Lecture 6 - Bayesian Binary Logistic Regression

Contents

Today's Learning Objectives

1. Extend the paradigm of Bayesian regression to Generalized Linear Models (GLMs), specifically Binary Logistic regression.
2. Explore and contrast Bayesian Binary Logistic regression versus its frequentist counterpart.
3. Illustrate the Bayesian modelling setup in Binary Logistic regression.
4. Compare the Bayesian prior and posterior sigmoid curves in simple Binary Logistic regression.
5. Compare the Bayesian prior and posterior predicted probabilities in simple Binary Logistic regression.

Loading R Packages

```
options(repr.matrix.max.rows = 6)
library(bayesrules)
library(tidyverse)
library(broom)
library(rstan)
library(bayesplot)
library(broom.mixed)
library(cowplot)
library(janitor)
library(wesanderson)
```

Previously...

We introduced Bayesian Normal regression, **comparable to Ordinary Least-Squares (OLS)**. Nonetheless, we tweaked the frequentist concepts to a Bayesian framework. For instance, we do not use p-values in a Bayesian framework to perform inference. Nevertheless, we can use posterior credible intervals to test whether a given regression coefficient is different from zero (i.e., **whether the credible interval contains zero or not**).

Furthermore, we know there are more regression models besides the classical OLS. As we saw in **DSCI 562**, we will encounter different types of responses. Hence, there is always a specific regression model for each possible case. Recall GLMs, such as Binary Logistic regression, are estimated via **maximum likelihood**.

GLMs also have a Bayesian version, and we can obtain an approximate posterior for the regression parameters via Markov Chain Monte Carlo (MCMC)!

1. Cards Against Humanity's "Pulse of the Nation" Survey

We will use the dataset `pulse_of_the_nation` found in the package `bayesrules` **to contrast Bayesian Binary Logistic regression versus its frequentist counterpart**.

According to the package's documentation, the description is the following:

> *Cards Against Humanity's "Pulse of the Nation" project (https://thepulseofthenation.com/) conducted monthly polls into people's social and*

> *political views, as well as some silly things. This data includes responses to a subset of questions included in the poll conducted in September 2017.*

The dataset contains $n = 1000$ observations (i.e., surveyed subjects) and fifteen variables.

```
pulse_of_the_nation
```

income	age	party	trump_approval	education	robots	climate_chang
<dbl>	<dbl>	<fct>	<fct>	<fct>	<fct>	<fct:
8	64	Democrat	Strongly disapprove	College degree	Unlikely	Real and Cause by Peopl
68	56	Democrat	Strongly disapprove	High school	Unlikely	Real and Cause by Peopl
46	63	Independent	Somewhat Approve	Some college	Unlikely	Real but no Caused b Peopl
⋮	⋮	⋮	⋮	⋮	⋮	
70	74	Republican	Strongly Approve	Some college	Unlikely	Real and Cause by Peopl
93	60	Independent	Strongly disapprove	Graduate degree	Unlikely	Real but no Caused b Peopl
35	46	Independent	Strongly Approve	High school	Likely	Not Real At A

We are interested in the following variables:

- `income`: Subject's annual income in thousand USD. It is a **continuous variable**.
- `climate_change`: Subject's belief in climate change. It is a **categorical variable** with three levels (`Not Real At All`, `Real and Caused by People`, `Real but not Caused by People`,).

```r
pulse_training <- pulse_of_the_nation |>
  select(income, climate_change)
pulse_training
levels(pulse_training$climate_change)
```

A tibble: 1000 × 2

income	climate_change
<dbl>	<fct>
8	Real and Caused by People
68	Real and Caused by People
46	Real but not Caused by People
⋮	⋮
70	Real and Caused by People
93	Real but not Caused by People
35	Not Real At All

'Not Real At All' · 'Real and Caused by People' · 'Real but not Caused by People'

Main statistical inquiries

Suppose we are interested in the following statistical inquiries:

1. How is continuous `income` associated with `climate_change`?
2. Is there a numerical result that can quantify this association?

Exercise 17

We will view `climate_change` as our response of interest and `income` as our feature. Thus, regression analysis will help you address these inquiries. Now, the question is: **what regression model should we use via a frequentist approach??**

A. A generalized linear model (GLM) such as Multinomial regression.

B. A GLM such as Ordinal Logistic regression.

C. A GLM such as Poisson regression.

D. Ordinary Least-squares (OLS).

Solution to Exercise 17

A straightforward answer to this question is using **Multinomial Logistic regression**. However, besides a frequentist approach, recall we will apply a Bayesian framework.

Of course there is a **Bayesian version of Multinomial Logistic regression**. **But this model implies using a more complex coding setup.**

We need to start with a simple approach as a Bayesian GLM. Therefore, let us convert `climate_change` to a binary response: `1` for `Real and Caused by People` and `Real but not Caused by People`, along with `0` for `Not Real At All`.

```
levels(pulse_training$climate_change)
```

'Not Real At All' · 'Real and Caused by People' · 'Real but not Caused by People'

```
pulse_training <- pulse_training |>
  mutate(climate_change = case_when(
    climate_change == "Real and Caused by People" ~ 1,
    climate_change == "Real but not Caused by People" ~ 1,
    climate_change == "Not Real At All" ~ 0
  ))
pulse_training
```

A tibble: 1000 × 2

income	climate_change
<dbl>	<dbl>
8	1
68	1
46	1
⋮	⋮
70	1
93	1
35	0

2. Exploratory Data Analysis

Let us make a quick exploratory data analysis on `pulse_training` for `climate_change` versus `income` via side-by-side boxplots. Moreover, **note 15% of the respondents believe climate change is not real!**

```
pulse_training |>
  tabyl(climate_change) |>
  adorn_pct_formatting()
```

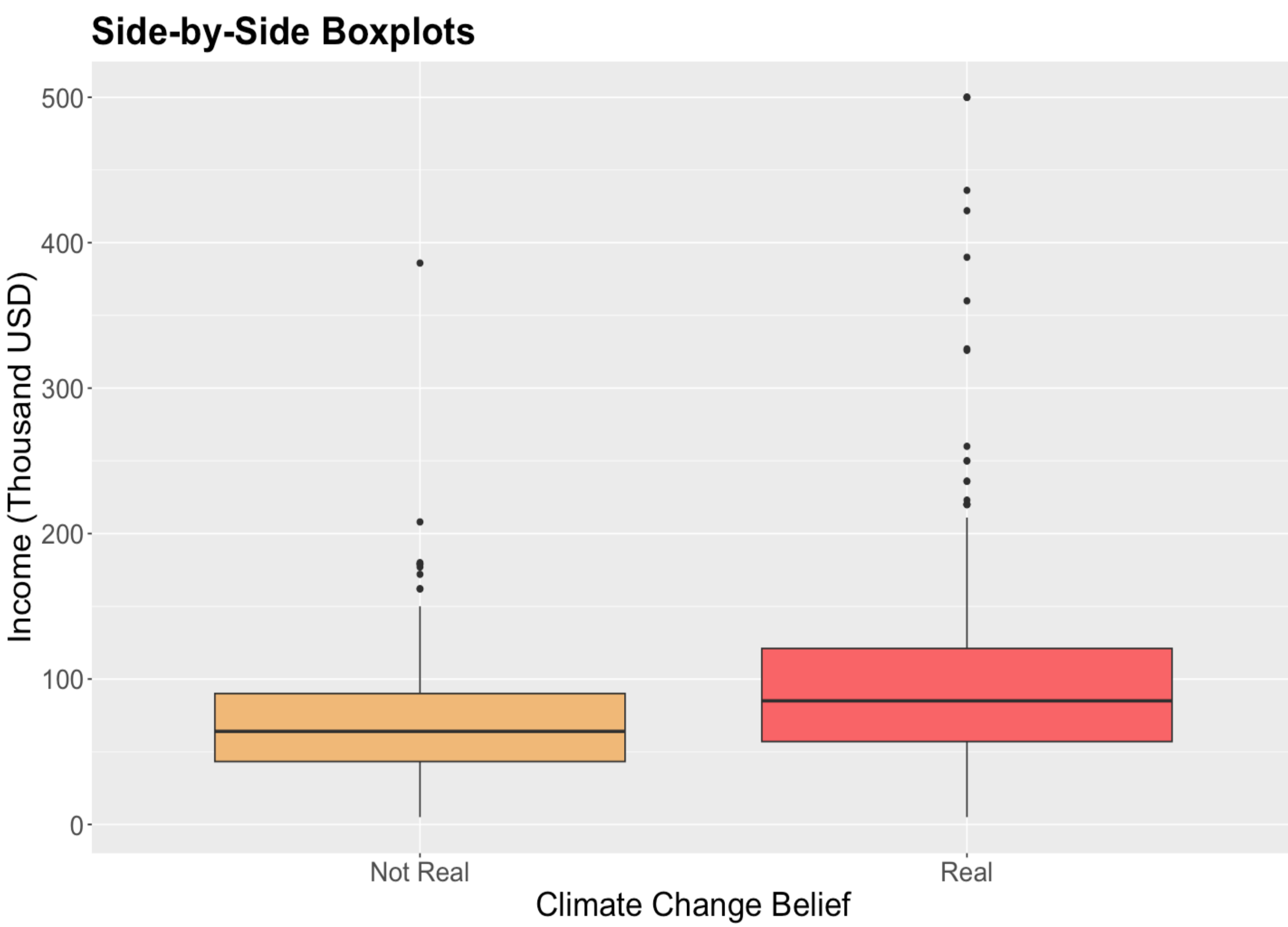
A tabyl: 2 × 3

climate_change	n	percent
<dbl>	<dbl>	<chr>
0	150	15.0%
1	850	85.0%

The below side-by-side boxplots show **graphical** evidence for lower incomes leaning more to the category **Not Real** in `climate_change`.

```
options(repr.plot.height = 8, repr.plot.width = 12)

climate_change_boxplots <- pulse_training |>
  ggplot(aes(as.factor(climate_change), income)) +
  geom_boxplot(aes(fill = as.factor(climate_change))) +
  labs(y = "Income (Thousand USD)", x = "Climate Change Belief") +
  ggtitle("Side-by-Side Boxplots") +
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21),
    legend.position = "none"
  ) +
  scale_x_discrete(labels=c("0" = "Not Real", "1" = "Real")) +
  scale_fill_manual(values = wes_palette(n = 3, name = "GrandBudapest1"))
```

```
climate_change_boxplots
```



