FOURIER TRANSFORM FOR CONTINUOUS TIME APERIODIC SIGNAL

3.1 Fourier Transform for Continuous Time A-periodic Signal

Let us consider an a-periodic signal $x(t)$ with finite duration as shown is Fig-3.1. From a-periodic signal, we can create a periodic signal x_P(t) with period T_P as shown in Fig-3.2. Clearly $x_P(\mathrm{t}) = x(t)$ in the limit as $T_P \to \infty$ that is

$$x(t) = \lim_{T_p \to \infty} x_P(t) \text{ -------------------- } (1)$$

$$x_P(\mathrm{t}) = \sum_{k=-\infty}^{\infty} c_k \, e^{j2\pi k F_o t} \text{ -------------------- } (2)$$

$$C_k = \frac{1}{T_P} \int_{-\frac{T_P}{2}}^{\frac{T_P}{2}} x_P(\, t) \, e^{-j2\pi k F_o t} \mathrm{dt} \text{ -------------------- } (3)$$

$$C_k = \frac{1}{T_p} \int_0^{T_p} x(t) \, e^{-j2\pi k F_o t} \mathrm{dt} \text{ -------------------- } (4)$$

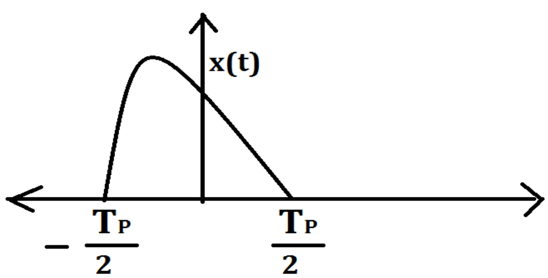


Fig-3.1 A-periodic signal $x(t)$

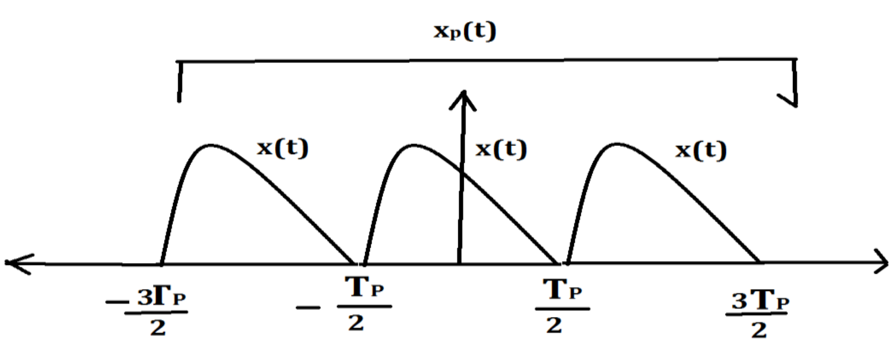


Fig-3.2: Periodic signal $x_p(t)$ constructed by three time repetition form of a-periodic signal $x(t)$ with period T_p

Since $x_P(t) = x(t)$ for $-\frac{T_P}{2} \leq t \leq \frac{T_P}{2}$ can be expressed as

$$C_k = \frac{1}{T_P} \int_{-\frac{T_P}{2}}^{\frac{T_P}{2}} x(t)\, e^{-j2\pi k F_o t}\, dt \text{ -- } (5)$$

It is also true that $x(t)$=0 for |t|> $\frac{T_P}{2}$,consequently the limits on the integral in (5) can be replaced by -∞ & ∞ Hence,

$$C_k = \frac{1}{T_P} \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi k F_0 t} dt \text{ -- } (6)$$

Let us now define a function $X(F)$, called the Fourier transform of $x(t)$ as

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi F t} dt \text{ -- } (7)$$

$X(F)$ is the function of the continuous variable F if we compare (6) and (7) then Fourier coefficients C_K an be expressed in terms of $X(F)$ as

$$C_K = \frac{1}{T_P} \text{ X}(kF_o) = \text{X}\left(\frac{K}{T_P}\right) \text{ -- } (8)$$

Those Fourier coefficient as sample of $X(F)$ taken at multiples of F_o and scaled by F_o (multiplied by $\frac{1}{T_P}$). Substitution for C_K from (7) in to (2) yields

$$x_P(t) = \frac{1}{T_P} \sum_{k=-\infty}^{\infty} X\left(\frac{K}{T_P}\right) e^{j2\pi k F_o t} \text{ --}(9)$$

If T_P is infinity then we write $\Delta F = \frac{1}{T_P}$ with substitution, equation (9) becomes as

$$x_P(t) = \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k \Delta F t} \; \Delta F \text{ -- } (10)$$

Now $\lim_{T_p \to \infty} x_P(t) = \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k \Delta F t} \; \Delta F$ ------------------------------ (11)

we get following equation as below

$x(t) = \lim_{T_p \to \infty} x_P(t) = \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k\Delta F t} \quad \Delta F$ ------------------------------ (12)

if $\Delta F \to 0$ then kΔFalso act as cautious variable (although k is integer). We are replacing kΔF by F which is act as continuous variable of frequency. So equation (6) and (7) can be written as given form

X(F) = $\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft}$ dt -- (13)

$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi Ft}$ dF -- (14)

Where X(F) is the Fourier transform of *x(t)*.

