6.52. Show that if $x[n]$ is real, then its DFT $X[k]$ satisfies the relation

$$X[N-k] = X^*[k] \tag{6.204}$$

where $*$ denotes the complex conjugate.

From Eq. (6.92)

$$X[N-k] = \sum_{n=0}^{N-1} x[n] W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)(N-k)n}$$

Now $$e^{-j(2\pi/N)(N-k)n} = e^{-j2\pi n} e^{j(2\pi/N)kn} = e^{j(2\pi/N)kn}$$

Hence, if $x[n]$ is real, then $x^*[n] = x[n]$ and

$$X[N-k] = \sum_{n=0}^{N-1} x[n] e^{j(2\pi/N)kn} = \left[\sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}\right]^* = X^*[k]$$

6.53. Show that

$$x[n] = \text{IDFT}\{X[k]\} = \frac{1}{N}[\text{DFT}\{X^*[k]\}]^* \tag{6.205}$$

where $*$ denotes the complex conjugate and

$$X[k] = \text{DFT}\{x[n]\}$$

We can write Eq. (6.94) as

$$x[n] = \frac{1}{N}\left[\sum_{n=0}^{N-1} X[k] e^{j(2\pi/N)kn}\right] = \frac{1}{N}\left[\sum_{n=0}^{N-1} X^*[k] e^{-j(2\pi/N)nk}\right]^*$$

Noting that the term in brackets in the last term is the DFT of $X^*[k]$, we get

$$x[n] = \text{IDFT}\{X[k]\} = \frac{1}{N}[\text{DFT}\{X^*[k]\}]^*$$

which shows that the same algorithm used to evaluate the DFT can be used to evaluate the IDFT.

Radix-2 DIT-FFT Algorithm:

DIT → Decimation in Time

FFT → Fast Fourier Transform.

$x(n) \rightarrow$ lenth N

$x(n) = \{x(0), x(1), x(2), x(3) \ldots . x(N-2), x(N-1)\}$

even indexed seq. : $\{x(0), x(2), x(4) \ldots . x(N-2)\}$

odd indexed seq : $\{x(1), x(3), x(5) \ldots . x(N-1)\}$

W.K.T. N-Point DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn} \quad ; \quad 0 \leq K \leq N-1 \longrightarrow ①$$

... even, odd seq.

eqn ① → Decimation

$$X(K) = \sum_{\substack{n=0 \\ n \rightarrow \text{even}}}^{N-2} x(n) W_N^{Kn} + \sum_{\substack{n=1 \\ n \rightarrow \text{odd}}}^{N-1} x(n) W_N^{Kn} \longrightarrow ②$$

$2r = N-2 \Rightarrow r = \frac{N}{2}-1$

$2r+1 = 1$, $2r = 0$, $r = 0$

$2r+1 = N-1$, $2r = N-2$, $r = \frac{N}{2}-1$

$2r = 0$, $r = 0$

Put $n = 2r$ in 1^{st} term, $n = 2r+1$ in 2^{nd} term.

$$X(K) = \sum_{r=0}^{N/2-1} \underbrace{x(2r)}_{g(r)} W_N^{2Kr} + \sum_{r=0}^{N/2-1} \underbrace{x(2r+1)}_{h(r)} W_N^{K(2r+1)} \longrightarrow ③$$

$$X(K) = \sum_{r=0}^{N/2-1} g(r) W_N^{2Kr} + \sum_{r=0}^{N/2-1} h(r) \boxed{W_N^{K(2r+1)}}$$

$W_N^{2Kr} \cdot W_N^{K}$

$$\because W_N = e^{\frac{-j2\pi}{N}} \Rightarrow W_N^2 = e^{\frac{-j2\pi \cdot 2}{N}} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

Rearrange,

$$X(K) = \underbrace{\sum_{r=0}^{N/2-1} g(r) W_{N/2}^{Kr}}_{N/2 \text{ DFT eqn even}} + W_N^K \underbrace{\sum_{r=0}^{N/2-1} h(r) W_{N/2}^{Kr}}_{N/2 \text{ DFT odd}} \rightarrow (4)$$

$$X(K) = \underline{G(K)} + W_N^K \underline{H(K)} \; ; \quad 0 \le K \le \frac{N}{2} - 1 \rightarrow (5)$$

$G(K)$ & $H(K) \rightarrow$ periodic with period $\frac{N}{2}$

$$X(K) = \underset{G(K-4)}{G\left(K - \frac{N}{2}\right)} + W_N^K \overset{H(K-4)}{H\left(K - \frac{N}{2}\right)} \; ; \quad \frac{N}{2} \le K \le N-1 \rightarrow (6)$$

<u>Ex</u>:- $N = 8$ $\quad \therefore K \rightarrow 0$ to 7

$K = 0$ to $3 \rightarrow (5)$ $\quad K = 4$ to 7 (6)

