Find the Curve of the Differential Equation Through the Given Point

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Problem Statement

Solve the differntial equation,

$$\frac{dy}{dx} + 2y = \sin x \tag{2.1}$$

with inital conditions $x = 0$ and $y = 0$

Laplace Transform in brief

Let, $f(t)$ be a piece-wise continuous function, then Laplace transform of the $f(t)$ is denoted as $\mathcal{L}\left\{f(t)\right\}$ and is defined as,

$$\mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-s\,t} f(t)\, dt = F(s) \tag{3.1}$$

It is a linear operator, meaning it follows superposition and homogeneity.
It has an inverse operation denoted as $\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$.
A property of Laplace transform we will use for solving the problem,

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \tag{3.2}$$

Theoretical Solution I

$$y' + 2y = \sin x \tag{3.3}$$
$$\tag{3.4}$$

Applying Laplace Transform,

$$\mathcal{L}\left\{y'\right\} + 2\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{\sin x\right\} \tag{3.5}$$
$$\left\{sY - y(0)\right\} + 2\left\{Y\right\} = \frac{1}{s^2+1} \tag{3.6}$$

Simplifying,

$$Y = \frac{1}{(s+1)(s^2+1)} \tag{3.7}$$
$$Y = \left(\frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s+2}\right)\frac{1}{5} \tag{3.8}$$

Theoretical Solution II

Applying inverse transform,

$$y = \frac{2\sin x - \cos x + e^{-2x}}{5} \tag{3.9}$$

Finite Difference Method

Difference equation can be written as,

$$\frac{dy}{dx} \approx \frac{y_{n+1} - y_n}{h} \tag{3.10}$$

$$y_{n+1} = y_n + \frac{dy}{dx}\, h \tag{3.11}$$

Substituting,

$$\frac{dy}{dx} = \sin x - 2y \tag{3.12}$$

we get,

$$y_{n+1} = y_n + \left(\sin x_n - 2y_n\right)\, h \tag{3.13}$$

Algorithm

Initial condition, $x_0 \leftarrow 0$
$y_0 \leftarrow 0$
Number of iterations, $iterations \leftarrow 20$
Step size, $h = 0.25$
for i in range(1, $iterations$) **do**
    $y_i = y_{i-1} + (\sin x_{i-1} - 2y_{i-1})\ h$
    $x_i \leftarrow x_{i-1} + h$
end for
plot(x, y)

Plotting the curve