CHAPTER 1

Continuous Functions in Euclidean Spaces

1. Continuous Functions

DEFINITION 1.1. Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$.

• f is *continuous* at x_0 if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in X$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0) < \epsilon$

• f is *continuous* if f is continuous at all $x_0 \in X$.

REMARK 1.2. We may consider the "more general" setting

$$f : D\ (\subseteq X) \to Y,\ x_0 \in D.$$

f is *continuous* at x_0 if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in D$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0) < \epsilon$.

But this is not more general than the definition since it coincides with the case when D is considered as a metric space with the metric $\rho|_{D\times D}$.

FACT 1.3. *Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$. TFAE:*

(i) *f is continuous at x_0*
(ii) *$\forall \epsilon > 0\ \exists \delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$*
(iii) *$\forall U \in \mathcal{V}(f(x_0))\ \exists V \in \mathcal{V}(x_0)$ such that $f(V) \subseteq U$.*
(iv) *$\forall U$ open in Y such that $f(x_0) \in U\ \exists V$ open in X such that $x_0 \in V$ and $f(V) \subseteq U$.*

The last characterization shows that continuity is a topological concept.

DEFINITION 1.4. Let $(X;\mathcal{T})$ and $(Y;\mathcal{Y})$ be topological spaces. Let $f : X \to Y$ be a function and $x_0 \in X$.

f is *continuous* at x_0 if $\forall U \in \mathcal{Y}$ such that $f(x_0) \in U\ \exists V \in \mathcal{T}$ such that $x_0 \in V$ and $f(V) \subseteq U$.

Continuity is related to the more general concept of limit.

DEFINITION 1.5. Let $(X;\rho)$ and $(Y;\eta)$ be topological spaces, $D \subseteq X$. Let $f : D \to Y$ be a function and $x_0 \in X, y_0 \in Y$.

f has limit y_0 at x_0 if:

• x_0 is an accumulation point for D

• $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

REMARK 1.6. If f has limit y_0 at x_0 then y_0 is unique, hence we can denote:

$$\lim_{x\to x_0} f(x) = y_0.$$

FACT 1.7.

(1) *$f : D\,(\subseteq X) \to Y$, metric spaces, $x_0 \in X$.*
  (i) *if x_0 is isolated in D then f is continuous at x_0*
  (ii) *if x_0 is an accum. point for D then f is cont. at x_0 iff*

$$\lim_{x\to x_0} f(x) = f(x_0).$$

(2)
(3) *$f : D \to Y$ function, $x_0 \in D$ accumulation point and $y_0 \in Y$. Then*

$$\begin{aligned}\lim_{x\to x_0} f(x) = y_0 \;\Leftrightarrow\; & \forall (x_n)_{n\geq 1} \textit{ with all } x_n \in D\\ & \textit{and } x_n \neq x_0 \;\forall n\\ & \textit{and } \lim_{n\to\infty} x_n = x_0\\ & \textit{we have } \lim_{n\to\infty} f(x_n) = y_0.\end{aligned}$$

PROOF. "$\Rightarrow$" $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

Take $(x_n)_{n\geq 1}$ a sequence in X, $x_n \neq x_0\ \forall n$ and $\lim_{n\to\infty} x_n = x_0$.

Then $\exists N_\delta \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N_\delta$ then $\rho(x_n, x_0) < \delta$ hence: $\eta(f(x_n), y_0) < \epsilon$.

"$\Leftarrow$" Assume that $\forall (x_n)_{n\geq 1}$ seq. with all elements in D such that $x_n \neq x_0\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = y_0$.

By contradiction assume that $f(x)$ does not converge to y_0 as x approaches x_0.

Then $\exists \epsilon_0 > 0$ such that $\forall \delta > 0\ \exists x \in D \setminus \{x_0\}$ with $\rho(x, x_0) < \delta$ and $\eta(f(x), y_0) \geq \epsilon_0$.

$\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$ hence $\exists x_n \in D \setminus \{x_0\}$ with $\rho(x_n, x_0) < \delta = \frac{1}{n}$ and $\eta(f(x_n), y_0) \geq \epsilon_0$

hence $x_n \xrightarrow[n\to\infty]{\rho} x_0$ but $f(x_n) \not\xrightarrow[n\to\infty]{\eta} y_0$.

