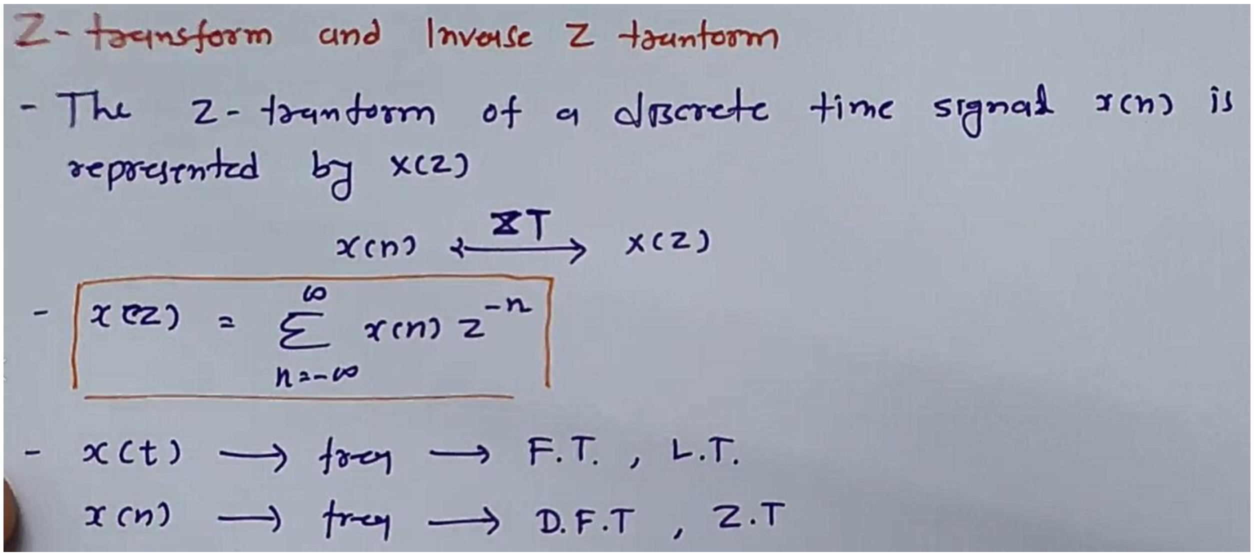
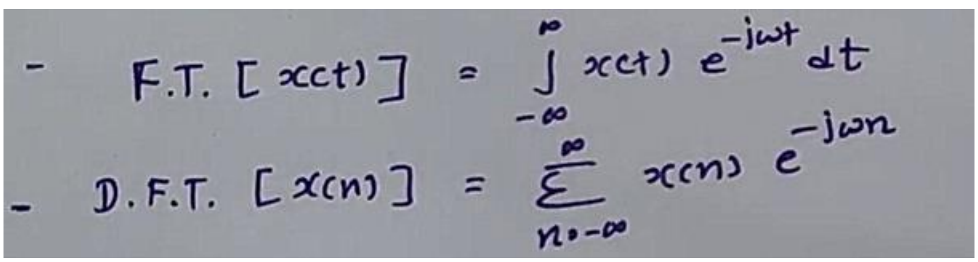
Z-transform:

In mathematics and signal processing, the **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain (**z-domain** or **z-plane**) representation.





The Z-transform can be defined as either a *one-sided* or *two-sided* transform.

Bilateral Z-transform

The *bilateral* or *two-sided* Z-transform of a discrete-time signal $x[n]$ is the formal power series $X(z)$ defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where n is an integer and z is, in general, a complex number:

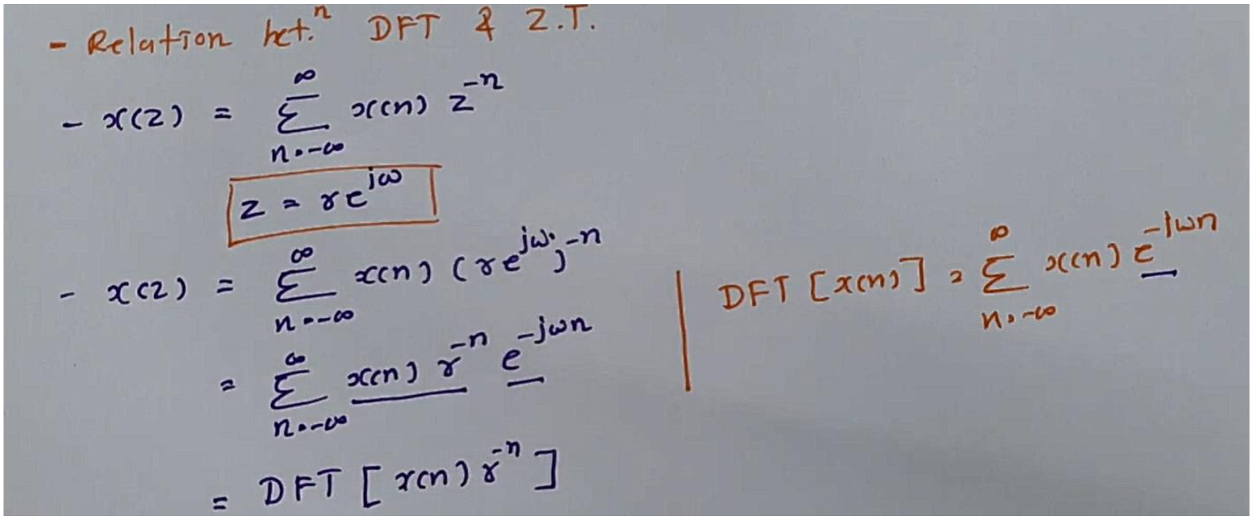
$$z = Ae^{j\phi} = A \cdot (\cos\phi + j\sin\phi)$$

