Locus Problems Related to Linear Combinations of Vectors

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Abstract

that utilize

This paper is a more in-depth discussions for selected problems discussed in [1], which are inspired by college entrance exam practice problems from China. When seeing the presented problem ~~at the first sight~~ during an exam, ~~readers~~ one may have no clue how the problem is created. In this paper, we ~~often first highlight~~ begin by presenting the essential algebraic manipulation skills that are required by high school students from China. ~~In the meantime, we explore~~ various scenarios if technological tools ~~are available to learners.~~ We shall see many unexpected surprising outcomes. More importantly, we summarize how a problem can be extended to a more general setting whenever is possible.

initially

We then continue the discussion by exploring

(Should there be some reference to supplemental materials?)

1 Introduction

such problems are

This paper is a more in depth discussion on selected problems from the published article [1]. Finding a curve defined by the locus of a moving point has been popular and often asked ~~in~~ on Gaokao (a college entrance exam) in China. There have been several exploratory activities (see [5] [7], and [9]) derived from Chinese college entrance exam practice problems (see [8]). In this article, we typically start with a practice problem originated from ([8]) to initiate our discussions. We demonstrate how a problem can be solved without the assistance of technological tools, which shall demonstrate those crucial algebraic manipulation skills that are required by high school students from China. From a content knowledge point of view, this paper is very accessible to those students who have learned parametric equations and have a basic ~~concept~~ understanding of linear algebra. Problems presented in this paper can be used as examples for professional training purposes.

In this paper, we stress that if problems are presented as explorations instead of in an examination setting, learners would ~~have~~ enjoy learning mathematics ~~than simply doing~~ assuming more than somewhat boring ~~setting~~ algebraic manipulations. We believe making conjectures by seeing possible solutions before asking complete analytic or algebraic solutions is much more accessible, convincing and intuitive dynamic geometry software (DGS) application such as [3].

to students. In addition to solving simple cases by hand, we typically construct a potential solution geometrically using the trace feature of a DGS. Finally, we use a CAS (such as [6]) to verify that our analytic solutions are identical to those obtained by using the DGS ~~[3]~~.

2 Locus From Shifting And Scaling

Here we are given two fixed points and a moving point on a smooth convex curve, and we need to find the locus of a point lying on the line segment connecting one fixed point and a moving point. We first present the original problem and solve it by hand ~~first~~ and the see how the problem can be extended to other scenarios in 2D. Next, we summarize how the problem is related to the translation and scaling of figures.

2.1 Generating a circle with two fixed points

The setting for our first has a fixed

~~The following~~ locus problem ~~is when we fix a~~ point on a circle and ~~fix~~ another fixed point that is not on the circle. In particular, we consider the following Example 1, which has been slightly modified from the original practice problem (see [8]).

Example 1 *We are given a fixed circle in blue (see Figure 1), the circle is of the form* $(x-a)^2+(y-b)^2=r^2$, *the moving point* $D=(x_0,y_0)$ *is on the circle. Furthermore if we choose the fixed point* $C=(c,d)$ *that is not on the circle and let* E *be the midpoint of* CD. *Find the locus of* E.

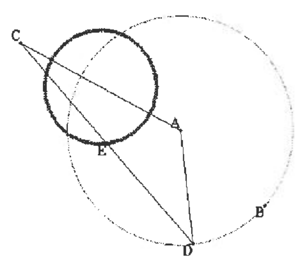


Figure 1. Circle and two fixed points

We first see how students solve this problem by hand in an exam. We note that the midpoint E of CD can be written as $E=\left(\frac{c+x_0}{2},\frac{d+y_0}{2}\right)$. Let $x=\frac{c+x_0}{2}$ and $y=\frac{d+y_0}{2}$. Then we see $((2x-c)-a)^2+((2y-d)-b)^2=r^2$, which implies that

$$\begin{aligned}(2x-c)^2-2a(2x-c)+a^2+(2y-d)^2-2b(2y-d)+b^2 &= r^2\\ 4x^2-4cx+c^2-4ax+2ac+a^2+4y^2-4dy+d^2-4by+2bd+b^2 &= r^2\end{aligned}$$

After simplifying, we see

$$\left(x-\left(\frac{a+c}{2}\right)\right)^2+\left(y-\left(\frac{b+d}{2}\right)\right)^2=\frac{1}{4}r^2,$$

Indeed the locus is a circle with center $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ and radius $\frac{r}{2}$.

Exploration. Following the discussions from Example 1, if we let the point E satisfying $\overrightarrow{CE} = s\overrightarrow{CD}$ with $s \in (0,1)$ and we would like to find the locus of E, then it is easy to verify that the locus for E will be as follows:

$$\left(x - s^2\left(a+c\right)\right)^2 + \left(y - s^2\left(b+d\right)\right)^2 = \left(sr\right)^2.$$

2.2 Locus as a result of simple translation and scaling

~~In fact,~~ we may view the discussions in the preceding Example 1 as a simple translation and scaling from a given curve to the other. For instance, if we consider the given fixed points $C = (c,d), A = (a,b)$ and a curve C_1 . We assume $C \neq A$. By taking the moving point D to be on the curve C_1, our objective is to find the locus of the midpoint F of CD. If O denotes the origin $(0,0)$,then the locus $\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD}$. In particular, if C_1 represents the circle centered at A and of radius r. The the locus F will be the circle centered at C and the radius is being scaled to $\frac{1}{2}$ of the original circle. (See Figure 2(a)). It is clear now that if we were to find the locus of the point F satisfying $\overrightarrow{CF} = s\overrightarrow{CD}$, where $s \in (0,1)$, it is equivalent to asking the locus of $\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF} = \overrightarrow{OC} + s\overrightarrow{CD}$, where $s \in [0,1)$. If the equation of C_1 is represented by $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$, then we write the locus of $\overrightarrow{OF}$ as

$$\begin{aligned}
\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} c \\ d \end{bmatrix} + s\begin{bmatrix} x_1(t) - c \\ y_1(t) - d \end{bmatrix} \\
&= (1-s)\begin{bmatrix} c \\ d \end{bmatrix} + s\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}.
\end{aligned}$$

If C_1 represents the circle centered at A and of radius r, the locus of $\overrightarrow{OF}$ can be viewed as a result of the combination of translation and scaling (see Figure 2(b) for translation and scaling of $s = \frac{1}{4}$).

It is clear that there can be more than one way of expression the locus F. For example, we may write $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CF} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CD} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\left(\overrightarrow{AD} - \overrightarrow{AC}\right) = \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{AD}\right)$. To find the locus of the point F satisfying $\overrightarrow{CF} = s\overrightarrow{CD}$, where $s \in (0,1)$, in this case, we note $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$, where

$$\begin{aligned}
\overrightarrow{AF} &= \overrightarrow{AC} + s\overrightarrow{CD} && (1) \\
&= \overrightarrow{AC} + s\left(\overrightarrow{AD} - \overrightarrow{AC}\right) \\
&= \overrightarrow{AC}\,(1-s) + s\overrightarrow{AD}, && (2)
\end{aligned}$$

If the equation of C_1 is represented by $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$, then the locus of $\overrightarrow{OF}$ can be written as

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + (1-s)\begin{bmatrix} c-a \\ d-b \end{bmatrix} + s\begin{bmatrix} x_1(t) - a \\ y_1(t) - b \end{bmatrix},$$

where $s \in \boxed{[0, 1]}$. We see that when $s = 0$, we get the point of C and, when $s = 1$, we get the parametric curve for C_1. In particular, ~~we see~~ Figure 2(b) shows the situation for translation and scaling ~~of~~ with the $s = \frac{1}{4}$ when C_1 is a circle.

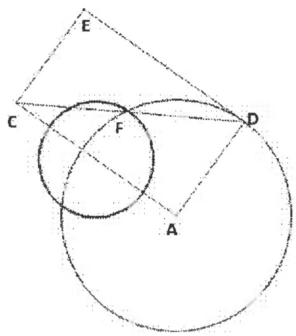


Figure 2(a). Locus F when $s = \frac{1}{2}$

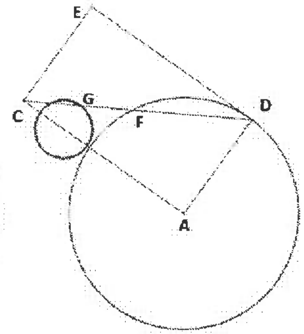


Figure 2(b). Locus G when $s = \frac{1}{4}$

With a CAS at hand such as Maple [6], we may validate further that the locus is a simple translation and scaling. If we set $A = (a, b) = (-.725, .15), r = .7685864$, and C ~~$= (c, d) =$~~ $(-1.8, .89)$. (2) ~~represents~~ shows the We see from the following screen shots (Figures 3(a)-(f)) that the loci are circles that are centered at points lying along the line segment AC (because of the factor $(1 - s)\overrightarrow{AC}$ ~~of 1~~) with a ~~and each~~ radius ~~of respective locus~~ that increases ~~when~~ as s increases (due to the factor of $s\overrightarrow{AD}$ ~~in 1~~). See [S1] for an exploration on this problem and other interesting translation and scaling problems.

