In mathematics and signal processing, the **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain (**z-domain** or **z-plane**) representation.

Z-transform and Inverse Z transform

- The Z-transform of a discrete time signal $x(n)$ is represented by $X(z)$

$$x(n) \xleftrightarrow{ZT} X(z)$$

- $$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$$

- $x(t) \longrightarrow$ freq $\longrightarrow$ F.T. , L.T.

  $x(n) \longrightarrow$ freq $\longrightarrow$ D.F.T , Z.T

- $$\text{F.T.}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- $$\text{D.F.T.}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

- Relation bet.^{n} DFT & Z.T.

- $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

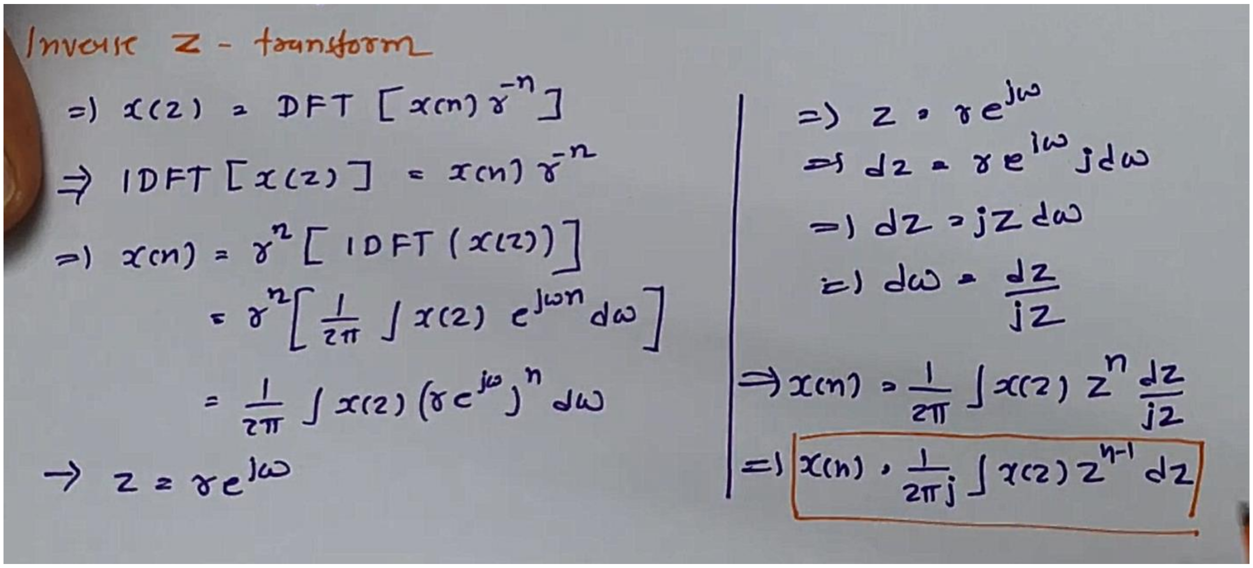
$$\boxed{z = r e^{j\omega}}$$

- $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, (r e^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, r^{-n}\, e^{-j\omega n}$$

$$= \text{DFT}[x(n)\, r^{-n}]$$

$$\text{DFT}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$



Region of Convergence

Region of Convergence is the range of complex variable Z in the Z-plane. The Z-transformation of the signal is finite or convergent. So, ROC represents those set of values of Z, for which X Z has a finite value.

Properties of ROC

- ROC does not include any pole.
- For right-sided signal, ROC will be outside the circle in Z-plane.
- For left sided signal, ROC will be inside the circle in Z-plane.
- For stability, ROC includes unit circle in Z-plane.
- For Both sided signal, ROC is a ring in Z-plane.
- For finite-duration signal, ROC is entire Z-plane.

Example

Let us find the Z-transform and the ROC of a signal given as x(n)={7,3,4,9,5}, where origin of the series is at 3.

Solution − Applying the formula we have -

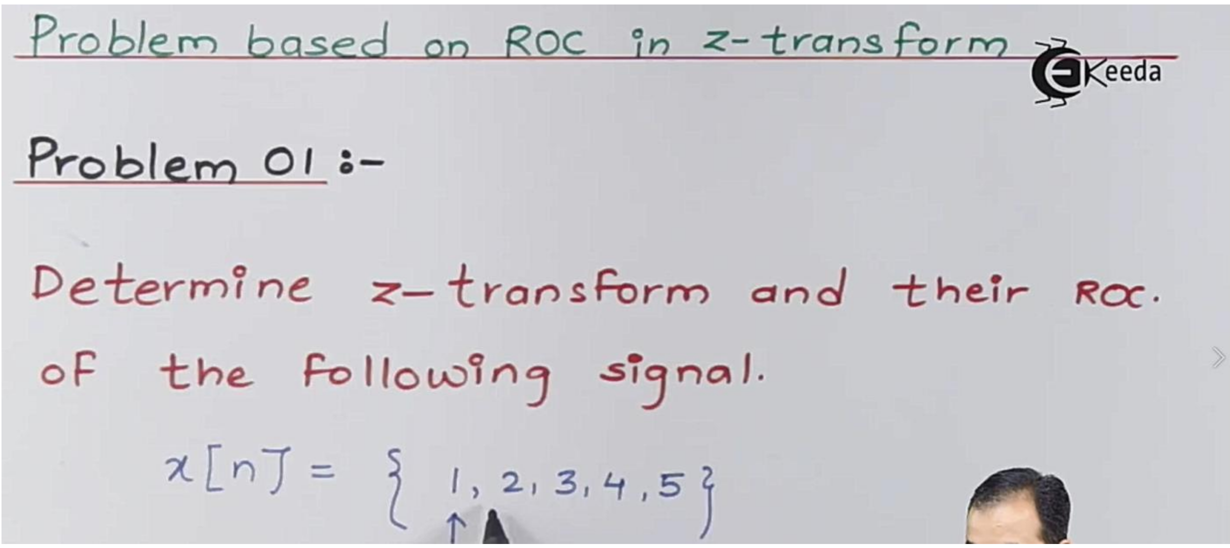
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-1}^{3} x(n)Z^{-n}$$

