Locus Problems Related to Linear Combinations of Vectors

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Abstract

This paper is a more in-depth discussions for selected problems discussed in [1], which are inspired by college entrance exam practice problems from China. When initially seeing the problem during an exam, one may have no clue how the problem is created. In this paper, we begin by presenting the essential algebraic manipulation skills that Chinese high school students are expected to apply to answer the question. We then continue the discussion by exploring various scenarios that utilize technological tools. We shall see many unexpected surprising outcomes. More importantly, we summarize how a problem can be extended to a more general setting whenever is possible.

1 Introduction

This paper is a more in depth discussion on selected problems from the published article [1]. Finding a curve defined by the locus of a moving point has been popular and such problems are often asked on Gaokao (a college entrance exam) in China. There have been several exploratory activities (see [5] [7], and [9]) derived from Chinese college entrance exam practice problems (see [8]). In this article, we typically start with a practice problem originated from ([8]) to initiate our discussions. We demonstrate how a problem can be solved without the assistance of technological tools, which shall demonstrate those crucial algebraic manipulation skills that are required by high school students from China. From a content knowledge point of view, this paper is very accessible to those students who have learned parametric equations and have a basic understaning of linear algebra. Problems presented in this paper can be used as examples for professional training purposes.

In this paper, we stress that if problems are presented as explorations instead of in an examination setting, learners would enjoy learning some new mathematics more than simply performing a collection of somewhat boring algebraic manipulations. We believe making conjectures by seeing possible solutions before asking complete analytic or algebraic solutions is much more accessible, convincing and intuitive to students. In addition to solving simple cases by hand, we typically construct a potential solution geometrically using the trace feature of dynamic geometry software (DGS) such as Geometry Expressions [3]. Finally, we use a computer algebra system (CAS) such as Maple [6] to verify that our analytic solutions are identical to those obtained by using the DGS.

2 Locus from Shifting and Scaling

In the first set of problems we consider we are given two fixed points and a moving point on a smooth convex curve, and need to find the locus of a point lying on the line segment connecting one fixed point and a moving point. We first present the original problem and solve it by hand and then see how the problem can be extended to other scenarios in 2D. Next, we summarize how the problem is related to the translation and scaling of figures.

2.1 Generating a circle with two fixed points

The setting for our first locus problem has a fixed point on a circle and another fixed point that is not on the circle. Example 1 is a slightly modified version of the original practice problem discussed in the Introduction (see [8]).

Example 1 *In Figure 1 we are given a fixed circle in blue,* $(x-a)^2+(y-b)^2=r^2$ *and the moving point* $D=(x_0,y_0)$ *is on the circle. Furthermore we choose the fixed point* $C=(c,d)$ *that is not on the circle and let* E *be the midpoint of* CD*. The goal is to find the locus of* E*.*

Figure 1: Circle and two fixed points.

We first see how students solve this problem by hand in an exam. We note that the midpoint E of CD can be written as $E=\left(\frac{c+x_0}{2},\frac{d+y_0}{2}\right)$. Let $x=\frac{c+x_0}{2}$ and $y=\frac{d+y_0}{2}$. Then we see $\left((2x-c)-a\right)^2+\left((2y-d)-b\right)^2=r^2$, which implies that

$$\begin{aligned}(2x-c)^2-2a\,(2x-c)+a^2+(2y-d)^2-2b\,(2y-d)+b^2&=r^2\\ 4x^2-4cx+c^2-4ax+2ac+a^2+4y^2-4dy+d^2-4by+2bd+b^2&=r^2\end{aligned}$$

After simplifying, we see

$$\left(x-\left(\frac{a+c}{2}\right)\right)^2+\left(y-\left(\frac{b+d}{2}\right)\right)^2=\frac{1}{4}r^2.$$

Indeed the locus of the point E is the circle with center $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ and radius $\frac{r}{2}$.

Exploration. Following the discussions from Example 1, if we let the point E satisfy $\overrightarrow{CE} = s\overrightarrow{CD}$ with $s \in (0,1)$ and we would like to find the locus of E, then it is easy to verify that the locus for E will be as follows:

$$\left(x - s^2\left(a+c\right)\right)^2 + \left(y - s^2\left(b+d\right)\right)^2 = \left(sr\right)^2.$$

To

2.2 Locus as a result of simple translation and scaling

We may view the discussions in the preceding Example 1 as a simple translation and scaling from a given curve to the other. For instance, if we consider the given fixed points $C = (c,d), A = (a,b)$ and a curve C_1 . We assume $C \neq A$. By taking the moving point D to be on the curve C_1, our objective is to find the locus of the midpoint E of CD. If O denotes the origin $(0,0)$,then the locus $\overrightarrow{OE} = \overrightarrow{OC} + \overrightarrow{CE} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD}$. In particular, if C_1 represents the circle centered at A and of radius r, then the locus of the point E will be the circle centered at C and the radius is being scaled to $\frac{1}{2}$ of the original circle. (See Figure 2(a).) It is clear now that if we were to find the locus of the point E satisfying $\overrightarrow{CE} = s\overrightarrow{CD}$, where $s \in [0,1]$, it is equivalent to asking the locus of $\overrightarrow{OE} = \overrightarrow{OC} + \overrightarrow{CE} = \overrightarrow{OC} + s\overrightarrow{CD}$, where $s \in [0,1]$. If the equation of C_1 is represented by $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$, then we can write the locus of $\overrightarrow{OE}$ as

$$\begin{aligned}
\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} c \\ d \end{bmatrix} + s \begin{bmatrix} x_1(t) - c \\ y_1(t) - d \end{bmatrix} \\
&= (1-s)\begin{bmatrix} c \\ d \end{bmatrix} + s \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}.
\end{aligned}$$

If C_1 represents the circle centered at A and of radius r, the locus of $\overrightarrow{OE}$ can be viewed as a result of the combination of translation and scaling. Figure 2(b) shows the translation and scaling when $s = \frac{1}{4}$.