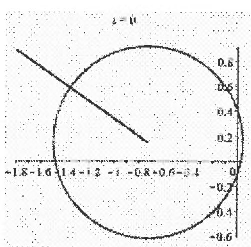


Figure 3(a). Locus of $s = 0$

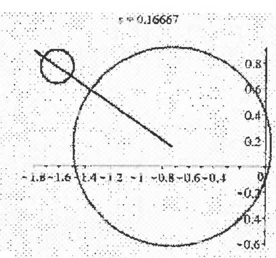


Figure 3(b). Locus of $s = 0.16667$

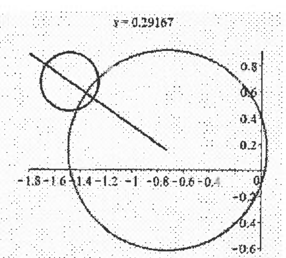


Figure 3(c). Locus of $s = 0.29167$

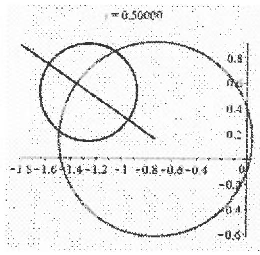


Figure 3(d). Locus of $s = 0.5$

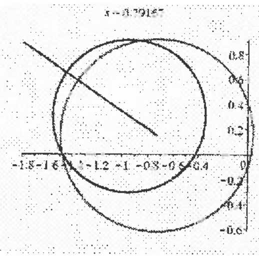


Figure 3(e). Locus of $s = 0.79167$

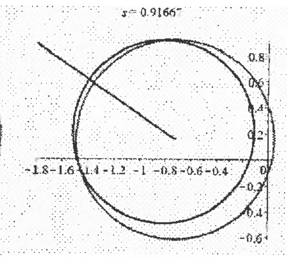


Figure 3(f). Locus of $s = 0.91667$

We encourage readers to explore when we replace the circle with an ellipse as follows:

Exercise 2 *Suppose we are given an ellipse and two fixed points A and C respectively (see Figures 4(a) or 4(b)), where A is the center of the ellipse and $C \neq A$. We let D be a moving point on the ellipse and E be a point such that $\overrightarrow{CE} = s\overrightarrow{CD}$, $s \in [0,1]$. ~~Then~~ find the locus $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$.*

It is clear that the locus in this case is a result of simple translation and scaling from the original ellipse. The Figures 4(a) and 4(b) shows the locus $\overrightarrow{OE}$ in red when $s = \frac{1}{2}$ and $\frac{1}{4}$ respectively. Here we use the ellipse of $\frac{x^2}{4} + y^1 = 1$ and the fixed point $C = [0, 1.3]$.

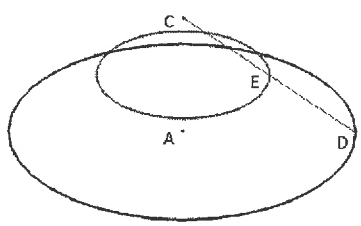


Figure 4(a). Ellipse, translation and $s = \frac{1}{2}$

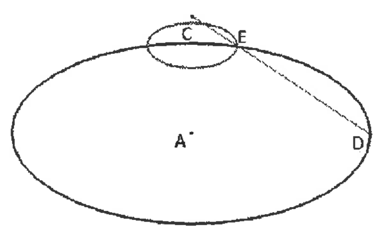


Figure 4(b). Ellipse, translation and $s = \frac{1}{4}$

The following observation is a simple result of ??. More exploration can be found in [S1].

Theorem 3 *Let $A = (a,b)$, $C = (c,d)$ be two fixed points and P represent a closed parametric curve that is represented by $[x(a,b,r,t), y(a,b,r,t)] = [a + f(r,t)\cos t, b + g(r,t)\sin t]$, where $f(r,0) = 0$ and $t \in [0, 2\pi]$. If D is a moving point on P, then the locus F satisfying $\overrightarrow{CF} = s\overrightarrow{CD}, s \in [0,1]$ is a simple translation of the parametric curve from point C to point A with a scaling factor of s. In case $f(r,t)$ and $g(r,t)$ are positive constants, then it is a translation and scaling of ellipse.*

As there can be more than one way of expressing the locus for the translation and scaling in 3D, we describe our 3D extension as follows:

Theorem 4 *Let* $A=(a,b,c)\,,C=(d,e,f)$ *be two fixed points and* P *represent a closed parametric surface that is represented by* $\overrightarrow{OA}+\begin{bmatrix} x_1(t_1,t_2)\\ y_1(t_1,t_2)\\ z_1(t_1,t_2)\end{bmatrix}$, *where* $t_1\in[0,2\pi]$ *and* $t_2\in[0,\pi]$. *If* D *is a moving point on* P, *then the locus* F *satisfying* $\overrightarrow{CF}=s\overrightarrow{CD}$, $s\in[0,1]$ *or*

$$\begin{bmatrix} x_2(t_1,t_2)\\ y_2(t_1,t_2)\\ z_2(t_1,t_2)\end{bmatrix}=(1-s)\begin{bmatrix} d\\ e\\ f\end{bmatrix}+s\begin{bmatrix} a+x_1(t_1,t_2)\\ b+y_1(t_1,t_2)\\ c+z_1(t_1,t_2)\end{bmatrix}$$

is a simple translation of the parametric surface $\begin{bmatrix} x_1(t_1,t_2)\\ y_1(t_1,t_2)\\ z_1(t_1,t_2)\end{bmatrix}$ *from point* C *to point* A *with a scaling factor of* s. *In particular, when* $s=0$, *we start with the point* C *and* $s=1$ *represents the surface of* $\overrightarrow{OA}+\begin{bmatrix} x_1(t_1,t_2)\\ y_1(t_1,t_2)\\ z_1(t_1,t_2)\end{bmatrix}$.

This is a simple observation that

$$\begin{aligned}\overrightarrow{OF} &= \overrightarrow{OC}+\overrightarrow{CF}=\overrightarrow{OC}+s\overrightarrow{CD}\\ &=\begin{bmatrix} d\\ e\\ f\end{bmatrix}+s\begin{bmatrix} a+x_1(t_1,t_2)-d\\ b+y_1(t_1,t_2)-e\\ c+z_1(t_1,t_2)-f\end{bmatrix}\\ &=(1-s)\begin{bmatrix} d\\ e\\ f\end{bmatrix}+s\begin{bmatrix} a+x_1(t_1,t_2)\\ b+y_1(t_1,t_2)\\ c+z_1(t_1,t_2)\end{bmatrix}.\end{aligned}$$

We provide an example on 3D exploration in [S1].

3 Locus From Linear Combinations

Suppose we are given three points A,B,C on three respective curves of C_1,C_2 and C_3. We would like to explore the locus of $r\overrightarrow{AB}+s\overrightarrow{AC}$, where r and $s\in(0,1)$. Similarly, if we are given four points A,B,C and D on four respective surfaces of S_1,S_2,S_3 and S_4, we can explore the locus of $r\overrightarrow{AB}+s\overrightarrow{AC}+t\overrightarrow{AD}$,where r,s and $t\in(0,1)$.

3.1 Locus of two moving points

We first discuss a locus generated by linear combinations of two vectors from two respective closed curves.

