Lecture 7 - Bayesian Hierarchical Models

Contents

Today's Learning Objectives

1. Explain the concept of the Bayesian hierarchical model.
2. Contrast the hierarchical model versus standard Bayesian models.
3. Demonstrate the advantages of hierarchical modelling in Bayesian inference.
4. Apply hierarchical modelling to predictive inquiries.

Loading R Packages

```
options(repr.matrix.max.rows = 6)
library(tidyverse)
library(rstan)
library(broom.mixed)
library(bayesplot)
library(cowplot)
library(bayesrules)
```

Previously...

We are in the last week of the block, and we have been exposed to **standard Bayesian models**. These models involve **a likelihood conditioned on parameters of interest, which are random variables**. These parameters have **prior distributions**. Furthermore, **these priors have fixed hyperparameters**.

Suppose you want to make inference on the **mean number of people** who stand in line, from 9:00 to 10:00 a.m. on Mondays, in the branches of a major Canadian bank. You collect data across different branches during that time slot and end up with n observations. Moreover, you decide to take a Bayesian approach for this inquiry.

Nonetheless, we can make **a flexible model in such a way we assign a mean by branch**. Hence, let S_i be the number of **people** standing in line from 9:00 to 10:00 a.m. on Mondays in the ith branch $(i = 1, \dots, n)$, and θ_i the mean by branch. Therefore, the formal Bayesian model will be the following:

$$
\begin{aligned}
\text{likelihood:} \qquad S_i \mid \theta_i &\overset{\text{ind}}{\sim} \text{Poisson}(\theta_i) \quad \text{for } i = 1, \dots, n \\
\text{priors:} \qquad \theta_i &\sim \text{Gamma}(\alpha, \beta) \\
\alpha &\sim \text{Exponential}(\eta = 0.1) \\
\beta &\sim \text{Exponential}(\gamma = 0.5).
\end{aligned}
$$

Note α and β are now random variables! This is a **hierarchical model**!

Exercise 19

Why do α and β have Exponential priors?

A. Both α and β are discrete and nonnegative in the Gamma prior distribution of θ_i.

B. Both α and β are continuous and unbounded in the Gamma prior distribution of θ_i.

C. Both α and β are continuous and nonnegative in the Gamma prior distribution of θ_i.

D. Both α and β are discrete and unbounded in the Gamma prior distribution of θ_i.

Solution to Exercise 19

α is the Gamma shape parameter, and β is the rate Gamma parameter. Therefore, the Gamma prior probability distribution function (PDF) for each θ_i $(i = 1, \ldots, n)$ is:

$$f(\theta_i) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta_i^{\alpha-1} \exp(-\beta\theta_i) \quad \text{for } \theta_i > 0,$$

where

$$\alpha > 0 \text{ and } \beta > 0.$$

The above mathematical definitions for α and β indicate they are nonnegative. Hence, they will have **nonnegative continuous priors** such as the Exponential in this hierarchical model. Other choices could be Gamma, Lognormal, etc.

> **Heads-up:** The cheatsheet has x as a variable in the Gamma's PDF instead of θ_i. Thus, we are adapting the above PDF notation to the context of our case. Moreover, we are reparameterizing the PDF with $\alpha = k$ and $\beta = 1/\theta$.

1. The Rockets Data

To illustrate the differences between the Bayesian models we have seen so far and the concept of **hierarchy**, we will retake the `rockets` problem from `lab2`. Then, throughout the lecture, we will build different Bayesian arrangements until we get to a hierarchical model. This will highlight these differences according to each model's specific inferential inquiry.

Important

Recall `rockets` contains data on 367 rockets. It has three columns:

- `LV.Type`: The rocket's type.
- `numberOfLaunches`: Number of launches.
- `numberOfFailures`: Number of failures.

```
rockets <- read_csv("../data/failure_counts.csv", show_col_types = FALSE)
rockets
```

A spec_tbl_df: 367 × 3

LV.Type	numberOfLaunches	numberOfFailures
<chr>	<dbl>	<dbl>
Aerobee	1	0
Angara A5	1	0
Antares 110	2	0
⋮	⋮	⋮
Zenit-3SL	36	3
Zenit-3SLB	6	0
Zenit-3SLBF	3	0

2. Single Rocket Model (Warmup!)

Let

$$X|\pi \sim \text{Binomial}(n, \pi)$$

be the number of successes that a **specific** rocket had in n launches. We will assume that the **probability of success** π is a continuous random variable that can take any value in the interval $[0, 1]$.

As our **prior distribution**, we will define a Beta PDF over the interval $[0, 1]$. Thus, the formal Bayesian model will be:

$$\begin{aligned} \text{likelihood:} \qquad & X|\pi \sim \text{Binomial}(n, \pi) \\ \text{prior:} \qquad & \pi \sim \text{Beta}(a = 1, b = 1). \end{aligned}$$

Let us call it `Single_Rocket`.

Attention

For a **GIVEN type of rocket** (i.e., a row in `rockets`), our goal with `Single_Rocket` will be to infer its **launch success probability** π.

2.1. Coding the Model

We start by coding `Single_Rocket` on `Stan`.

```
Single_Rocket <- "data {
int<lower=1> n;    // Number of trials in a given rocket type
int<lower=0> X;    // Number of successes in a given rocket type
real<lower=0> a;   // Hyperparameter for Beta prior
real<lower=0> b;   // Hyperparameter for Beta prior
}
parameters {
real <lower=0,upper=1> pi;    // Probability of launch success in a given rock
}
model {
pi ~ beta(a, b);
X ~ binomial(n, pi);
}"
```

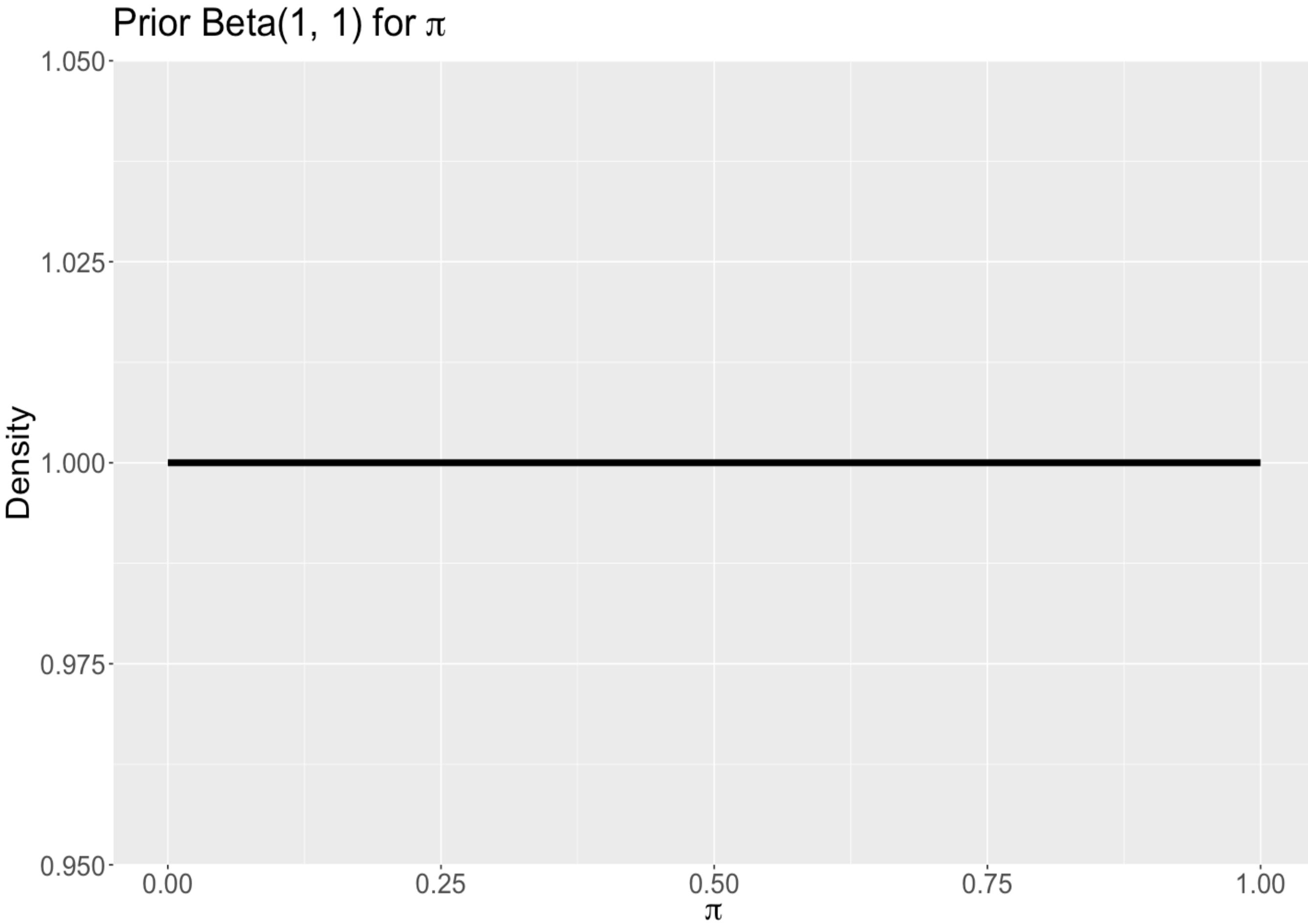
2.2. Running the MCMC Simulation

Suppose you are interested in making inference in the launch success probability π for the type `Ariane 1`. Hence, let us use that row from `rockets` as our **observed data**. First, we create the data dictionary and proceed with `stan()`. We use a prior $\mathrm{Beta}(a = 1, b = 1)$ which assumes all values of π as equally probable.

```
options(repr.plot.height = 8, repr.plot.width = 12)

# Plotting Beta(1,1) prior
prior_beta_1_1 <- ggplot() +
  xlim(0, 1) +
  geom_function(fun = dbeta, args = list(shape1 = 1, shape2 = 1), linewidth =
  theme(
    plot.title = element_text(size = 24),
    axis.text.x = element_text(size = 17, angle = 0),
    axis.text.y = element_text(size = 17, angle = 0),
    axis.title = element_text(size = 21),
  ) +
  labs(y = "Density", x = expression(pi)) +
  ggtitle(expression(paste("Prior Beta(1, 1) for ", pi)))
```

```
prior_beta_1_1
```




```
Ariane_1_data <- rockets |>
  filter(LV.Type == "Ariane 1")
Ariane_1_data

Single_Rocket_dictionary <- list(
  n = Ariane_1_data$numberOfLaunches,
  X = Ariane_1_data$numberOfLaunches - Ariane_1_data$numberOfFailures, # Succe
  a = 1, b = 1
)
```

A spec_tbl_df: 1 × 3

LV.Type	numberOfLaunches	numberOfFailures
<chr>	<dbl>	<dbl>
Ariane 1	11	2

```
posterior_Single_Rocket <- stan(
  model_code = Single_Rocket,
  data = Single_Rocket_dictionary,
  chains = 1,
  iter = 10000,
  warmup = 1000,
  thin = 5,
  seed = 553,
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1: 
Chain 1: Gradient evaluation took 7e-06 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.(
Chain 1: Adjust your expectations accordingly!
Chain 1: 
Chain 1: 
Chain 1: Iteration:    1 / 10000 [  0%]  (Warmup)
Chain 1: Iteration: 1000 / 10000 [ 10%]  (Warmup)
Chain 1: Iteration: 1001 / 10000 [ 10%]  (Sampling)
Chain 1: Iteration: 2000 / 10000 [ 20%]  (Sampling)
Chain 1: Iteration: 3000 / 10000 [ 30%]  (Sampling)
Chain 1: Iteration: 4000 / 10000 [ 40%]  (Sampling)
Chain 1: Iteration: 5000 / 10000 [ 50%]  (Sampling)
Chain 1: Iteration: 6000 / 10000 [ 60%]  (Sampling)
Chain 1: Iteration: 7000 / 10000 [ 70%]  (Sampling)
Chain 1: Iteration: 8000 / 10000 [ 80%]  (Sampling)
Chain 1: Iteration: 9000 / 10000 [ 90%]  (Sampling)
Chain 1: Iteration: 10000 / 10000 [100%]  (Sampling)
Chain 1: 
Chain 1:  Elapsed Time: 0.002 seconds (Warm-up)
Chain 1:                0.016 seconds (Sampling)
Chain 1:                0.018 seconds (Total)
Chain 1: 
```

2.3. Output Summary

With a left-skewed distribution, our posterior mean of π for `Ariane 1` is 0.767. Moreover, **the 95% credible interval shows high variability**. Therefore, we could either increase our sample size for `Ariane 1` (**to have a narrower likelihood**) or tune the Beta prior (**i.e., a prior with less variability**).

