^4 - x)\,dx$$

$$= \cancel{2}\left[\;2\int_0^1 x^4\,dx\right.$$

$$= 2/3$$

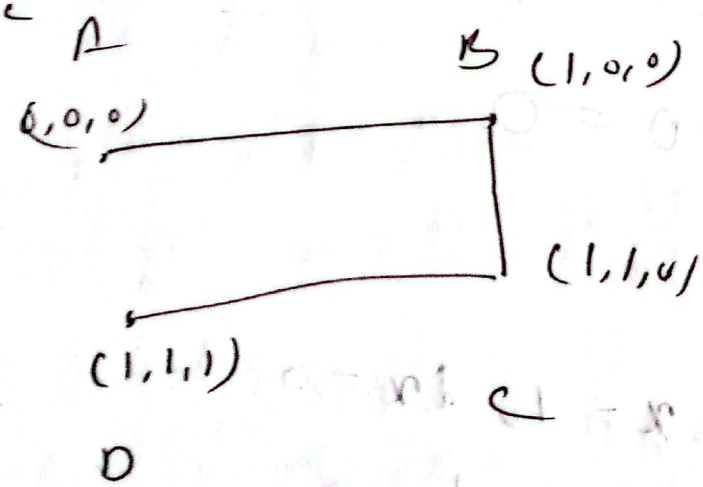
$$\therefore \int_C F\,dr = \int_{AB} F\,dr + \int_{BC} F\,dr + \int_{CD} F\,dr + \int_{DA} F\,dr$$

$$= 0$$

(5) Evaluate the line integral from $(0,0,0)$ to $(1,1,1)$ along path $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ ~~at~~ then to $(1,1,1)$

$$\bar{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$$

$$\rightarrow \int_C F\,dr = \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$



$$\int_C F\,dr = \int_{AB} F\,dr + \int_{BC} F\,dr + \int_{CD} F\,dr$$

Along AB, $y = 0$, $dy = 0$

$x = 0$ to 1

$z = 0$; $dz = 0$

$$\int_{AB} F\,dr = \int_0^1 3x^2\,dx$$

$$= \left[\frac{3x^3}{3}\right]_0^1 = 1$$

Along BC,

$x = 1$, $dx = 0$

$y = 0$ to 1

$z = 0$, $dz = 0$

$$\int_{BC} F\,dr = -\int_0^1 0 = 0$$

Along CD,

$x = 1$; $dx = 0$

$y = 1$, $dy = 0$

$z = 0$ to 1

$$\int_{CD} F\,dr = \int_0^1 20z^2\,dz = \left[20\,\frac{z^3}{3}\right]_0^1$$

$$= 20/3$$

$\therefore \int_c F\,dr = 1 + 0 + 20/3$

$= 23/3$

(8) If $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$

evaluate $\int_c A\,dr$ from $(0,0,0)$ to $(1,1,1)$

along the following paths c.

→ (a) $x = t,\ y = t^2,\ z = t^3$

→ $\int_c A\,dr = \int (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$ — (c)

$x = t$	$y = t^2$	$z = t^3$
$\Rightarrow dx = dt$	$\Rightarrow dy = 2t\,dt$	$\Rightarrow dz = 3t^2\,dt$

$$\int_c A\,dr = \int_0^1 \left[(3t^2 + 6t^2)\,dt - 14t^5\,2t\,dt + 20t^7\,3t^2\,dt \right]$$

$$= \int_0^1 \left[9t^2\,dt - 28t^6\,dt + 60t^9\,dt \right]$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10} \right]_0^1$$