3. Frequentist Logistic Regression

Let us review the frequentist Binary Logistic regression. For $i = 1, \ldots, n$ (where n is the training size) its response variable has the form:

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th observation is a success,} \\ 0 & \text{otherwise.} \end{cases}$$

As the response variable can only take the values 0 or 1, the key modelling term becomes the probability that Y_i takes on the value of 1, i.e. the probability of success, denoted as π_i. Hence:

$$Y_i \sim \text{Bernoulli}(\pi_i).$$

Specifically, π_i ($i = 1, 2, \ldots, n$) will depend on the values of the k regressors $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$ in the form:

$$\log \left(\frac{\pi_i}{1 - \pi_i} \right) = \beta_0 + \beta_1 X_{i,1} + \beta_1 X_{i,2} + \ldots + \beta_k X_{i,k},$$

or equivalently

$$\pi_i = \frac{\exp \left[\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \ldots + \beta_k X_{i,k} \right]}{1 + \exp \left[\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \ldots + \beta_k X_{i,k} \right]} \in [0, 1].$$

The log of the odds $\log \left(\frac{\pi_i}{1 - \pi_i} \right)$ is the link function in this GLM, i.e., **the logarithm of the ratio of the probability of the event to the probability of the non-event**.

In this example, let:

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th subject thinks climate change is real,} \\ 0 & \text{otherwise;} \end{cases}$$

and $X_{i,\text{income}}$ our single regressor.

Hence, the model becomes:

$$\log \left(\frac{\pi_i}{1 - \pi_i} \right) = \beta_0 + \beta_1 X_{i,\text{income}}.$$

Note we have two regression parameters: β_0 and β_1. **These parameters will be unrestricted on the real scale.**

Firstly, we run a frequentist Binary Logistic regression via `glm()` (i.e., **using maximum likelihood estimation**). Note the following:

- `income` is statistically significant for the log of the odds.
- The interpretation for $\hat{\beta}_1$ is: *"for each one thousand USD increase in* `income`*, a subject is 1.009 times more likely to believe in climate change than not to"*.

```
freq_log_reg <- glm(climate_change ~ income, data = pulse_training, family = "
summary_freq_log_reg <- tidy(freq_log_reg, conf.int = TRUE) |>
  mutate(
    exp.estimate = exp(estimate),
    exp.conf.low = exp(conf.low),
    exp.conf.high = exp(conf.high)
  ) %>%
  mutate_if(is.numeric, round, 3)
summary_freq_log_reg
```

A tibble: 2 × 10

term	estimate	std.error	statistic	p.value	conf.low	conf.high	exp.estima
<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dl
(Intercept)	0.965	0.184	5.248	0	0.606	1.327	2.6
income	0.009	0.002	4.376	0	0.005	0.014	1.0

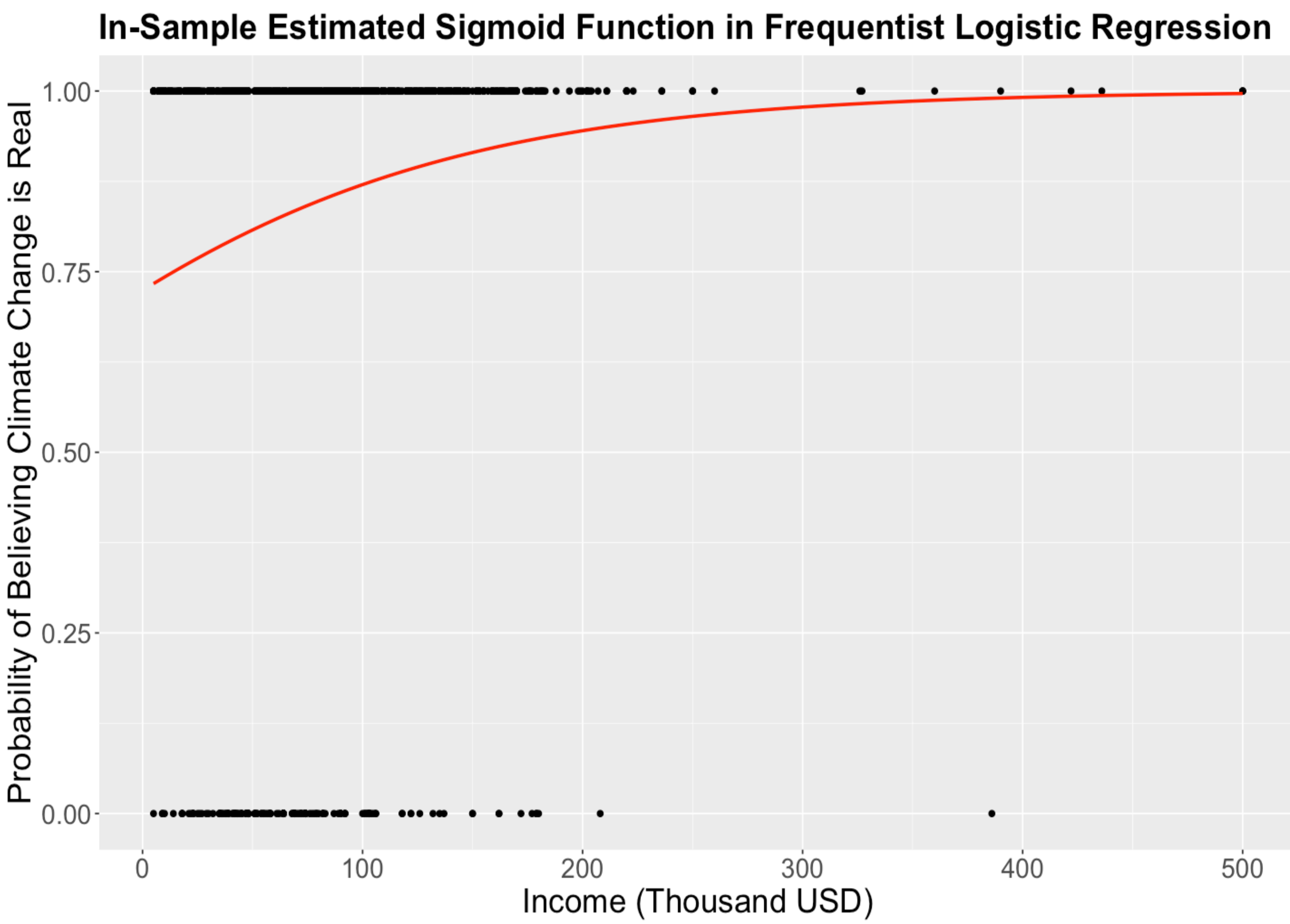
The log of the odds can be put on the probability scale as follows:

$$\pi_i = \frac{\exp\left[\beta_0 + \beta_1 X_{i,\text{income}}\right]}{1 + \exp\left[\beta_0 + \beta_1 X_{i,\text{income}}\right]} \in [0, 1].$$

This is the **sigmoid function**. The in-sample $\hat{\pi}_i$ values are shown below. There is no S-shaped in this case given the values for $\hat{\beta}_0$ and $\hat{\beta}_1$

```
plot_freq_log_reg <- pulse_training |>
  ggplot() +
  geom_point(aes(income, climate_change)) +
  geom_smooth(aes(income, climate_change),
    method = "glm", method.args = c(family = binomial), formula = "y ~ x", se
  labs(y = "Probability of Believing Climate Change is Real", x = "Income (Tho
  ggtitle("In-Sample Estimated Sigmoid Function in Frequentist Logistic Regres
  theme(
    plot.title = element_text(size = 22, face = "bold"),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21)
  )
```

```
plot_freq_log_reg
```



4. Bayesian Logistic Regression

4.1. The Likelihood

Let us start our Bayesian model. First off, our respose will be assumed as:

$$Y_i \mid \beta_0, \beta_1 \overset{\text{ind}}{\sim} \text{Bernoulli}(\pi_i).$$

