ME 257/357
Spring 2018
Mid-term Exam

ME 257/357 Gas Turbine Design: Mid-term
Thursday, 5/24/2018, 10:30 - 11:50 am

Write your name on this handout and sign the honor code:

Name: ....

Honor Code: *I have neither given nor received unauthorized aid on this examination, nor have I concealed any violations of the Honor Code.*

Signiture: ....

This exam consists of two parts. Part 1 contains 10 short-answer problems (*3 pts* each) and Part 2 consider a detailed analysis of a turbofan engine. You need to complete both parts for full credit. The exam is open-book/open-notes and no external resources are allowed. Plan on not taking more than 30 mins for Part 1.

PART 1: CONCEPTUAL QUESTIONS (30 pts)

This part tests your conceptual understanding of the course material. Provide a brief but clear explanation or sketch for each question. Provide answers to each question in the blue book.

(1) Write down the equation for the drag coefficient of a wing with *finite span*. Identify each term in the equation. Draw the drag polar.

Solution

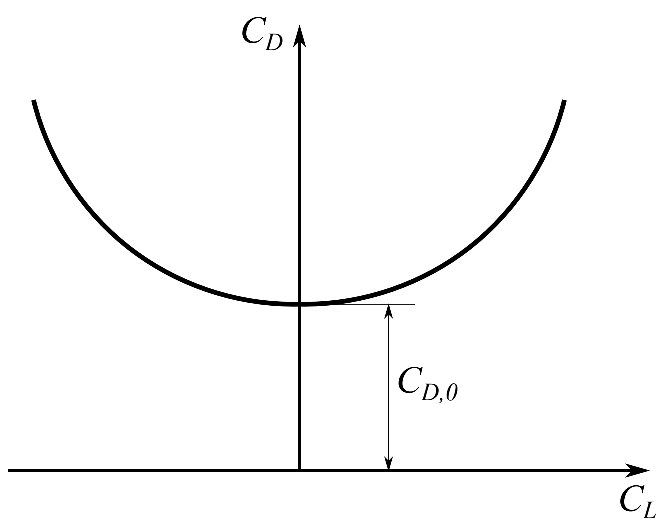

Figure 1: Drag polar of a wing with finite wing span.

$$C_D = D_{C,0} + \frac{C_L^2}{\pi e AR}, \tag{1}$$

where C_D is the total drag coefficient, $D_{C,0}$ is the profile drag due to viscous effects, $C_L^2/(\pi e AR)$ is the induced drag.

(2) Write down the thrust required for a steady level flight. Sketch T^* as a function of the dynamic pressure q_∞.

Solution

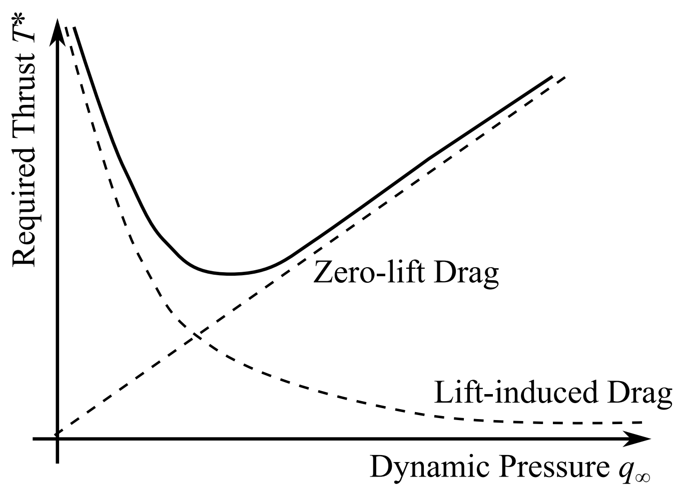

Figure 2: Required thrust versus dynamic pressure.

$$T_r = C_{D,0} q_\infty S + \frac{W^2}{q_\infty S \pi e AR}. \tag{2}$$

(3) What is the Breguet range equation? Sketch qualitatively the endurance factor as a function of the Mach number. Show curves at three different altitudes and explain why you drew them in this way.

Solution

Breguet range equation:

$$\frac{W(s)}{W_0} = \exp\left(-\frac{s}{RF}\right). \tag{3}$$

The range and endurance factors are:

$$RF = \frac{C_L}{C_D}\frac{U}{\mathrm{TSFC}g}, \tag{4}$$

$$EF = \frac{C_L}{C_D}\frac{1}{\mathrm{TSFC}g}. \tag{5}$$

The endurance factor is shown in Fig. 3. Note that TSFC = f(altitude, M), at each altitude there is an optimal Mach number that maximize the endurance factor.