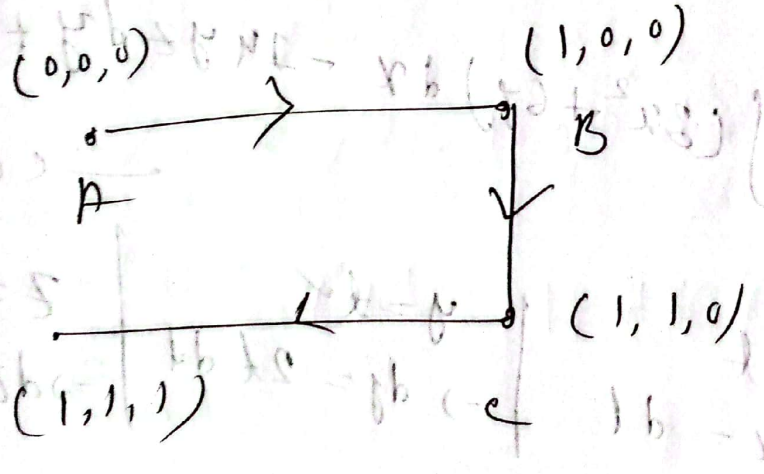
$= 3 - \frac{28}{7} + 6$

$= 3 - 4 + 6 = 5$

(b) The straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ & then to $(1,1,1)$

$\Rightarrow$

$$\int_c A\,dr = \int_c \left[(3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz \right] \quad \text{--- (1)}$$



$$\int_c A\,dr = \int_{AB} A\,dr + \int_{BC} A\,dr + \int_{CD} A\,dr$$

Along AB, $x = 0$ to 1

$y = 0, \quad dy = 0$

$z = 0, \quad dz = 0$

$$\therefore \int_{AD} A\,dr = \int_0^1 3x^2\,dx$$

$$= \left[\frac{3x^3}{3}\right]_0^1 = 1$$

Along BC, $x = 1,\ dx = 0$

$y = 0$ to 1

$z = 0,\ dz = 0$

$$\therefore \int_{BC} A\,dr = -\int_0^1 0 = 0$$

Along CD, $x = 1,\ dx = 0$

$y = 1,\ dy = 0$

$z = 0$ to 1

$$\therefore \int_{CD} A\,dr = \int_0^1 z^2\,dz = \left[z^3/3\right]_0^1 = 2\cancel{0}/3$$

$$\therefore \int_C A\,dr = 1 + 0 + 2\cancel{0}/3 = \cancel{\frac{3+1}{3}} = \cancel{4}/3$$

CamScanner

(8) Show that, $F = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\,k$ is conservative field.

Find its scalar potential & also the work done in moving particle from $(1,-2,1)$ to $(3,1,4)$

$\longrightarrow$

$$\bar{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$$

$$\text{curl } \bar{F} = \bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= 0$$

Hence, F is conservative vector field

let, ϕ is scalar potential of F

$$\bar{F} = \text{grad } \phi$$

$$\Rightarrow d\phi = (2xy + z^3)dx + x^2dy + 3z^2x\,dz$$

$$= (2xy\,dx + x^2dy) + (z^3dx + 3z^2x\,dz)$$

$$\int d\phi = \int d(x^2y) + \int d(xz^3) + c$$

$$\phi = x^2y + xz^3 + c$$

work done, $(\phi)_{(1,-2,1)}^{(3,1,2)}$

$$= \left[x^2y + xz^3\right]_{(1,-2,1)}^{(3,1,2)}$$

$$= 202$$

Sheet

(3)

(a) curl of $\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$

$= \hat{i}(0-0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x)$

$= 0$

Hence it is conservative

(b) From a, curl of F is $\vec{0}$ that's why there will be scalar potential.

so, let, ϕ be the scalar potential

$\bar{F} = \text{grad}(\phi)$

$\Rightarrow d\phi = (2xy + z^3)dx + x^2 dy + 3xz^2 dz$

$= \cancel{(2xy + x^2)}$

$= (2xy\,dx + x^2 dy) +$

$(z^3 dx + 3xz^2 dz)$

$$= d(x^2y) + d(z^{3x})$$

$$\int d\phi = \int \left[d(x^2y) + d(z^{3x}) \right]$$

$$\phi = x^2y + z^{3x}$$

(c) work done,

$$W = \left[\phi \right]_{(1,-2,1)}^{(3,1,4)}$$

$$= (9 + 192 + 2 - 1)$$

$$= 202$$

Surface Integral

$$\iint_S F \cdot d\mathbf{s} = \iint_S (F \cdot \hat{n})\, ds$$

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|}$$

1. If R be projection of S on x-y plane,

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

2. y-z plane, $ds = \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

3. z-x plane, $ds = \dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$

(8) Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=1$ in the first octant.

