CHAPTER 1

Continuous Functions in Euclidean Spaces

1. Continuous Functions

DEFINITION 1.1. Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$.

• f is *continuous* at x_0 if $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall x \in X$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0) < \epsilon$

• f is *continuous* if f is continuous at all $x_0 \in X$.

REMARK 1.2. We may consider the "more general" setting

$$f : D\ (\subseteq X) \to Y,\ x_0 \in D.$$

f is *continuous* at x_0 if $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall x \in D$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0) < \epsilon$.

But this is not more general than the definition since it coincides with the case when D is considered as a metric space with the metric $\rho|_{D\times D}$.

FACT 1.3. *Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$. TFAE:*

(i) *f is continuous at x_0*
(ii) *$\forall\epsilon > 0\ \exists\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$*
(iii) *$\forall U \in \mathcal{V}(f(x_0))\ \exists V \in \mathcal{V}(x_0)$ such that $f(V) \subseteq U$.*
(iv) *$\forall U$ open in Y such that $f(x_0) \in U\ \exists V$ open in X such that $x_0 \in V$ and $f(V) \subseteq U$.*

The last characterization shows that continuity is a topological concept.

DEFINITION 1.4. Let $(X;\mathcal{T})$ and $(Y;\mathcal{Y})$ be topological spaces. Let $f : X \to Y$ be a function and $x_0 \in X$.

f is *continuous* at x_0 if $\forall U \in \mathcal{Y}$ such that $f(x_0) \in U\ \exists V \in \mathcal{T}$ such that $x_0 \in V$ and $f(V) \subseteq U$.

Continuity is related to the more general concept of limit.

DEFINITION 1.5. Let $(X;\rho)$ and $(Y;\eta)$ be topological spaces, $D \subseteq X$. Let $f : D \to Y$ be a function and $x_0 \in X, y_0 \in Y$.

f has limit y_0 at x_0 if:

• x_0 is an accumulation point for D

• $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

REMARK 1.6. If f has limit y_0 at x_0 then y_0 is unique, hence we can denote:

$$\lim_{x \to x_0} f(x) = y_0.$$

FACT 1.7.

(1) *$f : D\,(\subseteq X) \to Y$, metric spaces, $x_0 \in X$.*
  (i) *if x_0 is isolated in D then f is continuous at x_0*
  (ii) *if x_0 is an accum. point for D then f is cont. at x_0 iff*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

(2) *$f : D \to Y$ function, $x_0 \in D$ accumulation point and $y_0 \in Y$. Then*

$$\begin{aligned} \lim_{x \to x_0} f(x) = y_0 \;\Leftrightarrow\; & \forall (x_n)_{n \geq 1} \textit{ with all } x_n \in D \\ & \textit{and } x_n \neq x_0 \; \forall n \\ & \textit{and } \lim_{n \to \infty} x_n = x_0 \\ & \textit{we have } \lim_{n \to \infty} f(x_n) = y_0. \end{aligned}$$

PROOF."$\Rightarrow$" $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.
Take $(x_n)_{n \geq 1}$ a sequence in X, $x_n \neq x_0 \; \forall n$ and $\lim_{n \to \infty} x_n = x_0$.
Then $\exists N_\delta \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N_\delta$ then $\rho(x_n, x_0) < \delta$ hence: $\eta(f(x_n), y_0) < \epsilon$.

"$\Leftarrow$" Assume that $\forall (x_n)_{n \geq 1}$ seq. with all elements in D such that $x_n \neq x_0 \; \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x_0$ we have $\lim_{n \to \infty} f(x_n) = y_0$.
By contradiction assume that $f(x)$ does not converge to y_0 as x approaches x_0.
Then $\exists \epsilon_0 > 0$ such that $\forall \delta > 0 \; \exists x \in D \setminus \{x_0\}$ with $\rho(x, x_0) < \delta$ and $\eta(f(x), y_0) \geq \epsilon_0$.
$\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$ hence $\exists x_n \in D \setminus \{x_0\}$ with $\rho(x_n, x_0) < \delta = \frac{1}{n}$ and $\eta(f(x_n), y_0) \geq \epsilon_0$
hence $x_n \xrightarrow[n \to \infty]{\rho} x_0$ but $f(x_n) \not\xrightarrow[n \to \infty]{\eta} y_0$.