It is clear that there can be more than one way of expressing the locus of E. For example, we may write $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CE} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CD} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\left(\overrightarrow{AD} - \overrightarrow{AC}\right) = \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{AD}\right)$. To find the locus of the point F satisfying $\overrightarrow{CF} = s\overrightarrow{CD}$, where $s \in (0,1)$, in this case, we note $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$, where

$$\begin{aligned}
\overrightarrow{AF} &= \overrightarrow{AC} + s\overrightarrow{CD} && (1) \\
&= \overrightarrow{AC} + s\left(\overrightarrow{AD} - \overrightarrow{AC}\right) \\
&= (1-s)\overrightarrow{AC} + s\overrightarrow{AD}, && (2)
\end{aligned}$$

If the equation of C_1 is represented by $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$, then the locus of $\overrightarrow{OE}$ can be written as

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + (1-s)\begin{bmatrix} c-a \\ d-b \end{bmatrix} + s\begin{bmatrix} x_1(t) - a \\ y_1(t) - b \end{bmatrix},$$

where $s \in [0,1]$. We see that when $s = 0$ we get the point C and when $s = 1$ we get the parametric curve C_1. In particular, Figure 2(b) shows the translation and scaling when $s = \frac{1}{4}$ and C_1 is a circle.

(a) (b)

Figure 2: Locus of the point F (a) when $s = \frac{1}{2}$ and (b) when $s = \frac{1}{4}$.

Analytic verification that the locus is a simple translation and scaling is easily completed with the help of a CAS such as Maple [6]. Figure 3 shows six snapshots of the translation and scaling for the case when $A = (a,b) = (-.725, .15), r = .7685864$, and $C = (c,d) = (-1.8, 0.89)$. In each frame the locus is a circle centered at a point lying along the line segment AC (because of the factor $(1-s)\overrightarrow{AC}$ in 1) and each radius increases as s increases (due to the factor of $s\overrightarrow{AD}$ in 1).

Supplemental resource [S1] provides a framework for further hands-on exploration on this problem and other interesting translation and scaling problems. In particular, we encourage readers to explore this problem when the circle is replaced with an ellipse as follows:

Exercise 2 ?? *Suppose we are given an ellipse and two fixed points A and C respectively where A is the center of the ellipse and $C \neq A$. (See Figure 4.) We let D be a moving point on the ellipse and E be a point such that $\overrightarrow{CE} = s\overrightarrow{CD}$, $s \in [0,1]$. Find the locus $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$.*

It is clear that the locus in this case is a result of simple translation and scaling from the original ellipse. The two scenarios in Figure 4 show the locus $\overrightarrow{OE}$ in red when $s = \frac{1}{2}$ and $\frac{1}{4}$ respectively. Here we use the ellipse of $\frac{x^2}{4} + y^1 = 1$ and the fixed point $C = [0, 1.3]$.

The following observation is a simple result of 1. More exploration can be found in supplemental resource [S1].

Theorem 3 *Let $A = (a,b)$, $C = (c,d)$ be two fixed points and P represent a closed parametric curve that is represented by $[x(a,b,r,t), y(a,b,r,t)] = [a + f(r,t)\cos t, b + g(r,t)\sin t]$ where $f(r,0) = 0$ and $t \in [0, 2\pi]$. If D is a moving point on P, then the locus F satisfying $\overrightarrow{CF} = s\overrightarrow{CD}$, $s \in [0,1]$, is a simple translation of the parametric curve from point C to point A with a scaling factor of s. In particular, when $f(r,t)$ and $g(r,t)$ are positive constants then P is an ellipse and the locus corresponds to a translation and scaling of this ellipse.*

(a) (b) (c)

(d) (e) (f)

Figure 3: Locus of the point E (a) when $s = 0$, (b) when $s = 0.16667$, (c) when $s = 0.29167$, (d) when $s = 0.5$, (e) when $s = 0.79167$, and (f) when $s = 0.91667$.

(a) (b)

Figure 4: Snapshots from the transition between two ellipses when (a) $s = \frac{1}{2}$ and (b) $s = \frac{1}{4}$.

In higher dimensions there can be more than one way of expressing the locus for the translation and scaling. We describe our 3D extension as follows:

Theorem 4 *Let $A = (a, b, c)$, $C = (d, e, f)$ be two fixed points and P represent a closed parametric surface that is represented by $\overrightarrow{OA} + \begin{bmatrix} x_1(t_1, t_2) \\ y_1(t_1, t_2) \\ z_1(t_1, t_2) \end{bmatrix}$, where $t_1 \in [0, 2\pi]$ and $t_2 \in [0, \pi]$. If D is a moving point on P, then the locus of the point E satisfying $\overrightarrow{CE} = s\overrightarrow{CD}$, $s \in [0, 1]$, which can be equivalently described in parametric form as*

$$\begin{bmatrix} x_2(t_1, t_2) \\ y_2(t_1, t_2) \\ z_2(t_1, t_2) \end{bmatrix} = (1 - s) \begin{bmatrix} d \\ e \\ f \end{bmatrix} + s \begin{bmatrix} a + x_1(t_1, t_2) \\ b + y_1(t_1, t_2) \\ c + z_1(t_1, t_2) \end{bmatrix}$$

, is a simple translation of the parametric surface $\begin{bmatrix} x_1(t_1, t_2) \\ y_1(t_1, t_2) \\ z_1(t_1, t_2) \end{bmatrix}$ from point C to point A with a scaling factor of s. In particular, $s = 0$ corresponds to the point C and $s = 1$ represents the surface of $\overrightarrow{OA} + \begin{bmatrix} x_1(t_1, t_2) \\ y_1(t_1, t_2) \\ z_1(t_1, t_2) \end{bmatrix}$.

