# Probability Cheatsheet

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# Complement of an Event

In general, for a given event $A$, the complement is the subset of other outcomes that do not belong to event $A$:

$$1 = P(A) + P(A^c),$$

where $^c$ means the complement (we read it as “not”).

# Conditional Independence

Random variables $X$ and $Y$ are conditionally independent given random variable $Z$ if and only if

$$P(X = x \cap Y = y \mid Z = z) = P(X = x \mid Z = z) \cdot P(Y = y \mid Z = z).$$

# Conditional Probability

In general, let $A$ and $B$ be two events of interest within the sample $S$, and $P(B) > 0$, then the conditional probability of $A$ given $B$ is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Note event $B$ is becoming the new sample space (i.e., $P(B \mid B) = 1$). The tweak here is that our original sample space $S$ has been updated to $B$.

# Covariance

Let $X$ and $Y$ be two numeric random variables; their covariance is defined as follows:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)],$$

where $\mu_X = \mathbb{E}(X)$ and $\mu_Y = \mathbb{E}(Y)$ are the respective means (or expected values) of $X$ and $Y$. After some algebraic and expected value manipulations, the above equation reduces to a more practical form to work with:

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - [\mathbb{E}(X)\mathbb{E}(Y)], \tag{43}$$

where $\mathbb{E}(XY)$ is the mean (or expected value) of the multiplication of the random variables $X$ and $Y$.

# Cumulative Distribution Function

Let $X$ be a continuous random variable with probability density function (PDF) $f_X(x)$. The cumulative distribution function (CDF) is usually denoted by $F(\cdot)$ and is defined as

$$F_X(x) = P(X \leq x).$$

We can calculate the CDF by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, \mathrm{d}t. \tag{44}$$

In order for $F_X(x)$ to be a valid CDF, the function needs to satisfy the following requirements:

1. Must never decrease.
2. It must never evalute to be $< 0$ or $> 1$.
3. $F_X(x) \rightarrow 0$ as $x \rightarrow -\infty$
4. $F_X(x) \rightarrow 1$ as $x \rightarrow \infty$.

# Entropy

Let $X$ be a random variable:

- If $X$ is discrete, with $P(X = x)$ as a probability mass function (PMF), then the entropy is defined as:

$$H(Y) = -\sum_x P(X = x) \log[P(X = x)].$$

- If $X$ is continuous, with $f_X(x)$ as a probability density function (PDF), then the entropy is defined as:

$$H(X) = -\int_x f_X(x) \log[f_X(x)] \mathrm{d}x.$$

Note that, in Statistics, the $\log(\cdot)$ notation implicates base $e$.

# Expected Value

Let $X$ be a numeric random variable. The mean $\mathbb{E}(X)$ (also known as expected value or expectation) is defined as:

- If $X$ is discrete, with $P(X = x)$ as a probability mass function (PMF), then

$$\mathbb{E}(X) = \sum_x x \cdot P(X = x). \tag{45}$$

- If $X$ is continuous, with $f_X(x)$ as a probability density function (PDF), then

$$\mathbb{E}(X) = \int_x x \cdot f_X(x) \mathrm{d}x. \tag{46}$$

In general for a function of $X$ such as $g(X)$, the expected value is defined as:

- If $X$ is discrete, with $P(X = x)$ as a PMF, then

$$\mathbb{E}\left[g(X)\right] = \sum_x g(X) \cdot P(X = x). \tag{47}$$

- If $X$ is continuous, with $f_X(x)$ as a PDF, then

$$\mathbb{E}\left[g(X)\right] = \int_x g(X) \cdot f_X(x) \mathrm{d}x. \tag{48}$$

# Inclusion-Exclusion Principle

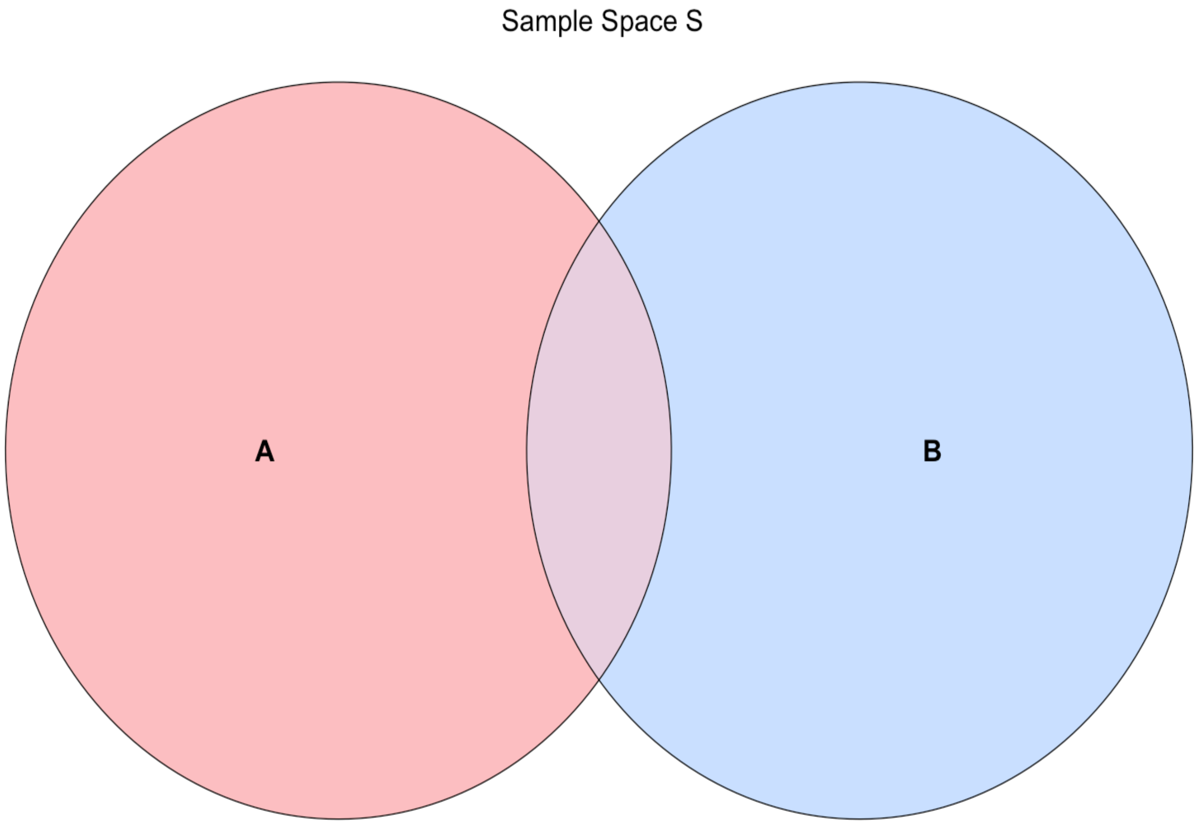
## Two Events

Let $A$ and $B$ be two events of interest in the sample space $S$. The probability of $A$ or $B$ occuring is denoted as $P(A \cup B)$, where $\cup$ means **"OR."** The Inclusion-Exclusion Principle allows us to compute this probability as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

where $P(A \cap B)$ denotes the probability of $A$ and $B$ occuring simultaneously ($\cap$ means **"AND"**).

$P(A \cup B)$ can be represented with the overall shaded area in the below Venn diagram.