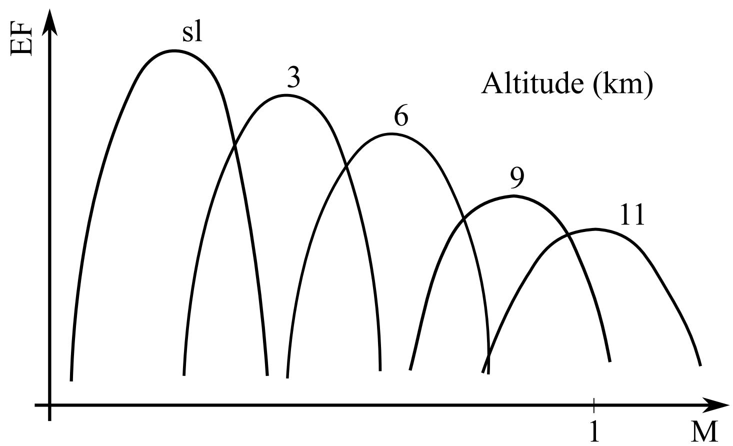

Figure 3: Endurance factor as a function of Mach number and altitude.

(4) Write down the thrust equation for an air-breathing propulsion system. Explain what will change when applying the thrust equation to a rocket engine. Show the final equation.

Solution

$$T = \dot{m}_a \left[(1+f)\, U_e - U_o \right] + A_e(p_e - p_0), \tag{6}$$

where the first term on the right-hand side is the jet thrust and the second term is the pressure thrust. For a rocket engine, there is no air inflow, i.e. $\dot{m}_a = 0$. The thrust equation of a rocket engine is:

$$T = \dot{m}_p U_e + A_e(p_e - p_0), \tag{7}$$

where $\dot{m}_p$ is the reaction product being expelled from the engine.

(5) What is a single crystal blade? Write down three advantages of using the single crystal technology for turbine blades.

Solution

A single crystal blade refers to a blade that is fabricated with one crystal grain. Using a single crystal blade has the following advantages:

- Due to the anisotropy of the material and the process of solidification, the strain and stress resistance is increased along the streamwise direction;
- Crack initiation due to grain boundary inside the blade is eliminated;
- Failure due to creeping is reliefed.

(6) Name three pollutants that are generated in the combustion process. Briefly explain how each pollutant forms and how to reduce it.

Solution

Some of the pollutants are:

- **Unburned Hydrocarbons (UHC)**: unburned fuel (fuel vapor, drops, spray) or partially dissociated fuel products due to poor fuel atomization and inadequate burning rate. *Solution*: combustion at lean conditions.
- **Smoke/Soot**: the soot is formed following the process of Fuel $\rightarrow$ Dissociation, pyrolysis, oxidation $\rightarrow$ Formation of precursor species $\rightarrow$ Particle inception $\rightarrow$ Surface growth and particle agglomeration $\rightarrow$ Particle oxidation. *Solution*: combustion at lean conditions.
- **Oxides of Nitrogen (NOx)**: formation mechanisms are thermal NO (most relevant), prompt NO, nitrous Oxide mechanism, and fuel NO. *Solution*: controlled flame temperature.

(7) Schematically draw a RQL combustor. Identify the combustion processes in each zone. Write down an equation (in symbolic form) that allows you to compute the density of the gas in the dilution region. Clearly state all assumptions.

Solution

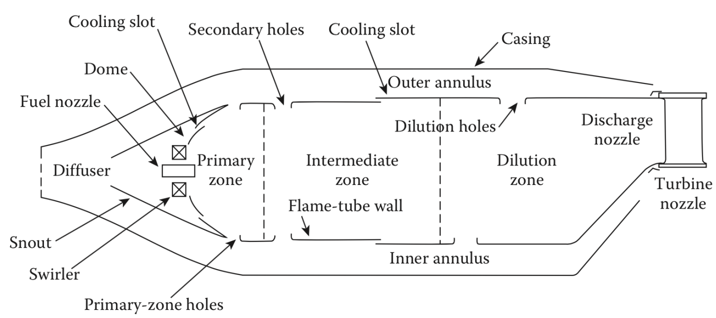

Figure 4: Schematic of a RQL cmobustor.

