Notes for Time-Series and Machine Learning Methods
Part A: Exponential Smoothing and SARIMAX

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Time-Series Data

- We collect data from observations over time
- We denote the observation numbers (i.e., *index*) as 1, 2, . . ., T
- What we observe is denoted as y_1, y_2, . . ., y_T (see below)
- So y_t for $t \in \{1, \ldots, T\}$ is the t^{th} observation
- For example, the monthly average number of calls, daily average sales per store across locations, amount of solar power generated, etc.
  see below
- We wish to forecast y_{T+h} for some pre-defined horizon h which is a positive integer
- That h-step ahead forecast for observation $T + h$ given we have observed vales from time 1 to T is represented as $\hat{y}_{T+h|T}$
- Below is a 15-minute average solar irradiance dataset sample

Index	Date	Time	Avg. Irradiance y
1	Apr-5-2023	11:00-11:14	632 $= y_1$
2	Apr-5-2023	11:15-11:29	641 $= y_2$
3	Apr-5-2023	11:30-11:44	579 $= y_3$
⋮	⋮	⋮	⋮
T	Apr-12-2023	11:15-11:29	645 $= y_T$

$T+1$
$T+2$ ... next day

Time-Series Forecast: Exponential Smoothing (ES)

$\hat{y}_{T+1|T} = y_T$ is persistent forecast

- Why at all exponential smoothing?
  - Handles seasonal data well
  - Unlike SARIMAX, ES very rarely throws errors
  - A special case is indeed persistent forecast
- We begin with the simple time series **without** any trend or seasonality
- A one-step ahead forecast for time $T + 1$ given we have observed vales from time 1 to T is represented as $\hat{y}_{T+1|T}$

$h=1$

- The most fundamental exponential smoothing method uses

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \ldots$$

$\alpha = 1 \Rightarrow$ persistent forecast

where $\alpha \in [0, 1]$ is called the smoothing parameter

- Note that larger weights are given to most recent observations
- The weights attached to the observations decrease exponentially as we go back in time, hence the name ES

We do not know α
y_0 is unknown

ES: Component Form for Simple Time Series

$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots$

For any t

$\hat{y}_{t+1|t} = \alpha y_t + \alpha(1-\alpha) y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \cdots$

- We can show that for any t,

$$\boxed{\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\hat{y}_{t|t-1}}$$

$\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\{\alpha y_{t-1} + \alpha(1-\alpha) y_{t-2} + \cdots\}$

$\hat{y}_{t|t-1}$

- The above is written in component form as
  - Forecast equation: $\hat{y}_{t+1|t} = \ell_t$
  - Smoothing equation: $\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$
- We call $y_1, y_2, \ldots, y_T$ as the training data
- There are two unknowns α and ℓ_0
- We select α and ℓ_0 by minimizing the sum of squared errors (SSE)

$$SSE = \sum_{t=1}^{T} (y_t - \hat{y}_{t|t-1})^2$$

- The minimization is usually done with an optimization tool

ES: Component Form for Time Series with Trend but no Seasonality

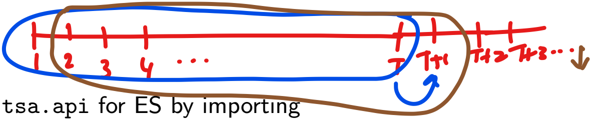
- Holt's linear trend method
  - Forecast equation: $\hat{y}_{t+1|t} = \ell_t + b_t$
  - Level equation: $\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
  - Trend equation: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

  ℓ_t is an estimate of the level of the series at time t, b_t is an estimate of the trend of the series at time t, α the smoothing parameter for level, and β^* the smoothing parameter for trend
- We could consider other aspects such as *damped* trend, and multiplicative trend (above is additive) $\ell_t * b_t$
- See Hyndman and Athanasopoulos' FPP3 book for details (same notation)
- We will not use damping in this course but is an effective tool that prevents extremely high forecast values
- In fact, this case of no seasonality itself is something we do not consider in this course

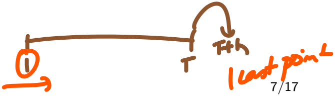
ES: Component Form for Time Series with Trend and Seasonality

- Results for any horizon h (earlier we had $h = 1$)
- The seasonality period is m and define $k = \lfloor (h-1)/m \rfloor$ = 0 { if $m > h-1$ i.e. predict less than a seasonal period
- Holt-Winters' seasonal method uses the iteration
  - Forecast equation: $\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$
  - Level equation: $\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$
  - Trend equation: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
  - Seasonal equation: $s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$
- Multiplicative seasonality (useful for solar forecast)
  - Forecast equation: $\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
  - Level equation: $\ell_t = \alpha \frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1}$
  - Trend equation: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
  - Seasonal equation: $s_t = \gamma \frac{y_t}{\ell_{t-1}+b_{t-1}} + (1-\gamma)s_{t-m}$
- We select α, β^*, γ, s_0, s_{-1} ..., s_{-m+1}, b_0 and ℓ_0 by minimizing the RMSE or maximizing the likelihood → Python code
- Software also allows for a Box-Cox transformation → Makes data normal

ES: Python Code



- We use `statsmodels.tsa.api` for ES by importing `ExponentialSmoothing`
- Refer to the documentation to see what all parameters can be set
- Check what the default values are for them
- In our case we may have historical data that is much larger than T
- We select T, train the model, and predict for $T + h$
- Next time we use data from index 2 to $T + 1$, again train the model, and predict for $T + 1 + h$
- We continue this process of sliding window
- We let the model choose α, β^*, γ, s_0, s_{-1} …, s_{-m+1}, b_0 and ℓ_0 by the default optimization method
- Technically speaking we may already know some of them as we are considering a sliding window
- We ignore that aspect but note that these can be provided to the function
- For point forecasts we will not do a Box-Cox transformation
- Let us use the code provided in the course