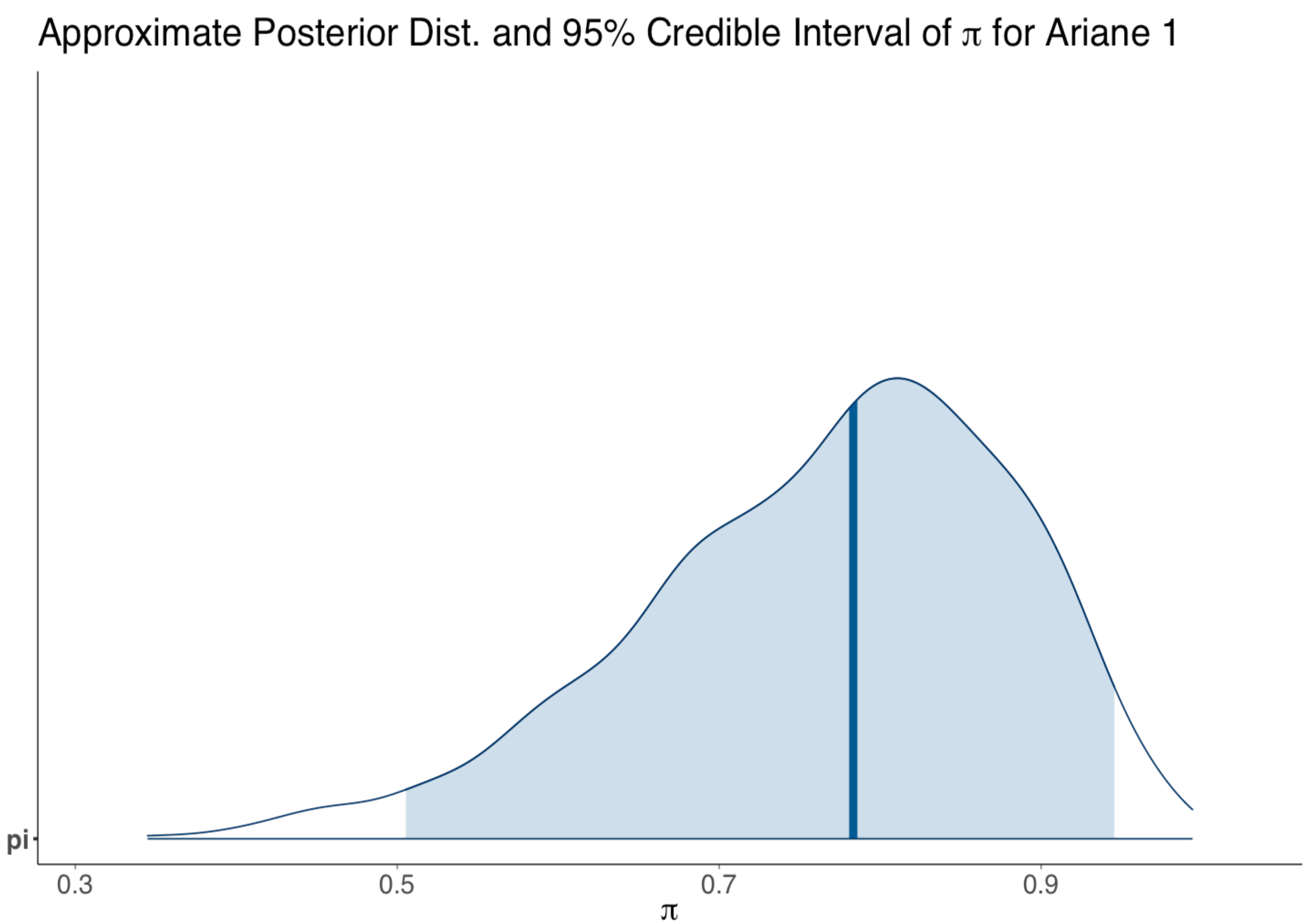
```
summary_Ariane_1 <- as.data.frame(summary(posterior_Single_Rocket)$summary)
summary_Ariane_1 <- summary_Ariane_1[1, c("mean", "sd", "2.5%", "97.5%")] %>%
  mutate_if(is.numeric, round, 3)
summary_Ariane_1
```

A data.frame: 1 × 4

	mean	sd	2.5%	97.5%
	<dbl>	<dbl>	<dbl>	<dbl>
pi	0.767	0.116	0.505	0.945

```
post_plot_Ariane_1 <- mcmc_areas(posterior_Single_Rocket, pars = "pi", prob =
  theme(
    plot.title = element_text(size = 24, family = "sans"),
    axis.text.x = element_text(size = 17, angle = 0, family = "sans"),
    axis.text.y = element_text(size = 17, angle = 0, family = "sans"),
    axis.title = element_text(size = 21, family = "sans")
  ) +
  labs(title = expression(paste("Approximate Posterior Dist. and 95% Credible
  xlab(expression(pi))
```

```
post_plot_Ariane_1
```



3. Complete Pooled Rocket Model

Our second Bayesian approach is called a **complete pooled model**. In this model, we will combine the data provided by **ALL** rockets into **one pool of information**. For the ith rocket, the model is:

$$\begin{aligned} \text{likelihood:} \qquad X_i \vert \pi &\sim \text{Binomial}(n_i, \pi) \quad \text{for } i = 1, \dots, 367 \\ \text{prior:} \qquad \pi &\sim \text{Beta}(a = 1, b = 1). \end{aligned}$$

Caution

Even though it might not look like it, this model is quite different from the `SingleRocket` model. The difference is that, in this complete pooled model, **we are considering all the rocket types**.

While we allow them to have different numbers of launches, we assume that all of them have the same probability of success π. **In this way, we will be estimating the OVERALL probability of success across all rockets.**

3.1. Coding the Model

We code up this `Complete_Pooled_Rocket` model on `Stan`. Note the `data` block includes one integer to define the number of rockets to use in the simulation (`num_rockets`), a vector of integers containing all launch successes by rocket (`X[num_rockets]`), and a vector of integers containing all trials by rocket (`n[num_rockets]`).

```
Complete_Pooled_Rocket <- "data {
int<lower=1> num_rockets;     // number of rockets
int<lower=0> X[num_rockets];  // vector with sucesses by rocket
int<lower=1> n[num_rockets];  // vector with trials by rocket
real<lower=0> a;
real<lower=0> b;
}
parameters {
real <lower=0,upper=1> pi;
}
model {
pi ~ beta(a, b);
for (i in 1:num_rockets){    // modelling the likelihood by rocket
X[i] ~ binomial(n[i], pi);
}
}"
```

3.2. Running the MCMC Simulation

To build the `Complete_Pooled_Rocket_dictionary` we need to use the corresponding columns from `rockets`. Recall `X` is provided by substracting two columns.

```
Complete_Pooled_Rocket_dictionary <- list(
  num_rockets = nrow(rockets),
  X = rockets$numberOfLaunches - rockets$numberOfFailures,
  n = rockets$numberOfLaunches,
  a = 1, b = 1
)
```

```
posterior_Complete_Pooled_Rocket <- stan(
  model_code = Complete_Pooled_Rocket,
  data = Complete_Pooled_Rocket_dictionary,
  chains = 1,
  iter = 10000,
  warmup = 1000,
  thin = 5,
  seed = 553,
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 2.7e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.2
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 10000 [  0%]  (Warmup)
Chain 1: Iteration: 1000 / 10000 [ 10%]  (Warmup)
Chain 1: Iteration: 1001 / 10000 [ 10%]  (Sampling)
Chain 1: Iteration: 2000 / 10000 [ 20%]  (Sampling)
Chain 1: Iteration: 3000 / 10000 [ 30%]  (Sampling)
Chain 1: Iteration: 4000 / 10000 [ 40%]  (Sampling)
Chain 1: Iteration: 5000 / 10000 [ 50%]  (Sampling)
Chain 1: Iteration: 6000 / 10000 [ 60%]  (Sampling)
Chain 1: Iteration: 7000 / 10000 [ 70%]  (Sampling)
Chain 1: Iteration: 8000 / 10000 [ 80%]  (Sampling)
Chain 1: Iteration: 9000 / 10000 [ 90%]  (Sampling)
Chain 1: Iteration: 10000 / 10000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 0.04 seconds (Warm-up)
Chain 1:                0.37 seconds (Sampling)
Chain 1:                0.41 seconds (Total)
Chain 1:
```

3.3. Output Summary

Unlike `Ariane 1`, the posterior distribution of the overall π is symmetric with a mean of 0.924. Moreover, **the 95% CI shows a low variability in this posterior distribution**.

```r
summary_Complete_Pooled_Rocket <- as.data.frame(summary(posterior_Complete_Poo
summary_Complete_Pooled_Rocket <- summary_Complete_Pooled_Rocket[1, c("mean",
  mutate_if(is.numeric, round, 3)
summary_Complete_Pooled_Rocket
```

