Lecture 4 - Linear Mixed-Effects Models

Contents

Today's Learning Goals

By the end of this lecture, you should be able to:

- Identify the model assumptions in a linear Mixed-Effects model.
- Associate a term (or combination of terms) in a Mixed-Effects model with the following quantities:
  - Fixed effect estimates.
  - Variances of the random effects.
  - Regression coefficients for each group and population.
  - Predictions on existing groups and a new group.
- Fit a linear Mixed-Effects model in `R`, and extract estimates of the above quantities.
- Identify the consequences of fitting a fixed-effects linear regression model when there are groups, whether a slope parameter is pooled or fit separately per group.
- Explain the difference between the distributional assumption on the random effects and the fixed effects estimates' sampling distribution.

Loading Libraries

```
options(repr.matrix.max.rows = 6)
library(AER)
library(tidyverse)
library(broom)
library(nlme)
library(lme4)
library(lmerTest)
```

1. Linear Fixed-Effects Model

So far, we have been working with regression models fitted with a **training set** of n **independent elements**.

Let us start with different modelling techniques from the ones you learned in **DSCI 561**. **Under a frequentist and classical Ordinary Least-squares (OLS) paradigm**, given a set of k regressors $X_{i,j}$ and a continuous response Y_i, we fit a model

$$Y_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \ldots + \beta_k X_{i,k} + \varepsilon_i \quad \text{for } i = 1, \ldots, n.$$

Note parameters $\beta_0, \beta_1, \ldots, \beta_k$ are **fixed** and **constant** for **all the observed values** $(x_{i,1}, \ldots, x_{i,k}, y_i)$.

> **Definition of Fixed Effects**
>
> OLS parameters $\beta_0, \beta_1, \ldots, \beta_k$ are called **fixed effects** in Regression Analysis. In an inferential framework, it is our interest to evaluate whether they are statistically significant for the response.

To illustrate today's topic, let us introduce a suitable dataset.

1.1. Grunfeld's Investment Dataset

Consider the following example: **to study how gross investment depends on the firm's value and capital stock, [Grunfeld (1958)](#) collected data from eleven different American companies over the years 1935-1954.**

The Grunfeld's Investment Dataset

The data frame `Grunfeld`, from package `AER`, contains 220 observations from a balanced panel of 11 **sampled American firms** from 1935 to 1954 (20 observations per `firm`). The dataset includes a continuous response `investment` subject to two explanatory variables, `market_value` and `capital`.

Definition of Balanced Panel

In statistical jargon, a **panel** refers to a dataset in which each **individual** (e.g., a firm) is observed within a timeframe. Furthermore, the term **balanced** indicates that we have the same number of observations per individual.

Firstly, we will load the data which has the following variables:

- `investment`: the gross investment in millions of American dollars (additions to plant and equipment along with maintenance), **a continuous response**.
- `market_value`: the firm's market value in millions of American dollars, **a continuous explanatory variable**.
- `capital`: stock of plant and equipment in millions of American dollars, **a continuous explanatory variable**.
- `firm`: **a nominal explanatory variable** with eleven levels indicating the firm (`General Motors`, `US Steel`, `General Electric`, `Chrysler`, `Atlantic Refining`, `IBM`, `Union Oil`, `Westinghouse`, `Goodyear`, `Diamond Match`, and `American Steel`).
- `year`: the year of the observation (*it will not be used in our analysis, but take it into account for our next in-class question*).

The code below only renames some columns from the original dataset.

```
data(Grunfeld)
Grunfeld <- Grunfeld |>
  rename(investment = invest, market_value = value)
Grunfeld
```

A data.frame: 220 × 5

	investment	market_value	capital	firm	year
	<dbl>	<dbl>	<dbl>	<fct>	<int>
1	317.6	3078.5	2.8	General Motors	1935
2	391.8	4661.7	52.6	General Motors	1936
3	410.6	5387.1	156.9	General Motors	1937
⋮	⋮	⋮	⋮	⋮	⋮
218	7.329	57.616	78.631	American Steel	1952
219	9.020	57.441	80.215	American Steel	1953
220	6.281	47.165	83.788	American Steel	1954

We will start with an in-class question via **iClicker**.

Exercise 12

What class of **data hierarchy** do you observe in this dataset? Do you expect any class of **correlation** within the data points?

A. Yes, we have a data hierarchy with one level: `firm`. Still, there will not be a correlation among subsets of data points.

B. Yes, we have a data hierarchy with one level: `firm`. Hence, there will be a correlation among subsets of data points.

C. There is no data hierarchy at all. All observations in the training set are independent.

D. Yes, we have a data hierarchy with two levels: `firm` (**level 1**) and the corresponding yearly observations (**level 2**). Hence, there will be a correlation among subsets of data points.

Solution to Exercise 12

We have **two hierarchical levels** in the data:

- Grunfeld first sampled the eleven firms (level 1).
- Then, for each firm, Grunfeld collected 20 data points (level 2).

Therefore, **we would expect data within a company to be correlated under the context of our data collection**.

Note we refer to **data hierarchy** when there are **sampling levels** on which we collected the data. For instance, in a Canadian political poll, data might be collected via a sampling scheme starting by province and then by electoral district. Then, your data hierarchy has two levels before getting to your corresponding **observational unit** (e.g., a potential voter).

Main Statistical Inquiries

We are interested in assessing the association of gross `investment` with `market_value` and `capital` **in the population of American firms**. Then, how can we fit a linear model to this data?