$\longrightarrow$ Let, $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$f = x^2+y^2+z^2-1$$

Surface integral of F over S is

$$\iint_S (\bar{F}\cdot\hat{n})\,ds$$

Now,

$$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{n} \cdot \hat{k} = z$$

$$\bar{F} \cdot \hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= xyz + xyz + xyz = 3xyz$$

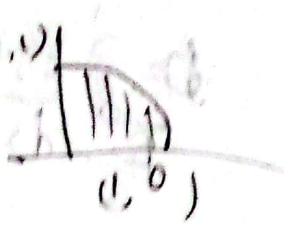
$\therefore$ surface is

$$\iint_S \bar{F} \cdot ds = \iint_S 3xyz \, \frac{dx\,dy}{z}$$

$$= \iint_S 3xy \, dx\,dy$$

Now the projection of S on xy-plane is the quadrant of the circle $x^2+y^2=1$ & $z=0$

$$\therefore \iint_S 3xy\,dx\,dy = 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dx\,dy$$



$x^2+y^2=1$

$\Rightarrow y=\sqrt{1-x^2}$

$$= 3\int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x(1-x^2)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = \frac{3}{2}\times\frac{1}{4} = 3/8$$

~~(8) Evaluate $\iint_S \bar{F}\cdot\hat{n}\,dS$ when $\bar{F}=($~~

(9) Evaluate (1) $\int_S \bar{F}\cdot d\bar{S}$ (1) $\int_S \bar{F}\times d\bar{S}$ when

$\bar{F}(y^2+z,\ z^2+x,\ x^2+y)$ & S is the square $0\le x\le 1,\ 0\le y\le 1,\ z=0$ positively oriented along the positive z axis.

① let,

$$\vec{F} = (y^2+z)\hat{i} + (z^2+x)\hat{j} + (x^2+y)\hat{k}$$

$$\vec{F}\cdot\vec{ds} = \vec{F}\cdot\hat{n}\,ds$$

$$= \vec{F}\cdot\hat{k}\,dx\,dy$$

Now, $F\cdot\hat{k} = x^2+y$

$$\therefore \int \vec{F}\cdot\vec{ds} = \int_0^1\int_0^1 (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + xy\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{3}+y\right) dy$$

$$= \left[\frac{1}{3}y + \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} = 5/6$$

(11) $F \times \bar{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y^2+z & z+x & x^2+y \\ 0 & 0 & 1 \end{vmatrix}$

$= \hat{i}(z+x) - \hat{j}(y^2+z)$

$\therefore \int F \times d\bar{s} = \int_0^1 \int_0^1 \{\hat{i}(z+x) - \hat{j}(y^2+z)\} dx\, dy$