$\square$

(4) *(Sequential Characterization of Continuity) Let $f : X \to Y$ function and $x_0 \in X$.*

$$\begin{aligned}\textit{Then } f \textit{ is continuous at } x_0 \Leftrightarrow\; & \forall (x_n)_{n\geq 1} \textit{ seq. in } X\\ & \textit{such that } x_n \xrightarrow[n]{\rho} x_0\\ & \textit{we have } f(x_n) \xrightarrow[n]{\eta} y_0.\end{aligned}$$

(5) *(Composition of Functions) Let $\underset{\rho}{X} \xrightarrow{f} \underset{\eta}{Y} \xrightarrow{g} \underset{\theta}{Z}$ functions between metric spaces.*

(i)
(ii)

$$\textit{Assume that } x_0 \in X$$
$$f \textit{ is cont. at } x_0$$
$$g \textit{ is cont. at } f(x_0)$$

Then $g \circ f$ is continuous at x_0.

(6) *(Functions Between Euclidean Spaces) Let $D \subseteq \mathbb{R}^p$ and $f : D \to \mathbb{R}^q$ be a function, hence $f(x) = (f_1(x), \ldots, f_q(x))$, $\forall x \in D$ where $f_j : D \to \mathbb{R}$ function $\forall j = 1, \ldots, q$.*

(i) *Let $x_0 \in D'$, i.e. x_0 is an accumulation point for D and $y^{(0)} = \left(y_1^{(0)}, \ldots, y_q^{(0)}\right) \in \mathbb{R}^q$. Then*

$$\lim_{x \to x_0} f(x) = y^{(0)} \Leftrightarrow \ \forall j = 1, \ldots, q \ \lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

(ii) *Let $x_0 \in D$. Then*

$$f \textit{ is cont. at } x_0 \Leftrightarrow \ \forall j = 1, \ldots, q \ , f_j \textit{ is cont. at } x_0.$$

Proof.

(i)
"$\Rightarrow$". Assume that $\lim_{x \to x_0} f(x) = y^{(0)}$ and use the $\| \cdot \|_\infty$.
$\forall \epsilon > 0 \ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\| < \delta$ then $\|f(x) - y^{(0)}\|_\infty < \epsilon$.
Let $j \in \{1, \ldots, q\}$, then

$$|f_j(x) - y_j^{(0)}| \leq \|f(x) - y^{(0)}\|_\infty < \epsilon.$$

"$\Leftarrow$". Assume that $\forall j = 1, \ldots, q$

$$\lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

Then $\forall \epsilon > 0 \ \exists \delta_j > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\|_\infty < \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$.
Take $\delta := \min\{\delta_1, \ldots, \delta_q\} > 0$.
Then $\forall j = 1, \ldots, q$, if $\|x - x_0\|_\infty < \delta \leq \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$ hence

$$\|f(x) - y^{(0)}\|_\infty = \max_{j=1}^{q}\{|f_j(x) - y_j^{(0)}|\} < \epsilon.$$

(ii) • if x_0 isolated, nothing to prove.
• if x_0 accum. point for D, we use (i).

∎

EXAMPLE 1.8. (1) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

• f continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$
• $(0, 0)$ is an accum. point for $\mathbb{R}^2$
and f does not have a limit $(x, y) \to (0, 0)$

$$\begin{aligned} x = 0,\ y \to 0 &\Rightarrow f(0, y) = 0 \to 0 \\ y = 0,\ x \to 0 &\Rightarrow f(x, 0) = 0 \to 0 \\ x = y \to 0 &\Rightarrow f(x, y) = \frac{1}{2} \to \frac{1}{2} \end{aligned}$$

(2) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

• f continuous on $\mathbb{R}^2$
• at $(x_0, y_0) \neq (0, 0)$, clear
• at $(0, 0)$

$$|f(x, y)| = \frac{|xy|}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{x^2 + y^2}}{2} = \frac{\|(x, y)\|_2}{2}$$

DEFINITION 1.9. • A *curve* in $\mathbb{R}^2$ is a continuous function $\gamma : I \to \mathbb{R}^q, q \geq 1$ where I is an interval.

• If the interval $I = [a, b]$ is compact, then the curve has *endpoints* $x = \gamma(a)$ and $y = \gamma(b)$. In this case we say that γ is a *path* joining x and y.