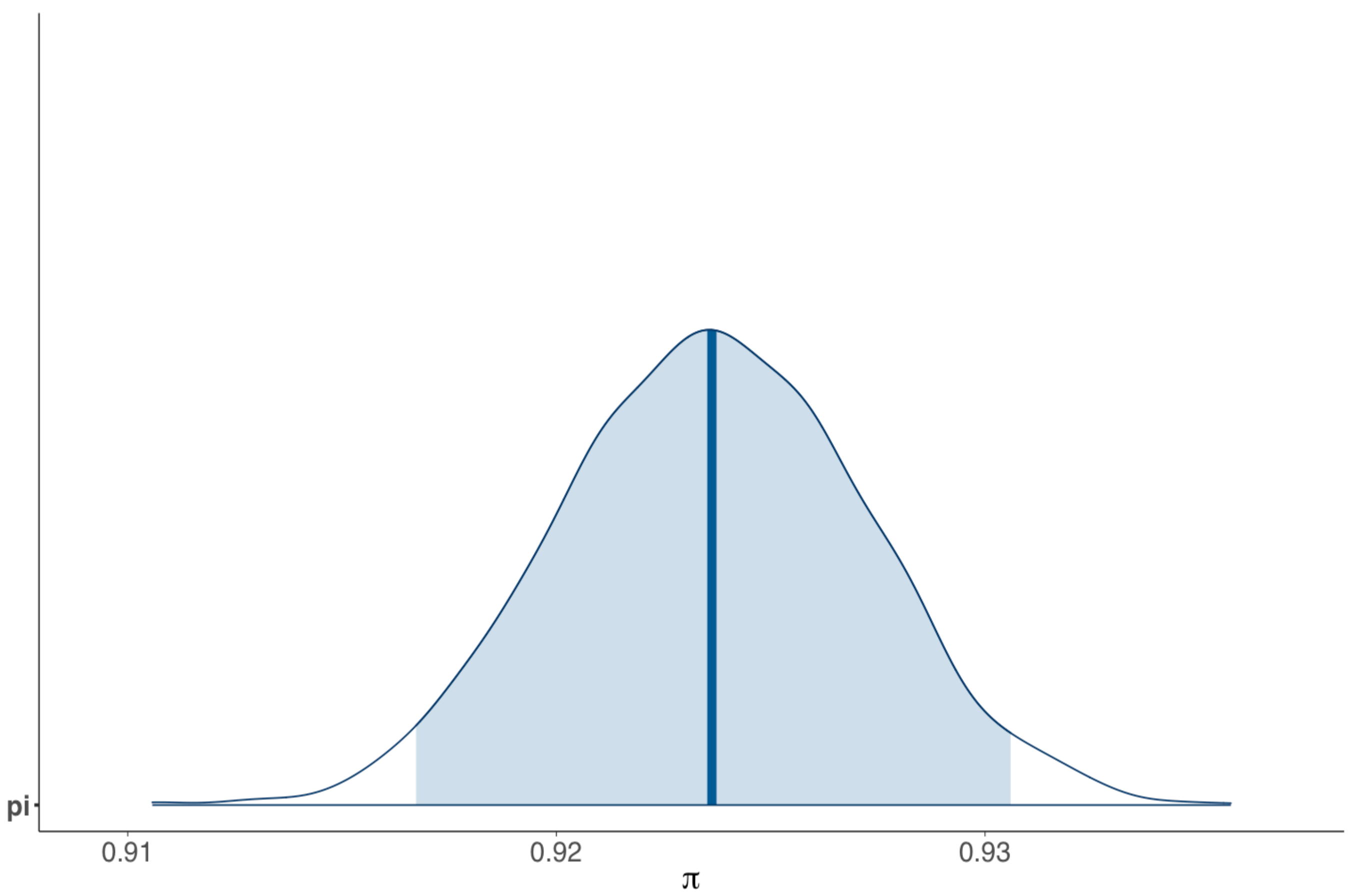
A data.frame: 1 × 4

	mean	sd	2.5%	97.5%
	<dbl>	<dbl>	<dbl>	<dbl>
pi	0.924	0.004	0.917	0.931

```
post_plot_Complete_Pooled_Rocket <- mcmc_areas(posterior_Complete_Pooled_Rocke
  theme(
    plot.title = element_text(size = 24, family = "sans"),
    axis.text.x = element_text(size = 17, angle = 0, family = "sans"),
    axis.text.y = element_text(size = 17, angle = 0, family = "sans"),
    axis.title = element_text(size = 21, family = "sans")
  ) +
  labs(title = expression(paste("Approximate Posterior Dist. and 95% Credible
  xlab(expression(pi))
```

```
post_plot_Complete_Pooled_Rocket
```

Approximate Posterior Dist. and 95% Credible Interval of π



Exercise 20

Note that the approximate 95% credible interval for π is $(0.917, 0.931)$. Is this interval an indicator of high precision on the estimation of π?

A. Yes.

B. No.

Solution to Exercise 20

Yes, **this low variability in the posterior distribution of π might indicate a precise estimation reflected in the approximate 95% credible interval**. Nonetheless, there are two critical things to consider in `Complete_Pooled_Rocket`:

- Each rocket is assumed to have **the same sampling variability** (we are using a common `pi` in the `binomial()` likelihood).
- **However, not all the rocket types are equal!** Therefore, our Bayesian model needs to consider this matter when making inference on the launch success probabilities.

4. Let Us Go to Space!

You told a certain CEO from a rocket company that you wanted to go to space, and they asked you to pick your preferred rocket model from the `rockets` dataset. **Let us try to decide which rocket we trust the most.** Therefore, the Bayesian `Complete_Pooled_Rocket` is not suitable for this purpose **since it makes inference on a single π**!

We will start with a similar approach such as in `lab1` for the helicopter problem.

Firstly, **we will only trust our observed data (i.e., the likelihood) without relying on the Bayes' rule**. This will implicate using **maximum likelihood estimation** to compute $\hat{\pi}_i$ for the ith rocket $(i = 1, \dots, 367)$. In a Binomial likelihood, we know the maximum likelihood estimate (MLE) is:

$$\hat{\pi}_i = \frac{x_i}{n_i},$$

where x_i and n_i are the respective **observed** numbers of successes and trials.

We will compute $\hat{\pi}_i$ in `rockets` **so we can select that rocket with the largest estimation for its launch success probability**.

```r
options(repr.matrix.max.rows = 60)

rockets <- rockets |>
  mutate(MLE = (numberOfLaunches - numberOfFailures) / numberOfLaunches)

rockets |>
  arrange(-MLE, LV.Type)
```

A tibble: 367 × 4

LV.Type	numberOfLaunches	numberOfFailures	MLE
<chr>	<dbl>	<dbl>	<dbl>
ARPA Taurus	2	0	1
Aerobee	1	0	1
Angara A5	1	0	1
Antares 110	2	0	1
Antares 120	2	0	1
Antares 230	1	0	1
Ariane 40	7	0	1
Ariane 42L	13	0	1
Ariane 44P	15	0	1
Ariane 5ES	1	0	1
Ariane 5ES/ATV	5	0	1
Ariane 5G+	3	0	1
Ariane 5GS	6	0	1
Athena-1	2	0	1
Atlas 3A	2	0	1
Atlas 3B	4	0	1
Atlas Agena D	15	0	1
Atlas B	1	0	1
Atlas Centaur D	7	0	1
Atlas E	15	0	1
Atlas E Altair	1	0	1
Atlas E/OIS	1	0	1
Atlas E/SGS-2	4	0	1
Atlas F/Agena D	1	0	1
Atlas F/MSD	3	0	1

LV.Type	numberOfLaunches	numberOfFailures	MLE
<chr>	<dbl>	<dbl>	<dbl>
Atlas F/OIS	1	0	1
Atlas F/PTS	1	0	1
Atlas F/SVS	7	0	1
Atlas H	5	0	1
Atlas II	10	0	1
⋮	⋮	⋮	⋮
Proton-M/DM-3	3	2	0.3333333
Scout X-2M	3	2	0.3333333
Thor Able I	3	2	0.3333333
Vostok-L 8K72	9	6	0.3333333
Vanguard	11	8	0.2727273
Lambda 4S	5	4	0.2000000
Antares 130	1	1	0.0000000
Atlas Able	3	3	0.0000000
Atlas E/MSD	1	1	0.0000000
Atlas E/SVS	1	1	0.0000000
Blue Scout II	1	1	0.0000000
Chang Zheng 2	1	1	0.0000000
Conestoga 1620	1	1	0.0000000
Energiya	1	1	0.0000000
Europa I	3	3	0.0000000
Europa II	1	1	0.0000000
KT-1	3	3	0.0000000
LLV-1	1	1	0.0000000
N-1 11A52	4	4	0.0000000
Paektusan 1	1	1	0.0000000

LV.Type	numberOfLaunches	numberOfFailures	MLE
<chr>	<dbl>	<dbl>	<dbl>
Project Pilot	6	6	0.0000000
SS-520	1	1	0.0000000
Scout X-2B	1	1	0.0000000
Start	1	1	0.0000000
Super Strypi	1	1	0.0000000
Taurus 3110	2	2	0.0000000
Unha-2	1	1	0.0000000
VLS-1	2	2	0.0000000
Volna	1	1	0.0000000
Zenit-2 11K77.05	1	1	0.0000000

We can see there is more than one rocket with a 100% success rate! Nevertheless, **they often could have a small number of launches**. So we might not trust those rockets all that much. **MLE is frequentist-based; thus, it heavily relies on large sample sizes!**

Bayesian inference can help us decide while considering sample size by rocket with the upcoming model.

5. Non-Pooled Rocket Model

Our third Bayesian approach is a **non-pooled model**. In this model, we will make inference on the 367 different success probabilities (one by rocket) in a single `Stan` model. This approach goes to the other extreme when compared to `Complete_Pooled_Rocket`!

For the ith rocket, the model is:

$$
\begin{aligned}
\text{likelihood:} \qquad X_i | \pi_i &\sim \text{Binomial}(n_i, \pi_i) \quad \text{for } i = 1, \dots, 367 \\
\text{prior:} \qquad \pi_i &\sim \text{Beta}(a = 1, b = 1).
\end{aligned}
$$

Exercise 21

In this **non-pooled model**, is it possible to make inference on an overall π?

A. Yes.

B. No.

Solution to Exercise 21

No, unlike the **complete pooled model**, the likelihood this **non-pooled model** indicates a subindex i in our parameters of interest(i.e., π_i for the ith rocket where $n = 1, \dots, 367$).

Therefore, we will have $n = 367$ approximate posterior distributions to make inference on (one per rocket!).