1.2. Exploratory Data Analysis

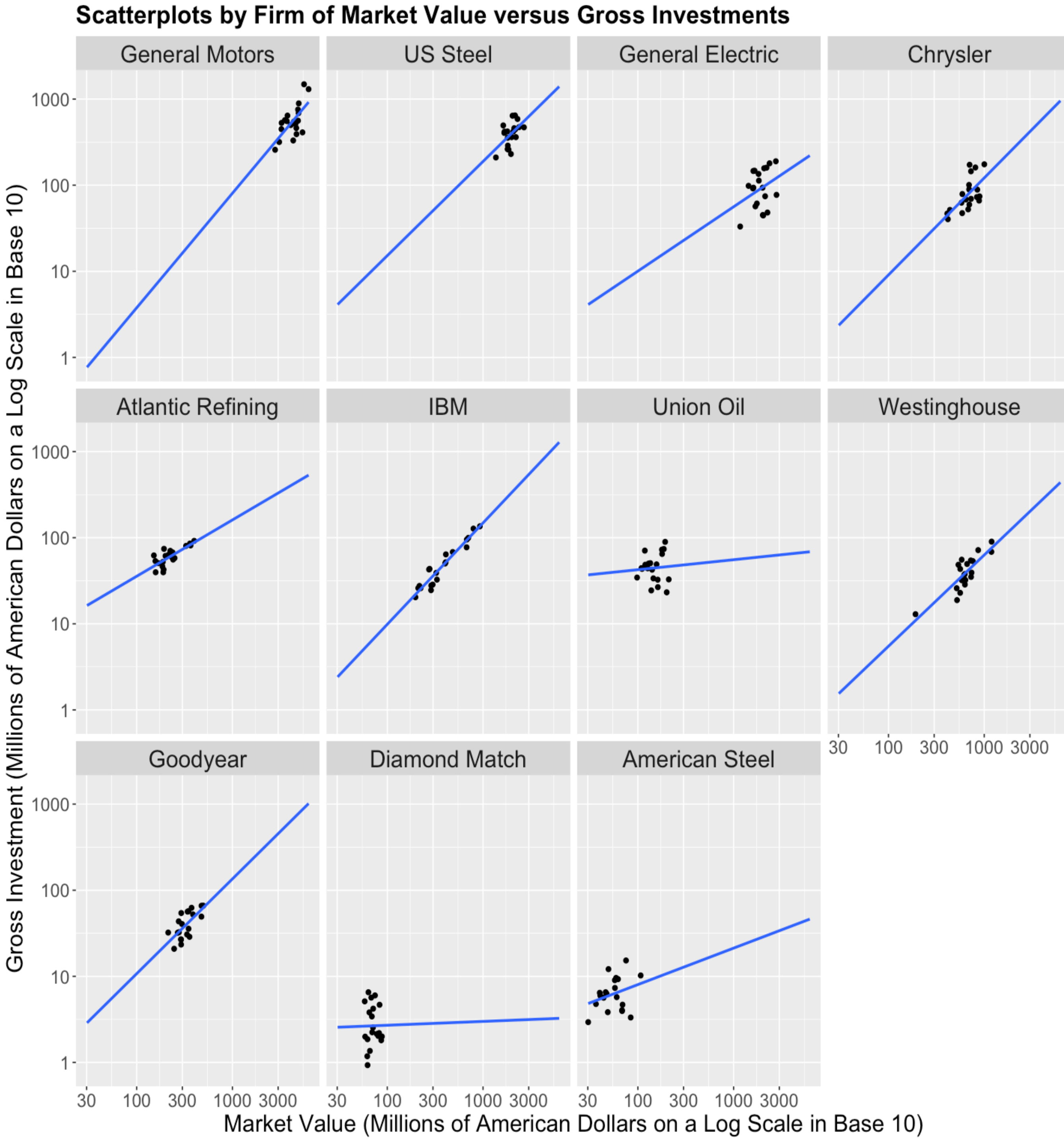
Let us plot the $n = 220$ data points of `investment` versus `market_value` and `capital` but **facetted** by `firm` and use `geom_smooth()` to fit sub-models by `firm`.

Important

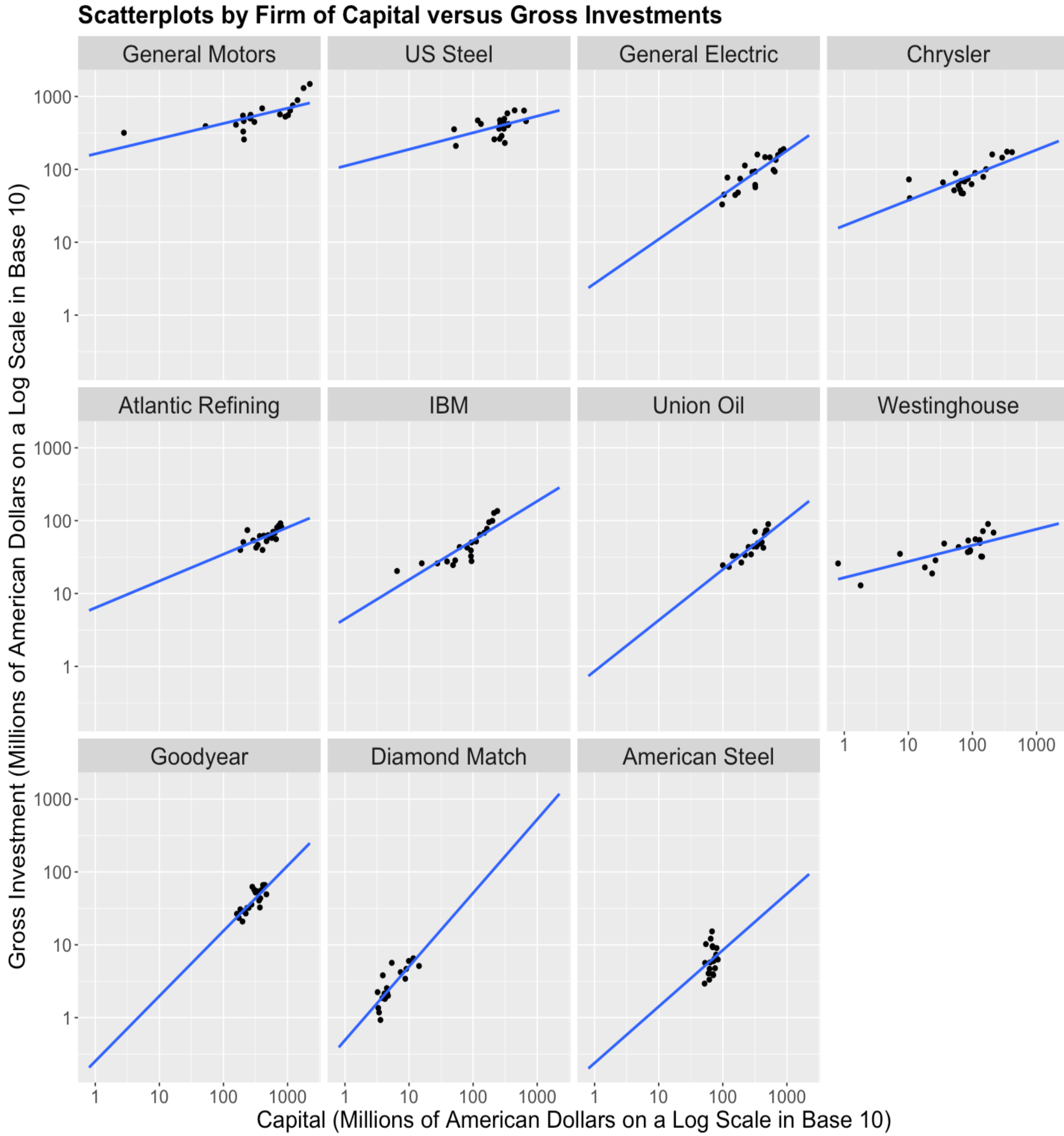
Only for plotting, we will transform both x and y-axes on the logarithmic scale in base 10 (`trans = "log10"`). This allows us to compare those firms under dissimilar market values, capital, and gross investments.

```
scatterplots_firm_market_value <- Grunfeld |>
  ggplot(aes(market_value, investment)) +
  geom_point() +
  geom_smooth(method = "lm", fullrange = TRUE, formula = y ~ x, se = FALSE) +
  facet_wrap(~firm) +
  labs(x = "Market Value (Millions of American Dollars on a Log Scale in Base
       y = "Gross Investment (Millions of American Dollars on a Log Scale in B
  ggtitle("Scatterplots by Firm of Market Value versus Gross Investments") +
  theme(
    plot.title = element_text(size = 19, face = "bold"),
    axis.text = element_text(size = 14),
    axis.title = element_text(size = 18),
    strip.text.x = element_text(size = 18),
  ) +
  scale_x_continuous(trans = "log10") +
  scale_y_continuous(trans = "log10")

scatterplots_firm_capital <- Grunfeld |>
  ggplot(aes(capital, investment)) +
  geom_point() +
  geom_smooth(method = "lm", fullrange = TRUE, formula = y ~ x, se = FALSE) +
  facet_wrap(~firm) +
  labs(x = "Capital (Millions of American Dollars on a Log Scale in Base 10)",
       y = "Gross Investment (Millions of American Dollars on a Log Scale in B
  ggtitle("Scatterplots by Firm of Capital versus Gross Investments") +
  theme(
    plot.title = element_text(size = 19, face = "bold"),
    axis.text = element_text(size = 14),
    axis.title = element_text(size = 18),
    strip.text.x = element_text(size = 18),
  ) +
  scale_x_continuous(trans = "log10") +
  scale_y_continuous(trans = "log10")
```

```
options(repr.plot.height = 12, repr.plot.width = 12)

scatterplots_firm_market_value
```




```
scatterplots_firm_capital
```