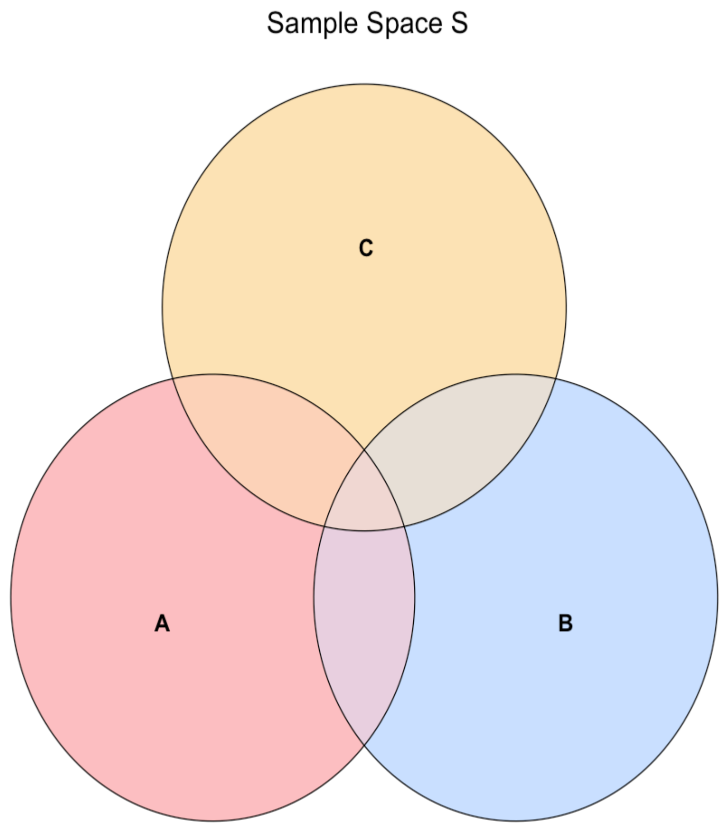

## Three Events

We can also extend this principle to three events ($A$, $B$, and $C$ in the sample space $S$):

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A$$

where $P(A \cap B \cap C)$ denotes the probability of $A$, $B$, and $C$ occuring simultaneously.

$P(A \cup B \cup C)$ can be represented with the overall shaded area in the below Venn diagram.




# Independent Events

Let $A$ and $B$ be two events of interest in the sample space $S$. These two events are independent if the occurrence of one of them does not affect the probability of the other. In probability notation, their intersection is defined as:

$$P(A \cap B) = P(A) \cdot P(B).$$

# Independence in Probability Distributions between Two Random Variables

Let $X$ and $Y$ be two independent random variables. Using their corresponding marginals, we can obtain their corresponding joint distributions as follows:

- **$X$ and $Y$ are discrete.** Let $P(X = x, Y = y)$ be the joint probability mass function (PMF) with $P(X = x)$ and $P(Y = y)$ as their marginals. Then, we define the joint PMF as:

$$P(X = x, Y = y) = P(X = x) \cdot (Y = y).$$

The term denoting a discrete joint PMF $P(X = x, Y = y)$ is equivalent to the intersection of events $P(X = x \cap Y = y)$.

- **$X$ and $Y$ are continuous.** Let $f_{X,Y}(x, y)$ be the joint probability density function (PDF) with $f_X(x)$ and $f_Y(y)$ as their marginals. Then, we define the joint PDF as:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y).$$

# Independent Random Variables

Let $X$ and $Y$ be two random variables. We say $X$ and $Y$ are independent if knowing something about one of them tells us nothing about the other. A definition of $X$ and $Y$ being independent is the following:

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y).$$

# Kendall's $\tau_K$

Let $X$ and $Y$ be two numeric random variables. Kendall's $\tau_K$ measures concordance between each pair of observations $(x_i, y_i)$ and $(x_j, y_j)$ with $i \neq j$:

- **Concordant**, which gets a positive sign, means

$$\begin{gathered} x_i < x_j \quad \text{and} \quad y_i < y_j, \\ \text{or} \\ x_i > x_j \quad \text{and} \quad y_i > y_j. \end{gathered}$$

- **Discordant**, which gets a negative sign, means

$$\begin{gathered} x_i < x_j \quad \text{and} \quad y_i > y_j, \\ \text{or} \\ x_i > x_j \quad \text{and} \quad y_i < y_j. \end{gathered}$$

Mathematically, we can set it up as:

$$\tau_K = \frac{\text{Number of concordant pairs} - \text{Number of discordant pairs}}{\binom{n}{2}},$$

with the "true" Kendall's $\tau_K$ value obtained by sending $n \to \infty$. Here, $n$ is the sample size (i.e., the number of data points). Note that:

$$-1 \leq \tau_K \leq 1.$$

# Law of Total Expectation

Let $X$ and $Y$ be two numeric random variables. Generally, a marginal mean $\mathbb{E}_Y(Y)$ can be computed from the conditional means $\mathbb{E}_Y(Y \mid X = x)$ and the probabilities of the conditioning variables $P(X = x)$:

$$\mathbb{E}_Y(Y) = \sum_x \mathbb{E}_Y(Y \mid X = x) \cdot P(X = x). \tag{49}$$

Or, it can also be written as:

$$\mathbb{E}_Y(Y) = \mathbb{E}_X[\mathbb{E}_Y(Y \mid X)].$$

Also, the previous result in Equation (49) extends to probabilities:

$$P(Y = y \cap X = x) = P(Y = y \mid X = x) \cdot P(X = x).$$

# Linearity of Expectations

If $a$ and $b$ are constants, with $X$ and $Y$ as numeric random variables, then we can obtain the expected value of the following expressions as:

$$\begin{gathered}\mathbb{E}(aX) = a\mathbb{E}(X) \\ \mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) \\ \mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y).\end{gathered}$$

