EXPOSURE VI

FIBER CATEGORIES AND DESCENT

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0. Introduction

Contrary to what had been announced in the introduction to the previous talk, it turned out to be impossible to descend into the pre-schema category, even in special cases, without having previously developed with sufficient care the language of the descent into general categories.

The notion of “descent” provides the general framework for all the processes of “recollement "of objects, and consequently of" reattachment "of categories. The most classical reattachment is relative to the data of a topological space X and of a overlap of X by openings X_i ; if we give ourselves a fiber space for all i (say) E_i above X_i , and for any pair (i, j) an isomorphism f_{ji} of $E_i | X_{ij}$ on $E_j | X_{ij}$ (where we set $X_{ij} = X_i \cap X_j$), satisfying a transitivity condition well-known (which we write in an abbreviated way $f_{kj} f_{ji} = f_{ki}$), we know that there is a space bundle E over X, defined up to isomorphism by the condition that we have isomorphisms $f_i : E | X_i \xrightarrow{\sim} E_i$, satisfying the relations $f_{ji} = f_j f_i^{-1}$ (with the abuse of writing usual). Let X be the sum space of X_i , which is therefore a fiber space above X (ie provided with a continuous map X → X). We can interpret more concise the data of E_i as a fiber space E on X, and the data of f_{ji} as an isomorphism between the two inverse images (by the two canonical projections)

E^1 summer of E on $X = X \times_X X$, the bonding condition then being able to be written as an identity between isomorphisms of fiber spaces E $_1$ summer $_3$ about the product triple bundle $X = X \times_X X \times_X X$ (where E_i denote the inverse image of E on X by the canonical projection of index i). The construction of E, from E and f, is a typical case of a "descent" process. Moreover, starting from a fiber space E 146 on X, we say that X is "locally trivial", of fiber F, if there is an overlap open (X_i) of X such that the $E \mid X_i$ are isomorphic to $F \times X_i$, or what amounts to same, such that the inverse image E of E on $X = \coprod_i X_i$ is isomorphic to $X \times F$.

Thus, the notion of "reattachment" of objects like that of "localization" of a property, are related to the study of certain types of "base changes" $X \to X$. In Algebraic geometry, many other types of base change, and in particular the morphisms $X \to X$ faithfully flat, must be considered as corresponding to a process of "localization" in relation to preschemas, or other objects, "au-above "of X. This type of localization is used just as much as the simple localization topological (which is a special case for that matter). It is the same (in a lesser extent) in Analytical Geometry.

Most of the proofs, being reduced to verifications, are omitted or simply sketched; if necessary we specify the less obvious diagrams which are introduced into a demonstration.

1. Universe, categories, equivalence of categories

To avoid certain logical difficulties, we will admit here the notion of Universe, which is a set "big enough" so that we do not leave it by operations practices of set theory; an "axiom of the Universes" guarantees that all object is in a Universe. For details, see a book in preparation by C. Chevalley and the lecturer (1) . Thus, the acronym **Ens** designates, not the category of all sets (notion which has no meaning), but the category of sets which are found in a given Universe (which we will not specify here in the notation). Of

even, **Cat will** designate the category of categories found in the Universe in question.
tion, the "morphisms" of an object X from **Cat** into another Y, being by definition the functors from X to Y.

147 If C is a category, we denote by Ob (C) the set of objects of C, by Fl (C) the set of arrows of C (or morphisms of C). So we will write X ∈ Ob (C) by avoiding the current abuse of notation X ∈ C. If C and C are two categories, a functor from C to C will always be what is commonly called a covariant functor from C to C; its data implies that of the category of arrival and the starting category, C and C. The functors from C to C form a set, denoted Hom (C, C), which is the set of objects of a category denoted **Hom** (C, C). By definition, a contravariant functor from C to C is a functor of the category opposite C ° from C to C.

We will admit the notion of projective limit and inductive limit of a functor F: I → C, and in particular the most common special cases of this notion: Cartesian products and fiber products, dual notions of direct sums and sums amalgamated, and the usual formal properties of these operations.

(1) The final authors are C. Chevalley and P. Gabriel. The book is due out in the year 3000. In the meantime, cf. also SGA 4 I.

For example, in the category **Cat** introduced above, the projective limits (relating to categories I found in the chosen Universe) exist; all of objects (resp. the set of arrows) of the projective limit category C of C_i, is ob-holds by taking the projective limit of the sets of objects (resp. of the sets of arrows) of categories C_i. The best-known case is that of the product of a family of categories. We will constantly use the fiber product of two categories thereafter. gories on a third.

For everything related to categories and functors, while waiting for the book in preparation already indicated, see [**1**] (which is necessarily very incomplete, even in this which concerns the generalities outlined in this issue).

Let us take this opportunity to explain the notion of equivalence of categories, which is not satisfactorily exposed in [**1**]. A functor F: C → C is said faithful (resp. fully faithful) if for any pair of objects S, T of C, the map $u \mapsto \to F(u)$ from Hom (S, T) to Hom (F (S), F (T)) is injective (resp. bijective). One says that F is an equivalence of categories if F is fully faithful, and if moreover all 148 object S of C is isomorphic to an object of the form F (S). We show that it comes back to even to say that there exists a functor G from C in C quasi-inverse of F, ie, such that GF is isomorphic to id_C. When this is the case, the data of a functor G: C → C and of an isomorphism ϕ: GF → id_C is equivalent to the data, for all $S \in Ob(C)$, of a pair (S, u) formed by an object S of C and an isomorphism u: F (S) → S, let (G (S), ϕ (S)). (With these notations, there exists a unique functor C → C having the given application $S \mapsto \to G(S)$ as application-objects, and such as the application $S \mapsto \to \phi(S)$ ie a homomorphism of functors FG → $id_{C'}$). Finally, if G is a quasi-inverse functor of F, and if we choose isomorphisms ϕ: FG $\xrightarrow{\sim} id_{C'}$ and ψ: GF $\xrightarrow{\sim} id_C$, then the two compatibility conditions on ϕ, ψ stated in [**1** , I.1.2] are in fact equivalent to each other, and for any isomorphism ϕ chosen, there exists a unique isomorphism ψ such that said conditions are satisfied.

2. Categories on another

Let E be a category in **Univ** , so it is an object of **Cat** , and we can consider the **Cat** $_{/E}$ category of " **Cat** objects above E". An object of this category is therefore a functor

$$p: F \dashrightarrow E$$

We also say that the category F, endowed with such a functor, is a category above of E, or an E -category. We will therefore call E-breaker of a category F on E in a category G on E, a functor

$$f: F \dashrightarrow G$$

such as

$$qf = p$$

where p and q are the projection functors for F resp. G. All E-breakers f from F to G is therefore in one-to-one correspondence with the set of arrows

149 of origin F and end G in $\mathbf{Cat}_{/E}$, without however having an identity (since the data of an f as above does not determine F and G as categories on E); but of course, as in any other $C_{/S}$ category, we will do commonly the abuse of language consisting in identifying E-breakers (in the explicit sense above) to arrows in a $\mathbf{Cat}_{/E}$ category .

We will denote by

$$\mathrm{Hom}_E(F, G)$$

the set of E-breakers from F to G. Of course, a compound of E-breakers is an E-breaker (the composition in question corresponding by definition to the position of arrows in $\mathbf{Cat}_{/E}$).

Now consider two E-breakers

$$f, g: F \dashrightarrow G$$

and a homomorphism of functors:

$$u: f \dashrightarrow g$$

We say that u is an E -homomorphism or a "homomorphism of E -functors", if for all $\xi \in \mathrm{Ob}(F)$, we have

$$q(u(\xi)) = \mathrm{id}_{p(\xi)},$$

in words: posing $S = p(\xi) = qf(\xi) = qg(\xi) \in \mathrm{Ob}(E)$, the morphism

$$u(\xi): f(\xi) \dashrightarrow g(\xi)$$

in G is an id_S -morphism. (In general, for any morphism $\alpha: T \to S$ in E, and any category G above E, a morphism v in G is called an α-morphism if $q(v) = \alpha$, q denoting the projection functor $G \to E$). If we have a third E -functionor $h: F \to G$ and an E -homomorphism $v: g \to h$, then seen

150 is also an E -homomorphism. Thus, the E-breakers from F to G, and the E - homomorphisms of such, form a subcategory of the category $\mathbf{Hom}(F, G)$ of all the functors from F to G, which we will call the category of E-functors of F in G and that we will denote

$$\mathbf{Hom}_{E/-}(F, G)$$

It is also the kernel subcategory of the couple of functors

$$R, S: \mathbf{Hom}(F, G) \quad //// \quad \mathbf{Hom}(F, E),$$

where R is the constant functor defined by the object p of $\mathbf{Hom}(F, E)$, and where S is the functor $f \mapsto q \circ f$ defined by $q: G \to E$.

To finish these generalities, it remains to define the natural couplings between the $\mathbf{Hom}_{E/-}(F, G)$ categories by composition of E-breakers. In other words, we wants to define a "composition functor":

(i) $\mathbf{Hom}_{E/-}(F, G) \times \mathbf{Hom}_{E/-}(G, H) \longrightarrow \mathbf{Hom}_{E/-}(F, H)$

when F, G, H are three categories on E, such that this functor induces, for objects, the application of composition $(f, g) \mapsto gf$ of E-breakers $f: F \to G$ and $g: G \to H$. For this, remember that we define a canonical functor:

(ii) $\mathbf{Hom}(F, G) \times \mathbf{Hom}(G, H) \longrightarrow \mathbf{Hom}(F, H)$

which, for objects, is none other than the application of composition $(f, g) \mapsto gf$ of functors, and which transforms an arrow (u, v), where

$$u: f \longrightarrow f, \quad v: g \longrightarrow g$$

are arrows in **Hom** (F, G) resp. in **Hom** (G, H), in the arrow

$$v * u: gf \longrightarrow gf$$

defined by the relation: 151

$$v * u(\xi) = v(f(\xi))\, g(u(\xi)) = g(u(\xi))\, v(f(\xi))$$

It is well known that we thus obtain a homomorphism from gf to gf, and that (for f, g and u, v variables) we thus obtain a functor (ii), ie we have

(I) $\mathrm{id}_g * \mathrm{id}_f = \mathrm{id}_{gf}$,

(II) $(v * u) \circ (v * u) = (v \circ v) * (u \circ u)$

Recall also that we have an associativity formula for the canonical couplings iques (ii), which is expressed on the one hand by the associativity $(hg) f = h (gf)$ of the composition of functors, and on the other hand by the formula

(III) $(w * v) * u = w * (v * u)$

for the composition products of homomorphisms of functors (where: $f \to f$ and $v: g \to g$ are as above, and where we further assume given a homomorphism $w: h \to h$ of functors $h, h: H \to K$). I say now that when F, G are E -categories, the canonical composition functor (ii) induces a functor (i). As we already know that the compound of two E-breakers is an E-breaker, that is to say that when $u: f \to f$ and $v: g \to g$ are homomorphisms of E-breakers, then $v * u: gf \to gf$ is also a homomorphism of E-breakers. This results in trivially effect of definitions. As the couplings (i) are induced by the couplings (ii), they satisfy the same property of associativity, also expressed in the formulas $(hg) f = h (gf)$ and $(w * v) * u = w * (v * u)$ for E-breakers and E -homomorphisms of E-breakers.