Note that the mass flow rate and temperature after the primary combustion zone is $\dot{m}_{a,1}+\dot{m}_f$ and T_b, the mass flow rate and temperature in the dilution stream is $\dot{m}_{a,2}$ and T_{03}. The temperature in the dilution region T_{dil} is found from the mass and energy conservations, which are:

$$\dot{m}_{\text{dil}} = \dot{m}_{a,1} + \dot{m}_{a,2} + \dot{m}_f = (1+f)\dot{m}_a, \tag{8}$$
$$(\dot{m}_{a,1} + \dot{m}_f)\, T_b + \dot{m}_{a,2} T_{03} = \dot{m}_{\text{dil}} T_{\text{dil}}. \tag{9}$$

T_{dil} is then computed as:

$$T_{\text{dil}} = \frac{(\dot{m}_{a,1} + \dot{m}_f)\, T_b + \dot{m}_{a,2} T_{03}}{(1+f)\dot{m}_a}. \tag{10}$$

Assuming that the efficiency of the combustor is 100%, the pressure at the dilution region is the same as the pressure at the compressor exit, i.e. $p_{\text{dil}} = p_{03}$, the density at the dilution region is:

$$\rho_{\text{dil}} = \frac{p_{03}}{RT_{\text{dil}}}. \tag{11}$$

(8) Define choke and stall in a compressor stage and explain their consequences.

Solution

Stall and choke refer to flow seperation on the low and high pressure side of the compressor blade. The occurrence of the stall and choke leads to compressor malfunctioning hence engine failure.

(9) Draw the compressor map qualitatively. Mark all important features on the graph.

Solution

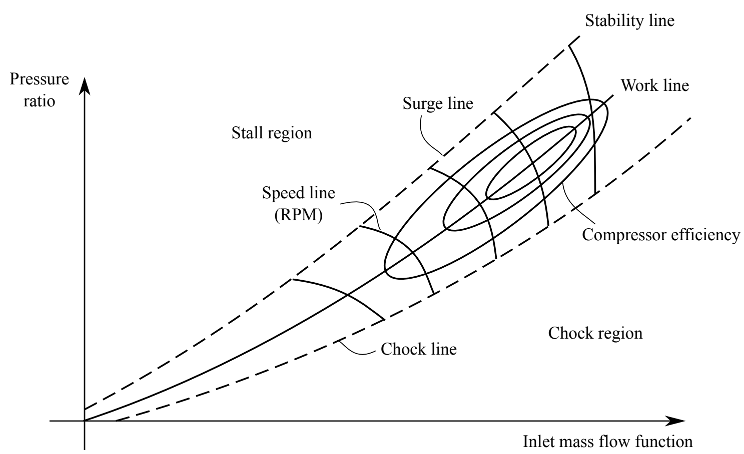

Figure 5: Compressor map.

(10) Briefly explain why the isentropic efficiency of a compressor is typically smaller than that of a turbine.

Solution

Because the compressor experiences inverse pressure gradient along the flow direction, there is a greater possibility for flow seperation as well as other entropy entropy generation process. In contrast, the pressure gradient associated with the turbine is along the streamwise direction. Therefore the isentropic efficiency for the compressor is lower than for the turbine.

PART 2: ENGINE ANALYSIS (70 pts)

In this part of the exam you are asked to provide a detailed analysis of a turbofan engine. Clearly write down your derivation steps and mark the final results.

Problem description

In this problem, you will analyze a GE 90 turbofan engine (as shown in Fig. 6) with efficiencies of each component listed in Table 1. The free stream Mach number is $M_0 = 0.85$, temperature is $T_0 = 220$ K, and pressure is $P_0 = 24$ kPa. The core mass flow rate of air is $\dot{m}_{a,c} = 3.0$ kg/s, the bypass ratio is 2.9, the fan pressure ratio is 2.0.

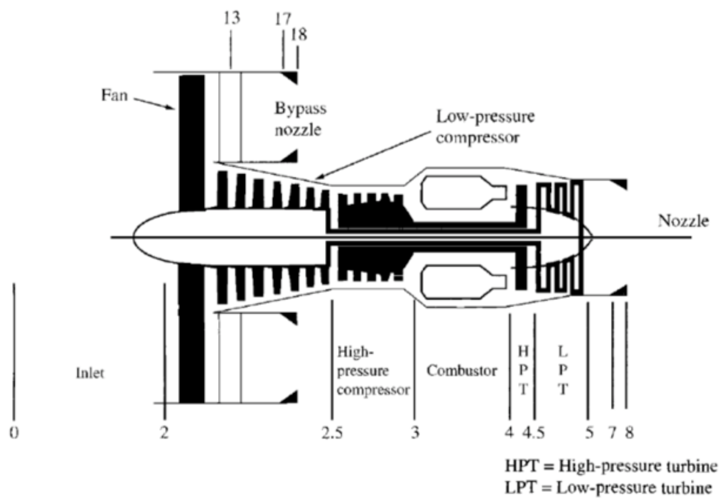

Figure 6: Schematic of the GE90 engine with station numbering.

