

# Locus Problems Related to Linear Combinations of Vectors

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## Abstract

This paper is a more in-depth discussions for selected problems discussed in [1], which are inspired by college entrance exam practice problems from China. When initially seeing the problem during an exam, one may have no clue how the problem is created. In this paper, we begin by presenting the essential algebraic manipulation skills that Chinese high school students are expected to apply to answer the question. We then continue the discussion by exploring various scenarios that utilize technological tools. We shall see many unexpected surprising outcomes. More importantly, we summarize how a problem can be extended to a more general setting whenever is possible.

## 1 Introduction

This paper is a more in depth discussion on selected problems from the published article [1]. Finding a curve defined by the locus of a moving point has been popular and such problems are often asked on Gaokao (a college entrance exam) in China. There have been several exploratory activities (see [5] [7], and [9]) derived from Chinese college entrance exam practice problems (see [8]). In this article, we typically start with a practice problem originated from ([8]) to initiate our discussions. We demonstrate how a problem can be solved without the assistance of technological tools, which shall demonstrate those crucial algebraic manipulation skills that are required by high school students from China. From a content knowledge point of view, this paper is very accessible to those students who have learned parametric equations and have a basic understanding of linear algebra. Problems presented in this paper can be used as examples for professional training purposes.

In this paper, we stress that if problems are presented as explorations instead of in an examination setting, learners would enjoy learning some new mathematics more than simply performing a collection of somewhat boring algebraic manipulations. We believe making conjectures by seeing possible solutions before asking complete analytic or algebraic solutions is

much more accessible, convincing and intuitive to students. In addition to solving simple cases by hand, we typically construct a potential solution geometrically using the trace feature of dynamic geometry software (DGS) such as Geometry Expressions [3]. Finally, we use a computer algebra system (CAS) such as Maple [6] to verify that our analytic solutions are identical to those obtained by using the DGS.

## 2 Locus from Shifting and Scaling

In the first set of problems we consider we are given two fixed points and a moving point on a smooth convex curve, and need to find the locus of a point lying on the line segment connecting one fixed point and a moving point. We first present the original problem and solve it by hand and then see how the problem can be extended to other scenarios in 2D. Next, we summarize how the problem is related to the translation and scaling of figures.

### 2.1 Generating a circle with two fixed points

The setting for our first locus problem has a fixed point on a circle and another fixed point that is not on the circle. Example 1 is a slightly modified version of the original practice problem discussed in the Introduction (see [8]).

**Example 1** *In Figure 1 we are given a fixed circle in blue,  $(x - a)^2 + (y - b)^2 = r^2$  and the moving point  $D = (x_0, y_0)$  is on the circle. Furthermore we choose the fixed point  $C = (c, d)$  that is not on the circle and let  $E$  be the midpoint of  $CD$ . The goal is to find the locus of  $E$ .*

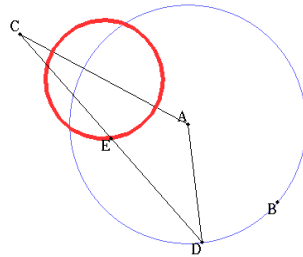


Figure 1: Circle and two fixed points.

We first see how students solve this problem by hand in an exam. We note that the midpoint  $E$  of  $CD$  can be written as  $E = \left(\frac{c+x_0}{2}, \frac{d+y_0}{2}\right)$ . Let  $x = \frac{c+x_0}{2}$  and  $y = \frac{d+y_0}{2}$ . Then we see  $((2x - c) - a)^2 + ((2y - d) - b)^2 = r^2$ , which implies that

$$\begin{aligned} (2x - c)^2 - 2a(2x - c) + a^2 + (2y - d)^2 - 2b(2y - d) + b^2 &= r^2 \\ 4x^2 - 4cx + c^2 - 4ax + 2ac + a^2 + 4y^2 - 4dy + d^2 - 4by + 2bd + b^2 &= r^2 \end{aligned}$$

After simplifying, we see

$$\left(x - \left(\frac{a+c}{2}\right)\right)^2 + \left(y - \left(\frac{b+d}{2}\right)\right)^2 = \frac{1}{4}r^2.$$

Indeed the locus of the point  $E$  is the circle with center  $(\frac{a+c}{2}, \frac{b+d}{2})$  and radius  $\frac{r}{2}$ .

**Exploration.** Following the discussions from Example 1, if we let the point  $E$  satisfy  $\overrightarrow{CE} = s\overrightarrow{CD}$  with  $s \in (0, 1)$  and we would like to find the locus of  $E$ , then it is easy to verify that the locus for  $E$  will be as follows:

$$(x - s^2(a + c))^2 + (y - s^2(b + d))^2 = (sr)^2.$$

To

## 2.2 Locus as a result of simple translation and scaling

We may view the discussions in the preceding Example 1 as a simple translation and scaling from a given curve to the other. For instance, if we consider the given fixed points  $C = (c, d)$ ,  $A = (a, b)$  and a curve  $C_1$ . We assume  $C \neq A$ . By taking the moving point  $D$  to be on the curve  $C_1$ , our objective is to find the locus of the midpoint  $E$  of  $CD$ . If  $O$  denotes the origin  $(0, 0)$ , then the locus  $\overrightarrow{OE} = \overrightarrow{OC} + \overrightarrow{CE} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD}$ . In particular, if  $C_1$  represents the circle centered at  $A$  and of radius  $r$ , then the locus of the point  $E$  will be the circle centered at  $C$  and the radius is being scaled to  $\frac{1}{2}$  of the original circle. (See Figure 2(a).) It is clear now that if we were to find the locus of the point  $E$  satisfying  $\overrightarrow{CE} = s\overrightarrow{CD}$ , where  $s \in [0, 1]$ , it is equivalent to asking the locus of  $\overrightarrow{OE} = \overrightarrow{OC} + \overrightarrow{CE} = \overrightarrow{OC} + s\overrightarrow{CD}$ , where  $s \in [0, 1]$ . If the equation of  $C_1$  is represented by  $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ , then we can write the locus of  $\overrightarrow{OE}$  as

$$\begin{aligned} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} c \\ d \end{bmatrix} + s \begin{bmatrix} x_1(t) - c \\ y_1(t) - d \end{bmatrix} \\ &= (1 - s) \begin{bmatrix} c \\ d \end{bmatrix} + s \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}. \end{aligned}$$

If  $C_1$  represents the circle centered at  $A$  and of radius  $r$ , the locus of  $\overrightarrow{OE}$  can be viewed as a result of the combination of translation and scaling. Figure 2(b) shows the translation and scaling when  $s = \frac{1}{4}$ .

It is clear that there can be more than one way of expressing the locus of  $E$ . For example, we may write  $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CE} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CD} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}(\overrightarrow{AD} - \overrightarrow{AC}) = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{AD})$ . To find the locus of the point  $F$  satisfying  $\overrightarrow{CF} = s\overrightarrow{CD}$ , where  $s \in (0, 1)$ , in this case, we note  $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$ , where

$$\overrightarrow{AF} = \overrightarrow{AC} + s\overrightarrow{CD} \tag{1}$$

$$\begin{aligned} &= \overrightarrow{AC} + s(\overrightarrow{AD} - \overrightarrow{AC}) \\ &= (1 - s)\overrightarrow{AC} + s\overrightarrow{AD}, \end{aligned} \tag{2}$$

If the equation of  $C_1$  is represented by  $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ , then the locus of  $\overrightarrow{OE}$  can be written as

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + (1 - s) \begin{bmatrix} c - a \\ d - b \end{bmatrix} + s \begin{bmatrix} x_1(t) - a \\ y_1(t) - b \end{bmatrix},$$

where  $s \in [0, 1]$ . We see that when  $s = 0$  we get the point  $C$  and when  $s = 1$  we get the parametric curve  $C_1$ . In particular, Figure 2(b) shows the translation and scaling when  $s = \frac{1}{4}$  and  $C_1$  is a circle.

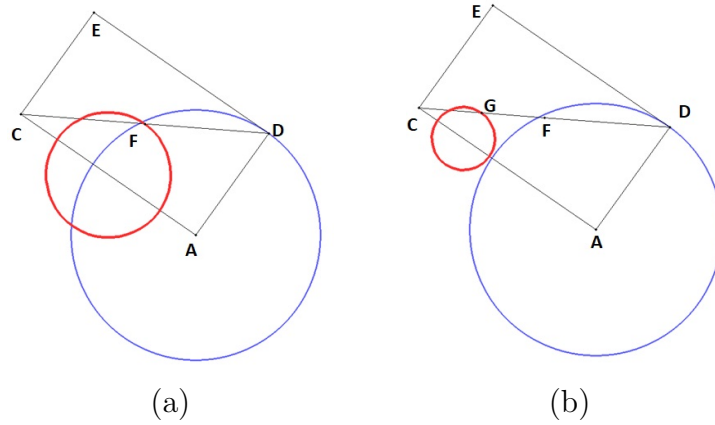


Figure 2: Locus of the point  $F$  (a) when  $s = \frac{1}{2}$  and (b) when  $s = \frac{1}{4}$ .