Summary 3(A): Fourier Transform for Continuous Time Aperiodic Signal

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi Ft} dF$$

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft} \mathrm{dt}$$

3.2 Energy Density and Power Density Spectrum for Continuous Time Aperiodic Signal

A continuous time aperiodic signal has infinite energy which is given as

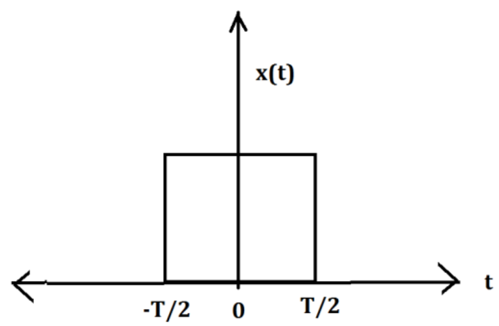
$E_X = \int_{-\infty}^{+\infty} [x(t)]^2 dt = \int_{-\infty}^{+\infty} [X(F)]^2 dF$-- (15)

The $[X(F)]^2$ is represented the energy density spectrum of continuous time aperiodic signal

In this case, we can't calculate power density spectrum of aperiodic signal due to unknown period.

Example-05: Determine the Fourier transform and energy density spectrum of rectangular pulse define as

$$x(t) = \begin{cases} 0, & \frac{T}{2} < [t] \\ 1, & \frac{T}{2} \geq [t] \end{cases}$$



Solution:

We know that for Continuous Time A-periodic Signal

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

$$X(F) = \int_{-\infty}^{-T/2} x(t)\, e^{-j2\pi Ft}\, dt + \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi Ft}\, dt + \int_{T/2}^{\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

$$X(F) = \int_{-\infty}^{-T/2} 0\, e^{-j2\pi Ft}\, dt + \int_{-T/2}^{T/2} 1\, e^{-j2\pi Ft}\, dt + \int_{T/2}^{\infty} 0\, e^{-j2\pi Ft}\, dt$$

$$X(F) = \int_{-T/2}^{T/2} 1\, e^{-j2\pi Ft}\, dt$$

$$X(F) = \frac{\sin(\pi FT)}{\pi F}$$

$$X(F) = T\,\frac{\sin(\pi FT)}{\pi FT}$$

Or $X(F) = T sinc(\pi FT)$

Fourier transform for given rectangular pulse is $T\,\frac{\sin(\pi FT)}{\pi FT}$

So energy density spectrum for given rectangular pulse is $X(F)^2 = [T\,\frac{\sin(\pi FT)}{\pi FT}]^2$

The graphical representation of energy density spectrum for given rectangular pulse as shown in fig-(3.3)

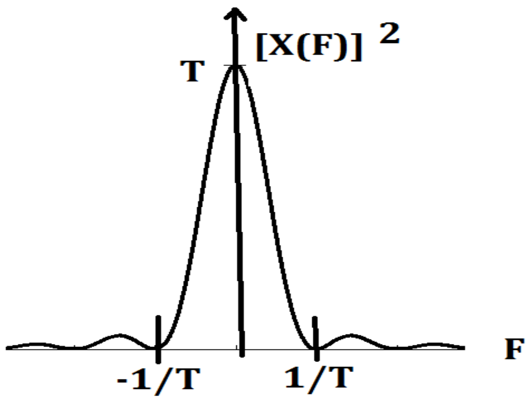


Fig-3.3: Energy density spectrum (EDS) for given rectangular pulse

Any continuous-time signal $x(t)$ that has finite "energy", i.e.,

$$\int_{-\infty}^{\infty} x^2(t)dt < +\infty,$$

can be represented in the frequency domain via the **Fourier transform**:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

We will also write

$$X(\omega) = \mathcal{F}[x(t)]$$

to denote the fact that $X(\omega)$ is the Fourier transform of $x(t)$.

Properties of the Fourier transform

The Fourier transform has many useful properties that make calculations easier and also help thinking about the structure of signals and the action of systems on signals.

