Distribution Cheatsheet

Contents

The below mindmap summarizes all the distributions to be covered in this course.

Discrete Distributions

Bernoulli

Process

It is a random variable X that is binary as follows

$$X = \begin{cases} 1 & \text{if there is a success,} \\ 0 & \text{otherwise.} \end{cases}$$

The value 1 has a probability of $0 \leq p \leq 1$, whereas the value 0 has a probability of $1 - p$.

Then, X is said to have a Bernoulli distribution:

$$X \sim \text{Bernoulli}(p).$$

PMF

A Bernoulli distribution is characterized by the PMF

$$P(X = x \mid p) = p^x (1-p)^{1-x} \quad \text{for} \quad x = 0, 1.$$

Mean

The mean of a Bernoulli random variable is defined as:

$$\mathbb{E}(X) = p.$$

Variance

The variance of a Bernoulli random variable is defined as:

$$\text{Var}(X) = p(1-p).$$

Binomial Distribution

Process

Let X be the number of succcesses after n independent Bernoulli trials with probability of success $0 \leq p \leq 1$.

Then, X is said to have a Binomial distribution:

$$X \sim \text{Binomial}\,(n, p).$$

Probability Mass Function (PMF)

A Binomial distribution is characterized by the PMF

$$P(X = x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for} \quad x = 0, 1, \ldots, n.$$

Term $\binom{n}{x}$ indicates the total number of combinations for x successes out of n trials:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}.$$

Mean

The mean of a Binomial random variable is defined as:

$$\mathbb{E}(X) = np.$$

Variance

The variance of a Binomial random variable is defined as:

$$\text{Var}(X) = np(1-p).$$

Geometric Distribution

Process

Let X be the number of failed independent Bernoulli trials before experiencing the first success.

Then, X is said to have a Geometric distribution:

$$X \sim \text{Geometric}(p).$$

PMF

A Geometric distribution is characterized by the PMF

$$P(X = x \mid p) = p(1-p)^x \quad \text{for} \quad x = 0, 1, \ldots$$

Mean

The mean of a Geometric random variable is defined as:

$$\mathbb{E}(X) = \frac{1-p}{p}.$$

Variance

The variance of a Geometric random variable is defined as:

$$\text{Var}(X) = \frac{1-p}{p^2}.$$

Negative Binomial Distribution (a.k.a. Pascal)

Process

Let X be the number of failed independent Bernoulli trials before experiencing k independent successes.

Then, X is said to have a Negative Binomial distribution:

$$X \sim \text{Negative Binomial}(k, p).$$

PMF

A Negative Binomial distribution is characterized by the PMF

$$P(X = x \mid k, p) = \binom{k - 1 + x}{x} p^k (1 - p)^x \quad \text{for} \quad x = 0, 1, \ldots$$

Mean

The mean of a Negative Binomial random variable is defined as:

$$\mathbb{E}(X) = \frac{k(1 - p)}{p}.$$

Variance

The variance of a Negative Binomial random variable is defined as:

$$\text{Var}(X) = \frac{k(1 - p)}{p^2}.$$

Poisson

Process

Let X be the number of events happening in a fixed interval of time or space at some average rate λ.

Then, X is said to have a Poisson distribution:

$$X \sim \text{Poisson}(\lambda).$$

PMF

A Poisson distribution is characterized by the PMF

$$P(X = x \mid \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!} \quad \text{for} \quad x = 0, 1, \dots \tag{55}$$

Mean

The mean of a Poisson random variable is defined as:

$$\mathbb{E}(X) = \lambda.$$

Variance

The variance of a Poisson random variable is defined as:

$$\text{Var}(X) = \lambda.$$

`R` Functions

The Poisson distribution has some handy `R` functions to perform different probabilistic computations. Let us check them via some quick example.

Suppose that we have the following random variable:

$$X = \text{Number of orders received at an online store during the weekend.}$$

Note we have a count-type random variable denoting the number of events (i.e., **orders**) happening in a fixed interval of time (i.e., **the weekend**). Let us assume that, in average during the weekend, we receive 15 orders. We can model X as a Poisson random variable:

$$X \sim \text{Poisson}(\lambda = 15).$$

`ppois()`

Given the above random variable modelling, let us answer the following:

1. What is the probability of getting more than 10 orders during the weekend, i.e., $P(X > 10)$?
2. What is the probability of getting between 12 and 16 orders during the weekend, i.e., $P(12 \leq X \leq 16)$?

We can manually compute these probabilities via Equation (55). Nevertheless, let us try a quicker way via `ppois()`. This function allows us to compute probabilities as follows:

- We must indicate an argument `q` for the quantile corresponding to $P(X \leq \texttt{q})$.
- Argument `lambda` corresponds to λ.

```
answer_ppois_1 <- 1 - ppois(q = 10, lambda = 15, lower.tail = TRUE) # lower.ta
answer_ppois_1 <- round(answer_ppois_1, 3) # Rounding to three decimal places
answer_ppois_1
```

0.882

The above code corresponds to:

$$\begin{aligned} P(X > 10) &= 1 - P(X \leq 10) \\ &= 0.882. \end{aligned}$$

Now, for the second question:

```
answer_ppois_2 <- ppois(q = 16, lambda = 15, lower.tail = TRUE) -
  ppois(q = 12, lambda = 15, lower.tail = TRUE)
answer_ppois_2 <- round(answer_ppois_2, 3)
answer_ppois_2
```

