Exploring Reflections That Are Inspired by A Chinese Exam Problem

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5. We will produce many nice geometric patterns using reflections over a circle.

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1. We replace the incoming and outgoing rays from lines to curves. As a result, we may create beautiful patterns involving curves.
2. To encourage beginners to appreciate how technological tools can inspire learning interesting mathematics, we use technological tools to explore three known facts about reflections along ellipses.
3. Many boring exercises can be made lively and appealing to broader students again.

1 A College Entrance Practice Problem From China

Example 1 *We refer to the following figure: A light beam starts from $M(x_0, 4)$ and follows the direction parallel to the $x-$ axis and hits $y^2 = 8x$ at P and reflects and touches the horizontal parabola at Q then the light beam touches the line* $\boxed{x - y - 10 = 0}$ *at the point N.* $\boxed{\textit{Find } x_0}$ *if the final reflection at N comes back to M.*

Line is fixed

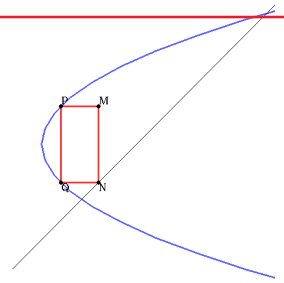

Figure 1. Reflections and a parabola

Explorations: To design an exam type of question, it is understandable that the problem cannot be too complicated and the answer has to be simple too. However, we may make a problem more realistic if technological tools are available to students. For example, one can explore the following scenarios:

1. Given point $M(x_0, y_0)$ is fixed. Now we make the line $ax + by + c = 0$ to be movable. Move the individual variable a, b or c, so that the final reflection comes back to the point M? <**Link to an exploration**>
   Netpad
2. Suppose the given point $M(x_0, y_0)$ and the line of $ax + by + c = 0$ are fixed. Make the point P on the curve $y^2 = 8x$ be movable, after the reflections of MP, PQ and QN, when will the last reflection NM come back to the point M? <**Link to an exploration**>

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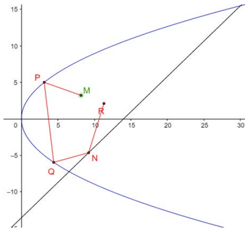
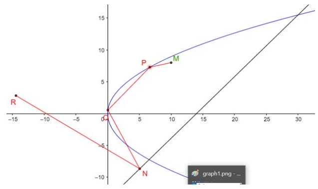
move P





#1 Netpad

We recall that if a straight line in the plane has the form of $ax + by + c = 0$ and if $(u, v) \in \mathbb{R}^2$, then the reflected point (u', v') of (u, v) with respect to the line $ax + by + c = 0$ will have the form of

$$
\begin{aligned}
u' &= u - \frac{2a\,(au + bv + c)}{a^2 + b^2}, \\
v' &= v - \frac{2b\,(au + bv + c)}{a^2 + b^2}.
\end{aligned}
\tag{1}
$$

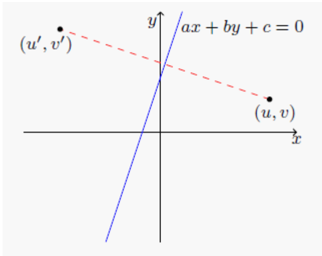


Figure 2. Reflection of a point with respect to a line

We recall a game called ‘Brick Breaker Arcade’ from the following videos, see [1] or [2].

Example 2 *Given a circle of* $x^2 + y^2 = 4$ *and a point* A =(0.385, 0.805) *in the interior of the curve. We start with the initial incidental ray of* $\overrightarrow{AB}$= (0.705 86, 0.871 32) *,where* B = (1.09086, 1.67632) *is a point on the circle. Select the proper point* D *and the line* L *so that the second reflection followed by* L *will come back to the starting point* A *after two reflections on the circle.* *<Link to an exploration>*