Table 1: Turbofan component efficiencies.

Component	$\eta_{\text{adiabatic}}$
Diffuser	95%
Fan	92%
Compressor	87%
Combustor	100%
Turbine (LPT & HPT)	91%
Core/Fan nozzle	98%

1 Compressor analysis (15 pts)

(a) (5 pts) For a single compressor stage (consider the first stage after the inlet guide vane) shown in Fig. 7, complete the velocity triangle across the rotor at the pitchline r_m. Assume the following quantities as known: the inlet velocity U_{c1}, the angle of attack of the flow after the inlet guide vane α_{c1}, the angular velocity of the compressor Ω, the relative angle at the rotor leading edge β_{c1}, and the relative angle at the rotor trailing edge β_{c2}. Express $U_{c1,\theta}$, $U_{c2,\theta}$, and α_{c2} using the known quantities.

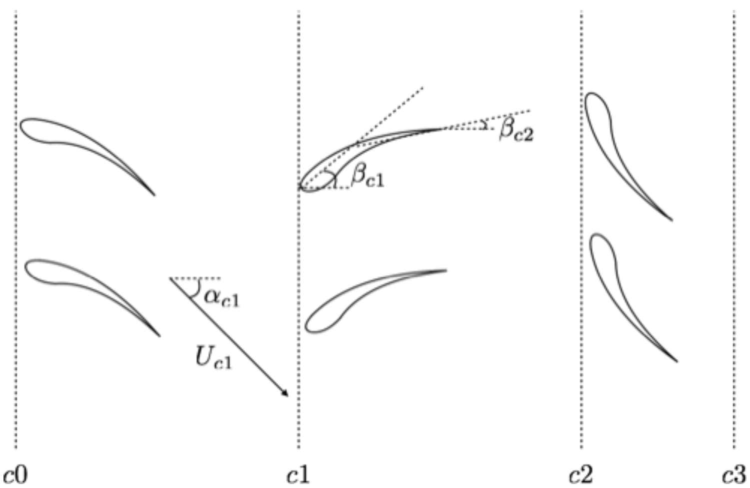

Figure 7: Schematic of the the first compressor stage with station numbering.

Solution

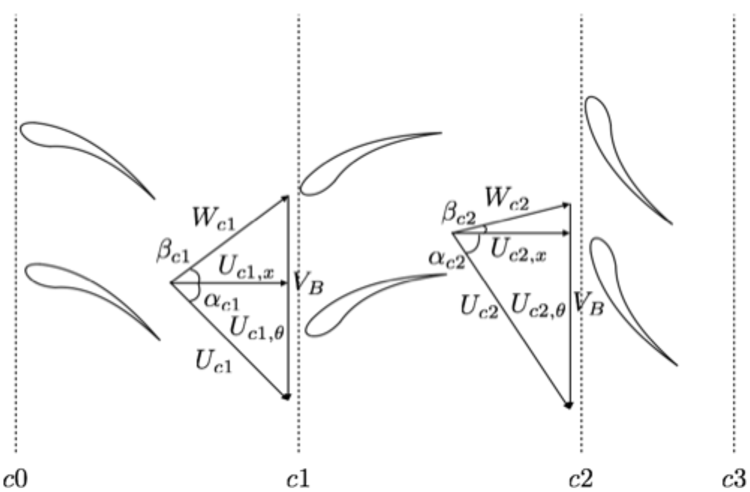

Figure 8: Schematic of the the first compressor stage with station numbering.

(b) (5 pts) What is the work consumed by this compressor stage? Assuming the inlet velocity $U_{c1} = 150$ m/s, the angular velocity is $\Omega = 24000$ rpm, the vortex design follows the *constant reaction* specification, the angle of attack at the hub and tip are $\alpha_{c1,\text{tip}} = 30°$ and $\alpha_{c1,\text{tip}} = 23.8°$, the radius at the hub and the tip are $r_{\text{hub}} = 0.9$ m and $r_{\text{tip}} = 1.0$ m, respectively.

