NCERT 10.3.2.3.4

EE24BTECH11053 - S A Aravind Eswar

Question: Solve the following equation,

$$5x - 3y = 11 \tag{1}$$
$$-10x + 6y = -22 \tag{2}$$

0.1 Theoretical Method:

If we try to solve the equation, we get infinite number of points as they are coincident.

0.2 LU decomposition:

The equations can be written as,

$$\begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = 11 \tag{3}$$
$$\begin{pmatrix} -10 & 6 \end{pmatrix} \mathbf{x} = -22 \tag{4}$$

Writing them together,

$$\begin{pmatrix} 5 & -3 \\ -10 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 \\ -22 \end{pmatrix} \tag{5}$$

which is in the form of

$$A\mathbf{x} = \mathbf{b} \tag{6}$$

We can perform LU decomposition to find $\mathbf{x}$
There are multiple way to perform LU decomposition.
0.2.1 Gaussian Elimination: Gaussian Elimination can be written in the following way.
For a matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{7}$$

Given that the diagonal elements aren't zero, we can reduce the first column as,

$$R_i = R_i - \frac{a_{i1}}{a_{11}} R_1 \tag{8}$$
$$R_i = R_i - l_{i1} R_1 \tag{9}$$

where $i > 1$

Similarly reduing other $n-2$ columns, A is transformed into an upper triangular matrix U.

In general, we can write it as,

$$R_i = R_i - \frac{a_{ij}}{a_{jj}} R_1 \tag{10}$$

$$R_i = R_i - l_{i1} R_1 \tag{11}$$

L is given by,

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{pmatrix} \tag{12}$$

Where,

$$l_{ij} = \frac{a_{ij}}{a_{ii}} \tag{13}$$

0.2.2 Doolittle's Algorithm: Doolittle's algorithm is given by,

For U

$$\forall j \tag{14}$$

$$i = 0 \rightarrow U_{ij} = A_{ij} \tag{15}$$

$$i > 0 \rightarrow U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \tag{16}$$

For L

$$\forall i \tag{17}$$

$$j = 0 \rightarrow L_{ij} = \frac{A_{ij}}{U_{jj}} \tag{18}$$

$$j > 0 \rightarrow L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \tag{19}$$

After performing LU decomposition, we get,

$$A = LU \tag{20}$$

where,

$$L = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \tag{21}$$

$$U = \begin{pmatrix} 5 & -3 \\ 0 & 0 \end{pmatrix} \tag{22}$$

Now, we can write the original equation as,

$$LU\mathbf{x} = \mathbf{b} \tag{23}$$

Taking,

$$U\mathbf{x} = \mathbf{y} \tag{24}$$

we get,

$$L\mathbf{y} = \mathbf{b} \tag{25}$$

We can find $\mathbf{y}$ using forward substitution, giving,

$$\mathbf{y} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{26}$$

Now substituting,

$$\mathbf{y} = U\mathbf{x} \tag{27}$$

We get,

$$\begin{pmatrix} 5 & -3 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{28}$$

As the rank of U is 1, we do not have a unique solution.
As b_{21} is also 0, we can write it as,

$$\begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = 11 \tag{29}$$

Implying there are infinite number of solutions for the given pair of equations.

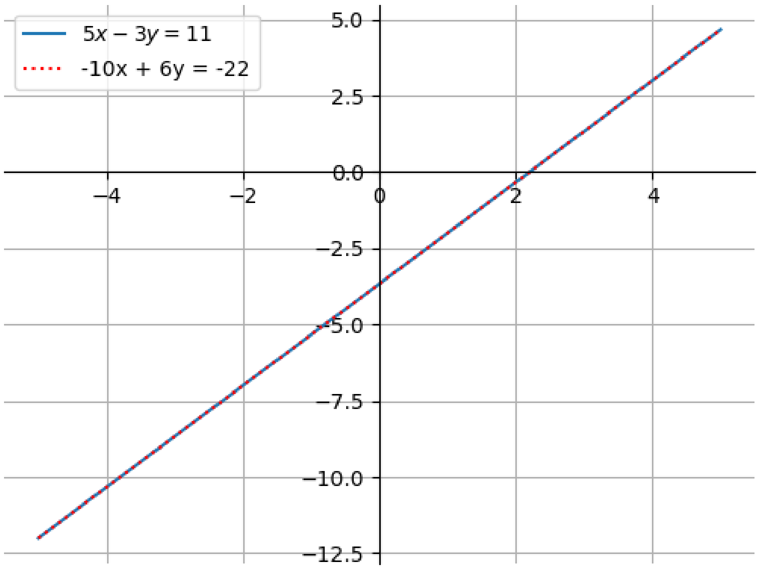

Fig. 0: Verification