$\square$

(3) *(Sequential Characterization of Continuity) Let $f : X \to Y$ function and $x_0 \in X$.*

$$\begin{aligned} \textit{Then } f \textit{ is continuous at } x_0 \Leftrightarrow\; & \forall (x_n)_{n \geq 1} \textit{ seq. in } X \\ & \textit{such that } x_n \xrightarrow[n]{\rho} x_0 \\ & \textit{we have } f(x_n) \xrightarrow[n]{\eta} y_0. \end{aligned}$$

(4) *(Composition of Functions) Let $\underset{\rho}{X} \xrightarrow{f} \underset{\eta}{Y} \xrightarrow{g} \underset{\theta}{Z}$ functions between metric spaces.*
  (i)

(ii)

$$\textit{Assume that } x_0 \in X$$
$$f \textit{ is cont. at } x_0$$
$$g \textit{ is cont. at } f(x_0)$$

Then $g \circ f$ is continuous at x_0.

(5) *(Functions Between Euclidean Spaces) Let $D \subseteq \mathbb{R}^p$ and $f : D \to \mathbb{R}^q$ be a function, hence $f(x) = (f_1(x), \dots, f_q(x))$, $\forall x \in D$ where $f_j : D \to \mathbb{R}$ function $\forall j = 1, \dots, q$.*

(i) *Let $x_0 \in D'$, i.e. x_0 is an accumulation point for D and $y^{(0)} = \left(y_1^{(0)}, \dots, y_q^{(0)}\right) \in \mathbb{R}^q$. Then*

$$\lim_{x \to x_0} f(x) = y^{(0)} \Leftrightarrow \ \forall j = 1, \dots, q \ \lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

(ii) *Let $x_0 \in D$. Then*

$$f \textit{ is cont. at } x_0 \Leftrightarrow \ \forall j = 1, \dots, q \ , f_j \textit{ is cont. at } x_0.$$

PROOF.

(i)

"$\Rightarrow$". Assume that $\lim_{x \to x_0} f(x) = y^{(0)}$ and use the $\| \cdot \|_\infty$.
$\forall \epsilon > 0 \ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\| < \delta$ then $\|f(x) - y^{(0)}\|_\infty < \epsilon$.
Let $j \in \{1, \dots, q\}$, then

$$|f_j(x) - y_j^{(0)}| \le \|f(x) - y^{(0)}\|_\infty < \epsilon.$$

"$\Leftarrow$". Assume that $\forall j = 1, \dots, q$

$$\lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

Then $\forall \epsilon > 0 \ \exists \delta_j > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\|_\infty < \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$.
Take $\delta := \min\{\delta_1, \dots, \delta_q\} > 0$.
Then $\forall j = 1, \dots, q$, if $\|x - x_0\|_\infty < \delta \le \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$ hence

$$\|f(x) - y^{(0)}\|_\infty = \max_{j=1}^{q}\{|f_j(x) - y_j^{(0)}|\} < \epsilon.$$

(ii)
- if x_0 isolated, nothing to prove.
- if x_0 accum. point for D, we use (i).

$\square$

EXAMPLE 1.8. (1) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

- f continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$

- $(0,0)$ is an accum. point for $\mathbb{R}^2$
and f does not have a limit $(x,y) \to (0,0)$

$$\begin{aligned} x = 0,\ y \to 0 \Rightarrow & f(0,y) = 0 \to 0 \\ y = 0,\ x \to 0 \Rightarrow & f(x,0) = 0 \to 0 \\ x = y \to 0 \Rightarrow & f(x,y) = \frac{1}{2} \to \frac{1}{2} \end{aligned}$$

(2) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

- f continuous on $\mathbb{R}^2$
- at $(x_0, y_0) \neq (0,0)$, clear
- at $(0,0)$

$$|f(x,y)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{2} = \frac{\|(x,y)\|_2}{2}$$

Definition 1.9. • A *curve* in $\mathbb{R}^2$ is a continuous function $\gamma : I \to \mathbb{R}^q, q \geq 1$ where I is an interval.

• If the interval $I = [a,b]$ is compact, then the curve has *endpoints* $x = \gamma(a)$ and $y = \gamma(b)$. In this case we say that γ is a *path* joining x and y.

• A curve with endpoints x and y is called *closed* if $x = y$.

