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Descent technique and existence theorems in algebraic geometry. I. General. Descent by faithfully flat morphisms

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TECHNIQUE OF DESCENT AND THEOREMS OF EXISTENCE IN ALIGNMENT ALGtBRIQUE

I. GENERAL. DESCENT BY FIDELY PIATS MORPHISMS by Alexander GROTHENDIECK

On the point of view technique, the present outlines, and those that it will result, can be considered ::: s as variations on the famous "Theorem 90" of Hilbert. The introduction of the method of descent in geometry algebraic seems due to A. Weil, under the name of "descentedu body of basis. " WEIL is restraint of garlic

their in the case of finite separable body extensions . If extensions radicals cial of height 1 has been then treated by P. CARI'IER. For lack of the language of the schells, and more particularly for the lack of nilpotent elements in the rings which one allowed oneself to consider, the essential identity between these two cases could not have been formulated by CARI'IER.

A time current, it seems qua art General descent to be exposed e (attached e l e ca s echean t at x theorem s fondamentau x d e sa ¹¹g e ome t ri e fonnel-
the " see [3J) is in the base of the majority of the theorems of one existence in geometry

algebraic . I the convien t d e Note r d 'aill eur s that has the has said e techniqu e little t certainly lies to transpose "geometrieanalytique ", and it is hoped qua in a pro che future, the specialists will demonstrate las similar "analytical" of theorems of 'existenc in geometry fonnelle who will give in exposes II. in any case, the recants work of Kodaira-Spencer, including the methods seem in-
likely has the definition and the study of "variates of modules" in the vicinity of their

singular points, show clearly enough the need methods closest to the theory of schemes (which will complete course las technical trans cendantes).

In the present exhibits I, we treat the case of lowering the more elementary (indicated in the title). The applications of Theorems 1, 2, 3 below, are however already very numerous. We are terminals to give some way of example, without trying to give here the maximum of generality pos- sible.

We freely use the language of schemes, for which we refer At exposes already cited, as has [2]. Note, however, expressly qua the pre-schemes envisaged in the present exposes only are not necessarily Noetherian, and the

morphisms do are not necessarily of the type done. Thus, if A is a local Noetherian ring , of complete Â, we will have to consider the non-Noetherian ring

1 Ai, and the morphisms of affine schemas corresponding to the inclusions between

the rings envisaged.

A. Preliminaries on the categories.

I. Categories fibered, data of descent, morphisms of ,: -descente.

at. DEFINITION 1.1. - A category fibered : J ' of base C is FORMED a category C, for all XE C a category : FX , for all- £ _-morphism.

f : X y of a n functu: - r * : J ' . . J y X not e also

for £ (X being cqnsidere as an "object to . £. above of y II.

ie as endowed with mornhism f), and finally for two composable morphisms

X - £. y Z , of an isomorphism of functors

cf, g : (gr) * r * g *

these data being subject to the following conditions

(i) id * = id

(ii) cf, g identical

es t 1 'is omo rphi m e identiqu e s i f

where g is an isomorphism

(iii)

As data three morphisms composable

, we have

commutativity in the diagram according isomorphism form the means of isomorphism of the form c u, v

(h (gf)) * = ((hg) rt r * (hg) "'

*

....

(gf) h (f g) h = f ( )

Example 1. - Let C a category or the products fibers exist, it defines then a category fibered ; f ' of base £ _. In posing :,; _ = category of objects of C above for X , the functor r * : . { corresp,; mdant has a mor- morphism f X-) Y being defined by the product fiber Z, vv \,) Z xy X •

EXAMPLE 2. - Let C be the category of pre-schemes, and for X f. £ , or the category of quasi-coherent sheaves of moduli on X • If f : X + Y is a morphism of pre-schemas, r " ': 'fy --- ¼ S:; _ is the functor picture reverse

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of beams of modules . It gets as a category fibered of basic b.DEFINITION 1.2. - A diagram

C.

E E'

v
- 4 E II
- -,
V2

mappings of sets is said to be exact if u is a bijection from E on the part of E ' formed by x'E E' such that v 1 (x ') = v 2 (x') •

DEFINITION 1.J. - Either diagram

f a category fibered of base

C, consider a

(+)

of morphisms in C such that o1 .. JJ 1 = c, (. To this diagrarom is said : F-exact if

for any pair S , I of elements of , the following diagram of sets of mappings

(+)

Hom (**1** , *1*)

this. *

Hom (**o(•.**

Cs)

(or

¥ = **"f t** 1 = ***Ql. . / 1*** 2) are t right.

D years this last diagram, we have identified ' grace has c / 3i' '' - ' to simplify.

A *.
1 ' **1**

" '" / • has

(**R ..r i**) , · =
R

v *
0

DEFINITION 1.4. - Let ***J:*** a category fibered of base C us consider two morphisms **fi** 1 , ***p*** 2 : S "- + S ' in C • Let • € F; ,, . We call <u>given of re- gluing</u> on ; ' (relative to the couple (**,, a** 1 ,**/ J** 2)), urt isomorphism d e ***ft ;*** , (5) su r ***/ 3 ;,*** ( ) . S i , , ?, , , His t provided s Chacu n a e Donnee of reattachment, a morphism u : **t'** --- *r* "? ' in ' is said to be <u>compatible with the data for pasting</u> if it is commutative in the diagram as follows:

Within this way is, the objects of , provided a given of reattachment (relatively vely has *fl* 1 , **j.3** 2) f rm then a <u>category.</u> If **oC** ..: S ' S is a morphism e te s that e *cl* ... 1 3 1 = ***o <. / 3*** 2 , Alor s pou r tou t **S** €, , 1 'ob j e t ' = **o(** * (**!**)

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of eet having a given of gluing canonical, since

e n posan t encor e ¥ = ***c*** (/ \ = ***cl..f has*** 2 ; d e more , s i u : **t** ' ? es t u n morphism

in

, r
JS,

so

o (* (u): **o <* ( ) - joe .. * ('7)**

is a morphism in ***g:; ,,*** consistent with the data of gluing canonical.

We thus obtain a <u>canonical functor</u> of the category in the category of objects of . *r* .*r* *r* S, provided a given reattachment relatl tively to couples (fi 1 , ***p*** 2). This said, we can also express the definition 3 saying that the diagram (+) of said is said : f..exact if the preceding functor is "full men t f idele ".i.e . Set t a e equivalenc e d e l a class to e ave c a e sub- category of the category of objects . provided with a reattachment data relative to *(fa* 1 , ***ft,*** 2).

DEFINITION 1.5. - We say that an <u>effective</u> (relatively to ***r ;; l.***) If • ( ) , with **f:** .

In the case where the diagram (+)

given to gluing on

{ ' Equipped with this given is isomorphic to a is : f '.. exact, the object t of the definition

precedent is therefore determined to a near unique isomorphism , and the category : J'S is equivalente has the category of objects : ;; ,, mrmis a given of harvested LEMENT effective.

vs. The case the most significant is the one or

the j3i being the two projections p 1

and p 2

of S ' x S ' on these two fac-
S

tors (we now assume that in C, the fiber products exist). We have then two terms necessary immediates to that given to recolla ent · Lf: p (') -) p; (f) on an effective softe.

We have

(i}

OR 6 : S'S' XS S' designates the morphism diagonal, and OR is a identifies fl * p (•) a (pi 6.) * (•) =

We have

(ii)

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or pij designs the projection canonical of S ' XS S' XS S ' on the product par- tial of the i-th and j-th factor.

DEFINITION 1.6. - We call given by descents on • €', relative to the morphisms c(.: S' --t S, a given of gluing on relatively the pair (p 1 , p 2) of the projections canonical S ' XS S' - + S ' , satisfying the

conditions (i) and (ii) above.