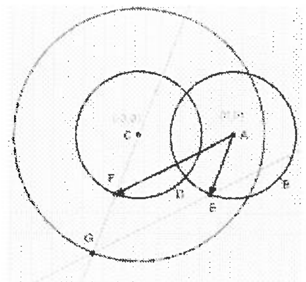


Figure 5 Locus of two moving points

We consider two circles C_1 and C_2, whose centers are at A and C, and the radii are r_1 and r_2 respectively. We let E and F be two moving points on C_1 and C_2 respectively. If G is ~~such a~~ the point so that $AEGF$ forms a parallelogram, ~~Then find~~ what is the locus of G? It is easy to see that the locus of $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \overrightarrow{OA} + \left(\overrightarrow{AE} + \overrightarrow{AF}\right)$ (see Figure 5). We remark that this problem can be solved much easier by using vectors instead of solving it algebraically. To simplify our problem a bit, we assume $A = (0, 0)$. The first glance of the locus to our surprises is that the new locus (in red of Figure 5(a) or 5(b)) is centered at the center C of the second circle but with larger radius. We note that the locus $\overrightarrow{OG} = \overrightarrow{AE} + \overrightarrow{AF}$ can be written as $\overrightarrow{AE} + \left(\overrightarrow{AC} + \overrightarrow{CF}\right) = \left(\overrightarrow{AE} + \overrightarrow{AC}\right) + \overrightarrow{CF}$. Note that $\left(\overrightarrow{AE} + \overrightarrow{AC}\right)$ is a translation from the circle C_1 to another circle centered at C of the same radius r_1. But the locus for $\left(\overrightarrow{AE} + \overrightarrow{AC}\right) + \overrightarrow{CF}$ becomes the circle centered at C and radius is $r_1 + r_2$. This is clear if we take C_1 to be $[a + r\cos t, b + r\sin t]$, C_2 to be $[c + r_2\cos t, d + r_2\sin t]$. If E and F are two moving points on C_1 and C_2 respectively, then it is clear that the locus $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AE} + \overrightarrow{AF} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} (a + r\cos t) - a \\ (b + r\sin t) - b \end{bmatrix} + \begin{bmatrix} (c + r_2\cos t) - a \\ (d + r_2\sin t) - b \end{bmatrix} = \begin{bmatrix} (c + (r + r_2)\cos t) \\ (d + (r + r_2)\sin t) \end{bmatrix}$. It is clear that the locus in this case is a circle centered at (c, d) and radius $r + r_2$. ~~We see an~~ The animation ~~from~~ in [S1] ~~that~~ shows the locus in green is indeed a translation from A to C with radius being the sum of the original two circles, we show a sequence of screen shots below when $A = (-1, 2), C = (2, 3), r_2 = 1$ and r is going from 0 to 1 as following Figures 6(a)-6(c). See

[S2] for ~~more~~ additional explorations.

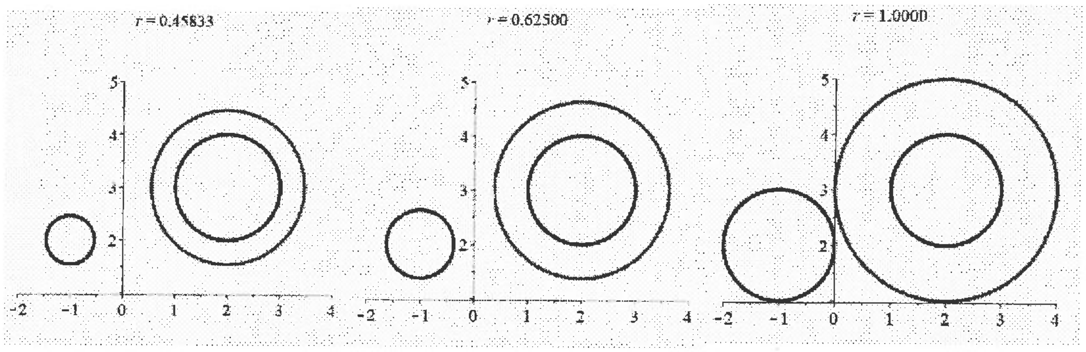


Figure 6(a). Locus of translation when $r = 0.45833$

Figure 6(b). Locus of translation when $r = 0.62500$

Figure 6(c). Locus of translation when $r = 1$

Remark: We encourage readers to explore the locus of $\overrightarrow{OG} = \overrightarrow{OA} + r\overrightarrow{AE} + s\overrightarrow{AF}$, where r and $s \in (0, 1)$ for the preceding example.

We now consider the scenario when we replace circles by ellipses:

Exercise 5 ~~*We are*~~ G*iven two fixed ellipses* a $\frac{x^2}{4} + y^2 = 1$ *(in blue) and* $\frac{(x+1)^2}{2} + (y-1)^2 = 1$ *(in black), centered at* O *and* P *respectively (see Figure* 7 *below). Let* A *and* B *be two moving points on these two ellipses respectively. Find the locus for* $\overrightarrow{OA} + \overrightarrow{OB}$.

We leave it to the reader to verify that the locus is an ellipse centered at P. ~~Furthermore~~ In fact the major and minor lengths of the locus will be the sums from the those respective lengths of the original two ellipses. In other words, the equation of the locus should be $\frac{(x+1)^2}{(2+\sqrt{2})^2} + \frac{(y-1)^2}{4} =$ 1.

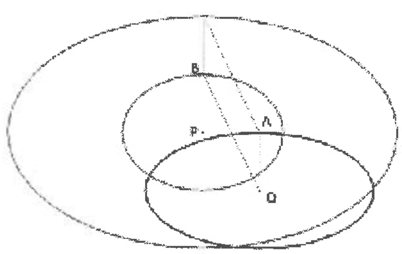


Figure 7. Locus and translation of an ellipse

Remarks:

1. The locus in the preceding exercise can be viewed as a result of a linear combinations from the original two ellipses. ~~We encourage~~ Readers are encouraged to explore finding the locus of $r\overrightarrow{OA} + s\overrightarrow{OB}$, where r and $s \in (0, 1)$.

??

2. It is clear that only when two moving points are on the two given ellipses, which are in the standard forms. Then the major and minor lengths of the locus will be the sums ~~from~~ of the those respective major and minor lengths of the original two ellipses.

Exercise 6 *We are given two fixed cardioid* $[x_1(a,r,t), y_1(b,r,t)] = [a+(r-\cos t)\cos t, b+(r-\cos t)\sin t]$ *and* $[x_2(c,r,t), y_2(d,r,t)] = [c+(r-\cos t)\cos t, d+(r-\cos t)\sin t]$. *Let* A *and* B *be two moving points on these two cardioids respectively. Find the locus for* $\overrightarrow{OA}+\overrightarrow{OB}$. *(Solution can be found at [S2], [S6] or [S9])*

3.2 Locus of three moving points

We now explore finding a locus if ~~we have~~ when there are three moving points on three ~~respective~~ separate closed curves, it is surprisingly simple to find the locus using vectors in this case once we know the given (parametric) equations of three ~~respective~~ closed curves. First, ~~we~~ recall a rotated ellipse ~~in rectangular form of~~ the implicit form of is

$$\frac{((x-c_x)\cos\overset{\alpha}{\cancel{\theta}}+(y-c_y)\sin\overset{\alpha}{\cancel{\theta}})^2}{r_x^2}+\frac{((x-c_x)\sin\overset{\alpha}{\cancel{\theta}}-(y-c_y)\cos\overset{\alpha}{\cancel{\theta}})^2}{r_y^2}=1$$

~~or its~~ The corresponding parametric form ~~of~~ is

$$\begin{aligned} x(\overset{\theta}{\cancel{\alpha}}) &= c_x + r_x\cos\overset{\alpha}{\cancel{\theta}}\cos\overset{\theta}{\cancel{\alpha}} - r_y\sin\overset{\alpha}{\cancel{\theta}}\sin\overset{\theta}{\cancel{\alpha}},\\ y(\overset{\theta}{\cancel{\alpha}}) &= c_y + r_x\sin\overset{\alpha}{\cancel{\theta}}\cos\overset{\theta}{\cancel{\alpha}} + r_y\cos\overset{\alpha}{\cancel{\theta}}\sin\overset{\theta}{\cancel{\alpha}}, \end{aligned}$$

$0 \le \theta \le 2\pi$

where (c_x, c_y) is the center of the ellipse, r_x is the major radius and r_y is the minor radius and $\overset{\alpha}{\cancel{\theta}}$ is the angle of ellipse rotation. So, for example;

Example 7 ~~*We consider a rotated ellipse. We*~~ *Rotate the ellipse* $\frac{(x+2)^2}{4}+(y-1)^2=1$ *by the angle of* $\alpha = 120$ *degrees. We obtain the following rotated ellipse of* C_1*, whose the center is at* $(-2,1)$*, the major radius is* 2 *and minor radius is* 1. *We write the parametric equation* C_1 *as follows:*

$$\begin{aligned} x_1(\theta) &= -2+2\cos\left(\frac{2\pi}{3}\right)\cos\theta - \sin\left(\frac{2\pi}{3}\right)\sin\theta,\\ y_1(\theta) &= 1+2\sin\left(\frac{2\pi}{3}\right)\sin\theta + \cos\left(\frac{2\pi}{3}\right)\sin\theta, \end{aligned}$$

You've swapped the roles of α & θ; change one

where $\theta \in [0, 2\pi]$. *The second curve is the cardioid* C_2 *of* $r = 1-\cos\theta$, *where* $\theta \in [0,2\pi]$. *Thus,* ~~*we*~~ *the parametric equation of* C_2 *is*

$$\begin{aligned} x_2(\theta) &= (1-\cos\theta)\cos\theta,\\ y_2(\theta) &= (1-\cos\theta)\sin\theta, \end{aligned}$$

where $\theta \in [0, 2\pi]$. *The third curve is the circle* C_3 *of the form*

$$\begin{aligned} x_3(\theta) &= 2+\cos\theta,\\ y_3(\theta) &= 2+\sin\theta, \end{aligned}$$

where $\theta \in [0, 2\pi]$. ~~*We*~~ *Let* I, F *and* G *be three moving points on* C_1, C_2 *and* C_3 *respectively.* ~~*Then*~~ *find the locus of* $\overrightarrow{IF}+\overrightarrow{IG}$.