0.397

The above code corresponds to:

$$\begin{aligned} P(12 \leq X \leq 16) &= P(X \leq 16) - P(X \leq 12) \\ &= 0.397. \end{aligned}$$

`qpois()`

It is also possible to obtain the p-quantile $Q(p)$ associated with the probability $P\left[X \leq Q(p)\right]$. Suppose we want to obtain the 0.6-quantile, i.e. $Q(0.6)$, for this specific example. Function `qpois()` allows us to compute this quantile as follows:

- We must indicate an argument `p` for the corresponding probability p.
- Argument `lambda` corresponds to λ.

```
answer_qpois <- qpois(p = 0.6, lambda = 15)
answer_qpois <- answer_qpois
answer_qpois
```

16

The above code corresponds to:

$$\begin{aligned} P\left[X \leq Q(0.6)\right] &= 0.6 \\ P\left[X \leq 16\right] &= 0.6. \end{aligned}$$

Uniform (Discrete)

Process

Let X be the random discrete outcome of a finite set of N outcomes. Suppose each outcome has a numeric label whose lower and upper bounds are a and b, respectively. Then, X is said to have a discrete Uniform distribution:

$$X \sim \text{Discrete Uniform}(a, b).$$

PMF

A discrete Uniform distribution is characterized by the PMF

$$P(X = x \mid a, b) = \frac{1}{N} \quad \text{for} \quad x = a, \dots, b.$$

Mean

The mean of a discrete Uniform random variable is defined as:

$$\mathbb{E}(X) = \frac{a + b}{2}.$$

Variance

The variance of a discrete Uniform random variable is defined as:

$$\text{Var}(X) = \frac{N^2 - 1}{12}.$$

Continuous Distributions

Beta

Process

The Beta family of distributions is defined for random variables taking values between 0 and 1, so is useful for modelling the distribution of proportions. This family is quite flexible, and has the Uniform distribution as a special case. It is characterized by two positive shape parameters, $\alpha > 0$ and $\beta > 0$.

The Beta family is denoted as

$$X \sim \text{Beta}(\alpha, \beta).$$

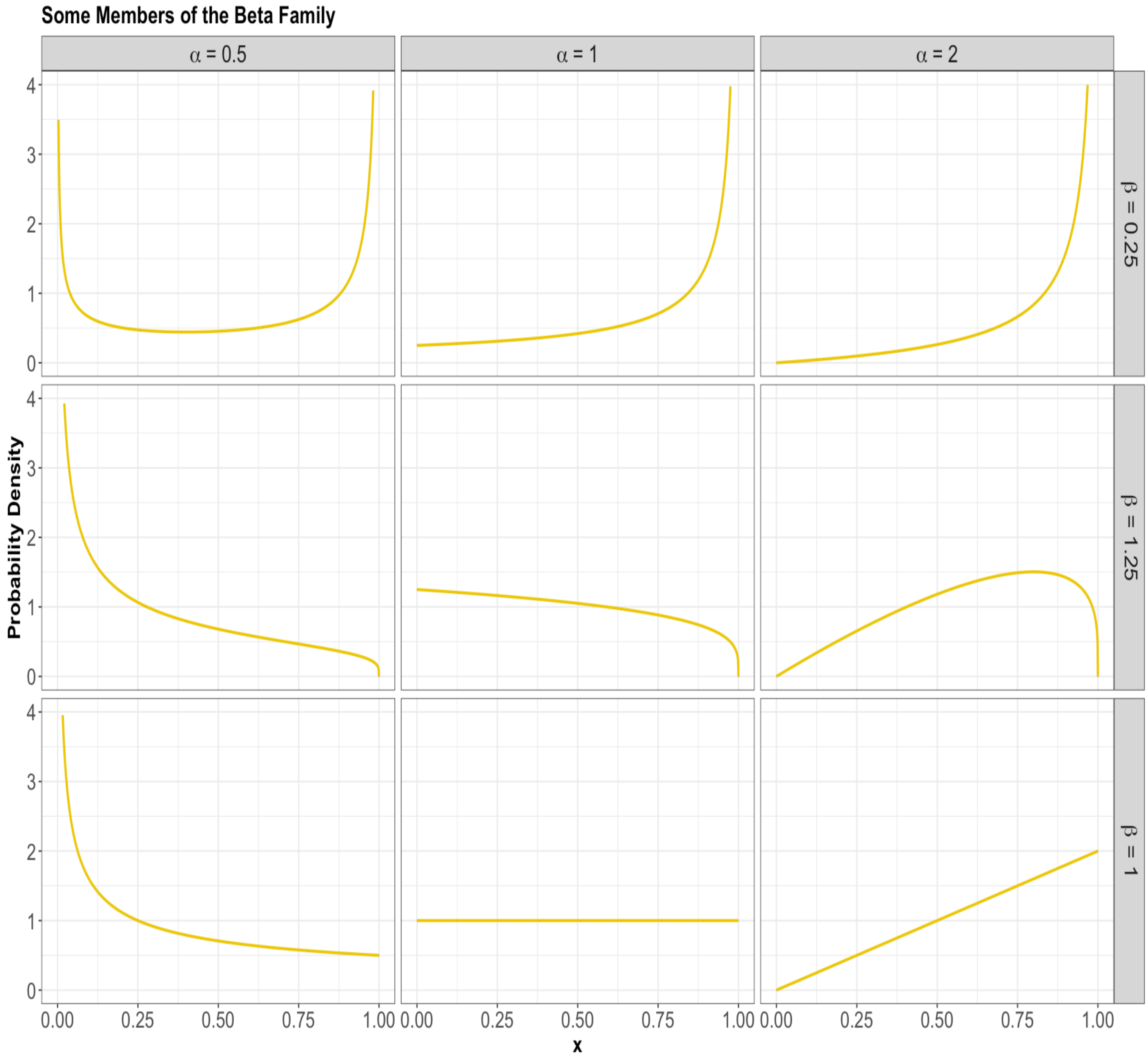
PDF

The density is parameterized as

$$f_X(x \mid \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad \text{for} \quad 0 \leq x \leq 1,$$