To complete form (I), (II), (III), let us also recall the formulas:

(IV) $v * \mathrm{id}_F = v$ and $\mathrm{id}_G * u = u$,

where for simplicity we write $v * f$ or $u * g$ instead of $v * u$, when u resp. v is 152

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the identical automorphism of f resp. g.

From the definition of couplings (i) results that $\mathbf{Hom}_{E/-}(F, G)$ is a functor in F, G, of product category $\mathbf{Cat}_{/E}$ $^{\circ} \times \mathbf{Cat}_{/E}$ in the category $\mathbf{Cat}$. If indeed $g: G \to G_1$ is an E-breaker, ie an object of $\mathbf{Hom}_{E/-}(G, G_1)$, then doing in (i) $H = G_1$, it corresponds to a functor

$$g_* : \mathbf{Hom}_{E/-}(F, G) \longrightarrow \mathbf{Hom}_{E/-}(F, G_1)$$

We define in an analogous way, for an E-breaker $f: F_1 \to F$, a functor

$$f^* : \mathbf{Hom}_{E/-}(F, G) \longrightarrow \mathbf{Hom}_{E/-}(F_1, G)$$

For short, we also denote these functors by the acronyms $f \mapsto g \circ f$ resp. $g \mapsto g \circ f$ (which designate only, in fact, the corresponding mappings on the sets objects). It results from the property of associativity indicated above that from this way, we get as announced a $\mathbf{Cat}$ functor $^{\circ}_{/E} \times \mathbf{Cat}_{/E} \to \mathbf{Cat}$.

3. Base change in categories on E

As in $\mathbf{Cat}$ the projective limits (relatively to categories I elements de $\mathbf{Univ}$) exist, it is the same in $\mathbf{Cat}_{/E}$, in particular Cartesian products exist there, which are interpreted as fibered products in $\mathbf{Cat}$. In accordance with general notations, if F and G are categories above E, we denote by

$$F \times_E G$$

153 their product in $\mathbf{Cat}_{/E}$, i.e. their fiber product above E in $\mathbf{Cat}$, as category above E. Thus, $F \times_E G$ is provided with two E-breakers pr_1 and pr_2 , who define, for any category H above E, a bijection

$$\mathrm{Hom}_E(H, F \times_E G) \xrightarrow{\sim} \mathrm{Hom}_E(H, F) \times \mathrm{Hom}_E(H, G).$$

This bijection also comes from an isomorphism of categories

$$\mathbf{Hom}_{E/-}(H, F \times_E G) \xrightarrow{\sim} \mathbf{Hom}_{E/-}(H, F) \times \mathbf{Hom}_{E/-}(H, G)$$

by taking the sets of objects of the two members, where the functor written is the one that has as components the functors $h \mapsto pr_1 \circ h$ and $h \mapsto pr_2 \circ h$ of the first member in the two factors of the second. We leave it to the reader to verify that we obtain well thus an isomorphism (the analogous fact being true, more generally, each once we have a projective limit of categories - and not only in the case of a fiber product).

Remember also that we (as stated in the N o 1):

$$\mathrm{Ob}(F \times_E G) = \mathrm{Ob}(F) \times_{\mathrm{Ob}(E)} \mathrm{Ob}(G)$$

$$Fl(F \times_E G) = Fl(F) \times_{Fl(E)} Fl(G),$$

the composition of the arrows being made component by component.

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In the following, we consider a functor

$$\lambda: E \dashrightarrow E$$

and for any category F above E, we consider $F \times_E E$ as a category above E thanks to pr_2 ; in other words, we interpret the operation "product fiber "as an operation ″rebasing ″, the functor $\lambda: E \to E$ taking the name of ″base change functor ″. In accordance with general facts well known, we thus obtain a functor, called base change functor for λ: 154

$$\lambda^* : \mathbf{Cat}_{/E} \dashrightarrow \mathbf{Cat}_{/E'},$$

(adjunct of the "base restriction" functor which, has any category F above of E, with projection functor p, associates F, considered as a category above of E by the functor $p = \lambda p$). As is well known in the general case of a base change functor in a category, the base change functor "Commutes to projective limits", and in particular "transforms" fiber products over E into fiber products over E.

Let F and G be two categories above E, we will define an isomorphism canonical:

$$\text{(i)} \quad \mathbf{Hom}_{E'/-}(F, G) \xrightarrow{\sim} \mathbf{Hom}_{E/-}(F \times_E E, G)$$

where $F = F \times_E E$, $G = G \times_E E$.

For this, consider the functor

$$pr_1 : G = G \times_E E \dashrightarrow G,$$

and define (i) by

$$F \longmapsto pr_1 \circ F,$$

which a priori designates a functor

(ii) **Hom** (F, G) - → **Hom** (F, G)

It is therefore necessary to check only that the latter induces a functor for the sub-categories (i), and that the latter is an isomorphism. That (ii) induce a bijection

$$\mathrm{Hom}_{E'/-}(F, G) \xrightarrow{\sim} \mathrm{Hom}_{E/-}(F \times_E E, G)$$

is the characteristic property of the base change functor. It therefore remains to prove 155 that if F, G are E-breakers F → G, then the application

$$u \longmapsto \mathrm{pr}_1 \circ u$$

induces a bijection

$$\mathrm{Hom}_{E'}(F, G) \xrightarrow{\sim} \mathrm{Hom}_E(\mathrm{pr}_1 \circ F, \mathrm{pr}_1 \circ G).$$

The verification of this fact is immediate, and left to the reader.

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It follows from this isomorphism (i), and from the end of the previous number, that

$$\mathbf{Hom}_{E'/-}(F \times_E E, G \times_E E)$$

can be considered as a functor in E, F, G, of the category $\mathbf{Cat}^{\circ}_{/E} \times \mathbf{Cat}^{\circ}_{/E} \times \mathbf{Cat}_{/E}$ in the category **Cat** , isomorphic to the functor defined by the expression

Hom $_{E/-}$ (F × $_E$ E, G). In particular, for F, G fixed, we obtain a functor in E, E → **Hom** $_{E'/-}$ (F, G) = **Hom** $_{E'/-}$ (F × $_E$ E, G × $_E$ E), and in particular the E-projection functor λ: E → E defines a morphism i.e. a functor

$$\lambda^*_{F,G} : \mathbf{Hom}_{E/-}(F, G) \longrightarrow \mathbf{Hom}_{E'/-}(F, G)$$

that we are going to explain. For the sets of objects of the two members, it is the application

$$f \longmapsto f = f \times_E E$$

which expresses the functorial dependence of F × $_E$ E of the object F on E. On the other hand, consider two E-breakers

$$f, g: F \longrightarrow G$$

and a homomorphism of E-breakers

$$u: f \longrightarrow g,$$

156 we will explain the corresponding homomorphism of E -functors:

$$u: f \dashrightarrow g.$$

For everything

$$\xi = (\xi, S) \in \mathrm{Ob}(F)$$

with

$$\xi \in \mathrm{Ob}(F), S \in \mathrm{Ob}(E), p(\xi) = \lambda(S) = S$$

morphism

$$u(\xi): f(\xi) = (f(\xi), S) \dashrightarrow g(\xi) = (g(\xi), S) \text{ in } G$$

is defined by the formula

$$u(\xi) = (u(\xi), \mathrm{id}_{S'})$$

(which is indeed an S -morphism in G, because $q(u(\xi)) = \lambda(\mathrm{id}_{S'}) = \mathrm{id}_S$).

Now consider any E-breaker

$$\lambda: E \dashrightarrow E$$

and the corresponding functor

$$\mathbf{Hom}_{E'/}(F \times_E E, G \times_E E) \dashrightarrow \mathbf{Hom}_{E''/}(F \times_E E, G \times_E E),$$

I say that this functor is none other than the functor that we obtain by the previous process, starting from F and G on E and considering E as a category on E, taking into account the isomorphisms of ″base change transitivity ″:

$$F \times_{E'} E \xrightarrow{\sim} F = F \times_E E \quad \text{and } G \times_{E'} E \xrightarrow{\sim} G = G \times_E E,$$

which imply a canonical isomorphism

$$\mathbf{Hom}_{E'/}(F \times_{E'} E, G \times_{E'} E) \xrightarrow{\sim} \mathbf{Hom}_{E'/}(F \times_E E, G \times_E E)$$

The verification of this compatibility is immediate, and left to the reader. 157

The functors we have just defined are compatible with the defined couplings to the previous number, precisely, if F, G, H are categories above E and if we ask

$$F = F \times_E E, G = G \times_E E, H = H \times_E E,$$

we have commutativity in the following functor diagram:

$$\begin{array}{ccc}
\mathbf{Hom}_{E/-}(F, G) \times \mathbf{Hom}_{E/-}(G, H) & \longrightarrow & \mathbf{Hom}_{E/-}(F, H) \\
\downarrow \lambda^*_{F,G} \times \lambda^*_{G,H} & & \downarrow \lambda^*_{F,H} \\
\mathbf{Hom}_{E'/-}(F, G) \times \mathbf{Hom}_{E'/-}(G, H) & \longrightarrow & \mathbf{Hom}_{E'/-}(F, H)
\end{array}$$

where the horizontal arrows are the composition-functors defined in the previous number. This commutativity is expressed by the formulas

$$(gf) = gf$$

for $f \in \mathrm{Hom}_E(F, G)$, $g \in \mathrm{Hom}_E(G, H)$, (formula which simply expresses the function toriality of base change), and the formula

$$(v * u) = v * u$$

when $u: f \to f_1$ is an arrow of $\mathbf{Hom}_{E/-}(F, G)$ and $v: g \to g_1$ an arrow of $\mathbf{Hom}_{E/-}(G, H)$. The verification of this formula results easily from the definitions tions.