The goal is to

To find the locus for $\overrightarrow{IF}+\overrightarrow{IG}$, which denote it as J, we need to find $\overrightarrow{OJ}=\overrightarrow{OI}+\overrightarrow{IJ}$, where $\overrightarrow{IJ}=\overrightarrow{IF}+\overrightarrow{IG}$, and

$$\overrightarrow{IF}=\begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix}=\begin{bmatrix} (1-\cos\theta)\cos\theta-\left[-2+2\cos\left(\frac{2\pi}{3}\right)\cos\theta-\sin\left(\frac{2\pi}{3}\right)\sin\theta\right] \\ (1-\cos\theta)\sin\theta-\left[1+2\sin\left(\frac{2\pi}{3}\right)\sin\theta+\cos\left(\frac{2\pi}{3}\right)\sin\theta\right] \end{bmatrix}$$

and

$$\overrightarrow{IG}=\begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix}=\begin{bmatrix} 2+\cos\theta-\left[-2+2\cos\left(\frac{2\pi}{3}\right)\cos\theta-\sin\left(\frac{2\pi}{3}\right)\sin\theta\right] \\ 2+\sin\theta-\left[1+2\sin\left(\frac{2\pi}{3}\right)\sin\theta+\cos\left(\frac{2\pi}{3}\right)\sin\theta\right] \end{bmatrix}.$$

We use Maple [6] to show the locus $\overrightarrow{OJ}$ in red in the Figures 8(a) and 8(b) at $t=0$ and π respectively. In addition, Figures 8(a) and (b) also demonstrate that $\overrightarrow{IJ}$ is a result of linear combination of $\overrightarrow{IF}$ and $\overrightarrow{IG}$. With the CAS Maple [6] at hand, it is easy to generate animations when the factor of r or s varies between 0 and 1. We demonstrate the locus when $(r,s)=(0.54167,1)$ and $(r,s)=(0.5,0.20833)$ in Figures 8(c) and 8(d) respectively.

why these values? why not 0.5 & 0.2?

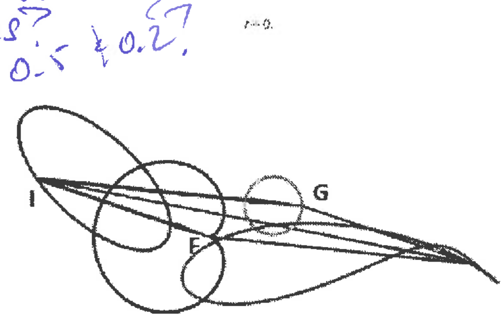


Figure 8(a). $\overrightarrow{OI}+\overrightarrow{IF}+\overrightarrow{IG}$, when $t=0$

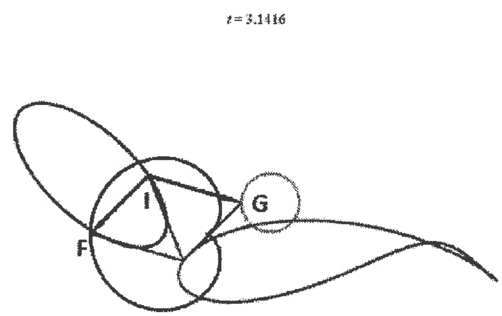


Figure 8(b). $\overrightarrow{OI}+\overrightarrow{IF}+\overrightarrow{IG}$, when $t=\pi$

these are in the caption, so remove from figures, so the rest can be enlarged.

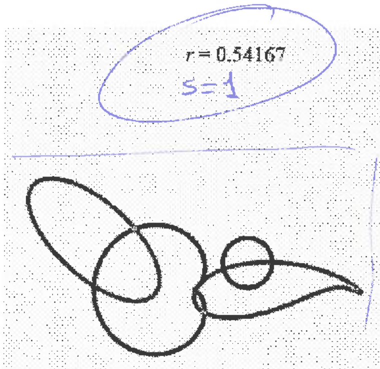


Figure 8(c).
$\overrightarrow{OI}+0.54167\overrightarrow{IF}+\overrightarrow{IJ}$

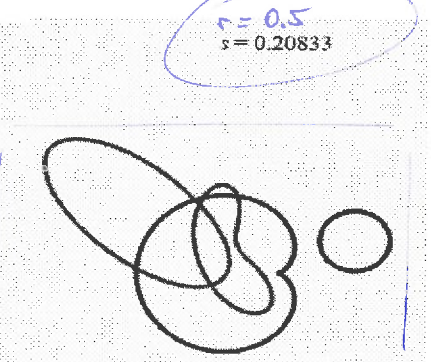


Figure 8(d).
$\overrightarrow{OI}+\frac{1}{2}\overrightarrow{IF}+0.20833\overrightarrow{IG}$

In addition, it is trivial to find the locus of a linear combination such as

$$\overrightarrow{OJ} = \overrightarrow{OI} + r\overrightarrow{IF} + s\overrightarrow{IG}, \tag{3}$$

where r and $s \in (0,1)$. We depict the locus in red when the values of r and s are set randomly in Figure 8(c) and Figure 8(d) respectively. Please see [S3] for more details and exploration.

Discussion: In the preceding example, the locus closed curve C_4 is determined once we are given three closed curves, C_1, C_2 and C_3, and properly setting up the linear combinations of vectors. One application of this will be using the light source at a point on either C_1, C_2 or C_3, and we need to find the caustic curve of C_4, which we call it C_5. We can continue this process by finding a sequence of closed curves, $C_4, C_5, \ldots$ so that C_{n+1} depends on C_n, where $n \geq 3$, which we can imagine finding each C_n becomes more computational intensive when n increases.

Now we consider the locus result~~ed~~ing from four closed surfaces in 3D, three of which are originated from the preceding 2D example. Suppose surface S_1 ~~of~~ is the parametric representation of the

$$\begin{aligned}
x_1(t_1,t_2) &= (x_1(t_1)+2)\sin t_2 - 2 \\
&= -2 + \left(2\cos\left(\frac{2\pi}{3}\right)\cos t_1 - \sin\left(\frac{2\pi}{3}\right)\sin t_1\right)\sin t_2, \\
y_1(t_1,t_2) &= (y_1(t_1)-1)\sin t_2 + 1 \\
&= 1 + \left(2\sin\left(\frac{2\pi}{3}\right)\sin\theta + \cos\left(\frac{2\pi}{3}\right)\sin\theta\right)\sin t_2, \\
z_1(t_1,t_2) &= \cos(t_2),
\end{aligned}$$

where $t_1 \in [0,2\pi]$ and $t_2 \in [0,\pi]$. The surface S_2 is given by rotating the curve $[x_2(t_1), y_2(t_1)]$ around the $x - axis$ as follows:

$$\begin{aligned}
x_2(t_1,t_2) &= x_2(t_1) \quad = (1-\cos t_1)\cos t_1 \\
y_2(t_1,t_2) &= y_2(t_1)\cos t_2 \\
&= \cancel{(1-\cos t_1)\sin t_1\cos t_2} \\
z_2(t_1,t_2) &= y_2(t_1)\sin t_2 \\
&= \cancel{(1-\cos t_1)\sin t_1\sin t_2},
\end{aligned}$$

where $t_1, t_2 \in [0,2\pi]$. We consider the surface S_3 to be the sphere with center at $(1,1,1)$ ~~with~~ and radius 1 and the surface S_4 to be the sphere with center at $(1,-1,-1)$ ~~with~~ and radius 1. Therefore S_3 and S_4 are written as follows:

$$\begin{aligned}
x_3(t_1,t_2) &= 1+\sin t_2\cos t_1, y_3(t_1,t_2) = 1+\sin t_2\sin t_1, z_3(t_1,t_2) = 1+\cos t_2, \\
x_4(t_1,t_2) &= 1+\sin t_2\cos t_1, y_4(t_1,t_2) = -1+\sin t_2\sin t_1, z_4(t_1,t_2) = -1+\cos t_2,
\end{aligned}$$

where $t_1, t_2 \in [0,2\pi]$. We let $A, B, C,$ and D ~~are~~ be four moving points on S_1, S_2, S_3 and S_4 respectively. Then the locus of $\overrightarrow{OE} = \overrightarrow{OA} + \left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right)$ can be easily calculated and plotted (see Figure 9(d)). We see S_1 is a rotated ellipsoid shown in yellow, S_2 is a surface shown in blue formed by rotating ~~which rotates~~ the cardioid $[x_2(t_1), y_2(t_1)]$ around the $x - axis$. ~~If~~ The sphere S_3 ~~is shown~~ in red and the sphere S_4 is shown in magenta in Figure 9(c). We depict the locus $\overrightarrow{OE} = \overrightarrow{OA} + \left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right)$ in green (see Figure 9(d)). ~~Please see [S3] for more details. We encourage~~

ok (keep)

as well as S_1 and S_2 are displayed

put italic x for x-axis

Readers are encouraged to spend some time using [S3]

~~the readers~~ to [S3] to explore the locus of linear combinations of vector $\overrightarrow{OE} = \overrightarrow{OA} + r\overrightarrow{AB} + s\overrightarrow{AC} + t\overrightarrow{AD}$, where r, s and $t \in [0, 1]$.

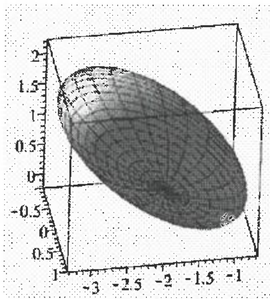


Figure 9(a). Surface of S_1

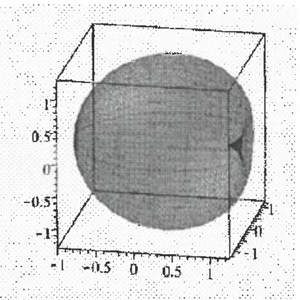


Figure 9(b). Surface of S_2

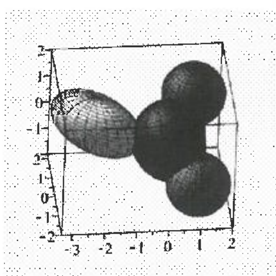


Figure 9(c). Surfaces of S_1, S_2, S_3 and S_4

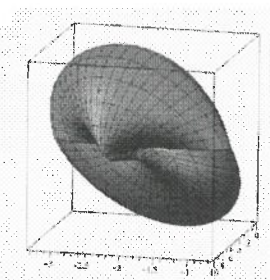


Figure 9(d). Locus of linear combinations

4 Locus When Fixing Two Points On A Curve

In this section, we discuss a locus problem that is inspired by the following college entrance exam practice problem from China (see [8]).

Example 8 *We are given a fixed ellipse, say* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *$BE$ is the major axis and F is a moving point on the ellipse. We construct two lines passing through B and E respectively and two lines intersect at I such that $\angle IBF = \angle FEI = 90°$. If J is the midpoint of BI, find the locus of J.[Note that the red curve* in Figure 10 *represents the scattered* Locus *plot of J that can be traced* ~~*by*~~ *using*

[2]]

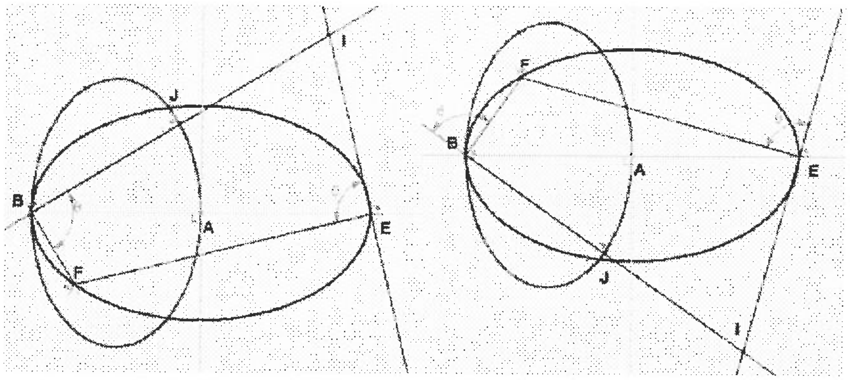


Figure 10(a). Locus, ellipse and two fixed points when $t = t_1$ *Figure 10(b). Locus, ellipse and two fixed points when $t = t_2$*

in the context of

We remark here that if this problem is presented ~~as~~ a mathematics experiment class instead, of as a question on an entrance exam more students would have enjoyed exploring it if technological tools are available to learners. For example, before answering this question analytically, they can play and learn how ~~a~~ the locus ~~might~~ looks ~~like~~, which makes the learning process much more enjoyable. In fact, one may adopt the following steps when exploring a problem:

1. Start with a DGS (say [2] in this case) for necessary geometric constructions and next use the scattered plot to conjecture what the locus should look like. Further experiments with a symbolic DGS such as [3] to generate a possible symbolic solution.

2. Solve the problem analytically by hand for simple scenario or solve it analytically with a CAS such as [6] if the problem becomes algebraically intensive.

We first present how one may solve this simple case by hand, without the ~~presence~~ use of technological tools. We let $B = (-a, 0)$, $E = (a, 0)$ and the moving point on the ellipse $F = (x_0, y_0)$. We denote the slopes of FB and FE to be k_{FB} and k_{FE} respectively, then $k_{FB} = \frac{y_0}{x_0+a}$ and $k_{FE} = \frac{y_0}{x_0-a}$. Thus, the ~~the line~~ lines equations for BI and EI are

$$y = -\frac{x_0 + a}{y_0}(x + a) \tag{4}$$

and

$$y = -\frac{x_0 - a}{y_0}(x - a), \tag{5}$$

respectively. We substitute (4) into (5) and yield the followings:

$$\begin{aligned} -\frac{x_0 + a}{y_0}(x + a) &= -\frac{x_0 - a}{y_0}(x - a) \\ (x_0 + a)(x + a) &= (x_0 - a)(x - a) \\ 2ax + 2ax_0 &= 0, \end{aligned}$$

When these two lines intersect:

*if $a=0$, then there's no ellipse

so that, since ~~assuming~~ $a \neq 0$, the point of intersection is

~~We see if $a \neq 0$,~~ $x = -x_0$. In other words, ~~if we assume $a \neq 0$, then~~ $I = \left(-x_0, -\frac{1}{y_0}(a+x_0)(a-x_0)\right)$, and The midpoint for BI is thus

$$J = (X, Y) = \left(\frac{-x_0 - a}{2}, -\frac{1}{2y_0}(a+x_0)(a-x_0)\right).$$

This implies that $Y = \frac{1}{y_0} X (a - x_0)$. To obtain the parametric form for the locus J, we note that $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, which implies that $x_0 = a\cos t$ and $y_0 = b\sin t$. Thus we obtain the following parametric equations for the locus ~~to be~~ of J:

$$\begin{cases} X(t) = \dfrac{-a\cos t - a}{2} = \dfrac{-a(\cos t + 1)}{2} \\ Y(t) = \dfrac{X(a - a\cos t)}{b\sin t} = \dfrac{aX(1-\cos t)}{b\sin t} \end{cases}.$$

Exploration 1. It is not difficult to extend our result if we ask for the locus $J = (X, Y)$ satisfying $\overrightarrow{BJ} = s\overrightarrow{BI}$ for some real number s. In view of $\overrightarrow{BJ} = s\overrightarrow{BI}$, we have

$$\begin{aligned} (X + a, Y) &= s\left(-x_0 + a, -\frac{1}{y_0}(a+x_0)(a-x_0)\right) \\ J &= (X, Y) = (-s(x_0 - a) - a), \frac{s}{y_0}(x_0 + a)(x_0 - a) \end{aligned}$$

Since $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, we set $x_0 = a\cos t$ and $y_0 = b\sin t$, where $t \in [0, 2\pi]$, to obtain the parametric equation for the locus J to be

$$\begin{aligned} X &= -s(a\cos t - a) - a \\ Y &= \frac{s\left((a\cos t)^2 - a^2\right)}{b\sin t}. \end{aligned} \tag{6}$$

Remarks:

1. We use the DGS Geometry Expressions [3] to construct the locus J above through geometric constructions. ~~We~~ depict some screen shots when $s = 0.25$ and 0.75 ~~in the following~~ Figures 11(a) and 11(b).
2. We use the CAS [6] and the analytic ~~derivation~~ equations (6) to verify that the Figures 11(a) and

11(b) obtained by using [3] are identical to the corresponding ones obtained ~~by using~~ with [6].