(5) $\Rightarrow$

$K=0; \; X(0) = G(0) + W_8^0 H(0)$

$K=1; \; X(1) = G(1) + W_8^1 H(1)$

$K=2; \; X(2) = G(2) + W_8^2 H(2)$

$K=3; \; X(3) = G(3) + W_8^3 H(3)$

(6) $\Rightarrow$

$K=4; \; X(4) = G(0) + W_8^4 H(0)$

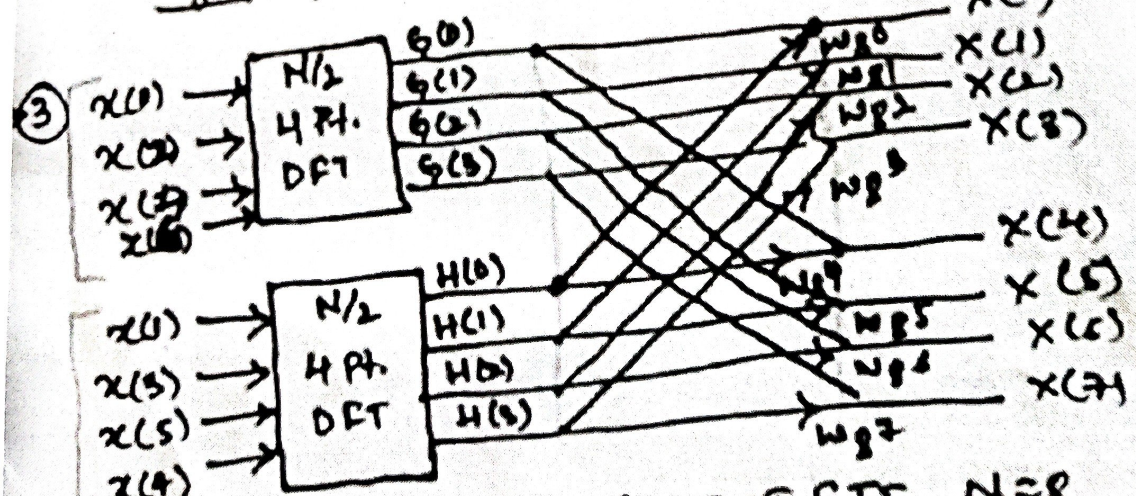
$K=5; \; X(5) = G(1) + W_8^5 H(1)$

$K=6; \; X(6) = G(2) + W_8^6 H(2)$

$K=7; \; X(7) = G(3) + W_8^7 H(3)$

$K=3; \; X(3) =$

<u>Signal flow graph</u>



First stage in DIT FFT $\underline{N=8}$

$G(K)$ & $H(K) \rightarrow \frac{N}{2}$ point

combination of two $\frac{N}{4}$ points. (8)

$$G(K) = \sum_{r=0}^{N/2-1} g(r)\, W_{N/2}^{Kr} \rightarrow (7)$$

$2l = \frac{N}{2}-2 \Rightarrow l = \frac{N}{4}-1$ $\quad$ $2l+1 = \frac{N}{2}-1 \Rightarrow l = \frac{N}{4}-1$

$$G(K) = \sum_{r=0}^{N/2-2} g(r)\, W_{N/2}^{Kr} + \sum_{r=1}^{N/2-1} g(r)\, W_{\frac{N}{2}}^{Kr} \quad (9) \Rightarrow$$

$2l = 0 \Rightarrow l = 0$ $\quad$ $2l+1 = 1 \Rightarrow l = 0$

$$g(r) = \{ g(0), g(1), g(2) \dots g(\tfrac{N}{2}-2), g(\tfrac{N}{2}-1) \}$$

Put $r = 2l$ in 1st term, $r = 2l+1$ in 2nd (7)

$$G(K) = \sum_{l=0}^{N/4-1} \underbrace{g(l)}_{a(l)}\, W_{N/2}^{2Kl} + \sum_{l=0}^{N/4-1} \underbrace{g(2l+1)}_{b(l)} \underbrace{\boxed{W_{N/2}^{K(2l+1)}}}_{W_{N/2}^{2Kl} \cdot W_{N/2}^{K}}$$

$$\Rightarrow G(K) = \underbrace{\sum_{l=0}^{N/4-1} a(l)\, W_{N/4}^{2Kl}}_{A(K)} + W_{N/2}^{K} \cdot \underbrace{\sum_{l=0}^{N/4-1} b(l)\, W_{N/4}^{2Kl}}_{B(K)} \quad \text{eqn } (10)$$

$$W_{N/2}^{2} = W_{N/4}$$

$$G(K) = \underbrace{A(K)}_{N/4} + W_{N/2}^{K} \underbrace{B(K)}_{N/4} ; \quad 0 \le K \le \frac{N}{4}-1 \rightarrow (8)$$

|||ly $H(K) = C(K) + W_{N/2}^{K} D(K) ; \quad 0 \le K \le \frac{N}{4}-1 \rightarrow (9)$

$A(K), B(K), C(K)$ & $D(K) \rightarrow$ periodic $\frac{N}{4}$.