- ✓ **Linearity:**

The Fourier transform is **linear**: if

$$x_1(t) \leftrightarrow X_1(\omega) \qquad \text{and} \qquad x_2(t) \leftrightarrow X_2(\omega),$$

then

$$c_1 x_1(t) + c_2 x_2(t) \leftrightarrow c_1 X_1(\omega) + c_2 X_2(\omega)$$

for any two numbers c_1 and c_2.

Proof: obvious –

$$\mathcal{F}\left[c_1 x_1(t) + c_2 x_2(t)\right] = \int_{-\infty}^{\infty} \left[c_1 x_1(t) + c_2 x_2(t)\right] e^{-j\omega t} dt$$

$$= c_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + c_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= c_1 X_1(\omega) + c_2 X_2(\omega)$$

Linearity

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Proof: obvious –

$$\mathcal{F}\left[c_1 x_1(t) + c_2 x_2(t)\right] = \int_{-\infty}^{\infty} \left[c_1 x_1(t) + c_2 x_2(t)\right] e^{-j\omega t} dt$$

$$= c_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + c_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= c_1 X_1(\omega) + c_2 X_2(\omega)$$

Time Shifting:

If $x(t) \leftrightarrow X(\omega)$, then $x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$ for any constant c.

Proof:

$$\begin{aligned}
\mathcal{F}\left[x(t-c)\right] &= \int_{-\infty}^{\infty} x(t-c)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega(t+c)}dt \\
&= e^{-j\omega c}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= X(\omega)e^{-j\omega c}.
\end{aligned}$$

- ✓ **Frequency Shifting:**

If $x(t) \leftrightarrow X(\omega)$, then $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$ for any real ω_0.

Proof:

$$\begin{aligned}
\mathcal{F}\left[x(t)e^{j\omega_0 t}\right] &= \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\
&= X(\omega-\omega_0).
\end{aligned}$$

Multiplication by a complex exponential

If $x(t) \leftrightarrow X(\omega)$, then $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$ for any real ω_0.

Proof:

$$
\begin{aligned}
\mathcal{F}\left[x(t)e^{j\omega_0 t}\right] &= \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\
&= X(\omega - \omega_0).
\end{aligned}
$$

Time Reversal

$$\boxed{f(-t) \xleftrightarrow{\mathcal{F}} F(-j\omega)}$$

Pf)

$$\mathcal{F}[f(-t)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt \quad = \int_{t=-\infty}^{t=\infty} f(-t)e^{-}$$

$$= \int_{-t=-\infty}^{-t=\infty} f(t)e^{j\omega t}d(-t) \quad = \int_{-t=-\infty}^{-t=\infty} f(t$$

$$= -\int_{t=\infty}^{t=-\infty} f(t)e^{j\omega t}dt \quad = \int_{t=-\infty}^{t=\infty} f(t)e^{j\omega}$$

$$= \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt \quad = F(-j\omega)$$

✓ **The Modulation Theorem:**

If $x(t) \leftrightarrow X(\omega)$, then $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$.

Proof: use linearity and the last property to get

$$\begin{aligned}\mathcal{F}\left[x(t)\cos(\omega_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}x(t)\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)\right] \\ &= \frac{1}{2}\mathcal{F}\left[x(t)e^{j\omega_0 t}\right]+\frac{1}{2}\mathcal{F}\left[x(t)e^{-j\omega_0 t}\right] \\ &= \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right].\end{aligned}$$

Multiplication by a cosine

If $x(t) \leftrightarrow X(\omega)$, then $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$.

Proof: use linearity and the last property to get

$$\begin{aligned}\mathcal{F}\left[x(t)\cos(\omega_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}x(t)\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)\right] \\ &= \frac{1}{2}\mathcal{F}\left[x(t)e^{j\omega_0 t}\right]+\frac{1}{2}\mathcal{F}\left[x(t)e^{-j\omega_0 t}\right] \\ &= \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right].\end{aligned}$$

If $x(t) \leftrightarrow X(\omega)$ and $v(t) \leftrightarrow V(\omega)$, then

$$x(t) \star v(t) \leftrightarrow X(\omega)V(\omega)$$

Proof:

$$\begin{aligned}
\mathcal{F}\left[x(t) \star v(t)\right] &= \int_{-\infty}^{\infty} [x(t) \star v(t)] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\lambda) v(t-\lambda) d\lambda \right) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} x(\lambda) \underbrace{\left(\int_{-\infty}^{\infty} v(t-\lambda) e^{-j\omega t} dt \right)}_{\mathcal{F}[v(t-\lambda)]} d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda) V(\omega) e^{-j\omega\lambda} d\lambda \\
&= V(\omega) \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\
&= X(\omega)V(\omega).
\end{aligned}$$

Let $x(t)$ and $v(t)$ be real-valued signals. Then

$$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)v(t)dt &= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)e^{j\omega t}d\omega\right)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)\left(\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)X(-\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega,
\end{aligned}$$

where we used the fact that, since $x(t)$ is real,

$$\overline{X(\omega)} = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-\omega).$$

An important consequence of Parseval's theorem is that

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

In other words, signal energy can be computed both in time domain and in frequency domain (up to a factor of $1/2\pi$).