Solution

From the vortex design of the *constant reaction* specification, i.e.

$$rU_{c1,\theta} = -b + ar^2, \tag{12}$$

$$rU_{c2,\theta} = b + ar^2. \tag{13}$$

Substituting r_{hub}, $\alpha_{c1,\text{hub}}$ and r_{tip}, $\alpha_{c1,\text{tip}}$ into Eq. (13), coefficients a and b are obtained as:

$$a = \frac{\sin\alpha_{c1,\text{tip}}/r_{\text{tip}} - \sin\alpha_{c1,\text{hub}}/r_{\text{hub}}}{1/r^2_{c1,\text{tip}} - 1/r^2_{c1,\text{hub}}} U_{c1} = 145.8, \tag{14}$$

$$b = \frac{r_{\text{tip}}\sin\alpha_{c1,\text{tip}} - r_{\text{hub}}\sin\alpha_{c1,\text{hub}}}{r^2_{\text{tip}} - r^2_{\text{hub}}} U_{c1} = -0.245. \tag{15}$$

Therefore the work across this stage is:

$$h_{0c3} - h_{0c1} = V_B \Delta U_{c,\theta} \tag{16}$$

$$= (r\Omega)(2a/r) \tag{17}$$

$$= 2a\Omega = 7 \text{ MJ}. \tag{18}$$

(c) (5 pts) What is the pressure ratio across this stage? Assuming there are three identical stages, what is the overall pressure ratio provided by the compressor?

Solution

The pressure ratio across this single stage is:

$$\frac{p_{0,c3}}{p_{0,c1}} = \frac{1}{\pi\left(r^2_{\text{tip}} - r^2_{\text{hub}}\right)} \int_{r_{\text{hub}}}^{r_{\text{tip}}} 2\pi r \left(1 + \eta_{\text{st}} \frac{V_B \Delta U_\theta}{CpT_{0,c1}}\right)^{\frac{\gamma}{\gamma-1}} dr \tag{19}$$

$$= \left(1 + \eta_{\text{st}} \frac{V_B \Delta U_\theta}{CpT_{0,c1}}\right)^{\frac{\gamma}{\gamma-1}} \tag{20}$$

$$= 2.71. \tag{21}$$

Assuming the stages are identical, the overall compression ratio is:

$$\frac{p_{03}}{p_{02.5}} = \left(\frac{p_{0,c3}}{p_{0,c1}}\right)^3 = 20. \tag{22}$$

(d) *bonus (5 pts)*: what is the degree of reaction for this compressor stage?

2 Combustor analysis (15 pts)

(a) (5 pts) n-Hexadecane ($C_{16}H_{34}$) is used in the current engine. Write down the reaction equation for the oxidation with air at stoichiometric conditions. Assuming the fuel-to-air ratio is $f = 0.0043$, what is the equivalence ratio in this combustor?

Solution

The reaction for n-Hexadecane and air is:

$$C_{16}H_{34} + 24.5(O_2 + 3.76N_2) = 16CO_2 + 17H_2O. \tag{23}$$

With this, it is found that:

$$f_{\text{st}} = \frac{W_{C_{16}H_{34}}}{(\nu_{O_2}(W_{O_2} + 4.76W_{N_2}))} = 0.0672 \tag{24}$$

The equivalence ratio is $\phi = f/f_{\text{st}} = 0.064$. In the fuel lean case, the reaction equation becomes:

$$\phi C_{16}H_{34} + 24.5(O_2 + 3.76N_2) = 16\phi CO_2 + 17\phi H_2O + 24.5 * (1-\phi)(O_2 + 3.76N_2). \tag{25}$$

(b) (5 pts) What is the lower heating value of the reaction for the given fuel-to-air ratio?

Solution

The heat of combustion is:

$$H_c^0 = \sum_{k \in \text{product}} \left(\nu_k h_{f,k}^0\right) - \sum_{k \in \text{reactant}} \left(\nu_k h_{f,k}^0\right) = -153.4 \text{ cal/mol}. \tag{26}$$

The lower heating value is $LHV = -H_c^0 = 153.4$ kcal/mol at reference state with $T_0 = 298.15$ K.

(c) (5 pts) Compute the adiabatic flame temperature assuming calorically perfect gas with $\gamma = 1.4$. Use this temperature as combustor exit temperature T_{04} in the following calculation.