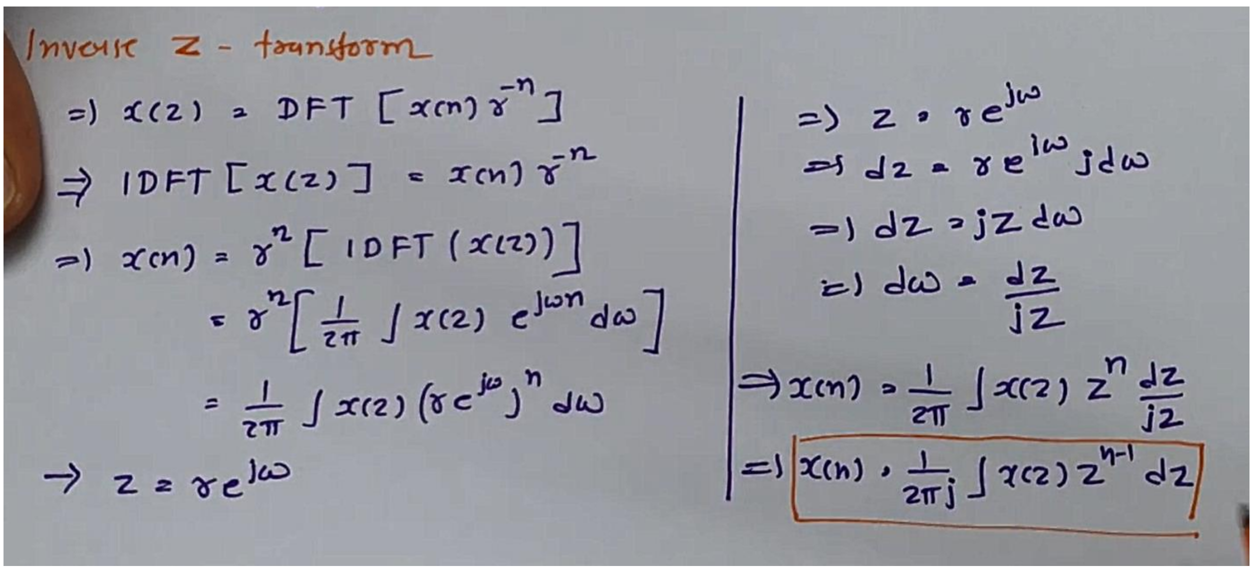
where A is the magnitude of z, j is the imaginary unit, and ϕ is the *complex argument* (also referred to as *angle* or *phase*) in radians.

Unilateral Z-transform

Alternatively, in cases where $x[n]$ is defined only for $n \geq 0$, the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$





Region of Convergence

Region of Convergence is the range of complex variable Z in the Z-plane. The Z-transformation of the signal is finite or convergent. So, ROC represents those set of values of Z, for which $X\,(Z)$ has a finite value.

Properties of ROC

- ROC does not include any pole.
- For right-sided signal, ROC will be outside the circle in Z-plane.
- For left sided signal, ROC will be inside the circle in Z-plane.
- For stability, ROC includes unit circle in Z-plane.
- For Both sided signal, ROC is a ring in Z-plane.
- For finite-duration signal, ROC is entire Z-plane.

Example

Let us find the Z-transform and the ROC of a signal given as x(n)={7,3,4,9,5}, where origin of the series is at 3.

Solution − Applying the formula we have -

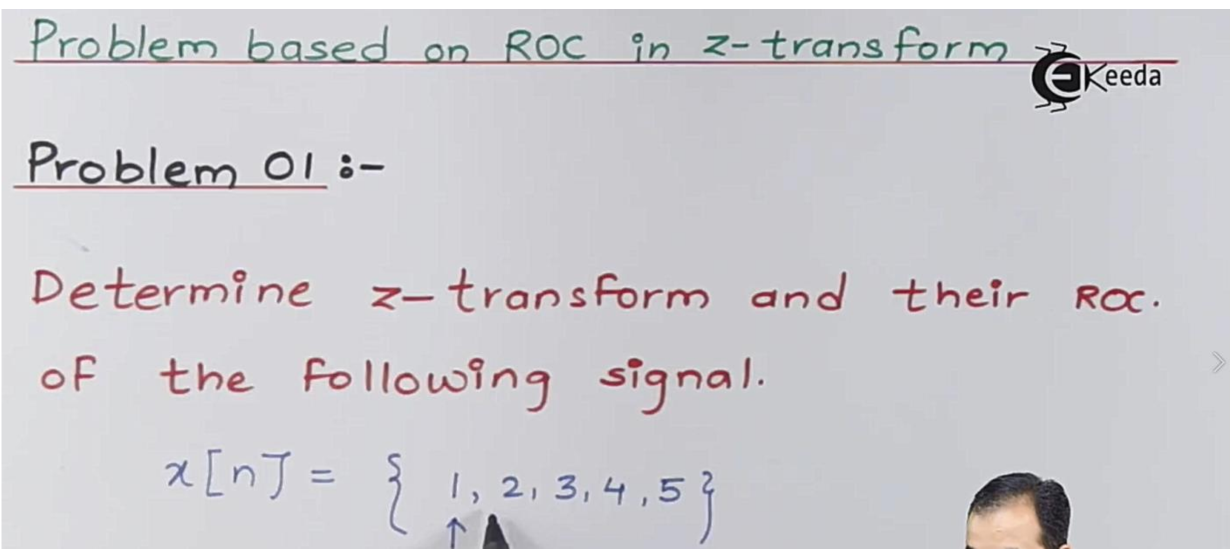
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-1}^{3} x(n)Z^{-n}$$

$$= x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$= 7Z + 3 + 4Z^{-1} + 9Z^{-2} + 5Z^{-3}$$

ROC is the entire Z-plane excluding Z = 0, ∞



Solution :-

$$x[n] = \{ \underset{\uparrow}{1}, 2, 3, 4, 5 \}$$
$$\quad\ \ 0 \ \ 1 \ \ 2 \ \ 3 \ \ 4$$

$$z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

$n = 0$ to 4

$$\therefore X(z) = \sum_{n=0}^{4} x(n) \cdot z^{-n}.$$

$$= x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + x(3) z^{-3} + x(4) \cdot z^{-4}.$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}.$$

$$\therefore X(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4}$$

at $z = 0$, $X(z) = \infty$.