Example 1.10. (1) $\gamma(t) = (\cos t, \sin t), \gamma : [0, 2\pi] \to \mathbb{R}^2$
$\gamma(0) = \gamma(2\pi)$ so γ is a closed curve.
(2) $\gamma(t) = (t^2, t^3), \gamma : [0,1] \to \mathbb{R}^2$
$\gamma(0) = (0,0)$ $\gamma(1) = (1,1)$ a curve with endpoints but not closed.
(3) $\gamma(t) = (t\cos t, t\sin t, t), \gamma : \mathbb{R} \to \mathbb{R}^3$
is a curve with no endpoints
a *spiral* inside $\{(x,y,z) | x^2 + y^2 = |z|\}$ a *cone*

Definition 1.11. A *surface* in $\mathbb{R}^q$ $(q \geq 2)$ is a continuous function $F : A \to \mathbb{R}^q$, D open, nonempty in $\mathbb{R}^2$, $D \subseteq A \subseteq \bar{D}$.

Example 1.12. (1) *2-dimensional sphere in* $\mathbb{R}^3$

$$F : [0, 2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$$
$$F(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$$

(2)

$$G : B \to \mathbb{R}^3$$
$$G(x,y,z) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$$
$$B = \left\{(x,y) | x^2 + y^2 \leq 1\right\}$$

2. Continuity and Topology

THEOREM 2.1. *(*Topological Characterization of Continuity*)*
Let $(X;\rho)$, $(Y;\eta)$ be metric spaces. $f : X \to Y$ a function. TFAE:

(i) *f is continuous.*
(ii) *For all U open in Y, $f^{-1}(U)$ is open in X.*
(iii) *For all F closed in Y, $f^{-1}(F)$ is closed in X.*

PROOF.

(i) $\Rightarrow$ (ii). Let U open in Y and $x \in f^{-1}(U)$, i.e. $f(x) \in U$.
Then $\exists \epsilon > 0$ s.t. $B_\epsilon(f(x)) \subseteq U$.
Since f is cont. at x $\exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(f(x)) \subseteq U$
hence $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))) \subseteq f^{-1}(U)$, i.e. $f^{-1}(U)$ is open.

(ii) $\Rightarrow$ (i). Let $x \in X$ and $\epsilon > 0$. Then $B_\epsilon(f(x))$ is open in Y
hence $f^{-1}(B_\epsilon(f(x)))$ is open in X.
Since $x \in f^{-1}(B_\epsilon(f(x)))$ it follows that $\exists \delta > 0$
s.t. $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$, i.e. $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.
Hence f is cont. at each $x \in X$.

(ii) $\Leftrightarrow$ (iii). Since $f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A)$.

$\square$

COROLLARY 2.2. *Let $\emptyset \neq D \subseteq \mathbb{R}^p$, $f : D \to \mathbb{R}^q$ a function. TFAE:*

(i) *f is continuous.*
(ii) *For all U open in $\mathbb{R}^q$, $f^{-1}(U)$ is relatively open in D.*
(iii) *For all F closed in $\mathbb{R}^q$, $f^{-1}(F)$ is relatively closed in D.*

COROLLARY 2.3. *Let $\emptyset \neq D \subseteq \mathbb{R}^p$ open, $f : D \to \mathbb{R}^q$ a function. TFAE:*

(i) *f is continuous.*
(ii) *For all U open in $\mathbb{R}^q$, $f^{-1}(U)$ is open in $\mathbb{R}^p$.*
(iii) *For all F closed in $\mathbb{R}^q$, $f^{-1}(F)$ is closed in $\mathbb{R}^p$.*

3. Continuity and Compactness

THEOREM 3.1. *Let $(X;\rho)$, $(Y;\eta)$ be metric spaces, $f : X \to Y$ a continuous function and K compact in X. Then $f(K)$ is compact.*

PROOF. Let $\{U_i | i \in \mathcal{J}\}$ be an open (in Y) covering of $f(K)$:

- $\forall i \in \mathcal{J}, U_i$ is open in Y;
- $f(K) \subseteq \bigcup_{i \in \mathcal{J}} U_i$.

Then $\forall i \in \mathcal{J}$, $f^{-1}(U_i)$ is open in X and

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(\bigcup_{i \in \mathcal{J}} U_i) = \bigcup_{i \in \mathcal{J}} f^{-1}(U_i)$$

hence $\{f^{-1}(U_i)|i \in \mathcal{J}\}$ is an open covering of K.