where $\Gamma(\cdot)$ is the Gamma function.

Here are some examples of densities:



Mean

The mean of a Beta random variable is defined as:

$$\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}.$$

Variance

The variance of a Beta random variable is defined as:

$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Bivariate Gaussian or Normal

Process

Members of this family need to have all Gaussian marginals, and their dependence has to be Gaussian dependence. Gaussian dependence is obtained as a consequence of requiring that any linear combination of Gaussian random variables is also Gaussian.

Parameters

To characterize the bivariate Gaussian family (i.e., $d = 2$ involved random variables), we need the following parameters:

- Mean for both X and Y denoted as $-\infty < \mu_X < \infty$ and $-\infty < \mu_Y < \infty$, respectively.
- Variance for both X and Y denoted as $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$, respectively.
- The covariance between X and Y, sometimes denoted σ_{XY} or, equivalently, the Pearson correlation (50) denoted $-1 \leq \rho_{XY} \leq 1$.

That is five parameters altogether; and only one of them, Pearson correlation or covariance (43), is needed to specify the dependence part in a bivariate Gaussian family.

Then, we can construct two objects that are useful for computations: a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$$

and

(56)

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}.$$

Note that the covariance matrix (56) is always defined as above. Even if we are given the correlation ρ_{XY} instead of the covariance σ_{XY}, we would then need to calculate the covariance as

$$\sigma_{XY} = \rho_{XY}\sigma_X\sigma_Y$$

before constructing the covariance matrix. However, there is another matrix that is sometimes useful, called the correlation matrix $\mathbf{P}$. Firstly, let us recall the formula of the Pearson correlation (50) between X and Y:

$$\rho_{XY} = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{\sigma_{XY}}{\sqrt{\sigma_X^2\sigma_Y^2}}.$$

It turns out that:

$$\rho_{XX} = \frac{\operatorname{Cov}(X, X)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(X)}} = \frac{\operatorname{Var}(X)}{\sqrt{\sigma_X^2\sigma_X^2}} = \frac{\sigma_X^2}{\sigma_X^2} = 1$$
$$\rho_{YY} = \frac{\operatorname{Cov}(Y, Y)}{\sqrt{\operatorname{Var}(Y)\operatorname{Var}(Y)}} = \frac{\operatorname{Var}(Y)}{\sqrt{\sigma_Y^2\sigma_Y^2}} = \frac{\sigma_Y^2}{\sigma_Y^2} = 1.$$

Thus, correlation matrix $\mathbf{P}$ is defined as:

$$\begin{aligned}\mathbf{P} &= \begin{pmatrix} \frac{\sigma_X^2}{\sigma_X^2} & \frac{\sigma_{XY}}{\sqrt{\sigma_X^2\sigma_Y^2}} \\ \frac{\sigma_{XY}}{\sqrt{\sigma_X^2\sigma_Y^2}} & \frac{\sigma_Y^2}{\sigma_Y^2} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{XX} & \rho_{XY} \\ \rho_{XY} & \rho_{YY} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{pmatrix}.\end{aligned}$$

PDF

The density can be parameterized as

$$f_{XY}\left(x, y \mid \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{XY}\right) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \times \exp\left\{-\frac{1}{2\left(1-\rho_{XY}^2\right)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\cdot\right.\right.\right. \tag{57}$$

Exponential

Process

The Exponential family is for positive random variables, often interpreted as **wait time** for some event to happen. Characterized by a **memoryless property**, where after waiting for a certain period of time, the remaining wait time has the same distribution.

The family is characterized by a single parameter, usually either the **mean wait time** $\beta > 0$, or its reciprocal, the **average rate** $\lambda > 0$ at which events happen.