Exercise 13

What do you observe in the plots above?

Solution to Exercise 13

- The estimated intercepts vary per firm.
- The effect of the `firm` is clear in how the data is distributed. Some firms have more variability than others.
- There is variability in the estimated slopes across all firms.

1.3. OLS Modelling Framework

Before digging into today's new modelling approach, let us try a classical approach via OLS for comparative purposes.

Important

Always keep in mind the main statistical inquiry when taking any given modelling approach.

1.3.1. Regression Alternatives

Based on what you have seen via OLS in **DSCI 561**, there might be four possible approaches:

- Take the average for each `firm`, and fit an OLS regression on the averages. **This is not an ideal approach**, since we would lose valuable data points if we summarize them using a summary statistic of central tendency such as the average. Recall the frequentist paradigm: the more data points, the better! Therefore, **we will not use this approach in this lecture**.
- **We could ignore `firm`**, and fit an OLS regression with the other columns in the training set. Nonetheless, **this is not an ideal approach** since we will not be properly defining the corresponding systematic component of our population of interest: American firms.
- **Allow different intercepts for each `firm`**. This might be an approach worth taking using OLS modelling techniques.
- **Allow a different slope and intercept for each `firm`** (i.e., and **interaction model**!). This might be an approach worth taking using OLS modelling techniques.

1.3.2. OLS Ignoring Firm

Let us start with this basic OLS model to warm up our modelling skills regarding setting up equations. Suppose we ignore factor `firm`. Then, we will estimate an OLS regression with `investment` as a response to `market_value` and `capital` as regressors.

The regression equation for the ith sampled observation will be:

$$\texttt{investment}_i = \beta_0 + \beta_1 \texttt{marketValue}_i + \beta_2 \texttt{capital}_i + \varepsilon_i \quad \text{for } i = 1, \ldots, 220.$$

We use `lm()` and `glance()` to get the corresponding outputs.

```
ordinary_model <- lm(formula = investment ~ market_value + capital, data = Gru
tidy(ordinary_model) |>
  mutate_if(is.numeric, round, 4)
glance(ordinary_model) |>
  mutate_if(is.numeric, round, 4)
```

A tibble: 3 × 5

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	-38.4101	8.4134	-4.5654	0
market_value	0.1145	0.0055	20.7534	0
capital	0.2275	0.0242	9.3904	0

A tibble: 1 × 12

r.squared	adj.r.squared	sigma	statistic	p.value	df	logLik	AIC
<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
0.8179	0.8162	90.2806	487.284	0	2	-1301.299	2610.598

With $\alpha = 0.05$, we have evidence to state see that `market_value` and `capital` are statistically associated to `investment` since the p-values $< .001$.

1.3.3. OLS Regression with Varying Intercept

Now, let us estimate another OLS regression model with `investment` as a response to `market_value` and `capital` as regressors **but with varying intercepts by each `firm`**.

We will do this with the `lm()` function by adding `- 1` on the right-hand side of the argument `formula`. This `- 1` will allow the baseline `firm` to have its intercept (i.e., renaming `(Intercept)` in column `estimate` with `firmCompanyName`). In this case, `General Motors` is the **baseline** (**it appears on the left-had side of the `levels()` output**).

```
levels(Grunfeld$firm)
```

'General Motors' · 'US Steel' · 'General Electric' · 'Chrysler' · 'Atlantic Refining' · 'IBM' · 'Union Oil' · 'Westinghouse' · 'Goodyear' · 'Diamond Match' · 'American Steel'

```
options(repr.matrix.max.rows = 33)

model_varying_intercept <- lm(formula = investment ~ market_value + capital +
tidy(model_varying_intercept) |>
  mutate_if(is.numeric, round, 4)
glance(model_varying_intercept) |>
  mutate_if(is.numeric, round, 4)
```

A tibble: 13 × 5

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
market_value	0.1101	0.0113	9.7461	0.0000
capital	0.3100	0.0165	18.7439	0.0000
firmGeneral Motors	-70.2991	47.3754	-1.4839	0.1394
firmUS Steel	101.9047	23.7687	4.2873	0.0000
firmGeneral Electric	-235.5694	23.2861	-10.1163	0.0000
firmChrysler	-27.8091	13.4186	-2.0724	0.0395
firmAtlantic Refining	-114.6025	13.5025	-8.4875	0.0000
firmIBM	-23.1602	12.0759	-1.9179	0.0565
firmUnion Oil	-66.5442	12.2420	-5.4357	0.0000
firmWestinghouse	-57.5465	13.3379	-4.3145	0.0000
firmGoodyear	-87.2145	12.2887	-7.0971	0.0000
firmDiamond Match	-6.5680	11.2736	-0.5826	0.5608
firmAmerican Steel	-20.5782	11.2978	-1.8214	0.0700