5.1. Coding the Model

We code up this `Non_Pooled_Rocket` model on `Stan`. The `data` block structure from `Complete_Pooled_Rocket` remains. However, **we tweak the `model` block to assign the same prior `beta(a,b)` by rocket probability**. Furthermore, **the `parameters` block is adapted to have a vector of probabilities `pi` of size `num_rockets`**.

```
Non_Pooled_Rocket <- "data {
int<lower=1> num_rockets;     // number of rockets
int<lower=0> X[num_rockets];  // vector with sucesses by rocket
int<lower=1> n[num_rockets];  // vector with trials by rocket
real<lower=0> a;
real<lower=0> b;
}
parameters {
vector<lower=0,upper=1>[num_rockets] pi;  // vector of 367 probabilities of su
}
model {
for (i in 1:num_rockets){
pi[i] ~ beta(a,b);            // prior for pi_i by rocket
X[i] ~ binomial(n[i],pi[i]); // modelling the likelihood by rocket
}
}"
```

5.2. Running the MCMC Simulation

To run the `Non_Pooled_Rocket` model, we need to use the corresponding columns from `rockets`. The dictionary is identical to `Complete_Pooled_Rocket_dictionary`.

```
Non_Pooled_Rocket_dictionary <- list(
  num_rockets = nrow(rockets),
  X = rockets$numberOfLaunches - rockets$numberOfFailures,
  n = rockets$numberOfLaunches,
  a = 1, b = 1
)
```

```
posterior_Non_Pooled_Rocket <- stan(
  model_code = Non_Pooled_Rocket,
  data = Non_Pooled_Rocket_dictionary,
  chains = 1,
  iter = 10000,
  warmup = 1000,
  thin = 5,
  seed = 553,
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 6e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.(
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 10000 [  0%]  (Warmup)
Chain 1: Iteration: 1000 / 10000 [ 10%]  (Warmup)
Chain 1: Iteration: 1001 / 10000 [ 10%]  (Sampling)
Chain 1: Iteration: 2000 / 10000 [ 20%]  (Sampling)
Chain 1: Iteration: 3000 / 10000 [ 30%]  (Sampling)
Chain 1: Iteration: 4000 / 10000 [ 40%]  (Sampling)
Chain 1: Iteration: 5000 / 10000 [ 50%]  (Sampling)
Chain 1: Iteration: 6000 / 10000 [ 60%]  (Sampling)
Chain 1: Iteration: 7000 / 10000 [ 70%]  (Sampling)
Chain 1: Iteration: 8000 / 10000 [ 80%]  (Sampling)
Chain 1: Iteration: 9000 / 10000 [ 90%]  (Sampling)
Chain 1: Iteration: 10000 / 10000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 0.834 seconds (Warm-up)
Chain 1:                5.939 seconds (Sampling)
Chain 1:                6.773 seconds (Total)
Chain 1:
```

5.3. Output Summary

We obtain the output from `Non_Pooled_Rocket`. Since `Stan` labels the parameters in the posterior samples as `pi[1]`, ..., `pi[367]`, we will add a new column with the actual `rocket` name using `rockets$LV.Type`.

```
options(repr.matrix.max.rows = 15)

summary_Non_Pooled_Rocket <- as.data.frame(summary(posterior_Non_Pooled_Rocket
summary_Non_Pooled_Rocket <- summary_Non_Pooled_Rocket[-368, c("mean", "sd", "
  mutate_if(is.numeric, round, 3)
summary_Non_Pooled_Rocket$rocket <- rockets$LV.Type
summary_Non_Pooled_Rocket |>
  arrange(-mean)
```

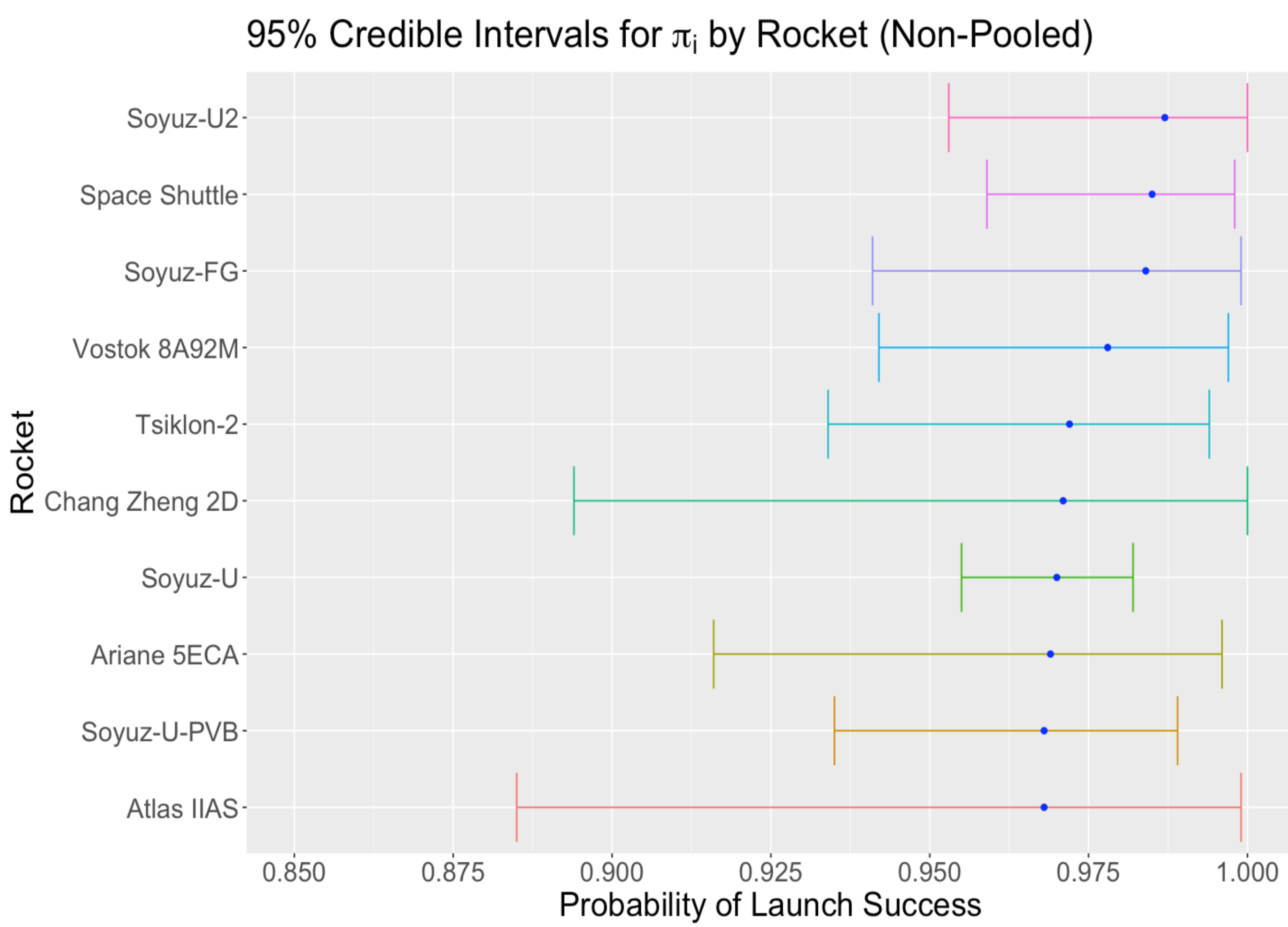
A data.frame: 367 × 5

	mean	sd	2.5%	97.5%	rocket
	<dbl>	<dbl>	<dbl>	<dbl>	<chr>
pi[266]	0.987	0.013	0.953	1.000	Soyuz-U2
pi[267]	0.985	0.011	0.959	0.998	Space Shuttle
pi[261]	0.984	0.016	0.941	0.999	Soyuz-FG
pi[357]	0.978	0.015	0.942	0.997	Vostok 8A92M
pi[343]	0.972	0.016	0.934	0.994	Tsiklon-2
pi[78]	0.971	0.029	0.894	1.000	Chang Zheng 2D
pi[264]	0.970	0.007	0.955	0.982	Soyuz-U
pi[16]	0.969	0.021	0.916	0.996	Ariane 5ECA
⋮	⋮	⋮	⋮	⋮	⋮
pi[281]	0.255	0.196	0.009	0.711	Taurus 3110
pi[353]	0.250	0.188	0.009	0.692	VLS-1
pi[28]	0.204	0.167	0.006	0.623	Atlas Able
pi[179]	0.203	0.165	0.008	0.601	KT-1
pi[157]	0.201	0.161	0.006	0.589	Europa I
pi[201]	0.165	0.140	0.005	0.507	N-1 11A52
pi[211]	0.125	0.113	0.003	0.422	Project Pilot

Note that the top 3 rockets in the posterior values for π_i do not correspond to the previous top 3 rockets in MLE from `rockets`. We can also plot the **top 10 rockets by posterior mean probability of launch** and the 95% credible intervals. Note the overlap across these ten rockets.

```
posterior_rocket_CIs_plot_Non_Pooled <- summary_Non_Pooled_Rocket |>
  arrange(-mean) |>
  slice(1:10) |>
  mutate(rocket = fct_reorder(rocket, mean)) |>
  ggplot(aes(x = mean, y = rocket)) +
  geom_errorbarh(aes(xmax = `2.5%`, xmin = `97.5%`, color = rocket)) +
  geom_point(color = "blue") +
  theme(
    plot.title = element_text(size = 24),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21),
    legend.position = "none"
  ) +
  ggtitle(expression(paste("95% Credible Intervals for ", pi[i], " by Rocket (
  labs(x = "Probability of Launch Success", y = "Rocket") +
  scale_x_continuous(limits=c(0.85, 1), breaks = seq(0.85, , 0.025))
```

```
posterior_rocket_CIs_plot_Non_Pooled
```



Let us add our Bayesian posterior means to `rockets` and sort them from the largest posterior mean to the lowest.

```
options(repr.matrix.max.rows = 20)
rockets$posterior_non_pooled_means <- summary_Non_Pooled_Rocket$mean
rockets |>
  arrange(-posterior_non_pooled_means)
```

A tibble: 367 × 5

LV.Type	numberOfLaunches	numberOfFailures	MLE	posterior_non_poole
<chr>	<dbl>	<dbl>	<dbl>	
Soyuz-U2	72	0	1.0000000	
Space Shuttle	135	1	0.9925926	
Soyuz-FG	59	0	1.0000000	
Vostok 8A92M	93	1	0.9892473	
Tsiklon-2	105	2	0.9809524	
Chang Zheng 2D	32	0	1.0000000	
Soyuz-U	633	18	0.9715640	
Ariane 5ECA	61	1	0.9836066	
Atlas IIAS	30	0	1.0000000	
Soyuz-U-PVB	154	4	0.9740260	
⋮	⋮	⋮	⋮	
Zenit-2 11K77.05	1	1	0.0000000	
Vanguard	11	8	0.2727273	
Lambda 4S	5	4	0.2000000	
Taurus 3110	2	2	0.0000000	
VLS-1	2	2	0.0000000	
Atlas Able	3	3	0.0000000	
KT-1	3	3	0.0000000	
Europa I	3	3	0.0000000	

LV.Type	numberOfLaunches	numberOfFailures	MLE	posterior_non_poole
<chr>	<dbl>	<dbl>	<dbl>	
N-1 11A52	4	4	0.0000000	
Project Pilot	6	6	0.0000000	

Exercise 22

From `rockets`, which one of the two estimators (MLE or the posterior mean with `Non_Pooled_Rocket`) do you think is more helpful in choosing your rocket?

Solution to Exercise 22

The Bayesian approach seems more adequate! This is because it considers the sample proportion (i.e., the MLE) **AND** the size of your sample. So, as far as the Bayesian approach is concerned, having 100% success in just a few attempts is not enough to have a large estimated probability of success π_i.

Based on our Bayesian findings, we would go with one of the top 2 rockets: `Soyuz-U2` or `Space Shuttle`.

6. Hierarchical Rocket Model

Except for the initial warmup model (which is a single iteration of `Non_Pooled_Rocket`!), we have tried different Bayesian models using two opposite approaches:

- `Complete_Pooled_Rocket`: We obtained approximate posterior samples for π, the overall probability of launch success. **However, this approach does not allow us to infer any probability of an individual rocket.**
- `Non_Pooled_Rocket`: We obtained approximate posterior samples for the probability of launch success π_i by **EACH** rocket type.