$$= x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$= 7Z + 3 + 4Z^{-1} + 9Z^{-2} + 5Z^{-3}$$

ROC is the entire Z-plane excluding Z = 0, ∞



Solution :-

$$x[n] = \{ \underset{\uparrow}{1}, 2, 3, 4, 5 \}$$
$\quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

$n = 0$ to 4

$$\therefore X(z) = \sum_{n=0}^{4} x(n) \cdot z^{-n}.$$

$$= x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + x(3) z^{-3} + x(4) \cdot z^{-4}.$$

$$\underline{X(z)} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}.$$

$$\therefore X(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4}$$

at $z = 0$, $X(z) = \infty$.

At $z = \infty$, $X(z) = 1 + 0 + 0 + 0 + 0 = 1$.

The ROC of $X(z)$ is available over the entire region of z-plane, except $z = 0$.

Problem 02 :-

Determine z-transform and their ROC of the following discrete time signal,

$$x[n] = \{1, 2, 3, 4, \underset{\uparrow}{5}\}$$

Solution :-

$$x[n] = \{\underset{-4}{1}, \underset{-3}{2}, \underset{-2}{3}, \underset{-1}{4}, \underset{\uparrow 0}{5}\}$$

By the definition of z-transform,

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$n = -4$ to 0.

$$\therefore\ X(z) = \sum_{n=-4}^{0} x(n) \cdot z^{-n}.$$

$$= x(-4) \cdot z^{4} + x(-3) \cdot z^{3} + x(-2) \cdot z^{2} + x(-1)\, z^{1} + x(0).$$

$$X(z) = 1 \cdot z^{4} + 2z^{3} + 3z^{2} + 4z + 5.$$

$$\therefore \underline{X(z)} = z^4 + 2z^3 + 3z^2 + 4z + 5$$

R.O.C.

At $\underline{z=0}$,

$$X(z) = 5$$

At $\underline{z=\infty}$,

$$X(z) = \infty$$

The ROC of $X(z)$ is available over entire region of z-plane, except $z=\infty$.

A finite sequence $x(n)$ is defined as $x(n) = \left\{ \begin{array}{cccccc} x(0) & x(1) & x(2) & x(3) & x(4) & x(5) \\ \underset{\uparrow}{5}, & 3, & -3, & 0, & 4, & -2 \end{array} \right\}$ find $X(z)$

$$Z.T\left[x(n)\right] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{5} x(n)\, z^{-n}$$

$$= x(0)\,z^{0} + x(1)\,z^{-1} + x(2)\,z^{-2} + \underbrace{x(3)\,z^{-3}}_{0} + x(4)\,z^{-4} + x(5)\,z^{-5}$$

$$\therefore X(z) = 5 + 3z^{-1} - 3z^{-2} + 4z^{-4} - 2z^{-5}$$

Z- transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, \underset{\uparrow}{-3}, 0, 4, -2\}$

find $X(z)$ of given sequence.

$$- \; x(n) = \{\overset{x(-2)}{5}, \overset{x(-1)}{3}, \underset{\uparrow}{\overset{x(0)}{-3}}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2) z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration Sequence $x(n) = \{\underset{\uparrow}{5}, 3, 0, 1, 2, 4\}$

Find Z- transform of $x(n)$.

$$- \; x(n) = \{\underset{\uparrow}{\overset{x(0)}{5}}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4}\}$$

$$- \; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Linearity

It states that when two or more individual discrete signals are multiplied by constants, their respective Z-transforms will also be multiplied by the same constants.