# Linearity of Variances with Two Independent Random Variables

If $a$ and $b$ are constants, with $X$ and $Y$ as independent numeric random variables, then we can obtain the variance of the following expressions as:

$$\begin{gathered}\text{Var}(aX) = a^2\text{Var}(X) \\ \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \\ \text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y).\end{gathered}$$

# Marginal (Unconditional) Probability

In general, the probability of an event $A$ occurring is denoted as $P(A)$ and is defined as

$$P(A) = \frac{\text{Number of times event } A \text{ is observed}}{\text{Total number of events observed}}.$$

# Median

Let $X$ be a numeric random variable. The median $\text{M}(X)$ is the outcome for which there is a 50-50 chance of seeing a greater or lesser value. So, its distribution-based definition satisfies

$$P[X \leq \mathrm{M}(X)] = 0.5.$$

# Mode

Let $X$ be a random variable:

- If $X$ is discrete, with $P(X = x)$ as a probability mass function (PMF), then the mode is the outcome having the highest probability.
- If $X$ is continuous, with $f_X(x)$ as a probability density function (PDF), then the mode is the outcome having the highest density. That is:

$$\text{Mode} = \arg\max_x f_X(x).$$

# Mutual Information

The mutual information between two random variables $X$ and $Y$ is defined as

$$H(X, Y) = \sum_x \sum_y P(X = x \cap Y = y) \log \left[ \frac{P(X = x \cap Y = y)}{P(X = x) \cdot P(Y = y)} \right].$$

# Mutually Exclusive (or Disjoint) Events

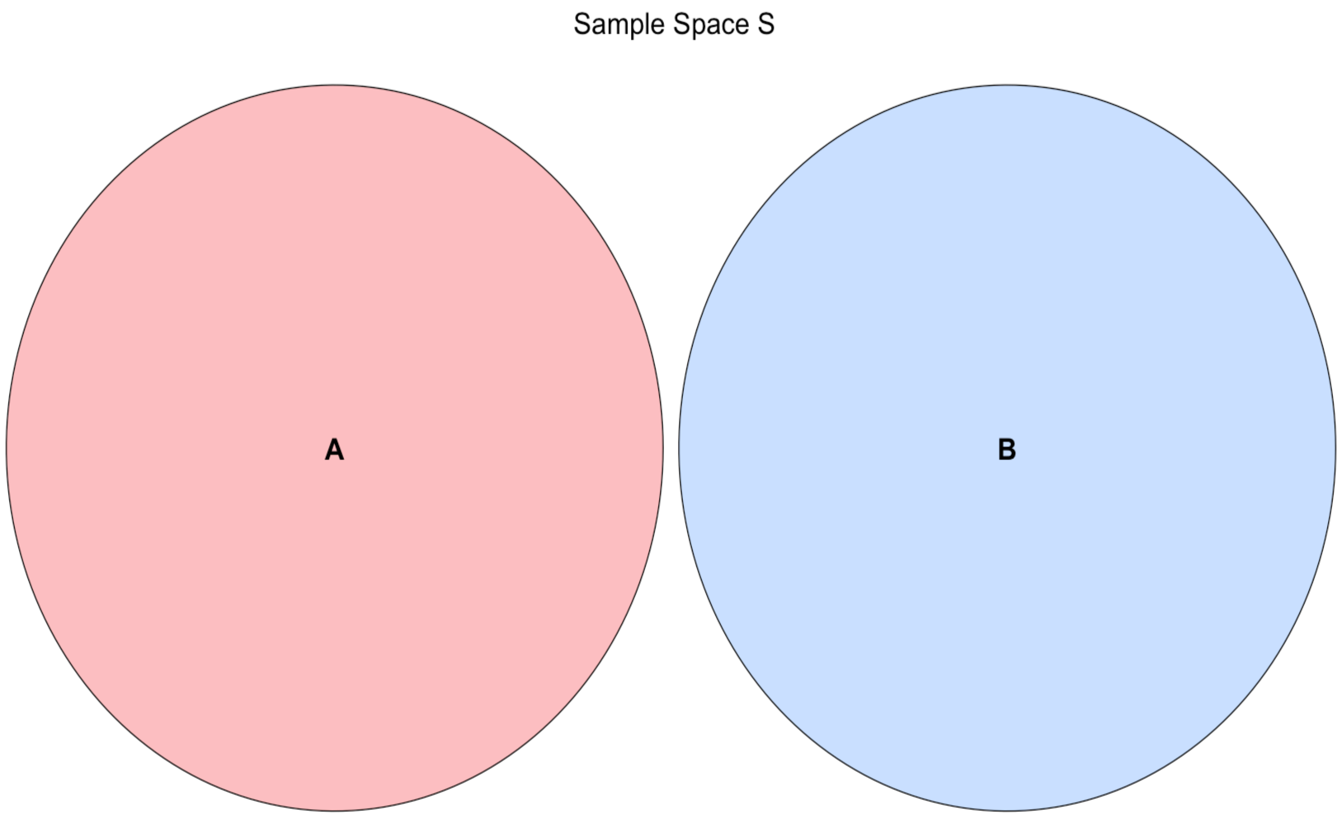
Let $A$ and $B$ be two events of interest in the sample space $S$. These events are mutually exclusive (or disjoint) if they cannot happen at the same time in the sample space $S$. Thus, in probability notation, their intersection will be:

$$P(A \cap B) = 0.$$

Therefore, by the Inclusion-Exclusion Principle, the union of these two events can be obtained as follows:

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - \underbrace{P(A \cap B)}_{0} \\
&= P(A) + P(B).
\end{aligned}
$$

These two events are shown in the below Venn diagram.




# Odds

Let $p$ be the probability of an event of interest $A$. The odds $o$ is the ratio of the probability of the event $A$ to the probability of the non-event $A$:

$$o = \frac{p}{1 - p}.$$

In plain words, the odds will tell how many times event $A$ is more likely compared to how unlikely it is.

# Pearson's Correlation

Let $X$ and $Y$ be two numeric random variables, whose respective variances are defined by Equation (51), with a covariance defined as in Equation (43). Pearson's correlation standardizes the distances according to the standard deviations $\sigma_X$ and $\sigma_Y$ of $X$ and $Y$, respectively. It is defined as:

$$\begin{aligned} \rho_{XY} = \operatorname{Corr}(X, Y) &= \mathbb{E}\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right] \\ &= \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}. \end{aligned} \tag{50}$$

As a result of the above equation, it turns out that

$$-1 \leq \rho_{XY} \leq 1.$$

# Probability of a Continuous Random Varible $X$ Being between $a$ and $b$

For a continuous random variable $X$ with probability density function (PDF) $f_X(x)$, the probability of $X$ being between $a$ and $b$ is

$$P(a \leq X \leq b) = \int_a^b f_X(x) \mathrm{d}x.$$

We can connect the dots with our new definition of a cumulative distribution function (CDF) from Equation (44). First,

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a)$$

because if $X \leq b$ but not $\leq a$ then it must be that $a \leq X \leq b$. But now we can write these two terms using the CDF:

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a).$$

Now, plugging in the definition of the CDF as the integral of the PDF,

$$P(a \leq X \leq b) = \int_{-\infty}^{b} f_X(x)\,\mathrm{d}x - \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x = \int_{a}^{b} f_X(x)\,\mathrm{d}x.$$

# Properties of the Bivariate Gaussian or Normal Distribution

Let $X$ and $Y$ be part of a bivariate Gaussian or Normal distribution with means $-\infty < \mu_X < \infty$ and $-\infty < \mu_Y < \infty$, variances $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$, and correlation coefficient $-1 \leq \rho_{XY} \leq 1$.

This bivariate Gaussian or Normal distribution has the following properties:

1. **Marginal distributions are Gaussian.** The marginal distribution of a subset of variables can be obtained by just taking the relevant subset of means, and the relevant subset of the covariance matrix.
2. **Linear combinations are Gaussian.** This is actually by definition. If $(X, Y)$ have a bivariate Gaussian or Normal distribution with marginal means $\mu_X$ and $\mu_Y$ along with marginal variances $\sigma_X^2$ and $\sigma_Y^2$ and covariance $\sigma_{XY}$; then $Z = aX + bY + c$ with constants $a, b, c$ is Gaussian. If we want to find the mean and variance of $Z$, we apply the linearity of expectations and variance rules:

$$\begin{aligned} \mathbb{E}(Z) &= \mathbb{E}(aX + bY + c) \\ &= \mathbb{E}(aX) + \mathbb{E}(bY) + \mathbb{E}(c) \\ &= a\mathbb{E}(X) + b\mathbb{E}(Y) + c \\ &= a\mu_X + b\mu_Y + c. \end{aligned}$$

$$\begin{aligned} \operatorname{Var}(Z) &= \operatorname{Var}(aX + bY + c) \\ &= \operatorname{Var}(aX) + \operatorname{Var}(bY) + \operatorname{Var}(c) + 2\operatorname{Cov}(aX, bY) \\ &= a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y) + 0 + 2ab\operatorname{Cov}(X, Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}. \end{aligned}$$

