Fourier analysis

Mainly two type signals are present in the world such as analog or continuous signal and digital or discrete signal. Both signals have two forms such as periodic and aperiodic. Fourier analysis is an excellent method to analysis every type of signal either continuous time or discrete time signal.

Fourier analysis for following case such as:

1. Continuous Time Periodic Signal
2. Discrete Time Periodic Signal
3. Continuous Time Aperiodic Signal
4. Discrete Time Aperiodic Signal

1. FOURIER SERIES FOR CONTINUOUS TIME PERIODIC

1.1 Fourier series for Continuous Time Periodic Signal in Trigonometric Form

Fourier given the concept all real periodic signal consists of dc signal which frequency is zero and one to infinity number of sinusoidal signal. Each sinusoidal wave has different amplitude and different frequency. In real world, sinusoidal signal means sine wave signal and cosine wave signal. This statement can be represented by mathematical form which is called Fourier series as given below.

$f(t)= a_0 +(a_1\cos w_0t + a_2 \cos 2w_0t + a_3\cos 3w_0t + \text{---}\infty)$

$+ (b_1\sin w_0t + b_2\sin 2w_0t+ b_3\sin 3w_0t + b_4\sin 4w_0t + \text{-------------------------------}\infty)$

Or $f(t)=a_0 + \sum_{n=1}^{\infty}(a_n\cos nw_0t + b_n\sin nw_0t)$---(1)

Or $f(\theta)=a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$---(2)

Calculation to find out the value of a_0:

We know that from equation –(2),

$f(\theta)=a_0 + \sum_{n=1}^{+\infty}(a_n cosn\theta + b_n sinn\theta)$

Or $\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} a_0 d\theta + \sum_{n=1}^{+\infty}(a_n\int_0^{2\pi} cosn\theta\, d\theta + b_n\int_0^{2\pi} sin\, n\theta\, d\theta)$

Or $\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} a_0 d\theta + 0 + 0$ *[used geometrical function in **Appendix –A]***

Or $\int_0^{2\pi} f(\theta)d\theta = a_0 2\pi$

Or, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ -- *(3)*

Calculation to find out the value of a_n:

We know that from initial (first) equation

$f(\theta)=a_0 + a_1 \cos\theta + a_2 \cos2\theta +$ -- $+a_n cosn\theta$

$+b_1 \sin\theta + b_2 \sin2\theta +$ -- $+b_n sinn\theta$

Or $\int_0^{2\pi} f(\theta) \cos n\theta\, d\theta = \int_0^{2\pi} a_0 \cos n\theta\, d\theta + a_1\int_0^{2\pi} \cos\theta \cos n\theta\, d\theta$

$+a_2\int_0^{2\pi} \cos2\theta \cos n\theta\, d\theta$ + -- +

$a_n\int_0^{2\pi} cosn\theta cosn\theta\, d\theta$

$+ b_1\int_0^{2\pi} cosn\theta\, sin\theta\, d\theta + b_2\int_0^{2\pi} sin2\theta\, cos2\theta\, d\theta +$ ------------------------------------- +

$b_n\int_0^{2\pi} \sin n\theta \cos n\theta\, d\theta$ -- (4)

(We know that from geometrical function in **Appendix –A,** $\int_0^{2\pi}$ cos nθ cos mθ dθ=0 [if m ≠ n] *and* $\int_0^{2\pi}$ cos nθ cos mθ dθ=π [if m=n] $\int_0^{2\pi}$ sin nθ cos mθ dθ=0 [either m ≠ n nor m=n]) Now apply the above this geometrical function in equation-(4) then we get

$Or \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = 0+0+0+$ ------------------$a_n\int_0^{2\pi} \cos n\theta \cos n\theta \, d\theta$

$+ 0 + 0 + 0 +$---+

$Or \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = a_n\int_0^{2\pi} \cos n\theta \cos n\theta \, d\theta$

$Or \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = a_n\pi$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$---(5)

Calculation to find out the value of bn:

We know that from initial (first) equation

$f(\theta) = a_0 + a_1 \cos \theta + a_2 \cos 2\theta +$ ---- --$+a_n \cos n\theta$

$+b_1 \sin \theta + b_2 \sin 2\theta +$ ---$+b_n \sin n\theta$

Now we are multiply both side of above equation by sinnɵ and apply integration on both side. So we get following equation

$Or\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = \int_0^{2\pi} a_0 \sin n\theta \, d\theta + a_1\int_0^{2\pi} \cos\theta \sin n\theta \, d\theta$

$+a_2\int_0^{2\pi} \cos 2\theta \sin n\theta \, d\theta +$ ------------- $+a_n\int_0^{2\pi} \cos n\theta \sin \theta \, d\theta$

$+ b_1\int_0^{2\pi} \cos\theta \sin n\theta \, d\theta + b_2\int_0^{2\pi} \sin 2\theta \sin n\theta \, d\theta +$ -----------+

$b_n\int_0^{2\pi} \sin n\theta \sin n\theta \, d\theta +$--- (6)