Mathematically,

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z)$$

Proof − We know that,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

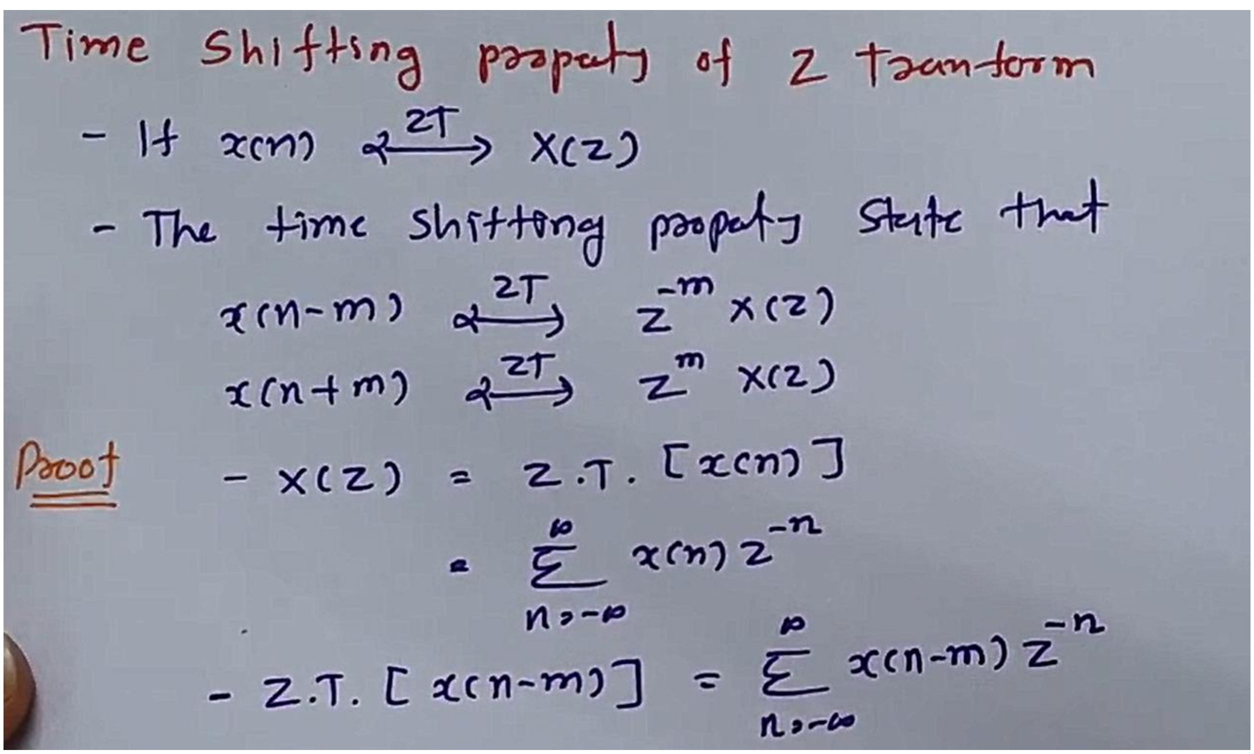
$$= \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))Z^{-n}$$

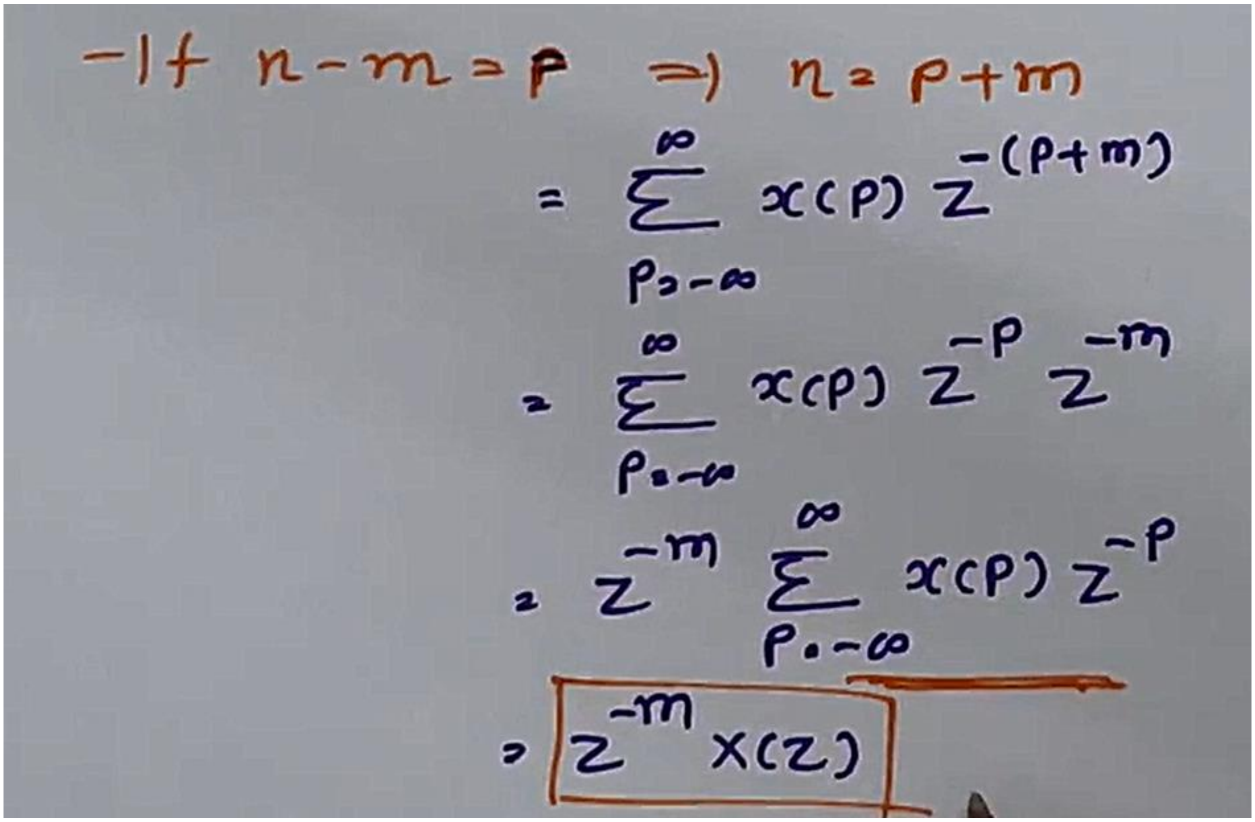
$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n)Z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)Z^{-n}$$