DEFINITION 1.7. - A morphisms o <. : S' -, S is called a morphisms of ! F' down if the diagram of morphisms

s s • s • x s •
- P2 s

is :: F-exact (definition 1.3.); we said that is a morphism of =; a.escente.? strict side over any given of descent (definition 1.6.) on the subject of : Dr. J ,
is effective.

This last requirement is also of Fagon more pictures to enunciate well: "It returns to the same "to be given a subject of , or an object fg, equipped, - - e given to descents.

Note that if a morphisms of : f..descent s S S' (ie a morphism s such that r;: - ',., -
me to u-down strict: if • £ 5 , is tively has o(, "he comes" in effect from

o{: S'S adme t un e "secti o "n
: (s = ids) then this is a morphis- having a given of descent re-
= s * (

d. It can present to Fagon more intuitive the notions preceding, in intro- duisant, for a parameter object T to C above of S, all

Homs (T, S ') = S' (T)

including the Elements will be designated by t , t ' , etc. An object t:. • 6 ! /. being S '
gives , has tou t t ' = S' (T) corre d alor s u n ob j e t t (') d e , that i will also i not e ft . A e provides e d e recollemen t su r ' (relativemen t has p_1, p_2)

is then defined by the data , for all T on S, and any pair of points t , t ' ES' (T) , of an isomorphisms

lft •, t: - + .

(satisfying the requirements obvious to fonctoralite, for T variable). The conditions (i) and (ii) of 1.) are then written

(ibis) for all T and all t £ S '(T)

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(I i bis) \ ft, t " = '-ft, t ' 'it • , t n, louse r tou t T e t t , t ' , t "E S '(T)

We see moreover that (ii bis), by making t = t ' = t ".implies Lf 2 - I..P
or to , SINCE e L i es t a isomorphism e pa r hypothesis , the has relatio n (ibis) , which
t t t
is therefore in fact a consequence of (ibis) (done (i) is a consequence of (ii)).
But if we do assume more a priori the lft, t ' of isomorphisms (i, e. That
L f : p(S')-) p ; (S ') are t u n isomorphisms) , alor s (i i bis) n 'Involved e more
necessarily (ibis) ; the combination of (ii bis) and (ia) involves however
that the 'f t, t' are the isomorphisms (because we will 1ft t ' ' -ft • t = 'ft t = id ,) •
, ' ' t

2. Exact diagrams and strict epimorphisms , morphisms of descents. Examples. a.DE ' FINISHING 2.1. - Let C be a category. A diagram.me

T T ' !!. 1 .. + T rr
/ 32

of morphisms in C is said - ' if for any Z E. £ , the chart correspond ing applications ensemblist

Hom (Z , T) --t Hom (Z , T')) Hom (Z , T ")

is correct (definition 1.2.). We then say that T) is a kernel of the pair of morphisms

(T , o <) (or by abuse of language,
(fi1 ' ; 32).

This nucleus is obviously determined up to single isomorphisms . If C is the cate- gory of sets, the definition preceding is consistent with the definition 1.2.
It defines of way is dual accuracy of a chart

al .. / J 1
S S't ===, S "
/ D2

of morphisms in _£ ; we then say that (S , of.) is a cokernel of the pair of morphisms (fl 1 , p 2)

DEFINITION 2.2. - A morphisms o(: S '-> S is called a strict epimorphism , if it is an epimorphism, and if for all morphisms u: S • Z, the following necessary condition is also sufficient for u to factor in S • S -"7Z for all S "E 2. and any pair of morphisms J 1 ,/ J 2 S "-> S ' such that
o < . p 1 = "j J 2 , where n has auss i u / J 1 = u f is 2 ,

If the fiber product S' x S S'

exists, it is the same to say that the diagram

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is exact, ie that S is a cokernel of the pair (p 1 , p 2) • In any case, a cokernel morphism is a strict epimorphism . Note also that an epimorphism
strict which is a monomorphism is an isomorphism. We leave to the reader to develop the dual notion of strict monomorphism .

To clarify the relations between the notion of morphism of
. '. f-descent, and the

notion of strict epimorphism , we again introduce the following definitions

DEFINITION 2.3. - A morphism d..: S ' -is is called an epimorphism universal (resp one. Epi Jlorphisme! Strict universal) if for every T in S, the pro- Duit fiber T' = S ' XS T exists, and the projection T ' T is an epimorphism (resp. A strict epimorphism).

In the very good categories (such as the category of sets, the category of beam sets on a space topological, the categories abelian, etc.) the four concepts of epijectivite thus introduced coincide; they are contra.ire all distinct in · category such as the category of pre-
diagrams, OR the category of pre- diagrams above 1011D d 1 tm pN = SOh &! n hOfi vlge d rttl
S mftine if it is terminal l of S-schemas finished on S.

DEFINITION 2.4. - A morphism and ..: S '-> S is called morphism of descent (resp. A morphism of down strict) if c 1
is a morphism of .1: aescente
(resp. a morphism of .1 "-ctescente strict) (cf. definition 1.7.), where : f ' denotes the
ca.t gory fibered of base C of the objects of C on the objects of C (No. 1, example 1).

PROPOSAL 2.1. - If in C the product finished and the product fibers (finished) exist, then there is identity in C between morphisms of desceute, and epimor- phismes strict universal.

b. EXAMPLES. - Let C be the category of pre-diagrams. Let Sc.C, be S ' and
S " deu x pre-schema s finish s su r S ,i.e . corresponding s a of s sheaf x d ıalge-

bra s A ' , A " su r S that i e n tan t that e beam x d e modulus s his t quasi-coherent

and of such finished (i.e. Coherent .if S is locally Noetherian). Is
c(:S '- ¼ S the structural morphism of S' , and let / 3 1 , ft 2 be two

S-morphisms of S "to S ' , defined by the homomorphisms of algebras A' A "
designes again by fii, j3 2 . Using the fact that a morphism finish is Fenne (first theorem of Cohen-Seidenberg) one proves easily that the diagram

c(/ Jl

(+) S S' t:; :: = S"

'fi2

in C is true if and only if the diagram of beams

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su r S es t true . I n particular , s i d..:S '-S es t u n morphism e end i corresponding sponding a bundle !J.. d
• algebras on S, then ol. is an epimorphism strict if and only if the diagram of beams

0 = A •
-S ..,, . -

P1
: = :::::: i A'

P2 ,...

g A'
A -

is exact (it is an epimorphism if and only if ! 'is injective). If S is refined to 'ring A, therefore S ' refines ring A' ended on A, then

S '-> S is a epimorphism strict if and only if A A' is a isomorphis- me of A on the subring of A ' forrne of x' A '! Such that

(it is an epimorphism if and only if A_ ,. A ' is injective). Like us
1 'avonsdeja signals, even if S is the diagram of a ring local artinian, a mor- finished morphism S '- ;. S is an epimorphism is not necessarily an epimorphism strict. However, we can prove that & S is a noethe pre-schema

nothing, any finite morphism S '- + S which is an epimorphism, is the sum of a finite sequence of strict epimorphisms (also finite). This shows , moreover, that the compound of two strict epimorphisms is not necessarily a strict epimorphism .

vs. If (+) is a diagram right of morphisms finite, then for any morphis- me flat T S of pre-patterns, the diagram transfonne of (+) by the Change- ment of basic T -) - S is still accurate. There in CLEAR that if X, Y are two