Since K is compact $\exists i_1, \ldots, i_n \in \mathcal{J}$ such that

$$K \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}) = f^{-1}(\bigcup_{k=1}^{n} U_{i_k})$$

hence $f(K) \subseteq f(f^{-1}(\bigcup_{k=1}^{n} U_{i_k})) \subseteq \bigcup_{k=1}^{n} U_{i_k}$. ☐

COROLLARY 3.2. *Let $f : X \to \mathbb{R}$ a continuous function and K compact, nonempty in X. Then:*

- *f is bounded on K, i.e. $f(K)$ is bounded in $\mathbb{R}$.*
- *The extreme values of f on K are attained, i.e. $\exists x_m, x_M \in K$ such that*

$$f(x_m) = \inf_K f,\ f(x_M) = \sup_K f.$$

PROOF. $f(K)$ is compact in $\mathbb{R}$, hence closed and bounded.

- $f(K)$ bounded $\Rightarrow \inf_K f, \sup_K f \in \mathbb{R}$.
- $\inf_K f \in_! f(\bar{K}) = f(K)$, hence $\exists x_m \in K$ such that $\inf_K f = f(x_m)$.
- $\sup_K f \in f(\bar{K}) = f(K)$, hence $\exists x_M \in K$ such that $\sup_K f = f(x_M)$. ☐

COROLLARY 3.3. *Let $f : X \to \mathbb{R}^q$ a continuous function. Then for all K nonempty and compact in X $\exists x_m, x_M \in K$ such that*

$$\|f(x_m)\| = \inf_K \|f\|,\ \|f(x_M)\| = \sup_K \|f\|.$$

PROOF.

$$\|f\| = \|\cdot\| \circ f : X \to \mathbb{R}^q \to \mathbb{R}.$$

☐

4. Continuity and Connectedness

THEOREM 4.1. *Let $(X;\rho)$, $(Y;\eta)$ be metric spaces, $f : X \to Y$ a continuous function, C connected in X. Then $f(C)$ is connected in Y.*

PROOF. By contrapositive, assume that f(C) is separated, hence: there exist U, V open in Y such that

- $f(C) \subseteq U \cup V$
- $f(C) \cap U \neq \emptyset$
- $f(C) \cap V \neq \emptyset$
- $f(C) \cap U \cap V = \emptyset$

Then $f^{-1}(U)$, $f^{-1}(V)$ are open in X.

- $C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
- $\emptyset \neq f^{-1}(f(C) \cap U) = f^{-1}(f(C)) \cap f^{-1}(U)$

- We prove $C \cap f^{-1}(U) \neq \emptyset$.
  Since $f(C) \cap U \neq \emptyset$, $\exists y \in f(C)$ and $y \in U$ hence $\exists x \in C$ such that $f(x) \in U$, hence $x \in f^{-1}(U)$, i.e. $x \in C \cap f^{-1}(U)$.
- Similarly $C \cap f^{-1}(V) \neq \emptyset$.
- Similarly $C \cap f^{-1}(U) \cap f^{-1}(V) = \emptyset$
  Assume $C \cap f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$, then $\exists x \in C$ such that $x \in f^{-1}(U) \cap f^{-1}(V)$ hence $f(x) \in f(C)$ and $f(x) \in f(f^{-1}(U)) \subseteq U$, $f(x) \in f(f^{-1}(V)) \subseteq V$ i.e. $f(x) \in f(C) \cap U \cap V$, contradiction!

Thus, $f^{-1}(U)$ and $f^{-1}(V)$ separate C, contradiction! $\square$

COROLLARY 4.2. (1) *Let $f : (X; \rho) \to \mathbb{R}$ continuous, C connected in X. Then $f(C)$ is an interval.*
(2) *Let I be an interval in $\mathbb{R}$ and $\gamma : I \to \mathbb{R}^d$ continuous (a curve). Then $\gamma(I)$ is connected.*

DEFINITION 4.3. A subset $S \subseteq (X; \rho)$ is called *pathwise connected* if for all $a, b \in S$ there exists a (continuous) path $\gamma : [0,1] \to S$ such that $a = \gamma(0)$ and $b = \gamma(1)$.