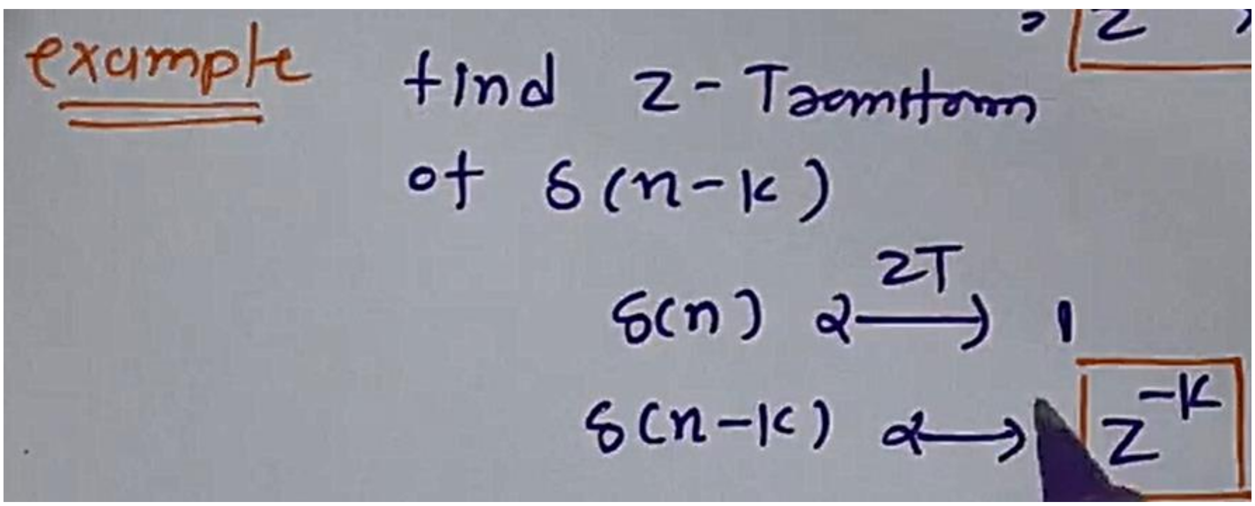
$$= a_1X_1(z) + a_2X_2(z) \qquad\qquad HenceProved$$

Here, the ROC is $ROC_1 \bigcap ROC_2$.

Time Shifting







Time shifting property depicts how the change in the time domain in the discrete signal will affect the Z-domain, which can be written as;

$$x(n-n_0) \longleftrightarrow X(Z)Z^{-n}$$

Or $$x(n-1) \longleftrightarrow Z^{-1}X(Z)$$

Proof −

Let $y(P) = X(P-K)$

$$Y(z) = \sum_{p=-\infty}^{\infty} y(p)Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} (x(p-k))Z^{-p}$$

Let s = p-k

$$= \sum_{s=-\infty}^{\infty} x(s)Z^{-(s+k)}$$

$$= \sum_{s=-\infty}^{\infty} x(s)Z^{-s}Z^{-k}$$

$$= Z^{-k}[\sum_{s=-\infty}^{\infty} x(m)Z^{-s}]$$

$$= Z^{-k}X(Z) \qquad HenceProved$$

Time Scaling

Scaling property of Z Transform

- $x(n) \xleftrightarrow{ZT} X(z)$

Then scaling property of Z Transform states that

$$a^n x(n) \xleftrightarrow{ZT} X[z/a]$$

Proof

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T.[a^n(x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$Z.T.[a^n(x(n))] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a z^{-1})^n$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{a z^{-1}}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$= \boxed{X(z/a)}$$

e.g. $x(n) = a^n u(n)$, find $X(z)$

$$u(n) \xleftrightarrow{ZT} z/z-1$$

$$a^n u(n) \xleftrightarrow{ZT} \frac{(z/a)}{\left(\frac{z}{a}\right)-1} = \frac{z}{z-a}$$

Time Scaling property tells us, what will be the Z-domain of the signal when the time is scaled in its discrete form, which can be written as;

$$a^n x(n) \longleftrightarrow X(a^{-1}Z)$$

Proof −

Let $\quad y(p) = a^p x(p)$

$$Y(P) = \sum_{p=-\infty}^{\infty} y(p) Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} a^p x(p) Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) [a^{-1} Z]^{-p}$$

$$= X(a^{-1} Z) \qquad\qquad Hence proved$$

Convolution

This depicts the change in Z-domain of the system when a convolution takes place in the discrete signal form, which can be written as −

$$x_1(n) * x_2(n) \longleftrightarrow X_1(Z).\,X_2(Z)$$

Proof −

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)]Z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)[\sum_{n}^{\infty} x_2(n-k)Z^{-n}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)[\sum_{n=-\infty}^{\infty} x_2(n-k)Z^{-(n-k)}Z^{-k}]$$