(2)

(8) $\Rightarrow G(K) = A\left(K-\frac{N}{4}\right) + W_{N/2}^{K}\, B\left(K-\frac{N}{4}\right)$; $\frac{N}{4} \le K \le \frac{N}{2}-1 \rightarrow$ (10)

$A(K-2)$ $\quad B(K-2)$

(9) $\Rightarrow H(K) = C\left(K-\frac{N}{4}\right) + W_{N/2}^{K}\, D\left(K-\frac{N}{4}\right)$; $\frac{N}{4} \le K \le \frac{N}{2}-1 \rightarrow$ (11)

$C(K-2)$ $\quad D(K-2)$

1)}

(7)

2l+1)

$K'.\ W_{N/2}^{K}$

/2

2Kl

N/4

$K \Rightarrow 0$ & $1 \rightarrow$ eqn (8) & (9)

2 & $3 \rightarrow$ eqn (10) & (11)

eqn (8) $\Rightarrow$

$K=0;\ G(0) = A(0) + W_4^0 B(0)$ $\qquad$ eqn (9) $K=0;\ H(0) = C(0) + W_4^0 D(0)$

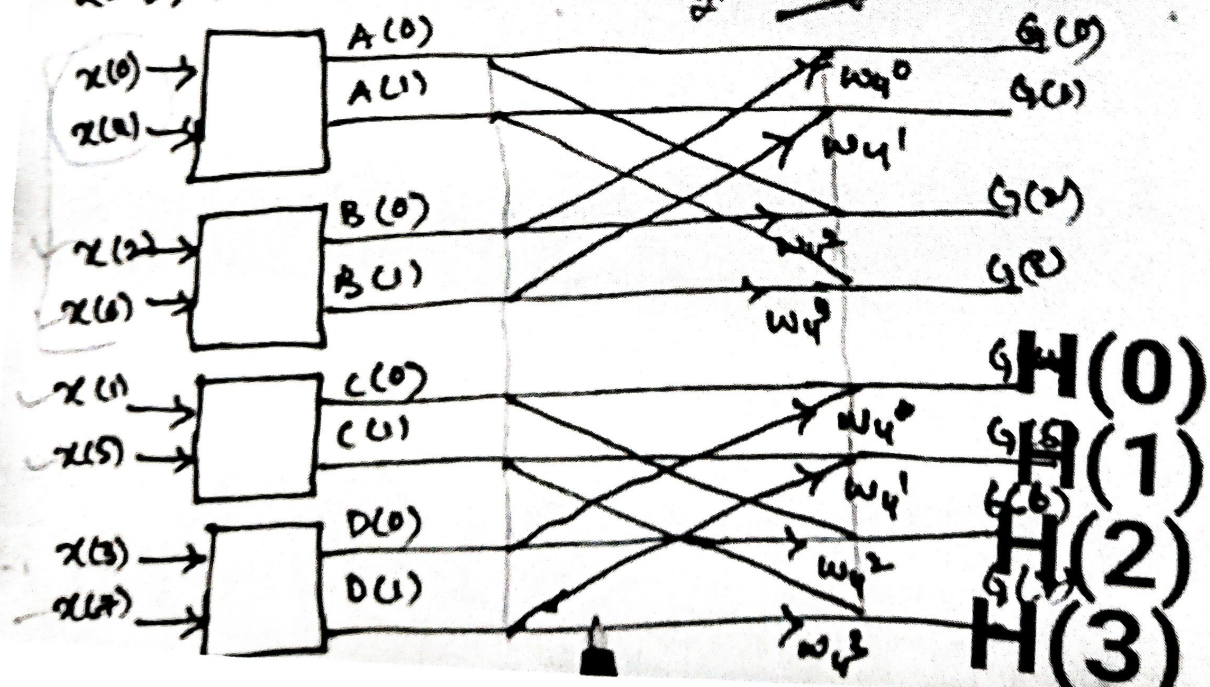
$K=1;\ G(1) = A(1) + W_4^1 B(1)$ $\qquad$ $K=1;\ H(1) = C(1) + W_4^1 D(1)$

eqn (10) $K=2;\ G(2) = A(0) + W_4^2 B(0)$ $\qquad$ eqn (11) $K=2;\ H(2) = C(0) + W_4^2 D(0)$

$K=3;\ G(3) = A(1) + W_4^3 B(1)$ $\qquad$ $K=3;\ H(3) = C(1) + W_4^3 D(1)$

$K=3;\ G(3) = A(1) + W_4^3$

2nd stage



Radix-2 DIT-FFT Algorithm:

Each $\frac{N}{4}$ DFT as two $\frac{N}{8}$ point DFTs.