S-pre-schemas, X being flat on S, then the chart application ensem- blistes following (or X ' , Y' are the images reciprocal of X, Y on S ' , and X " , Y" their pictures reciprocal on S ")

is correct. In pa.rticulier, if :;. Denotes the category fibered of base the category
C de s pr é-sch émas , tell e qu e pour r X i , vt k 'so i't1 a c a t'egor i'e d e s X -pr e'- s c h'emas
dishes, then the diagram (+) is : F'-exact. (This result becomes false if we do not
is not the assumption of flatness, in pa.rticulier an epimorphism tight finish is

not necessarily a descent morphism). One of spots same as (+) is

;; -_

exact if .: I designate the fibered category for which quasi-coherent and flat sheaves on the pre-diagram X

vX is the category of
(here again the hypothesis

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of flatness is essential). In one and the other cases, the issue of effectiveness of a given of reattachment (and more particularly, a given of descent, when S " = S ' x_8 S') on an object flat in -Dessus of S ' , is
delicate, (and its response in various cases individuals is a of objects main
the present exposed). The lecturer ignore if for any epimorphism strict finite S '- ¾S any given raid on a flat quasi-coherent beam S' is effective (even by assuming that S is the spectrum of a ring local ar- Justinian, and by limiting oneself to the locally free bundles of rank 1). Generis more rattle, are A a ring, A ' a A-algebra (everything is commutative) such that the diagram of applications

se t exact , c e that i equivau t auss i a u fai t that e l e mor p h i sm e S ' S c o r respondent for the
spectra of A , A ' is a morphism of -descent, where : J- is the , - . - , , . ,, . ..
fibered category of flat quasi-coherent sheaves . Let M ' a -module flat provided with a given of descent has A, ie an isomorphism

'f: M'iAA' .A.'iAM '

A 'A'-QA module, satisfying conditions (i) and (ii) 1. (c) {we let the reader the care to explain in terms of modules). This given is
it effective (relative to the fibered category of quasi-coherent sheaves
dishes) ? Let M the subset of M ' form of x' C = M ' such that

it is a sub-module of M • Injection canonical *H* • *M* to end a homo- morphism of A'-modules M A A', M • The effectiveness of means so that: M is a A-modulus flat, and the preceding homomorphism is an isomorphism.

REMA EU. - In the considerations preceding, we had done no hvpo- thesis flatness on 1ES morphisms diagram (+), which we obliged, for a technical of descent, to make the assumptions of flatness on the
objects above of S, S ' we considered. In the paragraph 2, -we will
a hypothesis of flatness on o<.:S ' -> S , that which will allow us to have a
Technical of descent to the items above of S, S ' which will be pius Sou set has no requirement of flatness. In all cases, there is a hypothesis of
platitude that intervenes. This is a major reason for the importance of the concept of flatness in geometry algebraic (including the role only could appear
as we are confined to the body of basis on which anyone what, in fact,
is flat!).

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J. Application to etalements.- Let A a ring local, B an algebra loca- the on A whose the ideal maximum armature that of A. We say that B is etale su r A (a u binds u d e "no n ra m ifi e'.used e pa r elsewhere) s'i the satisfai t the s con- ditions following:

(i) E is flat over A

(ii) B / 1 \\ B is a separable finite extension of A / "Cfl. = k (or m desi-
gne the maximal ideal of A •)

When A and B are noetherian and k algebraically closed, it means that the homomorphisms A -¼ ii on the completes which prolongs A - + B is an isomor
phism. A morphism of such finished f : T -4S is said etale in xE T , or even

T is said to be spread over S at x, if

O
- x

is spread over
Of ( ) , e t f es t say
-.

spread OR again f is called a spread, OR T is said spread over S' if f is spread in all x f.T • Note also that if S is locally noethe- nothing, all the issues of t or f is etale is open, and ! Use the "main theorem" of Zariski allows to specify the structure of T / S at voisina- ge of such a point (by an equation of a well- known type).

When S is a scheme of the type done on the body of the complex, there it corresponds
sponds analytical espa.ce *S* in the sense of Serre [5J, was it close as S may have Elements nilpotent in the beam structural, that which changes nothing essential in [5]. One sees so easily that f is a Spreading whether and on ONLY
also if f: T-> S 1 ¹ is, ie if any point of T admits a neighborhood on
which f induces an isomorphism on an open set of S. In particular, has all
coating etale T of S (ie a morphism finished etale f: T S) corre- pon d u n revetemen t etal e T d e S , that i es t connex e s i e t seulemen t s i T 1'e s t
[5]. We also see easily that if T, T ' are two schemes spread on S, then applying natural

is bijective, i.e. the functor T--, T in the category of schemes spread on
O dan s l a class to e of s espa.ce s analytical s etale s su r S are t ¹¹plei nem e n t faithful ", thus defines an equivalence of the first category -with a sub-category of
the second. A GRAUERT-REMMERT theorem [2] implies that if S is nennial, we thus obtain an equivalence of the category of etal- coatings of S and of
l has Categorí e of s coating s etale s (finished) d e S , l.th.that e ten t in t etale of S is S-isomorphic has a T'oil st a coating etale of S • We show
that the theorem of Grauert-Remmert remains valid without assumption of normality on
S. Either in effect first S '- + S an epimorphism strict · finished, let the

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theorem.a demonstrated for S ' , show that it will be true for S. In effect, either u n revetemen t and has the
e d e S' consideron s n n imag e reciproqu e " ' su r S.
which corresponds to a coherent analytical sheaf cl of algebras on S ', reciprocal image of the sheaf of ıalgebras *Cl* on S defining By hypothesis,
su r S ' , *C* • provien t u n revetemen t etal e T ' d e S • i , e . (2.'comes
u n faiscea u coheren t of algebra s A ' su r S ' • money Austria e by t d..'es t provided

a given of descent canonical relatively as - • S - , ie a isomor- morphism between its deu.x pictures reciprocal of S ' XS S • = (S ' XS S •) , (satisfai
sant of terms (i), (ii)) and this isomorphism comes, to after that which has been
words, an isomorphism on the faisceau.x algebraic corresponding, ie a given descent on A ' relative to S' --JS. Is easily verified that cett e last th es t effectiv e (ca r cell e su r f! L'is , e t the effectivit e a
{e data of descent, such that it has been EXPLAINED the number precedent, sere- known locally on the comprehensive of modules that used in the game). Hence a
faiscea u coheren t algebra s A su r S , definissnn t u n revetemen t T d e S,

which is the coating looking for. The previous result then remains obviously valid if S '- + S is only a compound of a finite number of finite strict epimorphisms , i. e. is a epimorphism finished one (of after the result of factorization indicates in paragraph 2). It follows that the theorem of Grauert-
Remmert remains valid if S is a scheme reduced, ie such as 2..S did not of elements nilpotent, as we saw in introducing his normalizes S ' • We pa.sse
easily from the the case generally.

A demonstration all nalogue, using even the result of factorization for epimorphisms finished strict and nature "formal "of the effectiveness of data are downhill, the result can prove this: either S a pre- scheme is locally Noetherian and S ' -) S a morphism finished surjective diciel (or, equivalently, a finite morphism such that for all T su r S , the e morphism e T ' = S 'x 5 T T se t u n homeomorphism , c e that o n ex prime again by saying that S ' ,,. S is a universal homeomorphism). For everything
T spread over s, let us consider its inverse image T ' = T XS S' , which is spread over S '• Al to the functor T T • is an equivalence of categories of the category of pre- schemes T etal over S with the category of pre- schemas T ' etales on S' • (We use the bijectivity of

Homs (Tl ' T2) - + Homs, (Ti , T2)

for deu.x pre-patterns T 1 , T 2 spread on S, fact of which the verification direct is easy, and the fact that the theorem utterance is true if S ' = (S, 2._sf: /)

OR;/is a coherent beam of nilpotent ideals of 2s ([4 J'18Jlllle 6)). Note d 'also qua we will assume not here las t envisaged finished on S , this re-
Result involved in pa.rticulier, qua the morphism S • ---, S induces an isomorphism
the group fundamental {] of S ' on one of S ("invariance topological the group e fondamenta l u n pre- schem is).