3. **Conditional distributions are Gaussian.** If $(X, Y)$ have a bivariate Gaussian or Normal distribution with marginal means $\mu_X$ and $\mu_Y$ along with marginal variances $\sigma_X^2$ and $\sigma_Y^2$

and covariance $\sigma_{XY}$; then the distribution of $Y$ given that $X = x$ is also Gaussian. Its distribution is

$$Y \mid X = x \sim \mathcal{N}\left(\mu_{Y|X=x} = \mu_Y + \frac{\sigma_Y}{\sigma_X}\rho_{XY}(x - \mu_X), \sigma^2_{Y|X=x} = \ (1 - \rho^2_{XY})\sigma^2_Y\right).$$

# Quantile

Let $X$ be a numeric random variable. A $p$-quantile $Q(p)$ is the outcome with a probability $p$ of getting a smaller outcome. So, its distribution-based definition satisfies

$$P[X \leq Q(p)] = p.$$

# Quantile Function

Let $X$ be a continuous random variable. The quantile function $Q(\cdot)$ takes a probability $p$ and maps it to the $p$-quantile. It turns out that this is the inverse of the cumulative distribution function (CDF) (44):

$$Q(p) = F^{-1}(p).$$

Note that this function does not exist outside of $0 \leq p \leq 1$. This is unlike the other functions (density, CDF, and survival function) which exist on all real numbers.

# Skewness

Let $X$ be a numeric random variable:

- If $X$ is discrete, with $P(X = x)$ as a probability mass function (PMF), then skewness can be defined as

$$\text{Skewness}(X) = \mathbb{E}\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^3\right] = \sum_x \left(\frac{x - \mu_X}{\sigma_X}\right)^3 \cdot P(X = x).$$

- If $X$ is continuous, with $f_X(x)$ as a probability density function (PDF), then

$$\text{Skewness}(X) = \mathbb{E}\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^3\right] = \int_x \left(\frac{x - \mu_X}{\sigma_X}\right)^3 \cdot f_X(x) \mathrm{d}x.$$

where $\mu_X = \mathbb{E}(X)$ as in Equations (45) if $X$ is discrete and (46) if $X$ is continuous. On the other hand, $\sigma_X = \text{SD}(X)$ as in Equation (52).

# Survival Function

Let $X$ be a continuous random variable. The survival function $S(\cdot)$ is the cumulative distribution function (CDF) (44) *"flipped upside down"*. For this random variable $X$, the survival function is defined as

$$S_X(x) = P(X > x) = 1 - F_X(x).$$

# Variance

Let $X$ be a numeric random variable. The variance, either for a discrete or continuous random variable, is defined as

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}\{[X - \mathbb{E}(X)]^2\} \\ &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2. \end{aligned} \tag{51}$$

For the continuous case with $f_X(x)$ as a probability density function (PDF), an alternative definition of the variance is

$$\text{Var}(X) = \mathbb{E}[(X - \mu_X)^2] = \int_x (x - \mu_X)^2 f_X(x) \mathrm{d}x.$$

The term $\mu_X$ is equal to $\mathbb{E}(X)$ from Equation (46).

Finally, either for a discrete or continuous random variable, the standard deviation is the square root of the variance:

$$\mathrm{SD}\left[\mathrm{Var}(X)\right] = \sqrt{\mathrm{Var}(X)}. \tag{52}$$

The above measure is more practical because it is on the same scale as the outcome, unlike the variance.

# Variance of a Sum Involving Two Non-Independent Random Variables

Suppose $X$ and $Y$ are not independent numeric random variables. Therefore, the variance of their sum is:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y). \tag{53}$$

Furthermore, if $X$ and $Y$ are independent, then

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y). \tag{54}$$

Therefore, using Equation (54), the sum (53) becomes:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$