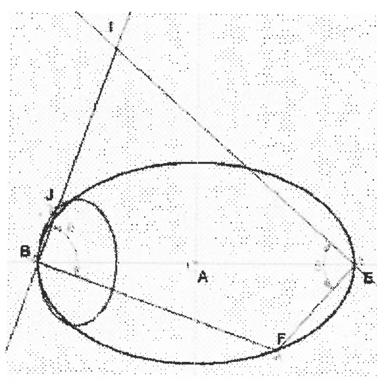


Figure 11(a). Locus when $s = 0.25$

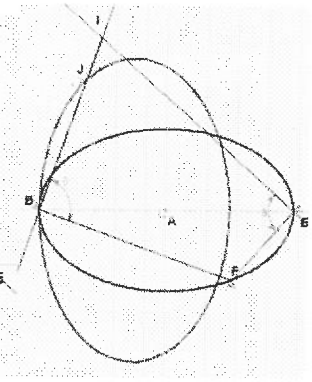


Figure 11(b). Locus when $s = 0.75$

Exploration 2. We discuss the scenario if we set $\measuredangle IBF = \measuredangle FEI = \theta$, for a fixed angle $\theta \neq \pi/2$. We let $B = (-a, 0)$, $E = (a, 0)$ and the moving point on the ellipse is called $F = (x_0, y_0)$. ~~We~~ denote the slopes of FB and FE to be k_{FB} and k_{FE}, respectively, then $k_{FB} = \frac{y_0}{x_0+a} = \tan\theta_1$, and $\theta_1 = \tan^{-1}\left(\frac{y_0}{x_0+a}\right)$. ~~In the meantime~~ Therefore, $k_{FE} = \frac{y_0}{x_0-a} = \tan\theta_2$, and $\theta_2 = \tan^{-1}\left(\frac{y_0}{x_0-a}\right)$.

As before, ~~We write the~~ The ~~line~~ lines equations for BI and EI to be

$$y = \tan\left(\tan^{-1}\left(\frac{y_0}{x_0+a}\right) + \theta\right)(x+a) \tag{7}$$

and

$$y = \tan\left(\tan^{-1}\left(\frac{y_0}{x_0-a}\right) - \theta\right)(x-a) \tag{8}$$

respectively. We ~~use~~ use Maple [6] to solve (7) and (8) ~~to solve~~ for x ~~and yield the following from Maple [6]~~:

$$x = \frac{a\tan\left(\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right) - \tan\left(\arctan\left(\frac{y_0}{x_0+a}\right)+\theta\right)}{\tan\left(\arctan\left(\frac{y_0}{x_0+a}\right)+\theta\right) + \tan\left(\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right)},$$

and use (7) or (8) to ~~figure out~~ determine y, which is too long to display ~~and~~ but can be found ~~at~~ in [S4]. To obtain parametric equations for the locus of $J = (X_1, Y_1)$ such that $\overrightarrow{BJ} = s\overrightarrow{BI}$, where $I = (x, y)$ is the intersection of BI and EI, $s \in [0,1]$, substitute ~~We further substitute~~ $x_0 = a\cos t$ and $y_0 = b\sin t$ ~~into~~ in x and y, respectively. ~~Next if we denote the intersection between BI and EI as $I = (x, y)$ and locus $J = (X_1, Y_1)$ satisfying $\overrightarrow{BJ} = s\overrightarrow{BI}$, where $s \in (0, 1)$~~. Then we see that the locus of J ~~satisfying~~ can be expressed as.

$$\begin{bmatrix} X_1(a,b,s,t,\theta) \\ Y_1(a,b,s,t,\theta) \end{bmatrix} = \begin{bmatrix} s\left(x(a,b,s,t,\theta)+a\right) - a \\ sy(a,b,s,t,\theta) \end{bmatrix} = s\begin{bmatrix} x(a,b,s,t,\theta) \\ y(a,b,s,t,\theta) \end{bmatrix} - (1-s)\begin{bmatrix} a \\ 0 \end{bmatrix}.$$

We display $X_1(a,b,s,t,\theta)$ and $Y_1(a,b,s,t,\theta)$ obtained from Maple [6] in the following screen shots respectively:

X1(a, b, s, t, theta);

$$s\left(\frac{a\left(\tan\left(\arctan\left(\frac{b\sin(t)}{-a\cos(t)+a}\right)+\theta\right)-\tan\left(\arctan\left(\frac{b\sin(t)}{a\cos(t)+a}\right)+\theta\right)\right)}{\tan\left(\arctan\left(\frac{b\sin(t)}{a\cos(t)+a}\right)+\theta\right)+\tan\left(\arctan\left(\frac{b\sin(t)}{-a\cos(t)+a}\right)+\theta\right)}+a\right)-a$$

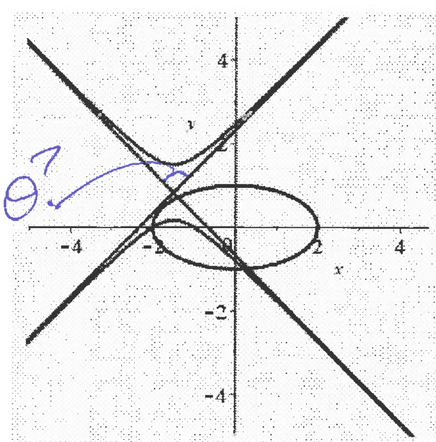
Y1(a, b, s, t, theta);

$$s\tan\left(\arctan\left(\frac{b\sin(t)}{a\cos(t)+a}\right)+\theta\right)\left(\frac{a\left(\tan\left(\arctan\left(\frac{b\sin(t)}{-a\cos(t)+a}\right)+\theta\right)-\tan\left(\arctan\left(\frac{b\sin(t)}{a\cos(t)+a}\right)+\theta\right)\right)}{\tan\left(\arctan\left(\frac{b\sin(t)}{a\cos(t)+a}\right)+\theta\right)+\tan\left(\arctan\left(\frac{b\sin(t)}{-a\cos(t)+a}\right)+\theta\right)}+a\right)$$

In the meantime, the parametric equation for the locus J can also be obtained from Geometry Expressions [3] as follows:

$$\left(\begin{array}{l} X=-(1-s)\,|a|-\dfrac{2a^2 s\sin(t)\cos(t)\,|b|}{-2a^2\sin(t)^2\sin(\theta)\cos(\theta)+2b^2\sin(t)^2\sin(\theta)\cos(\theta)-2\sin(t)\,|a|\,|b|+4\sin(t)\sin(\theta)^2\,|a|\,|b|} \\ Y=\dfrac{s\left(-a^2\sin(\theta)^2|a|+a^2\sin(\theta)^2\cos(t)^2|a|-b^2\sin(t)^2\cos(\theta)^2|a|-2a^2\sin(t)\sin(\theta)\cos(\theta)\,|b|\right)}{\left(-a^2\sin(t)\sin(\theta)\cos(\theta)+b^2\sin(t)\sin(\theta)\cos(\theta)+\left(-|a|+2\sin(\theta)^2|a|\right)|b|\right)\sin(t)} \end{array}\right).$$

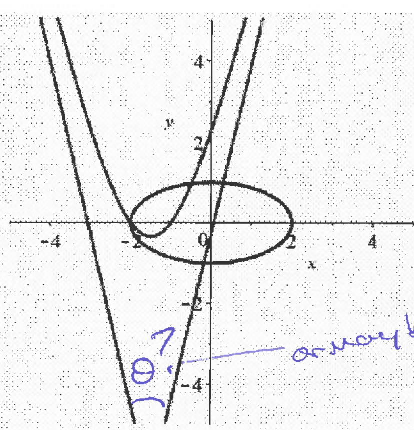
Remark: The DGS Geometry Expressions [3] has the capability of linking its outputs to a CAS such as [6] for further computation. Although it is not trivial to prove algebraically that the locus equation $[X_1, Y_1]$ obtained from the CAS [6] is identical to that of $[X, Y]$ obtained from the DGS [3], ~~We provide~~ the worksheets ~~in the~~ [S4], [S7] or [S10] ~~that~~ show that the plots from either system are identical when varying one parameter. Furthermore, we also show that two plots are identical when we use animations by varying one parameter and fixing the other parameters. In Figure 12(a), we show that the locus for $[X_1, Y_1]$ or $[X, Y]$ is the same for $a=2, b=2, s=\frac{1}{4}$ and $\theta=0.7853$. In Figure 12(b), we show that the locus for $[X_1, Y_1]$ or $[X, Y]$ is the same for $a=2, b=2, s=\frac{1}{4}$ and $\theta=2.0734512$.