In the following, we will be particularly interested in $\mathbf{Hom}_E(F, G)$ (and certain sub-remarkable categories of it) when $F = E$, and for this reason we introduce a special notation:

$$\mathbf{\Gamma}(G / E) = \mathbf{Hom}_E(E, G),\ \Gamma(G / E) = \mathrm{Ob}(\mathbf{\Gamma}(G / E)) = \mathrm{Hom}_E(E, G).$$

Remarks . - When E is a point category, ie Ob (E) and Fl (E) reduced to 158 only one element, which also means that E is a final object of the **Cat** category , then the data of a category on E is equivalent to the data of a simple category, (because there will be a unique functor from F to E). More specifically, $\mathbf{Cat}_{/E}$ is then isomorphic to **Cat** . Moreover, the categories $\mathbf{Hom}_{E/-}(F, G)$ are not then other than the **Hom** (F, G). Recall that the fundamental formula

$$\mathrm{Hom}(H, \mathbf{Hom}(F, G)) \xrightarrow{\sim} \mathrm{Hom}(F \times H, G)$$

(functorial isomorphism in the three arguments which appear there), allows to interpret **Hom** (F, G) axiomatically, in terms internal to the category **Cat** , so that the

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known form of categories **Hom** appears as a particular case of a valid in categories such as **Cat** , where " **Hom** objects " (defined by previous formula) exist. There is an analogous interpretation of **Hom** $_{E/-}$ (F, G) when we assume again that E is arbitrary, by the formula

$$\mathrm{Hom}(H, \mathbf{Hom}_{E/-}(F, G)) \xrightarrow{\sim} \mathrm{Hom}_E(F \times H, G)$$

(functorial isomorphism in the three arguments). In this way, the properties formalities exposed in N$_o$ 2, 3 are particular cases of more general results, valid in the categories where the objects **Hom** $_{E/-}$ (F, G) (when F, G are two category objects above a third E) exist.

4. Fiber categories; equivalence of E **-categories**

Let F be a category over E, and let S ∈ Ob (E). We call category-fiber of F
159 in S the subcategory F$_S$ of F reciprocal image of the point subcategory of E defined by S. Therefore the objects of F$_S$ are the objects ξ of F such that p (ξ) = S, its morphisms are the morphisms u of F such that p (u) = id$_S$, ie the S-morphisms in F. Of course, F$_S$ is canonically isomorphic to the fibered product $F \times_E \{S\}$, where {S} denotes the point subcategory of E defined by S, endowed with its functor inclusion in E. This results (taking into account the transitivity of the change of base) that if we make the change of base λ: E → E, then for all S ∈ Ob (E), the projection $pr_1 : F = F \times_E E \to F$ induces an isomorphism

$$F_{S'} \longrightarrow F_S \qquad \text{(where } S = \lambda(S)\text{)}.$$

Proposal 4.1 . - Let f: F → G be an E-breaker. If f is fully faithful, then for any change of basis E → E, the corresponding functor $f: F = F \times_E E \to G = G \times_E E$ is fully faithful.

The verification is immediate; more generally, we can show that any limit projective of fully faithful functors (here, f and identical functors in E, E) is a fully faithful functor.

Note that the assertion analogous to 4.1, where "fully faithful" is replaced by "Equivalence of categories", is false, already for G = E. However:

Proposal 4.2 . - Let f: F → G be an E-breaker. The following conditions are equivalent:

(i) There exists an E-breaker g: G → F and E -isomorphisms

$$gf \xrightarrow{\sim} \mathrm{id}_F, \quad fg \xrightarrow{\sim} \mathrm{id}_G .$$

(ii) Pout any category E on E, the functor

$$f = f \times_E E : F = F \times_E E \longrightarrow G = G \times_E E$$

is an equivalence of categories.

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(iii) f is an equivalence of categories, and for all $S \in Ob(E)$, the functor $f_S : F_S \to G_S$ induced by f is an equivalence of categories.

(iii bis) f is fully faithful, and for all $S \in Ob(E)$ and all $\eta \in Ob(G_S)$, it exists a $\xi \in Ob(F_S)$ and an S-isomorphism $u: f(\xi) \to \eta$.

Demonstration. - Obviously (i) implies that f is an equivalence of categories (notion which is defined by the same condition, but where the isomorphisms of functors are not required to be E -morphisms). On the other hand, it results from the rialities of the previous issue that condition (i) is kept after changing base $E \to E$. It follows that (i) ⇒ (ii). Obviously (ii) ⇒ (iii), because it suffices to do $E = E$ and $E = \{S\}$. It is even more trivial that (iii) ⇒ (iii bis), remains to prove that (iii bis) ⇒ (i). For this, let us choose for all $\eta \in Ob(G)$ a $g(\eta) \in Ob(F)$ and a isomorphism $u(\eta): f(g(\eta)) \to \eta$ which is such that $q(u(\eta)) = id_S$, where $S = q(\eta)$. This is possible thanks to the second condition (iii bis). As it is known and immediate, the fact that f is fully faithful implies that g can uniquely be considered as a functor from G to F, so that the $u(\eta)$ define a homomorphism (hence an isomorphism) functorial $u: fg \xrightarrow{\sim} id_G$. Moreover, by construction g is a E-breaker and u an E -homomorphism. The previous data then corresponds to a functorial isomorphism $v: gf \to id_F$, defined by the condition that $f * v = u * f$, and we immediately notice that it is also an E -morphism, cqfd.

Definition 4.3 . - If the previous conditions are true, we say that f is a equivalence of categories on E, or an E -equivalence.

Corollary 4.4 . - Suppose that the projection functor $p: F \to E$ is a functor transportable, ie that for any isomorphism $\alpha: T \to S$ in E and any object ξ in F_T, there exists an isomorphism u in F of source ξ such that $p(u) = \alpha$. So all E-breaker $f: F \to G$ which is an equivalence of categories, is an E -equivalence.

Results from criterion (iii bis). 161

Corollary 4.5 . - Let $f: F \to G$ be an E -equivalence. So for any category H on E, the corresponding functors:

$$\mathbf{Hom}_{E/-}(G, H) \longrightarrow \mathbf{Hom}_{E/-}(F, H)$$
$$\mathbf{Hom}_{E/-}(H, F) \longrightarrow \mathbf{Hom}_{E/-}(H, G)$$

(cf. N^o 2) are category equivalences.

This follows from criterion (i) by the usual reasoning.

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5. Cartesian morphisms, inverse images, Cartesian functors

Let F be a category on E, with functor-projection p.

Definition 5.1 . - Consider a morphism

$$\alpha : \eta \dashrightarrow \xi$$

in F, and let

$$S = p(\xi),\ T = p(\eta),\ f = p(\alpha).$$

We say that α is a Cartesian morphism if for all η ∈ Ob (F_T) and all f-morphism u: η → ξ, there exists a unique T-morphism u: η → η such that u = α ∘ u.

This therefore means that for all η ∈ Ob (F_T), the map v ↦ → α ∘ v:

(i) $$\mathrm{Hom}_T(\eta, \eta) \dashrightarrow \mathrm{Hom}_f(\eta, \xi)$$

is bijective. This also means that the pair (η, α) represents the functor in η F°_T → **Set** of the second member. If for a morphism f: T → S in E given,
162 and a given ξ ∈ Ob (F_S), there exists such a pair (η, α), ie a Cartesian morphism α in F of goal ξ, such that p (α) = f, then η is determined in F_T has isomorphism unique close. We then say that the inverse image of ξ by f exists, and an object η of F_T endowed with a Cartesian f-morphism α: η → ξ is called an inverse image of ξ by f. Often it is assumed that such an inverse image is chosen each time it exists (F being fixed); we will then denote the inverse image by symbols such as $f^*_F(\xi)$, or simplement $f^*(\xi)$ or $\xi \times_S T$ when these notations do not lead to confusion; the mor-canonical phism α: η → ξ will then be denoted, in what follows, by $\alpha_f(\xi)$. If for everything ξ ∈ Ob (F_S), the inverse image of ξ by f exists, we will also say that the image functor inverse by f in F exists, and $f^*(\xi)$ then becomes a covariant functor in ξ, of F_S in F_T . This comes from the fact that the second member in (i) depends in a way covariant of ξ, ie precisely denotes a functor of $F^{\circ}_T \times F_S$ in **Ens** . This functorial dependence for $f^*(\xi)$ can be explained as follows: consider f-morphisms Cartesians

$$\alpha : \eta \dashrightarrow \xi,\ \alpha : \eta \dashrightarrow \xi$$

and an S-morphism λ: ξ → ξ, then there exists a T-morphism and only one μ: η → η such that we have

$$\alpha\ \mu = \lambda\alpha$$

(as it follows from the fact that α is Cartesian).

Note also the following immediate fact: consider a commutative diagram

$$\begin{tikzcd} \xi & \eta \arrow[l, "\alpha"'] \\ \xi \arrow[u, "\lambda"] & \eta \arrow[l, "\alpha"'] \arrow[u, "\mu"'] \end{tikzcd}$$

163 in F, where α and α are f-morphisms, and λ an S-isomorphism, μ a T-isomorphism. For α to be Cartesian, it is necessary and sufficient that α is.

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Definition 5.2 . - An E-breaker F: F → G is called a Cartesian functor if it transforms Cartesian morphisms into Cartesian morphisms. We denote by **Hom** $_{\text{cart}}$ (F, G) the full subcategory of **Hom** $_{E/-}$ (F, G) formed by functors Cartesians.

For example, considering E as a category over E thanks to the identical functor, every morphism from E is Cartesian, therefore a Cartesian functor from E to F is a section functor F: E → F which transforms any morphism of E into a morphism Cartesian; such a functor is called a Cartesian section from F over E.

Proposition 5.3 . - (i) A functor F: F → G which is an E -equivalence, is a Cartesian functor. (ii) Let F, G be two isomorphic E-functors F → G. If one is Cartesian, the other is. (iii) The compound of two Cartesian functors F → G and G → H is a Cartesian functor.

The assertion (iii) is trivial on the definition, (ii) results from the previous remark-dant 5.2, (i) follows easily from definition and criterion 4.2 (iii); more precisely, a morphism α in F is Cartesian if and only if F (α) is.