At $z = \infty$, $X(z) = 1 + 0 + 0 + 0 + 0 = 1$.

The ROC of $X(z)$ is available over the entire region of z-plane, except $z = 0$.

Problem 02 :-

Determine z-transform and their ROC of the following discrete time signal,

$$x[n] = \{1, 2, 3, 4, \underset{\uparrow}{5}\}$$

Solution :-

$$x[n] = \{\underset{-4}{1}, \underset{-3}{2}, \underset{-2}{3}, \underset{-1}{4}, \underset{\uparrow 0}{5}\}$$

By the definition of z-transform,

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$$

$n = -4$ to 0.

$$\therefore X(z) = \sum_{n=-4}^{0} x(n) \cdot z^{-n}.$$

$$= x(-4) \cdot z^{4} + x(-3) \cdot z^{3} + x(-2) \cdot z^{2} + x(-1) z^{1} + x(0).$$

$$X(z) = 1 \cdot z^{4} + 2z^{3} + 3z^{2} + 4z + 5.$$

$$\therefore \underline{X(z)} = z^4 + 2z^3 + 3z^2 + 4z + 5$$

R.O.C.

At $\underline{z=0}$,

$X(z) = 5$

At $\underline{z=\infty}$,

$X(z) = \infty$

The ROC of $X(z)$ is available over entire region of z-plane, except $z=\infty$.

A finite sequence $x(n)$ is defined as $x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}}, \overset{x(1)}{3}, \overset{x(2)}{-3}, \overset{x(3)}{0}, \overset{x(4)}{4}, \overset{x(5)}{-2}\}$ find $X(z)$

$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{5} x(n)\, z^{-n}$$

$$= x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + \underbrace{x(3)z^{-3}}_{0} + x(4)z^{-4} + x(5)z^{-5}$$

$$\therefore X(z) = 5 + 3z^{-1} - 3z^{-2} + 4z^{-4} - 2z^{-5}$$

Z- transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, \underset{\uparrow}{-3}, 0, 4, -2\}$ find $X(z)$ of given sequence.

$$- \; x(n) = \{\overset{x(-2)}{5}, \overset{x(-1)}{3}, \underset{\uparrow}{\overset{x(0)}{-3}}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2) z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence $x(n) = \{\underset{\uparrow}{5}, 3, 0, 1, 2, 4\}$ Find Z- transform of $x(n)$.

$$- \; x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Linearity

It states that when two or more individual discrete signals are multiplied by constants, their respective Z-transforms will also be multiplied by the same constants.

Mathematically,

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z)$$

Proof − We know that,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

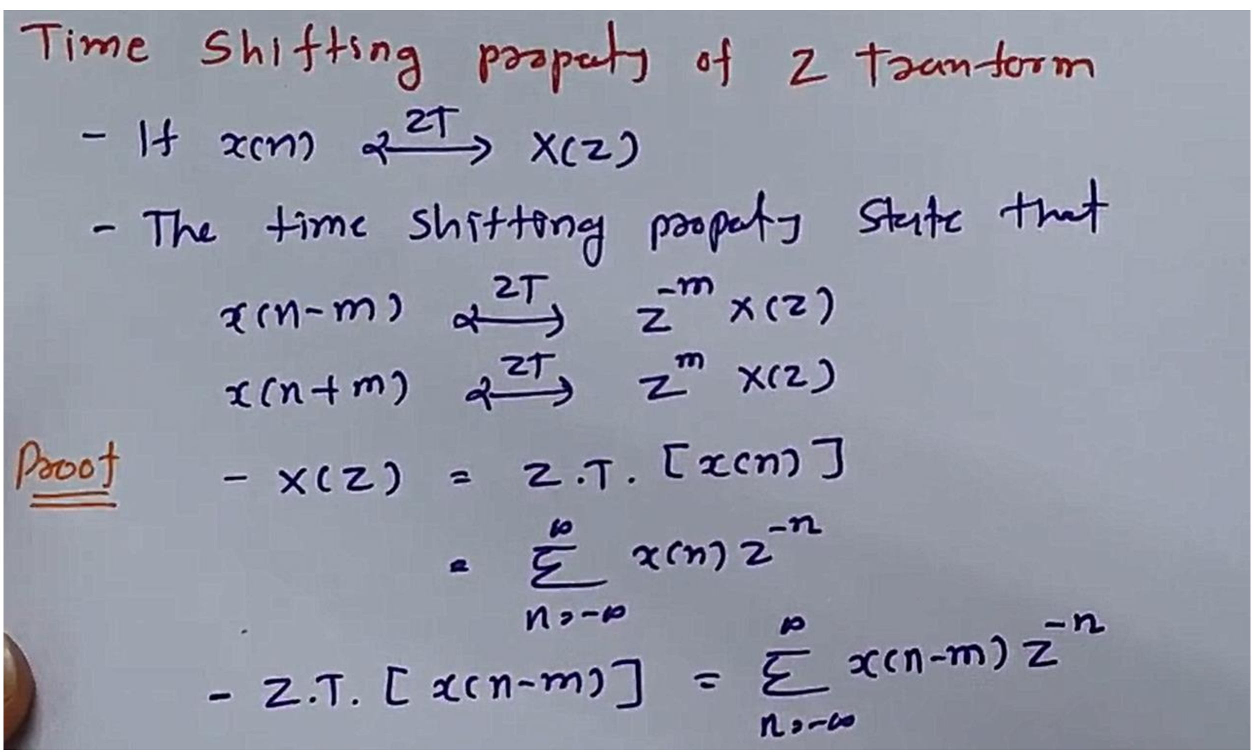
$$= \sum_{n=-\infty}^{\infty}(a_1x_1(n) + a_2x_2(n))Z^{-n}$$