θ?

Figure 12(a). ~~The locus for~~ The locus for $a=2, b=2, s=\frac{1}{4}$ and $\theta=0.7853$

where do I see this angle in the figure? This should be $\pi/4 = 45°$, right?



θ? or maybe be $\pi-\theta$? $\approx 60°$.

Figure 12(b). The locus for $a=2, b=2, s=\frac{1}{4}$ and $\theta=2.0734512$

Maybe I'll answer my own question when I explore the supplied files.

4.1 When we replace the ellipse by a cardioid

Assuming technological tools are available to learners, it is natural to ask what if the ellipse, discussed earlier, is replaced by another curve, say a cardioid. In particular, we consider the following

Example 9 *We are given the cardioid $r = 1 - \cos t, t \in [0, 2\pi]$ in Figure 13. Suppose the moving point C is on the cardioid and two lines passing through $B = (0,0)$ and $A = (a, 0)$, respectively, and intersect at G so that the angles $\measuredangle CAG = \measuredangle CBG = 90°$. ~~If~~ J ~~is~~ the midpoint of AG, ~~find the locus for J.~~ [Note the red curve is a ~~scattered~~ plot of the locus of J, when $A = (-2, 0)$~~, and has been~~ obtained using [2]]*

Find the locus of J,

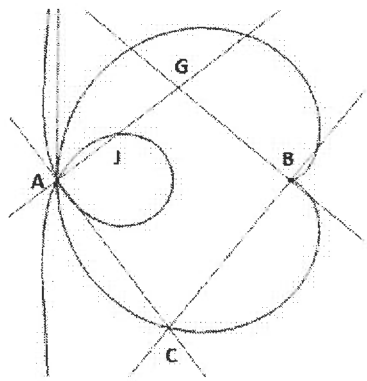


Figure 13. Locus and a cardioid

Since $A = (a, 0)$ and $B = (0,0)$, and the moving point $C = (x_0, y_0)$, we denote the slopes for CB and CA to be $k_{CB} = \frac{y_0}{x_0}$ and $k_{CA} = \frac{y_0}{x_0 - a}$ respectively. Thus, the ~~the line~~ equations for the lines CB and CA are ~~respectively~~

$$y = -\frac{x_0}{y_0}x, \tag{9}$$

and

$$y = -\frac{(x_0 - a)}{y_0}(x - a), \tag{10}$$

The intersection of these lines occurs when

respectively. ~~We substitute 9 into 10 and yield~~ $\frac{x_0}{y_0}x = \frac{(x_0 - a)}{y_0}(x - a)$. ~~By assuming~~ Since $a \neq 0$, we see $x = a - x_0$, then $y = \left(\frac{x_0}{y_0}\right)(x_0 - a)$, in other words, the intersection $G = \left(a - x_0, \left(\frac{x_0}{y_0}\right)(x_0 - a)\right)$. Then the midpoint for AG is

$$J = (X, Y) = \left(\frac{2a - x_0}{2}, \frac{x_0(x_0 - a)}{2y_0}\right).$$

Because ~~We note that~~ C is a point on $r = f(t) = 1 - \cos t$, we can write ~~which implies that~~ $x_0 = f(t)\cos t$ and $y_0 = f(t)\sin t$. Thus we obtain the parametric equation for the locus of J to be

$$X(t) = \frac{2a - (1 - \cos t)\cos t}{2}$$

$$Y(t) = \frac{(1 - \cos t)\cos t\,((1 - \cos t)\cos t - a)}{2(1 - \cos t)\sin t}.$$

Exploration 1. It is not difficult to extend our result if we ~~ask for~~ seek the locus $J = (X, Y)$ such that $\overrightarrow{AJ} = s\overrightarrow{AG}$ for some real number s. In view of $\overrightarrow{AJ} = s\overrightarrow{AG}$, we see $X = a - sx_0$ and $Y = s\left(\frac{x_0}{y_0}\right)(x_0 - a)$. Since (x_0, y_0) is a point on the cardioid $r = f(t) = 1 - \cos t$, the locus J in this case is

$$\begin{aligned} X(a,s,t) &= a - s\,(1-\cos t)\cos t \\ Y(a,s,t) &= \frac{s\,(1-\cos t)\cos t}{(1-\cos t)\sin t}\,((1-\cos t)\cos t - a) \end{aligned} \tag{11}$$

We show some screen shots when $a = -2$ and $s = 0.3, 0.7$ and 1.5 respectively in Figures 14(a)-(c) by using [3] below, which we have verified that they are identical to those corresponding ones when using the CAS [6].

~~Further~~ Remarks:

1. We notice that the curve of $r = 1 - \cos t$ has a point of non-differentiabilty at $B = (0, 0)$, what will be the corresponding point for the locus J?

2. In view of the derivation in equations (11), we encourage readers to explore how the graphs varies according to the parameters a, s, and t respectively.

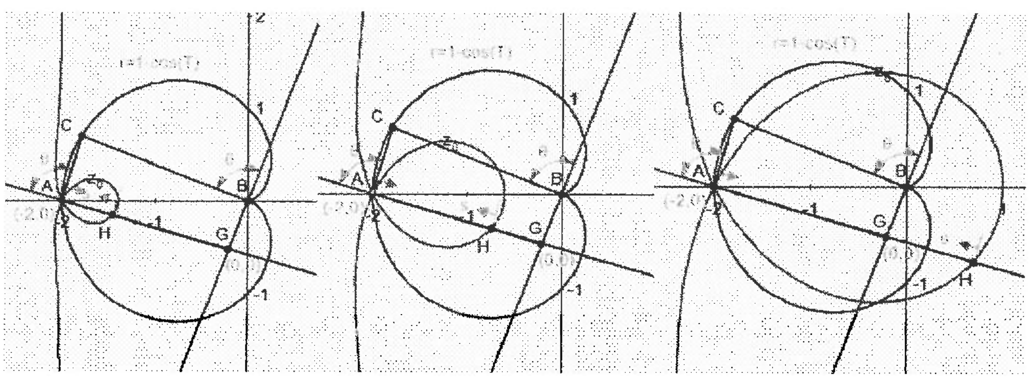


Figure 14(a). Locus and cardioid when $s = 0.3$ Figure 14(b). Locus and cardioid when $s = 0.7$ Figure 14(c). Locus and cardioid when $s = 1.5$

Exploration 2. To ~~We~~ discuss the scenario when ~~we set~~ $\measuredangle CAG = \measuredangle CBG = \pi - \theta$, ~~If we~~ denote the moving point $C = (x_0, y_0)$ and the intersection $G = (x, y)$ for the lines BG and AG. Since the slopes for CB and CA to be $k_{CB} = \frac{y_0}{x_0}$ and $k_{CA} = \frac{y_0}{x_0 - a}$, respectively, ~~We set~~ $\theta_1 = \arctan\left(\frac{y_0}{x_0}\right)$ and $\theta_2 = \arctan\left(\frac{y_0}{x_0 - a}\right)$. The equations for lines BG and AG can be written as

$$y = \tan(\theta_1 - \theta)\,x \tag{12}$$

and

$$y = \tan(\theta_2 + \theta)\,(x - a)\,. \tag{13}$$

When Maple [6] solves

~~We use~~ (12) and (13) ~~to solve~~ for x ~~and yield the following from Maple [6]~~ it reports:

$$x=\frac{a\tan\left(-\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right)}{\tan\left(-\arctan\left(\frac{y_0}{-x_0+a}\right)+\theta\right)+\tan\left(-\arctan\left(\frac{y_0}{x_0}\right)+\theta\right)}.$$