We need to set up the **systematic component** of the model. Thus, we will use the link function:

$$\log \left(\frac{\pi_i}{1 - \pi_i} \right) = \beta_0 + \beta_1 X_{i,\text{income}}.$$

This will be our likelihood:

$$Y_i \mid \beta_0, \beta_1 \overset{\text{ind}}{\sim} \text{Bernoulli}(\pi_i) \quad \text{where} \quad \log \left(\frac{\pi_i}{1 - \pi_i} \right) = \beta_0 + \beta_1 X_{i,\text{income}}.$$

4.2. The Priors

Unlike OLS, we do not have σ^2. Therefore, our parameters of interest are β_0 and β_1. For the prior specification, we will proceed as follows:

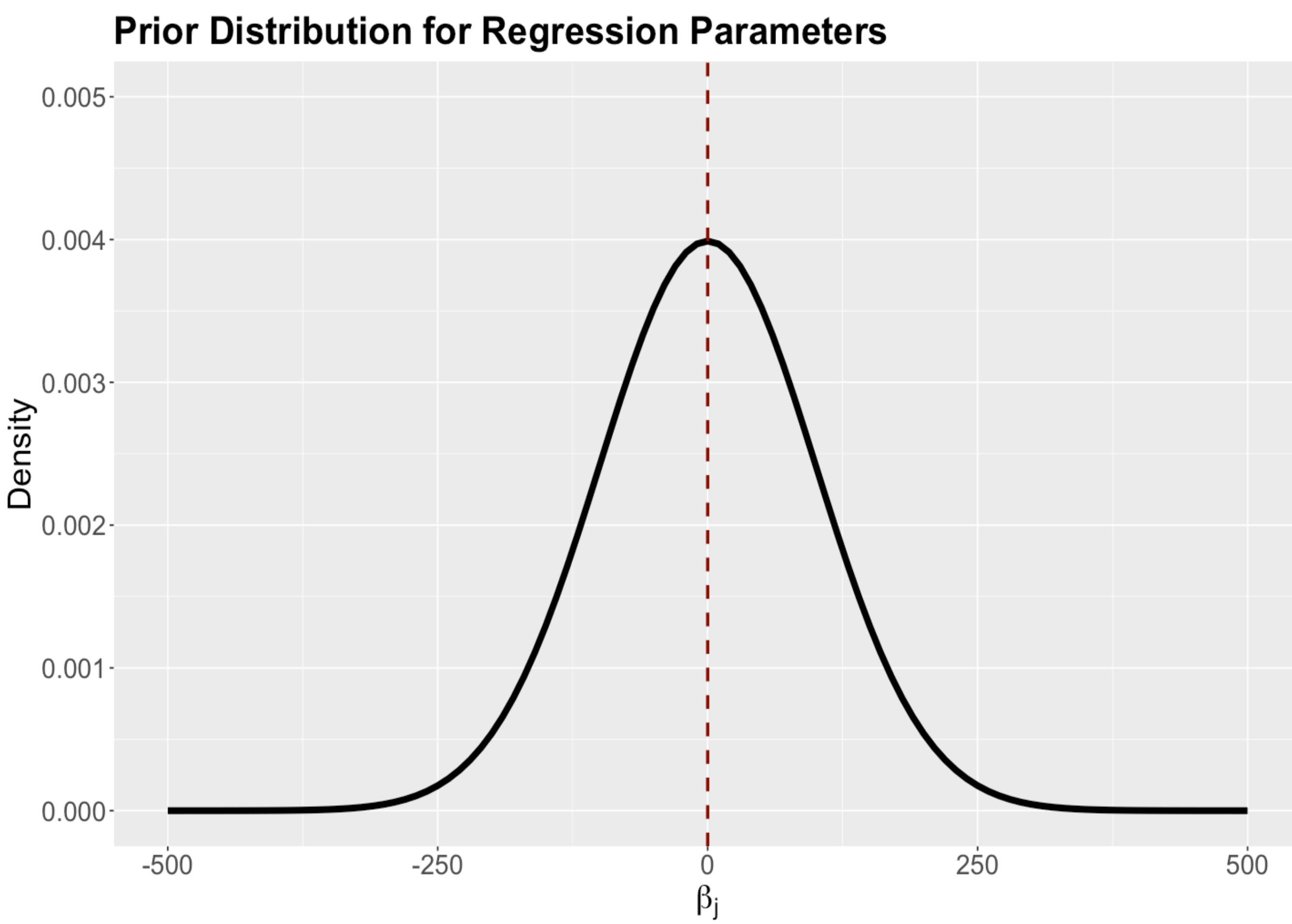
- $\beta_0 \sim \mathcal{N}(\mu_{\beta_0} = 0, \sigma^2_{\beta_0} = 100^2)$.
- $\beta_1 \sim \mathcal{N}(\mu_{\beta_1} = 0, \sigma^2_{\beta_1} = 100^2)$.

We have to stress that for these prior specifications, **we assume the prior mean of β_1 as 0, since we do not know if the variable $X_{i,\text{income}}$ has any association with Y_i**. On the other hand, **the intercept β_0 will be assumed as zero in our prior knowledge**. We will use a large variances (i.e, $\sigma^2_{\beta_j} = 100^2$ for $j = 0, 1$) **to reflect uncertainty**.

The density plot for these priors is shown below with the corresponding hyperparameter mean (equal to zero) indicated as red vertical dashed line.

```
# Plotting Normal prior for regression parameters
prior_gamma_beta_j <- ggplot() +
  xlim(-500, 500) +
  ylim(0, 0.005) +
  geom_function(fun = dnorm, args = list(mean = 0, sd = 100), linewidth = 2) +
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text.x = element_text(size = 17, angle = 0),
    axis.text.y = element_text(size = 17, angle = 0),
    axis.title = element_text(size = 21),
  ) +
  labs(y = "Density", x = expression(beta["j"])) +
  ggtitle("Prior Distribution for Regression Parameters") +
  geom_vline(xintercept = 0, colour = "darkred", linetype = "dashed", linewidt
```

```
prior_gamma_beta_j
```



4.3. Coding a Prior Bayesian Model on `Stan`

Before setting up our full Bayesian model, we will check why our likelihood (i.e., collected data) is essential. Suppose you set up a prior Bayesian model, i.e., you only code the prior distributions for β_0 and β_1 in `Stan`.

What does this mean? It means that we build a `Stan` code with the prior distribution of the `parameters` without specifying the likelihood and `data` block. Hence, the process becomes a Monte Carlo simulation in which we draw random values from the two prior normal distributions.

> **Caution**
>
> This is not MCMC since we are not applying the Bayes' rule.

```
prior_stan_climate_change <- "parameters {
real beta_0;
real beta_1;
}
model {
beta_0 ~ normal(0, 100);
beta_1 ~ normal(0, 100);
}"
```

Therefore, when using the function `stan()` (or `sampling()` if we are using the `R` markdown), using `thin` and `warmup` might not be necessary. Furthermore, you do not need a `data` dictionary in this prior model.

```
prior_climate <- stan(
  model_code = prior_stan_climate_change,
  chains = 1,
  iter = 1000,
  warmup = 0,
  thin = 1,
  seed = 553
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 2e-06 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.(
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: WARNING: No variance estimation is
Chain 1:          performed for num_warmup < 20
Chain 1:
Chain 1: Iteration:    1 / 1000 [  0%]  (Sampling)
Chain 1: Iteration:  100 / 1000 [ 10%]  (Sampling)
Chain 1: Iteration:  200 / 1000 [ 20%]  (Sampling)
Chain 1: Iteration:  300 / 1000 [ 30%]  (Sampling)
Chain 1: Iteration:  400 / 1000 [ 40%]  (Sampling)
Chain 1: Iteration:  500 / 1000 [ 50%]  (Sampling)
Chain 1: Iteration:  600 / 1000 [ 60%]  (Sampling)
Chain 1: Iteration:  700 / 1000 [ 70%]  (Sampling)
Chain 1: Iteration:  800 / 1000 [ 80%]  (Sampling)
Chain 1: Iteration:  900 / 1000 [ 90%]  (Sampling)
Chain 1: Iteration: 1000 / 1000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 0 seconds (Warm-up)
Chain 1:                0.132 seconds (Sampling)
Chain 1:                0.132 seconds (Total)
Chain 1:
```

The output `prior_climate` will contain **the prior samples for β_0 and β_1**. Note the summary statistics are close to the hyperparameters we set up in `prior_stan_climate_change`.

```
prior_climate_samples <- as.data.frame(round(summary(prior_climate)$summary, 3
prior_climate_samples <- prior_climate_samples[, c("mean", "sd")]
prior_climate_samples[1:2,]
```