Analytic verification that the locus is a simple translation and scaling is easily completed with the help of a CAS such as Maple [6]. Figure 3 shows six snapshots of the translation and scaling for the case when  $A = (a, b) = (-.725, .15)$ ,  $r = .7685864$ , and  $C = (c, d) = (-1.8, 0.89)$ . In each frame the locus is a circle centered at a point lying along the line segment  $AC$  (because of the factor  $(1 - s)\vec{AC}$  in 1) and each radius increases as  $s$  increases (due to the factor of  $s\vec{AD}$  in 1).

Supplemental resource [S1] provides a framework for further hands-on exploration on this problem and other interesting translation and scaling problems. In particular, we encourage readers to explore this problem when the circle is replaced with an ellipse as follows:

**Exercise 2** Suppose we are given an ellipse and two fixed points  $A$  and  $C$  respectively where  $A$  is the center of the ellipse and  $C \neq A$ . (See Figure 4.) We let  $D$  be a moving point on the ellipse and  $E$  be a point such that  $\vec{CE} = s\vec{CD}$ ,  $s \in [0, 1]$ . Find the locus  $\vec{OE} = \vec{OA} + \vec{AE}$ .

It is clear that the locus in this case is a result of simple translation and scaling from the original ellipse. The two scenarios in Figure 4 show the locus  $\vec{OE}$  in red when  $s = \frac{1}{2}$  and  $\frac{1}{4}$  respectively. Here we use the ellipse of  $\frac{x^2}{4} + y^2 = 1$  and the fixed point  $C = [0, 1.3]$ .

The following observation is a simple result of 1. More exploration can be found in supplemental resource [S1].

**Theorem 3** Let  $A = (a, b)$ ,  $C = (c, d)$  be two fixed points and  $P$  represent a closed parametric curve that is represented by  $[x(a, b, r, t), y(a, b, r, t)] = [a + f(r, t) \cos t, b + g(r, t) \sin t]$  where  $f(r, 0) = 0$  and  $t \in [0, 2\pi]$ . If  $D$  is a moving point on  $P$ , then the locus  $F$  satisfying  $\vec{CF} = s\vec{CD}$ ,  $s \in [0, 1]$ , is a simple translation of the parametric curve from point  $C$  to point  $A$  with a scaling factor of  $s$ . In particular, when  $f(r, t)$  and  $g(r, t)$  are positive constants then  $P$  is an ellipse and the locus corresponds to a translation and scaling of this ellipse.

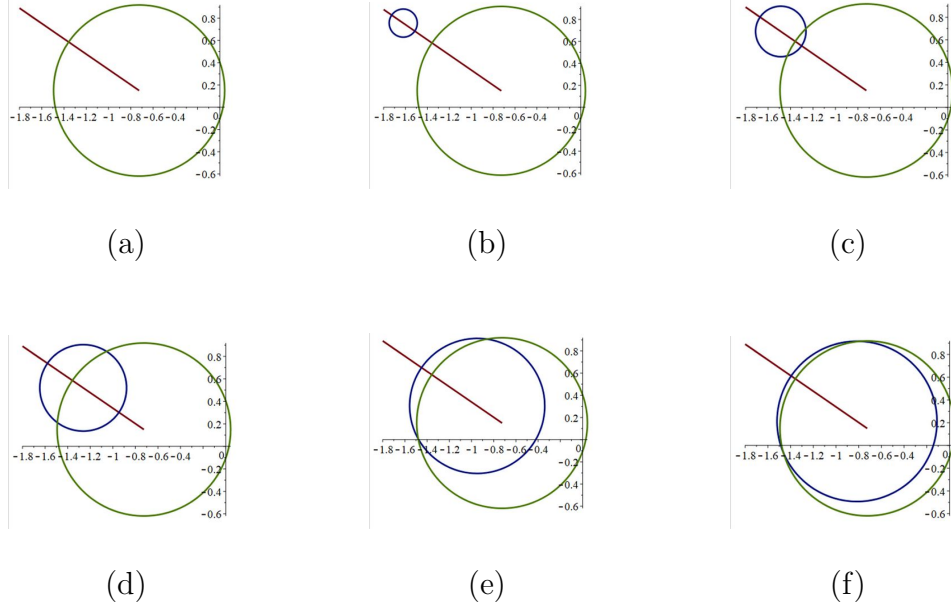


Figure 3: Locus of the point  $E$  (a) when  $s = 0$ , (b) when  $s = 0.16667$ , (c) when  $s = 0.29167$ , (d) when  $s = 0.5$ , (e) when  $s = 0.79167$ , and (f) when  $s = 0.91667$ .

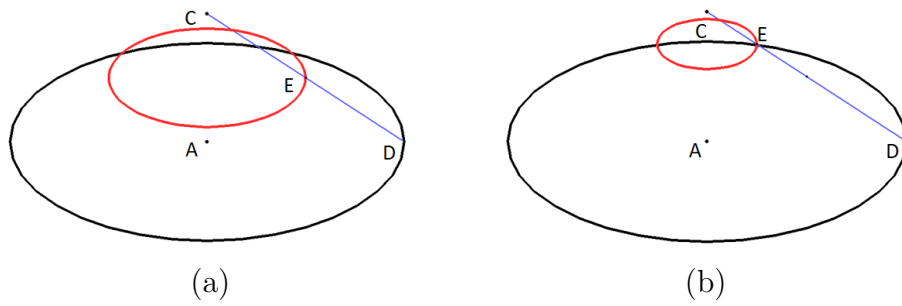


Figure 4: Snapshots from the transition between two ellipses when (a)  $s = \frac{1}{2}$  and (b)  $s = \frac{1}{4}$ .

In higher dimensions there can be more than one way of expressing the locus for the translation and scaling. We describe our 3D extension as follows:

**Theorem 4** Let  $A = (a, b, c)$ ,  $C = (d, e, f)$  be two fixed points and  $P$  represent a closed parametric surface that is represented by  $\overrightarrow{OA} + \begin{bmatrix} x_1(t_1, t_2) \\ y_1(t_1, t_2) \\ z_1(t_1, t_2) \end{bmatrix}$ , where  $t_1 \in [0, 2\pi]$  and  $t_2 \in [0, \pi]$ .

If  $D$  is a moving point on  $P$ , then the locus of the point  $E$  satisfying  $\overrightarrow{CE} = s\overrightarrow{CD}$ ,  $s \in [0, 1]$ , which can be equivalently described in parametric form as

$$\begin{bmatrix} x_2(t_1, t_2) \\ y_2(t_1, t_2) \\ z_2(t_1, t_2) \end{bmatrix} = (1 - s) \begin{bmatrix} d \\ e \\ f \end{bmatrix} + s \begin{bmatrix} a + x_1(t_1, t_2) \\ b + y_1(t_1, t_2) \\ c + z_1(t_1, t_2) \end{bmatrix}$$

, is a simple translation of the parametric surface  $\begin{bmatrix} x_1(t_1, t_2) \\ y_1(t_1, t_2) \\ z_1(t_1, t_2) \end{bmatrix}$  from point  $C$  to point  $A$  with a scaling factor of  $s$ . In particular,  $s = 0$  corresponds to the point  $C$  and  $s = 1$  represents the surface of  $\overrightarrow{OA} + \begin{bmatrix} x_1(t_1, t_2) \\ y_1(t_1, t_2) \\ z_1(t_1, t_2) \end{bmatrix}$ .

The proof of this result follows immediately from the simple observation that

$$\begin{aligned} \overrightarrow{OF} &= \overrightarrow{OC} + \overrightarrow{CF} = \overrightarrow{OC} + s\overrightarrow{CD} \\ &= \begin{bmatrix} d \\ e \\ f \end{bmatrix} + s \begin{bmatrix} a + x_1(t_1, t_2) - d \\ b + y_1(t_1, t_2) - e \\ c + z_1(t_1, t_2) - f \end{bmatrix} \\ &= (1 - s) \begin{bmatrix} d \\ e \\ f \end{bmatrix} + s \begin{bmatrix} a + x_1(t_1, t_2) \\ b + y_1(t_1, t_2) \\ c + z_1(t_1, t_2) \end{bmatrix}. \end{aligned}$$

We provide an example on 3D exploration in supplemental resource [S1].