The proof of this result follows immediately from the simple observation that

$$\begin{aligned} \overrightarrow{OF} &= \overrightarrow{OC} + \overrightarrow{CF} = \overrightarrow{OC} + s\overrightarrow{CD} \\ &= \begin{bmatrix} d \\ e \\ f \end{bmatrix} + s \begin{bmatrix} a + x_1(t_1, t_2) - d \\ b + y_1(t_1, t_2) - e \\ c + z_1(t_1, t_2) - f \end{bmatrix} \\ &= (1 - s) \begin{bmatrix} d \\ e \\ f \end{bmatrix} + s \begin{bmatrix} a + x_1(t_1, t_2) \\ b + y_1(t_1, t_2) \\ c + z_1(t_1, t_2) \end{bmatrix}. \end{aligned}$$

We provide an example on 3D exploration in supplemental resource [S1].

3 Locus From Linear Combinations

Suppose we are given three points A, B, C on three respective curves C_1, C_2, C_3. We would like to explore the locus of $r\overrightarrow{AB} + s\overrightarrow{AC}$, where r and $s \in (0, 1)$. Similarly, if we are given four points A, B, C and D on four respective surfaces of S_1, S_2, S_3 and S_4, we can explore the locus of $r\overrightarrow{AB} + s\overrightarrow{AC} + t\overrightarrow{AD}$,where r, s and $t \in (0, 1)$.

3.1 Locus of two moving points

We first discuss a locus generated by linear combinations of two vectors from two respective closed curves.

Figure 5: Circle and two fixed points.

We consider two circles C_1 and C_2, whose centers are at A and C, and the radii are r_1 and r_2 respectively. We let E and F be two moving points on C_1 and C_2 respectively. If G is the point so that $AEGF$ forms a parallelogram, what is the locus of G?

It is easy to see that the points G satisfy $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \overrightarrow{OA} + \left(\overrightarrow{AE} + \overrightarrow{AF}\right)$ (see Figure 5). We observe that it is easier to solve this problem using vectors than algebraically.

To simplify our problem a bit, assume $A = (0,0)$. The first glance of the locus contains a surprise. The new locus (in red of both parts of Figure 5) is centered at the center C of the second circle C_2 but with a radius larger than r_2. The locus $\overrightarrow{OG} = \overrightarrow{AE} + \overrightarrow{AF}$ can be written as $\overrightarrow{AE} + \left(\overrightarrow{AC} + \overrightarrow{CF}\right) = \left(\overrightarrow{AE} + \overrightarrow{AC}\right) + \overrightarrow{CF}$. Note that $\left(\overrightarrow{AE} + \overrightarrow{AC}\right)$ is a translation from the circle C_1 to another circle centered at C with the same radius r_1. But the locus for $\left(\overrightarrow{AE} + \overrightarrow{AC}\right) + \overrightarrow{CF}$ becomes the circle centered at C and radius is $r_1 + r_2$. This is clear if we take C_1 to be $[a + r\cos t, b + r\sin t]$ and C_2 to be $[c + r_2\cos t, d + r_2\sin t]$. If E and F are two moving points on C_1 and C_2 respectively, then it is clear that the locus $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AE} + \overrightarrow{AF} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} (a + r\cos t) - a \\ (b + r\sin t) - b \end{bmatrix} + \begin{bmatrix} (c + r_2\cos t) - a \\ (d + r_2\sin t) - b \end{bmatrix} = \begin{bmatrix} (c + (r + r_2)\cos t) \\ (d + (r + r_2)\sin t) \end{bmatrix}$. It is clear that the locus in this case is a circle centered at (c, d) with radius $r + r_2$. We see an animation from supplemental resource [S1] that the locus in green is indeed a translation from A to C with radius being the sum of the original two circles, we show a sequence of screen shots below when $A = (-1, 2), C = (2, 3), r_2 = 1$ and r is going from 0 to 1 as following Figure 6. Additional exploration is provided in supplemental resource [S2].

Remark: We encourage readers to explore the generalization of the previous example when the points G satisfy $\overrightarrow{OG} = \overrightarrow{OA} + r\overrightarrow{AE} + s\overrightarrow{AF}$, where r and $s \in (0, 1)$.

We now consider the scenario when the circles C_1 and C_2 are replaced by ellipses:

Exercise 5 *Given two fixed ellipses:* $\frac{x^2}{4} + y^2 = 1$ *(in blue) and* $\frac{(x+1)^2}{2} + (y-1)^2 = 1$ *(in black), centered at* O *and* P *respectively (see Figure 7 below). Let* A *and* B *be two moving points on these two ellipses respectively. Find the locus for* $\overrightarrow{OA} + \overrightarrow{OB}$.

We leave it to the reader to verify that the locus is an ellipse centered at P. In fact, the major and minor axes of the locus will be the sums from the the respective lengths of the original two ellipses. In other words, the equation of the locus should be $\dfrac{(x+1)^2}{\left(2+\sqrt{2}\right)^2} + \dfrac{(y-1)^2}{(1+1)^2} = 1.$

(a) (b) (c)

Figure 6: Snapshots of the translation from $A = (-1, 2)$ to $C = (2, 3)$ when (a) $r = 0.45833$, (b) $r = 0.625$, and (c) $r = 1$.