Solution

Note that the gas constant for species k is computed as:

$$R_k = R_u/W_k, \tag{27}$$

where $R_u = 8.314$ J/(mol K) is the universal gas constant. With is, the heat capacity for each species under the assumption of Calorically perfect gas is estimated as:

$$C_{p,k} = \frac{\gamma R_k}{\gamma - 1}. \tag{28}$$

The mixture averaged heat capacity of the reactants and products are obtained as:

$$C_{p,r} = \sum_{k\in\text{reactant}} Y_k C_{p,k}, \tag{29}$$

$$C_{p,p} = \sum_{k\in\text{product}} Y_k C_{p,k}, \tag{30}$$

where $Y_k = \nu_k W_k / \sum_{k\in\text{reactant}} (\nu_k W_k)$ is the mass fraction of species k. The adiabatic flame temperature is:

$$T_{04} = T_0 + \frac{C_{p,r} * (T_{03} - T_0) + LHV}{C_{p,p}} = 1600.3 \text{ K}. \tag{31}$$

3 Brayton Cycle Analysis (40 pts)

(a) (30 pts) Perform the Brayton cycle analysis to the engine using the T-s diagram shown in Fig. 9. Ignore the low-pressure compressor, i.e. assume that the low-pressure turbine drives the fan only. Assume the pressure at the core nozzle exit is the same as the free stream pressure. Complete the T-s diagram with the numbers obtained from your cycle analysis. Mark each station clearly and draw addition constant pressure lines if necessary. Station 0 has been marked for you.

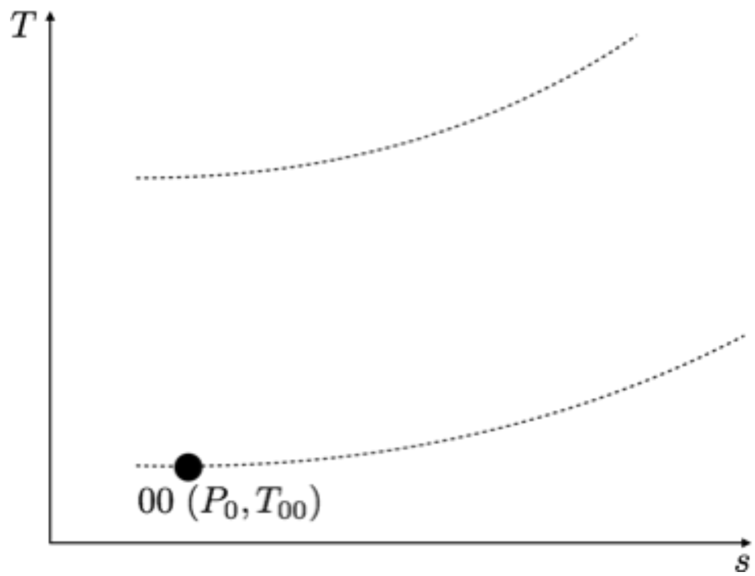

Figure 9: T-s diagram of the Brayton cycle analysis to the GE90 engine.

Solution

Note that the stagnation temperature is:

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2. \tag{32}$$

The isentropic state relationship is:

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}. \tag{33}$$

For real engines with none unity efficiencies, it is required to relate the real stagnation states with the isentropic states. Here, it is assumed that the gas is Calorically perfect, i.e. C_p is constant. It is also assumed that $p_0/p_{0s} = 1$. For components with pressure increases, e.g.

diffusor, fan, and compressor, the efficiency from station a to b:

$$\eta = \frac{h_{0bs} - h_{0a}}{h_{0b} - h_{0a}} \tag{34}$$

$$= \frac{T_{0bs} - T_{0a}}{T_{0b} - T_{0a}} \tag{35}$$

$$= \frac{T_{0bs}/T_{0a} - 1}{T_{0b}/T_{0a} - 1}. \tag{36}$$

With this,

$$T_{0b}/T_{0a} = 1 + \frac{1}{\eta}\left(\frac{T_{0bs}}{T_{0a}} - 1\right), \tag{37}$$

$$T_{0bs}/T_{0a} = 1 + \eta\left(\frac{T_{0b}}{T_{0a}} - 1\right). \tag{38}$$

If the pressure ratio p_{0b}/p_{0a} is known, the temperature ratio is found as:

$$T_{0b}/T_{0a} = 1 + \frac{1}{\eta}\left(\frac{T_{0bs}}{T_{0a}} - 1\right), \tag{39}$$

$$= 1 + \frac{1}{\eta}\left[\left(\frac{p_{0b}}{p_{0a}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]. \tag{40}$$