Corollary 5.4 . - Let F: F → G be an E -equivalence. So for any category H on E, the corresponding functors G ↦ → G∘F and G ↦ → F∘G induce equivalences of categories:

$$\mathbf{Hom}_{\text{cart}}(G, H) \xrightarrow{\approx} \mathbf{Hom}_{\text{cart}}(F, H)$$

$$\mathbf{Hom}_{\text{cart}}(M, F) \xrightarrow{\approx} \mathbf{Hom}_{\text{cart}}(H, G)$$

This is deduced in the usual way from 4.2 criterion (i) and 5.3 (i) (ii) (iii). We can 164
specify that the E-breaker G: G → H is Cartesian if and only if G ∘ F is, and similarly an E-breaker G: H → F is Cartesian if and only if F ∘ G is.

It follows from 5.4 (iii) that if we consider the subcategory $\mathbf{Cat}_{\mathrm{cart}\,/E}$ of $\mathbf{Cat}_{/E}$ whose objects are the same as those of $\mathbf{Cat}_{/E}$, and whose morphisms are the functors Cartesian so we like N o 2 couplings:

$$\mathbf{Hom}_{\mathrm{cart}}(F, G) \times \mathbf{Hom}_{\mathrm{cart}}(G, H) \dashrightarrow \mathbf{Hom}_{\mathrm{cart}}(F, H)$$

induced by those of N o 2, allowing to consider $\mathbf{Hom}_{\mathrm{cart}}(F, G)$ as a functor in F, G, of the category $(\mathbf{Cat}_{\mathrm{cart}/E})^{\circ} \times \mathbf{Cat}_{\mathrm{cart}/E}$ in $\mathbf{Cat}$. We will need this remark especially for the case where F = G:

Definition 5.5 . - Let F be a category over E. We denote by

$$\varprojlim F / E$$

the category of Cartesian E-breakers E → F, i.e. Cartesian sections of F safe .

From what we just said, $\varprojlim F / E$ is a functor in F, of the category $\mathbf{Cat}_{\mathrm{cart}/E}$ in the category $\mathbf{Cat}$.

We will see below the relations between this Lim operation $\varprojlim$ and the notion of limit projective categories, as well as numerous examples.

6. Fiber categories and pre-fiber categories. Products and change of base in them

Definition 6.1 . - A category F on E is called a fibered category (and we say while the functor F → E is fibrant) if it satisfies the following two axioms:

165 - Fib I For any morphism f: T → S in E, the inverse image functor by f in F exists.

- Fib II The compound of two Cartesian morphisms is Cartesian.

A category F on E satisfying the condition Fib I is called a prefiber category.

safe .

If F is a fibered (resp. Prefibrated) category over E, a subcategory G of F is called a fiber subcategory (resp. a prefiber subcategory) if it is a fibred category (resp. prefibrated) on E, and if moreover the inclusion functor is Cartesian. If for example G is a full subcategory of F, we see that this means that for any morphism f: T → S in E and for all $\xi \in \mathrm{Ob}(G_S)$, $f^*_F(\xi)$ is T-isomorphic `object has a G_T . Another interesting case is the following: F being a category fibered on E, consider the subcategory G of F having the same objects, and whose morphisms are Cartesian morphisms of F; in particular the morphisms of G_S are the isomorphism of F_S . We see immediately that it is indeed a fibered subcategory of F, because in the bijection

$$\mathrm{Hom}_T(\eta, \eta) \xrightarrow{\sim} \mathrm{Hom}_f(\eta, \xi)$$

relative to a Cartesian f-morphism α in F, to the T-isomorphisms of the first member correspond to the Cartesian morphisms of the second. By definition, the sec-Cartesian tions E → F then correspond one-to-one to E -functors some-conches E → G (but note that the natural functor

$$\mathbf{Hom}_{E/-}(E, G) \longrightarrow \mathbf{Hom}_{cart}(E, F) = \varprojlim (F/E)$$

is faithful, but in general is not fully faithful, ie is not an isomorphism).

Remarks . - Let F be a category over E. The following conditions are equivalent to slow: (i) All morphisms of F are Cartesian (ii) F is a fibered category on E, and the F_S are groupoids (i.e. any morphism in F_S is an isomor-
166 phism). We then say that F is a category fibered into groupoids over E. They are they that we find especially in "module theory". If E is a groupoid, we show that conditions (i) and (ii) are also equivalent to the following: (iii) F is a groupoid,

and the projection functor p: F → E is transportable (cf. 4.4). For example, if E and F are groupoids such that Ob (E) and Ob (F) are reduced to a point, so that E and F are defined, up to isomorphism, by groups E and F, and the functor p: F → E is defined by a group homomorphism p: F → E, then F is bundled on E if and only if p is surjective, i.e. if p defines an extension of the group E by the group G = Kerp.

Proposition 6.2 . - Let F: F → G be an E -equivalence. For F to be a category fibered (resp. prefibrated) on E, it is necessary and sufficient that G be it.

Easily follows from the definitions and the remark mentioned above that a morphism α in F is Cartesian if and only if F (α) is.

Proposition 6.3 . - Let F_1 , F_2 be two categories on E, and let $\alpha = (\alpha_1, \alpha_2)$ be morphism in $F = F_1 \times_E F_2$. For α to be Cartesian, it is necessary and sufficient that its components are.

Let ξ_i be the goal and η_i the source of α_i , and let f: T → S be the morphism of E such that α_1 and α_2 are f-morphisms. For all $\eta = (\eta_1, \eta_2)$ in F_T , we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_T(\eta, \eta) & \longrightarrow & \mathrm{Hom}_f(\eta, \xi) \\ \mathrm{Hom}_T(\eta_1, \eta_1) \times \mathrm{Hom}_T(\eta_2, \eta_2) & \longrightarrow & \mathrm{Hom}_f(\eta_1, \xi_1) \times \mathrm{Hom}_f(\eta_2, \xi_2) \end{array}$$

where the vertical arrows are bijections. So if one of the horizontal arrows is one bijection, it is the same for the other. This already shows that if α_1 , α_2 are Cartesians (hence the second horizontal bijective arrow) then α is. The reverse can be seen by doing in the diagram above $\eta_i = \eta_i$ where $\mathrm{Hom}_T(\eta_i, \eta_i) = \emptyset$, first for i = 2 which proves that α_1 is Cartesian, then for i = 1 which proves that α_2 is Cartesian.

Corollary 6.4 . - Let $F = F_1 \times_E F_2$, and let $F = (F_1, F_2)$ be an E-breaker G → F. 167 For F to be Cartesian, it is necessary and sufficient that F_1 and F_2 are. We thus obtain an isomorphism of categories

$$\mathbf{Hom}_{cart}(G, F_1 \times_E F_2) \xrightarrow{\sim} \mathbf{Hom}_{cart}(G, F_1) \times \mathbf{Hom}_{cart}(G, F_2)$$

and in particular (making G = E) an isomorphism of categories

$$\varprojlim (F_1 \times_E F_2 / E) \xrightarrow{\sim} \varprojlim (F_1 / E) \times \varprojlim (F_2 / E)$$

Corollary 6.5 . - Let F_1 and F_2 be two fibered (resp. Prefibrated) categories above of E, then their fiber product $F = F_1 \times_E F_2$ is a fiber category (resp. prefiber) safe .

These results also extend to the case of the fiber product of any family of categories on E.

Proposition 6.6 . - Let F be a category on E, with functor-projection p, and let λ: E → E a functor, consider F = F × E E as a category on E by the functor-projection p = p × E id E · . Let α be a morphism of F, so that α is a Cartesian morphism, it is necessary and sufficient that its image α in F be it.

The demonstration is immediate and left to the reader.

Corollary 6.7 . - For any Cartesian functor F: F → G of categories on E, the functor F = F × E E from F = F × E E to G = G × E E is Cartesian.

Consequently, the functor **Hom** E (F, G) → **Hom** E · (F, G) considered in N o 3 induce a functor

Hom cart (F, G) - → **Hom** cart (F, G);

in other words, for F, G fixed, we can consider

Hom cart (F × E E, G × E E)

168 like a functor in E, of the category **Cat** / E ◦ in **Cat** . If we let vary equal-ment F, G, we find a functor of the category **Cat** / E ◦ × (**Cat** cart / E) ◦ × **Cat** cart / E in **Cat** . When taking into account isomorphism

Hom E · (F, G) - ~→ Hom E (F × E E, G)

envisaged N o 3, then the Cartesian -foncteurs E F G match in E - functors F × E E → G which transform any morphism whose first projection is a Cartesian morphism of F, into a Cartesian morphism of G. Doing F = E, we find (after change of notation):

Corollary 6.8 . - Lim ←– (F / E) is isomorphic to the full subcategory of **Hom** E / - (E, F) formed by E-breakers E → F which transform morphisms arbitrary in Cartesian morphisms. In particular, if F is a fibered category and if F̃ is the subcategory of F whose morphisms are Cartesian morphisms of F, then we have a bijection

Ob Lim ←– (F / E) - ~→ Hom E / - (E, F̃).

This specifies how the expression Lim ←– (F × E E / E) should be considered as a functor in E and in F, of the category **Cat** / E ◦ × **Cat** cart / E in the category **Cat** . We will see later a more complete functorial dependence with respect to E, when F is required to be a fibered category.

Corollary 6.9 . - Let F be a fibered (resp. Prefibrated) category over E, then F = F × E E is a fibered (resp. Prefibrated) category over E.

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Proposition 6.10 . - Let F, G be prefiber categories on E, F an E-breaker

Cartesian of F in G. For F to be faithful, (resp. Fully faithful, resp. A 169
E -equivalence) it is necessary and sufficient that for all $S \in Ob(E)$, the induced functor
$F_S : F_S \to G_S$ be faithful (resp. Fully faithful, resp. An equivalence).

Immediate demonstration from the definitions.

To end this issue, we give some properties of the fibered categories, using
izing the Fib $_{II}$ axiom .

Proposal 6.11 . - Let F be a prefiber category on E. For F to be fibered, it
it is necessary and sufficient that it satisfies the following condition:

Fib $_{II}$: Let $\alpha: \eta \to \xi$ be a Cartesian morphism in F above the morphism
$f: T \to S$ of E. For any morphism $g: U \to T$ in E, and any $\zeta \in Ob(F_U)$,
the application $u \mapsto \alpha \circ u$:

$$Hom_g(\zeta, \eta) \longrightarrow Hom_{fg}(\zeta, \xi)$$

is bijective.

In other words, in a category fibered on E, the Cartesian diagrams
are characterized by a property, stronger a priori than that of the definition (which
obtains by making $g = id_T$ in the preceding statement).

Corollary 6.12 . - Let F be a category on E, α a morphism in F. For
that α be an isomorphism, it is necessary that $p(\alpha) = f$ be an isomorphism and that α be
Cartesian; the converse is true if F is fibred on E.