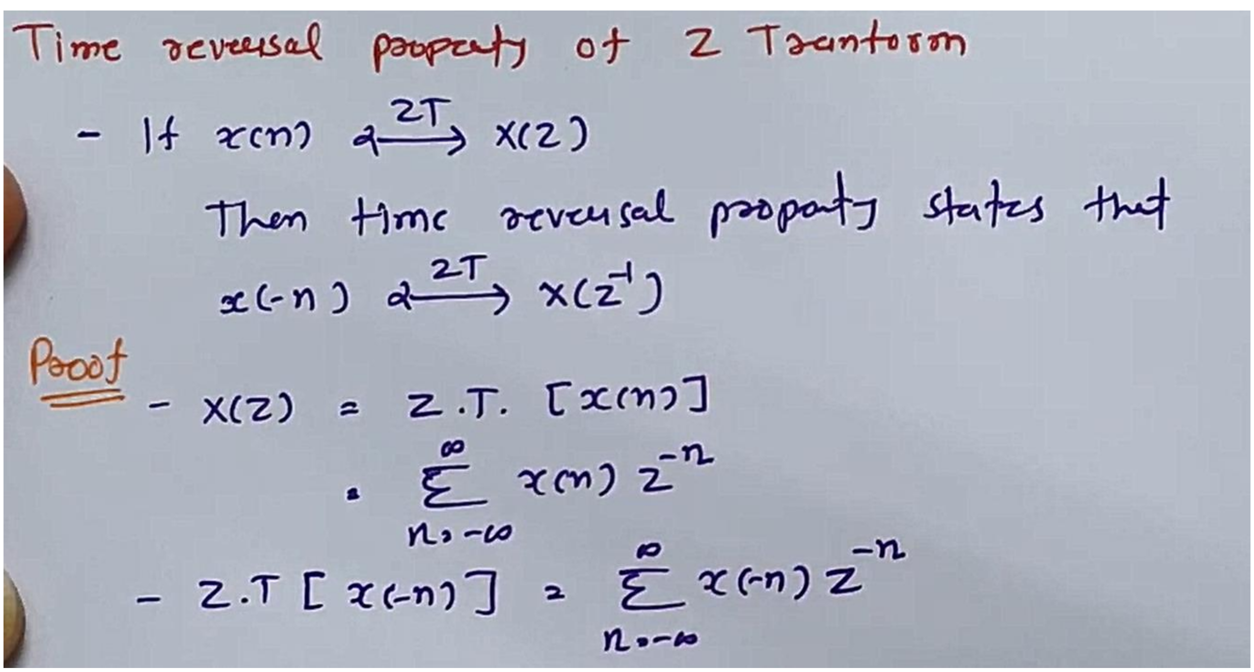
Let n-k = l, then the above equation cab be written as −

$$X(Z) = \sum_{k=-\infty}^{\infty} x_1(k)[Z^{-k}\sum_{l=-\infty}^{\infty} x_2(l)Z^{-l}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)X_2(Z)Z^{-k}$$

$$= X_2(Z)\sum_{k=-\infty}^{\infty} x_1(Z)Z^{-k}$$

$$= X_1(Z).\,X_2(Z) \qquad\qquad HenceProved$$



$$- \ Z.T[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$$

If $p = -n$

$$= \sum_{p=-\infty}^{\infty} x(p) Z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (Z^{-1})^{-p}$$

$$= X(Z^{-1})$$

e.g. $x(n) = a^{n} u(-n)$. find $X(Z)$

$$- \ x(n) = a^{n} u(-n)$$

$$= (1/a)^{-n} u(-n)$$

$$x(n) = (1/a)^{n} u(n) \xleftrightarrow{ZT} \frac{Z}{Z - 1/a}$$

$$(1/a)^{-n} u(-n) \xleftrightarrow{ZT} \frac{Z^{-1}}{Z^{-1} - 1/a} = \frac{1/Z}{1/Z - 1/a} = \boxed{\frac{a}{a-Z}}$$

Multiplication & Convolution property of Z-Transform

If $x_1(n) \xleftrightarrow{ZT} X_1(Z)$

$x_2(n) \xleftrightarrow{ZT} X_2(Z)$

- Multiplication property states that

$$x_1(n)\, x_2(n) \xleftrightarrow{ZT} x_1(Z) * x_2(Z)$$

- convolution property states that

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(Z)\, X_2(Z).$$

e.g. $u(n-1) * \delta(n)$, find Z-Transform

$$- \ x_1(n) = u(n-1) \xleftrightarrow{ZT} X_1(z) = \frac{1}{z-1}$$

$$- \ x_2(n) = \delta(n) \xleftrightarrow{ZT} X_2(z) = 1$$

$$- \ X(z) = ZT[x_1(n) * x_2(n)]$$

$$= X_1(z)\, X_2(z)$$

$$= \frac{1}{z-1}$$

e.g. $u(n-2) * \delta(n-3)$, find Z-Transform.

$$- \ x_1(n) = u(n-2) \xleftrightarrow{ZT} X_1(z) = z^{-2}\left(\frac{z}{z-1}\right) = \frac{1}{z}\left(\frac{1}{z-1}\right)$$

$$- \ x_2(n) = \delta(n-1) \xleftrightarrow{ZT} X_2(z) = z^{-1} = 1/z$$

$$- \ X(z) = ZT[x_1(n) * x_2(n)]$$

$$= X_1(z)\, X_2(z)$$

$$= \frac{1}{z}\left(\frac{1}{z-1}\right)\left(\frac{1}{z}\right)$$

$$= \frac{1}{z^2(z-1)}$$

Successive Differentiation

Successive Differentiation property shows that Z-transform will take place when we differentiate the discrete signal in time domain, with respect to time. This is shown as below.