the 2-point DFT of $x(0)$ & $x(4)$

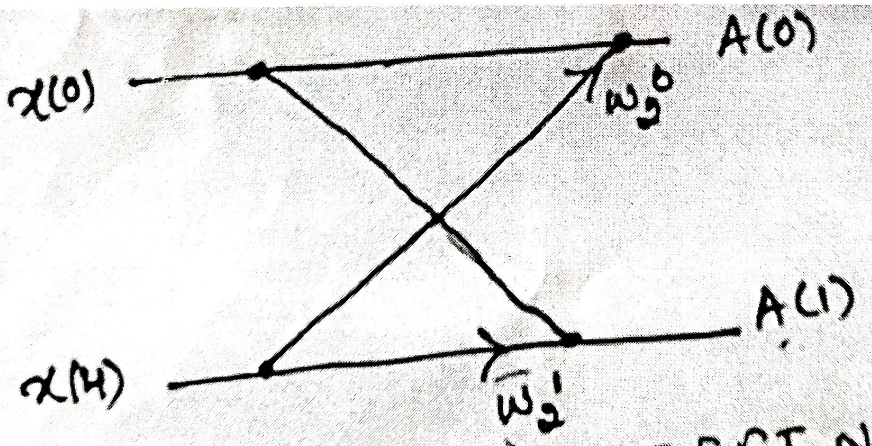
$$A(K) = \sum_{n=0}^{N/4-1} x(n)\, W_{N/4}^{Kn} \quad ; \; 0 \le K \le \frac{N}{4} - 1$$

$\boxed{N=8}$

$$A(K) = \sum_{n=0}^{1} x(n)\, W_{2}^{Kn} \quad ; \; 0 \le K \le 1$$

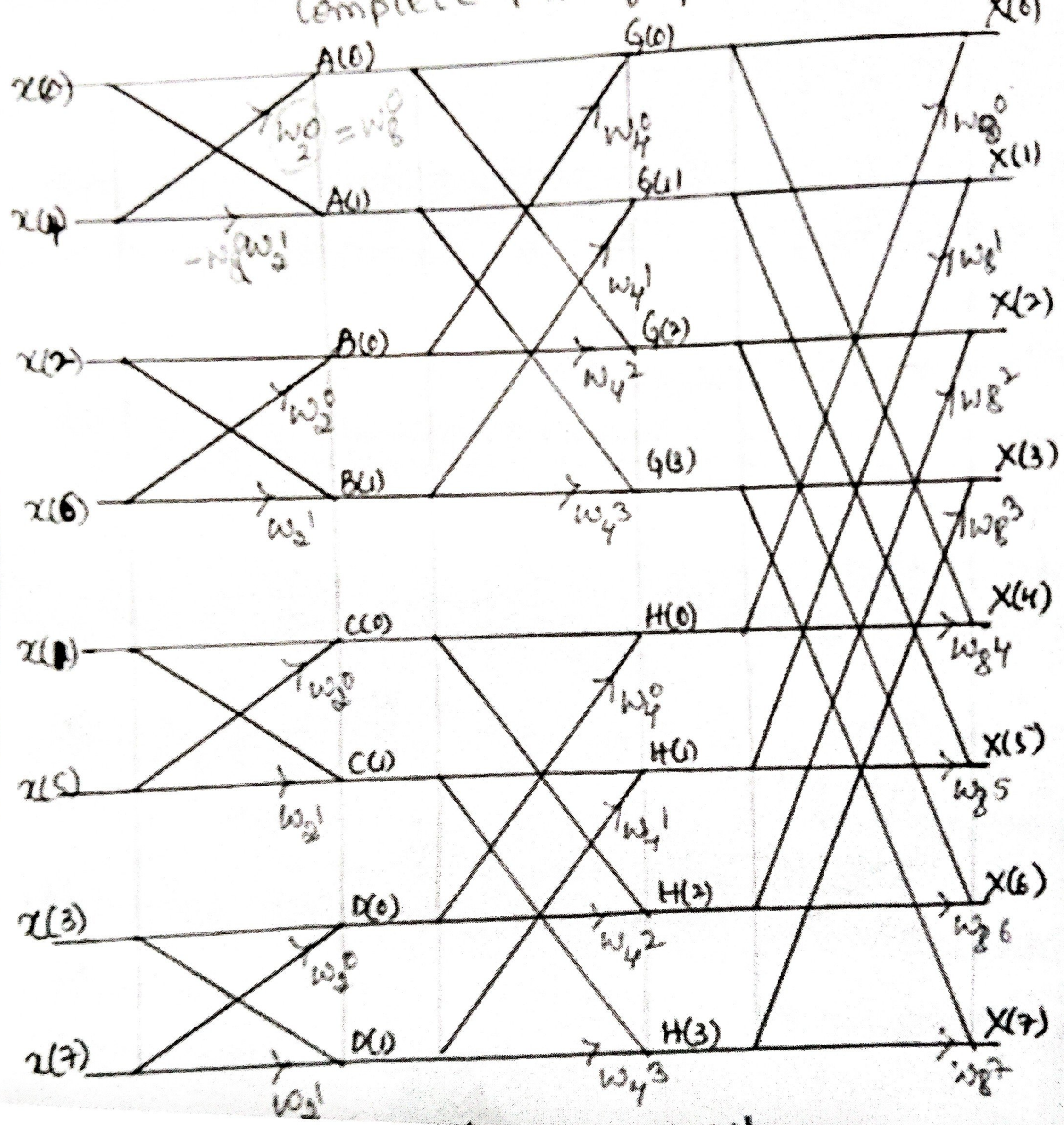
For ; $K=0 \Rightarrow A(0) = x(0) + W_2^0 x(4)$