### 3 Locus From Linear Combinations

Suppose we are given three points  $A, B, C$  on three respective curves  $C_1, C_2, C_3$ . We would like to explore the locus of  $r\overrightarrow{AB} + s\overrightarrow{AC}$ , where  $r$  and  $s \in (0, 1)$ . Similarly, if we are given four points  $A, B, C$  and  $D$  on four respective surfaces of  $S_1, S_2, S_3$  and  $S_4$ , we can explore the locus of  $r\overrightarrow{AB} + s\overrightarrow{AC} + t\overrightarrow{AD}$ , where  $r, s$  and  $t \in (0, 1)$ .

#### 3.1 Locus of two moving points

We first discuss a locus generated by linear combinations of two vectors from two respective closed curves.

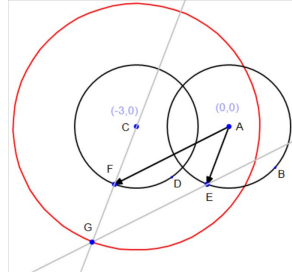


Figure 5: Circle and two fixed points.

We consider two circles  $C_1$  and  $C_2$ , whose centers are at  $A$  and  $C$ , and the radii are  $r_1$  and  $r_2$  respectively. We let  $E$  and  $F$  be two moving points on  $C_1$  and  $C_2$  respectively. If  $G$  is the point so that  $AEGF$  forms a parallelogram, what is the locus of  $G$ ?

It is easy to see that the points  $G$  satisfy  $\vec{OG} = \vec{OA} + \vec{AG} = \vec{OA} + (\vec{AE} + \vec{AF})$  (see Figure 5). We observe that it is easier to solve this problem using vectors than algebraically.

To simplify our problem a bit, assume  $A = (0,0)$ . The first glance of the locus contains a surprise. The new locus (in red of both parts of Figure 5) is centered at the center  $C$  of the second circle  $C_2$  but with a radius larger than  $r_2$ . The locus  $\vec{OG} = \vec{AE} + \vec{AF}$  can be written as  $\vec{AE} + (\vec{AC} + \vec{CF}) = (\vec{AE} + \vec{AC}) + \vec{CF}$ . Note that  $(\vec{AE} + \vec{AC})$  is a translation from the circle  $C_1$  to another circle centered at  $C$  with the same radius  $r_1$ . But the locus for  $(\vec{AE} + \vec{AC}) + \vec{CF}$  becomes the circle centered at  $C$  and radius is  $r_1 + r_2$ . This is clear if we take  $C_1$  to be  $[a + r \cos t, b + r \sin t]$  and  $C_2$  to be  $[c + r_2 \cos t, d + r_2 \sin t]$ . If  $E$  and  $F$  are two moving points on  $C_1$  and  $C_2$  respectively, then it is clear that the locus  $\vec{OG} = \vec{OA} + \vec{AE} + \vec{AF} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} (a + r \cos t) - a \\ (b + r \sin t) - b \end{bmatrix} + \begin{bmatrix} (c + r_2 \cos t) - a \\ (d + r_2 \sin t) - b \end{bmatrix} = \begin{bmatrix} (c + (r + r_2) \cos t) \\ (d + (r + r_2) \sin t) \end{bmatrix}$ . It is clear that the locus in this case is a circle centered at  $(c, d)$  with radius  $r + r_2$ . We see an animation from supplemental resource [S1] that the locus in green is indeed a translation from  $A$  to  $C$  with radius being the sum of the original two circles, we show a sequence of screen shots below when  $A = (-1, 2), C = (2, 3), r_2 = 1$  and  $r$  is going from 0 to 1 as following Figure 6. Additional exploration is provided in supplemental resource [S2].

**Remark:** We encourage readers to explore the generalization of the previous example when the points  $G$  satisfy  $\vec{OG} = \vec{OA} + r\vec{AE} + s\vec{AF}$ , where  $r$  and  $s \in (0, 1)$ .

We now consider the scenario when the circles  $C_1$  and  $C_2$  are replaced by ellipses:

**Exercise 5** Given two fixed ellipses:  $\frac{x^2}{4} + y^2 = 1$  (in blue) and  $\frac{(x+1)^2}{2} + (y-1)^2 = 1$  (in black), centered at  $O$  and  $P$  respectively (see Figure 7 below). Let  $A$  and  $B$  be two moving points on these two ellipses respectively. Find the locus for  $\vec{OA} + \vec{OB}$ .

We leave it to the reader to verify that the locus is an ellipse centered at  $P$ . In fact, the major and minor axes of the locus will be the sums from the the respective lengths of the original two ellipses. In other words, the equation of the locus should be  $\frac{(x+1)^2}{(2+\sqrt{2})^2} + \frac{(y-1)^2}{(1+1)^2} = 1$ .

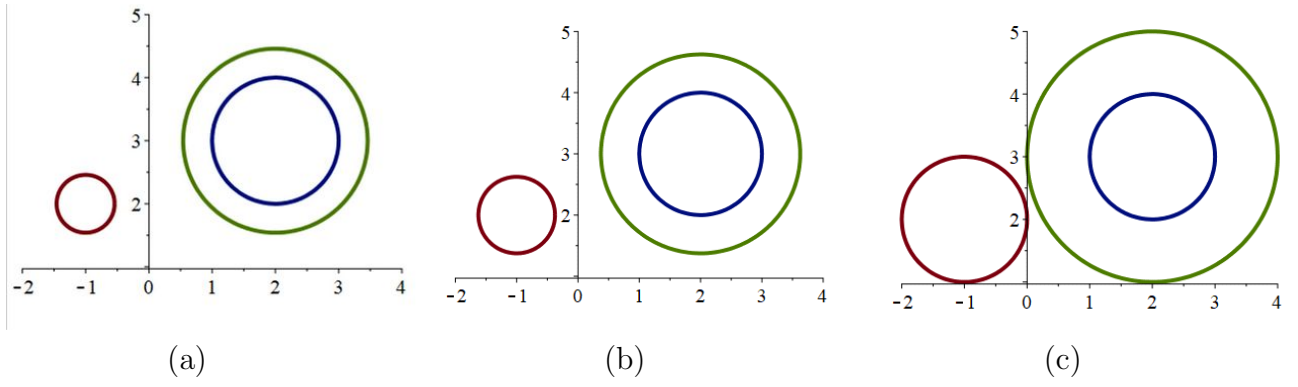


Figure 6: Snapshots of the translation from  $A = (-1, 2)$  to  $C = (2, 3)$  when (a)  $r = 0.45833$ , (b)  $r = 0.625$ , and (c)  $r = 1$ .

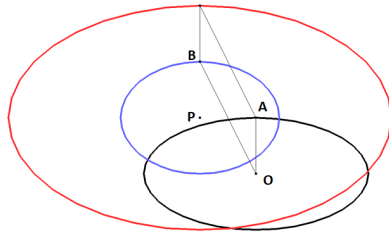


Figure 7: Locus and translation of an ellipse.



**Remark:** The locus in the preceding example can be viewed as a result of a linear combinations from the original two ellipses. Readers are encouraged to explore the problem of finding the locus of  $r\overrightarrow{OA} + s\overrightarrow{OB}$ , where  $r$  and  $s \in (0, 1)$ .

**Exercise 6** We are given two fixed cardioid  $[x_1(a, r, t), y_1(b, r, t)] = [a + (r - \cos t) \cos t, b + (r - \cos t) \sin t]$  and  $[x_2(c, r, t), y_2(d, r, t)] = [c + (r - \cos t) \cos t, d + (r - \cos t) \sin t]$ . Let  $A$  and  $B$  be two moving points on these two cardioids respectively. Find the locus for  $\overrightarrow{OA} + \overrightarrow{OB}$ . (A solution to this problem can be found in supplemental resources [S2], [S6] or [S9].)

### 3.2 Locus of three moving points

We now explore finding a locus when there are three moving points on three separate closed curves. It is surprisingly simple to find the locus using vectors in this case once we know the given (parametric) equations of the three closed curves.