(We know that from geometrical function $\int_0^{2\pi}$ sinnɵcosmɵ dɵ=0 [if m ≠ n] *and* $\int_0^{2\pi}$ sinnɵsinmɵ dɵ=π [if m=n] $\int_0^{2\pi}$ sinnɵcosmɵ dɵ=0 [either m ≠ n nor m=n]). Now apply the above this geometrical function in equation-(6) then we get

$Or \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = 0+0+0+0+0+0+$------------------------------$+ b_n\int_0^{2\pi} \sin^2 n\theta d\theta$

$Or \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = b_n\pi$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$--(7)

Summary 1.1(A): Fourier series for Continuous Time Periodic Signal

$$f(\theta)=a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

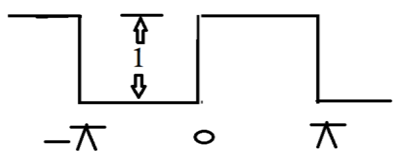
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

Example-01: Find the mathematical equation for given signal using Fourier series as trigonometric form.

$$f(x)=\begin{cases} o & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$



Solution:

We know that,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$=\frac{1}{2\pi}[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} 1\, dx$$

$$=\frac{1}{2}$$

$$Again \quad a_n = \frac{1}{\pi}\int_{\_\pi}^{\pi} f(x)\cos nx\, dx$$

[sin nπ=0 for n=1,2…..] [cos nπ=-1 for n=1,3,5…….. and =1 for n=2,4,….]

$$=\frac{1}{\pi}[\int_{\_\pi}^{0} 0\cos nx\, dx + \int_0^{\pi} 1\cos nx\, dx]$$

$$= 0$$

$$Aging\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} 0 \sin nx\, dx + \int_{0}^{\pi} 1 \sin nx\, dx$$

$$=\frac{1}{\pi}[\frac{-conx}{n}]_0^{\pi}$$

$$\frac{1}{n\pi}[1\text{-}cos\ n\pi]$$

$$= \begin{cases} 0 & for\ even\ \ n \\ \frac{2}{n\pi} & for\ odd\ n \end{cases}$$

Again we know that

$$f(x)=a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

Putting the values of a_0, a_n and b_n in above equation and we get below expression

$$f(x)=\frac{1}{2} + \sum_{n=1}^{\infty} 0cosnx \ + \ \sum_{n=1(odd)}^{\infty}\frac{2}{n\pi}\sin nx \ = \frac{1}{2} + 0 + \sum_{n=1(odd)}^{\infty}\frac{2}{n\pi}\sin nx$$

$$f(x)=\frac{1}{2} + \frac{2}{\pi}(\ \frac{sinx}{1} + \frac{sin3x}{3} + \frac{sin5x}{5} + \frac{sin7x}{7} + \text{---} + \infty)$$

The above this equation represent mathematical equation for example 01

1.2 Conversion of Trigonometric Form Fourier series in to Complex Form Fourier series for Continuous Time Periodic Signal

The article (1.1) represents trigonometrically form Fourier series but many books are explained the Fourier series in complex form because it is easily represent and analysis the different type of signals. It is remember that the complex form not complicated form but also simple form compare to others form such as trigonometric form, polar form etc. So we are convert trigonometrically form Fourier series in to a complex form Fourier series. However the trigonometrically form Fourier series and complex form Fourier series is same case but only representation form is different. Fourier series for periodic function as given by

$f(\theta)=a_0+\sum_{n=1}^{\infty}(a_n \cos\theta + b_n \sin\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$ ---------(8)

Or $f(\theta)=a_0+\sum_{n=1}^{\infty} a_n (e^{jn\theta}+e^{-jn\theta})/2 - j\sum_{n=1}^{\infty} b_n (e^{jn\theta}-e^{-jn\theta})/2$ -------------------------- (9)

[By using Appendix -(A)]

Or $f(\theta)=a_0+\sum_{n=1}^{\infty} e^{jn\theta}(a_n-jb_n)/2 + \sum_{n=1}^{\infty} e^{-jn\theta}(a_n+jb_n)/2$ ------------------------------- (10)

Or $f(\theta)= I_0 + I_1 + I_2$ --- (11)

Where $I_0 = a_0$

$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ **[Using Summery 1.1(A), page-5]**

$= \lim_{n\to 0} \frac{1}{2\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta\}$

$= \frac{1}{2}\lim_{n\to 0} \frac{1}{\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta\}$

$= \frac{1}{2}\sum_{n=0}^{0} \frac{1}{\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta\}$

$= \frac{1}{2}\sum_{n=0}^{0} e^{jn\theta}\frac{1}{\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta\}$

$= \frac{1}{2}\sum_{n=0}^{0} e^{jn\theta}\ \{a_n - jb_n\}$ **[Using Summery 1.1(A), page-5]**