(3) *If S is pathwise connected, then it is connected.*

PROOF. Assume S is not connected, let U, V open in X and separating S. Then there exist $a \in S \cap U$ and $b \in S \cap V$. Since S is pathwise connected, there exists $\gamma : [0,1] \to S$ continuous such that $\gamma(0) = a$ and $\gamma(1) = b$. But then U and V separate $\gamma([0,1])$, contradiction! $\square$

EXAMPLE 4.4. A set in $\mathbb{R}^2$ that is connected but not pathwise connected.

$$S = \{(0,y) | -1 \leq y \leq 1\} \cap \{(x, \sin\frac{1}{x}) | -\frac{1}{\pi} < x < \frac{1}{\pi}, x \neq 0\}$$

(4) *Assume that D is open in $\mathbb{R}^d$. Then D is connected iff D is pathwise connected.*

PROOF."$\Leftarrow$" Holds in general.
"$\Rightarrow$" On D we define a relation: $x \overset{p}{\sim} y$ if $\exists \gamma : [0,1] \to D$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$.

- $\overset{p}{\sim}$ is an equivalence relation on D.
- $\forall x \in D$ its equivalence class $[x]_p$ is an open set.
- If D is not pathwise connected then there exist at least two different cosets w.r.t. $\overset{p}{\sim}$, hence D is disconnected.

$\square$

5. Uniform Continuity

DEFINITION 5.1. Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces. A function $f : X \to Y$ is *uniformly continuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x_1, x_2 \in X$, if $\rho(x_1, x_2) < \delta$ then $\eta(f(x_1), f(x_2)) < \epsilon$.

THEOREM 5.2. *If $f : \underset{\rho}{X} \to \underset{\eta}{Y}$ is continuous and X is compact, then f is uniformly continuous.*

PROOF. Let $\epsilon > 0$. Since f is continuous on X, $\forall x \in X\ \exists \delta_x > 0$ such that $\forall z \in X$ with $\rho(x, z) < \delta_x \Rightarrow \eta(f(x), f(z)) < \epsilon/2$.

Since $\{B_{\delta_x/2}(x) | x \in X\}$ is an open covering of X compact, it follows that there exist $x_1, \dots, x_n \in X$ such that

$$X \subseteq \bigcup_{i=1}^{n} B_{\delta_{x_i}/2}(x_i)$$

Let $\delta := \min\{\frac{\delta_{x_i}}{2} | i = 1, \dots, n\} > 0$ and let $x, z \in X$ such that $\rho(x, z) < \delta$. Then $\exists j \in \{1, \dots, n\}$ such that $x \in B_{\delta_{x_j}/2}(x_j)$ i.e. $\rho(x, x_j) < \frac{\delta_{x_j}}{2}$. Then

$$\rho(z, x_j) \le \rho(z, x) + \rho(x, x_j) < \delta + \frac{\delta_{x_j}}{2} \le \frac{\delta_{x_j}}{2} + \frac{\delta_{x_j}}{2} = \delta_{x_j}$$

hence $\eta(f(x), f(x_j)) < \frac{\epsilon}{2}$ and $\eta(f(z), f(x_j)) < \frac{\epsilon}{2}$. Then

$$\eta(f(x), f(z)) \le \eta(f(x), f(x_j)) + \eta(f(z), f(x_j)) < \epsilon$$

□

THEOREM 5.3. *Let $(X;\rho)$ be a compact metric space, $(Y;\eta)$ be a complete metric space, $\emptyset \neq D \subseteq X$ and $f : D \to Y$ a function. TFAE:*

(i) *f is uniformly continuous on D*
(ii) *$\exists \bar{f} : \bar{D} \to Y$ such that $\bar{f}|_D = f$ and $\bar{f}$ is continuous on $\bar{D}$.*

PROOF(ii) $\Rightarrow$ (i) $\bar{D}$ closed in X compact hence $\bar{D}$ is compact $\Rightarrow \bar{f}$ unif. cont. on $\bar{D} \Rightarrow f = \bar{f}|_D$ unif. cont. on D.

(i) $\Rightarrow$ (ii) Assume that f is uniformly continuous.