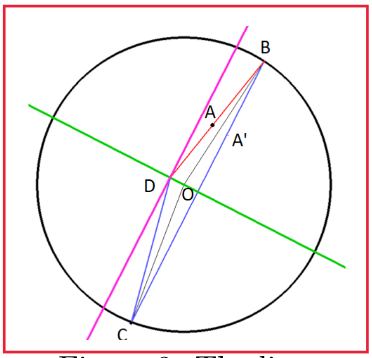


Figure 3. The line, second reflection and a circle

Step 1. The normal line at $B=(x_1,y_1)$ in rectangular form is $OB: y=1.536\,7x$,

Step 2. We find the reflection A with respect to OB, which we call it A'. The line equation $A'B$ is $y=1.954x-0.4557$. Next we find the proper intersection between $A'B$ and the circle to be $C=\begin{bmatrix} -0.7212729048 \\ -1.865412929 \end{bmatrix}$.

Step 3. We note the normal line at C is $y=2.586\,3x$. We next find the reflection of BC with respective to the line OC to be $y=3.685*x+0.7926$, which we call it L'.

Step 4. We find the intersection point between L' and AB to be $D=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -0.188\,84 \\ 0.096\,64\,7 \end{bmatrix}$.

Step 5. It suffices to find the line of angle bisector, L'', between DB and CD at the point D. This turns out to be $y=-0.5117x$, which is the green line in Figure 3.

Step 6. Finally, we find the desired line L''', which is perpendicular to L'' and passes through the point D, to be $y=1.954x+0.4657$.

1. Repeat the problem by choosing a different point A in the interior of the circle and different boundary point B on the circle. <Link to an exploration>

2. Repeat the problem by starting a proper interior point A and the boundary point B. Fnd the proper point D and the line L so that the **sixth** reflection followed by L will come back to the starting point A. <Link to an exploration>□

Remark: A game that can be easily linked to this problem and can be stated like this: We start with a point A within a given circle and start a random reflection along the circle, we are looking for a precise place within the circle (point D) and a proper line (L) so that the reflection will come back to the point A after finitely many reflections.

2 Geometric Patterns, Reflections and Circles

We now turn to a natural question one would ask by connecting an interior point of a given circle with another point that lies on the boundary of the circle. **The question we ask is if such initial starting ray will come back to the same starting point after finitely many reflections. In other words, we ask if the reflections will become periodic after finitely many steps.**

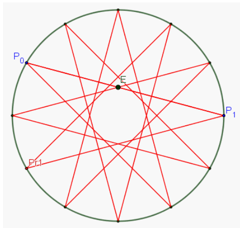


Figure 6(a) When $\theta = 15°$ and [6]

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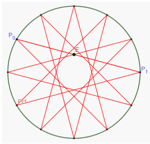


Figure 6(a) When θ $= 15°$ and [6]

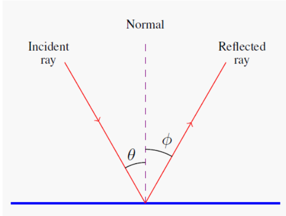


Figure 4. Law of reflection

1. We pick the starting point from an interior point E of a circle with the trajectory that hits a point P_1 on the boundary of the circle. At the point P_1, we define the angle θ of incidence as the angle between the inward pointing normal vector at point P_1 and the billiard trajectory EP_1.

2. Define the angle of reflection as the angle ϕ between the normal vector at P_1 and the billiard trajectory P_2P_3. We see the angle of incidence θ is same as the angle ϕ of refection. We first analyze the angle θ when the reflections form a regular polygon.

3. We continue with the reflection with the fixed angle θ and ask if there is a positive integer n so that the $P_n = E$. If such positive integer n exists, we call such reflection a periodic. In addition, theoretically we need to specify in advance how tow points can be numerically considered as the same point. For example, we may set a pre-determined numerical small error to be $\epsilon > 0$, and for the points $p = (x_1, y_1)$ and $q = (x_2, y_2) \in \mathbb{R}^2$ satisfying $\|p - q\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \epsilon$, we say p and q are identical within ϵ.

4. We first analyze the angle θ when the reflections form a regular polygon. With a dynamic geometry software (DGS) at hand, we start with a point $E \in \mathbb{R}^2$ with a fixed direction v, which forms a fixed angle θ with the normal vector at P_1 on the circle.Since this paper is meant for exploring new ideas, unless otherwise stated, we shall not get into the discussion of how ϵ is chosen.