$= \int_0^1 \{\hat{i}(zx + \frac{x^2}{2})]_0^1 - \hat{j}(y^2x + zx)]_0^1\} dy$

$= \int_0^1 [\hat{i}(z + 1/2) - \hat{j}(y^2+z)] dy$

$= [\hat{i}(zy + 1/2\, y) - \hat{j}(\frac{y^3}{3} + zy)]_0^1]$

$= \hat{i}(z + 1/2) - \hat{j}(1/3 + z)$

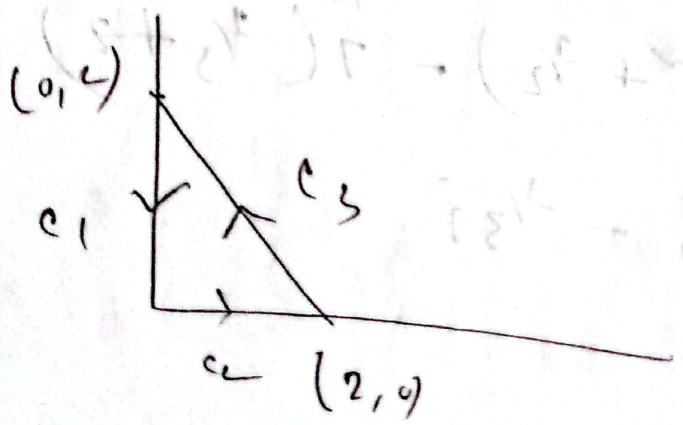
$= 1/2\, \hat{i} - 1/3\, \hat{j}$

Stokes's theorem

$$\oint_c f \cdot dr = \iint_s n \cdot \text{curl } f \, ds$$

$$= \iint_s \text{curl } f \cdot \hat{n} \, ds$$

Q1) Assume that a vector field $A = \hat{i}(2x^2 + y^2) + \hat{j}(xy - y^2)$, (a) find $\oint_c A \cdot dr$ around the triangular contour, (b) find $\int_s (\nabla \times A) \cdot ds$ over triangular arc (c) verify stokes's theorem.



$$\oint \bar{A}\cdot dr = \oint_{C_1} \bar{A}\cdot dr + \oint_{C_2} \bar{A}\cdot dr + \oint_{C_3} \bar{A}\cdot dr$$

Along C_1,

$x = 0,\ dx = 0$

$y = 2$ to 0

$$\oint_{C_3} \bar{A}\cdot dr = \int_2^0 (2x^2 + y^2)\,dx + \int_2^0 (0 - y^2)\,dy$$

$$= \int_0^2 y^2\,dy = \frac{y^3}{3} = 8/3$$

Along C_2,

$x = 0$ to 2

$y = 0,\ dy = 0$

$$\oint_{C_2} \bar{A}\,dr = \int_0^2 (2x^2)\,dx = 2\left[\frac{x^3}{3}\right]_0^2$$

$$= 16/3$$

Along C_3,

$\boxed{\cancel{x = 0 \text{ to } 0 \quad y = 0 \text{ to } 2}}$

$y = 2 - x$,

$dy = -dx$

$$\oint_{C_3} \bar{A}\cdot dr = \int_2^0 [2x^2 + (2-x)^2]\,dx - [x(2-x) - (2-x)^2]\,dx$$

$$= -\int_0^2 \left[2x^2 + (2-x)^2 - 2x + x^2 + (2-x)^2 \right] dx$$

$$= -\int_0^2 - \left[\frac{2x^3}{3} + \frac{(2-x)^3}{3} - \frac{2x^2}{2} + \frac{x^3}{3} \right]_0^2$$

$$= -\left[\frac{16}{3} + \frac{8}{3} - 4 + \frac{8}{3} - \frac{2}{3} \times \frac{8}{3} \right]$$

$$\int (2-x)^2 dx = \int (4 - 4x + x^2) dx$$

$$= -\left[\frac{2x^3}{3} - x^2 + \frac{x^3}{3} + 2\left(\frac{x^3}{3} - 2x^2 + 4x \right) \right]_0^2$$

$$= -\left[16/3 - 4 + 8/3 + 2\left(\frac{8}{3} - 8 + 8 \right) \right]$$

$$= -\left[\frac{16}{3} - 4 + \frac{8}{3} + \frac{16}{3} - \cancel{16} + \right]$$

$$= -28/3$$

$$\oint A \, dr = -9/3$$

(b)