- f maps Cauchy sequences from D to Cauchy sequences in Y:
  Let $\epsilon > 0$. Then $\exists \delta > 0$ such that $\forall x, z \in D$, if $\rho(x, z) < \delta$ then $\eta(f(x), f(z)) < \epsilon$.
  Let $(x_n)_{n\ge1}$ be a ρ-Cauchy sequence in D. $\forall \epsilon > 0$, with $\delta > 0$ as before, $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $m, n \ge \mathbb{N}$ then $\rho(x_m, x_n) < \delta$ hence $\eta(f(x_m), f(x_n)) < \epsilon$.
- Let $x \in \bar{D}$. Then $\exists (x_n)_{n\ge1}$ in D such that $x_n \xrightarrow[n]{\rho} x$ hence $(x_n)_{n\ge1}$ is ρ-Cauchy, hence $(f(x_n))_{n\ge1}$ is η-Cauchy hence, since $(Y;\eta)$ is complete, $\exists y_x \in Y$ such that $f(x_n) \xrightarrow[n]{\eta} y_x$. Define $\bar{f}(x) := y_x$. The definition of $\bar{f}$ is correct:

We use the interlacing method.

$$\text{If } (x_n)_{n\geq 1} \text{ is a seq. in D, } x_n \xrightarrow[n]{\rho} x$$
$$(z_n)_{n\geq 1} \text{ is a seq. in D, } z_n \xrightarrow[n]{\rho} x$$

Then $(t_n)_{n\geq 1}$ defined by

$$t_n = \begin{cases} x_{\frac{n}{2}}, & \text{if } n = 2k \\ z_{\frac{n-1}{2}}, & \text{if } n = 2k+1 \end{cases}$$

is a sequence in D and $t_n \xrightarrow[n]{\rho} x$.

Then $f(x_n) \xrightarrow[n]{\eta} y_x$, $f(z_n) \xrightarrow[n]{\eta} z_x$, $f(t_n) \xrightarrow[n]{\eta} t_x$. Since $(f(x_n))_{n\geq 1}$ and $(f(z_n))_{n\geq 1}$ are subsequences of $(f(t_n))_{n\geq 1}$, it follows that $y_x = t_x = z_x$.

- $\bar{f}$ is continuous on $\bar{D}$:
We use the sequential characterization of continuity. Let $\bar{x} \in \bar{D}$ and $(\bar{x}_n)_{n\geq 1}$ a sequence in $\bar{D}$ such that $\bar{x}_n \xrightarrow[n]{\rho} \bar{x}$.
Then $\forall n \geq 1\ \exists x_n \in D$ such that $\rho(\bar{x}_n, x_n) < \frac{1}{n}$ and $\rho(\bar{f}(\bar{x}_n), f(x_n)) < \frac{1}{n}$.
Then

$$\rho(x_n, \bar{x}) \leq \rho(x_n, \bar{x}_n) + \rho(\bar{x}_n, \bar{x}) < \frac{1}{n} + \rho(\bar{x}_n, \bar{x}) \xrightarrow[n]{} 0$$

hence $x_n \xrightarrow[n]{\rho} \bar{x}$, and then by def. of $\bar{f}$ we have $f(x_n) \xrightarrow[n]{\rho} \bar{f}(\bar{x})$, and then

$$\eta(\bar{f}(\bar{x}_n), \bar{f}(\bar{x})) \leq \eta(\bar{f}(\bar{x}_n), f(x_n)) + \eta(f(x_n), \bar{f}(\bar{x})) < \frac{1}{n} + \eta(f(x_n), \bar{f}(\bar{x})) \xrightarrow[n]{} 0$$

hence $\bar{f}(\bar{x}_n) \xrightarrow[n]{\eta} \bar{f}(\bar{x})$.

□

6. Sequences of Functions

Let X be a nonempty set and $(Y;\eta)$ a vector space. We consider sequences $(f_n)_{n\geq 1}$ of functions

$$f_n : X \to Y,\ n \in \mathbb{N}.$$

Definition 6.1.
- $(f_n)_{n\geq 1}$ *converges pointwise* to $f : X \to Y$ if
$\forall x \in X\ \forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N, \eta(f(x), f_n(x)) < \epsilon$,
i.e. $\forall x \in X$, $(f_n(x))_{n\geq 1}$ converges to $f(x)$ in Y with respect to η.
- $(f_n)_{n\geq 1}$ *converges uniformly* to $f : X \to Y$ if
$\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N, \forall x \in X, \eta(f(x), f_n(x)) < \epsilon$