2.1 Reflections and a regular polygon inscribed in a circle

We recall that *a convex polygon is a simple polygon (not self-intersecting) in which no line segment between two points on the boundary ever goes outside the polygon. A convex polygon is regular if each side is of equal length*; subsequently, each interior angle of a regular convex n-polygon has the measurement of $\left(1-\frac{2}{n}\right)\cdot 180^\circ = \left(1-\frac{2}{n}\right)\pi$. Therefore if the incidental angle for a reflection is

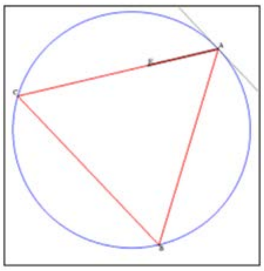
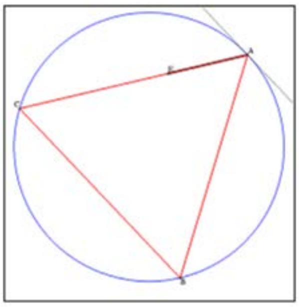


Figure 5. Reflections and an equilateral.

$$\theta = 90^\circ\left(1-\frac{2}{n}\right) = 90^\circ\left(\frac{n-2}{n}\right) = \frac{\pi}{2}\left(\frac{n-2}{n}\right), \tag{2}$$

where $n = 3, 4, \ldots$. Then the reflections become periodic and follow the path of a regular convex n-polygon.

For example, when $n = 3$ in (2) we see the inclination angle $\theta = 30°$, then we create an equilateral. In the Figure 5, we consider the circle $x^2 + y^2 = 4$ and start with the initial incoming ray of EA, with the interior point $E = (0.276886, 1.09285)$ and $A = (1.45596, 1.3712)$, which lies on the circle. We see the inclination angle θ between EA and the normal line at A is $\theta = 30°$. It follows that the $4-th$ reflection, the last reflection at the point C, will come back to the initial reflection of EA. In the meantime, we see the reflections form an equilateral triangle.



1. The reflections become periodic or not does not depend on the location or the size of the circle.

2. If we assume the initial ray starts with a point $E \in \mathbb{R}^2$ and ends with the point $P_1 = (a, 0)$ on the circle of $x^2 + y^2 = a^2$. Then the incidental angle $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For simplicity, we assume $\theta \in \left(0, \frac{\pi}{2}\right)$ in our discussions.

3. If we use $E = (0, e)$ and E is in interior of the circle, $x^2 + y^2 = a^2$, then $\theta \in (0, \frac{\pi}{4}]$, and if E is outside of the circle, then $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Without loss of generality, we may consider the initial incoming ray EP_1 is formed when $E = (0, e)$ lies on the $y - axis$ and $P_1 = (2, 0)$ lies on the circle of $x^2 + y^2 = 4$. For arbitrary interior point E and a point on the circle P_1, we may use rotation method to obtain the same results.

Next, we know that there are other scenarios where the reflections along a circle can be periodic. For example, we now consider *a regular star polygon, that is a self-intersecting, equilateral equiangular polygon.*

Definition 4 *We call caustic curve to be the curve such that each billiard trajectory is tangent to such a curve.*

We see the caustic curve when $\theta = 15^\circ$ as we see in Figures 6(a) or 6(b) is a regular convex 12-gons. We present another example as follows:

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Example 5 *Consider the incidental angle of $\theta = 5^\circ$ and $P_1 = (2, 0)$, then we obtain a regular star 36-polygons and $E = (0, 0.1749773271)$ in this case. We depict the regular 36-polygons using [6] and [7] respectively in Figures 7(a) and 7(b) respectively. We remark that the caustic curve in this case is a convex regular 36-polygons.*

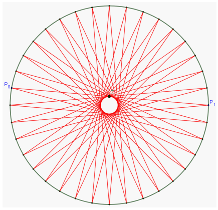


Figure 7(a)
When $\theta = 5^\circ$.