Indeed, if α is an isomorphism, it is obviously the same for $f = p(\alpha)$;
for all $\eta \in Ob(F_T)$, the map $u \mapsto \alpha \circ u$

$$Hom(\eta, \eta) \longrightarrow Hom(\eta, \xi)$$

is bijective; as f is an isomorphism, we see immediately that an element of the first
member is a T-morphism if and only if its image in the second is an f-
morphism, so we get a bijection

$$Hom_T(\eta, \eta) \longrightarrow Hom_f(\eta, \xi)$$

which proves the first assertion. Conversely, suppose that f is an iso- 170
morphism and that α satisfies the condition stated in Fib $_{II}$. (which therefore means,
when F is fibred on E, that α is Cartesian), then we see immediately that for
all $\zeta \in Ob\, F$, the map $u \mapsto \alpha \circ u$ from $Hom(\zeta, \eta)$ to $Hom(\zeta, \xi)$ is bijective,
therefore α is an isomorphism.

Corollary 6.13 . - Let $\alpha: \eta \to \xi$ and $\beta: \zeta \to \eta$ be two composable morphisms in
the category F fibered on E. If α is Cartesian then β is if and only if αβ is.

We use the definition of Cartesian morphisms in the reinforced form of 6.11.

7. Categories split on E

Definition 7.1 . - Let F be a category over E. We call cleavage of F on E a function which attaches to all $f \in Fl(E)$ an inverse image functor for f in F, let f^*. The cleavage is said to be normalized if $f = id_S$ implies $f^* Id = {}_{F_S}$. We call split category (resp. normalized split category) a category F on E with a cleavage (resp. of a normalized cleavage).

It is obvious that F admits a cleavage if and only if F is prefibrated on E, and then F admits a normalized cleavage. The set of cleavages on F corresponds to one-to-one dance with all the K parts of Fl (F) satisfying the conditions following:

a) The $\alpha \in K$ are Cartesian morphisms.
b) For any morphism $f: T \to S$ in E and any $\xi \in Ob(F_S)$, there exists a unique f-morphism in K, with goal ξ.

For the cleavage defined by K to be normalized, it is necessary and sufficient that K satisfies moreover the condition

c) Identical morphisms in F belong to K.

171 The element morphisms of K could be called the ″morphisms of transport ″ for the proposed cleavage.

The notion of isomorphism of cleaved categories on E is clear. More generalment, we can define the morphisms of split E -categories as the functors of E -categories $F \to G$ which apply transport morphisms to morphisms of transport. (They are in particular Cartesian functors). In this way the categories gories split on E are the objects of a category, the category of split categories safe . The reader will explain the existence of products, linked to the fact that if a category on E is a product of categories F_i on E each endowed with a cleavage, then F is provided with a corresponding natural cleavage. The reader is also left to explain the notion of basic change in split categories.

We denote by $\alpha_f(\xi)$ the canonical morphism

$$\alpha_f(\xi)\colon f^*(\xi) \dashrightarrow \xi.$$

It is, as we have said, functorial in ξ, ie we have a functorial homomorphism

$$\alpha_f\colon i_T f_* \dashrightarrow i_S,$$

where for all $S \in \mathrm{Ob}(E)$, i_S denotes the inclusion functor

$$i_S\colon F_S \dashrightarrow F$$

Now consider morphisms

$$f\colon T \dashrightarrow S \quad \text{and} \quad g\colon U \dashrightarrow T$$

in E, and let $\xi \in \mathrm{Ob}(F_S)$, then there exists a unique U-morphism

$$c_{f,g}(\xi)\colon g_* f_*(\xi) \dashrightarrow (fg)^*(\xi)$$

making the diagram commutative 172

$$\begin{tikzcd}
f_*(\xi) \arrow[d, "\alpha_f(\xi)"'] & g_*(f^*(\xi)) \arrow[l, "\alpha_g(f^*(\xi))"'] \arrow[d, "c_{f,g}(\xi)"] \\
\xi & (fg)^*(\xi) \arrow[l, "\alpha_{fg}(\xi)"]
\end{tikzcd}$$

(by definition of $(fg)^*(\xi)$). For ξ variable, this homomorphism is functoriel, ie: we have a homomorphism

$$c_{f,g}\colon g_* f_* \dashrightarrow (fg)_*$$

functor $F_S \to F_U$. Let us note immediately:

Proposal 7.2 . - For the cleaved category F on E to be fibered, it is necessary and sufficient let the $c_{f,g}$ be isomorphisms.

We conclude, taking for f an isomorphism, for g its inverse, and considering owing the isomorphisms $c_{f,g}$ and $c_{g,f}$:

Corollary 7.3 . - If F is a fibered category split over E, then for any isomorphism $f\colon T \to S$ in E, f^* is an equivalence of categories $F_S \to F_T$.

Proposition 7.4 . - Let F be a category split on E. We have

$$\text{AT)} \quad \begin{cases} c_{f, id_T}(\xi) = \alpha_{id_T}(f^*(\xi)) \\ c_{id_S, f}(\xi) = f^*(\alpha_{id_S}(\xi)) \end{cases}$$

$$\text{B)} \quad c_{f, gh}(\xi)\, c_{g, h}(f^*(\xi)) = c_{fg, h}(\xi)\, h^*(c_{f, g}(\xi))$$

(In these formulas, f, g, h denote morphisms 173

$$V \dashrightarrow U \dashrightarrow T \dashrightarrow S$$

and ξ an object of F_S).

The first and second relation, in the case of a normalized cleavage, take the simpler form

$$\text{AT)} \quad c_{f, id_T} = id_{f^*}, \; c_{id_S, f} = id_{f^*}.$$

As for the third, it is visualized by the commutativity of the diagram

$$\text{(D)} \quad \begin{tikzcd} h_* g_* f_*(\xi) \arrow[r, "c_{g,h}(f^*(\xi))"] \arrow[d, "h_*(c_{f,g}(\xi))"'] & (gh)^*(f^*(\xi)) \arrow[d, "c_{f,gh}(\xi)"] \\ h_*(fg)^*(\xi) \arrow[r, "c_{fg,h}(\xi)"'] & (fgh)_*(\xi) \end{tikzcd}$$

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In the case of fibered categories (where the $c_{f,g}$ are isomorphisms), this commutativity can be expressed intuitively by the fact that the successive use of isomorphisms of the form $c_{f,g}$ do not lead to ″contradictory identifications″.
We can also write this formula without argument ξ, by the use of the product of convolution of homomorphisms of functors:

$$c_{fg, h} \circ (h_* * c_{f,g}) = c_{f,gh} \circ (c_{g,h} * f_*).$$

The proof of the first two formulas 7.4 is trivial, let us sketch that of the third. For this, let us consider, in addition to the square (D), the square of homomorphisms:

$$g_* f_*(\xi) \xrightarrow{\alpha_g(f^*(\xi))} f_*(\xi)$$

(D) $c_{f,g}(\xi)$ $\alpha_f(\xi)$

$(fg)^*(\xi)$ $\alpha_{fg}(\xi)$ // ξ

174 which is commutative by definition of $c_{f,g}(\xi)$. Consider the diagram obtained by joining the vertices of (D) to the corresponding vertices of (D) by the homomorphisms of the form α:

$$\alpha_h(g_* f_*(\xi)), \alpha_{gh}(f^*(\xi))$$
$$\alpha_h((fg)^*(\xi)), \alpha_{fgh}(\xi).$$

The four side faces of the cube thus obtained are also commutative: for the one on the left, this comes from the fact that the left column of (D) is deduced from the left column of (D) by mapping h, and that α_h is a functoriel; for the other three, it is nothing other than the definition of the operations c of the three remaining sides of (D). So the five faces of the cube other than the upper face are commutative. It follows that the two (fgh) -morphisms $h_* g_* f_*(\xi) \to (fgh)^*(\xi)$ defined by (D) have the same compound with $\alpha_{fgh}(\xi)$: $(fgh)^*(\xi) \to \xi$, so they are equal by definition of $(fgh)^*$.

Let us confine ourselves for the following to standardized split categories. Such a category gives birth to the following objects:

a) A map $S \mapsto F_S$ from Ob (E) in **Cat** .

b) A map $f \mapsto f^*$, associating to any $f \in$ Fl (E), with source T and goal S, a functor $f^* F_S \to F_T$.

c) A map $(f, g) \mapsto c_{f,g}$, associating to any pair of arrows (f, g) of E, a functorial homomorphism $c_{f,g} : g_* f_* \to (fg)_*$.

Moreover, these data satisfy the conditions expressed in formulas A) and B) given above. (NB If one had not been limited to the case of a normalized cleavage, it would have been necessary to introduce an additional object, namely a function $S \mapsto \alpha_S$ which associates with any object S of E a functorial homomorphism $\alpha_S : (id_S)_* \to id_{F_S}$; the condition A) would then be replaced by condition A)).

175 We will now show how we can reconstitute (with isomorphism unique near) the normalized split category F over E using the previous objects.

8. Cleaved category defined by a pseudo-functor $E^{\circ} \to$ **Cat**

Let us call, for short, pseudo-functor of E° in **Cat** (we should say, pseudo-normalized functor), a dataset a), b), c) as above, satisfying conditions A) and B). In the previous number, we associated, to a category normalized split on E, a pseudo-functor $E^{\circ} \to$ **Cat**, here we will indicate the reverse construction. We will leave it to the reader to verify most of the details, as well as the fact that these constructions are indeed "inverse" to each other. In a way precise, it would be appropriate to consider the pseudo-functors $E^{\circ} \to$ **Cat** like objects of a new category, and to show that our constructions provide equivalences, quasi-inverse to each other, between the latter and the category of split categories above E, defined in the previous number.