If the temperature ratio T_{0b}/T_{0a} is known, the pressure ratio is found as:

$$p_{0b}/p_{0a} = \left(T_{0bs}/T_{0a}\right)^{\frac{\gamma}{\gamma-1}} \tag{41}$$

$$= \left[1 + \eta\left(\frac{T_{0b}}{T_{0a}} - 1\right)\right]^{\frac{\gamma}{\gamma-1}}. \tag{42}$$

For components with pressure decreases, e.g. turbine and nozzle, the efficiency from station a to b is defined as:

$$\eta = \frac{h_{0a} - h_{0b}}{h_{0a} - h_{0bs}} \tag{43}$$

$$= \frac{T_{0a} - T_{0b}}{T_{0a} - T_{0bs}} \tag{44}$$

$$= \frac{1 - T_{0b}/T_{0a}}{1 - T_{0bs}/T_{0a}}. \tag{45}$$

With this,

$$T_{0b}/T_{0a} = 1 - \eta\left(1 - T_{0bs}/T_{0a}\right), \tag{46}$$

$$T_{0bs}/T_{0a} = 1 - \frac{1}{\eta}\left(1 - \frac{T_{0b}}{T_{0a}}\right). \tag{47}$$

If the pressure ratio p_{0b}/p_{0a} is known, the temperature ratio is found as:

$$T_{0b}/T_{0a} = 1 - \eta\left(1 - T_{0bs}/T_{0a}\right), \tag{48}$$

$$= 1 - \eta\left[1 - \left(p_{0b}/p_{0a}\right)^{\frac{\gamma-1}{\gamma}}\right]. \tag{49}$$

If the temperature ratio T_{0b}/T_{0a} is known, the pressure ratio is found as:

$$p_{0b}/p_{0a} = (T_{0bs}/T_{0a})^{\frac{\gamma}{\gamma-1}} \tag{50}$$

$$= \left[1 - \frac{1}{\eta}\left(1 - \frac{T_{0b}}{T_{0a}}\right)\right]^{\frac{\gamma}{\gamma-1}}. \tag{51}$$

With these relationships, the Brayton cycle analysis for the turbofan engine is performed as follows.

- *Diffuser* ($0 \to 2$)
  Stagnation temperature at station 2:

$$\frac{T_{02}}{T_0} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2 = 1.1445. \tag{52}$$

  The pressure ratio is hence:

$$\frac{p_{02}}{p_0} = \left[1 + \eta\left(\frac{T_{02}}{T_0} - 1\right)\right]^{\frac{\gamma}{\gamma-1}} = 1.5687. \tag{53}$$

  The stagnation temperature and pressure are 251.79 K and 3.7648×10^4 Pa respectively.
- *Fan* ($2 \to 2.5/13$)
  Given pressure ratio across the fan $p_{02.5}/p_{02}$ (note that $p_{02.5} = p_{013}$), the temperature ratio is:

$$\frac{T_{02.5}}{T_{02}} = 1 + \frac{1}{\eta}\left[\left(\frac{p_{02.5}}{p_{02}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = 1.2381. \tag{54}$$

  The stagnation temperature and pressure are 311.73 K and 7.5296×10^4 Pa respectively.
- *High Pressure Compressor* ($2.5 \to 3$)
  The pressure ratio is obtained from previous problem as $p_{03}/p_{02.5} = 20$, the temperature ratio is:

$$\frac{T_{03}}{T_{02.5}} = 1 + \frac{1}{\eta}\left[\left(\frac{p_{03}}{p_{02.5}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = 2.5558. \tag{55}$$

  The stagnation temperature and pressure are 796.72 K and 1.5059×10^6 Pa respectively.
- *Combustor* ($3 \to 4$)
  The stagnation temperature at station 4 is obtained from previous problem as $T_{04} = T_{ad}$. With the assumption that the efficiency is 1 for the combustor, $p_{04}/p_{03} = 1$.
  The stagnation temperature and pressure are 1600.3 K and 1.5059×10^6 Pa respectively.
- *High-Pressure Turbine* ($4 \to 4.5$)
  From work balance between the high-pressure turbine and the high-pressure compressor:

$$(\dot{m}_{a,c} + \dot{m}_f)(T_{04} - T_{04.5}) = \dot{m}_{a,c}(T_{03} - T_{02.5}), \tag{56}$$

  the temperature ratio is obtained as:

$$\frac{T_{04.5}}{T_{04}} = 1 - \frac{1}{1+f}\frac{T_{03} - T_{02.5}}{T_{04}} = 0.6982. \tag{57}$$