A data.frame: 2 × 2

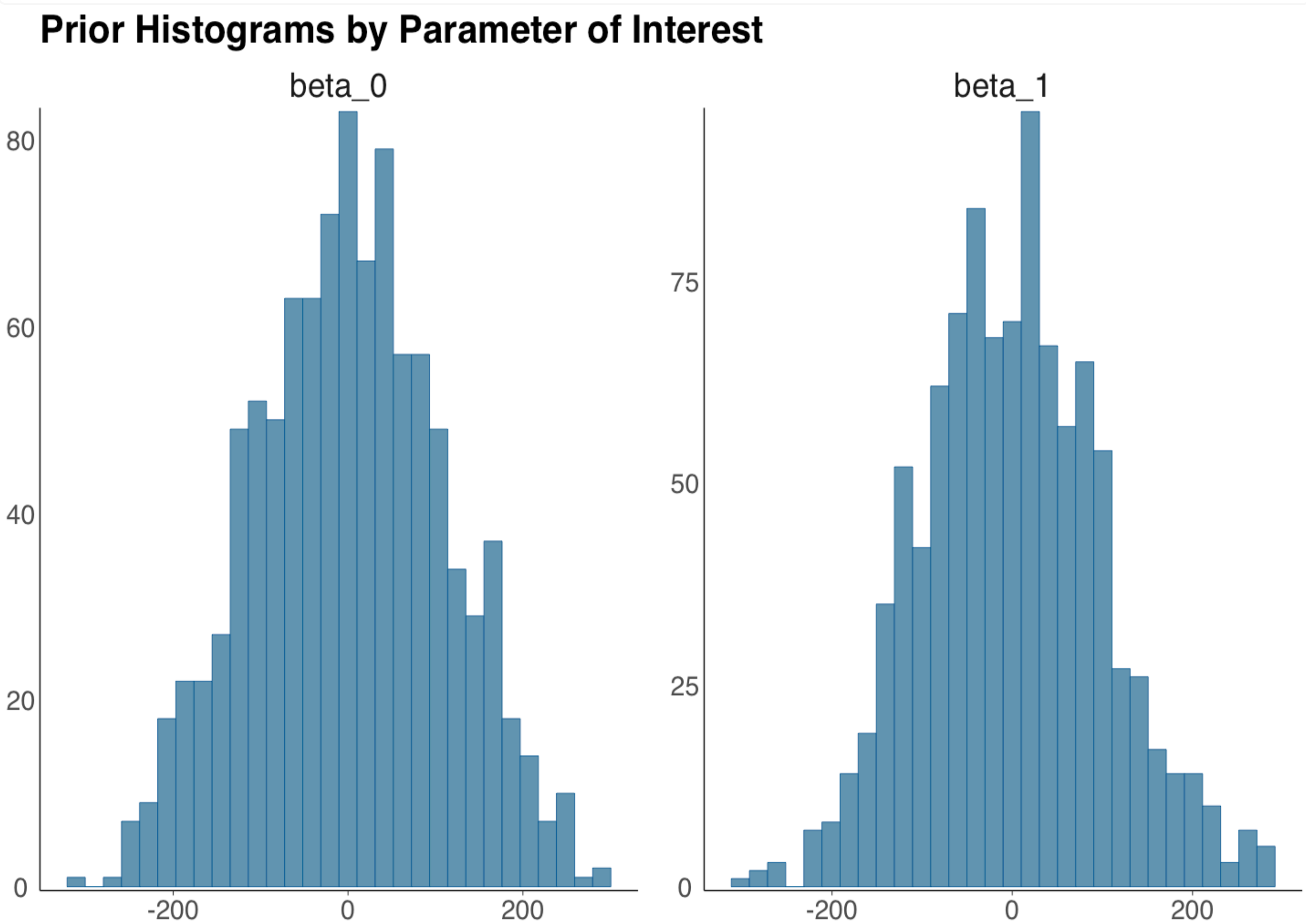
	mean	sd
	<dbl>	<dbl>
beta_0	0.732	108.576
beta_1	-0.062	99.967

We plot the prior histograms for these samples via `mcmc_hist()` from `bayesplot`. The distributions are symmetric and centred around zero.

```
prior_plots_betas <- mcmc_hist(prior_climate, pars = c("beta_0", "beta_1")) +
  theme(
    plot.title = element_text(size = 24, face = "bold", family = "sans"),
    axis.text.x = element_text(size = 17, family = "sans"),
    axis.text.y = element_text(size = 17, family = "sans"),
    strip.text.x = element_text(size = 21, family = "sans")
  ) +
  ggtitle("Prior Histograms by Parameter of Interest")
```

```
prior_plots_betas
```

```
`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



What is the purpose of getting these prior samples? These 1000 samples (pairs containing β_0 and β_1) will be use to plot 1000 different sigmoid functions:

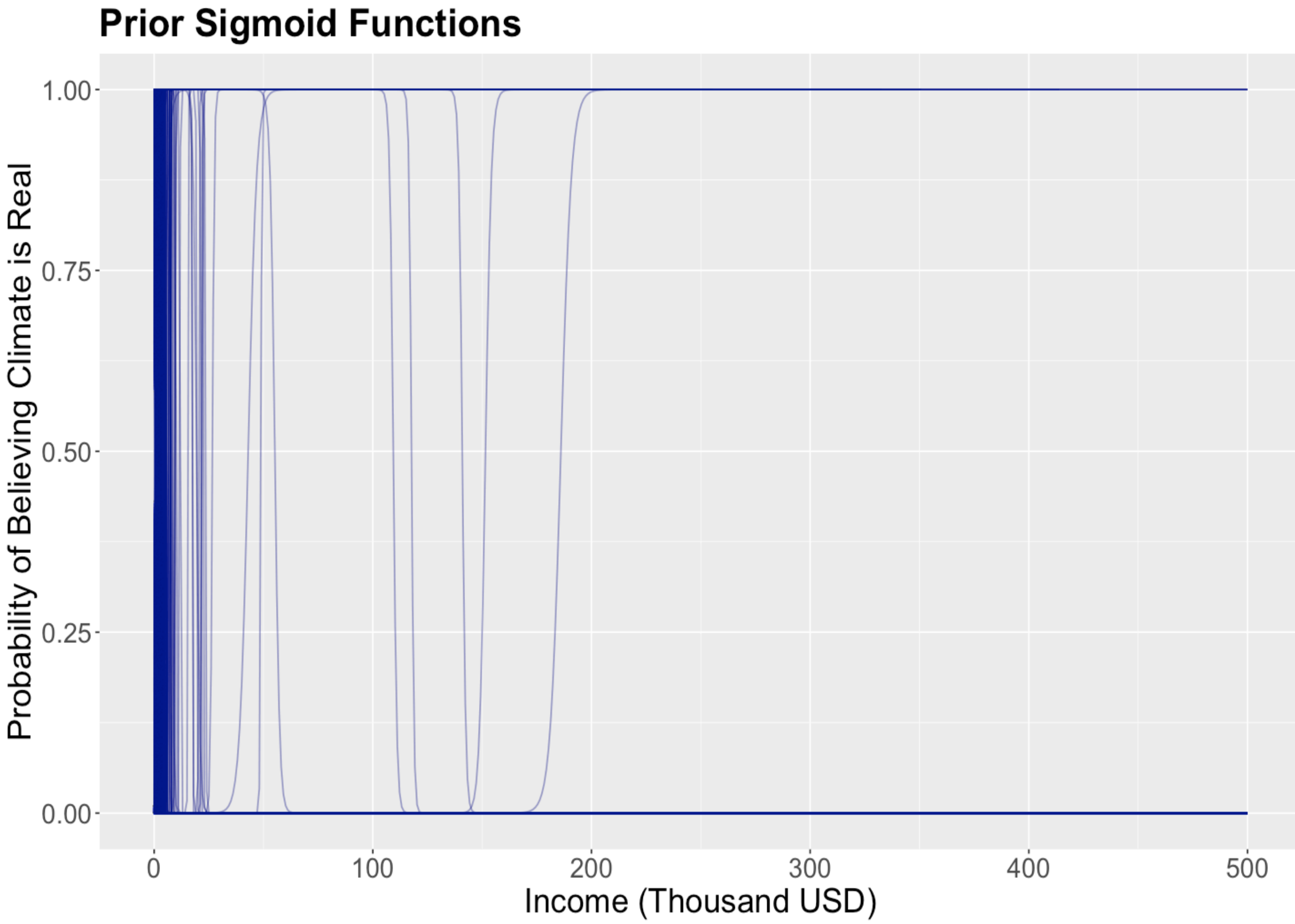
$$\pi_i = \frac{\exp\left[\beta_0 + \beta_1 X_{i,\text{income}}\right]}{1 + \exp\left[\beta_0 + \beta_1 X_{i,\text{income}}\right]} \in [0, 1]$$

The regressor income is set up as a grid from 0 to 500 thousand USD to plot these 1000 sigmoid functions.