We pose

$$F_{\circ} = \coprod_{S \in Ob(E)} Ob(F(S)),$$

set sum of sets Ob (F (S)) (NB we will denote here F (S) and not F_S the value in the object S of E of the given pseudo-functor, to avoid confusion of notation thereafter). We therefore have an obvious application:

$$p_{\circ} : F_{\circ} \longrightarrow ObE.$$

Be

$$\xi = (S, \xi), \eta = (T, \eta) \qquad \text{(with } \xi \in Ob\, F(S), \eta \in ObF(T))$$

two elements of $F_{\circ}$, and let $f \in Hom(T, S)$, we will set

$$h_f(\eta, \xi) = Hom_{F(T)}(\eta, f^*(\xi)).$$

If we also have a morphism g: U → T in E, and a $\zeta \in ObF(U)$, we define a 176 application, denoted $(u, v) \mapsto u \circ v$:

$$h_f(\eta, \xi) \times h_g(\zeta, \eta) \longrightarrow h_{fg}(\zeta, \xi),$$

ie an application

$$Hom_{F(T)}(\eta, f^*(\xi)) \times Hom_{F(U)}(\zeta, g^*(\eta)) \longrightarrow Hom_{F(U)}(\zeta, (fg)^*(\xi)),$$

by the formula

$$u \circ v = c_{f, g}(\xi) \cdot g^*(u)\, v,$$

ie $u \circ v$ is the compound of the sequence

$$\zeta \xrightarrow{u} g^*(\eta) \xrightarrow{g^*(u)} g^* f^*(\xi) \xrightarrow{c_{f,g}(\xi)} (fg)^*(\xi).$$

On the other hand we will ask

$$h(\eta, \xi) = \coprod_{f \in Hom(T, S)} h_f(\eta, \xi),$$

and previous couplings define couplings

$$h(\eta, \xi) \times h(\zeta, \eta) \dashrightarrow h(\zeta, \xi),$$

while the definition of $h(\eta, \xi)$ implies an obvious application:

$$p_{\eta,\xi} : h(\eta, \xi) \dashrightarrow \mathrm{Hom}(T, S).$$

That said, we check the following points:

177 1) The composition between elements of $h(\eta, \xi)$ is associative.

2) For all $\xi = (\xi, S)$ in $F_\circ$, consider the identity element of

$$h_{\mathrm{id}_S}(\xi, \xi) = \mathrm{Hom}_{F(S)}(\mathrm{id}^*_S(\xi), \xi) = \mathrm{Hom}_{F(S)}(\xi, \xi),$$

and its image in $h(\xi, \xi)$. This object is a left and right unit for comparison.
position between elements of $h(\eta, \xi)$.

This already shows that we obtain a category F, by setting

$$\mathrm{Ob} F = F_\circ, \ \mathrm{Fl}\, F = \coprod_{\xi,\eta \in F_\circ} h(\eta, \xi).$$

(NB we cannot simply take for FlF the union of the sets $h(\eta, \xi)$, because the latter are not necessarily disjoint). Furthermore :

3) The maps $p_\circ : \mathrm{Ob}\, F \to \mathrm{Ob}\, E$ and $p_1 = (p_{\eta,\xi}) : \mathrm{Fl}\, F \to \mathrm{Fl}\, E$ define a functor $p : F \to E$. In this way, F becomes a category over E, more the obvious map $h_f(\eta, \xi) \to \mathrm{Hom}(\eta, \xi)$ induces a bijection

$$h_f(\eta, \xi) \xrightarrow{\sim} \mathrm{Hom}_f(\eta, \xi).$$

4) The obvious applications

$$\mathrm{Ob} F(S) \dashrightarrow F_\circ = \mathrm{Ob}\, F, \qquad \mathrm{Fl}\, F(S) \dashrightarrow \mathrm{Fl}\, F,$$

where the second is defined by the obvious mappings

$$\mathrm{Hom}_{F(S)}(\xi, \xi) = h_{\mathrm{id}_S}(\xi, \xi) \dashrightarrow \mathrm{Hom}(\xi, \xi)$$

define an isomorphism

$$i_S : F(S) \xrightarrow{\sim} F_S.$$

5) For any object $\xi = (S, \xi)$ of F, and any morphism $f: T \to S$ of E, consider
178 the element $\eta = (T, \eta)$ of F_T, with $\eta = f^*(\xi)$, and the element $\alpha_f(\xi)$ of $\mathrm{Hom}(\eta, \xi)$, image of $\mathrm{id}_{f^*(\xi)}$ by the morphism $\mathrm{Hom}_{F(T)}(f^*(\xi), f^*(\xi)) = h_f(\eta, \xi) \to \mathrm{Hom}_f(\eta, \xi)$. This element ment is Cartesian, and it is the identity in ξ if $f = \mathrm{id}_S$, in other words, the set des $\alpha_f(\xi)$ defines a normalized cleavage of F on E. In addition, by construction, we have commutativity in the functor diagram

$$\begin{tikzcd} F(S) \arrow[r, "f^*"] \arrow[d, "i_S"] & F(T) \arrow[d, "i_T"] \\ F_S \arrow[r, "f^*_F"] & F_T \end{tikzcd}$$

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o`uf $^*_F$ is the inverse image functor by f, relating to the cleavage considered on F. Finally:

6) the homomorphisms $c_{f,g}$ given with the pseudo-functor are transformed, by the isomorphisms i_S, into the functorial homomorphisms $c_{f,g}$ associated with the cleavage of F.

We limit ourselves to giving the verification of 1) (which is, if possible, less trivial than the others). It suffices to prove the associativity of the composition between the objects of sets of the form $h_f(\eta, \xi)$. Let us therefore consider in E morphisms

$$S \xleftarrow{f} T \xleftarrow{g} U \xleftarrow{h} V$$

and objects

$$\xi, \eta, \zeta, \tau$$

in F (S), F (T), F (U), F (V), finally elements

$$u \in h_f(\eta, \xi) = \mathrm{Hom}_{F(T)}(\eta, f^*(\xi))$$
$$v \in h_g(\zeta, \eta) = \mathrm{Hom}_{F(U)}(\zeta, g^*(\eta))$$
$$w \in h_h(\tau, \zeta) = \mathrm{Hom}_{F(V)}(\tau, h^*(\zeta)).$$

We want to prove the formula 179

$$(u \circ v) \circ w = u \circ (v \circ w),$$

which is an equality in $\mathrm{Hom}_{F(V)}(\tau, (fgh)^*(\xi))$. By virtue of the definitions both members of this equality are obtained by composition following the upper contour and bottom of the diagram below:

$$\begin{tikzcd}
\tau \arrow[r, "w"] \arrow[drr, "v \circ w"'] & h^*(\zeta) \arrow[r, "h^*(v)"] \arrow[rrr, bend left, "h^*(u \circ v)"] & h^*g^*(\eta) \arrow[r, "h^*g^*(u)"] \arrow[d, "c_{g,h}(\eta)"] & h^*g^*f^*(\xi) \arrow[r, "h^*(c_{f,g}(\xi))"] \arrow[d, "c_{g,h}(f^*(\xi))"] & h^*(fg)^*(\xi) \arrow[d, "c_{fg,h}(\xi)"] \\
 & & (gh)^*(\eta) \arrow[r, "(gh)^*(u)"] & (gh)^*f^*(\xi) \arrow[r, "c_{f,gh}(\xi)"] & (fgh)^*(\xi)
\end{tikzcd}$$

Now the median square is commutative because $c_{g,h}$ is a functorial homomorphism, and the square on the right is commutative under condition B) for a pseudo-functor. Hence the announced result.

Of course, it remains to be specified, when the considered pseudo-functor already comes from

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of a normalized split category F on E, how we get an isomorphism
natural between F and F. We leave the details to the reader.

We also leave to the reader to interpret, in terms of pseudo-functors, the notion of inverse image of a category split F on E by a change functor basic E → E.

9. Example: split category defined by a functor $E^{\circ} \to$ **Cat; categories split on** E

Suppose we have a functor

$$\phi : E^{\circ} \dashrightarrow \mathbf{Cat},$$

he then defines a pseudo-functor by posing

$$F(S) = \phi(S), \quad f^{*} = \phi(f), \quad c_{f,g} = \mathrm{id}_{(fg)^{*}}.$$

180 So the construction of the previous number gives us a category F split on E, said associated with the functor ϕ. For a category split on E to be isomorphic to a split category defined by a functor $\phi : E^{\circ} \to$ **Cat**, it is obviously necessary and sufficient that it meets the conditions:

$$(fg)^{*} = g^{*} f^{*}, \quad c_{f,g} = \mathrm{id}_{(fg)^{*}}.$$

In terms of the set K of transport morphisms, this also simply means that the compound of two transport morphisms is a transport morphism. A cleavage of a category F on E satisfying the previous condition is called a split from F over E, and a category F over E with a split is called a category split on E. It is therefore a special case of the notion of split category. The category of categories split on E is therefore equivalent to **Hom** (E°, **Cat**). Note that a category split on E is a fortiori a category split on E.

If F is a fibered category on E, there does not always exist a splitting on F. Sup-let us assume for example that ObE and Ob F are reduced to one element, and that the set endomorphisms of said is a group E resp. F, so that the functor projects tion p is given by a homomorphism of groups p: F → E, surjective since p is fiber. We then check immediately that the set of cleavages of F on E is in corre-

one-to-one spondance with the set of maps s: $E \to F$ such that $ps = id_E$
(ie the set of " representative systems " for the classes mod the subgroup
G kernel of the surjective homomorphism p: $F \to E$). A cleavage is a split if and only if s is a group homomorphism. To say that there is a split means therefore that the extension of groups F of E by G is trivial, which is expressed when G is commutative, by the nullity of a certain class of cohomology in $H^2(E, G)$ (where G is considered as a group where E operates).

181 Suppose however that F is a fibered category on E such that the F_S are rigid categories, ie the group of automorphisms of any object of F_S is reduced to identity. It is then easy to prove that F admits a split on E. Indeed, we note first that the question of existence of a splitting is not modified if we replaces F by an E -equivalent category, which in this case brings us back to the case where the F_S are rigid and reduced categories (ie two isomorphic objects in F_S are the same). But if G is a rigid and reduced category, any isomorphism of two functors $H \to G$ (where H is any category) is an identity. It follows

that if F is a fibered category on E, such that the fiber-categories are rigid and reduced, then there exists a unique cleavage of F on E, which is necessarily a splitting. So F is isomorphic to the category defined by a functor ϕ: $E^\circ \to$ **Cat** , such that the $\phi(S)$ are rigid and discrete categories, and the functor ϕ is defined in isomorphism close.