Fact 6.2. (1) *If there exists $(\alpha_n)_{n\geq 1}$ such that $\alpha_n \geq 0\ \forall n \in \mathbb{N}$ and $\alpha_n \xrightarrow[n]{} 0$ and $\eta(f(x), f_n(x)) \leq \alpha_n\ \forall n \in \mathbb{N}$ and $\forall x \in X$, then $f_n \xrightarrow[n]{\text{unif.}} f$.*

(2) *If $(X;\rho)$ and $(Y;\eta)$ are metric spaces and functions $f_n : X \to Y$ continuous for all $n \in \mathbb{N}$, $f : X \to Y$ such that $f_n \xrightarrow[n]{\text{unif.}} f$, then f is continuous on X.*

PROOF. Let $x_0 \in X$ and $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$

$$\eta(f_n(x), f(x)) < \frac{\epsilon}{3}.$$

f_N is continuous at x_0, so there exists $\delta > 0$ such that for all $x \in X$, $\rho(x, x_0) < \delta$ implies $\eta(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$.

Then for all $x \in X$, if $\rho(x, x_0) < \delta$ then

$$\begin{aligned}\eta(f(x), f(x_0)) &\leq \eta(f(x), f_N(x)) + \eta(f_N(x), f_N(x_0)) + \eta(f_N(x_0), f(x_0))\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

□

EXAMPLE 6.3. (1) $f_n : \underbrace{\overline{B_1(0)}}_{\subseteq \mathbb{R}^d} \to \mathbb{R}$, $f_n(x) = \|x\|_2^n$,

$$f(x) = \begin{cases} 0, & x \in B_1(0) \\ 1, & x \in \partial B_1(0) \end{cases}$$

Then $f_n \xrightarrow[n]{\text{pointwise}} f$ but not uniformly.

(2) $B_1(0) \in \mathbb{R}^2$, $f_n : B_1(0) \to \mathbb{R}^2$, $f : B_1(0) \to \mathbb{R}^2$

$$f_n(x_1, x_2) = \left(\frac{x_1^2 - n x_2^2}{1 + n x_2^2}, \frac{n x_1}{1 + n x_1^2} \right)$$

$$f(x) = \begin{cases} (-1, \frac{1}{x_1}), & x_1 \neq 0, x_2 \neq 0 \\ (-1, 0, & x_1 = 0, x_2 \neq 0 \\ (x_1^2, \frac{1}{x_1}), & x_1 \neq 0, x_2 = 0 \\ (0,0), & x_1 = 0, x_2 = 0 \end{cases}$$

$f_n \xrightarrow[n]{\text{pointwise}} f$ but not uniformly.

DEFINITION 6.4. $f_n : \underset{\text{set}}{X} \to \underset{\eta}{Y}$, $\forall n \in \mathbb{N}$.

$(f_n)_{n\geq 1}$ is *uniformly Cauchy* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$\eta(f_n(x), f_m(x)) < \epsilon$$

.

FACT 6.5. (1) *If $f_n \xrightarrow[n]{\text{unif.}} f$ then $(f_n)_{n\geq 1}$ is uniformly Cauchy.*

(2) *If $(Y;\eta)$ is complete and $(f_n)_{n\geq 1}$ is uniformly Cauchy then $\exists f : X \to Y$ such that $f_n \xrightarrow[n]{\text{unif.}} f$.*

PROOF. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, $x \in X$

$$(*) \qquad \eta(f_m(x), f_n(x)) < \frac{\epsilon}{2}$$

Then, for each $x \in X$ $(f_n(x))_{n\geq 1}$ is Cauchy in Y complete, hence there exists $f(x) \in Y$ such that $f_n(x) \underset{n}{\to} f(x)$.

Due to the uniqueness of the limit in Y, there exists a function $f : X \to Y$ such that $f_n \xrightarrow[n]{\text{pointwise}} f$.

Letting $m \to +\infty$ in (**??**), for all $n \geq N$, $x \in X$

$$\eta(f(x), f_n(x)) \leq \frac{\epsilon}{2} < \epsilon$$

Hence $f_n \xrightarrow[n]{\text{unif.}} f$. □

REMARK 6.6. For all $z \in Y$ the map $Y \ni y \mapsto \eta(y, z) \in \mathbb{R}$ is continuous.