• A curve with endpoints x and y is called *closed* if $x = y$.

EXAMPLE 1.10. (1) $\gamma(t) = (\cos t, \sin t)\,, \gamma : [0, 2\pi] \to \mathbb{R}^2$
$\gamma(0) = \gamma(2\pi)$ so γ is a closed curve.
(2) $\gamma(t) = (t^2, t^3)\,, \gamma : [0, 1] \to \mathbb{R}^2$
$\gamma(0) = (0, 0)$ $\gamma(1) = (1, 1)$ a curve with endpoints but not closed.
(3) $\gamma(t) = (t \cos t, t \sin t, t)\,, \gamma : \mathbb{R} \to \mathbb{R}^3$
is a curve with no endpoints
a *spiral* inside $\{(x, y, z) | x^2 + y^2 = |z|\}$ a *cone*

DEFINITION 1.11. A *surface* in $\mathbb{R}^q$ $(q \geq 2)$ is a continuous function $F : A \to \mathbb{R}^q$, D open, nonempty in $\mathbb{R}^2$, $D \subseteq A \subseteq \bar{D}$.

EXAMPLE 1.12. (1) *2-dimensional sphere in* $\mathbb{R}^3$

$$F : [0, 2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$$

$$F(\theta, \phi) = (\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi)$$

(2)

$$G : B \to \mathbb{R}^3$$

$$G(x, y, z) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$$

$$B = \left\{(x, y) | x^2 + y^2 \leq 1\right\}$$

2. Continuity and Topology

THEOREM 2.1. *(*Topological Characterization of Continuity*)*
Let $(X; \rho)$, $(Y; \eta)$ be metric spaces. $f : X \to Y$ a function. TFAE:

(i) *f is continuous.*
(ii) *For all U open in Y, $f^{-1}(U)$ is open in X.*
(iii) *For all F closed in Y, $f^{-1}(F)$ is closed in X.*

PROOF.

(i) $\Rightarrow$ (ii). Let U open in Y and $x \in f^{-1}(U)$, i.e. $f(x) \in U$.
Then $\exists \epsilon > 0$ s.t. $B_\epsilon(f(x)) \subseteq U$.
Since f is cont. at x $\exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(f(x)) \subseteq U$
hence $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))) \subseteq f^{-1}(U)$, i.e. $f^{-1}(U)$ is open.
(ii) $\Rightarrow$ (i). Let $x \in X$ and $\epsilon > 0$. Then $B_\epsilon(f(x))$ is open in Y
hence $f^{-1}(B_\epsilon(f(x)))$ is open in X.
Since $x \in f^{-1}(B_\epsilon(f(x)))$ it follows that $\exists \delta > 0$
s.t. $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$, i.e. $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.
Hence f is cont. at each $x \in X$.
(ii) $\Leftrightarrow$ (iii). Since $f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A)$.

$\square$

COROLLARY 2.2. *Let $\emptyset \neq D \subseteq \mathbb{R}^p$, $f : D \to \mathbb{R}^q$ a function. TFAE:*

(i) *f is continuous.*
(ii) *For all U open in $\mathbb{R}^q$, $f^{-1}(U)$ is relatively open in D.*
(iii) *For all F closed in $\mathbb{R}^q$, $f^{-1}(F)$ is relatively closed in D.*

COROLLARY 2.3. *Let $\emptyset \neq D \subseteq \mathbb{R}^p$ open, $f : D \to \mathbb{R}^q$ a function. TFAE:*

(i) *f is continuous.*
(ii) *For all U open in $\mathbb{R}^q$, $f^{-1}(U)$ is open in $\mathbb{R}^p$.*
(iii) *For all F closed in $\mathbb{R}^q$, $f^{-1}(F)$ is closed in $\mathbb{R}^p$.*

3. Continuity and Compactness

THEOREM 3.1. *Let* $(X;\rho)$, $(Y;\eta)$ *be metric spaces,* $f : X \to Y$ *a continuous function and* K *compact in* X. *Then* $f(K)$ *is compact.*

PROOF. Let $\{U_i | i \in \mathcal{J}\}$ be an open (in Y) covering of $f(K)$:

- $\forall i \in \mathcal{J}, U_i$ is open in Y;
- $f(K) \subseteq \bigcup_{i\in\mathcal{J}} U_i$.