10. Co-fibered categories, bi-fibered categories

Consider a category F above E, with the projection functor

$$p: F \longrightarrow E,$$

it defines a category F° above E°, by the projection functor

$$p^\circ : F^\circ \longrightarrow E^\circ.$$

A morphism $\alpha: \eta \to \xi$ in F is said to be co-Cartesian if it is a Cartesian morphism for F° safe E°. Explicitly, we see that this means that for any object ξ of F_S, the application $u \mapsto u \circ \alpha$

$$\mathrm{Hom}_S(\xi, \xi) \dashrightarrow \mathrm{Hom}_f(\eta, \xi)$$

is bijective. We then also say that (ξ, α) is a direct image of η by f, in the category F on E. If it exists for all η in F_T, we say that the image functor direct by f exists, and we denote this functor f^F_* or f_*, once chosen. It is therefore defined by an isomorphism of bifunctors on $F^\circ_T \times F_S$: 182

$$\mathrm{Hom}_S(f_*(\eta), \xi) \xrightarrow{\sim} \mathrm{Hom}_f(\eta, \xi).$$

If therefore f_* exists, so that f^* exists, it is necessary and sufficient that f_* admits a functor Assistant, ie that there exists a functor $f^*: F_S \to F_T$ and an isomorphism of bifunctors

$$\mathrm{Hom}_S(f_*(\eta), \xi) \xrightarrow{\sim} \mathrm{Hom}_T(\eta, f^*(\xi)).$$

Let $g: U \to T$ be another morphism in E, and suppose that the inverse images and direct by f, g and fg exist. Consider then the functorial homomorphisms

$$c_{f,g}: f_* g_* \leftarrow (fg)_*$$
$$c_{f,g}: g^* f^* \dashrightarrow (fg)^*.$$

We see that if we consider $f_* g_*$ and $g^* f^*$ as a couple of adjunct functors, as well as $(fg)_*$ and $(fg)^*$, the two previous homomorphisms are appended to one of the other. So one is an isomorphism if and only if the other is. In particular :

Proposition 10.1 . - Suppose that the category F on E is prefibrated and co-prefibrated. For it to be fibered, it is necessary and sufficient for it to be co-fibered.

Of course, we say that F is co-prefibrated resp. co-fibered on E, if F° is pre-fiber resp. fibered on E. We will say that F is bi-fibered over E, if it is both fibred and co-fibered on E.

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11. Various examples

a) **Categories of arrows of** E. Let E be a category. Denote by $\mathbf{\Delta}^1$ the
183 category associated with the totally ordered set with two elements [0, 1]; so she has two objects 0 and 1, and in addition to the two identical morphisms an arrow (0, 1) of source 0 and goal 1. Let

$$\mathbf{Fl}(E) = \mathbf{Hom}(\mathbf{\Delta}_1, E)$$

we call it the category of arrows of E. Object 1 of **Δ** [1] defines a canon functor ic, called functor-goal

$$\mathbf{Fl}(E) \dashrightarrow E$$

(the functor defined by the object 0 of **Δ** [1] is called functor-source). For any object S of E, the fiber-category **Fl** (E) s is canonically isomorphic to the I / O category of objects of E above S.

Consider a morphism f: T → S in E, then it corresponds to a functor canonical

$$f_* : E_{/T} = F_T \dashrightarrow E_{/S} = F_S$$

and a functorial isomorphism

$$\mathrm{Hom}_S(f_*(\eta), \xi) \xrightarrow{\sim} \mathrm{Hom}_f(\eta, \xi)$$

which therefore makes f_* a direct image functor for f in F. We also have here

$$(\mathrm{id}_S)_* = \mathrm{id}_{F_S},\ (fg)_* = f_* g_*,\ c_{f,g} = \mathrm{id}_{(fg)},$$

ie F is endowed with a co-splitting on E. A fortiori, F is co-fibered on E. Note now that the set of morphisms in F is in one-to-one correspondence with the set of commutative square diagrams in E.

$$\begin{array}{ccc} X & \overset{f}{\longleftarrow} & Y \\ u & & v \\ S & \overset{f}{\longleftarrow} & T \end{array}$$

184 By definition, the morphism in question is Cartesian if the square is Cartesian in E, ie if it makes Y a fiber product of X and T over S. The inverse image functor f^* therefore exists if and only if for any object X on S, the fiber product $X \times_S T$ exists. It follows from 10.1 that if the product of two objects over a third always exists in E, ie if F is prefibrated on E, then F is even fibered on E.

b) **Category of pre-beams or beams over variable spaces**

Let E = **Top be** the category of topological spaces. If T is a topological space, we will denote by U (T) the category of the open ones of T, where the morphisms are the ap- inclusion plications. If C is a category, a functor U (T) ◦→ C is called a pre-beam on T with values in C, and a beam if it satisfies a condition of ex- an act on the left that we do not repeat here. The P (T) category of pre-beams

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on T with values in C, is by definition the category **Hom** (U (T) $^{\circ}$, C), and the category gorie F (T) of sheaves on T with values in C is the full subcategory whose the objects are the objects of **Hom** (U (T) $^{\circ}$, C) which are bundles. If f: T → S is a morphism in E, i.e. a continuous map of topological spaces, it corresponds by increasing map U ↦ → f^{-1} (U) a functor U (S) → U (T), hence a functor

$$f_* : \mathbf{Hom}(U(T)^{\circ}, C) \dashrightarrow \mathbf{Hom}(U(S)^{\circ}, C)$$

called a direct image functor of prefeams by f. We see as soon as the image direct of a sheaf is a sheaf, so the functor f_* : P (T) → P (S) induces a functor, also denoted f_* : F (T) → F (S). We also trivially verify (by the associativity of the composition of functors) that we have, for a second application continue g: U → T, the identity

$$(gf)_* = g_* f_* , \qquad \text{likewise } (id_S)_* = id_{P(S)} .$$

In this way, we got a functor

$$S \mapsto P(S)$$

resp.

$$S \mapsto F(S)$$

of E in **Cat** . In fact, we are interested in the corresponding functor 185

$$S \mapsto P(S)^{\circ} , \qquad \text{resp. } S \mapsto F(S)^{\circ} .$$

It defines a co-fibered category, and even co-split, on the category of spaces topological which we call the co-fibered category of the prefeaxes (resp. bundles) to values in C (implied: on variable spaces). Expliciting construction of N $_{o}$ 8, we see that a morphism of a pre-beam B on T in a pre-beam A on S is a pair (f, u) formed by a continuous map of T in S, and a morphism u: A → f_* (B) in the category P (S). This description also applies for beam morphisms, F being a full subcategory of P.

In the most important cases, category P and category F above E are also fibered categories, ie for any continuous map, the functors direct image P (T) → P (S) and F (T) → F (S) have an adjoint functor, which is then noted f^* and called inverse image functor of prefeacons resp. image functor inverse of beams, by continuous application f. This functor exists for example if C = **Ens** . We can show that the functor f^* : P (S) → P (T) exists whenever in C the inductive limits (relative to diagrams in the considered Universe) exist. The question is less easy for F; it will indeed be noted that (even in the case C = **Ens**) the inverse image of a pre-beam which is a beam is generally not not a beam, in other words the inverse image functor of beam is not isomorphic to the functor induced by the inverse image functor of pre-sheaves (despite the common notation f *). Thus, F is a co-fibered subcategory of P, but not a fibered subcategory, ie the inclusion functor F → P is not fibrant.

The co-fibered category P can be deduced from a co-fibered (or rather fibered) category more general, obtained in this way. For any category U (in the fixed Universe), we set

$$P(U) = \mathbf{Hom}(U, C)$$

186 and we note that U ↦ → P (U) is naturally a contravariant functor in U, of the **Cat** category in **Cat** . It therefore defines a split category above E = **Cat** , which we denote **Cat** $_{//C}$. The objects of this category are the pairs (U, p) of a category U and of a functor p: U → C, and a morphism from (U, p) into (V, q) is essentially a pair (f, u), where f is a functor U → V and u a homomorphism of functors u: p → qf. We leave it to the reader to explain the composition morphisms in **Cat** $_{//C}$. The functor-projection

$$F = \mathbf{Cat}_{//C} \dashrightarrow E = \mathbf{Cat}$$

associates with the pair (U, p) the object U; the U-grade fiber category is the **Hom** (U, C) (up to isomorphism). When in C the inductive limits exist, we easily show that the fibered category **Cat** $_{//C}$ on **Cat** is also co-fibered on **Cat** , ie we can define the notion of direct image of a functor p: U → C by a functor f: U → V. The category of pre-bundles is deduced from the fiber category previous by change of base

$$\mathbf{top}^{\circ} \dashrightarrow \mathbf{Cat}$$

(functor S ↦ → U (S) defined above), which gives a fibered category on **Top** °, and by passing to the opposite category, we obtain the co-fibered category P of the pre-bundles above **Top** . The notion of inverse image of a functor corresponds to that of image direct of preflight, the notion of direct image of a functor to that of inverse image of a pre-beam.

c) **Objects with operators above an object with operators**

Let F be a category over E, and let S be an object of E where a group G operates, on the left to fix ideas. This operator object can be interpreted as a corresponding to a functor λ: E → E of the category (to a single object, having G as group endomorphisms) E defined by G, in the category E, and thus defined by change- is based on a category F above E, which is fibered resp. co-fibered when
187 F is on E. A section of E over F (necessarily Cartesian, because E is a groupoid, and any isomorphism in F is Cartesian by virtue of 6.12), can also be interpreted as an E-breaker E → F above λ, or also as an object with operators ξ in F "above" the object with operators S.

d) **Quasi-inverse pairs of adjunct functors; autodualities**

When the base-category E is reduced to two objects a, b and, in addition to the arrows identical, to two isomorphisms f: a → b and g: b → a inverse to each other (i.e. E

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is a rigid connected groupoid with two objects), a normalized split category on E is essentially the same as the system formed by two categories F_a and F_b and a pair of adjunct functors $G: F_a \to F_b$ and $F: F_b \to F_a$, which are equivalences of categories (therefore almost the opposite of one another). We will take for F_a and F_b the fiber categories of F, for F and G the functors f^* and g^*, and both isomorphisms

$$u: FG \xrightarrow{\sim} id_{F_a} \qquad v: GF \xrightarrow{\sim} id_{F_b}$$

are $c_{g,f}$ and $c_{f,g}$. The two usual conditions of compatibility between u and v are not other than condition 7.4 B) for compounds fgf and gfg. It's easy to show that these conditions are sufficient to imply that we have a pseudo-functor $E^\circ \to$ **Cat**.