4. Relations with the 1-cohomology .
a. Either . £ a category or the product of two objects still exists , or
TE £. • For all together finished I #, sb can consider T 1 , for I variable is obtained as a functor covariate in the category of finite sets not empty s dan s £. , i.e . c e q u 'o n bit t called r u n obje t Simplicia. l d e . £ , not e K r • This last depends of fa it covariant of T; moreover if u, v aont two morphisms T → T ', then the corresponding morphisms K.r K.r , are homo-
12 - Let us say that T dominant T ' if Hom (T, T') # , it is a relationship filter preorder growing in _ £. It follows from what precedes almost T dominated e T ' , i s exist e a e class e (aun e homotopi e pres) canoniqu e d ' h omom o rph izat
of simplicial objects K.r KT ' , in particular if K.r and K.r • are such that
each dominates the other, then K.r and KT ' are homotopically equivalent. Let F now be a functor (contravariant to fix the ideas) of C in an abelian category C ' , then

c • (T, F) = F (KT)

is a cosimplicial object of C ', therefore defines in a well-known way a complex (of cochaines) in C' , of which we can take the cohomology

H • (T, F) = H * (c * (T, F)) = H • (F (Kr))

(o n Pourr has mettr e u n C e n indic e d u H * s'i the there is possibility e d e confusion) .

This is a cohomology functor in F, whose variance for T variable re- Sulte of that which has been said about las l \ r; of Fagon precise, for fixed F and T va- riable in C (preordered by the domination relationship) las H * (T, F) forming a u n system e inducti f d 'o b j and s graduated e s d e C. ' · E n pa.rticulier , s i t e t t ' are such qua each other · dominates, then H * (T , F) and H • (T, F) are canonically ment isomorphic.

Suppose that in C the fiber products exist, then we can, for S (.C
fixed, applying it which pricede has the category Cc of objects of C above for S , we will write c * (T / S , F) and H * (T / S , Ft .., in place of c "(T : F) and
H * (T, F) if we want to specify that we place ourselves in the category ; so,

c • (T / S F) is a complex of cochaines in . £ ' which, in dimension n is equal to F (T XS T XS ••• XS T) (where the parenthesis has n +1 factors).

Note that as usual, we can define H 0 (T / S, F) without assuming the category . £ ' abelian: it is the nucleus (definition 2.1), if it exists, of the neck
ple of morphisms F (pi) (i = 1, 2)

F {T) F(T XS T)

corresponding to the two projections p_1, p_2 : T x 8T :::; T • In particular, we will have a natural morphism (called augmentation)

F (S) H^0(T / S , F)

which is an isomorphism in the case favorable {in particular if T S is an epimorphism strict and if F is "right has left"). Likewise , when F takes its values in the category of groups in a category C ", we can also define H^1 (T / S, F); in the case where £ ' is the category of'; ' sets (ie F takes its values from the category of ordinary groups , not necessarily commutative), H^1 (T , F) is the quotient of the subgroup Z^1 (T / S , F) of C (T / S, F) = F (T xS T) form of g such that

F (pJ1) (g) = F (pJ2) (g) F {p21) (g)

by the group of operators F (T) , operating on c^1 (T / S , F) and in particular, on the subset Z^1 (T / S , F) by

b. Let for example ., R r -

p (g '). g = F (p2) {g') gF (pl) {g ') - 1

a fibered category of bi: i, se C. Let

and for all S ' on S, let

F (S ') = Ho m (xS', -, xS ') , Y) '> S

So, Ff ,? is a contravariant functor of 2S in the category of sets bles. This poses, say that the increase monnhism

F_f,? (S) -) H^0(S '/ S , Ff, '?)

is an isomorphism for any pair of elements S , 1 f's , means that IQ ... :S ' --t S is a morphism of J-descent (definition 1. 7).

c. Poson s d e m eme , for

f,I"'. L. ", s

and any object S ' of

C above of

we have thus defined

G\(S') = Jtut (xS S')
u n foncteu r control has varian t G 't- of)

C in the category of
- S

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groups. Caci poses, we see that Z^1 (S '/ S , G) is canonically identified with ! all the data of descent on • = S x_3 S ' relative to S' --- t S (Definition 1.6) and H^1 (S '/ S, G) is identified in the set of classes (has an iso- morphism pres) of bjets of J; ,, provided with a given lowering relatively to c :(:S • s , which, in both objects of J; ,, are isomorphic a ' = xs s'

If therefore o(:S ' -+ S is a morphism of t-descent (see (b)), then H^1 (S' / , G) contains comma subset all of the classes (in a isomor morphism near) objects ") : F; such as '7 xs S' is isomorphic in f's, a t x_5 S ' ; and this inciusion is an identity if and only if all given of descent on S ' = t xs S 'with respect to o{:S • S is effective. (This will be the case in particular if o/.:S • S is a morphism of S-down strict).

REMA EU. - complexes cochains type C * (T / S, F) contain such cases pa.rticuliers most complex standard known {cohomology Cech cohomology of groups, etc.), and play a role significant in geometry alge- brick , (especially in the "cohomology of e Weil " of the preschemas).

d. Example 1. - Let S ' item above S, C, and let I' a group of automorphisms dP. S ' such that S' is "fonnellement main on S, of group i. e. such as the natural morphism

L x S'- + S' x_S S'

OR r xS ' denotes the direct sum of r copies of S' ', ie an isomorphism.

(We suppose that in C the direct somrnes which intervene here exist). Let F be a contravariant functor of C in the category of abelian groups . So

c * {s' / S, F) is canonically isomorphic to grol.U) e aimElicial of cochaines **stan**
dard homogenees c * { **r,** F (S ' , , . one H (S '/ S, F) is canonically isomorphic to
< S , > > .

e. EXAMPLE 2 - Either _£ the category of pre-schemas. Is designated by Ga { "grou- p e addit i f, the e foncteu r contravarian t d e G dan s l has Categorí e of s group s abe links, **defined by**

$G(X) = H^0(X, OX>$

We define in the same way the functor G_m

$Gm(X)$

("group e m ult plica ti f ") by

$= H^0(X, 2x)^*$

{= group of invertible elements of the ring H^0{X, .Qx)), and more generally the functor Gl (n} (" linear group of order n ") by

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Gl (n) (X) = Gl (n , H^0(X , **_2x))**

which is a functor £. in the category of groups (not necessarily com- mutative if n > 1; for n = 1 include G_m).

One may also interpreted ter Gl (n) as a functor-automorphism (see (c)) by considering the category fibered *.! f "* of base . £ such that for **XE. £** , *fx* is the category of **fais-**
ceaux locally free on X: it is in effect Gl (n) (X) = Aut.1-i) • On after
(b) il follows that if **of.:** S • -, S is a morphism of fJescent (cf. **para.**
graph E 2 (c)) H^1(S• /S , Gl (n)) CONTAIN t ! 'ensembl e of s class s (a u n isomorphis-
I pres) of beams locally free on S whose image contrast on S ' is isomorphic to **_9,** , and this inclusion is an equality if and only if any given to descent on *2* , (relative to **o<**:S • - , S) is effective. When

that S is the spectrum of a local ring , this therefore means H^1 (S '/ S, Gl (n)) = (e) , since any locally free beam on S is then trivial.

Note the equivalence of the following conditions on a morphism **41<**:S '-. **S**

(i) . The homomorphism e of augmentatio n H^0 (S , O_{-S}) = G_a (S) - > M^0 (S '/ S , G_a) is an isomorphism

(Ii) • **S•---, S** is a morphism of V-descent (
of base . £ envisaged above).

v being the **fibered category**

If **S•** S is finished, these terms equivalent also has
(iii). S '-> S is a strict epimorphism (cf. paragraph 2. (c)).