Then $\forall i \in \mathcal{J}$, $f^{-1}(U_i)$ is open in X and

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(\bigcup_{i\in\mathcal{J}} U_i) = \bigcup_{i\in\mathcal{J}} f^{-1}(U_i)$$

hence $\{f^{-1}(U_i)|i \in \mathcal{J}\}$ is an open covering of K.

Since K is compact $\exists i_1, \dots, i_n \in \mathcal{J}$ such that

$$K \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}) = f^{-1}(\bigcup_{k=1}^{n} U_{i_k})$$

hence $f(K) \subseteq f(f^{-1}(\bigcup_{k=1}^n U_{i_k})) \subseteq \bigcup_{k=1}^n U_{i_k}$. □

COROLLARY 3.2. *Let* $f : X \to \mathbb{R}$ *a continuous function and* K *compact, nonempty in* X. *Then:*

- f *is bounded on* K, *i.e.* $f(K)$ *is bounded in* $\mathbb{R}$.
- *The extreme values of* f *on* K *are attained, i.e.* $\exists x_m, x_M \in K$ *such that*

$$f(x_m) = \inf_K f, \ f(x_M) = \sup_K f.$$

PROOF. $f(K)$ is compact in $\mathbb{R}$, hence closed and bounded.

- $f(K)$ bounded $\Rightarrow \inf_K f, \sup_K f \in \mathbb{R}$.
- $\inf_K f \in_! f(\bar{K}) = f(K)$, hence $\exists x_m \in K$ such that $\inf_K f = f(x_m)$.
- $\sup_K f \in f(\bar{K}) = f(K)$, hence $\exists x_M \in K$ such that $\sup_K f = f(x_M)$. □

COROLLARY 3.3. *Let* $f : X \to \mathbb{R}^q$ *a continuous function. Then for all* K *nonempty and compact in* X $\exists x_m, x_M \in K$ *such that*

$$\|f(x_m)\| = \inf_K \|f\|, \ \|f(x_M)\| = \sup_K \|f\|.$$

PROOF.

$$\|f\| = \|\cdot\| \circ f : X \to \mathbb{R}^q \to \mathbb{R}.$$

□

4. Continuity and Connectedness

THEOREM 4.1. *Let $(X;\rho)$, $(Y;\eta)$ be metric spaces, $f : X \to Y$ a continuous function, C connected in X. Then $f(C)$ is connected in Y.*

PROOF. By contrapositive, assume that f(C) is separated, hence: there exist U, V open in Y such that

- $f(C) \subseteq U \cup V$
- $f(C) \cap U \neq \emptyset$
- $f(C) \cap V \neq \emptyset$
- $f(C) \cap U \cap V = \emptyset$

Then $f^{-1}(U)$, $f^{-1}(V)$ are open in X.

- $C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
- $\emptyset \neq f^{-1}(f(C) \cap U) = f^{-1}(f(C)) \cap f^{-1}(U)$

- We prove $C \cap f^{-1}(U) \neq \emptyset$.
  Since $f(C) \cap U \neq \emptyset$, $\exists y \in f(C)$ and $y \in U$ hence $\exists x \in C$ such that $f(x) \in U$, hence $x \in f^{-1}(U)$, i.e. $x \in C \cap f^{-1}(U)$.
- Similarly $C \cap f^{-1}(V) \neq \emptyset$.
- Similarly $C \cap f^{-1}(U) \cap f^{-1}(V) = \emptyset$
  Assume $C \cap f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$, then $\exists x \in C$ such that $x \in f^{-1}(U) \cap f^{-1}(V)$ hence $f(x) \in f(C)$ and $f(x) \in f(f^{-1}(U)) \subseteq U$, $f(x) \in f(f^{-1}(V)) \subseteq V$ i.e. $f(x) \in f(C) \cap U \cap V$, contradiction!