Time-Series Forecasts: SARIMAX

- We begin by explaining ARIMA (Auto-Regressive Integrated Moving Average), i.e., without seasonality and exogenous data
- The approach taken in ARIMA is to first remove the effects of seasonality and trend using a process called differencing
- Then ARIMA simply uses recency to make the forecasts, and then re-introduce the seasonality and trend
- Hence, as an initial step, we try to create a *stationary* time-series
- A stationary time-series in an ideal condition is independent of the time stamp (index)
- Both trend and seasonality would render a time-series to be non-stationary and they would have to be removed
- ARIMA assumes the time-series to be stationary
- The approach used to create a "nearly" stationary time-series is called differencing

Differencing

- Recall that we have observations y_1, y_2, ..., y_T that we use for making the forecasts
- We denote the time series as $\{y_t : 1 \leq t \leq T\}$
- Now consider the time-series $\{y'_t : 2 \leq t \leq T\}$ which is called the first-order differencing term defined as

$$y'_t = y_t - y_{t-1}$$

  for all $t \in \{2, 3, \ldots, T\}$
- The differenced series has one fewer term than the original time-series
- Sometimes the series $\{y'_t : 1 \leq t \leq T\}$ may not be adequately stationary, and it may be required to do one more round of differencing
- For that, consider the time series $\{y''_t : 2 \leq t \leq T\}$ defined as

$$y''_t = y'_t - y'_{t-1}$$

  for all $t \in \{3, 4, \ldots, T\}$
- Note that the second order differenced series has one fewer term than the first-order time-series and two fewer than the original time series

Differencing Example

Differencing: Some Comments and Seasonal Differencing

- In the previous figure, the automated code using `pmdarima` picks first order
- Usually first order differencing is enough, rarely second order
- It is not recommended to go beyond second order differencing, although the series may not be stationary
- Sometimes that is because of lingering seasonal effects
- When there are seasonal effects, it may be worthwhile (although not guaranteed to remove) to perform seasonal differencing
- Let $\{y_t^s : m \leq t \leq T\}$ be lag-m differences, where m is the period of seasonality
- The seasonal differencing term is defined as

$$y_t^s = y_t - y_{t-m}$$

  for all $t \in \{m+1, m+2, \ldots, T\}$
- If needed, we could perform a second-differences (one against season and one against consecutive terms)

Time-Series Forecasts: Model for ARIMA

- We begin with ARIMA with no seasonal effects.
- We stationarize the time series using differencing
- The **ARIMA(p, d, q)** model has p as the autoregressive part, d the degree of differencing and q the moving average part
- This gives rise to the model (in terms of the differenced series y'_t)

$$y'_t = c + \phi_1 y'_{t-1} + \ldots + \phi_p y'_{t-p} + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q} + \epsilon_t$$

  where ϵ_t is from the original series

$$y_t = c + \phi_1 y_{t-1} \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \epsilon_t$$

- Also, the ϵ_t terms are a function of c, ϕ_1, ..., ϕ_p as well as the time series
- Note that c, ϕ_1, ..., ϕ_p, θ_1, ..., θ_q are all unknown

Time-Series Forecasts: Estimating the ARIMA Unknowns

- In the ARIMA model we can use MLE (default in the python library we use) to estimate c, ϕ_1, ..., ϕ_p, θ_1, ..., θ_q which are all unknown
- For the MLE, we maximize the likelihood of obtaining the series y'_t for all t assuming it is Gaussian and so are the ϵ_t values
- Another approach is to minimize the sum of squares of ϵ_t
- Either case would result in a complicated non-linear optimization problem that cannot be written in closed-form
- There are numerous software packages that produce the results

Time-Series Forecasts: Model for SARIMAX

- The key idea of seasonal-ARIMA is to add seasonal terms
- Extend the **ARIMA**(p, d, q) model to the **seasonal-ARIMA**$(p, d, q)(P, D, Q)$ with m seasons
- Although we do not present the equation, it is worthwile understanding that we add seasonality AR terms y_{t-m}, ... y_{t-Pm} with coefficients Φ_1, ..., Φ_P, as well as seasonality MA terms ϵ_{t-m}, ..., ϵ_{t-Qm} with coefficients Θ_1, ..., Θ_Q
- Now, if we were to add exogenous variables, say x_1, x_2, ..., x_r, then we essentially get the form

$$y'_t = c + b_1 x_1 + b_2 x_2 + \ldots + b_r x_r + \phi_1 y'_{t-1} + \ldots$$

Selecting Model Parameters for SARIMAX

- Next we need to determine how to get p, q, P, and Q
- They can be selected by searching through the space of values and find the one that minimizes an information cretrion
- The common choice of information criteria are AIC (Akaike's Information Criterion), AICc (i.e. AIC corrected), and BIC (Bayesian Information Criterion), see Hyndman and Athanasopoulos' FPP3 book
- We do not present the expressions here but they are typically functions of likelihood of obtain the data (that was used in MLE), T, p, q, P, Q, and whether or not c was used
- There is also an automated way (although many do not recommend) to do this using `pmdarima`

SARIMAX Forecasts: Python Code

- Import various python tools and libraries
- Select params viz. test data, sample size, etc.
- Get data and aggregate, if necessary
- Clean up data by imputing
- Merge with exogenous data (if we use them) to get required dataset
- Visualize the data for a few days
- Fit auto-arima model, or other methods for (p, d, q) and (P, D, Q)
- Use sliding window for predictions
- Test predictions using various metrics