Now suppose that S = Spec (A) , S ' = Spec (A') ; so we have

n + 1

Cn (S '/ S ,

$G_{a'a}$) = Cn (.A '/ A G) = @'A: A•

1 'operator coboundary Cn (A '/ A , *GJ-t* cn + l (A' / A , *G)* being sum Alternating d.es Op- tors faces

c \ (.x_0:

9 x_1 i...ixn) = x_0 9 •• • ix_{1-1} 9 lA , ix i 9 •. • 9 xn

Of same, Cn (S '/ S, Gm) = Cn (A' / A, Gm) identifies a (nit A ') •, the operations s1mpliciale s dan s c • (A' / A , Gm) etan t induced s pa r that s d e C • (S '/ S , Ga) •
Is explicit from even the operations simplicial in c • '/ **A** , Gl (n)) • tousles case has the knowledge of the speaker, it is Hi (A '/ A, G) = O : RQUr
a -
i > 0, and if **A** is local, we have **H** ^{1} **(A '/ A** , G) = 0 and more generally
H^1 (A '/ , Gl (n)) = (e) (when S'-, S is m a morphism of . - f-descent, ie the diagram **A** A' == *t* A ' iA A' is exact , compare with the paragraph 2 (c)),
We note that the "theorem 90 "of Hilbert is no other that the relationship

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H^1 (S '/ S, Gm) = 0 when A is a body and A' extension Galois fi denies of this latter (see example 1), and can still be expressed by saying that in the cases contemplated, S '--4S e t a morphism of descent strictly to the category fibered of beams locally free of rank 1 • it is under this last for- me he agrees to generalize the theorem of Hilbert, by varying the assumptions on both the morphism S '- tS on the beams almost Coherent environ- wise.

Note finally the equivalence of properties following, when **A** is a ring local artinian of ideal maximum m, an A- algebra (by **designating**, for any integer k > 0, by Ak (resp. **Ak)** the rings A / -cn .k + l (resp. A '/' 1'1 \ .k + l A '))

(i). H 1 (A / Ak , G) ::: O pourtout k.

(.ii). H 1 (A / Ak , Gm) = 0 for all k .

(iii). H 1 (Ak "'Ak , Gl (n)) · = (e) for all k and all n •

If S ' S is an epimorphism strict, while in previous terms IMPLIED Quent even that is a morphism of descent strictly for the modules free (of such finished or not) on A' .

NOTE. - The definition of the groups Hi (S '/ S, G) , in the case where S, S' m ? are the patterns of body A , A ' , is due **AMITSUR.** The group H- (s' / s , G m)

es t particula t attraction and t comm e variant e "gl o b a s e "d u group e d e Brauer, variant to which it may be refer to **[1],** chapter VII.

B. Descent by faithfully flat morphisms .

1. Sets out the theorems of descent.

DEFINITION 1.1. - A morphism **o'** S ' **->** S of pre-schemas is said flat if for all x ' ES' , **2x'** is a modulus flat on the ring **_Eo <' (x ')** (i. E. 0 , @M is an exact functor in the Oc <(x ') -module M) • A morphism -x o ((x ')

is said to be faithfully flat if it is flat and surjective.

For example, if S = Spec (A), S '= Spec (A') , then S ' is flat over S if and only if A' is a flat A-modulus , and S ' is faithfully flat over , **S** if and only if A ' is a faithfully flat A-module (ie the functor · A' AM in the A- module M is exact and faithful) ; this means also, in the terminology of SERRE [5], that the couple (A, A ') is flat. If S ' is faithfully flat over *S*, then the functor picture inverse of beams almost Coherent on S is true and faithful, as of autrestermes, for a result homomorphisms of beams

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Coherent almost on S is correct, it must and it is sufficient that the picture reciprocity on S ' on either n particular, to a homomorphism beams almost Coherent on S is a monomorphism, resp. an epimorphism, resp. isomor- a morphism, it is necessary and it is enough that its picture contrast on S ' on either). This pro- priete remains true if it is restraint over d 'a open any of S, and in this form characterizes the morphisms faithfully dishes.

DEFINITION 1.2. - A morphism **o** (: S'---- J S is said to be quasi-compact if **the** inverse **image** of any open quasi-compact part U of S is quasi-compact (ie finite union of affine openings).

Obviously, it suffices to verify this property for open affines of S. For example, an affine morphism (i, e. Such that the inverse image of an affine open is affine) is quasi-compact.

The class of flat morphisms , resp. faithfully flat, resp. Almost compact is stable, by composition and by "extension of the base," and contains well enten- of the isomorphisms.

T Om: ME 1. - Let **cl ...** • S'S be a pre-schema morphism , faithfully flat and quasi-compact. Then **o(** is a morphism of descent **strict** · (A, definition 1.7) for the category fibered **: J '** of beams quasi-coherent (A, paragraph 1, exem ple 2).

This utterance means two things :

(i) If F and **G** are two quasi-coherent sheaves on S' F ' their inverse images on S • ' then the natural homomorphism

Hom (], i:, L) Hom (F ' 'G ')

and G '

is a bijection of the first member of the subgroup of the second form of homo- morphisms F '- + G' which are compatible with the canonical descent data on these beams, i. e. which the image reverse by the two projections of S "= S ' **XS** S' on S ' give an even homomorphism F "--4 in •

(ii) All quasi-coherent beam F ' on S', provided with a given lowering rela.tivement the morphism **o (S'-4** S (A, definition 1.6) is isomorphic (provided for this given) has the inverse image of a quasi-coherent beam F on S.

Pheasant t F = 0 - S

in (i), we find

COROLIAIRE 1., - Let G a beam quasi-coherent on S , are G ' and G " its image inverted on S' and on S "= S ' x_8 S' , are p $_1$, p $_2$ the two pro- jections of S "on S ' , then the following diagram of applications of sets

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is exact (A, definition 1. (a)).

Furthermore, the combination of (i), (ii) of the definition 1.1 gives COROLIAIRE 2. - Either G as in the corollary 1. While it is correspondan-
this biunique between the sub-beams quasi-coherent of G, and the sub-beams quasi-coherent of G ' which the image inverse of S ".by the two projections p_1, p_2, give the same beam-sous of G ".

Of course, it has an utterance equivalent in terms of bundles quotients. As we have seen (A, paragraph 4 (e)), the theorem 1 is to be regarded as a ge- neralisatio n d u ''theorem e 9 0"d e Hilbert , e t Involved e comm e ca s particular s di
verse formulations in terms of 1-cohomology. For the demonstration, it is brought back easily to the case or S = Spec (A) S ' = Spec (A'), and (i) is brought back facilernent to prove the corollary 1, ie the accuracy of the chart

M = A AM ---> A ' ®AM A' tii! AA ' AM

for any A-module M, that which APPEARS of Lemma more general

LEMMA 1.1. - Let A 'be a faithfully flat A-algebra . Then for any A-modulus M, the M-augment complex C (A '/ A, Ga) AM (cf. A, paragraph 4, (e)) is a resolution of M.