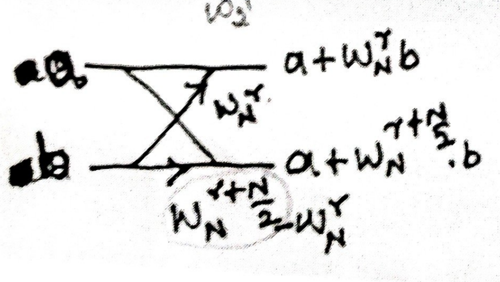
For ; $K=1 \Rightarrow A(1) = x(1) + W_2^1 x(4)$



3rd stage in DIT FFT N=8

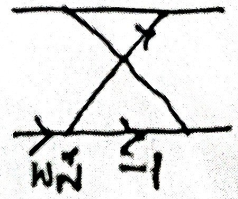
Complete Flow graph DIT-FFT ; N=8



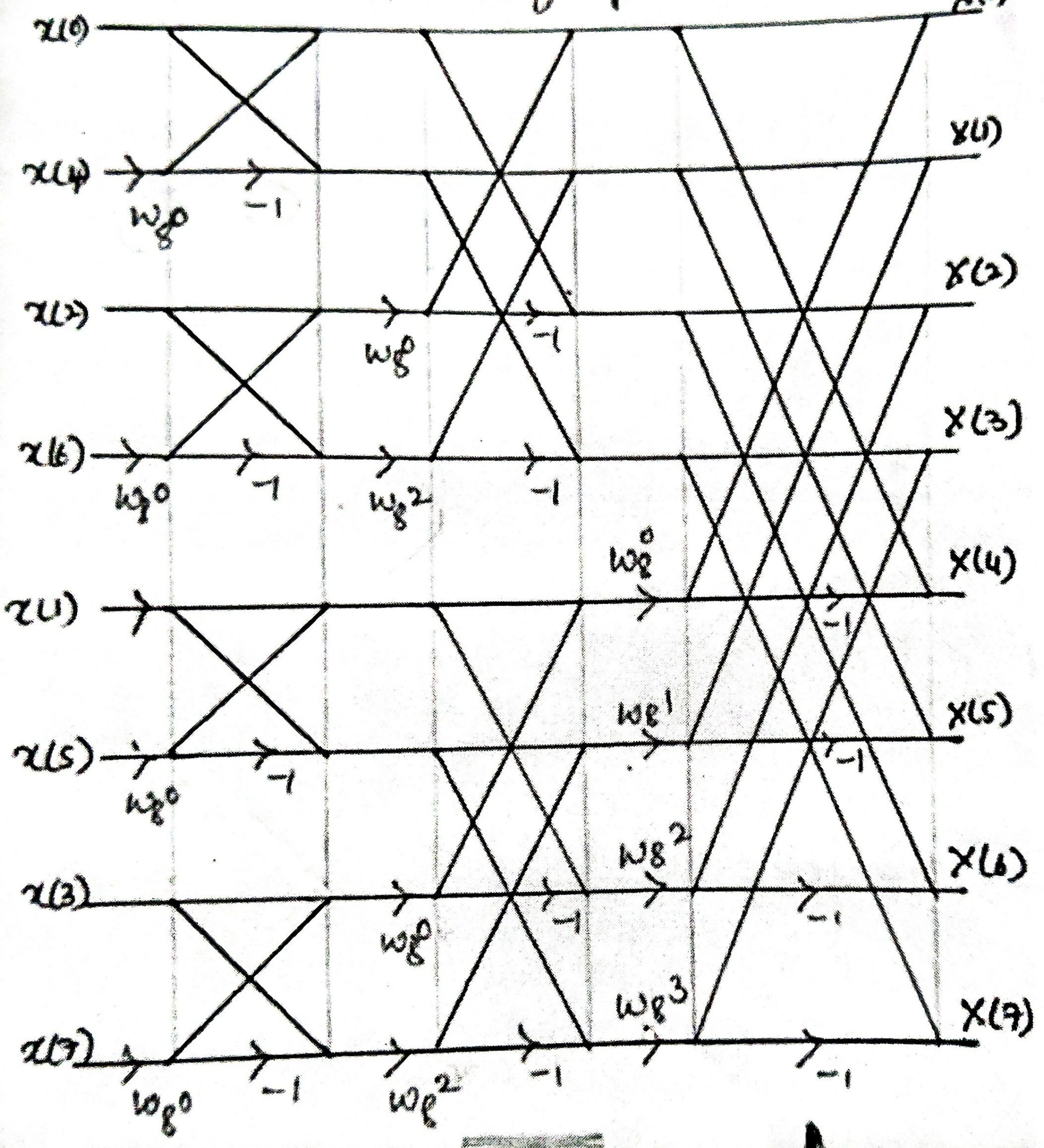


$$W_N^{r+\frac{N}{2}} = -W_N^r$$

$\hookrightarrow$ Symmetry



Reduced Flow graph: DIT-FFT N=8



[redacted]