Then, using ~~and use~~ (7) or (8) to find y, which is again too long to display but can be found in the worksheet at [S5]. More explorations can be found in [S8] or [S11]. We further substitute $x_0=(1-\cos t)\cos t$ and $y_0=(1-\cos t)\sin t$ into x and y respectively ~~to find~~ the G point of intersection between AG and BG. ~~which we denote it as~~ $G=(x,y)$. ~~Since the~~ locus $J=(X_1,Y_1)$ ~~is~~ such that $\overrightarrow{AJ}=s\overrightarrow{AG}$ for some real number s, the ~~locus J satisfying~~ is: $J=(X_1,Y_1)$

$$\begin{bmatrix} X_1(a,s,t,\theta)\\ Y_1(a,s,t,\theta)\end{bmatrix}=\begin{bmatrix} s\left(x(a,s,t,\theta)-a\right)\\ sy(a,s,t,\theta)\end{bmatrix}=s\begin{bmatrix} x(a,s,t,\theta)-a\\ y(a,s,t,\theta)\end{bmatrix}+\begin{bmatrix} a\\ 0\end{bmatrix}.$$

We display $X_1(a,s,t,\theta)$ and $Y_1(a,s,t,\theta)$ obtained from Maple [6] in the following screen shots respectively:

X1(*a*, *s*, *t*, theta):

$$s\left(\frac{\tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right)+\theta\right)a}{\tan\left(-\arctan\left(\frac{\sin(t)}{\cos(t)}\right)+\theta\right)+\tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right)+\theta\right)}-a\right)+a$$

Y1(*a*, *s*, *t*, theta):

$$\frac{s\tan\left(-\arctan\left(\frac{\sin(t)}{\cos(t)}\right)+\theta\right)\tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right)+\theta\right)a}{\tan\left(-\arctan\left(\frac{\sin(t)}{\cos(t)}\right)+\theta\right)+\tan\left(-\arctan\left(\frac{(1-\cos(t))\sin(t)}{-(1-\cos(t))\cos(t)+a}\right)+\theta\right)}$$

In the meantime, the parametric equation for the locus J can be obtained from Geometry Expressions [3], which can be exported to Maple [6], which we show below and allows us for further investigation.

```
[X=(a+(s*a*(-1))+(((sin(theta)*sin(t)*(1+(cos(t)*(-1)))*(-1))+(cos(theta)*cos(t)*(1+(cos(t)*(-1)))*(-1)))*((sin
(theta)*(a+(cos(t)*(1+(cos(t)*(-1)))*(-1)))*(-1))+(cos(theta)*sin(t)*(1+(cos(t)*(-1))))))*(((sin(t)*a)+(cos(t)*sin
(t)*a*(-1))+(cos(theta)*sin(theta)*(-2))+(cos(theta)*cos(t)*sin(theta)*4)+(cos(theta)*cos(t)*sin(theta)*a*2)+(cos
(theta)*(cos(t))^(2)*sin(theta)*(-2))+(cos(theta)*(cos(t))^(2)*sin(theta)*a*(-2))+((cos(theta))^(2)*sin(t)*a*(-2)
)+((cos(theta))^(2)*cos(t)*sin(t)*a*2)))^((-1))*s*a)),Y=(((cos(t)*sin(theta))+(cos(theta)*sin(t)*(-1)))*((sin
(theta)*a*(-1))+(cos(t)*sin(theta))+((cos(t))^(2)*sin(theta)*(-1))+(cos(theta)*(sin(t)+(cos(t)*sin(t)*(-1))))))*((
(sin(t)*a*(-1))+(cos(theta)*((sin(theta)*2)+(cos(t)*sin(theta)*(-2))+(cos(t)*sin(theta)*a*(-2))))+((cos(theta))^
(2)*sin(t)*a*2)))^((-1))*s*a*(-1))]
```

~~We show in the~~ Worksheets [S5], [S8] or [S11] show that the family of locus plots for the parametric equations obtained from [6], $[X_1(-2,s,t,\theta),Y_1(-2,s,t,\theta]$ are almost identical to those obtained from [3], $[X(-2,s,t,\theta),Y(-2,s,t,\theta)]$. 0.7 0.7 ~~except a~~ line segment, which continues to puzzle the author. ~~We~~ Figures 15(a) and 15(b) depict the locus in red when $a=-2$, $s=0.7$ and $\theta=2.1048671$ and $t\in[0,2\pi]$ ~~in Figures 15(a) and 15(b)~~ for $[X(-2,s,t,\theta),Y(-2,s,t,\theta)]$ and $[X_1(-2,s,t,\theta),Y_1(-2,s,t,\theta)]$ respectively. Furthermore, we plot $[X(-2,s,t,\theta),Y(-2,s,t,\theta)]$ and $[X_1(-2,s,t,\theta),Y_1(-2,s,t,\theta)]$ together with the cardioid of $r=1-\cos t$ in blue together in the Figure 15(c), which we see the extra

The difference is a

red line segment due to the DGS [3].

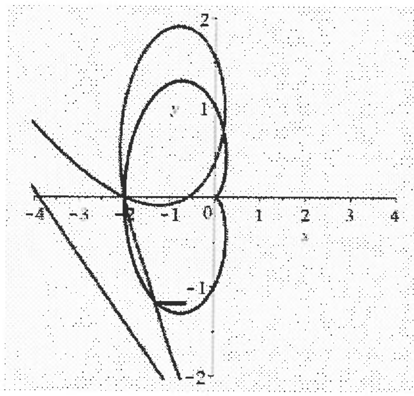


Figure 15(a).
$[X(-2, s, t, \theta), Y(-2, s, t, \theta)]$

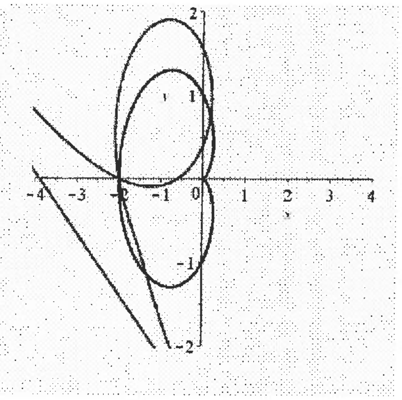


Figure 15(b).
$[X_1(-2, s, t, \theta), Y_1(-2, s, t, \theta)]$

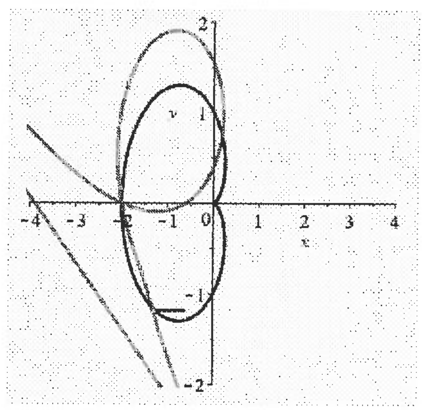


Figure 15(c). Two loci are almost identical except the red line segment

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5 Acknowledgements

The author would like to thank Dao Hoang and Phil Todd (founder of [3]) for creating instrumental dynamic figures using Geometry Expressions ([3]). The author also would like to thank Qiuxia Li for providing the proofs of some examples and numerous inspiring discussions about the level of high school math content knowledge from her teaching experiences in China.

after Conclusion

and supplementary materials [S6] - [S11].

5
~~6~~ Conclusion

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS such as [2] or [3]. In the meantime, we use a CAS such as Maple [6], for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to students from high school. Others require more advanced mathematics such as university levels, which are excellent examples for professional trainings for future teachers.

Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

7 Supplementary Electronic Materials

[S1] Maple worksheet for Section 2.2.

[S2] Maple worksheet for Section 3.1.

[S3] Maple worksheet for Example 6.

[S4] Maple worksheet for Example 7.

[S5] Maple worksheet for Example 8.

[S6] Geometry Expressions worksheet for Section 3.1.

[S7] Geometry Expressions worksheet Example 7.

[S8] Geometry Expressions worksheet Example 8.

[S9] The interactive html file of the Geometry Expressions worksheet for Section 3.1.

[S10] The interactive html file of the Geometry Expressions worksheet for Example 7.

[S11] The interactive html file of the Geometry Expressions worksheet for Example 8.

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[2] ClassPad Manager, a product of CASIO, https://edu.casio.com/freetrial/freetrial_list.php

[3] Geometry Expressions, see http://www.geometryexpressions.com/.

[4] Geometry Interactive Mathematical Arts (GInMA): A Dynamic Geometry System, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[5] Yang, W.-C., (2017) From Static Locus Problems to Exploring Mathematics with Technological Tools, *The Electronic Journal of Mathematics and Technology*, Volume 11, Number 2, pp. 67-88, 2017.

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[7] McAndrew, A., Yang, W.-C. (2016). Locus and Optimization Problems in Lower and Higher Dimensions. The Electronic Journal of Mathematics and Technology, 10(2), 69-83, https://php.radford.edu/~ejmt/deliveryBoy.php?paper=eJMT_v10n2p1.

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