$$\frac{dx(n)}{dn} = (1 - Z^{-1})X(Z)$$

Proof –

Consider the LHS of the equation – $\frac{dx(n)}{dn}$

$$= \frac{[x(n) - x(n-1)]}{[n - (n-1)]}$$

$$= x(n) - X(n-1)$$

$$= x(Z) - Z^{-1}x(Z)$$

$$= (1 - Z^{-1})x(Z) \qquad HenceProved$$

Multiplication in Time

It gives the change in Z-domain of the signal when multiplication takes place at discrete signal level.

$$x_1(n).\, x_2(n) \longleftrightarrow (\tfrac{1}{2\Pi j})[X1(Z) * X2(Z)]$$

Conjugation in Time

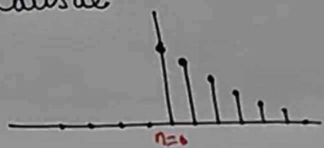
This depicts the representation of conjugated discrete signal in Z-domain.

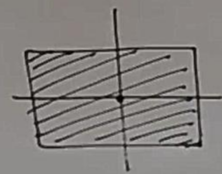
$$X^*(n) \longleftrightarrow X^*(Z^*)$$

Region of Convergence (RoC)

Finite Duration Signals

Causal

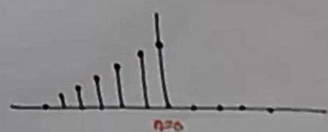


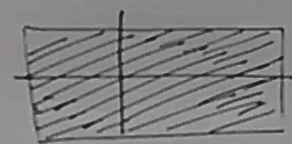


ROC

Entire z-plane except $|z| = 0$

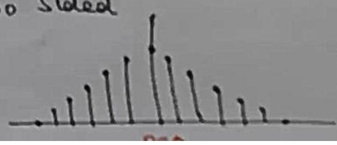
Anti Causal

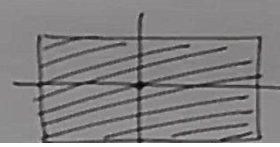




Entire z-plane except $|z| = \infty$

Two Sided

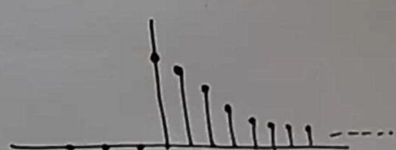


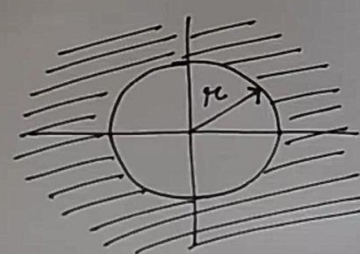


Entire z-plane except $|z| = 0$ & $|z| = \infty$

Infinite Duration Signals

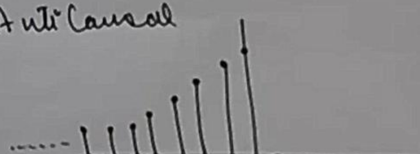
Causal

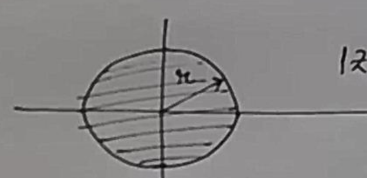




$|z| > r$

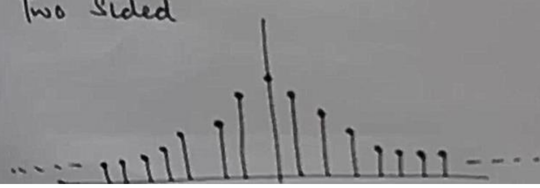
Anti Causal

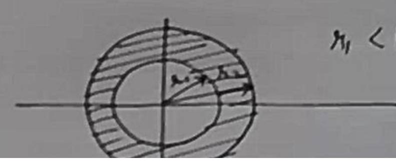




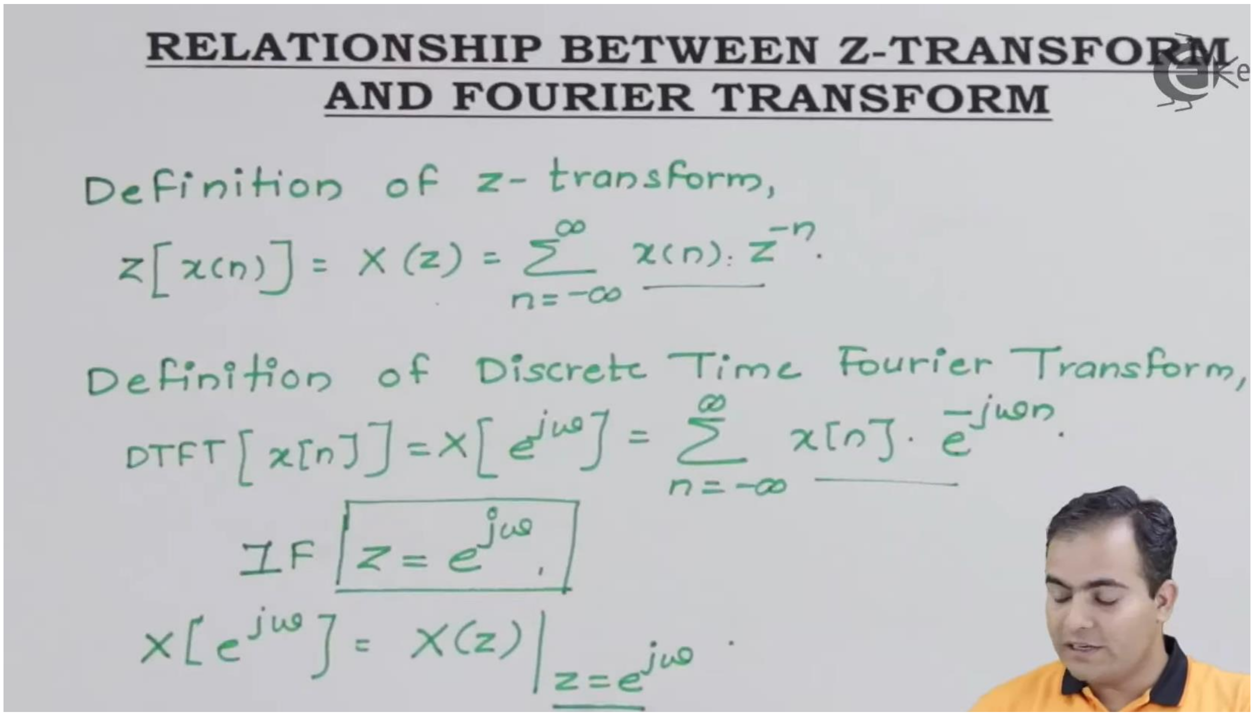
$|z| < r$

Two Sided





$r_1 < |z| < r_2$



57. Determine the z-transform of the signal $x(n) = \alpha^n u(n)$ and also the ROC and pole & zero locations of $X(z)$ in the z-plane.

Solution:

Given $x(n) = \alpha^n u(n)$

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

Using geometric series,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\therefore X(z) = \frac{1}{1-\alpha z^{-1}}; |z| > |\alpha|$$

$$= \frac{z}{z-\alpha}; |z| > |\alpha|$$

There is a pole at z = α & zero at z = 0.

ROC of Impulse function in Z-Transform

1. $\delta(n) \overset{ZT}{\longleftrightarrow} 1$

ROC: Entire Z plane, including $z=0$ & $z=\infty$

2. $\delta(n-k) \overset{ZT}{\longleftrightarrow} z^{-k}$

ROC: Entire Z plane, including $z=\infty$ & excluding $z=0$