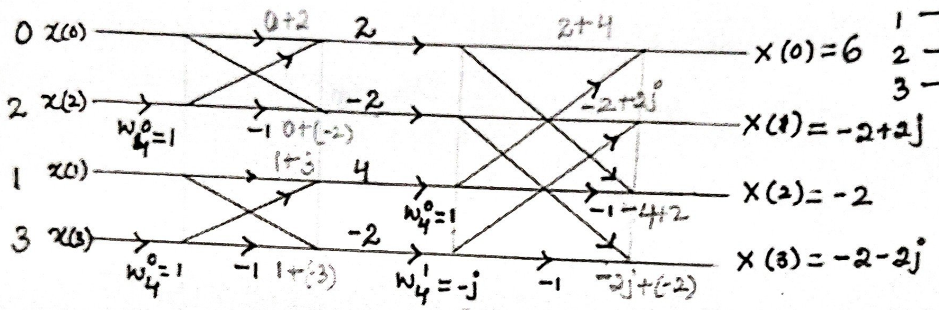
Given $x(n) = \{0, 1, 2, 3\}$, find $X(K)$ using DIT-FFT Algorithm.

$\therefore N = 4$ $\qquad W_4^0 = 1 \qquad W_4^1 = -j$

bit reversal

$4 = 2^{2}$

	BR
$0 \to 00$	$00 \to 0$ ✓
$1 \to 01$	$10 \to 2$ ✓
$2 \to 10$	$01 \to 1$ ✓
$3 \to 11$	$11 \to 3$ ✓