A tibble: 1 × 12

r.squared	adj.r.squared	sigma	statistic	p.value	df	logLik	AIC
<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
0.9616	0.9591	50.2995	398.2336	0	13	-1167.426	2362.851

By checking the `adj.r.squared`, we see that `model_varying_intercept` has a larger value (0.96) than `ordinary_model` (0.82) (i.e., the first fitted model without `firm` as a regressor). **This indicates that a model with estimated intercepts by firm fits the data better than a model without taking `firm` into account (at least by looking at the metrics!).**

Note that `model_varying_intercept` is equivalent to just fitting the OLS model using `formula = investment ~ market_value + capital + firm`. However, in this `primary_OLS`, `General Motors` estimated intercept would be called `(Intercept)` whereas the remaining ten firms will have their corresponding estimated intercepts as `(Intercept) + firmCompany` as in the `primary_OLS`. Moreover, the regression parameter estimates for `market_value` and `capital` (along with their corresponding standard errors, test statistics, and p-values) stay the same.

> **Caution**
>
> Even though the intercept estimates by `firm` can be easily computed using column `estimate` from `primary_OLS`; standard errors, test statistics, and p-values will be different given the model parametrization in the regression equation.

```
primary_OLS <- lm(investment ~ market_value + capital + firm , data = Grunfeld
tidy(primary_OLS) |>
  mutate_if(is.numeric, round, 4)
```

A tibble: 13 × 5

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	-70.2991	47.3754	-1.4839	0.1394
market_value	0.1101	0.0113	9.7461	0.0000
capital	0.3100	0.0165	18.7439	0.0000
firmUS Steel	172.2038	29.7001	5.7981	0.0000
firmGeneral Electric	-165.2703	30.2853	-5.4571	0.0000
firmChrysler	42.4900	41.8499	1.0153	0.3112
firmAtlantic Refining	-44.3034	48.1222	-0.9206	0.3583
firmIBM	47.1389	44.6144	1.0566	0.2919
firmUnion Oil	3.7548	48.1918	0.0779	0.9380
firmWestinghouse	12.7526	41.9860	0.3037	0.7616
firmGoodyear	-16.9155	46.1794	-0.3663	0.7145
firmDiamond Match	63.7310	47.9689	1.3286	0.1854
firmAmerican Steel	49.7209	48.2801	1.0298	0.3043

Going back to `model_varying_intercept` and `ordinary_model`, we can test if there is a gain in considering a varying intercept versus fixed intercept. Hence, we will make a formal F

-test to check whether the `model_varying_intercept` fits the data better than the `ordinary_model`.

```
anova(ordinary_model, model_varying_intercept) |>
  mutate_if(is.numeric, round, 4)
```

A anova: 2 × 6

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	217	1768678.4	NA	NA	NA	NA
2	207	523718.7	10	1244960	49.2071	0

We obtain a $p\text{-value} < .001$. Thus, with $\alpha = 0.05$, we have evidence to conclude that `model_varying_intercept` fits the data better than the `ordinary_model`.

However, this costs us one extra degree of freedom per `firm` **except for the baseline**. Therefore, we lose another 10 **degrees of freedom** (column `DF` in the `anova()` output).

Important

In this case, losing 10 degrees of freedom is not a big deal with 220 data points. Nonetheless, in other cases, when data is scarce, this could be an issue regarding inferential inquiries since we need them to perform the corresponding hypothesis testing.

Exercise 14

What is the **sample's regression equation** for `model_varying_intercept`?

A.

$$\begin{gathered}\texttt{investment}_{i,j} = \beta_0 + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\ \text{for } i = 1, \dots, 20 \text{ and } j = 1, \dots, 11.\end{gathered}$$

B.

$$\begin{gathered}\texttt{investment}_{i} = \beta_0 + \beta_1 \texttt{marketValue}_{i} + \beta_2 \texttt{capital}_{i} + \varepsilon_{i} \\ \text{for } i = 1, \dots, 220.\end{gathered}$$

C.

$$\begin{gathered}\texttt{investment}_{i,j} = \beta_{0,j} + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\ \text{for } i = 1, \dots, 20 \text{ and } j = 1, \dots, 11.\end{gathered}$$

Solution to Exercise 14

The regression equation, for the ith sampled observation in the jth `firm`, will be:

$$\begin{gathered}\texttt{investment}_{i,j} = \beta_{0,j} + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\ \text{for } i = 1, \dots, 20 \text{ and } j = 1, \dots, 11.\end{gathered}$$

Note that there is a varying term $\beta_{0,j}$ which is indexed with the jth `firm`.

1.3.4. OLS Regression for Each Firm

We can make the model more complex **with two interactions** (`market_value * firm` and `capital * firm`). **This will estimate a linear regression by `firm` with its own slopes.**

```
model_by_firm <- lm(investment ~ market_value * firm + capital * firm, data =
tidy(model_by_firm) |>
  mutate_if(is.numeric, round, 2)
glance(model_by_firm) |>
  mutate_if(is.numeric, round, 2)
```

A tibble: 33 × 5

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	-149.78	48.07	-3.12	0.00
market_value	0.12	0.01	10.17	0.00
firmUS Steel	100.58	80.04	1.26	0.21
firmGeneral Electric	139.83	67.16	2.08	0.04
firmChrysler	143.59	64.09	2.24	0.03
firmAtlantic Refining	172.49	57.52	3.00	0.00
firmIBM	141.10	53.48	2.64	0.01
firmUnion Oil	145.28	69.34	2.10	0.04
firmWestinghouse	149.27	58.14	2.57	0.01
firmGoodyear	142.06	64.41	2.21	0.03
firmDiamond Match	149.94	92.75	1.62	0.11
firmAmerican Steel	147.14	102.37	1.44	0.15
capital	0.37	0.02	22.06	0.00
market_value:firmUS Steel	0.06	0.03	1.63	0.11
market_value:firmGeneral Electric	-0.09	0.03	-3.56	0.00
market_value:firmChrysler	-0.04	0.06	-0.65	0.52
market_value:firmAtlantic Refining	0.04	0.26	0.16	0.87
market_value:firmIBM	0.01	0.16	0.08	0.94
market_value:firmUnion Oil	-0.03	0.29	-0.11	0.91
market_value:firmWestinghouse	-0.07	0.07	-1.02	0.31
market_value:firmGoodyear	-0.04	0.16	-0.28	0.78
market_value:firmDiamond Match	-0.11	1.04	-0.11	0.91
market_value:firmAmerican Steel	-0.05	0.55	-0.10	0.92
firmUS Steel:capital	0.02	0.06	0.29	0.78
firmGeneral Electric:capital	-0.22	0.04	-5.24	0.00