First, recall that the implicit form for the ellipse centered at  $(c_x, c_y)$  with major axis  $r_x$  and minor axis  $r_y$  and rotated by the angle  $\alpha$  is

$$\frac{((x - c_x) \cos \alpha + (y - c_y) \sin \alpha)^2}{r_x^2} + \frac{((x - c_x) \sin \alpha - (y - c_y) \cos \alpha)^2}{r_y^2} = 1.$$

The corresponding parametric representation of this rotated ellipse is, for each  $0 \leq \theta \leq 2\pi$ ,

$$\begin{aligned} x(\theta) &= c_x + r_x \cos \alpha \cos \theta - r_y \sin \alpha \sin \theta, \\ y(\theta) &= c_y + r_x \sin \alpha \cos \theta + r_y \cos \alpha \sin \theta. \end{aligned}$$

So, for example

**Example 7** Let  $C_1$  be the ellipse  $\frac{(x+2)^2}{4} + (y-1)^2 = 1$  rotated by the angle of  $\alpha = 120$  degrees. The center of this ellipse is  $(-2, 1)$ , the major radius is 2 and minor radius is 1. We write the parametric equation  $C_1$  as follows:

$$\begin{aligned} x_1(\theta) &= -2 + 2 \cos \left( \frac{2\pi}{3} \right) \cos \theta - \sin \left( \frac{2\pi}{3} \right) \sin \theta, \\ y_1(\theta) &= 1 + 2 \sin \left( \frac{2\pi}{3} \right) \sin \theta + \cos \left( \frac{2\pi}{3} \right) \sin \theta, \end{aligned}$$

where  $\theta \in [0, 2\pi]$ . Let the second curve be the cardioid  $C_2$ :  $r = 1 - \cos \theta$ , for  $\theta \in [0, 2\pi]$ . Thus, a parametric representation of  $C_2$  is

$$\begin{aligned} x_2(\theta) &= (1 - \cos \theta) \cos \theta, \\ y_2(\theta) &= (1 - \cos \theta) \sin \theta, \end{aligned}$$

where  $\theta \in [0, 2\pi]$ . And, let the third curve  $C_3$  be the unit circle centered at the point  $(2, 2)$ :

$$\begin{aligned} x_3(\theta) &= 2 + \cos \theta, \\ y_3(\theta) &= 2 + \sin \theta, \end{aligned}$$

where  $\theta \in [0, 2\pi]$ . Let  $I$ ,  $F$ , and  $G$  be three moving points on  $C_1$ ,  $C_2$  and  $C_3$ , respectively. The goal is to find the locus of  $\overrightarrow{IF} + \overrightarrow{IG}$ .

If the locus points are called  $J$ , then the problem is to describe the vectors  $\overrightarrow{OJ} = \overrightarrow{OI} + \overrightarrow{IJ}$ , where  $\overrightarrow{IJ} = \overrightarrow{IF} + \overrightarrow{IG}$ ,

$$\overrightarrow{IF} = \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = \begin{bmatrix} (1 - \cos \theta) \cos \theta - \left[-2 + 2 \cos\left(\frac{2\pi}{3}\right) \cos \theta - \sin\left(\frac{2\pi}{3}\right) \sin \theta\right] \\ (1 - \cos \theta) \sin \theta - \left[1 + 2 \sin\left(\frac{2\pi}{3}\right) \sin \theta + \cos\left(\frac{2\pi}{3}\right) \sin \theta\right] \end{bmatrix},$$

and

$$\overrightarrow{IG} = \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = \begin{bmatrix} 2 + \cos \theta - \left[-2 + 2 \cos\left(\frac{2\pi}{3}\right) \cos \theta - \sin\left(\frac{2\pi}{3}\right) \sin \theta\right] \\ 2 + \sin \theta - \left[1 + 2 \sin\left(\frac{2\pi}{3}\right) \sin \theta + \cos\left(\frac{2\pi}{3}\right) \sin \theta\right] \end{bmatrix}.$$

We use Maple [6] to show the locus of the points  $J$  in red, emphasizing the construction of the points corresponding to  $t = 0$  and to  $t = \pi$  in Figures 8(a) and 8(b), respectively. In particular, these snapshots clearly demonstrate that  $\overrightarrow{IJ}$  is a result of linear combination of  $\overrightarrow{IF}$  and  $\overrightarrow{IG}$ .

With the use of a CAS, such as Maple [6], it is easy to generate animations when the weights in the linear combination are allowed to be  $r$  or  $s$ , where each value is between 0 and 1. For example, Figures 8(c) and 8(d) demonstrate the locus when  $(r, s) = (0.54167, 1)$  and  $(r, s) = (0.5, 0.20833)$ , respectively. Please see supplemental resource [S3] for more details and explorations.

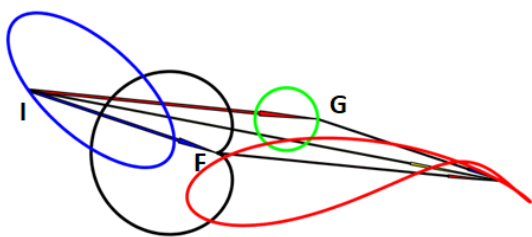
**Discussion:** In the preceding example, the locus is a closed curve  $C_4$  that can be determined once we are given the three closed curves,  $C_1$ ,  $C_2$ , and  $C_3$ , and properly setting up the linear combinations of vectors. One application of this will be using the light source at a point on either  $C_1$ ,  $C_2$ , or  $C_3$ , and we need to find the caustic curve of  $C_4$ , which we call it  $C_5$ . We can continue this process by finding a sequence of closed curves,  $C_4$ ,  $C_5$ , ... so that  $C_{n+1}$  depends on  $C_n$ , where  $n \geq 3$ , which we can imagine finding each  $C_n$  becomes more computational intensive when  $n$  increases.

Now we consider the locus resulting from four closed surfaces in 3D, three of which are originated from the preceding 2D example. Suppose surface  $S_1$  has a parametric representation as

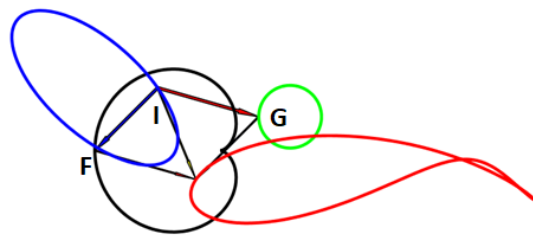
$$\begin{aligned} x_1(t_1, t_2) &= (x_1(t_1) + 2) \sin t_2 - 2 \\ &= -2 + \left(2 \cos\left(\frac{2\pi}{3}\right) \cos t_1 - \sin\left(\frac{2\pi}{3}\right) \sin t_1\right) \sin t_2, \\ y_1(t_1, t_2) &= (y_1(t_1) - 1) \sin t_2 + 1 \\ &= 1 + \left(2 \sin\left(\frac{2\pi}{3}\right) \sin t_1 + \cos\left(\frac{2\pi}{3}\right) \cos t_1\right) \sin t_2, \\ z_1(t_1, t_2) &= \cos(t_2), \end{aligned}$$

where  $t_1 \in [0, 2\pi]$  and  $t_2 \in [0, \pi]$ . The surface  $S_2$  is given by rotating the curve  $[x_2(t_1), y_2(t_1)]$  around the  $x$ -axis as follows:

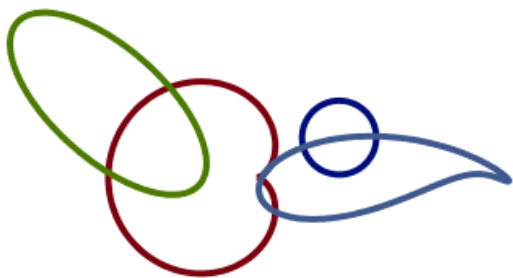
$$\begin{aligned} x_2(t_1, t_2) &= x_2(t_1) \\ y_2(t_1, t_2) &= y(t_1) \cos t_2 \\ &= (1 - \cos t_1) \sin t_1 \cos t_2 \\ z_2(t_1, t_2) &= y(t_1) \sin t_2 \\ &= (1 - \cos t_1) \sin t_1 \sin t_2, \end{aligned}$$



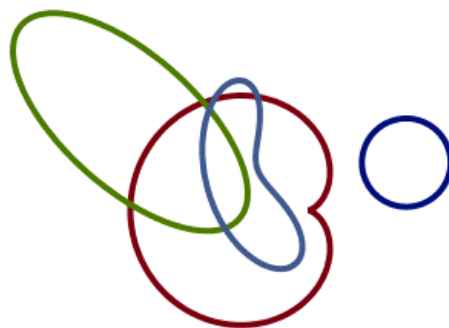
(a)



(b)



(c)



(d)

Figure 8: Snapshots showing the construction of the point  $J$  at (a)  $t = 0$  and (b)  $t = \pi$ . Also, the entire locus of the point  $J$  when (c)  $r = 0.54167$  and  $s = 1$  and (d)  $r = \frac{1}{2}$  and  $s = 0.20833$ .