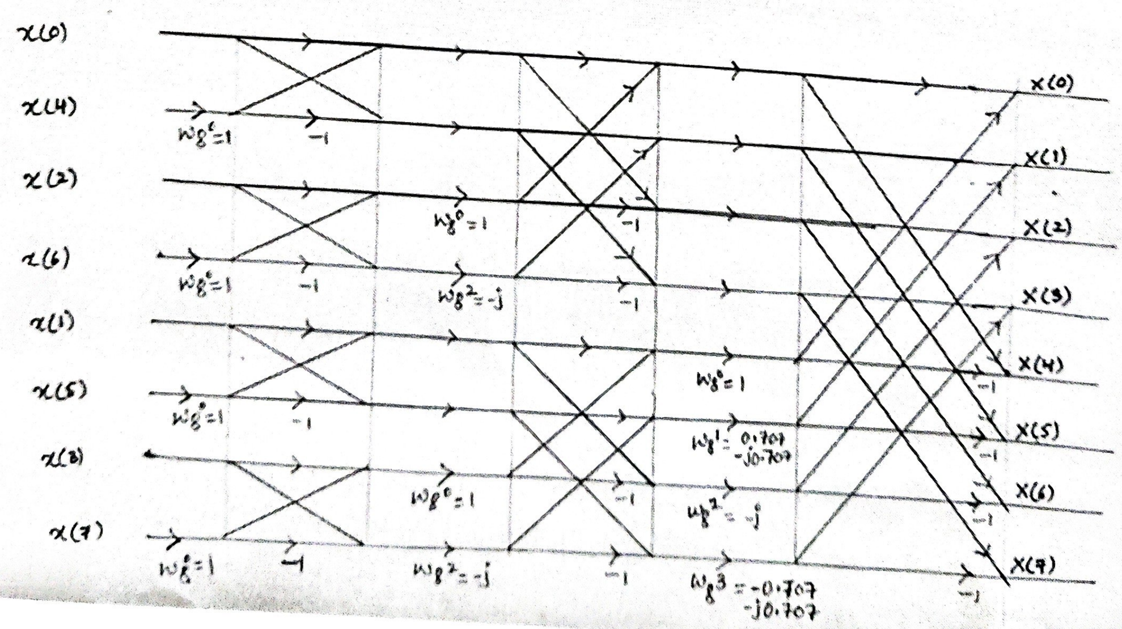


Flow-graph for DIT-FFT : N=4

$\therefore X(K) = \{6, -2+2j, -2, -2-2j\}$

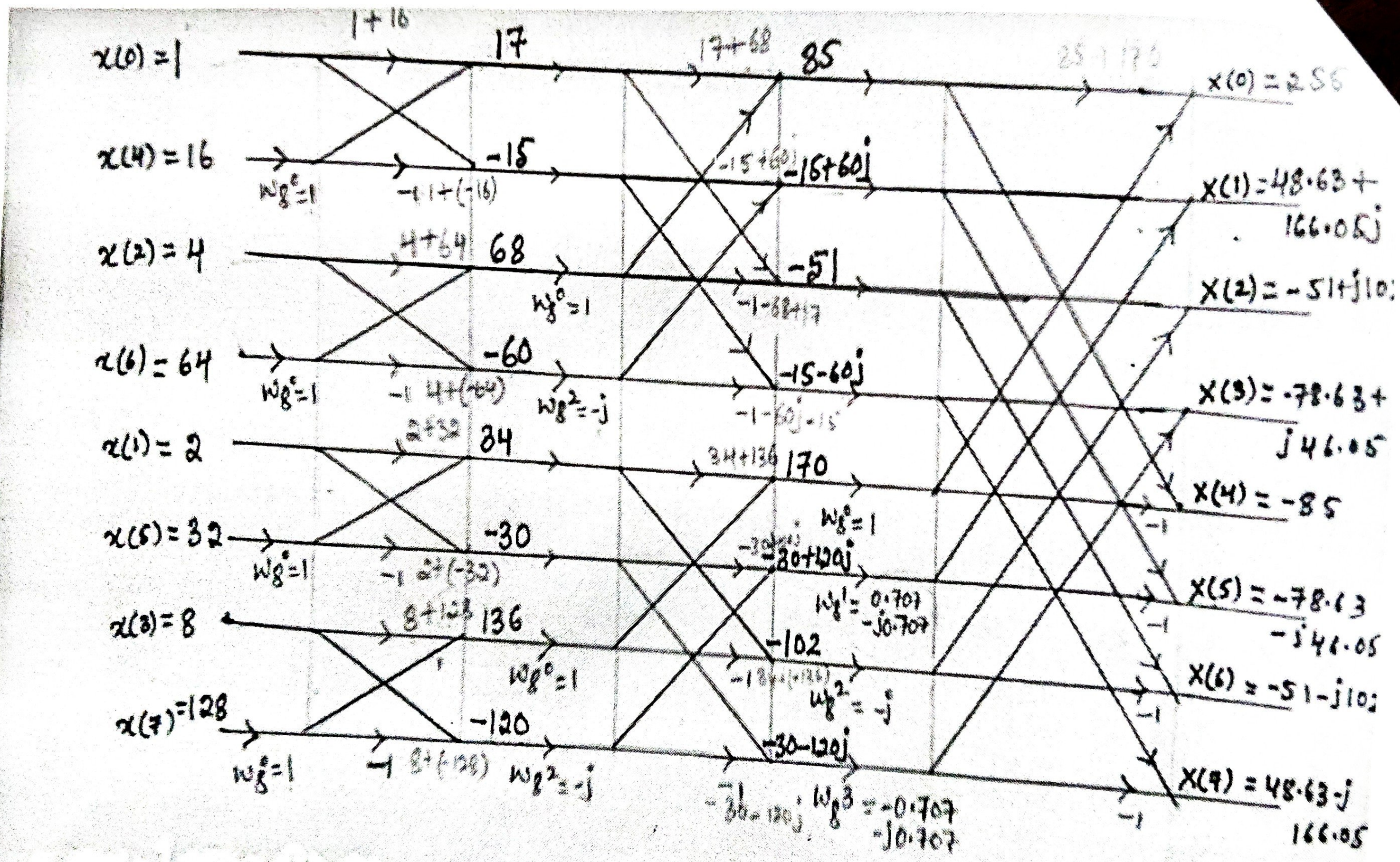
Given $x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$

Find $X(K)$ using DIT-FFT.



BIT REVERSAL

000 --> 000 --> 0
001 --> 100 --> 4
010 --> 010 --> 2
011 --> 110 --> 6
100 --> 001 --> 1
101 --> 101 --> 5
110 --> 011 --> 3
111 --> 111 --> 7



$\therefore X(k) = \{256, 48.63+j166.05, -51+j102, -78.63+j46.05, -85, -78.63-j46.05, -51-j102, 48.63-j166.05\}$

6.57. Consider a sequence

$$x[n]=\{1,1,-1,-1,-1,1,1,-1\}$$

Determine the DFT $X[k]$ of $x[n]$ using the decimation-in-time FFT algorithm.

From Figs. 6-38(a) and (b), the phase factors W_4^k and W_8^k are easily found as follows:

$$W_4^0=1 \qquad W_4^1=-j \qquad W_4^2=-1 \qquad W_4^3=j$$

and

$$W_8^0=1 \qquad W_8^1=\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} \qquad W_8^2=-j \qquad W_8^3=-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}$$

$$W_8^4=-1 \qquad W_8^5=-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \qquad W_8^6=j \qquad W_8^7=\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}$$

Next, from Eqs. (6.215a) and (6.215b)

$$f[n]=x[2n]=\{x[0],x[2],x[4],x[6]\}=\{1,-1,-1,1\}$$
$$g[n]=x[2n+1]=\{x[1],x[3],x[5],x[7]\}=\{1,-1,1,-1\}$$

Then, using Eqs. (6.206) and (6.212), we have

$$\begin{bmatrix} F[0]\\ F[1]\\ F[2]\\ F[3]\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\2+j2\\0\\2-j2\end{bmatrix}$$

$$\begin{bmatrix} G[0]\\ G[1]\\ G[2]\\ G[3]\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\4\\0\end{bmatrix}$$