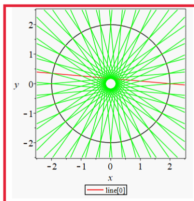


Figure 7(b)
When $\theta = 5^\circ$.

It is clear that we need to rely on a CAS ([7] in this case) to find the relationship between the incidental angle θ and the number of regular polygon it may create if the reflections become periodic. <Link to an exploration> <Link to a Maple exploration>

Example 6 *If we start with* $E = (0, e)$*, the point* $P_1 = (2, 0)$*. If we set* $\theta = \frac{\pi}{180}$ *or* 1°*. Then find the number of points needed to make the reflections periodic.*

We use Maple to compute that 179 points are needed to make the refection recursive. In such case, we get a regular star 180−polygons. We show the initial and final reflections as follows in Figure 8(a) and Figure 8(b) respectively. <Link to an exploration>, <Link to a Maple exploration>

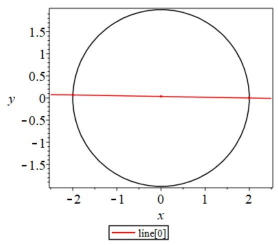


Figure 8(a) Initial ray when $\theta = 1^\circ$

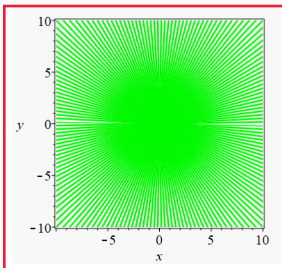


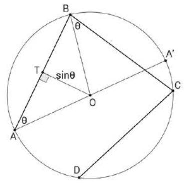
Figure 8(b) Final reflection when $\theta = 1^\circ$

3 Future Investigations

Theorem 1 *Let P_1 be a fixed point on the circle C of $x^2 + y^2$ and $E \neq P_1$ be a point on C. If θ represents the angle of incidence between EP_1 and the normal vector at P_1. Then the distance each reflection moving along the circle is $\pi - 2\theta$.*

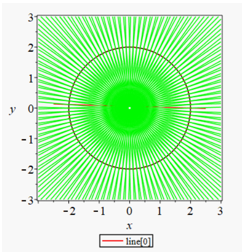
Theorem 2 *If there exists a positive integer n satisfying $\boxed{n(\pi - 2\theta) = 2m\pi}$ for some positive integer m, then the circle reflections become periodic when n is the smallest such positive integer.*

Theorem 3 *Let the incidental angle θ be a rational number, $\frac{p}{q}$, $(p, q) = 1$. Then the circle reflections become periodic after finitely times. The caustic curve of such reflections is a convex regular $n-$polygon, which circumscribes a concentric circle of radius $\sin\theta$. On the other hand, if θ is an irrational number, then the reflections will become infinite. The caustic curve of such reflections is a concentric circle of radius $\sin\theta$.*

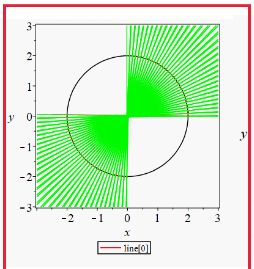


It is clear that in theory, if $\theta \in \mathbb{Q}^+$, then the needed n follows the equation $n(\pi - 2\theta) = 2m\pi$ for some positive integer m. We shall see how Maple handles the situations when θ is a repeating decimal.

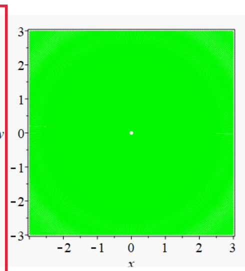
Angle θ	Needed nth reflections ($n-1$ point) (by Maple)
$\frac{1}{3}$	540
$\frac{2}{3}$	137
$\frac{4}{3}$	270
$\frac{5}{3}$	108
$\frac{7}{3}$	540



$\theta = \frac{5}{3}$ degree, 108 gons



$\theta = \frac{2}{3}$ degree, 137-gons



$\theta = \frac{7}{3}$ degree, 540-gons

3.1 Future Work

Find an explicit formula for needed n when a rational angle θ in degrees is given.