The Exponential family is denoted as

$$X \sim \text{Exponential}(\beta),$$

or

$$X \sim \text{Exponential}(\lambda).$$

PDF

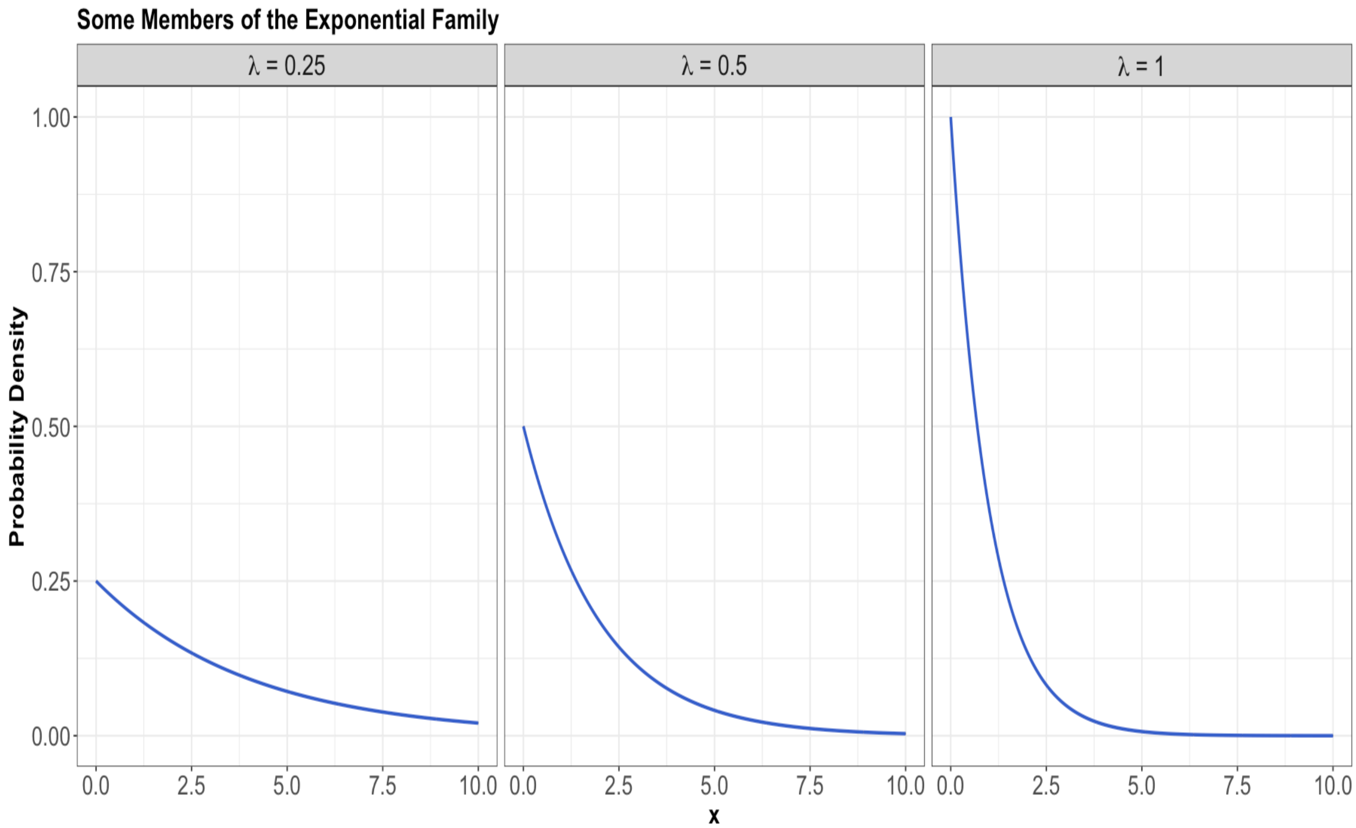
The density can be parameterized as

$$f_X(x \mid \beta) = \frac{1}{\beta}\exp(-x/\beta) \qquad \text{for} \quad x \geq 0$$

or

$$f_X(x \mid \lambda) = \lambda \exp(-\lambda x) \qquad \text{for} \quad x \geq 0.$$

The densities from this family all decay starting at $x = 0$ for rate λ:



Mean

Using a β parameterization, the mean of an Exponential random variable is defined as:

$$\mathbb{E}(X) = \beta.$$

On the other hand, **using a λ parameterization**, the mean of an Exponential random variable is defined as:

$$\mathbb{E}(X) = 1/\lambda.$$

Variance

Using a β parameterization, the variance of an Exponential random variable is defined as:

$$\text{Var}(X) = \beta^2.$$

On the other hand, **using a λ parameterization**, the variance of an Exponential random variable is defined as:

$$\text{Var}(X) = 1/\lambda^2.$$

Gamma

Process

Another useful two-parameter family with support on non-negative numbers. One common parameterization is with a shape parameter $k > 0$ and a scale parameter $\theta > 0$.