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
firmChrysler:capital	-0.06	0.09	-0.61	0.55
firmAtlantic Refining:capital	-0.37	0.10	-3.60	0.00
firmIBM:capital	-0.29	0.52	-0.55	0.58
firmUnion Oil:capital	-0.25	0.08	-3.20	0.00
firmWestinghouse:capital	-0.28	0.23	-1.22	0.23
firmGoodyear:capital	-0.29	0.13	-2.24	0.03
firmDiamond Match:capital	0.07	3.06	0.02	0.98
firmAmerican Steel:capital	-0.29	1.10	-0.26	0.79

A tibble: 1 × 12

r.squared	adj.r.squared	sigma	statistic	p.value	df	logLik	AIC	B
<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<db
0.97	0.96	41.68	168.84	0	32	-1114.91	2297.81	241

In this case, we are fitting eleven linear regressions, each with 20 points.

> **Important**
>
> We have plenty of data points in this case for all the degrees of freedom required to estimate each parameter. Nonetheless, this might not be the case with other datasets.

Exercise 15

What is the sample's regression equation for `model_by_firm`?

A.

$$\begin{gathered}\texttt{investment}_{i,j} = \beta_{0,j} + \beta_{1,j}\texttt{marketValue}_{i,j} + \beta_{2,j}\texttt{capital}_{i,j} + \varepsilon_{i,j} \\ \text{for } i = 1, \ldots, 20 \text{ and } j = 1, \ldots, 11.\end{gathered}$$

B.

$$\begin{gathered}\texttt{investment}_{j} = \beta_{0} + \beta_{1}\texttt{marketValue}_{j} + \beta_{2}\texttt{capital}_{j} + \varepsilon_{j} \\ \text{for } j = 1, \ldots, 11.\end{gathered}$$

C.

$$\begin{gathered}\texttt{investment}_{i,j} = \beta_{0,i} + \beta_{1,i}\texttt{marketValue}_{i,j} + \beta_{2,i}\texttt{capital}_{i,j} + \varepsilon_{i,j} \\ \text{for } i = 1, \ldots, 20 \text{ and } j = 1, \ldots, 11.\end{gathered}$$

Solution to Exercise 15

The regression equation for the ith sampled observation in the jth `firm` will be:

$$\begin{gathered}\texttt{investment}_{i,j} = \beta_{0,j} + \beta_{1,j}\texttt{marketValue}_{i,j} + \beta_{2,j}\texttt{capital}_{i,j} + \varepsilon_{i,j} \\ \text{for } i = 1, \ldots, 20 \text{ and } j = 1, \ldots, 11.\end{gathered}$$

Note that the terms $\beta_{0,j}$, $\beta_{1,j}$, and $\beta_{2,j}$ (indexed with the jth) vary by `firm`.

We went as far as fitting a **separate linear regression for each `firm`**.

How to interpret the coefficients in this `model_by_firm`?

> Each regression coefficient is associated with a `firm`. For example, $\texttt{firmUS Steel:capital} = 0.02$ means that the variable `capital` has a slope of $0.02 + 0.37 = 0.39$ for `US Steel`. We can double-check this by estimating an individual linear regression for `US Steel`:

```
tidy(lm(investment ~ market_value + capital,
  data = Grunfeld |> filter(firm == "US Steel")
)) |>
  mutate_if(is.numeric, round, 2)
```

A tibble: 3 × 5

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	-49.20	148.08	-0.33	0.74
market_value	0.17	0.07	2.36	0.03
capital	0.39	0.14	2.74	0.01

Exercise 16

Now, after all these OLS modelling approaches, are we in line with the main modelling objective with all these linear regression models?

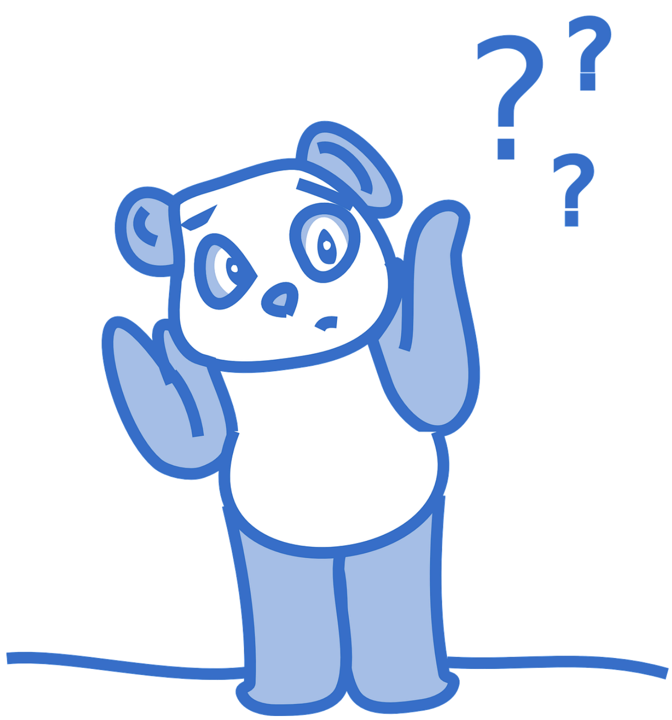


Fig. 10 The Questioning Panda, again!

Solution to Exercise 16

Actually, not!

Grunfeld wanted to see how capital and market value affect gross investment among the American companies. Not for **one specific** `firm`! The conclusions must be valid for "all" the American companies. Nevertheless, **we only have a sample of 11 companies!**

2. Linear Mixed-Effects Model

The fact that we have **a sample of 11 companies** and our inquiry aims to make inference on the **population** of American companies, along with the correlated data structure in this **panel**, paves the way to linear Mixed-Effects modelling.

Important

In linear Mixed-Effects modelling, the n rows in our training set will not be independent anymore. Let us view it this way: possibly, your data subsets of elements share a **correlation structure** (e.g., observations over time on a given response and regressors for a specific subject in a panel). **This is a fundamental idea in Mixed-Effects regression.**

Let us take a step back and think about a population of companies. For instance, all American companies. Grunfeld did not collect data on all the American companies but sampled 11 companies from this population. The author was interested in assessing whether `market_value` and `capital` were related to `investment` and by how much.