Figure 7: Locus and translation of an ellipse.

Remark: The locus in the preceding example can be viewed as a result of a linear combinations from the original two ellipses. Readers are encouraged to explore the problem of finding the locus of $r\overrightarrow{OA} + s\overrightarrow{OB}$, where r and $s \in (0, 1)$.

Exercise 6 *We are given two fixed cardioid* $[x_1(a, r, t), y_1(b, r, t)] = [a+(r-\cos t)\cos t, b+(r-\cos t)\sin t]$ *and* $[x_2(c, r, t), y_2(d, r, t)] = [c+(r-\cos t)\cos t, d+(r-\cos t)\sin t]$. *Let* A *and* B *be two moving points on these two cardioids respectively. Find the locus for* $\overrightarrow{OA} + \overrightarrow{OB}$. *(Solution can be found at [S2], [S6] or [S9])*

3.2 Locus of three moving points

We now explore finding a locus when there are three moving points on three separate closed curves. It is surprisingly simple to find the locus using vectors in this case once we know the given (parametric) equations of the three closed curves.

First, recall that the implicit form for the ellipse centered at (c_x, c_y) with major axis r_x and minor axis r_y and rotated by the angle α is

$$\frac{((x-c_x)\cos\alpha + (y-c_y)\sin\alpha)^2}{r_x^2} + \frac{((x-c_x)\sin\alpha - (y-c_y)\cos\alpha)^2}{r_y^2} = 1.$$

The corresponding parametric representation of this rotated ellipse is, for each $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} x(\theta) &= c_x + r_x\cos\alpha\cos\theta - r_y\sin\alpha\sin\theta, \\ y(\theta) &= c_y + r_x\sin\alpha\cos\theta + r_y\cos\alpha\sin\theta. \end{aligned}$$

So, for example

Example 7 *Let* C_1 *be the ellipse* $\frac{(x+2)^2}{4} + (y-1)^2 = 1$ *rotated by the angle of* $\alpha = 120$ *degrees. The center of this ellipse is* $(-2, 1)$, *the major radius is* 2 *and minor radius is* 1. *We write the parametric equation* C_1 *as follows:*

$$\begin{aligned} x_1(\theta) &= -2 + 2\cos\left(\frac{2\pi}{3}\right)\cos\theta - \sin\left(\frac{2\pi}{3}\right)\sin\theta, \\ y_1(\theta) &= 1 + 2\sin\left(\frac{2\pi}{3}\right)\sin\theta + \cos\left(\frac{2\pi}{3}\right)\sin\theta, \end{aligned}$$

where $\theta \in [0, 2\pi]$. *Let the second curve be the cardioid* C_2: $r = 1 - \cos\theta$, *for* $\theta \in [0, 2\pi]$. *Thus, a parametric representation of* C_2 *is*

$$\begin{aligned} x_2(\theta) &= (1-\cos\theta)\cos\theta, \\ y_2(\theta) &= (1-\cos\theta)\sin\theta, \end{aligned}$$

where $\theta \in [0, 2\pi]$. *And, let the third curve* C_3 *be the unit circle centered at the point* $(2, 2)$:

$$\begin{aligned} x_3(\theta) &= 2 + \cos\theta, \\ y_3(\theta) &= 2 + \sin\theta, \end{aligned}$$

where $\theta \in [0, 2\pi]$. *Let* I, F, *and* G *be three moving points on* C_1, C_2 *and* C_3, *respectively. The goal is to find the locus of* $\overrightarrow{IF} + \overrightarrow{IG}$.

If the locus points are called J, then the problem is to describe the vectors $\overrightarrow{OJ} = \overrightarrow{OI} + \overrightarrow{IJ}$, where $\overrightarrow{IJ} = \overrightarrow{IF} + \overrightarrow{IG}$,

$$\overrightarrow{IF} = \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = \begin{bmatrix} (1-\cos\theta)\cos\theta - \left[-2 + 2\cos\left(\frac{2\pi}{3}\right)\cos\theta - \sin\left(\frac{2\pi}{3}\right)\sin\theta\right] \\ (1-\cos\theta)\sin\theta - \left[1 + 2\sin\left(\frac{2\pi}{3}\right)\sin\theta + \cos\left(\frac{2\pi}{3}\right)\sin\theta\right] \end{bmatrix},$$

and

$$\overrightarrow{IG} = \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = \begin{bmatrix} 2 + \cos\theta - \left[-2 + 2\cos\left(\frac{2\pi}{3}\right)\cos\theta - \sin\left(\frac{2\pi}{3}\right)\sin\theta\right] \\ 2 + \sin\theta - \left[1 + 2\sin\left(\frac{2\pi}{3}\right)\sin\theta + \cos\left(\frac{2\pi}{3}\right)\sin\theta\right] \end{bmatrix}.$$

We use Maple [6] to show the locus $\overrightarrow{OJ}$ in red in the Figures 8(a) and 8(b) at $t = 0$ and $t = \pi$, respectively. In addition, Figures 8(a) and 8(b) clearly demonstrate that $\overrightarrow{IJ}$ is a result of linear combination of $\overrightarrow{IF}$ and $\overrightarrow{IG}$.

With the use of a CAS, such as Maple [6], it is easy to generate animations when the weights in the linear combination are allowed to be r or s, where each value is between 0 and 1. For example, Figures 8(c) and 8(d) demonstrate the locus when $(r, s) = (0.54167, 1)$ and $(r, s) = (0.5, 0.20833)$, respectively. Please see supplemental resource [S3] for more details and explorations.