It suffices to prove that the complex increases deduced from the preceding by extension
from the basis of A to A ' satisfies the same conclusions. This leads to verify the statement when we replace A by A ' and A' by A ' , A' , therefore brings us back
a case where there exists a homomorphism of A-algebras A '-> A (or, in geometric terms , in the case where S' on S has a section). In this case, it follows from the general points of A, paragraphs 4, (a). Note in passing the corollary sui- efore, which generalizes an utterance well known cohornologie Galois (compare A, paragraph 4(e))

COROLIARY. - If A '
Hi (A '/ A , G_a) = 0 for

is faithfully flat on A, we have H^0(A '/ A , G_a) = A and
i ? 1

To prove part (ii) of Theorem 1, it cornme METHOD for (i) by rame- ing the case or S ' on S admits a section, or it FOLLOWS of (i) (cf. A, pa- ragraphs 1 (vs)).
Can obviously vary ad libitum Theorem 1 and its corollaries intro- in duisant additonal various structures on the beams (or systems of beams) quasi-coherent envisaged. For example, the given on S of a beam

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Almost coheren t of algebra s commutative s ''equivau t i'l.1 was given e su r S ' u n such beam, fitted with a given of descent relative to <":S' --- t S. In light of the correspondence functorial between of such beams almost Coherent
on S, and the pre-schemas affine above of S, is obtained the second as- serting the theorem following
Theorem 2.- Let o(:S -) O as in Theorem 1. Then d.. is a morphism of descent {not strict in general) (A definition 2.4), and is a morphism descent strict for the category fibered diagrams: affine on top of pre-schemas, (A definition 1.?).
the first assertion of the theorem means this : let X, Y be two pre-schemes

above of S,

X', Y'

their inverse images on S '

and X ".Y" their ima-

inverse ges on
S " = S ' x_8 S' , then the following diagram of natural applications

r- "

Ho m_5 (X , Y) Ho m_8 , (X '

p

Y') -.!. + Hom (X ", Y ") P2
311

is , ie

o4 • is a bijection from Hom $_5$ (X , Y) on the part of

Hom $_8$,(X', Y')

crazy about homomorphisms which are compatible with the data of

canonical descent on X ', Y' (ie whose inverse images by the two projections of 5" on S'are equal). This follows easily from Theorem 1,
Corollary 1 when if terminal has Y affine on S; in the general case , it
must combine the theorem 1 with the results as follows:
LEMMA 1.2. - Let c(. :S'-+S be a faithfully flat and quasi-compact morphism .

Abrs S

identifies with a quotient topological space of

S '.ie . all e part

U of

S such that o (- 1 (u) is open, is open.

To complete the theorem 2, it must give the criteria of effectiveness for a given for descent on a S'-pre-schema X ' (in the case or X' does not SUP- poses refines on S '). Note first that such a given descent is ws necessarily effectiv , even if S is the spectrum of a body k, S ' the spectrum of an extension quadratic k ' of this latter, and S' a diagram Ugebrique own of dimension 2 on S ' (as we can see, from after GREENHOUSE, by using the surface of non- projective of NA.GATA). To that 'a given of descent

X ' / S' relatively to o(: S • S (faithfully , Elat and quasi-compact) actually , i the fau t e t i s enough t qu e X ' se t reunio n open s X ! , affi n e s of S ' , that i Be Free t 11 Stable s 1 by · s has given e d e descent e su r X ' • I l e n es t certainly ment as well (what that is X '/ S' and the given of descent on X ') if the morphis- me ac : S' --- + S is p-radical (ie injective, and has extensions residual that

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are radial). On This printer can show as it in is still well if ot. S '+ --- S is finished, and all finite subset of X', contained in a fiber of X ' on S, is contained in an open of X' affine on S (that is, the criterion of Weil). It in is in particular and, if X '/ S' is almost projective e t dan s c e case , o n bit t shows r that e l e pre-schem has ''descen du "x / S es t auss i quasi- projective (and projective if X '/ S' east) . In summary

THEOREM *3.* - Let c,t.: S ' S a pre-schema morphism, faithfully flat and quasi-compact. If oC is p-radical, that is a morphism of descent strict. Yes

d. is finished, it is a morphism of descent strict relativem.ent has the category fibered of pre-schemas almost projective (or projective) on the pre-schemas.

REMAIQUES. - I do not know if in the second statement above, the assumption that o { is a morphism finish is very necessary; we checked in all cases of Fagon "fonnelle" we can to replace with the hypothesis next, more weak in appearance : for all points of S is a neighborhood open U, a U finished and faithfully flat over U, and an S-morphism from U ' to S' • A typical case which does not fit into the previous one is that where S = Spec (A) S ' = Spec (A) where A is a local Noetherian ring and A is its complete; or even one or S ' is almost finished on S (ie locally isomorphic is an open one S-schema finished) and not finished. In these two cases, the lecturer ignores succeeded the response has the following question: let X be an S-diagram such that X ' = X S S' is projective on S ' , is it true that X is projective on S?

2. 4J> plication is the descent of certain properties of morphisms. - Either P a class of morphisms of pre-schemas. Or Ct(: S '----, S a morphism of pre- patterns, and either f : X - + Y a morphism of S-pre-schemas, f' : X '--- tY' 1 'Image reciprocal of f by o(• You may be asking then if the relationship '' f ' p "involves "f E P "·I1 appears that the answer is yes in many important cases when we assume that o1,.is faithfully flat and quasi-compact (the latter assumption being sometimes superabundant). This can be seen directly without difficulty if P is the class of surjective morphisms , resp. radials (these cases resulting from the surjectivity of oC.), resp. dishes, resp. fide- LEMENT dishes, resp. simple (these cases resulting from the faithful flatness of- o ( ) . resp of kind . finished Using the theorems 1, 2 and the lemma 1.2, we see also that in is of even if P is one of the classes following : isomorphism, Dumping open, Dumping closed immersions (if f is of the type finish and Y lo- cally noetherian), morphisms affine morphisms finished, morphisms almost finished, morphisms open morphisms fames, homeomorphisms, morphisms separated,

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proper morphisms . The only important unclear case is that of projective and quasi-projective morphisms , already pointed out in the remark of paragraph 1.

J. Descent i: nr finite morphisms faithfully flat. - Let o (S '...S un mor- morphism finished, corresponding to a algebras beam • on S which is locally free of such finished in so that beam of modules, and all 0; then o (is a faithfully flat and quasi-compact morphism , to which we can therefore apply

the results precedents. The given a beam quasi-coherent F ' on S ' equi- is has the given the beam quasi-coherent ow,. (F ') on S , provided of its struc- ture of A'-module (noting that A' = o((0,)). To simplify, this sheaf on S will also be denoted by F ' • The two inverse images p (F') of F ' on S ' X S S ' corresponden t d e mem e at x beam x virtually Coherent - d e (A ' & 0 A ') -Modules

. . . . S . . .

F ' Q A ' e t A ' i F ' . L is given e u n (S ' X S S ') -hom om o r p HISM e d u first

2s- - 2s

the second is equivalent to the given of a homomorphism of (A' & A') -module, and considering that • is locally free, this is equivalent to the given a homomorphism of ' i!') -modules :

U = Hom (A ' , _A ') = _A ' 0 A ' Hom
- 2 s - - -E
-

(F ' , F ')

S

i.e. is the given, to any section t of JI .... on an open V, a homomorphism morphism of .Es-modules T : F ' IV + F ---' I V, satisfying the requirements

(J.1)

or f, x are sections respectively of •, F ' on an open S contained naked in V. The conditions (i) and {ii) of a given descent (A, µ: i.ragraphe 1 (c)) can be written then respectively

(J.2) ie

(3.3)

In other tenns, a given for descent on F ' is equivalent to a representatiop

of, beam of

O S-algebra s U
...

= H o m O (A ' , A ') dan s l e faiscea u d e O -al g e b ras
-S - ,, .., - . .. . _

H om (F ' , F') , satisfactor t the s deu x proviso s d e linearity _ (.3 .1) . S i o n has a coupling of quasi-
- 2 s
coherent sheaves on S '

Fl x F2 --- + F3

(which can be interpreted as a coupling of bundles of A • -modules on S),

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e t of s given s d e recollemen t su r a s F ! , Defined s pa r of s homomorphism s T .
(i = 1 , 2 , .3) U-, Hom 0 (F! , F!) , then these data are comp.i.tibles with

coupling gives the obvious sense of the term, if and only if the condi- tion following is verified:

Pou r all e sectio n f d e .£ _ su r u n open , designan t pa r = E t i i ... , 1 the section of . !! QA ' 1L (£. Being considered examined as A i -module for its structure has left) defined by the formula

(or f and g are two sections of A ' on an open smaller) was the f9r- mule

(.3.4)

for every pair of sections x, y of A ' on a open more small. (We can express this property by saying that the "; homomorphisms T (i) are compatible with the diagonal application of . £., relative to the given coupling). In par- ticular, the formulas (3.1) to (.3.4) allow us to interpret in terms of representations of algebras has applications diagonals, the data of descent on a quasi-coherent beam of algebras on S ', therefore also (if restricting the algebras commutative) · the data of descent on a S'-schema refined.