To choose the most successful rocket, `Non_Pooled_Rocket` worked reasonably better than MLE. However, recall the formal modelling:

$$\begin{aligned} \text{likelihood:} \qquad X_i | \pi_i &\sim \text{Binomial}(n_i, \pi_i) \quad \text{for } i = 1, \dots, 367 \\ \text{prior:} \qquad \pi_i &\sim \text{Beta}(a = 1, b = 1). \end{aligned}$$

> ⚠ **Caution**
>
> **Why should we restrict our prior for each π_i to fixed hyperparameters a and b for the Beta distribution?** We have only been using $\text{Beta}(a = 1, b = 1)$ thorughout the whole lecture!

There are other important drawbacks in `Non_Pooled_Rocket`:

- Firstly, **we cannot generalize this model to some new rocket**.
- Secondly, **we would not take any valuable information provided by some rocket type to infer on another type in `rockets`**.

Here comes another Bayesian twist! To make the model even more flexible, let us make a and b random too in the Beta prior. Hence, a and b will become parameters too (**and not hyperparameters!**).

Important

This is the foundational point of a **hierarchical model**. This hierarchical model will take into account two sources of variability (different to the predictive sources of variability we saw in Lecture 5 - Bayesian Normal Linear Regression and Hypothesis Testing):

- **Within-group variability:** How much variability we have among the observations within each group of interest. In this case, the group is a rocket type, while the observations are different trials.
- **Between-group variability:** How much variability we have from group to group. In this example, we can assess this matter from one rocket type to another.

Essentially, the hierarchical model takes the best of `Non_Pooled_Rocket` and `Complete_Pooled_Rocket`. **Note it will be able to infer on the following:**

- It will use valuable information from all the `rockets` in the dataset to infer the launch success probability on **a specific rocket**.
- We can obtain the posterior predictive distribution of the launch success probability **for a new rocket**.

Our hierarchical model is formally defined as:

$$
\begin{gathered}
\text{likelihood:} \qquad X_i \mid \pi_i \sim \text{Binomial}(n_i, \pi_i) \quad \text{for } i = 1, \dots, 367 \\
\text{priors:} \qquad \pi_i \sim \text{Beta}(a, b) \\
a \sim \text{Gamma}(0.001, 0.001) \\
b \sim \text{Gamma}(0.001, 0.001).
\end{gathered}
$$

What is the big picture here?

First, **this hierarchical model learns a different π_i from each rocket type**. The key here is that the data from all rockets contribute to our inferences about a and b. Thus, **it is like our prior for each rocket is being set by all the rockets!**

6.1. The Priors

Why a Gamma distribution for a and b?

Recall the probability density function (PDF) of the Beta distribution:

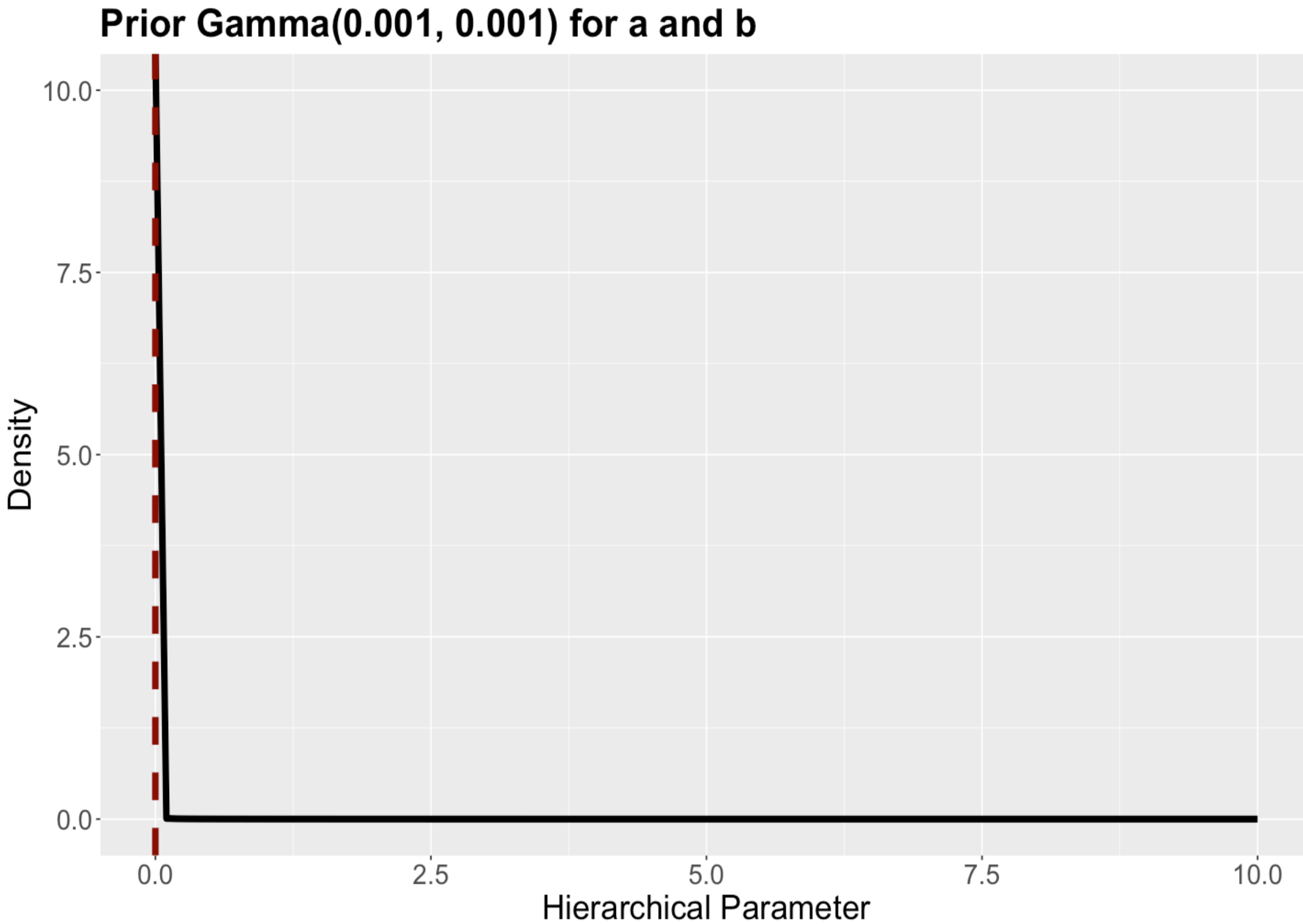
$$f(\pi_i) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi_i^{a-1} (1-\pi_i)^{b-1} \quad \text{for} \quad 0 \leq \pi_i \leq 1.$$

In this PDF, we have that $a > 0$ and $b > 0$ are continuous. Thus, **a nonnegative continuous Gamma is suitable in this case**.

```
# Plotting Gamma prior
prior_gamma <- ggplot() +
  xlim(0, 10) +
  ylim(0, 10) +
  geom_function(fun = dgamma, args = list(shape = 0.001, rate = 0.001), linewi
  geom_vline(xintercept = 0.00001, colour = "darkred", linetype = "dashed", li
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text.x = element_text(size = 17, angle = 0),
    axis.text.y = element_text(size = 17, angle = 0),
    axis.title = element_text(size = 21),
  ) +
  labs(y = "Density", x = "Hierarchical Parameter") +
  ggtitle("Prior Gamma(0.001, 0.001) for a and b" )
```

The corresponding Gamma prior distribution for a and b are plotted below (the prior mean is the vertical dashed red line).

```
prior_gamma
```



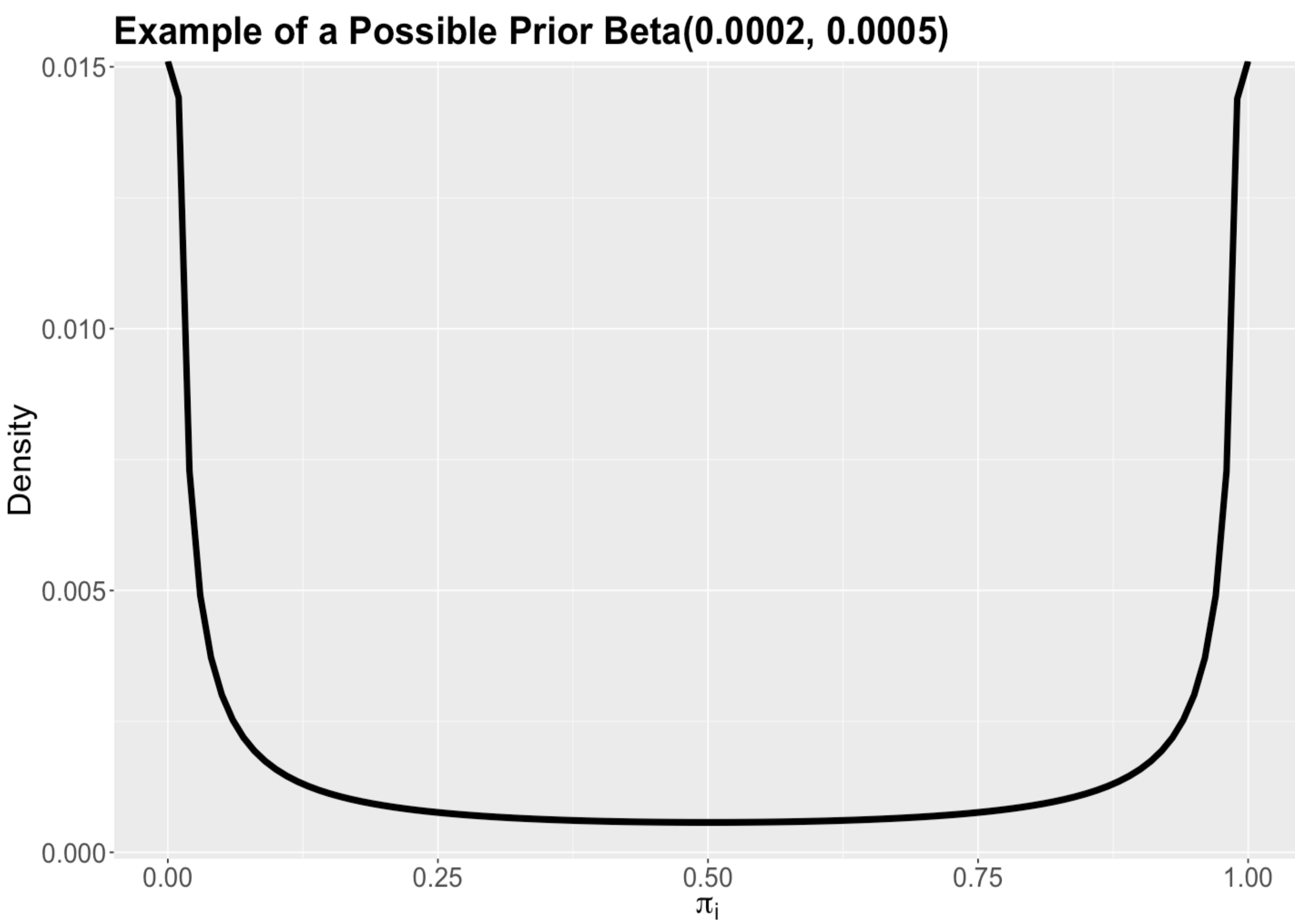
Note the Gamma priors **tend to assign small values** for a and b in

$$\pi_i \sim \text{Beta}(a, b).$$

As an example, for a given π_i, suppose its Beta prior has $a = 0.0002$ and $b = 0.0005$. This Beta prior for π_i is plotted below. We can see these prior beliefs for a and b basically put the success probability π_i close to either its lower or upper bound. Translating this into our prior beliefs, **we think that any rocket would be either good or bad (we do not want any intermediate point here since we are using that rocket to go to space!).**

```
# Plotting Beta prior
prior_beta <- ggplot() +
  xlim(0, 1) +
  geom_function(fun = dbeta, args = list(shape1 = 0.0002, shape2 = 0.0005), li
  theme(
    plot.title = element_text(size = 24, face = "bold"),
    axis.text.x = element_text(size = 17, angle = 0),
    axis.text.y = element_text(size = 17, angle = 0),
    axis.title = element_text(size = 21),
  ) +
  labs(y = "Density", x = expression(pi["i"])) +
  ggtitle("Example of a Possible Prior Beta(0.0002, 0.0005)")
```

```
prior_beta
```



6.2. Coding the Model

We code up this `Hierarchical_Rocket` model on `Stan`:

- The `data` block structure will not have `a` and `b` since they are random now.