```
prior_climate <- as.data.frame(prior_climate) # prior_climate as data frame

income <- seq(0, 500, length.out = 500) # Setting up grid for income

beta_0 <- as.numeric(prior_climate[1, "beta_0"]) # First sample
beta_1 <- as.numeric(prior_climate[1, "beta_1"]) # First sample

logistic_like <- data.frame(income = income) |> # Computing probabilities with
  mutate(values = exp(beta_0 + beta_1 * income) / (1 + exp(beta_0 + beta_1 * i

plot_logistic_like_prior_stanfit <- ggplot(data = logistic_like, na.rm = TRUE)
  geom_line(aes(x = income, y = values), alpha = 0.1, color = "darkblue", na.r
  labs(y = "Probability of Believing Climate is Real", x = "Income (Thousand U
  ggtitle("Prior Sigmoid Functions") +
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21)
  ) # First ggplot layer

for (j in 2:nrow(prior_climate)) # Loop from the second sample to the last sam
{
  beta_0 <- as.numeric(prior_climate[j, "beta_0"])
  beta_1 <- as.numeric(prior_climate[j, "beta_1"])

  logistic_like <- data.frame(income = income) |>
    mutate(values = exp(beta_0 + beta_1 * income) / (1 + exp(beta_0 + beta_1 *

  plot_logistic_like_prior_stanfit <- plot_logistic_like_prior_stanfit +
    geom_line(data = logistic_like, aes(x = income, y = values),
              alpha = 0.4, color = "darkblue", na.rm = TRUE) # Next ggplot lay
}
```

The plot below shows these 1000 sigmoid functions. We can think of this as **a prior distribution over curves** (since a curve is defined by β_0 and β_1). Hence, these blue sigmoid curves are samples from this distribution.

```
plot_logistic_like_prior_stanfit
```



Exercise 18

Having explained all this prior model, do you see a clear distribution on these fitted curves?

A. Yes.

B. No.

Solution to Exercise 18

There is no clear pattern in this distribution of curves since we need our data (i.e., likelihood) to obtain adequate posterior samples. Therefore, let us code up the complete `posterior_stan_climate_change`.

4.4. Coding the Posterior Bayesian Model on

`Stan`

The full formal Bayesian logistic model is set up as follows:

$$
\begin{gathered}
\text{likelihood:} \qquad Y_i \mid \beta_0, \beta_1 \overset{\text{ind}}{\sim} \text{Bernoulli}(\pi_i) \\
\text{where} \quad \log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 X_{i,\text{income}} \\
\text{priors:} \quad \beta_0 \sim \mathcal{N}(\mu_{\beta_0} = 0, \sigma^2_{\beta_0} = 100^2) \\
\beta_1 \sim \mathcal{N}(\mu_{\beta_1} = 0, \sigma^2_{\beta_1} = 100^2).
\end{gathered}
$$

Now, we will build a complete `posterior_stan_climate_change` with a `data` block and a likelihood in our `model` block. The vector for the response variable `climate_change` is set up as integer-type with lower and upper bounds.

Moreover we introduce a new modelling function `bernoulli_logit()`. This will indicate our logit function.

```
posterior_stan_climate_change <- "data {
int<lower=0> n;                                     // number of observatio
vector[n] income;                                   // regressor income
int<lower=0,upper=1> climate_change[n];             // setting the response
}
parameters {
real beta_0;
real beta_1;
}
model {
beta_0 ~ normal(0, 100);
beta_1 ~ normal(0, 100);
climate_change ~ bernoulli_logit(beta_0 + beta_1 * income);
}"
```

Since we have a `data` block in `posterior_stan_climate_change`, we need a `climate_dictionary` as the one below.

```
climate_dictionary <- list(
  n = nrow(pulse_training),
  income = pulse_training$income,
  climate_change = as.integer(pulse_training$climate_change)
)
```

Finally, we run an MCMC simulation with `posterior_stan_climate_change` and `climate_dictionary`. This time we use appropriate values for `iter`, `warmup`, and `thin`.

```
posterior_climate <- stan(
  model_code = posterior_stan_climate_change,
  data = climate_dictionary,
  chains = 1,
  iter = 21000,
  warmup = 1000,
  thin = 20,
  seed = 553
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 3.5e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:     1 / 21000 [  0%]  (Warmup)
Chain 1: Iteration:  1001 / 21000 [  4%]  (Sampling)
Chain 1: Iteration:  3100 / 21000 [ 14%]  (Sampling)
Chain 1: Iteration:  5200 / 21000 [ 24%]  (Sampling)
Chain 1: Iteration:  7300 / 21000 [ 34%]  (Sampling)
Chain 1: Iteration:  9400 / 21000 [ 44%]  (Sampling)
Chain 1: Iteration: 11500 / 21000 [ 54%]  (Sampling)
Chain 1: Iteration: 13600 / 21000 [ 64%]  (Sampling)
Chain 1: Iteration: 15700 / 21000 [ 74%]  (Sampling)
Chain 1: Iteration: 17800 / 21000 [ 84%]  (Sampling)
Chain 1: Iteration: 19900 / 21000 [ 94%]  (Sampling)
Chain 1: Iteration: 21000 / 21000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 0.41 seconds (Warm-up)
Chain 1:                3.764 seconds (Sampling)
Chain 1:                4.174 seconds (Total)
Chain 1:
```

We obtain the posterior summary and compare it versus the `tidy()` output from the frequentist `glm()`.

```
posterior_climate_samples <- as.data.frame(summary(posterior_climate)$summary)
posterior_climate_samples <- posterior_climate_samples[, c("mean", "sd", "2.5%
posterior_climate_samples <- posterior_climate_samples[1:2,]
posterior_climate_samples |>
  mutate(exp_mean = exp(mean), exp_2.5 = exp(`2.5%`), exp_97.5 = exp(`97.5%`))
  mutate_if(is.numeric, round, 3)
```

A data.frame: 2 × 7

	mean	sd	2.5%	97.5%	exp_mean	exp_2.5	exp_97.5
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
beta_0	0.967	0.186	0.595	1.317	2.63	1.814	3.734
beta_1	0.009	0.002	0.006	0.014	1.01	1.006	1.014