Let us assume that the jth sampled `firm` has its own intercept $b_{0,j}$ and the **overall fixed intercept** is β_0 **for all American companies**. Therefore, for the jth firm, we define the following **mixed** intercept:

$$\beta_{0,j} = \beta_0 + b_{0,j}.$$

The intercept $b_{0,j}$ is specifically for the jth `firm` that was sampled. It will change due to chance since it is linked to the jth sampled `firm` which would make it a **random effect**. This is the deviation of the jth `firm` from the overall fixed intercept β_0.

The **regression paradigm** of estimating a fixed unknown intercept β_0 will change now. Moreover, the intercept $\beta_{0,j}$ is what we call a **mixed effect**:

$$\begin{aligned}\texttt{investment}_{i,j} &= \overbrace{\beta_{0,j}}^{\text{Mixed Effect}} + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\ &= (\beta_0 + b_{0,j}) + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\ &\qquad \text{for } i = 1, \ldots, n_j \ \text{ and } \ j = 1, \ldots, 11.\end{aligned}$$

Note that n_j is making the model even more flexible by allowing different numbers of observations n_j in each jth firm.

Now,

$$b_{0,j} \sim \mathcal{N}(0, \sigma_0^2)$$

is called a **random effect** and we assume it is independent of the error component

$$\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma^2).$$

The observations for the same `firm` (group) share the same random effect **making a correlation structure**.

The variance of the ith response for the jth `firm` will be

$$\operatorname{Var}(\texttt{investment}_{i,j}) = \operatorname{Var}(b_{0,j}) + \operatorname{Var}(\varepsilon_{i,j}) = \sigma_0^2 + \sigma^2.$$

For the kth and lth responses, within the jth firm, the correlation is given by:

$$\operatorname{Corr}(\texttt{investment}_{k,j}, \texttt{investment}_{l,j}) = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2}.$$

We could even go further and model random slopes, along with the existing fixed ones, as follows:

$$
\begin{aligned}
\texttt{investment}_{i,j} &= \overbrace{\beta_{0,j}}^{\text{Mixed Effect}} + \overbrace{\beta_{1,j}}^{\text{Mixed Effect}} \times \texttt{marketValue}_{i,j} + \overbrace{\beta_{2,j}}^{\text{Mixed Effect}} \times \texttt{capital}_{i,j} \\
&= (\beta_0 + b_{0,j}) + (\beta_1 + b_{1,j}) \times \texttt{marketValue}_{i,j} + (\beta_2 + b_{2,j}) \times \texttt{capital} \\
&\qquad \text{for } i = 1, \ldots, n_j \text{ and } j = 1, \ldots, 11;
\end{aligned}
$$

with $(b_{0,j}, b_{1,j}, b_{2,j})^T \sim \mathcal{N}_3(\mathbf{0}, \mathbf{D})$, where $\mathbf{0} = (0, 0, 0)^T$ and $\mathbf{D}$ is a generic covariance matrix.

> **Important**
>
> Note that the above random effects $b_{0,j}$, $b_{1,j}$, and $b_{2,j}$ have a multivariate Gaussian (or Normal) distribution of $d = 3$. You can review further details about this distribution in the appendix Fundamentals of the Multivariate Normal Distribution.

2.1. What is this Generic Covariance Matrix $\mathbf{D}$?

This is a **standard form** in linear Mixed-Effects modelling. Hence, this matrix becomes:

$$
\mathbf{D} = \begin{bmatrix} \sigma_0^2 & \rho_{01}\sigma_0\sigma_1 & \rho_{02}\sigma_0\sigma_2 \\ \rho_{01}\sigma_0\sigma_1 & \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{02}\sigma_0\sigma_2 & \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_0^2 & \sigma_{0,1} & \sigma_{0,2} \\ \sigma_{0,1} & \sigma_1^2 & \sigma_{1,2} \\ \sigma_{0,2} & \sigma_{1,2} & \sigma_2^2 \end{bmatrix},
$$

where σ_0^2, σ_1^2, and σ_2^2 are the variances of $b_{0,j}$, $b_{1,j}$, and $b_{2,j}$ respectively.

Moreover $\rho_{uv} \in [0, 1]$ is the correlation between the uth and the vth random effects. We can reexpress the covariances as $\sigma_{u,v}$.

Important

- ρ_{uv} indicates a Pearson correlation, which you can also review **here**.
- While the **random effects are assumed to follow a mulivariate Normal distribution**, this is different from the **sampling distribution** of the estimates of the fixed effects.
- The multivariate Normal distribution explains the variability of **random** regression intercepts and coefficients. The spread does not change when we collect more data.
- The sampling distribution explains the uncertainty in the **fixed** regression estimates and **gets narrower** as we collect more data.

2.2. Model Fitting, Inference, and Coefficient Interpretation

Let us estimate the regression model with a mixed intercept only (`mixed_intercept_model`) via the function `lmer()` from package `lme4`. **Note that `(1 | firm)` allows the model to have a random intercept by `firm`.**

```
suppressWarnings(suppressMessages(print(mixed_intercept_model <- lmer(investme
  capital + (1 | firm),
data = Grunfeld
))))
```

```
Linear mixed model fit by REML ['lmerModLmerTest']
Formula: investment ~ market_value + capital + (1 | firm)
   Data: Grunfeld
REML criterion at convergence: 2394.616
Random effects:
 Groups   Name        Std.Dev.
 firm     (Intercept) 82.10
 Residual             50.27
Number of obs: 220, groups:  firm, 11
Fixed Effects:
 (Intercept)  market_value       capital
    -54.0318        0.1094        0.3082
fit warnings:
Some predictor variables are on very different scales: consider rescaling
```

Recall the regression equation for `mixed_intercept_model`:

$$
\begin{aligned}
\texttt{investment}_{i,j} &= \overbrace{\beta_{0,j}}^{\text{Mixed Effect}} + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\
&= (\beta_0 + b_{0,j}) + \beta_1 \texttt{marketValue}_{i,j} + \beta_2 \texttt{capital}_{i,j} + \varepsilon_{i,j} \\
&\qquad \text{for } i = 1, \ldots, n_j \; \text{ and } \; j = 1, \ldots, 11;
\end{aligned}
$$

where

$$b_{0,j} \sim \mathcal{N}(0, \sigma_0^2)$$

and

$$\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma^2).$$

The section `Random effects` will show the estimated standard deviations $\hat{\sigma}_0$ and $\hat{\sigma}$ as `82.10` and `50.27`, respectively. Estimated `Fixed effects` $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$ are `-54.0318`, `0.1094`, and `0.3082` respectively.