$$= a_1\sum_{n=-\infty}^{\infty} x_1(n)Z^{-n} + a_2\sum_{n=-\infty}^{\infty} x_2(n)Z^{-n}$$

$$= a_1X_1(z) + a_2X_2(z) \qquad HenceProved$$

Here, the ROC is $ROC_1 \bigcap ROC_2$.

Time Shifting



- If $n - m = p \Rightarrow n = p + m$

$$= \sum_{p=-\infty}^{\infty} x(p)\, z^{-(p+m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p)\, z^{-p}\, z^{-m}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)\, z^{-p}$$

$$= \boxed{z^{-m} X(z)}$$

example find Z-Transform of $\delta(n-k)$

$$\delta(n) \xleftrightarrow{ZT} 1$$

$$\delta(n-k) \longleftrightarrow \boxed{z^{-k}}$$

Time shifting property depicts how the change in the time domain in the discrete signal will affect the Z-domain, which can be written as;

$$x(n-n_0) \longleftrightarrow X(Z)Z^{-n}$$

Or $$x(n-1) \longleftrightarrow Z^{-1}X(Z)$$

Proof −

Let $y(P) = X(P-K)$

$$Y(z) = \sum_{p=-\infty}^{\infty} y(p)Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} (x(p-k))Z^{-p}$$

Let s = p-k

$$= \sum_{s=-\infty}^{\infty} x(s)Z^{-(s+k)}$$

$$= \sum_{s=-\infty}^{\infty} x(s)Z^{-s}Z^{-k}$$

$$= Z^{-k}[\sum_{s=-\infty}^{\infty} x(m)Z^{-s}]$$

$$= Z^{-k}X(Z) \qquad HenceProved$$

Time Scaling

Scaling property of Z Transform

- $x(n) \xleftrightarrow{ZT} X(z)$

Then Scaling property of Z Transform states that

$$a^n x(n) \xleftrightarrow{ZT} X[z/a]$$

Proof

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T.[a^n(x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

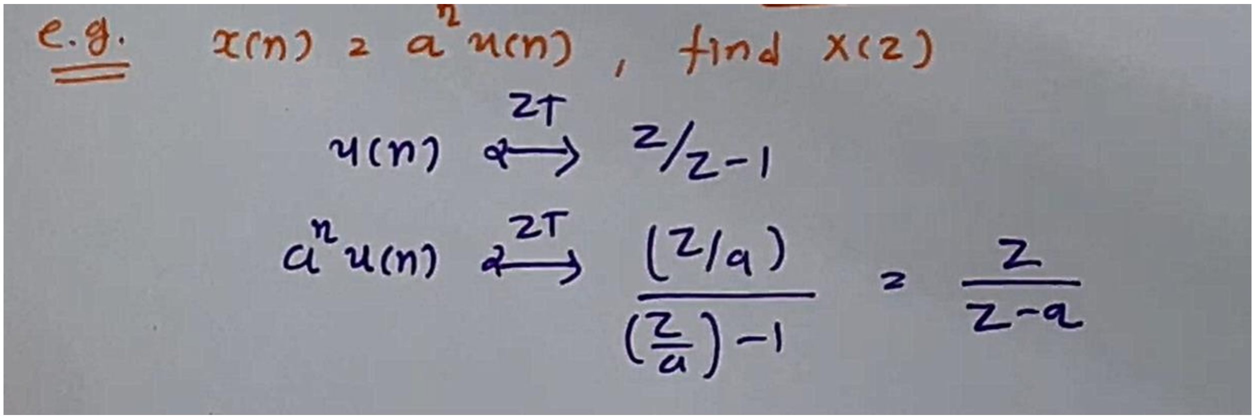
$$Z.T.[a^n(x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (az^{-1})^n$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{az^{-1}}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= \boxed{X(z/a)}$$



Time Scaling property tells us, what will be the Z-domain of the signal when the time is scaled in its discrete form, which can be written as;

$$a^n x(n) \longleftrightarrow X(a^{-1}Z)$$

Proof −

Let $\quad y(p) = a^p x(p)$

$$Y(P) = \sum_{p=-\infty}^{\infty} y(p)Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} a^p x(p)Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} x(p)[a^{-1}Z]^{-p}$$

$$= X(a^{-1}Z) \qquad\qquad Hence proved$$

Convolution

This depicts the change in Z-domain of the system when a convolution takes place in the discrete signal form, which can be written as −

$$x_1(n) * x_2(n) \longleftrightarrow X_1(Z).\,X_2(Z)$$

Proof −

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)]Z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)[\sum_{n}^{\infty} x_2(n-k)Z^{-n}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)[\sum_{n=-\infty}^{\infty} x_2(n-k)Z^{-(n-k)}Z^{-k}]$$