- Furthermore, we add the Gamma priors for a and b in the `model` block while keeping the Beta prior for each π_i.
- The `parameters` block has new variables: `a` and `b`.
- Besides the posterior sampled probabilities by rocket type, we will also obtain posterior samples for a and b.

```
Hierarchical_Rocket <- "data {
int<lower=1> num_rockets;    // number of rockets
int<lower=0> X[num_rockets];  // vector with sucesses by rocket
int<lower=1> n[num_rockets];  // vector with trials by rocket
}
parameters {
vector<lower=0,upper=1>[num_rockets] pi; // vector of 367 probabilities of lau
real<lower=0> a; // beta parameter
real<lower=0> b; // beta parameter
}
model {
a ~ gamma(0.001, 0.001);     // prior for a
b ~ gamma(0.001, 0.001);     // prior for b

for (i in 1:num_rockets){
pi[i] ~ beta(a,b);           // modelling the pi_i by rocket
X[i] ~ binomial(n[i],pi[i]); // modelling the likelihood by rocket
}
}"
```

6.3. Running the MCMC Simulation

To run the `Hierarchical_Rocket` model, we need to use the corresponding columns from `rockets`. This dictionary will not have `a` and `b`.

```
Hierarchical_Rocket_dictionary <- list(
  num_rockets = nrow(rockets),
  X = rockets$numberOfLaunches - rockets$numberOfFailures,
  n = rockets$numberOfLaunches
)
```

```
posterior_Hierarchical_Rocket <- stan(
  model_code = Hierarchical_Rocket,
  data = Hierarchical_Rocket_dictionary,
  chains = 1,
  iter = 10000,
  warmup = 1000,
  thin = 5,
  seed = 553,
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 7.8e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.7
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 10000 [  0%]  (Warmup)
Chain 1: Iteration: 1000 / 10000 [ 10%]  (Warmup)
Chain 1: Iteration: 1001 / 10000 [ 10%]  (Sampling)
Chain 1: Iteration: 2000 / 10000 [ 20%]  (Sampling)
Chain 1: Iteration: 3000 / 10000 [ 30%]  (Sampling)
Chain 1: Iteration: 4000 / 10000 [ 40%]  (Sampling)
Chain 1: Iteration: 5000 / 10000 [ 50%]  (Sampling)
Chain 1: Iteration: 6000 / 10000 [ 60%]  (Sampling)
Chain 1: Iteration: 7000 / 10000 [ 70%]  (Sampling)
Chain 1: Iteration: 8000 / 10000 [ 80%]  (Sampling)
Chain 1: Iteration: 9000 / 10000 [ 90%]  (Sampling)
Chain 1: Iteration: 10000 / 10000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 1.932 seconds (Warm-up)
Chain 1:                11.696 seconds (Sampling)
Chain 1:                13.628 seconds (Total)
Chain 1:
```

6.4. Output Summary

We obtain the output from `posterior_Hierarchical_Rocket`. Since `Stan` labels the parameters in the posterior samples as `pi[1]`, ..., `pi[367]`, we will add a new column with the actual `rocket` name using `rockets$LV.Type`.

```
options(repr.matrix.max.rows = 12)
summary_Hierarchical_Rocket <- as.data.frame(summary(posterior_Hierarchical_Ro
summary_Hierarchical_Rocket <- summary_Hierarchical_Rocket[1:367, c("mean", "s
  mutate_if(is.numeric, round, 3)
summary_Hierarchical_Rocket$rocket <- rockets$LV.Type
summary_Hierarchical_Rocket |>
  arrange(-mean)
```

A data.frame: 367 × 5

	mean	sd	2.5%	97.5%	rocket
	<dbl>	<dbl>	<dbl>	<dbl>	<chr>
pi[266]	0.991	0.011	0.961	1.000	Soyuz-U2
pi[261]	0.989	0.013	0.956	1.000	Soyuz-FG
pi[267]	0.988	0.010	0.963	0.999	Space Shuttle
pi[357]	0.983	0.013	0.950	0.998	Vostok 8A92M
pi[78]	0.982	0.021	0.923	1.000	Chang Zheng 2D
pi[54]	0.980	0.023	0.915	1.000	Atlas IIAS
⋮	⋮	⋮	⋮	⋮	⋮
pi[192]	0.545	0.074	0.398	0.685	Molniya 8K78
pi[184]	0.544	0.155	0.241	0.821	Lambda 4S
pi[360]	0.529	0.125	0.284	0.759	Vostok-L 8K72
pi[201]	0.496	0.162	0.187	0.800	N-1 11A52
pi[351]	0.470	0.123	0.242	0.701	Vanguard
pi[211]	0.405	0.145	0.147	0.699	Project Pilot

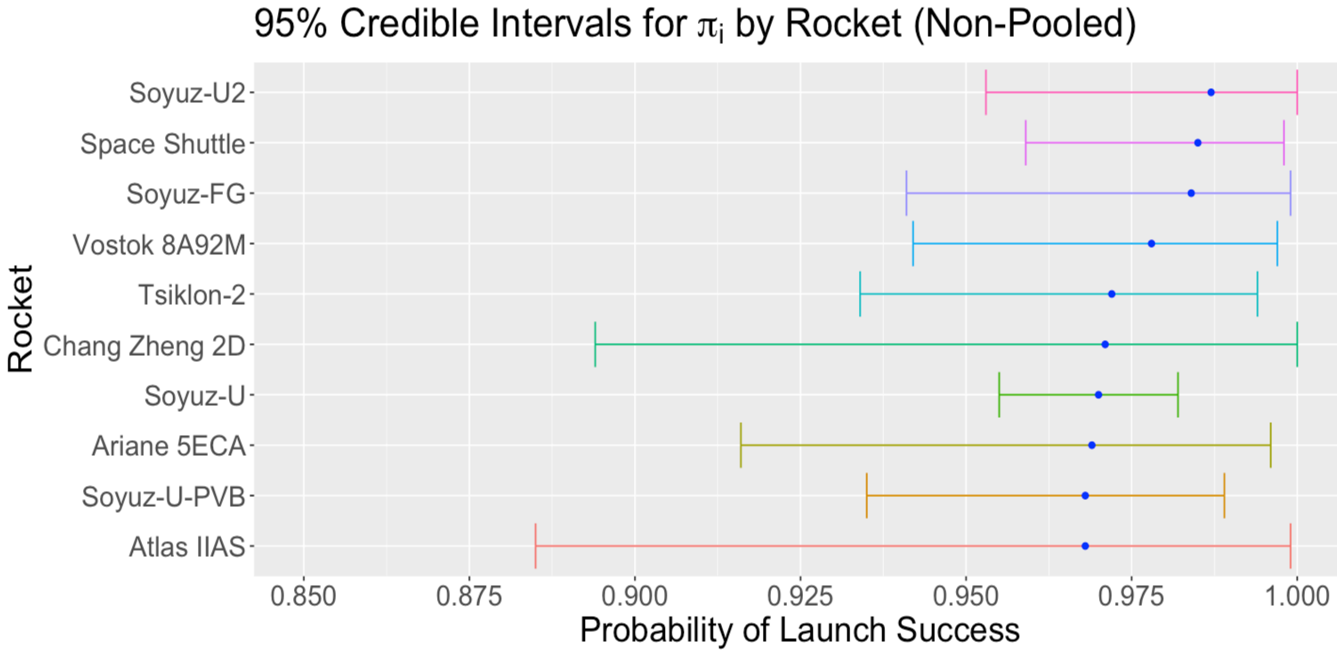
We plot the top 10 rockets and the posterior 95% credible intervals from `posterior_Hierarchical_Rocket`. Then we compare this top 10 versus the ones obtained with `posterior_Non_Pooled_Rocket`. The credible interval overlapping is still there.