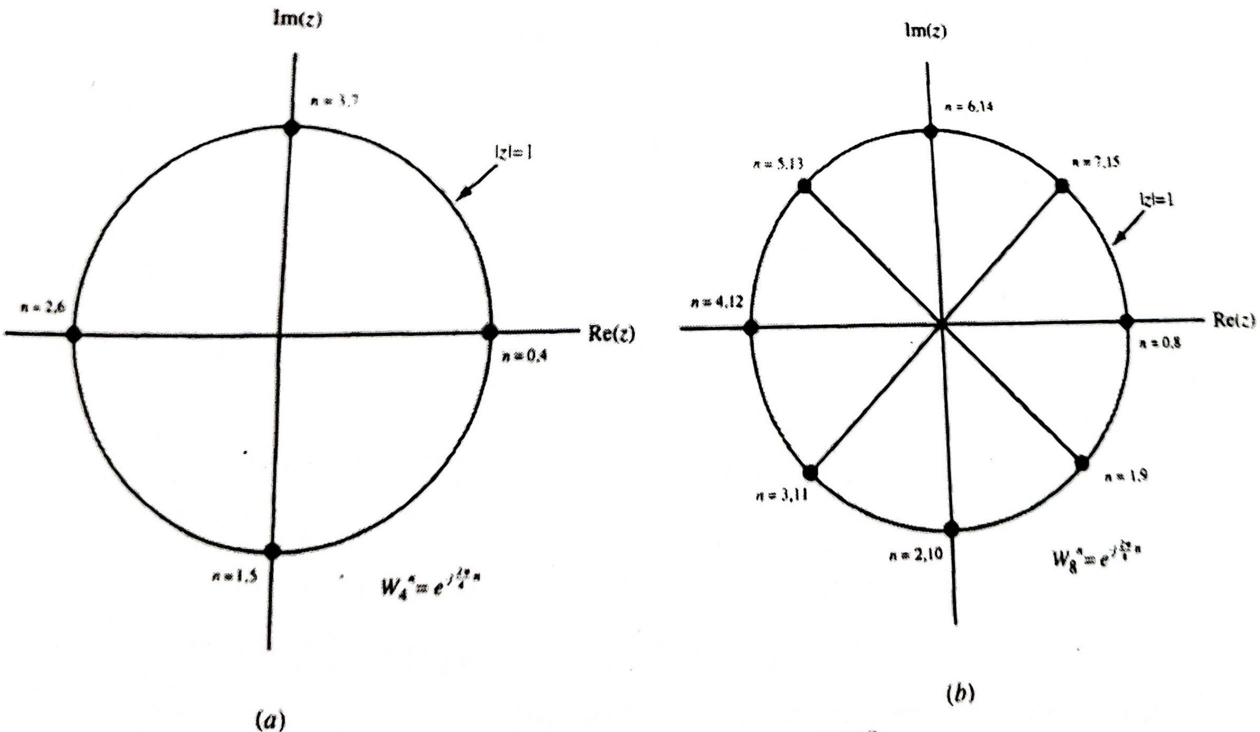


Fig. 6-38 Phase factors W_4^n and W_8^n.

and by Eqs. (6.217a) and (6.217b) we obtain

$$X[0] = F[0] + W_8^0 G[0] = 0 \qquad X[4] = F[0] - W_8^0 G[0] = 0$$
$$X[1] = F[1] + W_8^1 G[1] = 2 + j2 \qquad X[5] = F[1] - W_8^1 G[1] = 2 + j2$$
$$X[2] = F[2] + W_8^2 G[2] = -j4 \qquad X[6] = F[2] - W_8^2 G[2] = j4$$
$$X[3] = F[3] + W_8^3 G[3] = 2 - j2 \qquad X[7] = F[3] - W_8^3 G[3] = 2 - j2$$

Noting that since $x[n]$ is real and using Eq. (6.204), $X[7]$, $X[6]$, and $X[5]$ can be easily obtained by taking the conjugates of $X[1]$, $X[2]$, and $X[3]$, respectively.

6.59. Using the decimation-in-frequency FFT technique, redo Prob. 6.57.

From Prob. 6.57

$$x[n] = \{1, 1, -1, -1, -1, 1, 1, -1\}$$

By Eqs. (6.225a) and (6.225b) and using the values of W_8^n obtained in Prob. 6.57, we have

$$p[n] = x[n] + x\left[n + \frac{N}{2}\right]$$
$$= \{(1-1), (1+1), (-1+1), (-1-1)\} = \{0, 2, 0, 2\}$$
$$q[n] = \left(x[n] - x\left[n + \frac{N}{2}\right]\right) W_8^n$$
$$= \{(1+1)W_8^0, (1-1)W_8^1, (-1-1)w_8^2, (-1+1)W_8^3\}$$
$$= \{2, 0, j2, 0\}$$

Then using Eqs. (6.206) and (6.212), we have

$$\begin{bmatrix} P[0] \\ P[1] \\ P[2] \\ P[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -j4 \\ 0 \\ j4 \end{bmatrix}$$

$$\begin{bmatrix} Q[0] \\ Q[1] \\ Q[2] \\ Q[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ j2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+j2 \\ 2-j2 \\ 2+j2 \\ 2-j2 \end{bmatrix}$$

and by Eqs. (6.226a) and (6.226b) we get

$$X[0] = P[0] = 0 \qquad X[4] = P[2] = 0$$
$$X[1] = Q[0] = 2 + j2 \qquad X[5] = Q[2] = 2 + j2$$
$$X[2] = P[1] = -j4 \qquad X[6] = P[3] = j4$$
$$X[3] = Q[1] = 2 - j2 \qquad X[7] = Q[3] = 2 - j2$$

which are the same results obtained in Prob. 6.57.