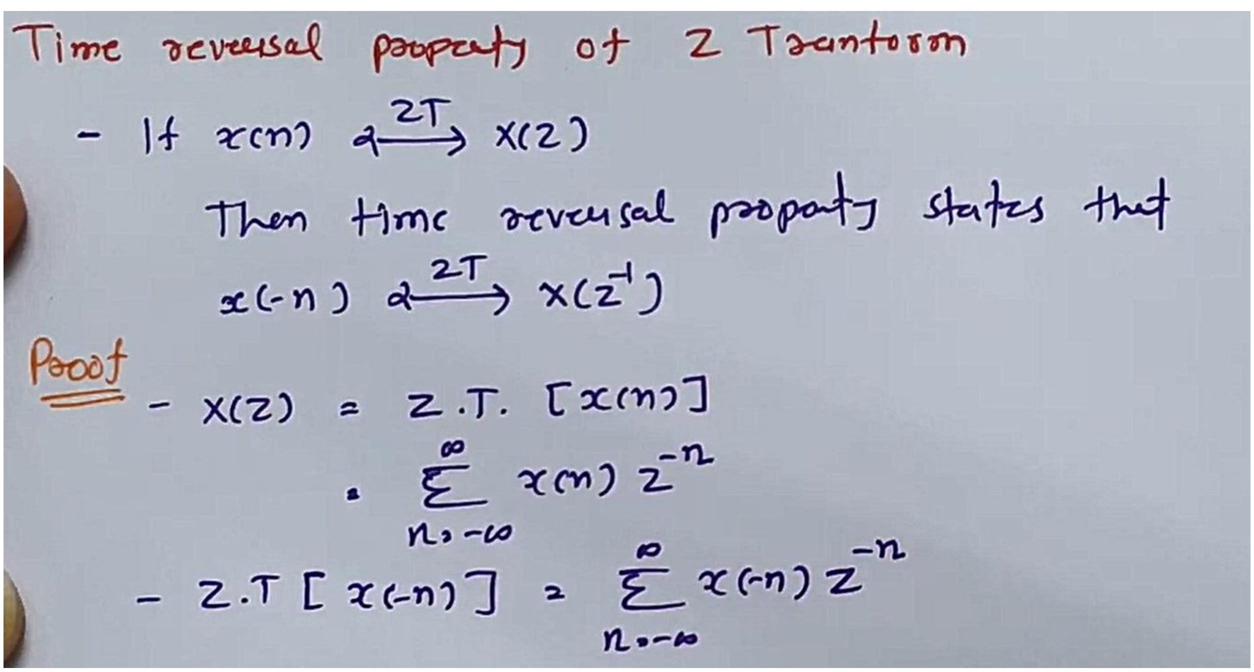
Let n-k = l, then the above equation cab be written as −

$$X(Z) = \sum_{k=-\infty}^{\infty} x_1(k)[Z^{-k}\sum_{l=-\infty}^{\infty} x_2(l)Z^{-l}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)X_2(Z)Z^{-k}$$

$$= X_2(Z)\sum_{k=-\infty}^{\infty} x_1(Z)Z^{-k}$$

$$= X_1(Z).\,X_2(Z) \qquad\qquad HenceProved$$



Time reversal property of Z Transform

- If $x(n) \xleftrightarrow{ZT} X(Z)$

Then time reversal property states that

$x(-n) \xleftrightarrow{ZT} X(Z^{-1})$

Proof

- $X(Z) = Z.T.\,[x(n)]$

$= \sum_{n=-\infty}^{\infty} x(n)\,Z^{-n}$

- $Z.T\,[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\,Z^{-n}$

$$- Z.T[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$$

If $p = -n$

$$= \sum_{p=-\infty}^{\infty} x(p) Z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (Z^{-1})^{-p}$$

$$= X(Z^{-1})$$

e.g. $x(n) = a^{n} u(-n)$. find $X(Z)$

$$- x(n) = a^{n} u(-n)$$

$$= (1/a)^{-n} u(-n)$$

$$x(n) = (1/a)^{n} u(n) \xleftrightarrow{ZT} \frac{Z}{Z - 1/a}$$

$$(1/a)^{-n} u(-n) \xleftrightarrow{ZT} \frac{Z^{-1}}{Z^{-1} - 1/a} = \frac{1/Z}{1/Z - 1/a} = \boxed{\frac{a}{a-Z}}$$

Multiplication & Convolution property of Z-Transform

If $x_1(n) \xleftrightarrow{ZT} X_1(Z)$

$x_2(n) \xleftrightarrow{ZT} X_2(Z)$

- Multiplication property states that

$$x_1(n)\, x_2(n) \xleftrightarrow{ZT} x_1(Z) * x_2(Z)$$

- convolution property states that

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(Z)\, X_2(Z).$$

e.g. $u(n-1) * \delta(n)$, find Z-Transform

$$- \; x_1(n) = u(n-1) \xleftrightarrow{ZT} X_1(z) = \frac{1}{z-1}$$

$$- \; x_2(n) = \delta(n) \xleftrightarrow{ZT} X_2(z) = 1$$

$$- \; X(z) = ZT[x_1(n) * x_2(n)]$$

$$= X_1(z) X_2(z)$$

$$= \frac{1}{z-1}$$

e.g. $u(n-2) * \delta(n-3)$, find Z-Transform.

$$- \; x_1(n) = u(n-2) \xleftrightarrow{ZT} X_1(z) = z^{-2}\left(\frac{z}{z-1}\right) = \frac{1}{z}\left(\frac{1}{z-1}\right)$$

$$- \; x_2(n) = \delta(n-1) \xleftrightarrow{ZT} X_2(z) = z^{-1} = 1/z$$

$$- \; X(z) = ZT[x_1(n) * x_2(n)]$$

$$= X_1(z) X_2(z)$$

$$= \frac{1}{z}\left(\frac{1}{z-1}\right)\left(\frac{1}{z}\right)$$

$$= \frac{1}{z^2(z-1)}$$

Successive Differentiation

Successive Differentiation property shows that Z-transform will take place when we differentiate the discrete signal in time domain, with respect to time. This is shown as below.