Angle θ	Needed reflections n (by Maple)
$\frac{7}{4}$	720
$\frac{7}{2}$	360
$\frac{5}{4}$	144
$\frac{11}{2}$	360
$\frac{5}{8}$	288

4 Replace Incoming And Outgoing Line Segments With Symmetric Curves

Mathematically, an incoming ray and an outgoing ray is symmetric to a normal line at a point on the circle. In other words, we may say that the outgoing ray is the inverse of the incoming ray with respect to the normal line. Now, suppose we replace the incoming ray by a smooth curve with proper starting and terminating points, and we would like to find the general inverse of this smooth curve with respect to a normal line at a point on a circle. Since circles are symmetric, we expect to create nice patterns of graphs.

$$
\begin{aligned}
\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \frac{1}{a^2+b^2} \begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix},
\end{aligned} \tag{3}
$$

Example 9 *Here we consider the circle* $x^2+y^2=4$, *centered at the point* O *(see Figure 9 in green). The triangle* $\triangle\ MNL$ *is an equilateral inscribed in the circle, where* $M = ([1.45596, 1.3712])$, $N = (0.45951, -1.9465)$ *and* $L = (-1.91547, 0.575301)$. *(See Figure 9) We describe how we construct three ellipses that passes through* M, N *and* L *respectively. Consequently, then we construct a curve (part of the first ellipse) that is symmetric to another curve (part of the second ellipse) with respective to the normal line at the point* M. *Similarly, we can construct two symmetric curves with respective to the normal lines at* N *and* L *respectively.*

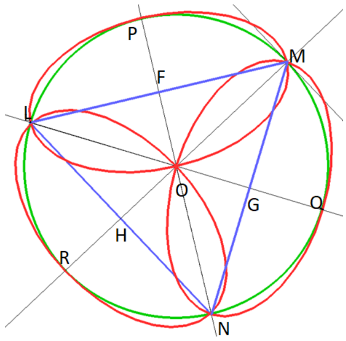


Figure 9. Reflections using smooth curves

Step 1. We find the midpoint F of ML and construct a perpendicular line l_1 to ML using F as the perpendicular foot. We call the intersection between l_1 and the circle to be P.

Step 2. We construct an ellipse using F as its center and FM and FP as its major and minor axes respectively.

Step 3. We proceed to construct the second and third ellipses analogously.

Step 4. It is easy to see that the curve, which is the portion of the ellipse passing through L, P and M, and another curve, which is the corresponding portion of the ellipse passing through M, Q and N, are symmetric with respect to the normal line at M.

Step 5. Analogously, we can construct two symmetric curves with respective to the normal lines at N and L respectively.

Remark: Incidentally, we ran into a construction of the rose with three leaves, where the angles between each leave is $\frac{2\pi}{3}$.

Discussions: As we see that the curve $\widehat{LPM}$ is symmetric to the curve $\widehat{MON}$ with respective to the normal line at M. Mathematically, we may ask to find the general inverse $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ for a given parametric equation $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ with respect to a line of $ax + by + d = 0$ (i.e., $y = \frac{-a}{b}x + \frac{-d}{b}$). We see the slope of this line to be $\frac{-a}{b}$ and we set $\theta = \tan^{-1}(\frac{-a}{b})$. According to [10], we have

$$\begin{aligned} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \frac{1}{a^2+b^2} \begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix}, \end{aligned} \tag{4}$$

we leave it as an exercise to readers to explore finding nice patterns when replacing lines by curves with respective to proper normal lines at points along a circle.