Discussion: In the preceding example, the locus is a closed curve C_4 that can be determined once we are given the three closed curves, C_1, C_2, and C_3, and properly setting up the linear combinations of vectors. One application of this will be using the light source at a point on either C_1, C_2, or C_3, and we need to find the caustic curve of C_4, which we call it C_5. We can continue this process by finding a sequence of closed curves, C_4, C_5, ... so that C_{n+1} depends on C_n, where $n \geq 3$, which we can imagine finding each C_n becomes more computational intensive when n increases.

Now we consider the locus resulting from four closed surfaces in 3D, three of which are originated from the preceding 2D example. Suppose surface S_1 has a parametric representation as

$$\begin{aligned} x_1(t_1, t_2) &= (x_1(t_1) + 2)\sin t_2 - 2 \\ &= -2 + \left(2\cos\left(\frac{2\pi}{3}\right)\cos t_1 - \sin\left(\frac{2\pi}{3}\right)\sin t_1\right)\sin t_2, \\ y_1(t_1, t_2) &= (y_1(t_1) - 1)\sin t_2 + 1 \\ &= 1 + \left(2\sin\left(\frac{2\pi}{3}\right)\sin\theta + \cos\left(\frac{2\pi}{3}\right)\sin\theta\right)\sin t_2, \\ z_1(t_1, t_2) &= \cos(t_2), \end{aligned}$$

where $t_1 \in [0, 2\pi]$ and $t_2 \in [0, \pi]$. The surface S_2 is given by rotating the curve $[x_2(t_1), y_2(t_1)]$ around the $x - axis$ as follows:

$$\begin{aligned} x_2(t_1, t_2) &= x_2(t_1) \\ y_2(t_1, t_2) &= y(t_1)\cos t_2 \\ &= (1 - \cos t_1)\sin t_1 \cos t_2 \\ z_2(t_1, t_2) &= y(t_1)\sin t_2 \\ &= (1 - \cos t_1)\sin t_1 \sin t_2, \end{aligned}$$

(a) (b)

(c) (d)

Figure 8: Locus

(a) (b)

(c) (d)

Figure 9: (a) The surface S_1. (b) The surface S_2. (c) The surfaces S_1 (yellow), S_2 (blue), S_3 (red), and S_4 (magenta). (d) Locus of linear combinations of surfaces S_1, S_2, S_3, and S_4.

where $t_1, t_2 \in [0, 2\pi]$. And, the surfaces S_3 and S_4 are taken to be spheres with radius 1 and centers $(1, 1, 1)$ and $(1, -1, -1)$, respectively. As such, S_3 and S_4 are written parametrically as:

$$\begin{aligned} x_3(t_1, t_2) &= 1 + \sin t_2 \cos t_1, & y_3(t_1, t_2) &= 1 + \sin t_2 \sin t_1, & z_3(t_1, t_2) &= 1 + \cos t_2, \\ x_4(t_1, t_2) &= 1 + \sin t_2 \cos t_1, & y_4(t_1, t_2) &= -1 + \sin t_2 \sin t_1, & z_4(t_1, t_2) &= -1 + \cos t_2, \end{aligned}$$

where $t_1, t_2 \in [0, 2\pi]$. If we let A, B, C, D denote the four moving points on S_1, S_2, S_3, and S_4, respectively, then the locus of interest consists of the points E such that $\overrightarrow{OE} = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD})$. In this form the locus is easily calculated and plotted (see Figure 9(d)).

We see S_1 is a rotated ellipsoid shown in yellow (see Figure 9(a)), S_2 is a surface shown in blue which rotates the cardioid $[x_2(t_1), y_2(t_1)]$ around the x-axis (see Figure 9(b)). Figure 9(c) shows all four surfaces, including the spheres S_3 and S_4 in red and magenta, respectively. We depict, in Figure 9(d), the locus of the points E satisfying $\overrightarrow{OE} = \overrightarrow{OA} + \left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right)$ in green. Please see supplemental resource [S3] for more details. We encourage the readers to explore the locus of linear combinations of vector $\overrightarrow{OE} = \overrightarrow{OA} + r\overrightarrow{AB} + s\overrightarrow{AC} + t\overrightarrow{AD}$, where r, s, and $t \in (0, 1)$.

4 Locus When Fixing Two Points On A Curve

In this section, we discuss a locus problem that is inspired by the following college entrance exam practice problem from China (see [8]).

Example 8 *We are given a fixed ellipse, say* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, BE *is the major axis and* F *is a moving point on the ellipse. We construct two lines passing through* B *and* E *respectively and two lines intersect at* I *such that* $\measuredangle IBF = \measuredangle FEI = 90°$*.If* J *is the midpoint of* BI, *find the locus of* J*.[Note that the red curve represents the scattered plot of* J *that can be traced by using [2]]*

(a) (b)

Figure 10: Ellipse, two fixed points, and locus of the point J when (a) $t = t_1$ and (b) $t = t_2$.

We remark here that if this problem is presented as a mathematics experiment class instead, more students would have enjoyed exploring it if technological tools are available to learners. For example, before answering this question analytically, they can play and learn how a locus might look like, which makes the learning process much more enjoyable. In fact, one may adopt the following steps when exploring a problem:

1. Start with a DGS (say [2] in this case) for necessary geometric constructions and next use the scattered plot to conjecture what the locus should look like. Further experiment with a symbolic DGS such as [3] to generate a possible symbolic solution.

2. Solve the problem analytically by hand for simple scenario or solve it analytically with a CAS such as [6] if the problem becomes algebraically intensive.