The pressure ratio is then:

$$p_{04.5}/p_{04} = \left[1 - \frac{1}{\eta}\left(1 - \frac{T_{05}}{T_{04}}\right)\right]^{\frac{\gamma}{\gamma-1}} = 0.2441. \tag{58}$$

The stagnation temperature and pressure are 1117.3 K and 3.6761×10^5 Pa respectively.

– *Low-Pressure Turbine* $(4.5 \to 5)$

From work balance between the low-pressure turbine and the fan:

$$(\dot{m}_{a,c} + \dot{m}_f)(T_{04.5} - T_{05}) = \beta \dot{m}_{a,c}(T_{02.5} - T_{02}), \tag{59}$$

the temperature ratio is obtained as:

$$\frac{T_{05}}{T_{04.5}} = 1 - \frac{\beta}{1+f}\frac{T_{02.5} - T_{02}}{T_{04.5}} = 0.8451. \tag{60}$$

The pressure ratio is then:

$$p_{05}/p_{04.5} = \left[1 - \frac{1}{\eta}\left(1 - \frac{T_{05}}{T_{04.5}}\right)\right]^{\frac{\gamma}{\gamma-1}} = 0.5204. \tag{61}$$

The stagnation temperature and pressure are 944.26 K and 1.9131×10^5 Pa respectively.

– *Core Nozzle* $(5 \to 8)$

Note that $p_{08}/p_{05} = p_{00}/p_{05} = 0.1254$ is known, the temperature ratio is found as:

$$T_{08}/T_{05} = 1 - \eta\left[1 - (p_{08}/p_{05})^{\frac{\gamma-1}{\gamma}}\right] = 0.5616. \tag{62}$$

The stagnation temperature and pressure are 530.26 K and 2.4×10^4 Pa respectively.

– *Fan/Bypass Nozzle* $(13 \to 18)$

Note that $p_{018}/p_{02} = p_{00}/p_{02} = 0.6375$ is known, the temperature ratio is found as:

$$T_{018}/T_{02} = 1 - \eta\left[1 - (p_{018}/p_{02})^{\frac{\gamma-1}{\gamma}}\right] = 0.8817. \tag{63}$$

The stagnation temperature and pressure are 222.00 K and 2.4×10^4 Pa respectively.

(b) (10 pts) Compute the following engine performance quantities:

(i) (5 pts) What is the thrust generated by the engine? What is the TSFC?

Solution

The exit velocity of the fan and core nozzle are:

$$U_{1e} = \sqrt{2\eta_f c_p T_{013}\left[1 - \left(\frac{p_0}{p_{013}}\right)^{\frac{\gamma-1}{\gamma}}\right]} = 399.3105 \text{ m/s}, \tag{64}$$

$$U_e = \sqrt{2\eta_n c_p T_{05}\left[1 - \left(\frac{p_0}{p_{05}}\right)^{\frac{\gamma-1}{\gamma}}\right]} = 908.8108 \text{ m/s}. \tag{65}$$

The thrust generated by the engine is:

$$T = \dot{m}_{a,c}\left[(1+f)U_e + \beta U_{1e} - (1+\beta)U_0\right] = 3.2657 \times 10^3 \text{ N}. \tag{66}$$

The thrust specific fuel consumption is:

$$TSFC = \frac{f}{(1+f)U_e + \beta U_{1e} - (1+\beta)U_0} = 3.9502 \times 10^{-6}. \tag{67}$$

(ii) (5 pts) What is the propulsive efficiency of the engine?

Solution

The propulsive efficiency is:

$$\eta_p = \frac{\dot{m}_{a,c}U_0\left[(1+f)U_e + \beta U_{1e} - (1+\beta)U_0\right]}{\frac{1}{2}\dot{m}_{a,c}\left[(1+f)U_e^2 + \beta U_{1e}^2 - (1+\beta)U_0^2\right]} \tag{68}$$

$$= \frac{2U_0\left[(1+f)U_e + \beta U_{1e} - (1+\beta)U_0\right]}{\left[(1+f)U_e^2 + \beta U_{1e}^2 - (1+\beta)U_0^2\right]} = 0.5249. \tag{69}$$