Of the, one pass has an interpretation analogous to data of descent on a S'-pre-schema X ' one: the given one such X' is equivalent to the given a pre-schema X ' on S, provided with 'a homomorphism of 2 8 -algebras

! ' .2x •

and a given of descent on X ' ceaux (compatible with the morphism

is equivalent to the given a homomorphism of fais-
h: X'-tS '):

U -, Hom I (OX ' , OX')
- -h- (0) - - -S

satisfying the conditions analogous to conditions (3.1) a (J.4) above.

EXAMPLE 1 ('' Weil "). - Let S i / O a coating etale Galois of group Galois r {see A, paragraph 3, ot 4 (d)). Then a given for descent on

a beam almost coherent F ' on S' (resp. on a $ '- pre-schema X') equi- worth has the given a representation of r by automorphisms (S ', F') (resp. to { S ' , X')) compatible with the operations of r on S ' • This result

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es t ''f o: n n e l ''·ie . s e demontr e e n term s of th categories , May s d u poin t d e seen th of this nu.mere
CLEAR also of the structure explicit in .!! _, (fitted to its structure
ring , the homomorphism e ring x A ' U e t the applicatio n diagonal) , com-

fully known thanks to the following result :
on the left, a base formed by sections of U

r.

EXEMPLE 2 (''C A RI ' IE R''). Se t p u n number Os is of characteristic p), A Pc. Bone = A

!!, admits, as A'-module which correspond to the elements of

first , suppose s p O ... s = 0 (ie (ie S '/ S is p-radical of heights

1) and that the sheaf of algebras • on A locally admits a p- ,
ie a family (x.) of sections such as **A** ' or generates comma algebra by the **x**., subject to ux only conditions x =
−0. We assume the set of i finite, with cardinal n. Let be the sheaf of A-derivations of **A** ' ,

it is a bundle of A'-modules locally free of rank n, in addition it is a beam of p-algebras Lie on A (but not A
•) satisfying the condi- tion

(3.5) [X, fY] = X (f) Y + f [X, YJ

LEMMA . - U = Hom (A' , **A'**) are t generates , e n tan t that e _ ç -alge b r e mun i a
- -2 s - -
homomorphism of algebras i' .. !!, , by the sub--module has left , with the relationship additional:

(3.6)

xr - fX = X (f)
{ XY - YX = [X, Y]
xP = x (p)

It follows the precedent lemma given a descent on the quasi beam coherent F 'S' is equivalent to the given, for any X **Ed** ,, a 2 3 endo morphism X of F ' , satisfying the requirements

(3.7)
(3.8)
(3.9)
(3.10)

a = rx
X (fx) = X (f) x + fX (x)
[X, Y] = [x, Y]
= xP

(This is what we could call a linear connection on F ' , without curvature
and compatible with the p-th power). It explicitly to even the notion of given for descent on a S'-pre-schema X ' ; the relation (J.4) is replaced here by
the requirement that the X are the derivations of .2x • • Comma the morphiSine S • S
is p-radical, the theorem *3* ensures that any such given of descent is

effective, therefore defines a S-pre-schema X.
We note that we have not had to make a hypothesis of flatness, of non- singularity or of finitude any on F ' respectively. X ' •

4 Application to the criteria of rationality. - Let X be an S-pre-schema such that the direct image of *Ex* on S is *2s;* this property will remain true then
by any flat extension S'--4- S of the base S • If F is an inverted beam
sible (i.e. locally free of rank 1) on X, the automorphisms of F identifying with the invertible sections of *_2x,* correspond one- to- one to the
sections reversal of 2s. Is then s a section of X above of s; we call for Fagon imaged, section of F over to s,
a section of beam reversal s "" (F) on S. CLEAR of this that preceded that if F
(i = 1, 2) are two beams invertible on X, provided each with a section above of s, and if F_1 and F_2 are isomorphic, it is an isomorphism
and a single of F_1 F_2 consistent with the sections in issue (i. e, trans
forming the first into the second). Moreover, and independent of the section
s, agree to regard as equivalent two beams invertible F_1 and F_2 on X such that every point of S has an open
neighborhood U as the restriction tions of F_1 and F_2 is X \ U are isomorphic. So any invertible beam

F on X is equivalent to an invertible beam F $_1$ muhi of a section mar quee above of s (it takes F $_1$ = Fs "" (F) -l), and F $_1$ is determined in an iso morphism pres. In other words, the classification of the beams invertible on X up to an equivalence is the same as the classification at one isomorphism near the invertible beams provided with a marked section .

Examined as these properties remain true by extension platform of .. S • S of the base (by substituting the section s by its image reversed s' by cl-), we concluded account terlU of theorerne 1

The ; ere-schema X / S being as above and admitting a section s , that is

0 (:S '-,) S a faithfully flat morphism and 9.uasi-comE!: Ct ;is

}-¹

a do-

CWater invertible on X ' = X ^{x} .c.. S ' . So that F ' is equivalent to the inverse image

on X ' of a beam reversal F on X it is necessary and it is sufficient that its ima es

inverse s p (F ') e t p ;(F ') su r X ' $_x$ X ' -

X

XX $_s$ (S ' "S S') are equivalent.

If so, F is determined at an equivalence ; eres. (We will then say that F ' is rational over S).

Inspired by this principle in the case or o (: S '--- JS is examined as in Example 1 OR Example 2 of the number preceding, we find the criteria of rationality de Weil or de Cartier. (We note that its author is confined to cases where S and S '

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are the spectra of the body; a fortiori, S is then the spectrum of a ring lo cal, and the equivalence relation introduced above is none other than the isomorphic relation). In the first case, F ' is rational over S if and only if

its transfonne s by r are equivalent to Fl • To express the criterion of

rationality in the second case, we consider, in general, the diagonal morphism x •-X " = X ' xx X' of X'/X, the corresponding bundle of ideals I

su r X ' $_x$ X ' e t l e faiscea u *l (! _* ^{2}, that i is identified e a n n imag e reciprocal

1

X

1) X (faiscea u from s ideals X ' by ratio X). As

the restrictions of the F!' = p. (F') (i = 1 , 2) at the diagonal are isomorphic (because isomorphic at F'), i.e. F 1 F 2 = F " has a restriction in the diagonal that

is trivial, it follows that the restriction of F "

has an isomorphism fathers, by tih element well determined

a (X ", xf :.

S of

, / £) are t given,

H ^{1} (X " J / I ^{2}) - H ^{1} (X ' 11 ^{1})

Moreover, in this case, was {) i • / x. = N;, / S i S2x, and consequently, if n;,; s is locally liber of S (in the case of cpmme. Cartier), S defines a section of e R ^{1} f ' (EX') i'/ S S'(called e class e d e Atiyah-Cartie r d u beam

inversibl e F'su r X '/ S) den t 1'annulatio n es t necessair e e t sufficient e louse r that the pictures reciprocal of F ' by the two projections of

on X ' are equivalent (or defined by the diagonal morphism

J is the sheaf of ideaux on S " = S ' x_8 S' S'---1 S ' xc S'). Cett e annulatio n es t don e tri-

"'

vialement necessary for the inverse images of F ' on X " = X x_8 S "itself are equivalent, thus also for that F is equivalent to 1 picture

Conversely a beam reversal F on X. Moreover, the class of Atiyah Cartier can also be interpreted as the obstruction has existence, locally over

of S ' , a connection of F' relatively to derivations of X '/ X, one such

connection being of more deteninee when we know the derivations of F ' corre- sponding to extensions natural has X' of derivations of S '/ S. From this, and

of developments of the preceding issue, it is easily concluded that in the case of Example 2 of said, and when X / S admits a section, the cancellation of the class of Atiyah Cartier is also sufficient for that F ' is rational on S .