3. $\delta(n+k) \overset{ZT}{\longleftrightarrow} z^{k}$

ROC: Entire Z plane, Including $z=0$ & excluding $z=\infty$

$\delta(n-3)$ – Entire Z plane.
Includes $z=\infty$
Exclude $z=0$.

ROC of discrete time sequence in Z Transform

1. If Sequence is purely right sided or causal, then

   ROC : Entire z plane except z = 0

   $$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \xleftrightarrow{ZT} X(z) = 1 + 2z^{-1} + 3z^{-2} + 4$$

2. If Sequence is purely Left sided or Anticausal, Then

   ROC : Entire z plane except at z = 0

   Check may be ∞

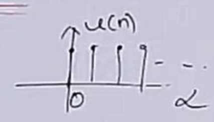
   $$x(n) = \{1, 2, 3, \underset{\uparrow}{4}\} \xleftrightarrow{ZT} X(z) = z + 2z + 3z + 4$$

3. If Sequence is a two sided, Then

   ROC : Entire z plane except at z = 0 & z = ∞

   $$x(n) = \{1, 2, \underset{\uparrow}{1}, 3, 1\} \xleftrightarrow{ZT} X(z) = 2z^{2} + 2z + 1 + 3z^{-1} + z^{-2}$$

* Z-Transform of Discrete unit step function. u(n) *



$$Z.T[u(n)]$$

$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$Z.T[u(n)] = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

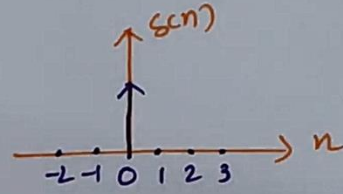
$$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^{n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\boxed{\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}}$$

Z- Transform of Unit Impulse function

- $x(n) = \delta(n)$
- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$



$$= \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

$$= \dots + \delta(-2) z^{2} + \delta(-1) z^{1} + \underline{\delta(0)} z^{0} + \delta(1) z^{-1} + \delta(2) z^{-2} + \dots$$

$$= 0 \times z^{2} + 0 \times z^{1} + 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \dots$$

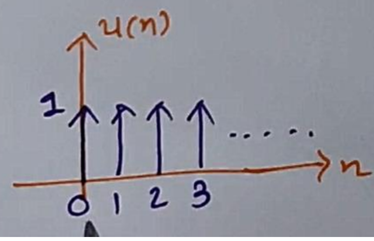
$$= 1$$

Z- Transform of Standard basic signals

- Find The Z- transform of the signals $a^{n} u(n)$ & $a^{-n} u(n)$
- $x(n) = a^{n} u(n)$ $\qquad y(n) = a^{-n} u(n)$
- $X(z) = Z.T.[x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{n} u(n)\, z^{-n}$$



0 1 2 3

$$- X(z) = \sum_{n=0}^{\infty} (a^{n})(z^{-n})$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^{n}$$