The Gamma family can be denoted as

$$X \sim \text{Gamma}(k, \theta).$$

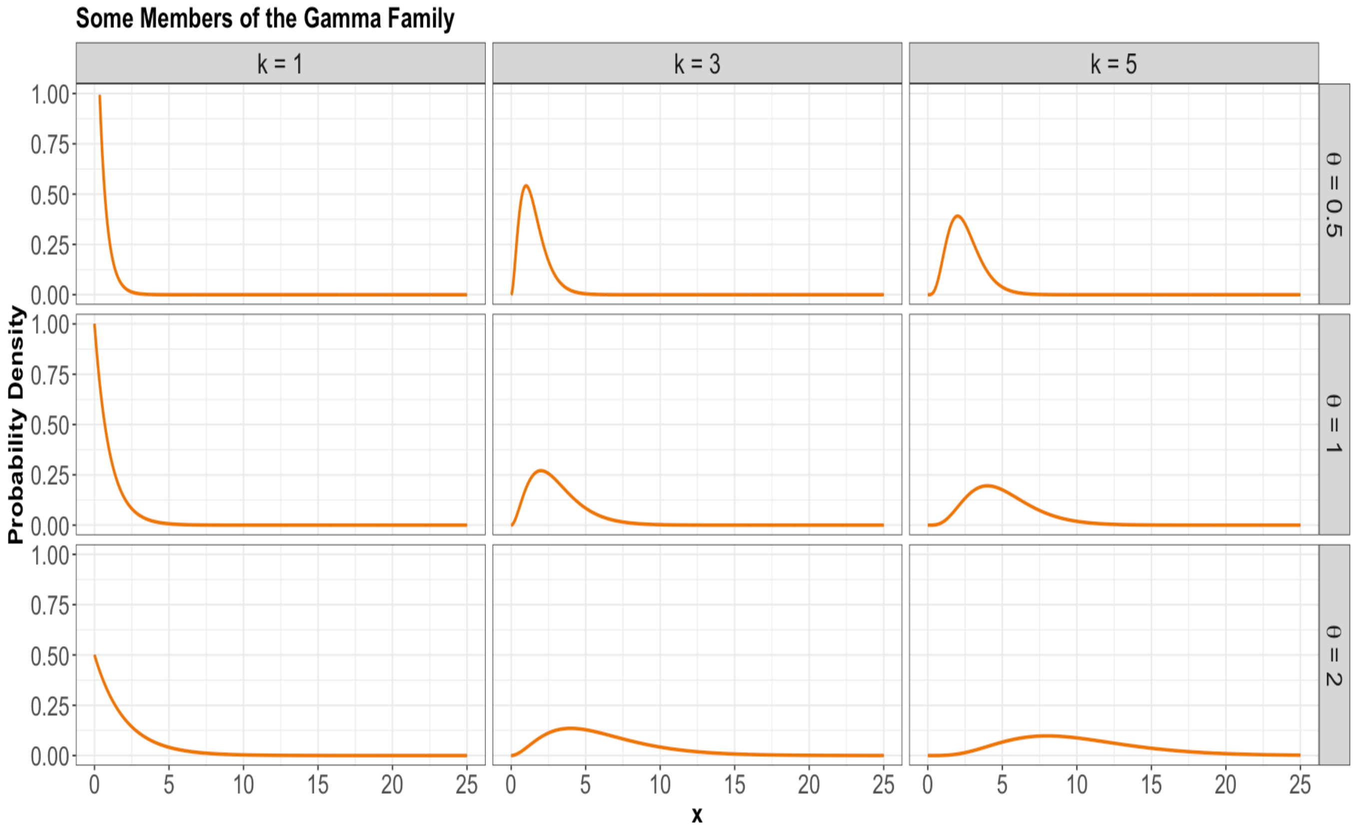
PDF

The density is parameterized as

$$f_X(x \mid k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} \exp(-x/\theta) \qquad \text{for} \quad x \geq 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Here are some densities:



Mean

The mean of a Gamma random variable is defined as:

$$\mathbb{E}(X) = k\theta.$$

Variance

The variance of a Gamma random variable is defined as:

$$\text{Var}(X) = k\theta^2.$$

Gaussian or Normal

Process

Probably the most famous family of distributions. It has a density that follows a **“bell-shaped”** curve. It is parameterized by its mean $-\infty < \mu < \infty$ and variance $\sigma^2 > 0$. A Normal distribution is usually denoted as

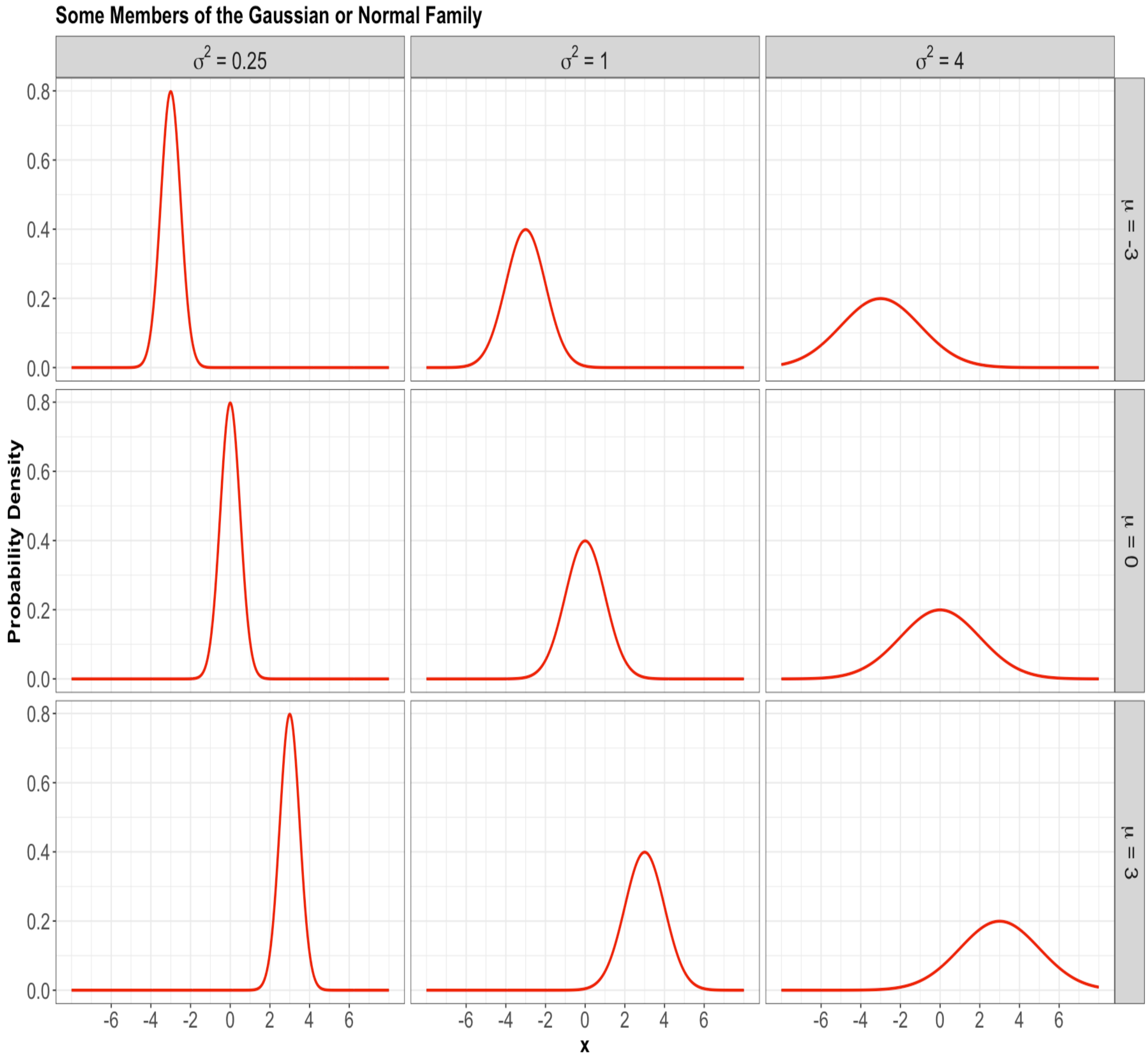
$$X \sim \mathcal{N}\left(\mu, \sigma^2\right).$$