5. Application to the restriction of the scheme based on an abelian scheme. - Let S be a pre-schema. We call an abelian schema on S a simple and clean schema X

on S whose fibers in the point x ES are the patterns of variates abelien- nes on the x)• Suppose S noetherian and regular (ie its local rings regular), then one can show in using the theorem of connection of Murre [4] (d u moin s dan s l e ca s "equal to s Specifica e's. Or the e theorem e cit e es t ac- tuellement demonstrated) that any section rational of X wire !!: S is pa.rtout of finished (ie is a section) (that which generalizes a theorem vector of WEIL).

It follows, more generally than if any rational s-application of X '

X ' is a single scheme on S, then in X is pa.rtout defined, it in CLEAR

this, which generalizes a result of ! GUSA-I.ANG: S being noetherian and regular K designating its ring functions sound (composed live body), either X an abelian scheme above of K..; if X is isomorphic has a K-schema of the Fonne X XS Spec (K)' OR X is a scheme abelian on S, _alors XO is determined to a near single isomorphism .

Using the result of uniqueness precedent, we see that the issue of restrictions of the base in X is local on S (and by result, it suffices to know to make the restriction to Spec (0), with x E S) • We see in the same way is that if S '--- i S is a morphism easy to kind over, if K' is the ring of functions sound of S ' , and if X eK K' is of the form X x 8 , Spec (K ') , then X is provided with a given of descent canonical relatively a o(. In view from Theorem 3, we conclude

PROPOSAL 5.1. - Let S a pre-noetherian scheme and regular, irreducible, body of functions rational K, or K ' an extension finite of K, BRANCHED on S, S' the normalizes of S in K ' (which is therefore a coating etale of S), X a scheme abelian over K such that X&KK'is of the fome X xS' Spec (K) , or X is a scheme abelian projective over S ' • So X is of the form X xs Spec (K), or X is a scheme abelian projective on S.

REMA UES. - The lecturer is not known whether it can replace hyrothese that S • S is a surjective etale coating (for using the tbeoreme 3) by hypo- thesis that it is a morphism type of finish single and surjective (even if it Assumed that it is a spreading), or if the proposal remains valid without assuming X pro- jective over- S ' (if that is perhaps filled automatically).

6. Application to the criteria of local triviality and isotrivialite. - Let S be a pre-schema, G a "pre-schema in groups " above S, P a pre-schema on S on which "G operates " (right). We say that P is fomellement princi- pal homogeneous under G if the morphism well known

G XS P -.- + P XS p

deduced from the operations of G on P, is an isomor: phism. We assume dore- nbefore G flat on S (done faithfully flat over S), and we reserved for you the name from homogeneous principal fiber under G to a formally homogeneous principal fiber P which is faithfully flat and guasi-comul.ct on S. It is im. immediately that it is the same to say that we can find an extension faithfully flat and almost compact S • S of the base S, such that the forrnally homogeneous principal fiber p • = p x S ' under G' = G x S ' is trivial, ie isomorphic to G' (ie admits a section) ; we can take in particular S ' = P. Note also that if S is locally noetherian, then the assumption of faithful platitude on P is equivalent to the hypothe.se that P s = P X S Spe c (6 S) es t fidelemen t pla t on 5s for any s S (or 6 s designates the comolete of the ring local O), as it FOLLOWS the fact that O s is faithfully flat on 0-• Moreover, if P is of finite type

on S locally Noetherian all points s satisfying a condition above is buildable., therefore, if S is a "pre-schema of Jacobson" (for exam- ple a diagram of such finished on a body, or a ring of Jacobson more generally), it suffices to verify the condition in question for the firm points of S. This brings us back to the case where the basis is the spectrum of a complete local ring A. when
S = Spec (A)(A complete noetherian local ring) and that P is of finite type on
S, the faithful flatness of P / S is equivalent also has the existence of an S ' finished and flat over S such that P ' is trivial, and side plus G is simple over S, we can assume S' spreads over S. Consequently, side plus the residual field of A is algebraically closed ("geometric case "), P is faithfully flat over A if and on ONLY LEMENT if it is trivial. Done, if S is an algebraic pre-schema on a field algebraically closed, and G simple of finite type on S, we see that the condition of faithful flatness over S is equivalent to the condition of analytical triviality (SLF)
de SERRE {(6] p. 1-12).

We can introduce other types of more strong in terms of P, with the na- ture of a "triviality local". We will say in particular, that P is isotrivial (resp. Strictly isotrivial) if for all sf S , there exists an open neighborhood
U of s, and a morphism finished and faithfully flat (resp. A étale sur- jective) U • U as P ' = p $\times_S$ U' is trivial. (We are deviating from the terms of Serre [1J, which calls locally isotrivial this that we call strictly isotrivial). The strict isotrivialite is especially useful if G is sim- ple on S, but is a concept inadequate by the center in the other cases.

If G is affine on S, any fiber main homogeneous P in G is affine of Apre s l e paragraph e 2 , o u l a possibility , grac e a u theorem e 2 , d e "d e CSNDT r e

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of such fibers by the morphisms faithfully flat and quasi-compact. Consider, in particular, G = Gl (n) $_5$, defined by the condition that the functor of S-pre-schemas S ' Homs (S' , G) (a value in the category of groups) is identified with the func- teu r Gl (n) (S= Gl (n , H^0(S ' , 2 $_8$,)) d u A , paragraph e 4 , (e) . Utilisan t s e made
(i) that any homogeneous main fiber under G (resp. any locally free sheaf of rank n on S) becomes isomorphic to the "trivial" object G (resp.
by extension faithfully flat and quasi. compact suitable for S, (ii) that can be down objects of the type contemplated (fibers main homogeneous under G, resp. bundles locally free of rank n) f, lr of such morphisms , and en- end (iii) that the group of automorphisms of fiber trivial on a S '/ S is func- toriellement isomorphic to the group of automorphisms of the beam locally free of rank n trivial on S' , we concluded "formally "that "even returns "from
s e gives r su r S (o u su r u n 8 '/ S) u n fibr e Principa the homogen e d e group e G , or of it give a beam locally free of rank n. (From Fagon more precise,
o n a a e equivalenc e d e category s f i . O n e n concluded t e n particular

PROPOSAL 6.1. - All fiber main homogeneous group Gl (n) $_5$ is locally ment trivial.

By the arguments known, is in concludes the same results for the groups struc- Turaux such that Sl (n) S, Sp (n) $_5$ and the products of such groups. We in conclusion

also that, if F is a closed subgroup of G = Gl (n) $_u$

, flat on S, such that

the quotient G / F exists, and that G is a homogeneous main fiber isotrivial (resp. strictly isotrivial) on G / F , of group F $\times_8$ (G / F), then fiber main homogeneous of group F is isotrivial (resp. strictly isotrivial).
This applies a tousles "groups linear "on S that have been used up to present ' and in particular, u case OR G = S xk r' S being a pre-schema on the body k , and r a group linear in the classical sense , and in particular sim

ple over k This solves therefore, for of such groups, an issue of Serre (loc. cit.).

Note also that, for the majority of groups (linear or not) single on S known, and in any case those of the form S x k as above, we can MON trate that all fiber main homogeneous isotrivial is strictly isotrivial, this that i resou d e n particular , an e AUTR e questio n d e SERR E (loc . cit . 1-14), has held a fiber main homogeneous obtained by a descent has the CARTIER (cf. pa paragraph 3, example 2) is clearly isotrivial.

REMAIQUE. - One of the difficulties essential in these matters (misea hand the question of the existence of schemes quotients G / F) is the lack of criteria ct'effectivite for a given of descent by a morphism faithfully flat not finished.

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