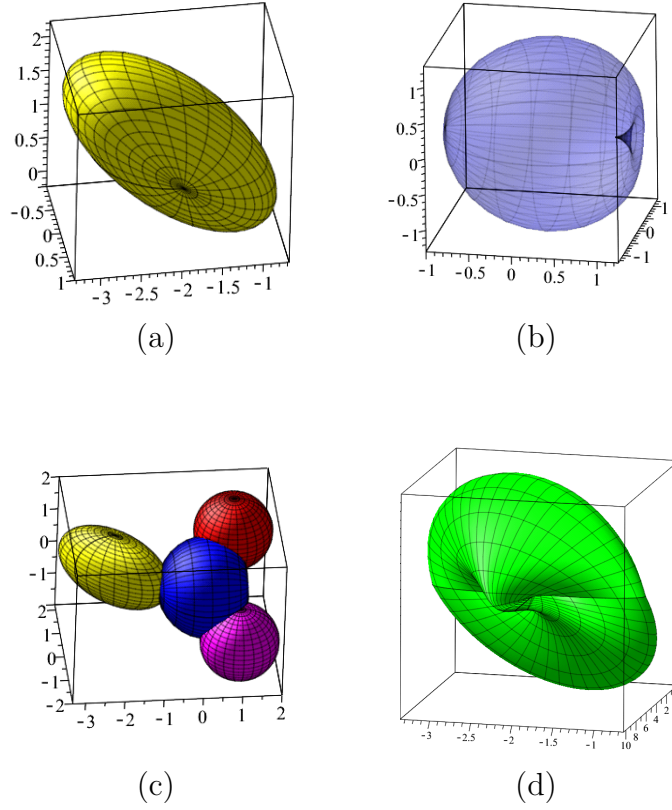


Figure 9: (a) The surface  $S_1$ . (b) The surface  $S_2$ . (c) The surfaces  $S_1$  (yellow),  $S_2$  (blue),  $S_3$  (red), and  $S_4$  (magenta). (d) Locus of linear combinations of surfaces  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$ .

where  $t_1, t_2 \in [0, 2\pi]$ . And, the surfaces  $S_3$  and  $S_4$  are taken to be spheres with radius 1 and centers  $(1, 1, 1)$  and  $(1, -1, -1)$ , respectively. As such,  $S_3$  and  $S_4$  are written parametrically as:

$$\begin{aligned} x_3(t_1, t_2) &= 1 + \sin t_2 \cos t_1, & y_3(t_1, t_2) &= 1 + \sin t_2 \sin t_1, & z_3(t_1, t_2) &= 1 + \cos t_2, \\ x_4(t_1, t_2) &= 1 + \sin t_2 \cos t_1, & y_4(t_1, t_2) &= -1 + \sin t_2 \sin t_1, & z_4(t_1, t_2) &= -1 + \cos t_2, \end{aligned}$$

where  $t_1, t_2 \in [0, 2\pi]$ . If we let  $A, B, C, D$  denote the four moving points on  $S_1, S_2, S_3$ , and  $S_4$ , respectively, then the locus of interest consists of the points  $E$  such that  $\overrightarrow{OE} = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD})$ . In this form the locus is easily calculated and plotted (see Figure 9(d)).

We see  $S_1$  is a rotated ellipsoid shown in yellow (see Figure 9(a)),  $S_2$  is a surface shown in blue which rotates the cardioid  $[x_2(t_1), y_2(t_1)]$  around the  $x$ -axis (see Figure 9(b)). Figure 9(c) shows all four surfaces, including the spheres  $S_3$  and  $S_4$  in red and magenta, respectively. We depict, in Figure 9(d), the locus of the points  $E$  satisfying  $\overrightarrow{OE} = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD})$  in green. Please see supplemental resource [S3] for more details. We encourage the readers to explore the locus of linear combinations of vector  $\overrightarrow{OE} = \overrightarrow{OA} + r\overrightarrow{AB} + s\overrightarrow{AC} + t\overrightarrow{AD}$ , where  $r, s$ , and  $t \in (0, 1)$ .

## 4 Locus When Fixing Two Points On A Curve

In this section we discuss a locus problem that is inspired by the following college entrance exam practice problem from China (see [8]).

**Example 8** Given a fixed ellipse, say  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  ( $a > 0$  and  $b > 0$ ), where  $BE$  is the major axis and  $F$  is a moving point on the ellipse, construct two lines passing through  $B$  and  $E$ , respectively, that intersect at the point  $I$  such that  $\angle IBF = \angle FEI = 90^\circ$  (see Figure 10). Let  $J$  be the midpoint of  $BI$ , find the locus of  $J$ . [Note that the red curve represents the scattered plot of  $J$  that can be traced by using ClassPad [2].]

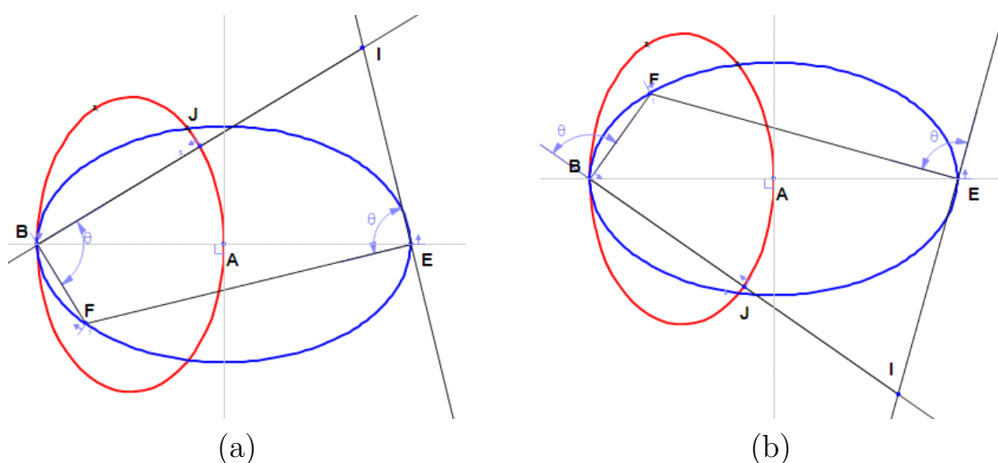


Figure 10: Ellipse, two fixed points, and the points  $J$  when (a)  $t = t_1$  and (b)  $t = t_2$ .

We remark here that if, instead of appearing as a question on an entrance exam, this problem is presented in the context of a mathematics experiment class with access to technological tools to enhance visualization, more students would have enjoyed the experience, and they would actually learn some useful mathematics. For example, before deriving an analytic answer to this question they can play and learn what a locus might look like. In this way the learning process becomes much more enjoyable. Here are some explicit steps showing ways to explore this problem.

1. Start with a DGS (say [2] in this case) for necessary geometric constructions and next use the scattered plot to conjecture what the locus should look like. Further experiment with a symbolic DGS such as [3] to generate a possible symbolic solution.
2. Solve the problem analytically by hand for simple scenario or solve it analytically with a CAS such as [6] as the problem becomes more algebraically intensive.