Thus, $f^{-1}(U)$ and $f^{-1}(V)$ separate C, contradiction! □

COROLLARY 4.2. (1) *Let $f : (X;\rho) \to \mathbb{R}$ continuous, C connected in X. Then $f(C)$ is an interval.*
(2) *Let I be an interval in $\mathbb{R}$ and $\gamma : I \to \mathbb{R}^d$ continuous (a curve). Then $\gamma(I)$ is connected.*

DEFINITION 4.3. A subset $S \subseteq (X;\rho)$ is called *pathwise connected* if for all $a, b \in S$ there exists a (continuous) path $\gamma : [0,1] \to S$ such that $a = \gamma(0)$ and $b = \gamma(1)$.

(3) *If S is pathwise connected, then it is connected.*

PROOF. Assume S is not connected, let U, V open in X and separating S. Then there exist $a \in S \cap U$ and $b \in S \cap V$. Since S is pathwise connected, there exists $\gamma : [0,1] \to S$ continuous such that $\gamma(0) = a$ and $\gamma(1) = b$. But then U and V separate $\gamma([0,1])$, contradiction! □

EXAMPLE 4.4. A set in $\mathbb{R}^2$ that is connected but not pathwise connected.

$$S = \{(0,y) | -1 \leq y \leq 1\} \cap \{(x, \sin\frac{1}{x}) | -\frac{1}{\pi} < x < \frac{1}{\pi}, x \neq 0\}$$

(4) *Assume that D is open in $\mathbb{R}^d$. Then D is connected iff D is pathwise connected.*

PROOF."$\Leftarrow$" Holds in general.

"$\Rightarrow$" On D we define a relation: $x \overset{p}{\sim} y$ if $\exists \gamma : [0,1] \to D$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$.

- $\overset{p}{\sim}$ is an equivalence relation on D.
- $\forall x \in D$ its equivalence class $[x]_p$ is an open set.
- If D is not pathwise connected then there exist at least two different cosets w.r.t. $\overset{p}{\sim}$, hence D is disconnected.

$\square$

5. Uniform Continuity

DEFINITION 5.1. Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces. A function $f : X \to Y$ is *uniformly continuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x_1, x_2 \in X$, if $\rho(x_1, x_2) < \delta$ then $\eta(f(x_1), f(x_2)) < \epsilon$.

THEOREM 5.2. *If $f : \underset{\rho}{X} \to \underset{\eta}{Y}$ is continuous and X is compact, then f is uniformly continuous.*

PROOF. Let $\epsilon > 0$. Since f is continuous on X, $\forall x \in X\ \exists \delta_x > 0$ such that $\forall z \in X$ with $\rho(x,z) < \delta_x \Rightarrow \eta(f(x), f(z)) < \epsilon/2$.

Since $\{B_{\delta_x/2}(x) | x \in X\}$ is an open covering of X compact, it follows that there exist $x_1, \ldots, x_n \in X$ such that

$$X \subseteq \bigcup_{i=1}^{n} B_{\delta_{x_i}/2}(x_i)$$

Let $\delta := \min\{\frac{\delta_{x_i}}{2} | i = 1, \ldots, n\} > 0$ and let $x, z \in X$ such that $\rho(x,z) < \delta$. Then $\exists j \in \{1, \ldots, n\}$ such that $x \in B_{\delta_{x_j}/2}(x_j)$ i.e. $\rho(x, x_j) < \frac{\delta_{x_j}}{2}$. Then

$$\rho(z, x_j) \leq \rho(z,x) + \rho(x, x_j) < \delta + \frac{\delta_{x_j}}{2} \leq \frac{\delta_{x_j}}{2} + \frac{\delta_{x_j}}{2} = \delta_{x_j}$$

hence $\eta(f(x), f(x_j)) < \frac{\epsilon}{2}$ and $\eta(f(z), f(x_j)) < \frac{\epsilon}{2}$. Then

$$\eta(f(x), f(z)) \leq \eta(f(x), f(x_j)) + \eta(f(z), f(x_j)) < \epsilon$$

$\square$

THEOREM 5.3. *Let $(X;\rho)$ be a compact metric space, $(Y;\eta)$ be a complete metric space, $\emptyset \neq D \subseteq X$ and $f : D \to Y$ a function. TFAE:*

(i) *f is uniformly continuous on D*
(ii) *$\exists \bar{f} : \bar{D} \to Y$ such that $\bar{f}|_D = f$ and $\bar{f}$ is continuous on $\bar{D}$.*

PROOF(ii) $\Rightarrow$ (i) $\bar{D}$ closed in X compact hence $\bar{D}$ is compact $\Rightarrow \bar{f}$ unif. cont. on $\bar{D} \Rightarrow f = \bar{f}|_D$ unif. cont. on D.