```
summary_freq_log_reg
```

A tibble: 2 × 10

term	estimate	std.error	statistic	p.value	conf.low	conf.high	exp.estima
<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dl
(Intercept)	0.965	0.184	5.248	0	0.606	1.327	2.6
income	0.009	0.002	4.376	0	0.005	0.014	1.0

When comparing both approaches, we highlight the following:

- The estimates (**maximum likelihood-based** in `glm()` versus **the posterior means in the Bayesian approach**) are practically equal!
- The same situation happens regarding variability (column `std.error` in `glm()`). **Recall `sd` in the Bayesian output refers to the standard deviation from posterior MCMC samples.**

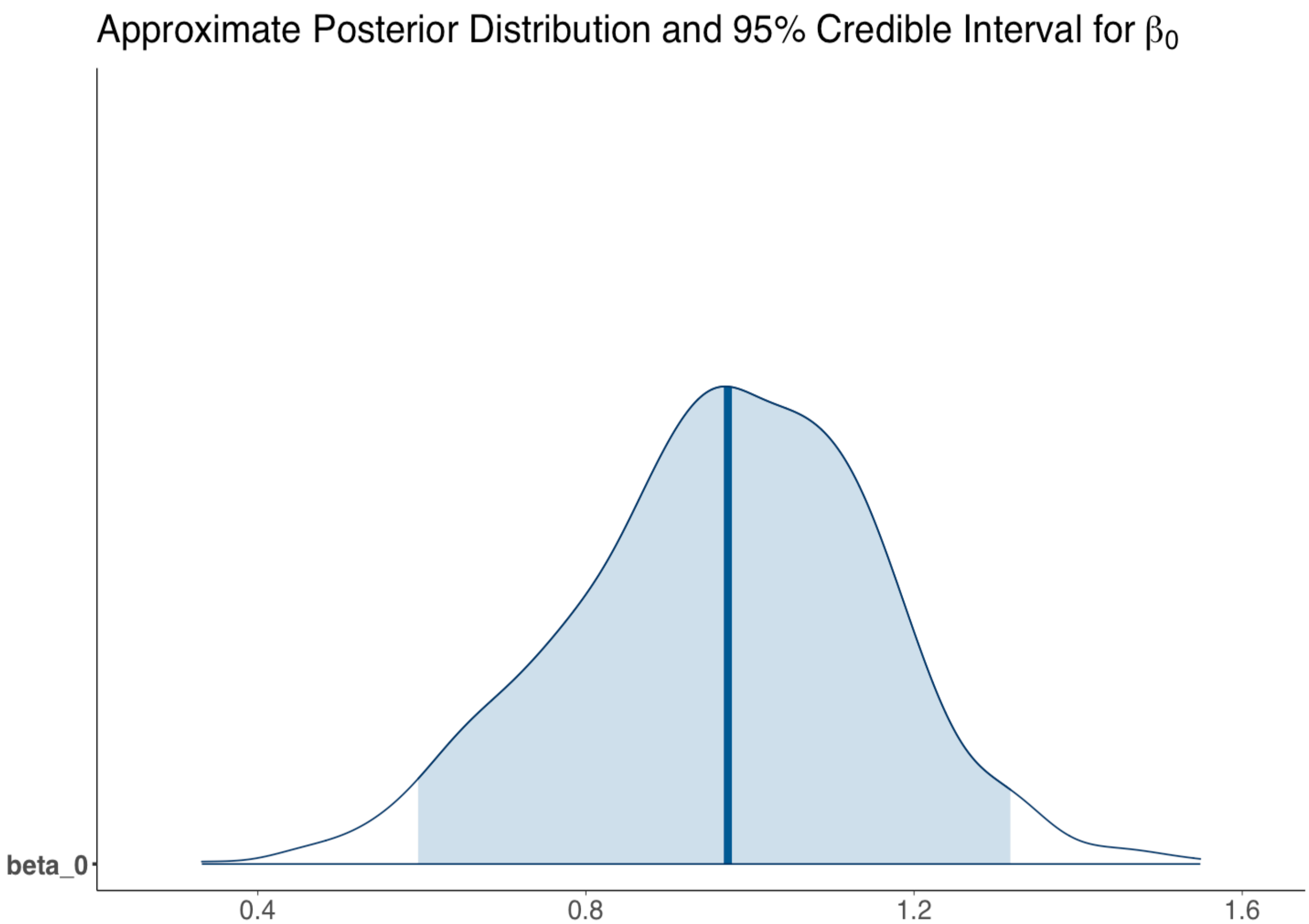
There is a big advantage in the Bayesian estimation: **we did not need to derive any maximum likelihood steps!**

Our Bayesian estimate (i.e., mean of the MCMC posterior samples) for β_1 indicates that `income` is associated with the log of the odds for believing in climate change versus not believing. **Since zero is not included in the 95% posterior credible interval, we can conclude this.**

The interpretation for the **posterior mean** of $\exp(\beta_1)$ is: "*for each one thousand USD increase in `income`, a subject is 1.01 times more likely to believe in climate change than not to*".

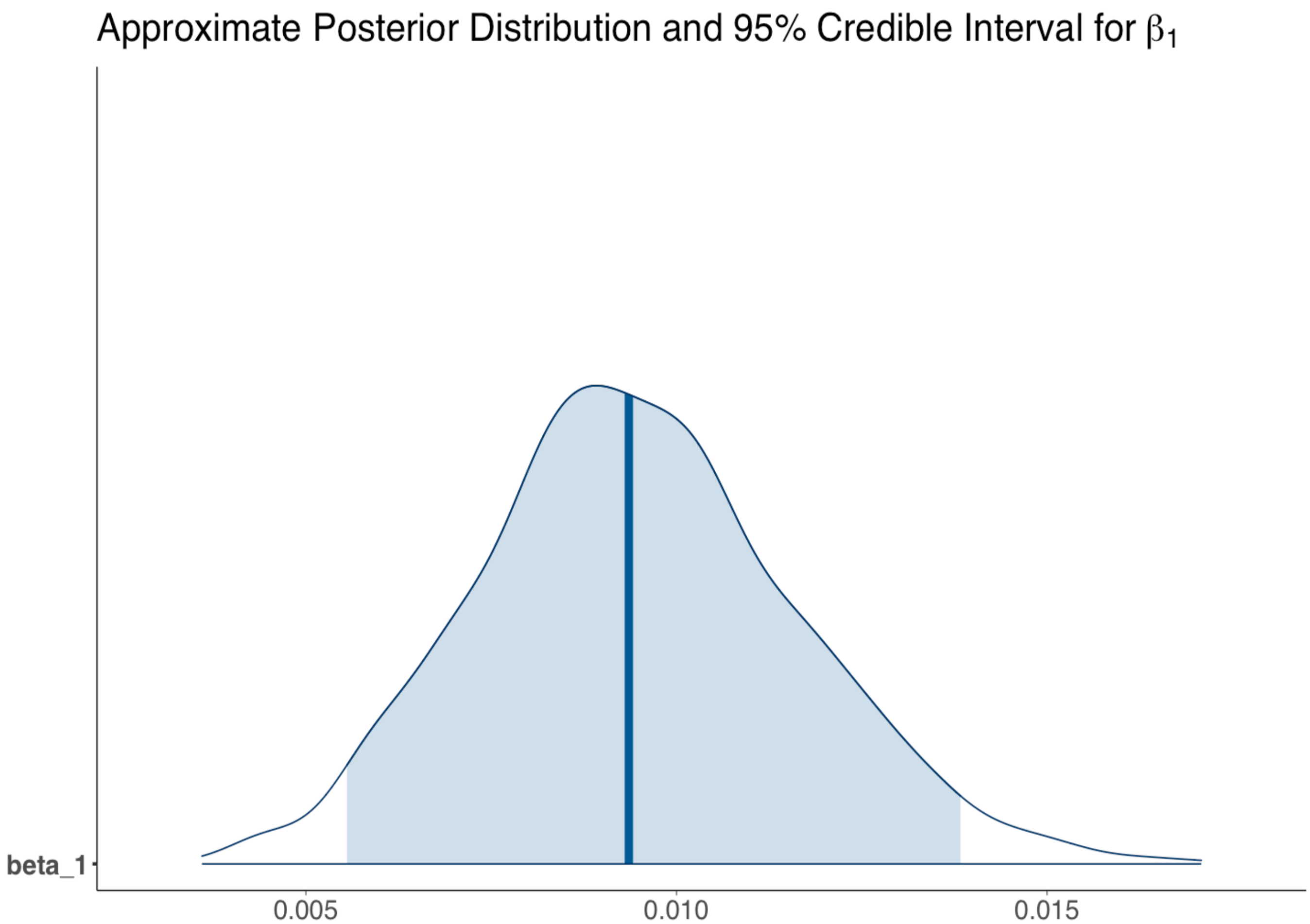
```r
beta_0_post_dist <- mcmc_areas(posterior_climate, pars = c("beta_0"), prob = 0
  theme(
    plot.title = element_text(size = 24, family = "sans"),
    axis.text.x = element_text(size = 17, angle = 0, family = "sans"),
    axis.text.y = element_text(size = 17, angle = 0, family = "sans"),
    axis.title = element_text(size = 21, family = "sans")
  ) +
  labs(title = expression("Approximate Posterior Distribution and 95% Credible
```

```r
beta_0_post_dist
```




```r
beta_1_post_dist <- mcmc_areas(posterior_climate, pars = c("beta_1"), prob = 0
  theme(
    plot.title = element_text(size = 24, family = "sans"),
    axis.text.x = element_text(size = 17, angle = 0, family = "sans"),
    axis.text.y = element_text(size = 17, angle = 0, family = "sans"),
    axis.title = element_text(size = 21, family = "sans")
  ) +
  labs(title = expression("Approximate Posterior Distribution and 95% Credible
```

```
beta_1_post_dist
```



Again, the 1000 posterior samples (pairs containing β_0 and β_1) will be use to plot 1000 different sigmoid functions:

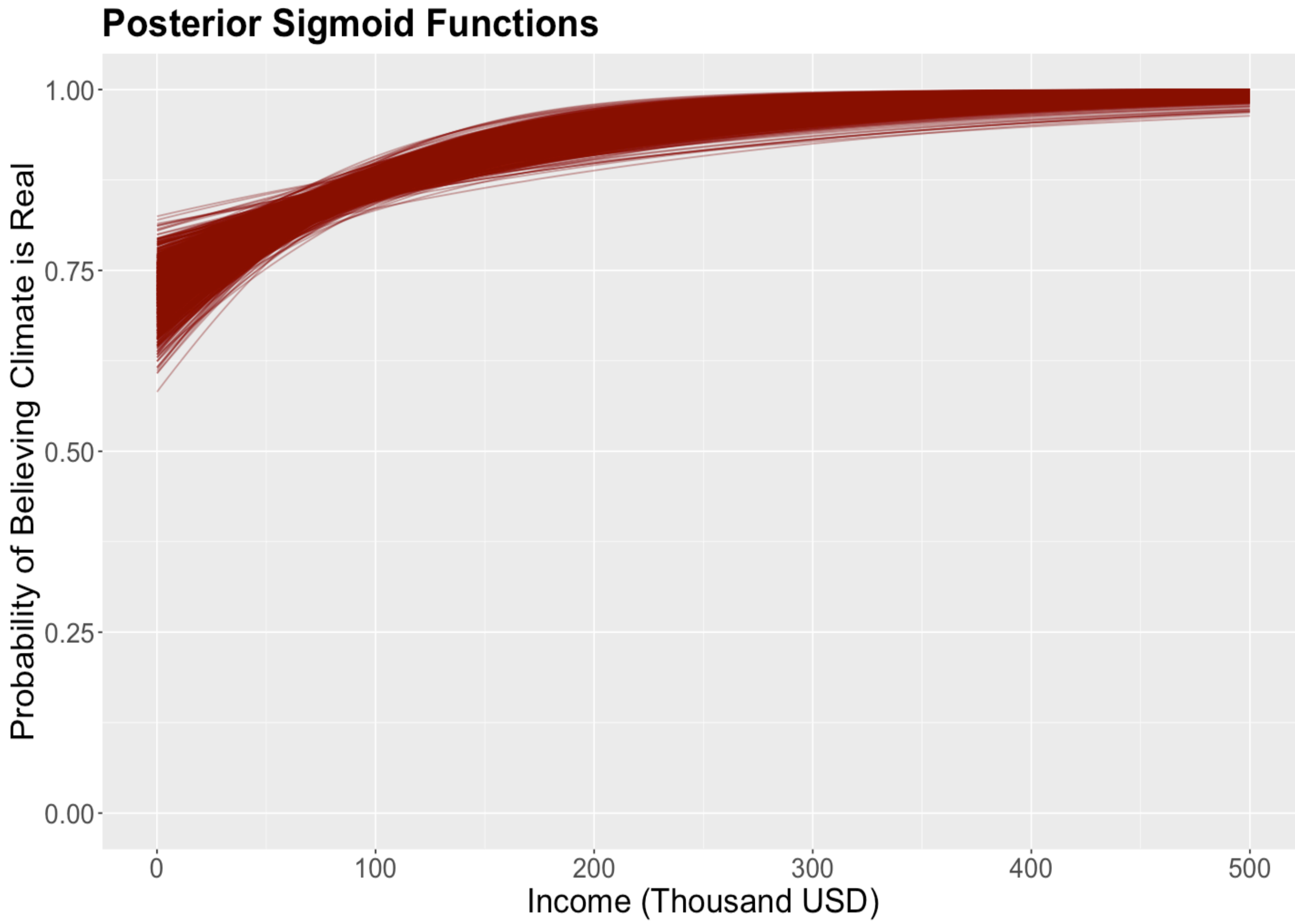
$$\pi_i = \frac{\exp\left[\beta_0 + \beta_1 X_{i,\text{income}}\right]}{1 + \exp\left[\beta_0 + \beta_1 X_{i,\text{income}}\right]} \in [0, 1].$$

The regressor income is set up as a grid from 0 to 500 thousand USD to obtain these 1000 sigmoid functions.