DEFINITION 6.7. Let $X \neq \emptyset$ and $(V, \|\cdot\|)$ a normed space.

$$\mathcal{B}(X; V) := \{f : X \to V | f \text{bounded function}\}$$

For all $f \in \mathcal{B}(X; V)$, $\|f\|_X := \sup_{x\in X} \|f(x)\|$.

FACT 6.8. (1) *$\mathcal{B}(X; V)$ is a vector space. $\|\cdot\|_X$ is a norm, called the* sup norm *on X.*
(2) *If $(V; \|\cdot\|)$ is complete, then the normed space $(\mathcal{B}(X; V); \|\cdot\|_X)$ is complete.*

PROOF. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{B}(X; V)$ with respect to the norm $\|\cdot\|_X$. Hence:

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N, \|f_m - f_n\|_X < \frac{\epsilon}{2}.$$

$$(*) \qquad \text{hence } \forall x \in X \quad \|f_m(x) - f_n(x)\| \leq \|f_m - f_n\|_X < \frac{\epsilon}{2}$$

Then for all $x \in X$ the sequence $(f_n(x))_{n\geq 1}$ is Cauchy in V complete, hence there exists $f(x) \in V$ such that

$$\|f_n(x) - f(x)\| \xrightarrow[n\to\infty]{} 0$$

Then $X \ni x \mapsto f(x) \in V$ is a function.

- $f \in \mathcal{B}(X; V)$ and $\|f_n - f\|_X \underset{n}{\to} 0$.
- For all $x \in X$ let $n \to \infty$ in (**??**) hence

$$(**) \qquad \|f(x) - f_n(x)\| \leq \frac{\epsilon}{2} < \epsilon$$

hence

$$(***) \qquad \sup_{x\in X} \|f(x) - f_n(x)\| \leq \frac{\epsilon}{2} < \epsilon$$

Also for all $x \in X$, letting $n = N$ in (**??**) we have

$$\|f(x)\| \leq \|f(x) - f_N(x)\| + \|f_N(x)\| \leq \frac{\epsilon}{2} + \|f_N\|_X < +\infty$$

hence $f \in \mathcal{B}(X;V)$.

On the other hand, from (**??**), for all $n \geq N$ $\|f - f_n\|_X < \epsilon$, hence $f_n \xrightarrow[n]{\|\cdot\|_X} f$. □

(3) *Let $(X;\rho)$ be a compact metric space, $(V;\|\cdot\|)$ Banach space and*

$$\mathcal{C}(X;V) := \{f : X \to V | f \text{ continuous}\}.$$

Then:

- $\mathcal{C}(X;V) \subseteq \mathcal{B}(X;V)$ *as a vector subspace*
- $(\mathcal{C}(X;V);\|\cdot\|_X)$ *is a complete normed space.*

7. Series of Functions

Let $\emptyset \neq X$ be a set and $(V, \|\cdot\|_V)$ be a normed space and $(f_n)_{n\geq 1}$ a sequence of functions $f_n : X \to V$.

Definition 7.1.

- A formal sum $\sum_{n=1}^{\infty} f_n$ is called a *series of functions* on X and valued in V.

$$\forall n \in \mathbb{N} \quad s_n = \sum_{k=1}^{n} f_n,$$

$(s_n)_{n=1}^{\infty}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$.
- The series $\sum_{n=1}^{\infty} f_n$ *pointwise converges* if the sequence $(s_n)_{n=1}^{\infty}$ converges pointwise to a function $f : X \to V$.
- The series $\sum_{n=1}^{\infty} f_n$ *uniformly converges* if the sequence $(s_n)_{n=1}^{\infty}$ converges uniformly to a function $f : X \to V$.
- The series $\sum_{n=1}^{\infty} f_n$ *absolutely* converges if $\sum_{n=1}^{\infty} \|f_n\|_V$ converges. This can be pointwise or uniformly.

Fact 7.2. *Assume that the normed space $(V;\|\cdot\|)$ is complete.*

(1) *If $\sum_{n=1}^{\infty} f_n$ absolutely converges then it converges.*
(2) *If $\|f_n(x)\| \leq \alpha_n$ for all $x \in X, n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \alpha_n < +\infty$ then $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly.*