We first present how one may solve this simple case by hand — without the use of technology. We let  $B = (-a, 0)$ ,  $E = (a, 0)$ , and the moving point on the ellipse  $F = (x_0, y_0)$ . We denote

the slopes of  $FB$  and  $FE$  to be  $k_{FB}$  and  $k_{FE}$ , respectively; then  $k_{FB} = \frac{y_0}{x_0+a}$  and  $k_{FE} = \frac{y_0}{x_0-a}$ . Thus, the equations for lines  $BI$  and  $EI$  are

$$y = -\frac{x_0 + a}{y_0} (x + a) \quad (3)$$

and

$$y = -\frac{x_0 - a}{y_0} (x - a), \quad (4)$$

respectively. Substituting (3) into (4) yields:

$$\begin{aligned} -\frac{x_0 + a}{y_0} (x + a) &= -\frac{x_0 - a}{y_0} (x - a) \\ (x_0 + a)(x + a) &= (x_0 - a)(x - a) \\ 2ax + 2ax_0 &= 0. \end{aligned}$$

Since  $a > 0$ , this requires  $x = -x_0$  and so the point of intersection is  $I = (-x_0, -\frac{1}{y_0} (a + x_0)(a - x_0))$ . The midpoint for  $BI$  is thus

$$J = (X, Y) = \left( \frac{-x_0 - a}{2}, -\frac{1}{2y_0} (a + x_0)(a - x_0) \right).$$

This implies that  $Y = \frac{a-x_0}{y_0} X$ . To obtain the parametric form for the locus  $J$ , note that since  $(x_0, y_0)$  is a point on the ellipse,  $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$ , and so  $x_0 = a \cos t$  and  $y_0 = b \sin t$  for some  $t \in [0, 2\pi)$ . The parametric equations for the locus of  $J$  is found to be

$$\begin{cases} X(t) = \frac{-a \cos t - a}{2} = \frac{-a(\cos t + 1)}{2} \\ Y(t) = \frac{(a - a \cos t)}{b \sin t} = \frac{a(1 - \cos t)}{b \sin t} X \end{cases}.$$

**Exploration 1:** It is not difficult to generalize this problem to ask for the locus  $J = (X, Y)$  satisfying  $\overrightarrow{BJ} = s\overrightarrow{BI}$  for some real number  $s \in [0, 1]$ . In view of  $\overrightarrow{BI} = s\overrightarrow{BI}$ , we have

$$\begin{aligned} (X + a, Y) &= s \left( -x_0 + a, -\frac{1}{y_0} (a + x_0)(a - x_0) \right) \\ J = (X, Y) &= (-s(x_0 - a) - a), \frac{s}{y_0} (x_0 + a)(x_0 - a). \end{aligned}$$

As before,  $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$  and we set  $x_0 = a \cos t$  and  $y_0 = b \sin t$ , where  $t \in [0, 2\pi)$ , to obtain the parametric equations for the locus  $J$  to be

$$\begin{aligned} X &= -s(a \cos t - a) - a \\ Y &= \frac{s((a \cos t)^2 - a^2)}{b \sin t}. \end{aligned} \quad (5)$$

**Remarks:**

1. We can use the DGS Geometry Expressions [3] to construct the locus  $J$  above through geometry constructions. We depict some screen shots when  $s = 0.25$  and  $0.75$  in the following Figures 11(a) and 11(b).
2. The analytic derivation leading to 5 should be equivalent to the geometric construction obtained using Geometry Expressions [3]. The two representations are graphed in Figure 11. The similarities in these figures suggest that these representations are equivalent. The CAS Maple [6] is useful in verifying that the two representations are, in fact, equivalent.

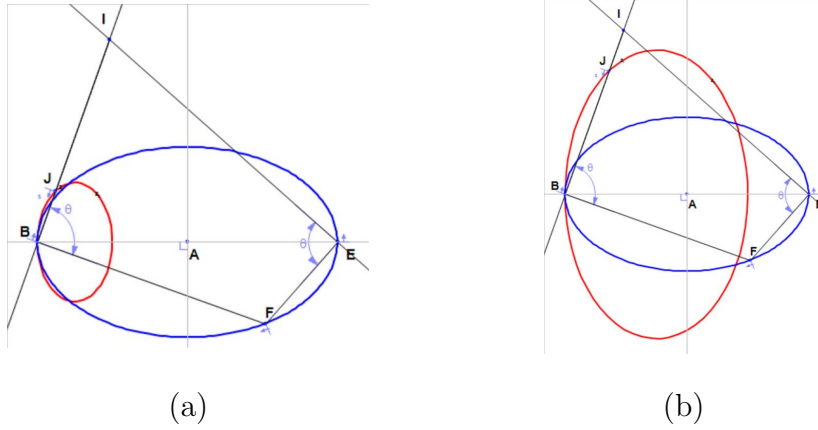


Figure 11: Locus of the points  $J$  for (a)  $s = \frac{1}{4}$  and (b)  $s = \frac{3}{4}$ .

**Exploration 2:** Consider the scenario in which instead of requiring the  $\angle IBF = \angle FEI = \theta$  for some given angle  $\theta$  (not necessarily a right angle). Let  $B = (-a, 0)$ ,  $E = (a, 0)$ , and denote the moving point on the ellipse by  $F = (x_0, y_0)$ . Denoting the slopes of  $FB$  and  $FE$  as  $k_{FB}$  and  $k_{FE}$ , respectively, then  $k_{FB} = \frac{y_0}{x_0 + a} = \tan \theta_1$  and  $\theta_1 = \tan^{-1} \left( \frac{y_0}{x_0 + a} \right)$ . In the meantime,  $k_{FE} = \frac{y_0}{x_0 - a} = \tan \theta_2$  and  $\theta_2 = \tan^{-1} \left( \frac{y_0}{x_0 - a} \right)$ .

The equations for lines  $BI$  and  $EI$  are found, with the help of Maple [6], to be:

$$y = \tan \left( \tan^{-1} \left( \frac{y_0}{x_0 + a} \right) + \theta \right) (x + a) \quad (6)$$

and

$$y = \tan \left( \tan^{-1} \left( \frac{y_0}{x_0 - a} \right) - \theta \right) (x - a) \quad (7)$$

respectively. Using (6) and (7) to solve for  $x$  yields

$$x = \frac{a \tan \left( \arctan \left( \frac{y_0}{-x_0 + a} \right) + \theta \right) - \tan \left( \arctan \left( \frac{y_0}{x_0 + a} \right) + \theta \right)}{\tan \left( \arctan \left( \frac{y_0}{x_0 + a} \right) + \theta \right) + \tan \left( \arctan \left( \frac{y_0}{-x_0 + a} \right) + \theta \right)},$$

and use (6) or (7) to find an expression for  $y$ , which is too long to display and can be found in supplemental resource [S4]. To obtain parametric equations for the locus of  $J = (X_1, Y_1)$  such that  $\overrightarrow{BJ} = s\overrightarrow{BI}$ ,  $s \in [0, 1]$ , where  $I = (x, y)$  is the intersection of  $BI$  and  $EI$ , substitute  $x_0 = a \cos t$  and  $y_0 = b \sin t$  for  $x$  and  $y$ , respectively. Then we see that the locus of  $J$  can be expressed as

$$\begin{aligned} \begin{bmatrix} X_1(a, b, s, t, \theta) \\ Y_1(a, b, s, t, \theta) \end{bmatrix} &= \begin{bmatrix} s(x(a, b, s, t, \theta) + a) - a \\ sy(a, b, s, t, \theta) \end{bmatrix} \\ &= s \begin{bmatrix} x(a, b, s, t, \theta) \\ y(a, b, s, t, \theta) \end{bmatrix} - (1 - s) \begin{bmatrix} a \\ 0 \end{bmatrix}. \end{aligned}$$

The actual expressions for  $X_1(a, b, s, t, \theta)$  and  $Y_1(a, b, s, t, \theta)$  produced by Maple [6] are shown in the following screenshots:

The screenshot shows the Maple command `X1(a, b, s, t, theta);` followed by a complex expression for  $X_1$ . The expression is 
$$s \left( \frac{a \left( \tan \left( \arctan \left( \frac{b \sin(t)}{-a \cos(t) + a} \right) + \theta \right) - \tan \left( \arctan \left( \frac{b \sin(t)}{a \cos(t) + a} \right) + \theta \right) \right)}{\tan \left( \arctan \left( \frac{b \sin(t)}{a \cos(t) + a} \right) + \theta \right) + \tan \left( \arctan \left( \frac{b \sin(t)}{-a \cos(t) + a} \right) + \theta \right)} + a \right) - a$$

The screenshot shows the Maple command `Y1(a, b, s, t, theta);` followed by a complex expression for  $Y_1$ . The expression is 
$$s \tan \left( \arctan \left( \frac{b \sin(t)}{a \cos(t) + a} \right) + \theta \right) \left( \frac{a \left( \tan \left( \arctan \left( \frac{b \sin(t)}{-a \cos(t) + a} \right) + \theta \right) - \tan \left( \arctan \left( \frac{b \sin(t)}{a \cos(t) + a} \right) + \theta \right) \right)}{\tan \left( \arctan \left( \frac{b \sin(t)}{a \cos(t) + a} \right) + \theta \right) + \tan \left( \arctan \left( \frac{b \sin(t)}{-a \cos(t) + a} \right) + \theta \right)} + a \right)$$

For comparison, Geometry Expressions [3] reports the corresponding parametric equations for the locus of the point  $J$  as, after simplification,

$$\begin{aligned} X &= s \frac{2a^2b \cos(t)}{(a^2 - b^2) \sin(2\theta) \sin(t) + 2ab \cos(2\theta)} - (1 - s)a \\ Y &= s \frac{2a(a^2 \sin^2(\theta) + b^2 \cos^2(\theta)) \sin(t) + ab \sin(2\theta)}{(a^2 - b^2) \sin(2\theta) \sin(t) + 2ab \cos(2\theta)} \end{aligned}$$

**Remark:** The DGS Geometry Expressions [3] has the capability of linking its outputs to a CAS such as [6] for further computation. Although it is not trivial to prove algebraically that the locus equation  $[X_1, Y_1]$  obtained from Maple is identical to that of  $[X, Y]$  obtained from the Geometry Expressions, the Maple worksheet used by the authors to check these results is provided as supplemental resources [S4]; see all [S7] or [S10] for additional hands-on explorations. These also provide graphical verifications under different scenarios: varying one parameter, varying one parameter and fixing the other. The two parts of Figure 12 show both  $[X_1, Y_1]$  and  $[X, Y]$  when  $a = 2$ ,  $b = 2$ ,  $s = \frac{1}{4}$ , and (a)  $\theta = \frac{\pi}{4}$  and (b)  $\theta = \frac{2\pi}{3}$ .