```
posterior_climate <- as.data.frame(posterior_climate) # posterior_climate as d

income <- seq(0, 500, length.out = 500) # Setting up grid for income

beta_0 <- as.numeric(posterior_climate[1, "beta_0"]) # First sample
beta_1 <- as.numeric(posterior_climate[1, "beta_1"]) # First sample

logistic_like <- data.frame(income = income) |> # Computing probabilities with
  mutate(values = exp(beta_0 + beta_1 * income) / (1 + exp(beta_0 + beta_1 * i

plot_logistic_like_posterior_stanfit <- ggplot(data = logistic_like, na.rm = T
  geom_line(aes(x = income, y = values), alpha = 0.1, color = "darkred", na.rm
  labs(y = "Probability of Believing Climate is Real", x = "Income (Thousand U
  ggtitle("Posterior Sigmoid Functions") +
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21)
  ) # First ggplot layer

for (j in 2:nrow(posterior_climate)) # Loop from the second sample to the last
{
  beta_0 <- as.numeric(posterior_climate[j, "beta_0"])
  beta_1 <- as.numeric(posterior_climate[j, "beta_1"])

  logistic_like <- data.frame(income = income) |>
    mutate(values = exp(beta_0 + beta_1 * income) / (1 + exp(beta_0 + beta_1 *

  plot_logistic_like_posterior_stanfit <- plot_logistic_like_posterior_stanfit
    geom_line(data = logistic_like, aes(x = income, y = values),
              alpha = 0.4, color = "darkred", na.rm = TRUE) # Next ggplot laye
}
```

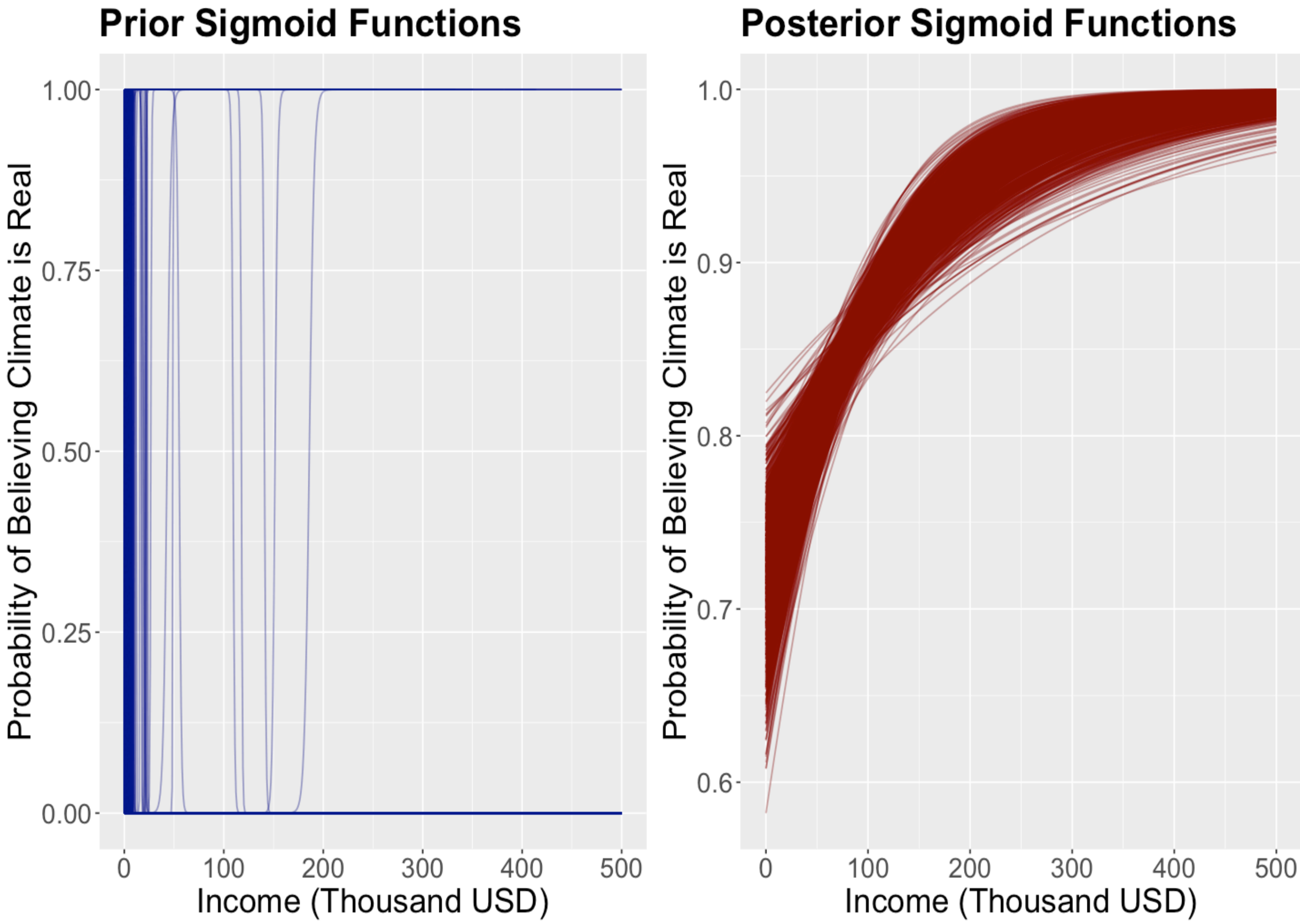
The plot below shows these 1000 sigmoid functions. **This is a posterior distribution over curves (since a curve is defined by β_0 and β_1). Hence, these red sigmoid curves are samples from this distribution.**

```
plot_logistic_like_posterior_stanfit + ylim(0, 1)
```



Now, we have a clear pattern in this posterior distribution! This is the likelihood (i.e., your data) in conjunction with the prior distribution via the Bayes' rule.

```
plot_grid(plot_logistic_like_prior_stanfit, plot_logistic_like_posterior_stanf
```



4.5. Distributions of Predicted Probabilities

Suppose you want to obtain **the prior distribution of predicted probabilities of success** (i.e., **a subject believing in climate change**) with an income of 100 thousand USD. We will use our 1000 prior samples in `prior_climate` for β_0 and β_1 to do so.

Note you will find `NaN`s in some rows since the predicted probabilities are so small that we encounter numerical errors.