PDF

The density is

$$f_X(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \text{for} \quad -\infty < x < \infty.$$

Here are some densities from members of this family:



Mean

The mean of a Normal random variable is defined as:

$$\mathbb{E}(X) = \mu.$$

Variance

The variance of a Normal random variable is defined as:

$$\operatorname{Var}(X) = \sigma^2.$$

Log-Normal

Process

A random variable X is a Log-Normal distribution if the transformation $\log(X)$ is Normal. This family is often parameterized by the mean $-\infty < \mu < \infty$ and variance $\sigma^2 > 0$ of $\log X$. The Log-Normal family is denoted as

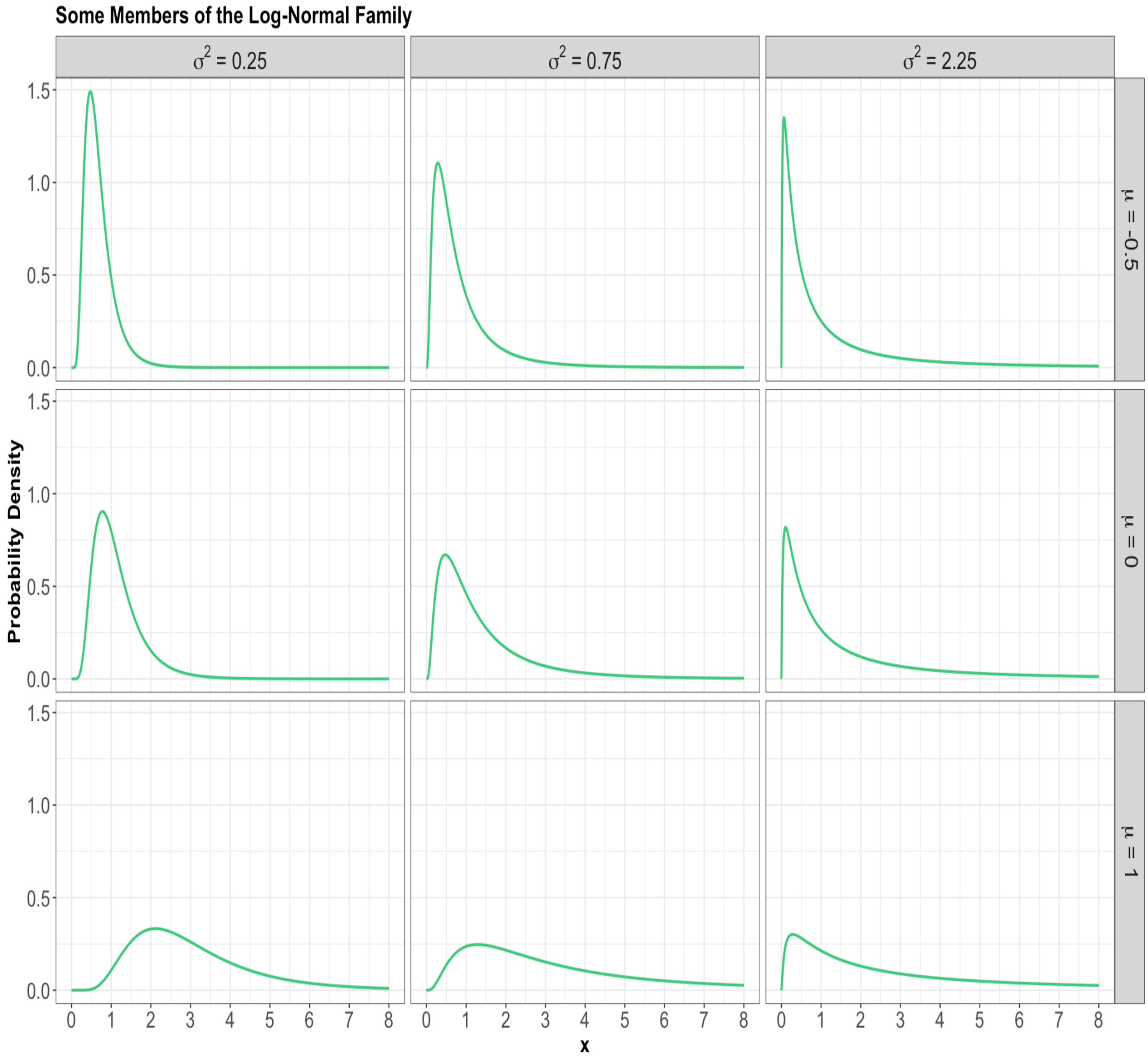
$$X \sim \text{Log-Normal}\left(\mu, \sigma^2\right).$$