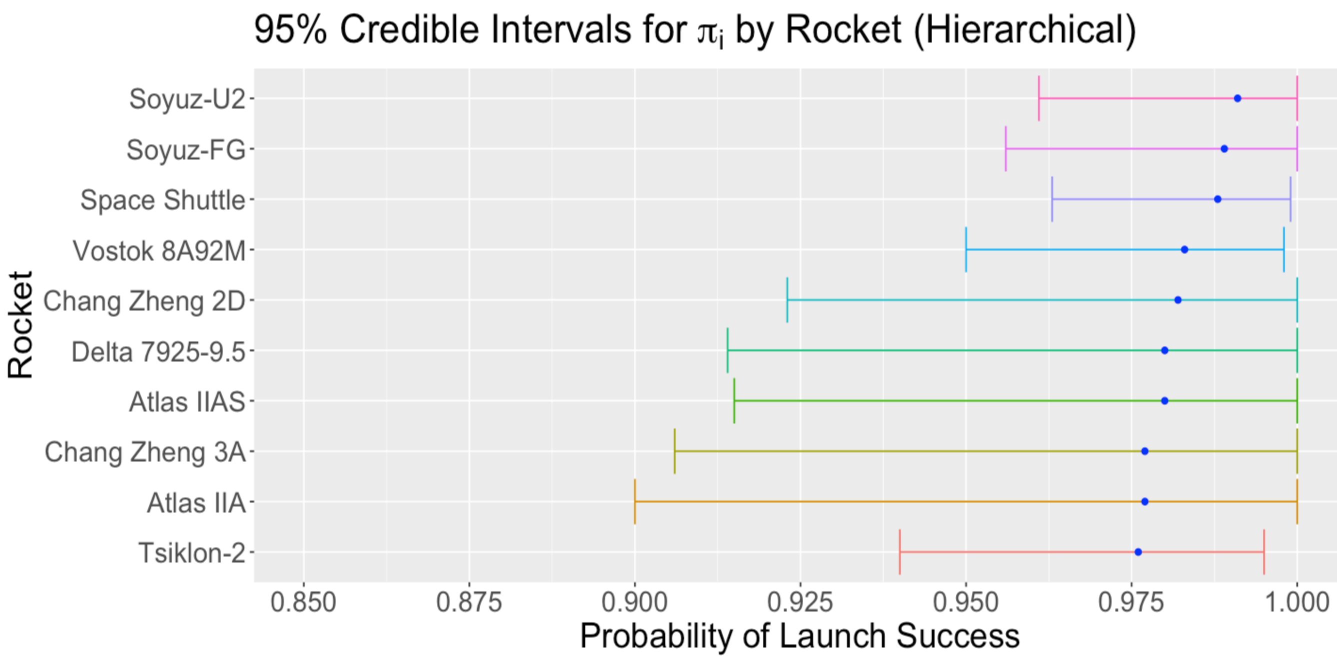
Nonetheles, for those rockets still present in the top 10 from `posterior_Hierarchical_Rocket`, the credible intervals are narrower! **We have more precise results.**

```
posterior_rocket_CIs_plot_Hierarchical <- summary_Hierarchical_Rocket |>
  arrange(-mean) |>
  slice(1:10) |>
  mutate(rocket = fct_reorder(rocket, mean)) |>
  ggplot(aes(x = mean, y = rocket)) +
  geom_errorbarh(aes(xmax = `2.5%`, xmin = `97.5%`, color = rocket)) +
  geom_point(color = "blue") +
  theme(
    plot.title = element_text(size = 24),
    axis.text = element_text(size = 17),
    axis.title = element_text(size = 21),
    legend.position = "none"
  ) +
  ggtitle(expression(paste("95% Credible Intervals for ", pi[i], " by Rocket (
  labs(x = "Probability of Launch Success", y = "Rocket") +
  scale_x_continuous(limits = c(0.85, 1), breaks = seq(0.85, , 0.025))
```

```
options(repr.plot.height = 11, repr.plot.width = 12)

plot_grid(posterior_rocket_CIs_plot_Non_Pooled, posterior_rocket_CIs_plot_Hier
```





So, what rocket should we choose with a hierarchical model?

If we had to choose based on the **hierarchical** Bayesian results, we would still take `Soyuz-U2` or `Space Shuttle` (now with narrower 95% credible intervals!).

Why are our estimates more precise?

A useful property of hierarchical models is the notion of **"borrowing strength"**, where a rocket uses the information about the probability of success from other rockets. As we saw above, this affects the mean of our posterior distributions, but it also affects the variance.

This helps us to learn the parameters and actually reduce the posterior variance of our Bayesian estimates!

How about the posterior results for a and b?

Also, we have posterior summaries for `a` and `b`.

```
summary_Hierarchical_Rocket_ab <- as.data.frame(summary(posterior_Hierarchical
summary_Hierarchical_Rocket_ab <- summary_Hierarchical_Rocket_ab[368:369, c("m
  mutate_if(is.numeric, round, 3)
summary_Hierarchical_Rocket_ab
```

A data.frame: 2 × 4

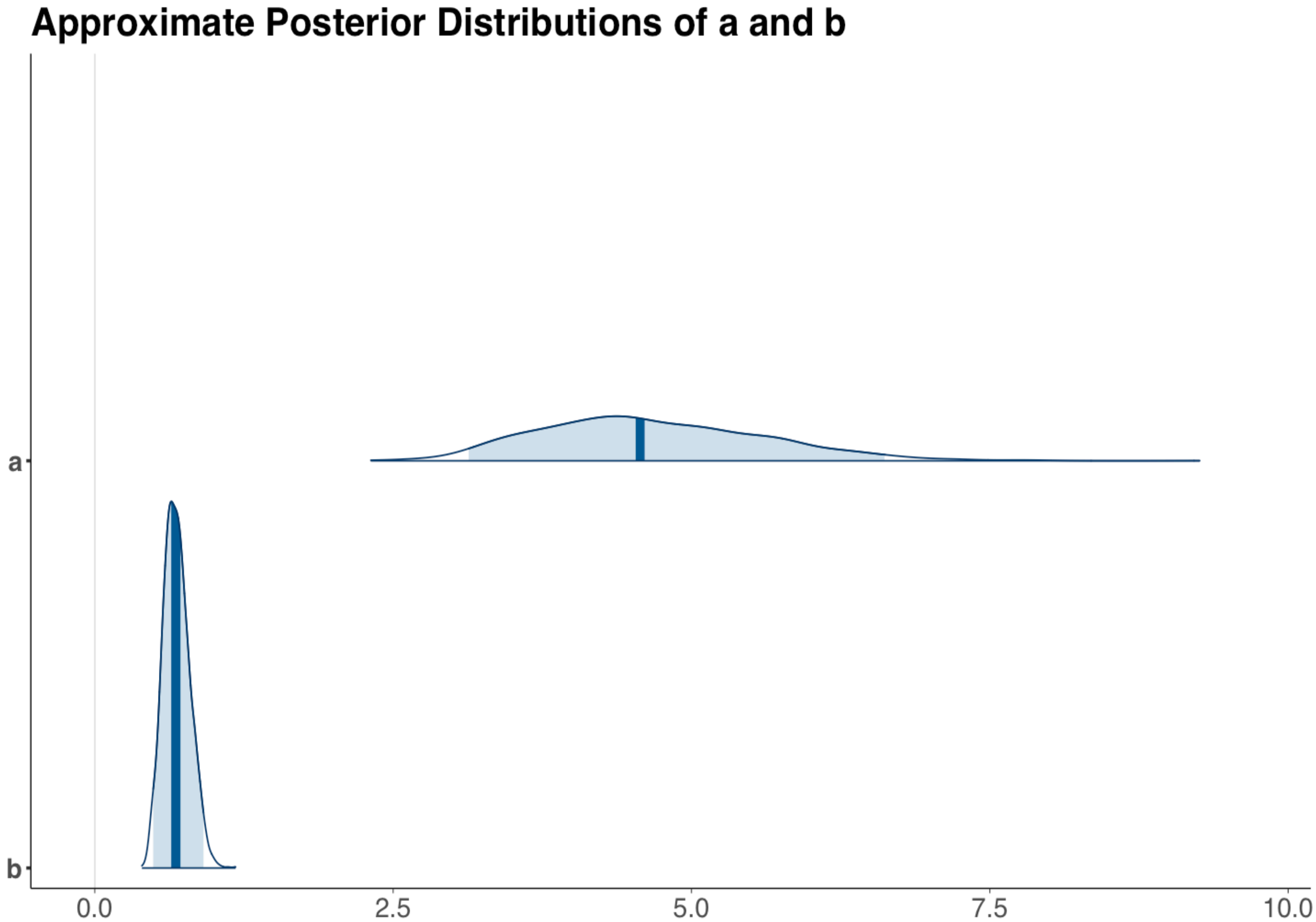
	mean	sd	2.5%	97.5%
	<dbl>	<dbl>	<dbl>	<dbl>
a	4.673	0.933	3.133	6.616
b	0.685	0.109	0.490	0.911

Note the posterior distributions of a and b are right-skewed.

```
post_plot_Hierarchical_Rocket_ab <- mcmc_areas(posterior_Hierarchical_Rocket,
  theme(
    plot.title = element_text(size = 24, face = "bold", family = "sans"),
    axis.text.x = element_text(size = 17, angle = 0, family = "sans"),
    axis.text.y = element_text(size = 17, angle = 0, family = "sans"),
    axis.title = element_text(size = 21, family = "sans"),
    strip.text.x = element_text(size = 17, family = "sans")
  ) +
  ggtitle("Approximate Posterior Distributions of a and b")
```

```
options(repr.plot.height = 8, repr.plot.width = 12)

post_plot_Hierarchical_Rocket_ab
```



How is the average launch success probability for all the `rockets`?

We will use the posterior means of a and b as parameters of a Beta distribution to report posterior metrics on the **average launch probability of success** π

```
summary_Hierarchical_Rocket_ab
```

A data.frame: 2 × 4

	mean	sd	2.5%	97.5%
	<dbl>	<dbl>	<dbl>	<dbl>
a	4.673	0.933	3.133	6.616
b	0.685	0.109	0.490	0.911

Using the posterior means for `a` and `b` from `summary_Hierarchical_Rocket_ab`, let us deliver the summary statistics for a theoretical Beta distribution via `summarize_beta()` from package `bayesrules`.

```
summarize_beta(alpha = summary_Hierarchical_Rocket_ab$mean[1], beta = summary_
  mutate_if(is.numeric, round, 3)
```

A data.frame: 1 × 4

mean	mode	var	sd
<dbl>	<dbl>	<dbl>	<dbl>
0.872	1	0.018	0.132

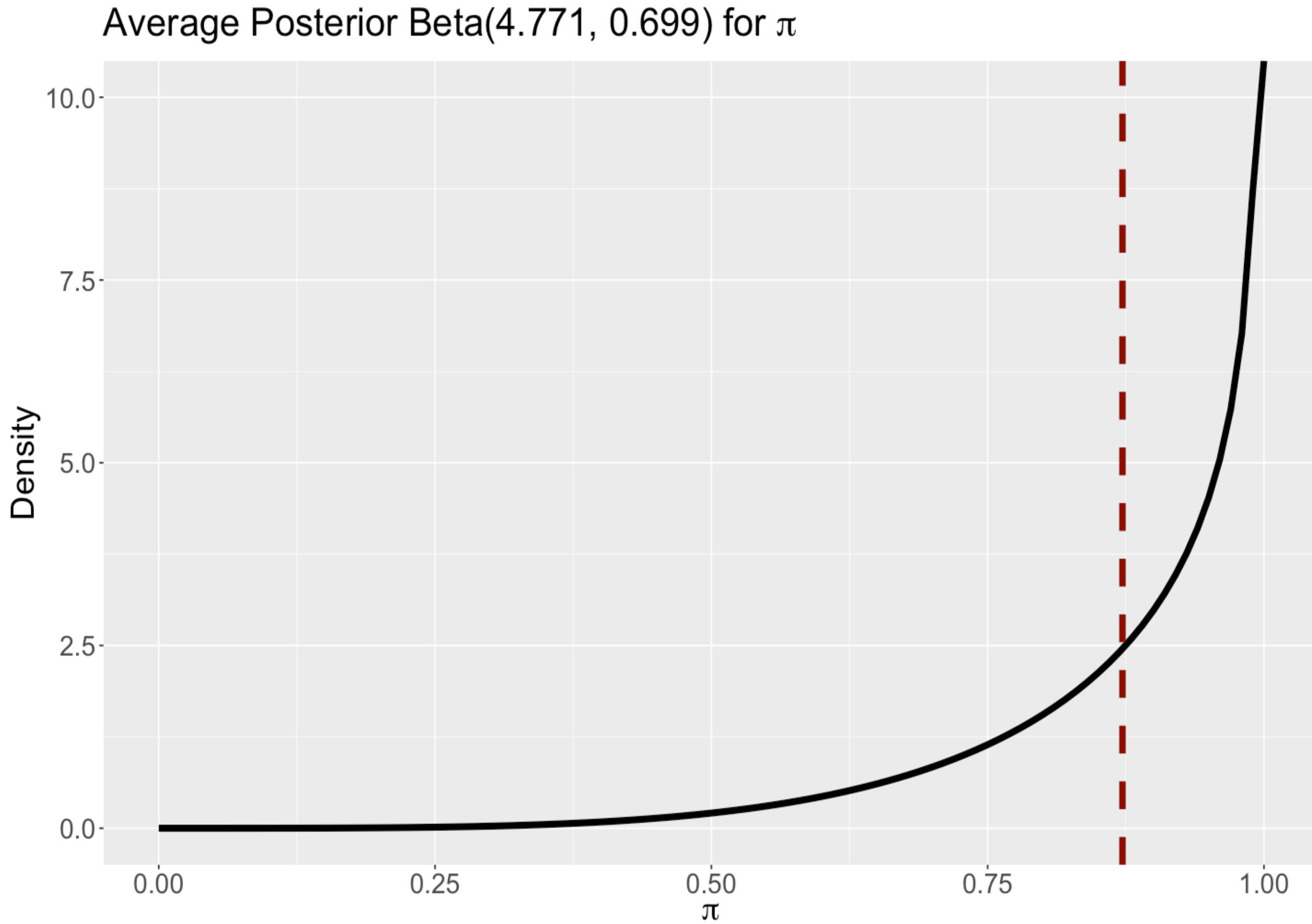
Then, using posterior means $a = 4.673$ and $b = 0.685$, we plot this **average posterior theoretical Beta**. Note that its corresponding mean 0.872 is indicated as a vertical dashed line.

```
# Plotting average Beta posterior
average_beta_posterior <- ggplot() +
  xlim(0, 1) +
  ylim(0, 10) +
  geom_function(fun = dbeta, args = list(shape1 = 4.673, shape2 = 0.685), line
  geom_vline(xintercept = 0.872, colour = "darkred", linetype = "dashed", line
  theme(
    plot.title = element_text(size = 24),
    axis.text.x = element_text(size = 17, angle = 0),
    axis.text.y = element_text(size = 17, angle = 0),
    axis.title = element_text(size = 21),
  ) +
  labs(y = "Density", x = expression(pi)) +
  ggtitle(expression(paste("Average Posterior Beta(4.771, 0.699) for ", pi)))
```