We first present how one may solve this simple case by hand without the presence of technological tools. We let $B = (-a, 0)$, $E = (a, 0)$ and the moving point on the ellipse $F = (x_0, y_0)$. We denote the slopes of FB and FE to be k_{FB} and k_{FE} respectively, then $k_{FB} = \frac{y_0}{x_0+a}$ and $k_{FE} = \frac{y_0}{x_0-a}$. Thus, the the line equations for BI and EI are

$$y = -\frac{x_0 + a}{y_0}(x + a) \tag{3}$$

and

$$y = -\frac{x_0 - a}{y_0}(x - a) \tag{4}$$

respectively. We substitute 3 into 4 and yield the followings:

$$\begin{aligned} -\frac{x_0 + a}{y_0}(x + a) &= -\frac{x_0 - a}{y_0}(x - a) \\ (x_0 + a)(x + a) &= (x_0 - a)(x - a) \\ 2ax + 2ax_0 &= 0. \end{aligned}$$

We see if $a \neq 0, x = -x_0$. In other words, if we assume $a \neq 0$, then $I = (-x_0, -\frac{1}{y_0}(a + x_0)(a - x_0)$. The midpoint for BI is thus

$$J = (X, Y) = \left(\frac{-x_0 - a}{2}, -\frac{1}{2y_0}(a + x_0)(a - x_0)\right).$$

This implies that $Y = \frac{1}{y_0} X (a - x_0)$. To obtain the parametric form for the locus J, we note that $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, which implies that $x_0 = a\cos t$ and $y_0 = b\sin t$. Thus we obtain the parametric equation for the locus to be

$$\begin{cases} X = \dfrac{-a\cos t - a}{2} = \dfrac{-a(\cos t + 1)}{2} \\ Y = \dfrac{X(a - a\cos t)}{b\sin t} = \dfrac{aX(1 - \cos t)}{b\sin t} \end{cases}.$$

Exploration 1. It is not difficult to extend our result if we ask for the locus $J = (X, Y)$ satisfying $\overrightarrow{BJ} = s\overrightarrow{BI}$ for some real number s. In view of $\overrightarrow{BJ} = s\overrightarrow{BI}$, we have

$$\begin{aligned} (X + a, Y) &= s\left(-x_0 + a, -\frac{1}{y_0}(a + x_0)(a - x_0)\right) \\ J = (X, Y) &= (-s(x_0 - a) - a), \frac{s}{y_0}(x_0 + a)(x_0 - a) \end{aligned}$$

Since $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, we set $x_0 = a\cos t$ and $y_0 = b\sin t$, where $t \in [0, 2\pi]$, to obtain the parametric equation for the locus J to be

$$\begin{aligned} X &= -s(a\cos t - a) - a \\ Y &= \frac{s\left((a\cos t)^2 - a^2\right)}{b\sin t}. \end{aligned} \tag{5}$$

Remarks:

1. We use the DGS Geometry Expressions [3] to construct the locus J above through geometry constructions. We depict some screen shots when $s = 0.25$ and 0.75 in the following Figures 11(a) and 11(b).

2. We use the CAS [6] and the analytic derivation 5 to verify that the Figures 11(a) and 11(b) obtained by using [3] are identical to the corresponding ones obtained by using [6].

(a) (b)

Figure 11: Locus of the points J for (a) $s = \frac{1}{4}$ and (b) $s = \frac{3}{4}$.

Exploration 2. We discuss the scenario if we set $\measuredangle IBF = \measuredangle FEI = \theta$. We let $B = (-a, 0)$, $E = (a, 0)$ and the moving point on the ellipse $F = (x_0, y_0)$. We denote the slopes of FB and FE to be k_{FB} and k_{FE} respectively, then $k_{FB} = \frac{y_0}{x_0+a} = \tan\theta_1$, and $\theta_1 = \tan^{-1}\left(\frac{y_0}{x_0+a}\right)$.In the meantime, $k_{FE} = \frac{y_0}{x_0-a} = \tan\theta_2$, and $\theta_2 = \tan^{-1}\left(\frac{y_0}{x_0-a}\right)$.

We write the line equations for BI and EI to be

$$y = \tan\left(\tan^{-1}\left(\frac{y_0}{x_0+a}\right) + \theta\right)(x+a) \tag{6}$$

and

$$y = \tan\left(\tan^{-1}\left(\frac{y_0}{x_0-a}\right) - \theta\right)(x-a) \tag{7}$$

respectively. We use 6 and 7 to solve for x and yield the following from Maple [6]

$$x = \frac{a\tan\left(\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right) - \tan\left(\arctan\left(\frac{y_0}{x_0+a}\right)+\theta\right)}{\tan\left(\arctan\left(\frac{y_0}{x_0+a}\right)+\theta\right) + \tan\left(\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right)},$$

and use 6 or 7 to figure out y, which is too long to display and can be found at [S4]. We further substitute $x_0 = a\cos t$ and $y_0 = b\sin t$ into x and y respectively. Next if we denote the intersection between BI and EI as $I = (x, y)$ and locus $J = (X_1, Y_1)$ satisfying $\overrightarrow{BJ} = s\overrightarrow{BI}$, where $s \in (0, 1)$. Then we see that the locus J satisfying

$$\begin{bmatrix} X_1(a,b,s,t,\theta) \\ Y_1(a,b,s,t,\theta) \end{bmatrix} = \begin{bmatrix} s(x(a,b,s,t,\theta)+a)-a \\ sy(a,b,s,t,\theta) \end{bmatrix} = s\begin{bmatrix} x(a,b,s,t,\theta) \\ y(a,b,s,t,\theta) \end{bmatrix} - (1-s)\begin{bmatrix} a \\ 0 \end{bmatrix}.$$