$$\frac{dx(n)}{dn} = (1 - Z^{-1})X(Z)$$

Proof –

Consider the LHS of the equation – $\frac{dx(n)}{dn}$

$$= \frac{[x(n) - x(n-1)]}{[n - (n-1)]}$$

$$= x(n) - X(n-1)$$

$$= x(Z) - Z^{-1}x(Z)$$

$$= (1 - Z^{-1})x(Z) \qquad Hence Proved$$

Multiplication in Time

It gives the change in Z-domain of the signal when multiplication takes place at discrete signal level.

$$x_1(n).\,x_2(n) \longleftrightarrow (\tfrac{1}{2\Pi j})[X1(Z) * X2(Z)]$$

Conjugation in Time

This depicts the representation of conjugated discrete signal in Z-domain.

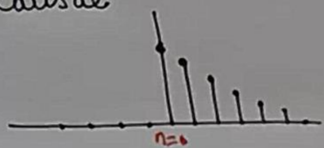
$$X^*(n) \longleftrightarrow X^*(Z^*)$$

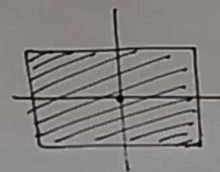
Region of Convergence (RoC)

Finite Duration Signals ✓

RoC

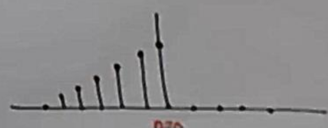
Causal

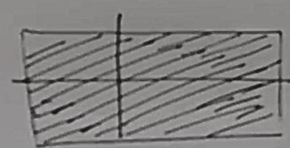




Entire z-plane except $|z| = 0$

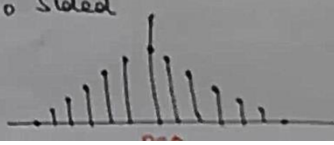
Anti Causal

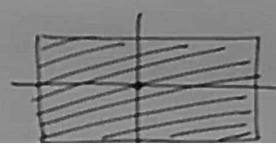




Entire z-plane except $|z| = \infty$

Two Sided

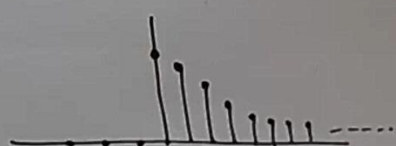


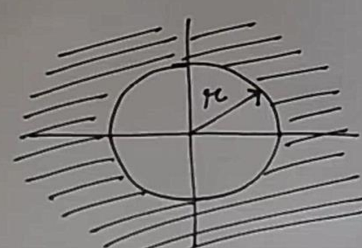


Entire z-plane except $|z| = 0$ & $|z| = \infty$ ✓

Infinite Duration Signals

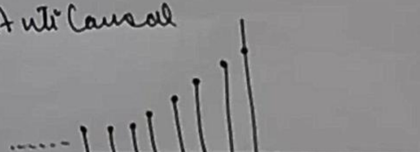
Causal

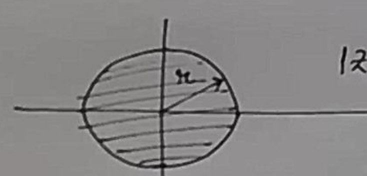




$|z| > r$

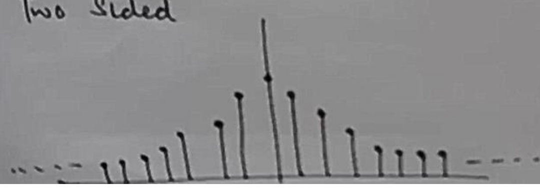
Anti Causal

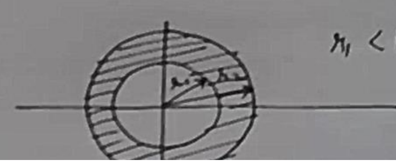




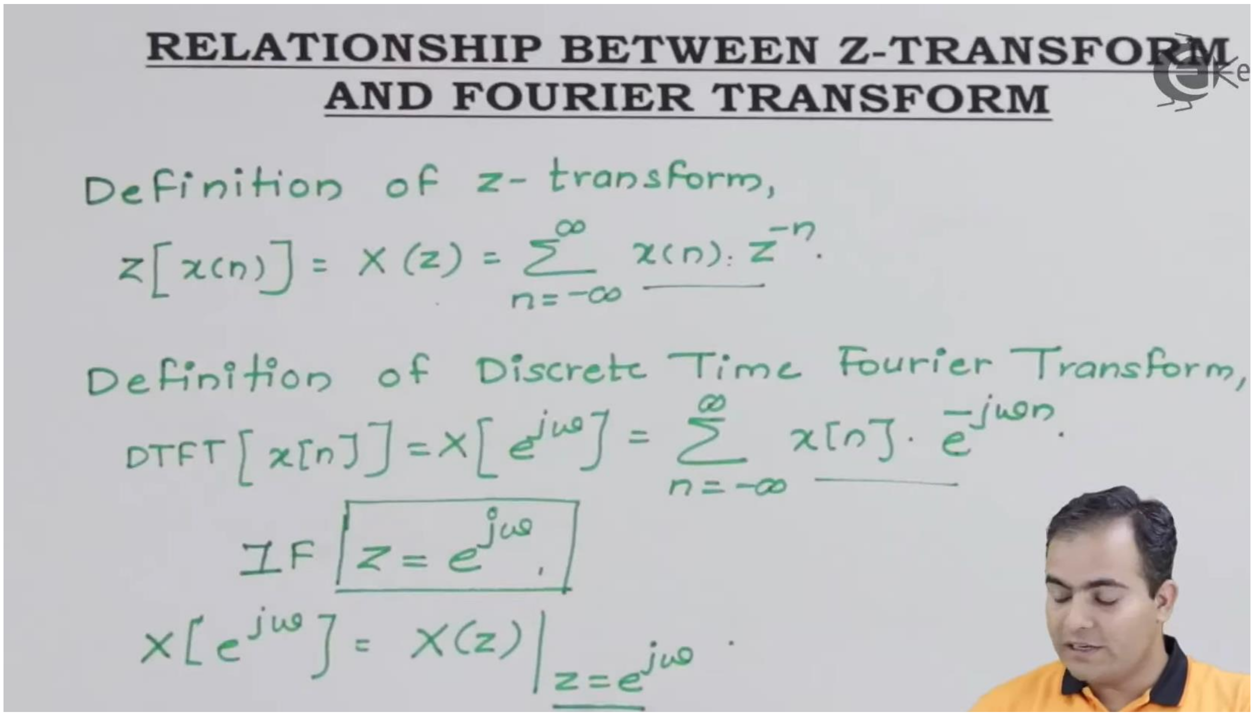
$|z| < r$

Two Sided





$r_1 < |z| < r_2$



57. Determine the z-transform of the signal x(n) = α^n u(n) and also the ROC and pole & zero locations of X(z) in the z-plane.

Solution:

Given x(n) = α^n u(n)

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

Using geometric series,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\therefore X(z) = \frac{1}{1-\alpha z^{-1}}; |z| > |\alpha|$$

$$= \frac{z}{z-\alpha}; |z| > |\alpha|$$

There is a pole at z = α & zero at z = 0.

ROC of Impulse function in Z-Transform

1. $\delta(n) \xleftrightarrow{ZT} 1$

ROC: Entire Z plane, including $z=0$ & $z=\infty$

2. $\delta(n-k) \xleftrightarrow{ZT} z^{-k}$

ROC: Entire Z plane, including $z=\infty$ & excluding $z=0$