Or $I_0 = \sum_{n=0}^{0} e^{jn\theta} \frac{\{a_n - jb_n\}}{2}$

$I_1 = \sum_{n=1}^{\infty} e^{jn\theta} \frac{\{a_n - jb_n\}}{2}$

$I_2 = \sum_{n=1}^{\infty} e^{-jn\theta} \frac{\{a_n + jb_n\}}{2}$

$= \sum_{n=1}^{\infty} e^{-jn\theta} \frac{1}{2}\{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta + j\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta\}$

[**Using Summery 1.1(A), page-5]**

$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2}\{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(-n\theta) d\theta + j\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(-n\theta) d\theta\}$

$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2}\{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta) d\theta + j\frac{1}{\pi}\int_0^{2\pi} -f(\theta)\sin(n\theta) d\theta\}$

$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2}\{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta) d\theta - j\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta) d\theta\}$

$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2}\{a_n - jb_n\}$ [**Using Summery 1.1(A), page-5]**

$I_2 = \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{\{a_n - jb_n\}}{2}$

Now, the value of I_0, I_1 and I_2 put in equation (11) and we get following expression

Or $f(\theta) = \sum_{n=0}^{0} e^{jn\theta}(a_n - jb_n)/2 + \sum_{n=1}^{\infty} e^{jn\theta}(a_n - b_n)/2 + \sum_{n=-1}^{-\infty} e^{jn\theta}(a_n - jb_n)/2$

Or $f(\theta) = \sum_{n=-\infty}^{-\infty} e^{jn\theta}(a_n - jb_n)/2$

Or $f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$ -- (12)

Or $f(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t}$

Or $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi F0t}$ $[\omega_0 = 2\pi F_0]$ -- (13)

Or $f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk2\pi F0t}$ [just n replace by k]---------------------------------- (14)

Where $C_n = (a_n - jb_n)/2$

$$C_n = 1/2\{1/\pi\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j1/\pi \int_0^{2\pi} f(\theta)\sin n\theta d\theta\}$$

$$C_n = 1/2\pi \int_0^{2\pi} f(\theta)\, e^{-jn\theta} d\theta \text{ -- (15)}$$

$$C_n = 1/T\int_0^{T} f(t)\, e^{-jn2\pi F0t}dt \quad [w_0=2\pi F_0 \text{ and } F_0=1/T] \text{ --------------------------- (16)}$$

$$C_k = 1/T\int_0^{T} f(t)\, e^{-jk2\pi F0t}dt \quad [\text{just n replace by k}] \text{ ------------------------------ (17)}$$

Summary 1.1(B): Fourier series for Continuous Time Periodic Signal

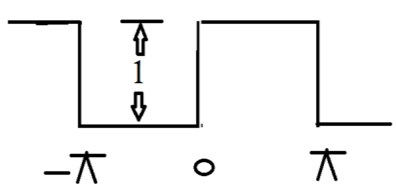
$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\theta} \quad \text{and} \quad C_n = 1/2\pi \int_0^{2\pi} f(\theta)\, e^{-jn\theta} d\theta$$

$$\text{Or } f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn2\pi F0t} \quad \text{and} \quad C_n = 1/T\int_0^{T} f(t)\, e^{-jn2\pi F0t}dt$$

$$\text{Or } f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk2\pi F0t} \quad \text{and} \quad C_k = 1/T\int_0^{T} f(t)\, e^{-jk2\pi F0t}dt$$

Example-02: Find the mathematical equation for given signal using Fourier series as complex form.

$$f(x) = \begin{cases} o & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$



Solution:

We know that complex form Fourier series for continuous time periodic signal is given by following expression

$$f(x) = \sum_{n=-\infty}^{n=\infty} C_n\, e^{inx} \quad [x=\theta]$$ **[See summary 1.1(B), page-9]**

Where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \int_{0}^{\pi} f(x)e^{-inx}\,dx\right]$$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\, e^{-inx}\,dx + \int_{0}^{\pi} 1\, e^{-inx}\,dx \quad (*)\right.$$

$$C_n = -\frac{1}{2in\pi}\left(e^{-in\pi} - 1\right)$$

If n is odd then $C_n = \frac{1}{in\pi}$

If n is even then $C_n = 0 \quad [n \neq 0]$

(*) If n=0 then $C_n = \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{2}$

Putting the value of C_n in equation (12) and we get

$$\Rightarrow f(x) = \frac{1}{2} + \frac{1}{i\pi}\left(\frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \frac{e^{i5x}}{5} + \ldots\ldots\right)$$
$$+ \frac{1}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-i3x}}{-3} + \frac{e^{-i5x}}{-5} + \ldots\ldots\right)$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{1}{1}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) + \frac{1}{3}\left(\frac{e^{3ix}-e^{-3ix}}{2i}\right) + \frac{1}{5}\left(\frac{e^{5ix}-e^{-5ix}}{2i}\right) + \ldots\ldots\right]$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \text{-----------}\right)$$

The above this equation represent mathematical equation for example-02

NOTE: Example-01and Example-02 are same problem. Example-01 solved by using of trigonometrical form Fourier series and Example-02 solved by using of complex form Fourier series but we get same result. So it is clear that the trigonometrically form Fourier series and complex form Fourier series is same case but only representation form is different.