We know $\displaystyle\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

$$y(n) = a^{-n} u(n)$$

$$- Y(z) = \text{Z.T.}\,[y(n)]$$

$$= \sum_{n=-\infty}^{\infty} y(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{-n} u(n)\, z^{-n}$$

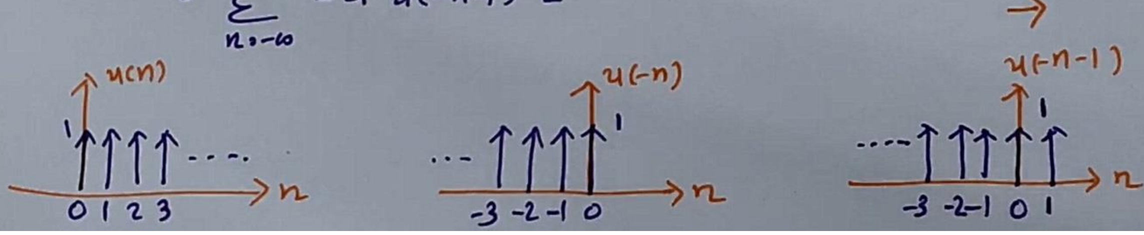
Z- Transform of standard basic signal

Find the Z- transform of the sequence $x(n) = -a^n u(-n-1)$

$$X(z) = Z.T.[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, z^{-n}$$



$$X(z) = \sum_{n=-\infty}^{1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{1} (az^{-1})^n \qquad \left[\ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\ \right]$$

- To swept limits place $n = -n$.

$$X(z) = -\sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$= -\sum_{n=1}^{\infty} (za^{-1})^n$$

$$= -\left[\sum_{n=0}^{\infty} (z a^{-1})^{n} - 1\right]$$

$$= -\left[\frac{1}{1 - z a^{-1}} - 1\right]$$

$$= -\left[\frac{a}{a - z} - 1\right]$$

$$= -\left[\frac{\cancel{a} - \cancel{a} + z}{a - z}\right]$$

$$= \boxed{\frac{z}{z - a}}$$

$$x(n) = -a^{n} u(-n-1)$$

$$a^{n}u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$-a^{n}u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$a^{-n}u(n) \xleftrightarrow{ZT} \frac{az}{az-1}$$

$$-a^{-n}u(-n-1) \xleftrightarrow{ZT} \frac{az}{az-1}$$

n

$$Y(z) = \sum_{n=0}^{\infty} a^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (az)^{-n}$$

we know $\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$

$$-Y(z) = \frac{1}{1-(az)^{-1}} = \frac{1}{1-1/az}$$

$$= \boxed{\frac{az}{az-1}}$$

↑

$y(n) = a^{-n} u(n)$

Z- transform for finite sequence

A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, \underset{\uparrow}{-3}, 0, 4, -2\}$ find $X(z)$ of given sequence.

$$\begin{array}{ccccccc} & x(-2) & x(-1) & x(0) & x(1) & x(2) & x(3) \\ - x(n) = \{ & 5, & 3, & \underset{\uparrow}{-3}, & 0, & 4, & -2 \} \end{array}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 3z^{1} - 3z^{0} + 0 \times z^{-1} + 4z^{-2} + (-2) z^{-3}$$

$$\boxed{X(z) = 5z^{2} + 3z^{1} - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence $x(n) = \{\underset{\uparrow}{5}, 3, 0, 1, 2, 4\}$ Find Z- transform of $x(n)$.

$$\begin{array}{ccccccc} & x(0) & x(1) & x(2) & x(3) & x(4) & x(5) \\ - x(n) = \{ & \underset{\uparrow}{5}, & 3, & 0, & 1, & 2, & 4 \} \end{array}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5z^{0} + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

Examples of Inverse Z transform

Find the signal $x(n)$ for which the z transform is

$$X(z) = 4z^{4} - z^{3} - 3z + 4z^{-1} + 3z^{-2}$$

- $x(n) = Z^{-1}[X(z)] = \boxed{\frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz}$

$$X(z) = \underset{x(-4)}{4}z^{4} - \underset{x(-3)}{1}z^{3} + \underset{x(-2)}{0}z^{2} - \underset{x(-1)}{3}z + \underset{x(0)}{0}z^{0} + \underset{x(1)}{4}z^{-1} + \underset{x(2)}{3}z^{-2}$$

- $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-4}^{2} x(n)\, z^{-n}$$

- $\boxed{x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}}$

If $x(z) = 2z^{2} + 3$, find $x(n)$

- $X(z) = \underset{-2}{2z^{2}} + \underset{-1}{0 \times z^{1}} + \underset{0}{3 \times z^{0}}$
- $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-2}^{0} x(n)\, z^{-n}$$

- $x(n) = \{2, 0, \underset{\uparrow}{3}\}$

If $x(z) = 3 + 2z^{-1} + 3z^{-3}$, find $x(n)$

- $x(z) = \underbrace{3 \times z^{0}}_{x(0)} + \underbrace{2}_{x(1)} \times z^{-1} + \underbrace{0}_{x(2)} \times z^{-2} + \underbrace{3}_{x(3)} \times z^{-3}$

- $x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{3} x(n)\, z^{-n}$

- $x(n) = \{\underset{\uparrow}{3}, 2, 0, 3\}$