3. $\delta(n+k) \xleftrightarrow{ZT} z^{k}$

ROC: Entire Z plane, Including $z=0$ & excluding $z=\infty$

$\delta(n-3)$ – Entire Z plane.
Includes $z=\infty$
Exclude $z=0$.

ROC of discrete time sequence in Z Transform

1. If Sequence is purely right sided or causal, then
   ROC : Entire Z plane except $z = 0$
   $$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \xleftrightarrow{ZT} X(z) = 1 + 2z^{-1} + 3z^{-2} + 4$$

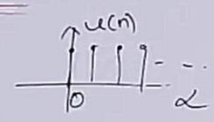
2. If Sequence is purely Left sided or Anticausal, Then
   ROC : Entire Z plane except at $z = 0$

   Check may be ∞

   $$x(n) = \{1, 2, 3, \underset{\uparrow}{4}\} \xleftrightarrow{ZT} X(z) = z + 2z + 3z + 4$$

3. If Sequence is a two sided, Then
   ROC : Entire Z plane except at $z = 0$ & $z = \infty$
   $$x(n) = \{1, 2, \underset{\uparrow}{1}, 3, 1\} \xleftrightarrow{ZT} X(z) = 2z^{2} + 2z + 1 + 3z^{-1} + z^{-2}$$

* Z-Transform of Discrete unit step function. $u(n)$ *



$$Z.T[u(n)]$$

$$Z.T[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

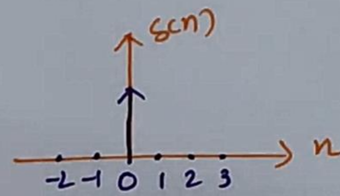
$$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Z- Transform of Unit Impulse function

- $x(n) = \delta(n)$

- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$



$$= \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

$$= \dots + \delta(-2)z^{2} + \delta(-1)z^{1} + \underline{\delta(0)}z^{0} + \delta(1)z^{-1} + \delta(2)z^{-2} + \dots$$

$$= 0 \times z^{2} + 0 \times z^{1} + 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \dots$$

$$= 1$$

Z- Transform of Standard basic signals

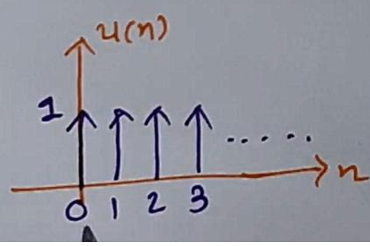
- Find The Z- transform of the signals $a^{n}u(n)$ & $a^{-n}u(n)$

- $x(n) = a^{n}u(n)$ $\qquad y(n) = a^{-n}u(n)$

- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{n}u(n)\, z^{-n}$$



$0 \quad 1 \quad 2 \quad 3$

$$- X(z) = \sum_{n=0}^{\infty} (a^{n})(z^{-n})$$

$$= \sum_{n=0}^{\infty} (az^{-1})^{n}$$

We know $\displaystyle\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

$$y(n) = a^{-n} u(n)$$

$$- Y(z) = \text{Z.T.}\,[y(n)]$$

$$= \sum_{n=-\infty}^{\infty} y(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{-n} u(n)\, z^{-n}$$