Fourier Transform--Delta Function

The Fourier transform of the delta function is given by

$$\begin{aligned}
\mathcal{F}_x\left[\delta(x-x_0)\right](k) &= \int_{-\infty}^{\infty} \delta(x-x_0)\, e^{-2\pi i k x}\, dx \\
&= e^{-2\pi i k x_0}.
\end{aligned}$$

Fourier Transform--Cosine

$$\mathcal{F}_x[\cos(2\pi k_0 x)](k) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\left(\frac{e^{2\pi i k_0 x} + e^{-2\pi i k_0 x}}{2}\right) d x$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[e^{-2\pi i (k-k_0) x} + e^{-2\pi i (k+k_0) x}\right] d x$$
$$= \tfrac{1}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right],$$

where $\delta(x)$ is the delta function.

Fourier Transform--Sine

$$\mathcal{F}_x[\sin(2\pi k_0 x)](k) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\left(\frac{e^{2\pi i k_0 x} - e^{-2\pi i k_0 x}}{2 i}\right) d x$$
$$= \tfrac{1}{2}\, i \int_{-\infty}^{\infty}\left[-e^{-2\pi i (k-k_0) x} + e^{-2\pi i (k+k_0) x}\right] d x$$
$$= \tfrac{1}{2}\, i\left[\delta(k+k_0) - \delta(k-k_0)\right],$$

where $\delta(x)$ is the delta function.

The Fourier Transform of the Box Function

Continuing the study of the Fourier Transform, we'll look at the box function (also called a square pulse or square wave):

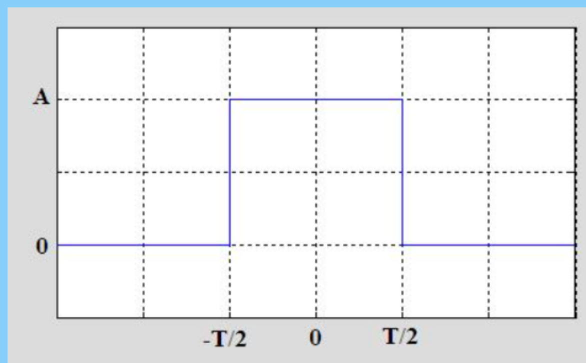

Figure 1. The box function.

In Figure 1, the function g(t) has amplitude of *A*, and extends from t=-T/2 to t=T/2. For |t|>T/2, g(t)=0.

Using the definition of the Fourier Transform (Equation [1] above), the integral is evaluated:

$$\begin{aligned}\mathcal{F}\{g(t)\} = G(f) &= \int_{-\infty}^{\infty} g(t)e^{-2\pi ift}\,dt \\ &= \int_{-T/2}^{T/2} Ae^{-2\pi ift}\,dt = \frac{A}{-2\pi if}\left[e^{-2\pi ift}\Big|_{-T/2}^{T/2}\right] \\ &= \frac{A}{-2\pi if}\left[e^{-\pi ifT} - e^{\pi ifT}\right] = \frac{AT}{\pi fT}\left[\frac{e^{\pi ifT} - e^{-\pi ifT}}{2i}\right] \\ &= \frac{AT}{\pi fT}\sin(\pi fT) = AT\left[\operatorname{sinc}(fT)\right]\end{aligned} \qquad \textbf{[Equation 4]}$$

The solution, G(f), is often written as the sinc function, which is defined as:

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \qquad \textbf{[Equation 5]}$$

The Fourier Transform of g(t) is G(f),and is plotted in Figure 2 using the result of equation [2].

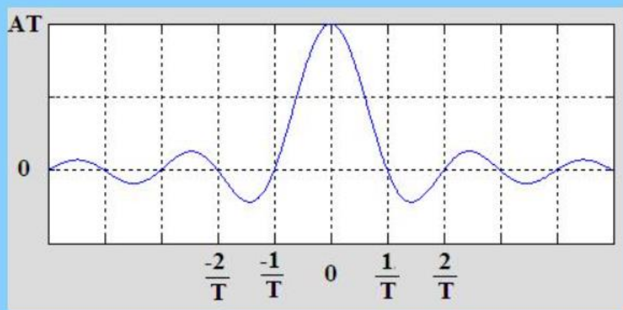

Figure 2. The sinc function is the Fourier Transform of the box function.