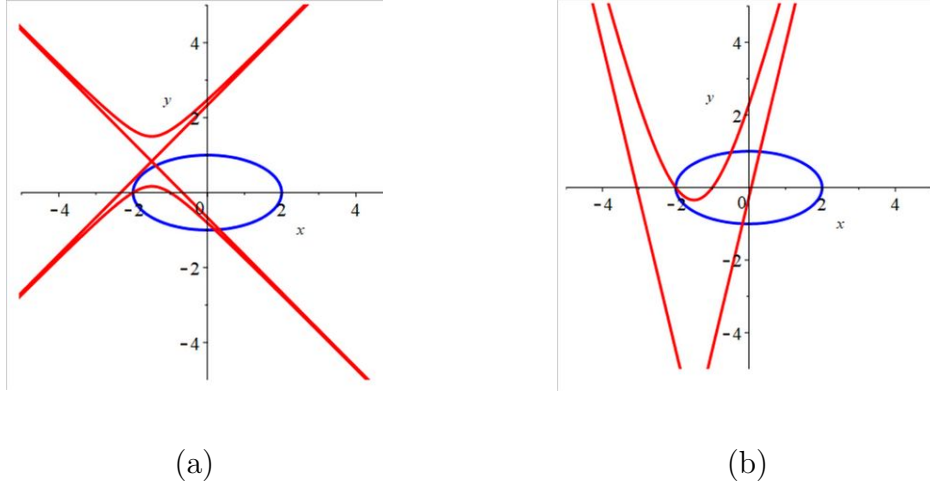


Figure 12: Parametric curves  $[X(2, 2, \frac{1}{4}, t, \theta), Y(2, 2, \frac{1}{4}, t, \theta)] = [X_1(2, 2, \frac{1}{4}, t, \theta), Y_1(2, 2, \frac{1}{4}, t, \theta)]$  for (a)  $\theta = \frac{\pi}{4}$  and (b)  $\theta = \frac{2\pi}{3}$

#### 4.1 Another Generalization: Replacing the Ellipse by a Cardioid

Assuming technological tools are available to learners, it is natural to ask what if the ellipse, discussed earlier, is replaced by another curve, say a cardioid. In particular, we consider the following

**Example 9** We are given the cardioid  $r = 1 - \cos t, t \in [0, 2\pi]$  in Figure 13. Suppose the moving point  $C$  is on the cardioid and two lines passing through  $B = (0, 0)$  and  $A = (a, 0)$  respectively, and intersect at  $G$  so that the angles  $\angle CAG = \angle CBG = 90^\circ$ . Find the locus of the midpoint,  $J$ , of  $AG$ . [Note, in Figure 13, the red curve is a scattered plot of the locus of  $J$ , when  $A = (-2, 0)$ , and has been obtained using [2]]

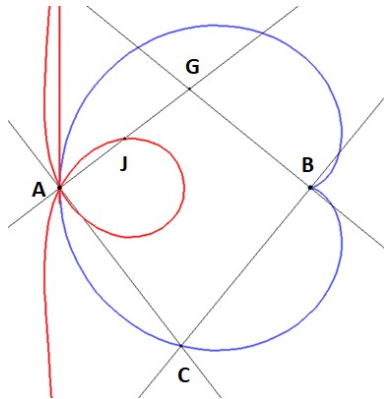


Figure 13: Locus of midpoints  $J$  of segment  $AG$  for a cardioid.

When  $A = (a, 0)$  ( $a \neq 0$ ) and  $B = (0, 0)$  denote the moving point  $C = (x_0, y_0)$ , the slopes for  $CB$  and  $CA$  are  $k_{CB} = \frac{y_0}{x_0}$  and  $k_{CA} = \frac{y_0}{x_0 - a}$ , respectively. The corresponding equations for  $CB$  and  $CA$  are

$$y = -\frac{x_0}{y_0}x, \quad (8)$$

and

$$y = -\frac{(x_0 - a)}{y_0}(x - a), \quad (9)$$

respectively. The intersection of these two lines occurs when  $\frac{x_0}{y_0}x = \frac{(x_0 - a)}{y_0}(x - a)$ . Since  $a \neq 0$  we see  $x = a - x_0$  so that  $y = \left(\frac{x_0}{y_0}\right)(x_0 - a)$ . In other words, the intersection is  $G = (a - x_0, \left(\frac{x_0}{y_0}\right)(x_0 - a))$  and the midpoint for  $AG$  is

$$J = (X, Y) = \left(\frac{2a - x_0}{2}, \frac{x_0(x_0 - a)}{2y_0}\right).$$

But, since  $C$  is a point on the cardioid  $r = f(t) = 1 - \cos t$  we can write  $x_0 = (1 - \cos t) \cos t$  and  $y_0 = (1 - \cos t) \sin t$ . Thus we obtain the following parametric equations for the locus of  $J$ :

$$\begin{aligned} X(t) &= \frac{2a - (1 - \cos t) \cos t}{2} \\ Y(t) &= \frac{(1 - \cos t) \cos t ((1 - \cos t) \cos t - a)}{2(1 - \cos t) \sin t}. \end{aligned}$$

**Exploration 1.** It is not difficult to generalize this result to the problem of identifying the locus of  $J = (X, Y)$  such that  $\overrightarrow{AJ} = s\overrightarrow{AG}$  for some real number  $s$ .

The equation  $\overrightarrow{AJ} = s\overrightarrow{AG}$  clearly defines the point  $J = (X, Y)$ . The same process as used above shows that  $X = a - sx_0$  and  $Y = s\left(\frac{x_0}{y_0}\right)(x_0 - a)$ . Since it is still the case that  $(x_0, y_0)$  is a point on the cardioid  $r = f(t) = 1 - \cos t$ , the locus  $J$  in this case is

$$\begin{aligned} X(a, s, t) &= a - s(1 - \cos t) \cos t \\ Y(a, s, t) &= \frac{s(1 - \cos t) \cos t}{(1 - \cos t) \sin t} ((1 - \cos t) \cos t - a) \end{aligned} \quad (10)$$

Figure 14 shows screenshots produced with Geometry Expressions [3] of the construction when  $a = -2$  and  $s = 0.3, 0.7$ , and  $1.5$ , respectively. Maple [6] has been used to verify that the expressions, and figures, produced with Geometry Expressions [3] are consistent with those obtained directly by Maple.

#### Further Remarks:

1. Notice that the cardioid  $r = 1 - \cos t$  has a point of non-differentiability at  $B = (0, 0)$ . What point on the locus of  $J$  corresponds to this point on the cardioid?
2. In view of the derivation in equations 10, we encourage readers to explore how the graphs varies according to the parameters  $a$ ,  $s$ , and  $t$ , respectively.