We display $X_1(a,b,s,t,\theta)$ and $Y_1(a,b,s,t,\theta)$ obtained from Maple [6] in the following screen shots respectively:

In the meantime, the parametric equation for the locus J can also be obtained from Geometry Expressions [3] as follows:

$$\left(\begin{array}{c} X=-(1-s)\,|a|-\\ \dfrac{2a^2 s\sin(t)\cos(t)\,|b|}{-2a^2\sin(t)^2\sin(\theta)\cos(\theta)+2b^2\sin(t)^2\sin(\theta)\cos(\theta)-2\sin(t)\,|a|\,|b|+4\sin(t)\sin(\theta)^2\,|a|\,|b|}\\ Y=\dfrac{s\left(-a^2\sin(\theta)^2\,|a|+a^2\sin(\theta)^2\cos(t)^2\,|a|-b^2\sin(t)^2\cos(\theta)^2\,|a|-2a^2\sin(t)\sin(\theta)\cos(\theta)\,|b|\right)}{\left(-a^2\sin(t)\sin(\theta)\cos(\theta)+b^2\sin(t)\sin(\theta)\cos(\theta)+\left(-|a|+2\sin(\theta)^2\,|a|\right)|b|\right)\sin(t)} \end{array}\right).$$

Remark: The DGS Geometry Expressions [3] has the capability of linking its outputs to a CAS such as [6] for further computation. Although it is not trivial to prove algebraically that the locus equation $[X_1, Y_1]$ obtained from the CAS [6] is identical to that of $[X, Y]$ obtained from the DGS [3]. We provide the worksheet in the [S4], [S7] or [S10] that show that the plots from either system are identical when varying one parameter. Furthermore, we also show that two plots are identical when we use animations by varying one parameter and fixing the other parameters. In Figure 12(a), we show that the locus for $[X_1, Y_1]$ or $[X, Y]$ is the same for $a = 2, b = 2, s = \frac{1}{4}$ and $\theta = 0.7853$. In Figure 12(b), we show that the locus for $[X_1, Y_1]$ or $[X, Y]$ is the same for $a = 2, b = 2, s = \frac{1}{4}$ and $\theta = 2.0734512$.

4.1 When we replace the ellipse by a cardioid

Assuming technological tools are available to learners, it is natural to ask what if the ellipse, discussed earlier, is replaced by another curve, say a cardioid. In particular, we consider the following

Example 9 *We are given the cardioid $r = 1 - \cos t, t \in [0, 2\pi]$ in Figure 13. Suppose the moving point C is on the cardioid and two lines passing through $B = (0, 0)$ and $A = (a, 0)$ respectively, and intersect at G so that the angles $\measuredangle CAG = \measuredangle CBG = 90°$. If J is the midpoint of AG, find the locus for J. [Note the red curve is a scattered plot of the locus of J, when $A = (-2, 0)$, and has been obtained using [2]]*

Since $A = (a, 0)$ and $B = (0, 0)$, and the moving point $C = (x_0, y_0)$, we denote the slopes for CB and CA to be $k_{CB} = \frac{y_0}{x_0}$ and $k_{CA} = \frac{y_0}{x_0-a}$ respectively. Thus, the the line equations for CB and CA are respectively

$$y = -\frac{x_0}{y_0}x, \tag{8}$$

(a) (b)

Figure 12: Parametric curves $[X(2,2,\frac{1}{4},t,\theta),Y(2,2,\frac{1}{4},t,\theta)] = [X_1(2,2,\frac{1}{4},t,\theta),Y_1(2,2,\frac{1}{4},t,\theta)]$ for (a) $\theta = \frac{\pi}{4}$ and (b) $\theta = \frac{2\pi}{3}$

Figure 13: Circle and two fixed points.

and

$$y = -\frac{(x_0 - a)}{y_0}(x - a) \tag{9}$$

respectively. We substitute 8 into 9 and yield $\frac{x_0}{y_0}x = \frac{(x_0-a)}{y_0}(x-a)$. By assuming $a \neq 0$, we see $x = a - x_0$, then $y = \left(\frac{x_0}{y_0}\right)(x_0 - a)$, in other words, the intersection $G = (a - x_0, \left(\frac{x_0}{y_0}\right)(x_0 - a))$. Then midpoint for AG is

$$J = (X, Y) = \left(\frac{2a - x_0}{2}, \frac{x_0 (x_0 - a)}{2y_0}\right).$$

We note that C is a point on $r = f(t) = 1 - \cos t$, which implies that $x_0 = f(t)\cos t$ and $y_0 = f(t)\sin t$. Thus we obtain the parametric equation for the locus to be

$$\begin{aligned} X &= \frac{2a - (1-\cos t)\cos t}{2} \\ Y &= \frac{(1-\cos t)\cos t\,((1-\cos t)\cos t - a)}{2(1-\cos t)\sin t}. \end{aligned}$$

Exploration 1. It is not difficult to extend our result if we ask for the locus $J = (X, Y)$ such that $\overrightarrow{AJ} = s\overrightarrow{AG}$ for some real number s. In view of $\overrightarrow{AJ} = s\overrightarrow{AG}$, we see $X = a - sx_0$ and $Y = s\left(\frac{x_0}{y_0}\right)(x_0 - a)$. Since (x_0, y_0) is a point on the cardioid $r = f(t) = 1 - \cos t$, the locus J in this case is

$$\begin{aligned} X(a,s,t) &= a - s(1-\cos t)\cos t \\ Y(a,s,t) &= \frac{s(1-\cos t)\cos t}{(1-\cos t)\sin t}((1-\cos t)\cos t - a) \end{aligned} \tag{10}$$

We show some screen shots when $a = -2$ and $s = 0.3$, 0.7 and 1.5 respectively in Figures 14(a)-(c) by using [3] below, which we have verified that they are identical to those corresponding ones when using the CAS [6].