PDF

The density is

$$f_X(x \mid \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[\log(x) - \mu]^2}{2\sigma^2}\right\} \qquad \text{for} \quad x \geq 0.$$

Here are some densities from members of this family:



Mean

The mean of a Log-Normal random variable is defined as:

$$\mathbb{E}(X) = \exp\left[\mu + \left(\sigma^2/2\right)\right].$$

Variance

The variance of a Log-Normal random variable is defined as:

$$\mathrm{Var}(X) = \exp\left[2\left(\mu + \sigma^2\right)\right] - \exp\left(2\mu + \sigma^2\right).$$

Uniform (Continuous)

Process

A continuous Uniform distribution has an equal density in between two points a and b (for $a < b$), and is usually denoted by

$$X \sim \text{Continuous Uniform}(a, b).$$

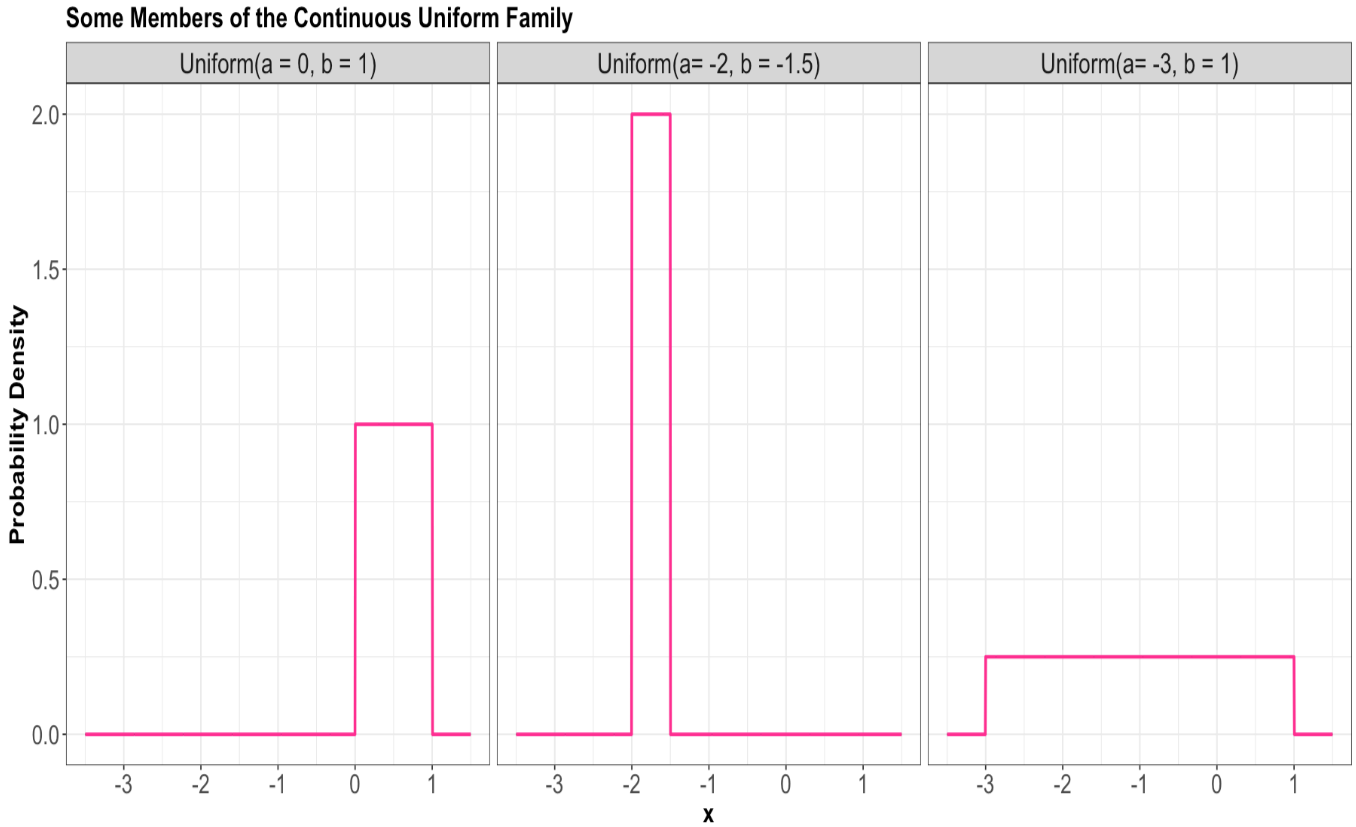
That means that there are **two parameters**: **one for each end-point**. A reference to a "*Uniform distribution*" usually implies **continuous uniform**, as opposed to discrete uniform.