(i) $\Rightarrow$ (ii) Assume that f is uniformly continuous.

- f maps Cauchy sequences from D to Cauchy sequences in Y:
  Let $\epsilon > 0$. Then $\exists \delta > 0$ such that $\forall x, z \in D$, if $\rho(x,z) < \delta$ then $\eta(f(x), f(z)) < \epsilon$.
  Let $(x_n)_{n\geq 1}$ be a ρ-Cauchy sequence in D. $\forall \epsilon > 0$, with $\delta > 0$ as before, $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $m, n \geq \mathbb{N}$ then $\rho(x_m, x_n) < \delta$ hence $\eta(f(x_m), f(x_n)) < \epsilon$.
- Let $x \in \bar{D}$. Then $\exists (x_n)_{n\geq 1}$ in D such that $x_n \underset{n}{\overset{\rho}{\to}} x$ hence $(x_n)_{n\geq 1}$ is ρ-Cauchy, hence $(f(x_n))_{n\geq 1}$ is η-Cauchy hence, since $(Y;\eta)$ is complete, $\exists y_x \in Y$ such that $f(x_n) \underset{n}{\overset{\eta}{\to}} y_x$. Define $\bar{f}(x) := y_x$. The definition of $\bar{f}$ is correct:
  We use the interlacing method.

$$\text{If } (x_n)_{n\geq 1} \text{ is a seq. in D, } x_n \underset{n}{\overset{\rho}{\to}} x$$
$$(z_n)_{n\geq 1} \text{ is a seq. in D, } z_n \underset{n}{\overset{\rho}{\to}} x$$

  Then $(t_n)_{n\geq 1}$ defined by

$$t_n = \begin{cases} x_{\frac{n}{2}}, & \text{if } n = 2k \\ z_{\frac{n-1}{2}}, & \text{if } n = 2k+1 \end{cases}$$

  is a sequence in D and $t_n \underset{n}{\overset{\rho}{\to}} x$.
  Then $f(x_n) \underset{n}{\overset{\eta}{\to}} y_x$, $f(z_n) \underset{n}{\overset{\eta}{\to}} z_x$, $f(t_n) \underset{n}{\overset{\eta}{\to}} t_x$. Since $(f(x_n))_{n\geq 1}$ and $(f(z_n))_{n\geq 1}$ are subsequences of $(f(t_n))_{n\geq 1}$, it follows that $y_x = t_x = z_x$.
- $\bar{f}$ is continuous on $\bar{D}$:
  We use the sequential characterization of continuity. Let $\bar{x} \in \bar{D}$ and $(\bar{x}_n)_{n\geq 1}$ a sequence in $\bar{D}$ such that $\bar{x}_n \underset{n}{\overset{\rho}{\to}} \bar{x}$.
  Then $\forall n \geq 1$ $\exists x_n \in D$ such that $\rho(\bar{x}_n, x_n) < \frac{1}{n}$ and $\rho(\bar{f}(\bar{x}_n), f(x_n)) < \frac{1}{n}$.
  Then

$$\rho(x_n, \bar{x}) \leq \rho(x_n, \bar{x}_n) + \rho(\bar{x}_n, \bar{x}) < \frac{1}{n} + \rho(\bar{x}_n, \bar{x}) \underset{n}{\to} 0$$

  hence $x_n \underset{n}{\overset{\rho}{\to}} \bar{x}$, and then by def. of $\bar{f}$ we have $f(x_n) \underset{n}{\overset{\rho}{\to}} \bar{f}(\bar{x})$, and then

$$\eta(\bar{f}(\bar{x}_n), \bar{f}(\bar{x})) \leq \eta(\bar{f}(\bar{x}_n), f(x_n)) + \eta(f(x_n), \bar{f}(\bar{x})) < \frac{1}{n} + \eta(f(x_n), \bar{f}(\bar{x})) \underset{n}{\to} 0$$

  hence $\bar{f}(\bar{x}_n) \underset{n}{\overset{\eta}{\to}} \bar{f}(\bar{x})$.

$\square$

6. Sequences of Functions

7. Series of Functions