Further Remarks:

1. We notice that the curve of $r = 1 - \cos t$ has a point of non-differentiabilty at $B = (0, 0)$, what will be the corresponding point for the locus J?
2. In view of the derivation in equations 10, we encourage readers to explore how the graphs varies according to the parameters $a, s,$ and t respectively.

Exploration 2. We discuss the scenario when we set $\measuredangle CAG = \measuredangle CBG = \pi - \theta$. If we denote the moving point $C = (x_0, y_0)$ and the intersection $G = (x, y)$ for the lines BG and AG. Since the slopes for CB and CA to be $k_{CB} = \frac{y_0}{x_0}$ and $k_{CA} = \frac{y_0}{x_0 - a}$ respectively. We set $\theta_1 = \arctan\left(\frac{y_0}{x_0}\right)$ and $\theta_2 = \arctan\left(\frac{y_0}{x_0-a}\right)$. The equation for BG and AG can be written as

$$y = \tan(\theta_1 - \theta)\,x \tag{11}$$

and

$$y = \tan(\theta_2 + \theta)(x - a). \tag{12}$$

(a) (b) (c)

Figure 14: Locus of the point F (a) when $s = \frac{1}{2}$ and (b) when $s = \frac{1}{4}$.

We use 11 and 12 to solve for x and yield the following from Maple [6]

$$x = \frac{a \tan\left(-\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right)}{\tan\left(-\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right)+\tan\left(-\arctan\left(\frac{y_0}{x_0}\right)+\theta\right)},$$

and use 6 or 7 to find y, which is too long to display but can be found in the worksheet at [S5]. More explorations can be found in [S8] or [S11]. We further substitute $x_0 = (1-\cos t)\cos t$ and $y_0 = (1-\cos t)\sin t$ into x and y respectively to find the intersection between AG and GG, which we denote it as $G = (x, y)$. Since the locus $J = (X, Y)$ is such that $\overrightarrow{AJ} = s\overrightarrow{AG}$ for some real number s, the locus J satisfying

$$\begin{bmatrix} X_1(a,s,t,\theta) \\ Y_1(a,s,t,\theta) \end{bmatrix} = \begin{bmatrix} s\left(x(a,s,t,\theta)-a\right) \\ sy(a,s,t,\theta) \end{bmatrix} = s\begin{bmatrix} x(a,s,t,\theta)-a \\ y(a,s,t,\theta) \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

We display $X_1(a,s,t,\theta)$ and $Y_1(a,s,t,\theta)$ obtained from Maple [6] in the following screen shots respectively:

In the meantime, the parametric equation for the locus J can be obtained from Geometry Expressions [3], which can be exported to Maple [6], which we show below and allows us for further investigation.

We show in the worksheet [S5], [S8] or [S11] that the family of locus plots for the parametric equations obtained from [6], $[X_1(-2, s, t, \theta), Y_1(-2, s, t, \theta]$ are almost identical to those obtained from [3], $[X(-2, s, t, \theta), Y(-2, s, t, \theta)]$ except a line segment, which puzzles the author. We depict the locus in red when $a = -2, s = 0.7$ and $\theta = 2.1048671$ and $t \in [0, 2\pi]$ in Figures 15(a) and 15(b) for $[X(-2, s, t, \theta), Y(-2, s, t, \theta)]$ and $[X_1(-2, s, t, \theta), Y_1(-2, s, t, \theta)]$ respectively. Furthermore, we plot $[X(-2, s, t, \theta), Y(-2, s, t, \theta)]$ and $[X_1(-2, s, t, \theta), Y_1(-2, s, t, \theta)]$ together with the cardioid of $r = 1 - \cos t$ in blue together in the Figure 15(c), which we see the extra red line segment due to the DGS [3].

5 Conclusion

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS such as [2] or [3]. We also showed the utility of a CAS, such as Maple [6], for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to students from high school. Others require more advanced mathematics often taught in universities. The more advanced problems are excellent examples for future teachers to explore and to understand before they try to explain the simpler problems to their students.

Evolving technological tools definitely have made mathematics fun and accessible. They also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, more students will be inspired to investigate problems ranging from the simple to the more challenging, and even open questions. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

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(a) (b)

(c)

Figure 15: Comparison of the parametric curves $[X(-2, 0.7, t, 2.1048671), Y(-2, 0.7, t, 2.1048671)]$ obtained (a) from Maple and (b) from Geometry Expressions. The results are identical, except for the spurious segment highlighted in red (c).

7 Supplementary Electronic Materials

[S1] Maple worksheet for Section 2.2.

[S2] Maple worksheet for Section 3.1.

[S3] Maple worksheet for Exercise 6.

[S4] Maple worksheet for Example 7.

[S5] Maple worksheet for Example 8.

[S6] Geometry Expressions worksheet for Section 3.1.

[S7] Geometry Expressions worksheet Example 7.

[S8] Geometry Expressions worksheet Example 8.

[S9] The interactive HTML version of the Geometry Expressions worksheet for Section 3.1.

[S10] The interactive HTML version of the Geometry Expressions worksheet for Example 7.

[S11] The interactive HTML version of the Geometry Expressions worksheet for Example 8.

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