```
prior_climate <- prior_climate |>
  mutate(probability_believing = exp(beta_0 + beta_1 * 100) / (1 + exp(beta_0
head(prior_climate)
```

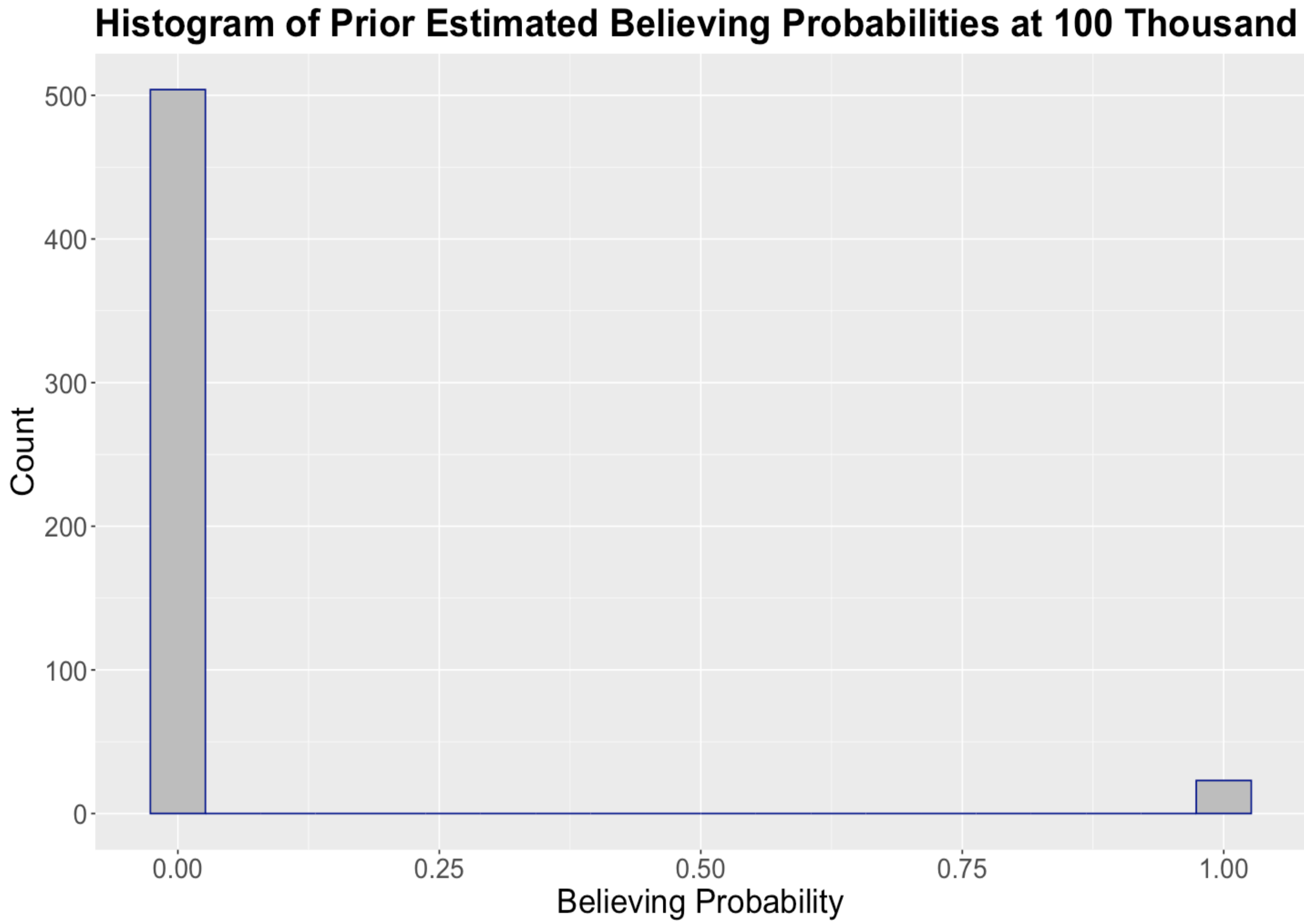
A data.frame: 6 × 4

	beta_0	beta_1	lp__	probability_believing
	<dbl>	<dbl>	<dbl>	<dbl>
1	-1.496540	130.56652	-0.8524928	NaN
2	24.444430	155.43653	-1.2379023	NaN
3	59.631513	-21.02121	-0.1998904	0
4	-4.196843	69.27654	-0.2408426	NaN
5	40.825609	28.84540	-0.1249394	NaN
6	-48.995753	84.73686	-0.4790460	NaN

We were not able to compute the 1000 prior probabilities. Nonetheless, the ones we could calculate are zero or close to zero. Therefore, this is not an informative predictive distribution for an income of 100 thousand USD.

```
hist_100_prior <- prior_climate |>
  ggplot() +
  geom_histogram(aes(probability_believing), fill = "grey", color = "darkblue"
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21)
  ) +
  ggtitle("Histogram of Prior Estimated Believing Probabilities at 100 Thousan
  labs(x = "Believing Probability", y = "Count")
```

```
hist_100_prior
```



Now, let us follow the same process with `posterior_climate`.

```
posterior_climate <- posterior_climate |>
  mutate(probability_believing = exp(beta_0 + beta_1 * 100) / (1 + exp(beta_0
head(posterior_climate)

round(mean(posterior_climate$probability_believing), 2) # Mean predicted proba
```

A data.frame: 6 × 4

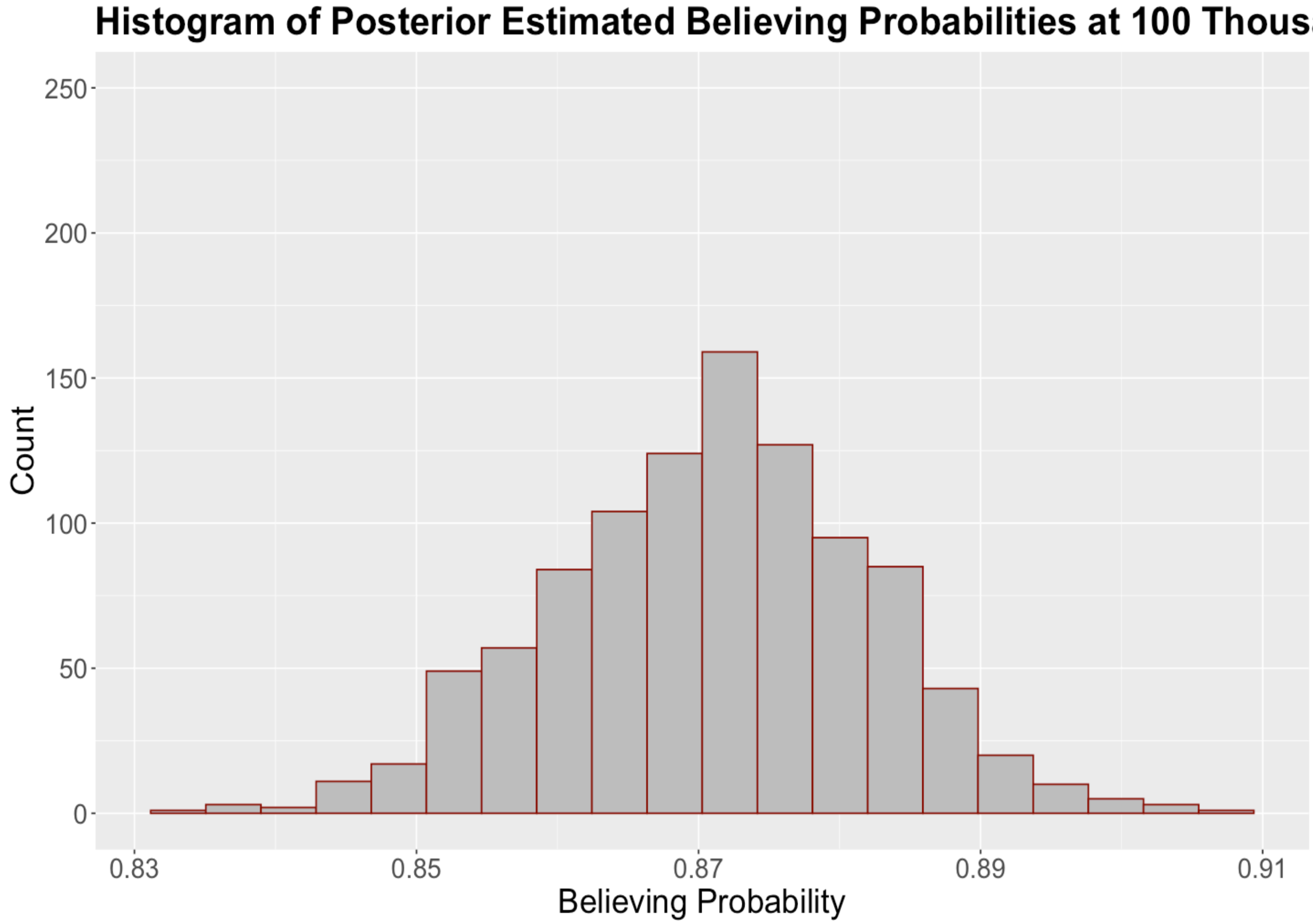
	beta_0	beta_1	lp__	probability_believing
	<dbl>	<dbl>	<dbl>	<dbl>
1	0.9704702	0.007264117	-413.4656	0.8451271
2	0.7640832	0.012281058	-412.2110	0.8799745
3	0.8504562	0.010322106	-411.5658	0.8679171
4	1.0570489	0.009500551	-411.9428	0.8815409
5	0.7329129	0.011229681	-412.2951	0.8648161
6	1.0641366	0.009201498	-411.7768	0.8791374

0.87

The predictive posterior distribution for the probability of success is more reasonable! We have a mean prediction of 0.87. **The distribution is relatively symmetric.** With an income of 100 thousand USD, the posterior predicted probability leans more towards believing in climate change.

```
hist_100_posterior <- posterior_climate |>
  ggplot() +
  geom_histogram(aes(probability_believing), fill = "grey", color = "darkred",
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21)
  ) +
  ggtitle("Histogram of Posterior Estimated Believing Probabilities at 100 Tho
  labs(x = "Believing Probability", y = "Count")
```

```
hist_100_posterior + ylim(0, 250)
```



5. Wrapping Up

- We extended Bayesian regression to a GLM approach while keeping one of the core components: the link function.
- The inferential conclusions are done as in Bayesian Normal regression.
- We can also build posterior predictive distributions in a Binary Logistic regression framework.
- This Bayesian GLM approach can be applied to other models of this class with the proper arrangements (e.g., count regression, multinomial logistic regression, etc.).