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -(x^2+2 & 0 \\ 2x^2+ & xy- & \\ y^2 & y^2 & \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(y-2y)$$

$$= -\hat{k}y$$

$$\therefore \int_S (\nabla \times A) \cdot ds = -\int_0^2 \int_0^{2-x} y \, dx \, dy$$

$$= -\int_0^2 \left[\frac{y^2}{2} \right]_0^{2-x} dx$$

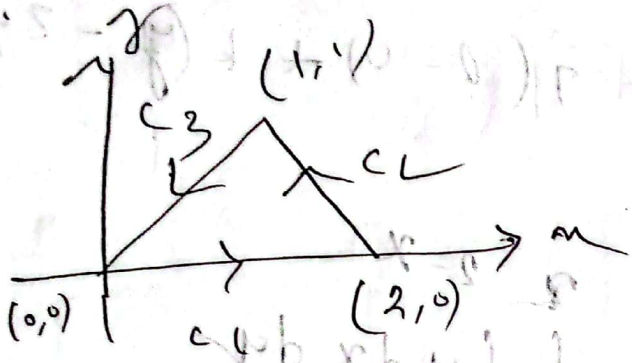
$$= -\int_0^2 \left[(2-x)^2 \right] dx$$

$$= -\int_0^2 (4 - 4x + x^2) \, dx$$

$$= -8/3 - 4/3$$

(c) Stokes theorem is verified

(9) Assume that a vector field $A = \hat{i}xy - \hat{j}(x^2+2y^2)$, (a) find $\oint \bar{A}\cdot dr$ around the triangular contour (b) find $\int (\nabla\times A)\cdot d$ over triangular arc (c) verify stoke's theorem.



$\frac{x-1}{1-2} = \frac{y-1}{1-0}$

$\frac{x-1}{-1} = 1-$

$x-1 = -1$

$x+y =$

$\rightarrow$

(a) $\oint_C \bar{A}\cdot d\bar{r} = \oint_{C_1} A\cdot dr + \int_{C_2} \bar{A}\cdot dr + \int_{C_3} \bar{A}\cdot dr$

$\frac{x-1}{1-0} =$

Along C_1, $x = 0$ to 2

$y = 0$, $dy = 0$

$x-1 = 1$

$x-y = 0$

$\oint_{C_1} \bar{A}\cdot dr = \int_0^2 xy\,dx = y\left[\frac{x^2}{2}\right]_0^2$

$= 0$

Along C_2,

$x = 2$ to 1

~~$y = 0$ to 1~~

Here, $y = 2 - x$, $dy = -dx$

$$\oint_{C_2} \bar{A} \cdot d\bar{r} = \int_2^1 x(2-x)\,dx + (x^2 + 2(2-x)^2)\,dx$$

$$= -\int_1^2 \left[2x - x^2 + x^2 + 2(4 - 4x + x^2)\right] dx$$

$$= -\int_1^2 \left[2x + 8 - 8x + 2x^2\right] dx$$

$$= -\int_1^2 \left[2x^2 - 6x + 8\right] dx$$

$$= -\int_1^2 = -11/3$$

Along C_3, $x = 1$ to 0

$y = x$, $dy = dx$

$$\oint_{C_3} \bar{A}\,d\bar{r} = 2/3$$

$$\oint_{C_3} \bar{A} \cdot d\bar{r} = -3$$

(b) $\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -(x^2+2y^2) & 0 \end{vmatrix}$

$= -3x\hat{k}\, ds$ $\cancel{= -3x\hat{k}\, dx\, dy}$

$\int (\nabla \times A) \cdot ds = \cancel{-3x\, dx\, dy}$

$= -\int_0^1 \int_y^{2-y} 3x\, dx\, dy$

$= -\frac{3}{2} \int_0^1 \left[(2-y)^2 - y^2 \right] dy$

$= -\frac{3}{2} \int_0^1 \left[4 - 4y \right] dy$

$= -\frac{3}{2} \left[4y - \frac{4y^2}{2} \right]_0^1$

$= -6 \left[1 - 1/2 \right]$

$= -3$

(c) Stokes's theorem is verified

Green's theorem:

If c be the regular closed curve in any plane bounding any region R & $P(x,y)$ & $Q(x,y)$ are continuous on c & in interior then,

$$\oint_c (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

(Q) Find $\oint_c (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$. where c is a rectangle whose vertices are $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$