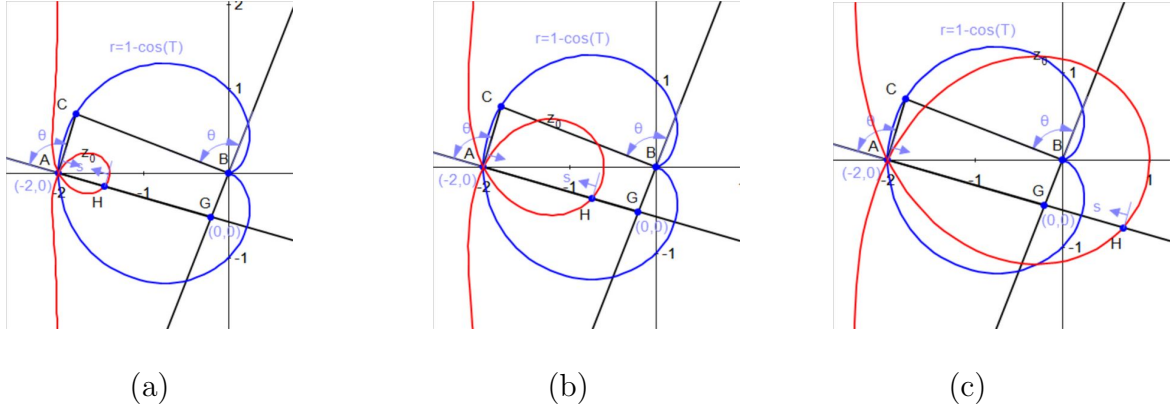


Figure 14: Locus of the point  $J$  (a) when  $s = \frac{1}{2}$  and (b) when  $s = \frac{1}{4}$ .

**Exploration 2.** To discuss the scenario when  $\angle CAG = \angle CBG = \pi - \theta$ . If we denote the moving point on the cardioid as  $C = (x_0, y_0)$  and the intersection of lines  $BG$  and  $AG$  as  $G = (x, y)$ . The slopes for  $CB$  and  $CA$  are  $k_{CB} = \frac{y_0}{x_0}$  and  $k_{CA} = \frac{y_0}{x_0 - a}$ , respectively, so that  $\theta_1 = \arctan\left(\frac{y_0}{x_0}\right)$  and  $\theta_2 = \arctan\left(\frac{y_0}{x_0 - a}\right)$ . The equations for  $BG$  and  $AG$  can be written as

$$y = \tan(\theta_1 - \theta) x \quad (11)$$

and

$$y = \tan(\theta_2 + \theta) (x - a). \quad (12)$$

When Maple [6] solves equations (11) and (12) it produces

$$x = \frac{a \tan\left(-\arctan\left(\frac{y_0}{-x_0 + a}\right) + \theta\right)}{\tan\left(-\arctan\left(\frac{y_0}{-x_0 + a}\right) + \theta\right) + \tan\left(-\arctan\left(\frac{y_0}{x_0}\right) + \theta\right)}.$$

Then, using equation (6) or (7) to find  $y$ , the result is too long to display but can be found in the supplemental resource [S5]. More explorations can be found in supplemental resources [S8] and [S11]. Continuing, we substitute  $x_0 = (1 - \cos t) \cos t$  and  $y_0 = (1 - \cos t) \sin t$  into the expressions for  $x$  and  $y$  to find the intersection between  $AG$  and  $BG$ , which we denote it as  $G = (x, y)$ . Since the locus  $J = (X, Y)$  is such that  $\vec{AJ} = s\vec{AG}$  for some real number  $s$ , the locus of the points  $J = (X_1, Y_1)$  is found to satisfy

$$\begin{bmatrix} X_1(a, s, t, \theta) \\ Y_1(a, s, t, \theta) \end{bmatrix} = \begin{bmatrix} s(x(a, s, t, \theta) - a) \\ sy(a, s, t, \theta) \end{bmatrix} = s \begin{bmatrix} x(a, s, t, \theta) - a \\ y(a, s, t, \theta) \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

The final expressions obtained by Maple [6] for  $X_1(a, s, t, \theta)$  and  $Y_1(a, s, t, \theta)$  are shown in the

following screenshots:

$$\begin{aligned}
 &XI(a, s, t, \theta); \\
 &s \left( \frac{\tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right) + \theta\right) a}{\tan\left(-\arctan\left(\frac{\sin(t)}{\cos(t)}\right) + \theta\right) + \tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right) + \theta\right)} - a \right) + a \\
 \\
 &YI(a, s, t, \theta); \\
 &\frac{s \tan\left(-\arctan\left(\frac{\sin(t)}{\cos(t)}\right) + \theta\right) \tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right) + \theta\right) a}{\tan\left(-\arctan\left(\frac{\sin(t)}{\cos(t)}\right) + \theta\right) + \tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right) + \theta\right)}
 \end{aligned}$$

The corresponding parametric equations for the locus of  $J$  obtained from Geometry Expressions [3] is, when converted into the Maple syntax:

```
[X=(a+(s*a*(-1))+((sin(theta)*sin(t)*(1+(cos(t)*(-1))*(-1))+cos(theta)*cos(t)*(1+(cos(t)*(-1))*(-1))*((sin(theta)*(a+(cos(t)*(1+(cos(t)*(-1))*(-1))*(-1))+cos(theta)*sin(t)*(1+(cos(t)*(-1))*(-1))))*((sin(t)*a)+(cos(t)*sin(t)*a*(-1))+cos(theta)*sin(theta)*(-2))+cos(theta)*cos(t)*sin(theta)*4)+(cos(theta)*cos(t)*sin(theta)*a*2)+(cos(theta)*(cos(t))^2*sin(theta)*(-2))+cos(theta)*(cos(t))^2*sin(theta)*a*(-2))+((cos(theta))^2*sin(t)*a*(-2))+((cos(theta))^2*cos(t)*sin(t)*a*2))^((-1))*s*a),Y=((cos(t)*sin(theta))+cos(theta)*sin(t)*(-1))*((sin(theta)*a*(-1))+cos(t)*sin(theta))+((cos(t))^2*sin(theta)*(-1))+cos(theta)*(sin(t)+(cos(t)*sin(t)*(-1))))*((sin(t)*a*(-1))+cos(theta)*((sin(theta)*2)+cos(t)*sin(theta)*(-2))+cos(t)*sin(theta)*a*(-2))+((cos(theta))^2*sin(t)*a*2))^((-1))*s*a*(-1)]
```

Supplemental resources [S5], [S8] and [S11] show that the family of locus plots for the parametric equations  $[X_1(-2, s, t, \theta), Y_1(-2, s, t, \theta)]$  obtained from Maple [6] are identical to  $[X(-2, s, t, \theta), Y(-2, s, t, \theta)]$  obtained from Geometry Expressions [3]. For the specific situation when  $a = -2$ ,  $s = 0.7$ , and  $\theta = \frac{2\pi}{3}$ , Figure 15 shows locus produced by Maple with blue diamonds and the locus produced by Geometry Expressions with red crosses (and the original cardioid in cyan). The actual proof that the two representations are equivalent is not difficult, but it is tedious and the final expressions are not immediately illuminating, so these details are omitted here.

## 5 Conclusion

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS such as [2] or [3]. We also showed the utility of a CAS, such as Maple [6], for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to students from high school. Others require more advanced mathematics often taught in

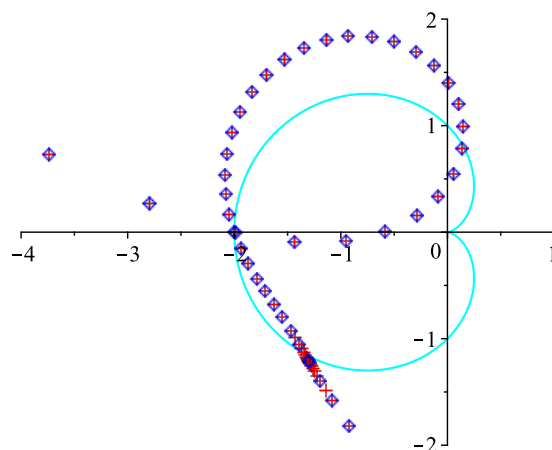


Figure 15: Comparison of the parametric curves  $[X_1(-2, 0.7, t, \frac{2\pi}{3}), Y_1(-2, 0.7, t, \frac{2\pi}{3})]$  from Maple (in blue diamonds) and  $[X(-2, 0.7, t, \frac{2\pi}{3}), Y(-2, 0.7, t, \frac{2\pi}{3})]$ , from Geometry Expressions (in red crosses).

universities. The more advanced problems are excellent examples for future teachers to explore and to understand before they try to explain the simpler problems to their students.

Evolving technological tools definitely have made mathematics fun and accessible. They also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, more students will be inspired to investigate problems ranging from the simple to the more challenging, and even open questions. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

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## 7 Supplementary Electronic Materials

Content	Maple	Geometry Expressions	HTML
Section 2.2	[S1]	—	—
Section 3.1	[S2]	[S6]	[S9]
Exercise 6	[S3]	—	—
Example 7	[S4]	[S7]	[S10]
Example 8	[S5]	[S8]	[S11]

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- [3] Geometry Expressions, see <http://www.geometryexpressions.com/>.
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