An interesting case is the one where we have

$$F_b = F^\circ_a, \; G = F^\circ, \; v = u^\circ.$$

We call autoduality in a category C, the data of a functor $D: C \to C^\circ$, and of an isomorphism $u: DD^\circ \xrightarrow{\sim} id_C$, such that u and the isomorphism $u^\circ: D^\circ D \xrightarrow{\sim} id_{C^\circ}$ do (D, D°) a pair of adjunct functors, (necessarily quasi-inverses the one the other). This condition is written:

$$D(u(x)) = u(D(x)) \qquad \text{for all } x \in Ob(C).$$

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e) **categories over a discr`ete category** E **.** We say that E is a discrete category if every arrow y is an identical arrow, so that E is defined Up to unique isomorphism by the knowledge of the set I = Ob (E). The data of a category F above E is therefore equivalent (up to a single isomorphism) to the data of a family of categories F_i ($i \in I$), the fiber categories. Any category F on E is fibered, every E-functor $F \to G$ is Cartesian, we have an isomorphism canonical

$$\mathbf{Hom}_{E/-}(F, G) \xrightarrow{\sim} \prod_i \mathbf{Hom}(F_i, G_i).$$

In particular, we obtain

$$\mathbf{\Gamma}(F/E) = \varprojlim{}_{F/E} \xrightarrow{\sim} \prod_i E_i.$$

f) **Suppose that** E **has exactly two objects** S **and** T **,** and in addition to the **mor- identical phisms, a morphism** f: T → S. Then a category F above of E is defined, up to E -isomorphism unique, by the data of two categories F_S and F_T and a bifunctor H (η, ξ) on $F^{\circ}_T \times F_S$, has values in **Set** . Indeed, if F is a category above E, we associate the two fiber categories F_S and F_T and the bifunctor H (η, ξ) = Hom_f (η, ξ). It is left to the reader to explain the construction in reverse. So that the category considered is fibered (or prefibrated, this is the same) it is necessary and sufficient that the functor H be representable with respect to to the argument ξ. For it to be co-fibered, it is necessary and sufficient that H be representable with respect to the argument η.

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g) Let F = C × E, considered as a category above E thanks to pr_2 . So F is fibered and co-fibered on E, and is even provided with a split and a co-split canonical, corresponding to the constant functor on E, resp. safe $^{\circ}$, has values in **Cat** , of C value. We have

$$\mathbf{\Gamma}(F / E) \simeq \mathbf{Hom}(E, C)$$

189 and $\varprojlim$ F / E corresponds to the full subcategory formed of the functors F: E → C transforming any morphisms into isomorphisms.

12. Functions on a split category

Let F be a split category normalized on E. For any object S of E we denote through

$$i_S : F_S \longrightarrow F$$

the inclusion functor. We therefore have a functorial homomorphism, for any morphism f: T → S in E:

$$\alpha_f : i_T f_* \longrightarrow i_S ,$$

o`uf * is the change of basis functor $F_S \to F_T$ for f defined by the cleavage. Is now

$$F: F \longrightarrow C$$

a functor of F in a category C, let us set, for all S ∈ Ob (E),

$$F_S = F \circ i_S : F_S \dashrightarrow C$$

and for all f: T → S in E,

$$\phi_f = F * \alpha_f : F_T f_* \dashrightarrow F_S$$

We have thus, to any functor F: F → C, associated a family (F_S) of functors $F_S \to C$, and a family (φ_f) of homomorphisms functors $F_T f^* \to F_S$. These families satisfy under the following conditions:

a) $\varphi_{id_S} = id_{F_S}$.

b) For two morphisms f: T → S and g: U → T in E, we have commutativity in the square of functorial homomorphisms:

$$\begin{tikzcd} F_U g_* f_* \arrow[r, "F_U * c_{f,g}"] \arrow[d, "\phi_g * f_*"'] & F_U (fg)^* \arrow[d, "\phi_{fg}"] \\ F_T f_* \arrow[r, "\phi_f"] & F_S . \end{tikzcd}$$

The first relation is trivial, and the second relation is obtained by applying the

functor F in the commutative diagram

$$\begin{tikzcd} g_* f_* (\xi) \arrow[r, "c_{f,g} (\xi)"] \arrow[d, "\alpha_g (f^* (\xi))"'] & (fg)^* (\xi) \arrow[d, "\alpha_{fg} (\xi)"] \\ f_* (\xi) \arrow[r, "\alpha_f (\xi)"] & \xi \end{tikzcd}$$

for an object in variable ξ F_S.

If G is a second functor F → C, giving rise to functors $G_S : F_S \to C$ and functorial homomorphisms $\psi_f : G_T f \to G_S$, and if u: F → G is a functorial homomorphism, then it corresponds to functorial homomorphisms $u * i_S$:

$$u_S : F_S \dashrightarrow G_S$$

and we immediately note that for any morphism f: T → S in E, we have commuta-
activity in squares

vs)
$$\begin{array}{ccc} F_T f_* & \xrightarrow{\varphi_f} & F_S \\ u_T * f_* \downarrow & & \downarrow u_S \\ G_T f_* & \xrightarrow{\psi_f} & G_S \end{array}$$

Proposal 12.1 . - Let H (F, C) be the category whose objects are the pairs of families (F_S) (S ∈ Ob (F)) of functors F_S → C, and families (ϕ_f) (f ∈ Fl (F)) of functorial homomorphisms $F_T f_*$ → F_S, satisfying the conditions a) and b), and where the morphisms are the families (u_S) (S ∈ Ob (F)) of homomorphisms F_S → G_S, 191 satisfying the commutativity condition c) written above, (the composition of the morphisms being made by the composition of homomorphisms of functors F_S → C). Then the two laws explained above define an isomorphism K of the category **Hom** (F, C) with the category H (F, C).

It is trivial that we have a functor of the first category in the second. This functor is fully faithful, because for F, G given, Hom (F, G) → Hom (K (F), K (G)) is trivially injective; to show that it is surjective, it suffices to note that the commutativity condition c) expresses the functoriality of the applications u (ξ) = u_S (ξ): F (S) = F_S (ξ) → G (ξ) = G_S (ξ) for the homomorphisms of the form α_f (ξ) in F, on the other hand we have the functoriality on each fiber category ie for the morphisms in F which are T-morphisms (T ∈ Ob (E)), hence the functoriality for any morphism in F, since an f-morphism (where f: T → S is a morphism in E is uniquely) a compound of a morphism α_f (ξ) and a T-morphism. It therefore remains to prove that the functor K is bijective for the objects. The previous argument already shows that K is injective for objects, remains at

prove that it is surjective, i.e. if we start from a system (F_S), (ϕ_f), satisfying a) and B) ; and if we define an ObF → Ob C application by

$$F(\xi) = F_S(\xi) \text{ for } \xi \in \mathrm{Ob}\, F_S \subset \mathrm{Ob}F$$

and a map Fl (F) → Fl (C) by

$$F(\alpha_f(\xi)\, u) = \phi_f(\xi)\, F_T(u)$$

for any morphism f: T → S in E, any object ξ of F_S and any T-morphism
u of goal f_* (ξ), then we get a functor F from F to C. Indeed, the relation
$F(\mathrm{id}_\xi) = \mathrm{id}_{F(\xi)}$ is trivial, it remains to prove the multiplicativity F (uv) = F (u) F (v)
when we have an f-morphism u: η → ξ and a g-morphism v: ζ → ν, with f: T → S
and g: U → T of the morphisms of E. Setting w = uv, we will have

$$u = \alpha_f(\xi)\, u,\quad v = \alpha_g(\eta)\, v,\quad w = \alpha_{fg}(\xi)\, w$$

192 with

$$w = c_{f,g}(\xi)\, g^*(u)\, v \qquad (\text{see N}^{\circ}\, 8).$$

With these notations, it is necessary to prove the commutativity of the outer contour of the diameter gram below:

$F(w)$

GF $F_U(\zeta)$ $F_u(v)$ // $F_U\, g_*(\eta)$ $F_u\, g_*(u)$ // $F_U\, g_*\, f_*(\xi)$ $F_U(c_{f,g}(\xi))$ ED // $F_U(fg)^*(\xi)$

$\phi_g(\eta)$ $\phi_g(f^*(\xi))$ $\phi_{fg}(\xi)$

QQQQQQQQQQQQQ(($F_T(\eta)$ @AT $F_T(u)$ // $F_T\, f_*(\xi)$ $\phi_f(\xi)$ // $F_S(\xi)$ BCOO

$F(u)$

Now the left triangle is commutative by definition of F (v), the median square is com-
mutative because deduced from the homomorphism u: ξ → f (η) by the functorial homomorphism
α_g, finally the square on the right is commutative by virtue of condition b). The conclusion
desired result.

Suppose now that C is also a split category normalized on E,
that from now on we will call G, and that we are interested in E-breakers
from F to G. If F is such a functor, it induces functors

$$F_S : F_S \dashrightarrow G_S$$

for the fiber categories. On the other hand, for any morphism f: T → S in E, and
any object ξ in F_S, the f-morphism $F(\alpha_f(\xi))$ is uniquely factored by a
T-morphism

$$\phi_f(\xi) : F_T(f^*_F(\xi)) \dashrightarrow f^*_G(F_S(\xi))$$

(where the F or the G in index indicates the split category for which we take the functor inverse image), hence a functorial homomorphism of functors from F_S to G_T :

$$\phi_f : F_T f^*_F \longrightarrow f^*_G F_S .$$

The two systems (F_S) and (ϕ_f) satisfy the following conditions: 193

a ') $\varphi_{id_S} = id_{F_S}$.

b ') For two morphisms f: T → S and g: U → T in E, we have commutativity in the following functorial homomorphism diagram:

$$\begin{tikzcd}
F_U g^*_F f^*_F \arrow[rr, "F_U * c^F_{f,g}"] \arrow[d, "\phi_g * f^*_F"'] & & F_U (fg)^*_F \arrow[dd, "\phi_{fg}"] \\
g^*_G F_T f^*_F \arrow[d, "g^*_G * \phi_f"'] & & \\
g^*_G f^*_G F_S \arrow[rr, "c^G_{f,g} * F_S"] & & (fg)^*_G F_S .
\end{tikzcd}$$

We leave the verification to the reader, as well as the statement and the demonstration of the analog of Proposition 12.1, implying that we thus obtain a correspond- one-to-one dance between the set of E-breakers from F to G, and the set of systems (F_S), (ϕ_f) satisfying the conditions a ') and b') above. Of course, in this correspondence, Cartesian functors are characterized by the property that the homomorphisms ϕ_f are isomorphisms.

Note . - Of course, there is usually an interest in reasoning directly about fibered categories without using explicit cleavages, which in particular dispenses to appeal, for the simple notion of E-breaker or Cartesian E-breaker, to a heavy interpretation as above. It is to avoid heaviness insupport- ables, and to obtain more intrinsic statements, that we had to give up 194 start as in [**2**] from the notion of cleaved category (called “fibered category” in loc. cit.), which takes second place in favor of that of the fibered category. It is moreover probable that, contrary to the still preponderant usage now, linked to old habits of thought, it will eventually prove to be more convenient in the l`emes universal, not to focus on a supposed solution chosen once for all, but to put all solutions on an equal footing.

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