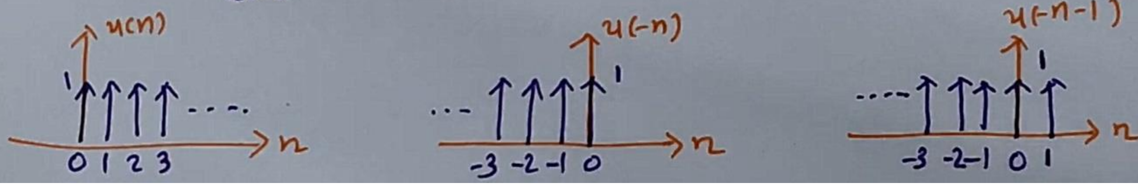
Z-Transform of standard basic signal

Find the Z-transform of the sequence $x(n) = -a^n u(-n-1)$

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$



$$X(z) = \sum_{n=-\infty}^{1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{1} (az^{-1})^n$$

$$\left[\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right]$$

- To swept limits place $n = -n$.

$$X(z) = -\sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$= -\sum_{n=1}^{\infty} (za^{-1})^n$$

$$= -\left[\sum_{n=0}^{\infty} (z a^{-1})^n - 1 \right]$$

$$= -\left[\frac{1}{1 - z a^{-1}} - 1 \right]$$

$$= -\left[\frac{a}{a - z} - 1 \right]$$

$$= -\left[\frac{\cancel{a} - \cancel{a} + z}{a - z} \right]$$

$$= \boxed{\frac{z}{z - a}}$$

$$x(n) = -a^n u(-n-1)$$

$$a^{n}u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$-a^{n}u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$a^{-n}u(n) \xleftrightarrow{ZT} \frac{az}{az-1}$$

$$-a^{-n}u(-n-1) \xleftrightarrow{ZT} \frac{az}{az-1}$$

n

$$Y(z) = \sum_{n=0}^{\infty} a^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (az)^{-n}$$

we know $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$$\therefore Y(z) = \frac{1}{1-(az)^{-1}} = \frac{1}{1-1/az}$$

$$= \boxed{\frac{az}{az-1}}$$

$y(n) = a^{-n} u(n)$

Z- transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, -3, 0, 4, -2\}$
find $X(z)$ of given sequence.

$$- \; x(n) = \{\overset{x(-2)}{5}, \overset{x(-1)}{3}, \overset{x(0)}{-3}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2) z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence $x(n) = \{5, 3, 0, 1, 2, 4\}$
Find Z- transform of $x(n)$.

$$- \; x(n) = \{\overset{x(0)}{5}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Examples of Inverse Z transform

Find the signal $x(n)$ for which the z transform is

$$X(z) = 4z^{4} - z^{3} - 3z + 4z^{-1} + 3z^{-2}$$

- $x(n) = Z^{-1}[x(z)] = \boxed{\frac{1}{2\pi j} \int x(z)\, z^{n-1}\, dz}$

$$x(z) = \underset{x(-4)}{4z^{4}} - \underset{x(-3)}{1z^{3}} + \underset{x(-2)}{0z^{2}} - \underset{x(-1)}{3z} + \underset{x(0)}{0z^{0}} + \underset{x(1)}{4z^{-1}} + \underset{x(2)}{3z^{-2}}$$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-4}^{2} x(n) z^{-n}$$

- $\boxed{x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}}$

If $x(z) = 2z^{2} + 3$, find $x(n)$

- $x(z) = \underset{-2}{\underline{2z^{2}}} + \underset{-1}{\underline{0 \times z^{1}}} + \underset{0}{\underline{3 \times z^{0}}}$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-2}^{0} x(n) z^{-n}$$

- $x(n) = \{2, 0, \underset{\uparrow}{3}\}$

If $x(z) = 3 + 2z^{-1} + 3z^{-3}$, find $x(n)$

- $x(z) = \underbrace{3 \times z^{0}}_{x(0)} + \underbrace{2}_{x(1)} \times z^{-1} + \underbrace{0}_{x(2)} \times z^{-2} + \underbrace{3}_{x(3)} \times z^{-3}$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{3} x(n)\, z^{-n}$

- $x(n) = \{\underset{\uparrow}{3}, 2, 0, 3\}$

4.18. Using the power series expansion technique, find the inverse z-transform of the following $X(z)$:

(a) $X(z) = \dfrac{z}{2z^2 - 3z + 1} \qquad |z| < \dfrac{1}{2}$

(b) $X(z) = \dfrac{z}{2z^2 - 3z + 1} \qquad |z| > 1$

(a) Since the ROC is $|z| < \frac{1}{2}$, $x[n]$ is a left-sided sequence. Thus, we must divide to obtain a series in power of z. Carrying out the long division, we obtain

$$
\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
\hline
1 - 3z + 2z^2 & z \\
 & \underline{z - 3z^2 + 2z^3} \\
 & \quad 3z^2 - 2z^3 \\
 & \quad \underline{3z^2 - 9z^3 + 6z^4} \\
 & \qquad 7z^3 - 6z^4 \\
 & \qquad \underline{7z^3 - 21z^4 + 14z^5} \\
 & \qquad\qquad 15z^4 \cdots
\end{array}
$$

Thus,

$$X(z) = \cdots + 15z^4 + 7z^3 + 3z^2 + z$$

and so by definition (4.3) we obtain

$$x[n] = \{\ldots, 15, 7, 3, 1, \underset{\uparrow}{0}\}$$

(b) Since the ROC is $|z| > 1$, $x[n]$ is a right-sided sequence. Thus, we must divide so as to obtain a series in power of z^{-1} as follows:

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \\
\hline
2z^2 - 3z + 1 & z \\
 & \underline{z - \frac{3}{2} - \frac{1}{2}z^{-1}} \\
 & \quad \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \quad \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\
 & \qquad \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \qquad\qquad \vdots
\end{array}
$$

Thus,

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots$$

and so by definition (4.3) we obtain

$$x[n] = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$