Now, let us estimate the Mixed-Effects regression model with mixed intercept and slopes (`full_mixed_model`). **Note that `(market_value + capital | firm)` allows the model to have a random intercept and slopes by `firm`.**

```
suppressWarnings(suppressMessages(print(full_mixed_model <- lmer(investment ~
  capital + (market_value + capital | firm),
data = Grunfeld
))))
```

```
Linear mixed model fit by REML ['lmerModLmerTest']
Formula: investment ~ market_value + capital + (market_value + capital |
    firm)
   Data: Grunfeld
REML criterion at convergence: 2299.116
Random effects:
 Groups   Name         Std.Dev. Corr
 firm     (Intercept)  15.15612
          market_value  0.05235 -1.00
          capital       0.12291 -0.85  0.85
 Residual              40.77647
Number of obs: 220, groups:  firm, 11
Fixed Effects:
 (Intercept)  market_value       capital
    -7.79756       0.06118       0.22694
fit warnings:
Some predictor variables are on very different scales: consider rescaling
optimizer (nloptwrap) convergence code: 0 (OK) ; 0 optimizer warnings; 1 lme4 w
```

Recall the regression equation for `full_mixed_model`:

$$\begin{aligned}\texttt{investment}_{i,j} &= \overbrace{\beta_{0,j}}^{\text{Mixed Effect}} + \overbrace{\beta_{1,j}}^{\text{Mixed Effect}} \times \texttt{marketValue}_{i,j} + \overbrace{\beta_{2,j}}^{\text{Mixed Effect}} \times \texttt{capital}_{i,j} \\ &= (\beta_0 + b_{0,j}) + (\beta_1 + b_{1,j}) \times \texttt{marketValue}_{i,j} + (\beta_2 + b_{2,j}) \times \texttt{capital} \\ &\qquad \text{for } i = 1, \dots, n_j \;\text{ and }\; j = 1, \dots, 11;\end{aligned}$$

with $(b_{0,j}, b_{1,j}, b_{2,j})^T \sim \mathcal{N}_3(\mathbf{0}, \mathbf{D})$, where $\mathbf{0} = (0, 0, 0)^T$ and $\mathbf{D}$ is a generic covariance matrix.

The section `Random effects` will show the estimated standard deviations $\hat{\sigma}_0$, $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\sigma}$ as `15.15612`, `0.05235`, `0.12291`, and `40.77647` respectively. Estimated `Fixed effects` $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$ are `-7.79756`, `0.06118`, and `0.22694` respectively.

Let us proceed with inference using `mixed_intercept_model`. We now assess whether the fixed effects are statistically associated with `investment` in each model via `summary()`. We will use the package `lmerTest` along with function `summary()`.

```
library(lmerTest)
summary(mixed_intercept_model)
```

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method [
lmerModLmerTest]
Formula: investment ~ market_value + capital + (1 | firm)
   Data: Grunfeld

REML criterion at convergence: 2394.6

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.6018 -0.3138  0.0139  0.3148  5.0604

Random effects:
 Groups   Name        Variance Std.Dev.
 firm     (Intercept) 6741     82.10
 Residual             2528     50.27
Number of obs: 220, groups:  firm, 11

Fixed effects:
               Estimate Std. Error         df t value Pr(>|t|)
(Intercept)  -54.031821  26.623551  12.069048  -2.029   0.0651 .
market_value   0.109352   0.009987 129.346325  10.950   <2e-16 ***
capital        0.308200   0.016368 213.247094  18.830   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Correlation of Fixed Effects:
            (Intr) mrkt_v
market_valu -0.312
capital     -0.021 -0.369
fit warnings:
Some predictor variables are on very different scales: consider rescaling
```

We do the same with `full_mixed_model`.

```
summary(full_mixed_model)
```

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method [
lmerModLmerTest]
Formula: investment ~ market_value + capital + (market_value + capital |
    firm)
   Data: Grunfeld

REML criterion at convergence: 2299.1

Scaled residuals:
    Min      1Q  Median      3Q     Max
-4.5030 -0.1660  0.0100  0.1819  4.1389

Random effects:
 Groups   Name         Variance  Std.Dev. Corr
 firm     (Intercept)  2.297e+02 15.15612
          market_value 2.741e-03  0.05235 -1.00
          capital      1.511e-02  0.12291 -0.85  0.85
 Residual              1.663e+03 40.77647
Number of obs: 220, groups:  firm, 11

Fixed effects:
             Estimate Std. Error       df t value Pr(>|t|)
(Intercept)  -7.79756    7.79890  7.01489  -1.000 0.350626
market_value  0.06118    0.01927  7.74466   3.175 0.013660 *
capital       0.22694    0.04422  9.52493   5.132 0.000517 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Correlation of Fixed Effects:
            (Intr) mrkt_v
market_valu -0.691
capital     -0.547  0.564
fit warnings:
Some predictor variables are on very different scales: consider rescaling
optimizer (nloptwrap) convergence code: 0 (OK)
boundary (singular) fit: see help('isSingular')
```

We can see that `market_value` and `capital` are significant with $\alpha = 0.05$ in both models. **Moreover, the regression coefficients' interpretation for the fixed effects will be on the effect these regressors have on the population `investment` mean of the American companies.**

We can obtain the estimated coefficients by `firm` along with the intercepts for both models via `coef()`.

```
coef(mixed_intercept_model)$firm
```

A data.frame: 11 × 3

	(Intercept)	market_value	capital
	<dbl>	<dbl>	<dbl>
General Motors	-65.526515	0.109352	0.3081997
US Steel	101.069974	0.109352	0.3081997
General Electric	-230.027388	0.109352	0.3081997
Chrysler	-27.544640	0.109352	0.3081997
Atlantic Refining	-112.435102	0.109352	0.3081997
IBM	-23.220327	0.109352	0.3081997
Union Oil	-65.632801	0.109352	0.3081997
Westinghouse	-56.815866	0.109352	0.3081997
Goodyear	-85.813152	0.109352	0.3081997
Diamond Match	-7.376705	0.109352	0.3081997
American Steel	-21.027505	0.109352	0.3081997

Column `(Intercept)` is the sum $\hat{\beta}_0 + \hat{b}_{0,j}$. Note the estimated regression coefficients for `market_value` and `capital` are the same since `mixed_intercept_model` only has β_1 and β_2 as its general modelling setup.