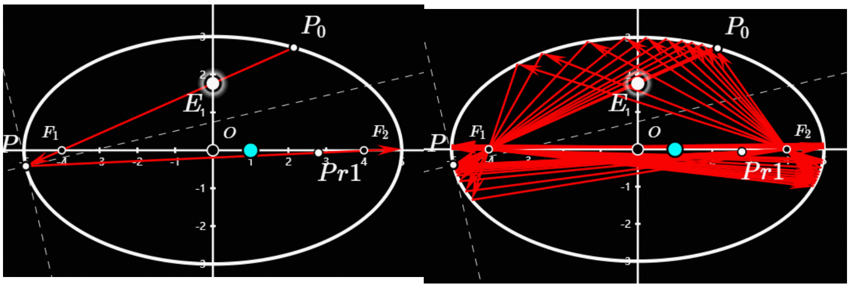
5 Exploration: Reflections and Ellipses

We certainly can extend the reflections over a circle to an ellipse. That is in an elliptical billiards: If a trajectory closes after a finite number of bounces. The history of Poncelet's Theorem is very interesting and there are many deep mathematical results in connection with this theorem, including the conditions for periodicity obtained by Cayley. We refer readers to [4] in exploring several interesting scenarios regarding the elliptical billiards when technological tools are implemented. In what follows, we use a DGS and CAS [8], which developed by a Chinese research group, to explore the following three known facts which are proved by [9]. In the following demonstrations with technological tools, we shall see that even though the proofs in [9] are evidently non-trivial; however, technological tools can indeed be effectively implemented for making complex mathematical concepts more accessible.

Example 10 *Consider the ellipse of* $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ *with the foci of* $(-4, 0)$ *and* $(4, 0)$. *Let* P *be a point on the ellipse. We explore if the incidental ray* EP *passes one of the foci, then the reflected ray will pass the other foci. It can also be shown theoretically that the trajectory of the billiard converges to the major axis of the ellipse.*

Readers can explore this example through

$$https://www.netpad.net.cn/svg.html\#posts/137109.$$



Example 11 *Consider the ellipse of* $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ *with the foci of* $(-4,0)$ *and* $(4,0)$. *Let* P *be a point on the ellipse. We explore if the incidental ray* EP *crosses the* $x-axis$ *between the two foci. Then we can show theoretically that the caustic forms a hyperbola.*

Readers can explore this example by modifying the example from

$$https://www.netpad.net.cn/svg.html\#posts/137109.$$

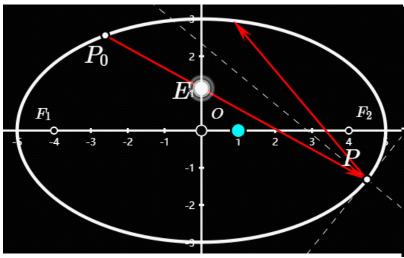


Figure 11(a). Incidental ray crosses between the two foci.

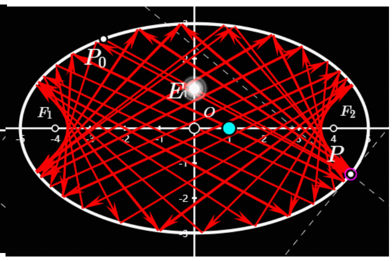


Figure 11(b). The caustic forms a hyperbola.

Example 12 *Consider the ellipse of* $\frac{x^2}{4.5^2}+\frac{y^2}{4.3^2}=1$ *with their respective foci. Let* P *be a point on the ellipse. We explore if the incidental ray* EP *does not intersect with the line segment between the two foci of the ellipse. Then every trajectory of the billiard is tangent to the ellipse which shares the same foci with the ellipse. In other words, the trajectory has a caustic which is an ellipse confocal to the elliptical billiard table.*

Readers can explore this example by modifying the example from

$$https://www.netpad.net.cn/svg.html\#posts/137109.$$

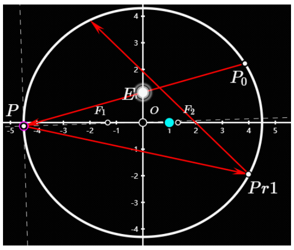


Figure 12(a) Incidental ray does not intersect the line segment containing two foci.

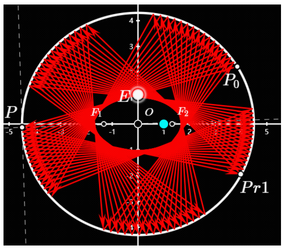


Figure 12(b). A caustic confocal to the elliptical billiard table

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