$\rightarrow$ $P = e^{-x}\sin y$, $Q = e^{-x}\cos y$

Now, $\frac{\partial Q}{\partial x} = -e^{-x}\cos y$

$\frac{\partial P}{\partial y} = e^{-x}\cos y$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -2\iint_R e^{-x}\cos y \, dx\,dy$$

$$= -2\int_0^{\pi}\int_0^{\pi/2} e^{-x}\cos y \, dx\,dy$$

$$= -2\int_0^{\pi} e^{-x}\left[\sin y\right]_0^{\pi/2} dx$$

$$= -2\int_0^{\pi} e^{-x}\, dx$$

$$= 2\left[e^{-x}\right]_0^{\pi} = 2\left[e^{-\pi} - 1\right]$$

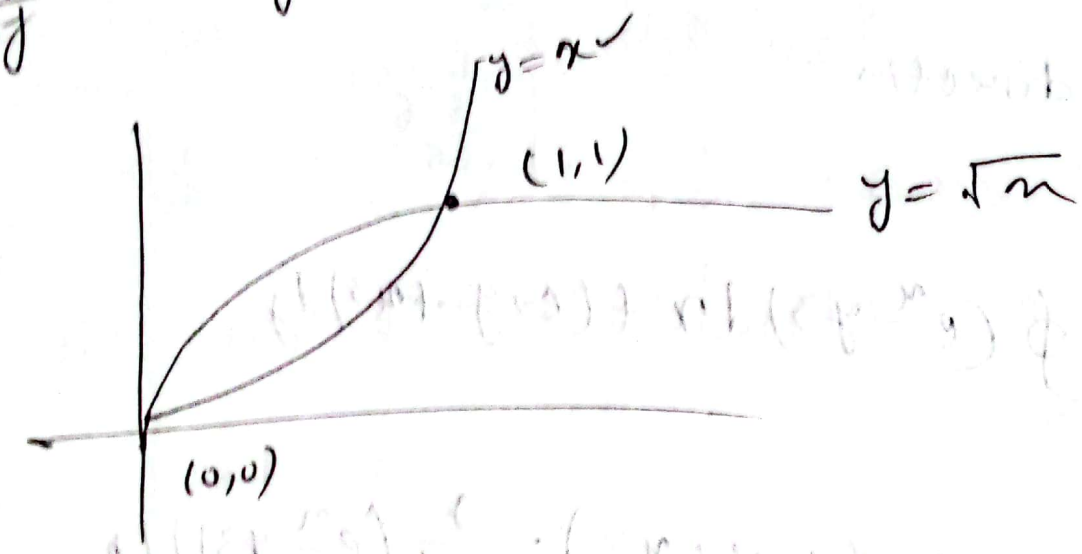
(3) Find $\iint_R (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

where C is the boundary of the region bounded by the parabola $y = \sqrt{x}$ & $y = x^2$

$\rightarrow \quad P = 3x - 8y$

$Q = 4y - 6xy$

$\frac{\partial P}{\partial y} = -16y \quad , \quad \frac{\partial Q}{\partial x} = -6y$



$$\int_0^1 \int_{x}^{\sqrt{x}} 10y \, dx \, dy$$

$$= \frac{10}{2} \int_0^1 (y^2)_{x}^{\sqrt{x}} \, dx$$

$$= 5\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = 3/2$$

(9) Find the work done by the force field $F(x,y) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around the unit circle $x^2+y^2=1$ in the counter clockwise direction

$\rightarrow$

$$\oint F \cdot d\bar{r} = \oint_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\right) dA$$

$$= \iint_R (3x^2 + 3y^2)\, dA$$

$$= 3\iint_R (x^2+y^2)\, dA \text{ — (e)}$$

Now,

$x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$$dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= r\,dr\,d\theta$$

$$(1) \Rightarrow \quad W = 3\int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta$$

$$= \frac{3}{4}\int_0^{2\pi} \left[r^4\right]_0^1 d\theta = \frac{3}{4}\int_0^{2\pi} 1\, d\theta$$

$$= 3/4 \left[\theta\right]_0^{2\pi}$$

$$= \frac{3\pi}{2}$$