This average Beta posterior leans more towards high probabilities for launch success **given the evidence provided by all `rockets` along with our priors**.

```
options(repr.plot.height = 8, repr.plot.width = 12)

average_beta_posterior
```



6.5. Prediction for a New Rocket

We will modify our `Hierarchical_Rocket` model to incorporate the block `generated quantities` for a predicted probability `pi_pred`. We use the same process described in Lecture 5 - Bayesian Normal Linear Regression and Hypothesis Testing to obtain predictions. This process considers the **posterior variability in `parameters`** and the **sampling variability**. The predictive samples will yield a **posterior predictive distribution** for `pi_pred`.

```
pred_Hierarchical_Rocket <- "data {
int<lower=1> num_rockets;    // number of rockets
int<lower=0> X[num_rockets];  // vector with sucesses by rocket
int<lower=1> n[num_rockets];  // vector with trials by rocket
}
parameters {
vector<lower=0,upper=1>[num_rockets] pi; // vector of 367 probabilities of lau
real<lower=0> a; // beta parameter
real<lower=0> b; // beta parameter
}
model {
a ~ gamma(0.001, 0.001);     // prior for a
b ~ gamma(0.001, 0.001);     // prior for b

for (i in 1:num_rockets){
pi[i] ~ beta(a,b);           // modelling the pi_i by rocket
X[i] ~ binomial(n[i],pi[i]); // modelling the likelihood by rocket
}
}
generated quantities {
  real<lower=0,upper=1> pi_pred = beta_rng(a, b);
}"
```

The MCMC simulation uses the same `Hierarchical_Rocket_dictionary` but the modified `pred_Hierarchical_Rocket`.

```
pred_posterior_Hierarchical_Rocket <- stan(
  model_code = pred_Hierarchical_Rocket,
  data = Hierarchical_Rocket_dictionary,
  chains = 1,
  iter = 10000,
  warmup = 1000,
  thin = 5,
  seed = 553,
)
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 8.7e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.8
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 10000 [  0%]  (Warmup)
Chain 1: Iteration: 1000 / 10000 [ 10%]  (Warmup)
Chain 1: Iteration: 1001 / 10000 [ 10%]  (Sampling)
Chain 1: Iteration: 2000 / 10000 [ 20%]  (Sampling)
Chain 1: Iteration: 3000 / 10000 [ 30%]  (Sampling)
Chain 1: Iteration: 4000 / 10000 [ 40%]  (Sampling)
Chain 1: Iteration: 5000 / 10000 [ 50%]  (Sampling)
Chain 1: Iteration: 6000 / 10000 [ 60%]  (Sampling)
Chain 1: Iteration: 7000 / 10000 [ 70%]  (Sampling)
Chain 1: Iteration: 8000 / 10000 [ 80%]  (Sampling)
Chain 1: Iteration: 9000 / 10000 [ 90%]  (Sampling)
Chain 1: Iteration: 10000 / 10000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 1.896 seconds (Warm-up)
Chain 1:                11.803 seconds (Sampling)
Chain 1:                13.699 seconds (Total)
Chain 1:
```

We only need the posterior metrics for `pi_pred`. **Note the mean is extremely similar to the overall mean of π!** Moreover, the posterior predictive distribution in `post_hist_Hierarchical_Rocket_pi_pred` is similar to `average_beta_posterior`.

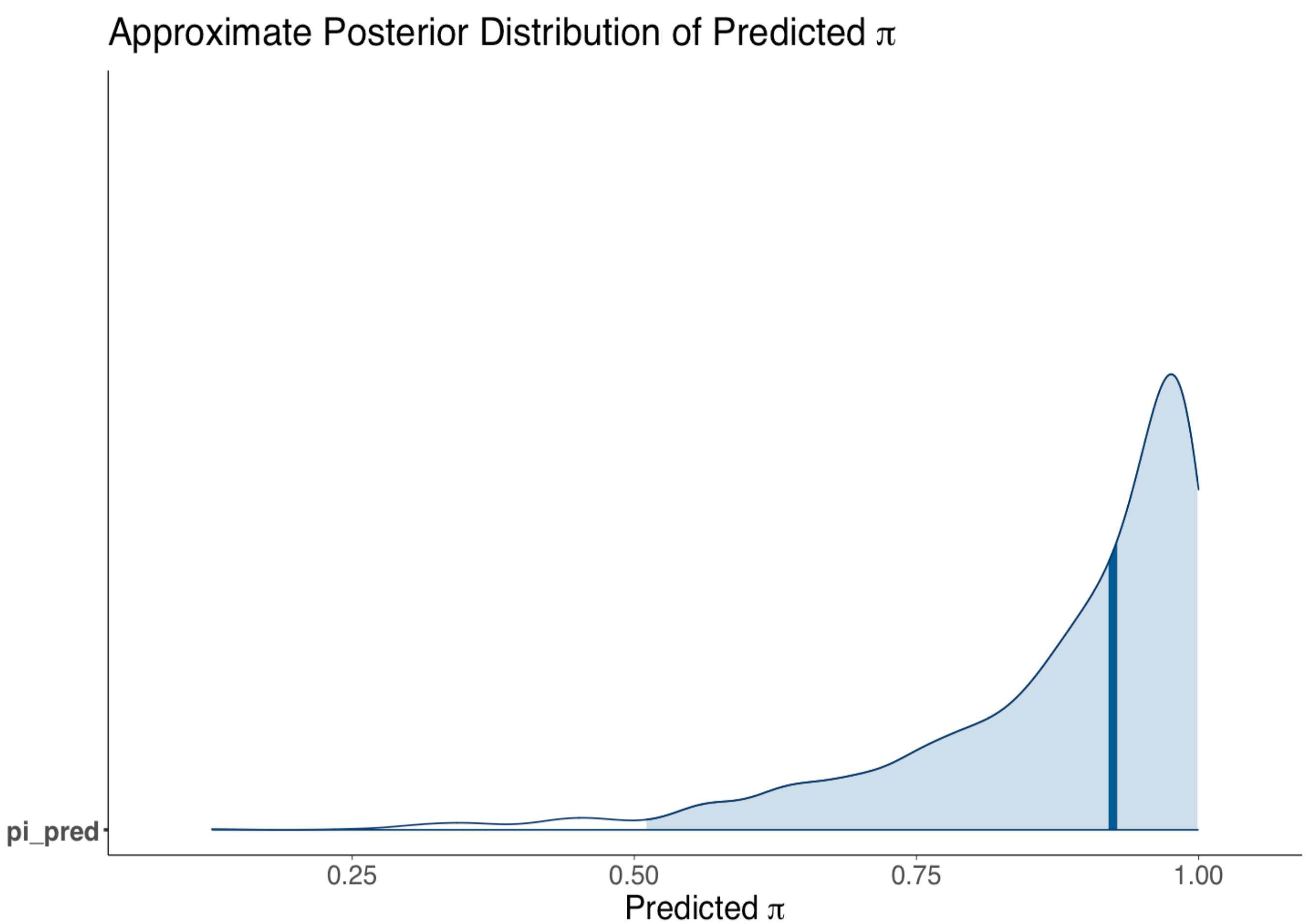
```
summary_Hierarchical_Rocket_pi_pred <- as.data.frame(summary(pred_posterior_Hi
summary_Hierarchical_Rocket_pi_pred <- summary_Hierarchical_Rocket_pi_pred[370
  mutate_if(is.numeric, round, 3)
summary_Hierarchical_Rocket_pi_pred
```

A data.frame: 1 × 4

	mean	sd	2.5%	97.5%
	<dbl>	<dbl>	<dbl>	<dbl>
pi_pred	0.876	0.135	0.511	0.999

```
post_hist_Hierarchical_Rocket_pi_pred <- mcmc_areas(pred_posterior_Hierarchica
  theme(
    plot.title = element_text(size = 24, family = "sans"),
    axis.text.x = element_text(size = 17, angle = 0, family = "sans"),
    axis.text.y = element_text(size = 17, angle = 0, family = "sans"),
    axis.title = element_text(size = 21, family = "sans"),
    strip.text.x = element_text(size = 21, family = "sans")
  ) +
  ggtitle(expression(paste("Approximate Posterior Distribution of Predicted ",
  xlab(expression(paste("Predicted ", pi)))
```

```
post_hist_Hierarchical_Rocket_pi_pred
```



> **Why do we have these similarities?**
>
> Our launch success mean posterior prediction for a new rocket is similar to the posterior mean of π because **this is the best Bayesian model we can obtain without further covariates (or features)**! So, this is a mean probability prediction for the rocket population for a launch success.

7. Wrapping Up

- Even though hierarchical models involve more Bayesian complexity, they are helpful to take the best from non-pooled and pooled models.
- These hierarchical models are also flexible enough to provide predictions for new observations.
- We saw a model without covariates. Nevertheless, the hierarchical approach can be extended to Bayesian regression models. These are the counterpart of frequentist mixed-effects models.