PDF

The density is

$$f_X(x \mid a, b) = \frac{1}{b-a} \qquad \text{for} \quad a \leq x \leq b.$$

Here are some densities from members of this family:



Mean

The mean of a continuous Uniform random variable is defined as:

$$\mathbb{E}(X) = \frac{a+b}{2}.$$

Variance

The variance of a continuous Uniform random variable is defined as:

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}.$$

Weibull

Process

A generalization of the Exponential family, which allows for an event to be more likely the longer you wait. Because of this flexibility and interpretation, this family is used heavily in **survival analysis** when modelling **time until an event**.

This family is characterized by two parameters, a **scale parameter** $\lambda > 0$ and a **shape parameter** $k > 0$ (where $k = 1$ results in the Exponential family).

The Weibull family is denoted as

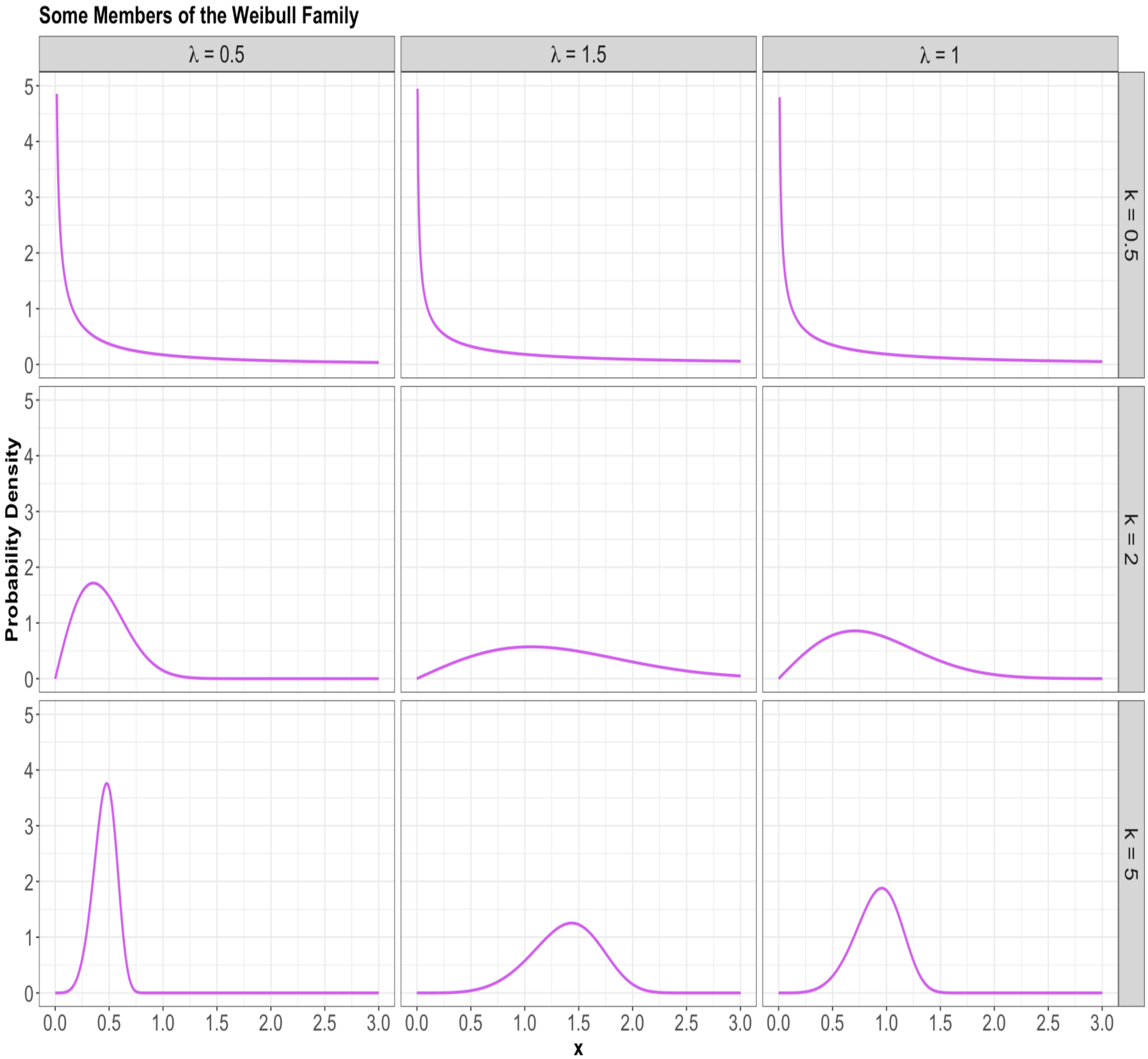
$$X \sim \text{Weibull}(\lambda, k).$$

PDF

The density is parameterized as

$$f_X(x \mid \lambda, k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp^{-(x/\lambda)^k} \qquad \text{for} \quad x \geq 0.$$

Here are some examples of densities:



Mean

The mean of a Weibull random variable is defined as:

$$\mathbb{E}(X) = \lambda\Gamma\left(1 + \frac{1}{k}\right).$$

Variance

The variance of a Weibull random variable is defined as:

$$\mathrm{Var}(X) = \lambda^2 \left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right].$$