The coefficient summary changes in `full_mixed_model` given that we also include random effects for `market_value` and `capital`, as shown below. Columns `market_value` and `capital` are the sums $\hat{\beta}_1 + \hat{b}_{1,j}$ and $\hat{\beta}_2 + \hat{b}_{2,j}$, respectively. Column `(Intercept)` is the sum $\hat{\beta}_0 + \hat{b}_{0,j}$.

```
coef(full_mixed_model)$firm
```

A data.frame: 11 × 3

	(Intercept)	market_value	capital
	<dbl>	<dbl>	<dbl>
General Motors	-15.9552583	0.08936163	0.3746270
US Steel	-37.3981285	0.16343240	0.4206200
General Electric	3.7069150	0.02144198	0.1453769
Chrysler	-16.9962883	0.09295769	0.3041952
Atlantic Refining	6.5953958	0.01146421	0.1027958
IBM	-14.2108882	0.08333600	0.2855949
Union Oil	0.9675705	0.03090458	0.1371497
Westinghouse	-2.7054789	0.04359251	0.1826596
Goodyear	3.9889246	0.02046783	0.1116272
Diamond Match	-3.5486176	0.04650499	0.1985645
American Steel	-10.2172843	0.06954077	0.2331584

Exercise 17

Note the standard errors for the estimated slopes in `market_value` and `capital` behave in a really particular way when comparing the OLS `model_varying_intercept` and the Mixed-Effects `full_mixed_model`.

Therefore, **what are the advantages of a Mixed-Effects model over an OLS model with fixed-effects only?**

```
tidy(model_varying_intercept) |>
  mutate_if(is.numeric, round, 4)
```

A tibble: 13 × 5

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
market_value	0.1101	0.0113	9.7461	0.0000
capital	0.3100	0.0165	18.7439	0.0000
firmGeneral Motors	-70.2991	47.3754	-1.4839	0.1394
firmUS Steel	101.9047	23.7687	4.2873	0.0000
firmGeneral Electric	-235.5694	23.2861	-10.1163	0.0000
firmChrysler	-27.8091	13.4186	-2.0724	0.0395
firmAtlantic Refining	-114.6025	13.5025	-8.4875	0.0000
firmIBM	-23.1602	12.0759	-1.9179	0.0565
firmUnion Oil	-66.5442	12.2420	-5.4357	0.0000
firmWestinghouse	-57.5465	13.3379	-4.3145	0.0000
firmGoodyear	-87.2145	12.2887	-7.0971	0.0000
firmDiamond Match	-6.5680	11.2736	-0.5826	0.5608
firmAmerican Steel	-20.5782	11.2978	-1.8214	0.0700

```
summary(full_mixed_model)
```

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method [
lmerModLmerTest]
Formula: investment ~ market_value + capital + (market_value + capital |
    firm)
   Data: Grunfeld

REML criterion at convergence: 2299.1

Scaled residuals: 
    Min      1Q  Median      3Q     Max 
-4.5030 -0.1660  0.0100  0.1819  4.1389 

Random effects:
 Groups   Name         Variance  Std.Dev. Corr       
 firm     (Intercept)  2.297e+02 15.15612            
          market_value 2.741e-03  0.05235 -1.00      
          capital      1.511e-02  0.12291 -0.85  0.85
 Residual              1.663e+03 40.77647            
Number of obs: 220, groups:  firm, 11

Fixed effects:
             Estimate Std. Error       df t value Pr(>|t|)    
(Intercept)  -7.79756    7.79890  7.01489  -1.000 0.350626    
market_value  0.06118    0.01927  7.74466   3.175 0.013660 *  
capital       0.22694    0.04422  9.52493   5.132 0.000517 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Correlation of Fixed Effects:
            (Intr) mrkt_v
market_valu -0.691       
capital     -0.547  0.564
fit warnings:
Some predictor variables are on very different scales: consider rescaling
optimizer (nloptwrap) convergence code: 0 (OK)
boundary (singular) fit: see help('isSingular')
```

Solution to Exercise 17

This is related to **estimation efficiency** for the model parameters and is entirely in function of our inference/prediction objectives. A Mixed-Effects model saves degrees of freedom by not estimating unnecessary fixed effects in high-dimensional scenarios.

Secondly, the Mixed-Effects model would avoid underestimating standard errors of our fixed-effects estimates. This will decrease our chances of **committing Type I error** in our hypothesis tests for the fixed-effects.

2.3. Estimation

The previous section skipped the estimation phase. However, we must point out that **maximum likelihood estimation (MLE)** is **still present** in Mixed-Effects modelling. Nevertheless, MLE is performed in a different way. Recall that OLS only has a single variance component σ^2, but in Mixed-Effects modelling, we have the variance component from each random effect.

Having said that, Mixed-Effects modelling performs what we call **restricted MLE (R-MLE)**. Overall, R-MLE partions the joint likelihood of the observed data in two parts:

1. One related to the fixed effects, which yields the estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k$ when we have k regressors.
2. Another related to the variance components of the random effects and the overall variance σ^2.

Overall, this likelihood restriction will allow us to deliver unbiased estimates of the variance components in (2) because the fixed effects' contribution is isolated in (1).

2.4. Prediction

We can make two classes of predictions with Mixed-Effects models:

1. **To predict on an existing group**, we find that group's regression coefficients (and therefore model function) by summing the fixed effects and (if present) the random effects, then use that model function to make predictions.
2. **To predict on a new group (using a mean prediction)**, we use the fixed effects as the regression coefficients (because the random effects are assumed to have a mean equal to zero) and use that model function to make predictions.

For **predictions on an existing group in our training set** we have:

```
round(predict(full_mixed_model, newdata = tibble(
  firm = "General Motors",
  market_value = 2000, capital = 1000
)), 2)
```

1: 537.4

If we wanted **to predict the `investment` for `General Motors`** with a `market_value` of USD $\$2,000$ million and capital of USD $\$1,000$ million, then our answer would be USD $\$537.4$ million. This prediction uses $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{b}_{0,j}$, $\hat{b}_{1,j}$, and $\hat{b}_{2,j}$.

Whereas that for **predictions on American companies in general**, we have:

```
round(predict(full_mixed_model,
  newdata = tibble(
    firm = "New Company",
    market_value = 2000, capital = 1000
  ),
  allow.new.levels = TRUE
), 2)
```

1: 341.51

If we wanted **to predict the MEAN `investment` for American companies** with a `market_value` of USD $\$2,000$ million and capital of USD $\$1,000$ million, then our answer would be USD $\$341.51$ million. This prediction only uses $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$.

3. Wrapping Up on Mixed-Effects Modelling

- In many different cases, when there is a correlation structure in our observations, OLS models are not suitable for our inferential or predictive inquiries.
- Therefore, linear Mixed-Effects models are suitable for correlated observations. Nonetheless, the model's complexity will also be in function of our specific inquiries.
- We can even extend the Mixed-Effects approach to generalized linear models (GLMs)!