CHAPTER 1

Continuous Functions in Euclidean Spaces

1. Continuous Functions

DEFINITION 1.1. Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$.

• f is *continuous* at x_0 if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in X$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0) < \epsilon$

• f is *continuous* if f is continuous at all $x_0 \in X$.

REMARK 1.2. We may consider the "more general" setting

$$f : D\ (\subseteq X) \to Y,\ x_0 \in D.$$

f is *continuous* at x_0 if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in D$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0) < \epsilon$.

But this is not more general than the definition since it coincides with the case when D is considered as a metric space with the metric $\rho|_{D\times D}$.

FACT 1.3. *Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$. TFAE:*

(i) *f is continuous at x_0*
(ii) *$\forall \epsilon > 0\ \exists \delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$*
(iii) *$\forall U \in \mathcal{V}(f(x_0))\ \exists V \in \mathcal{V}(x_0)$ such that $f(V) \subseteq U$.*
(iv) *$\forall U$ open in Y such that $f(x_0) \in U\ \exists V$ open in X such that $x_0 \in V$ and $f(V) \subseteq U$.*

The last characterization shows that continuity is a topological concept.

DEFINITION 1.4. Let $(X;\mathcal{T})$ and $(Y;\mathcal{Y})$ be topological spaces. Let $f : X \to Y$ be a function and $x_0 \in X$.

f is *continuous* at x_0 if $\forall U \in \mathcal{Y}$ such that $f(x_0) \in U\ \exists V \in \mathcal{T}$ such that $x_0 \in V$ and $f(V) \subseteq U$.

Continuity is related to the more general concept of limit.

DEFINITION 1.5. Let $(X;\rho)$ and $(Y;\eta)$ be topological spaces, $D \subseteq X$. Let $f : D \to Y$ be a function and $x_0 \in X, y_0 \in Y$.

f has limit y_0 at x_0 if:

• x_0 is an accumulation point for D

• $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

REMARK 1.6. If f has limit y_0 at x_0 then y_0 is unique, hence we can denote:

$$\lim_{x \to x_0} f(x) = y_0.$$

FACT 1.7.

(1) *$f : D\,(\subseteq X) \to Y$, metric spaces, $x_0 \in X$.*
  (i) *if x_0 is isolated in D then f is continuous at x_0*
  (ii) *if x_0 is an accum. point for D then f is cont. at x_0 iff*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

(2)
(3) *$f : D \to Y$ function, $x_0 \in D$ accumulation point and $y_0 \in Y$. Then*

$$\begin{aligned} \lim_{x \to x_0} f(x) = y_0 \;\Leftrightarrow\; & \forall (x_n)_{n \geq 1} \text{ with all } x_n \in D \\ & \text{and } x_n \neq x_0 \ \forall n \\ & \text{and } \lim_{n \to \infty} x_n = x_0 \\ & \text{we have } \lim_{n \to \infty} f(x_n) = y_0. \end{aligned}$$

PROOF. "$\Rightarrow$" $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

Take $(x_n)_{n \geq 1}$ a sequence in X, $x_n \neq x_0\ \forall n$ and $\lim_{n \to \infty} x_n = x_0$.

Then $\exists N_\delta \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N_\delta$ then $\rho(x_n, x_0) < \delta$ hence: $\eta(f(x_n), y_0) < \epsilon$.

"$\Leftarrow$" Assume that $\forall (x_n)_{n \geq 1}$ seq. with all elements in D such that $x_n \neq x_0\ \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x_0$ we have $\lim_{n \to \infty} f(x_n) = y_0$.

By contradiction assume that $f(x)$ does not converge to y_0 as x approaches x_0.

Then $\exists \epsilon_0 > 0$ such that $\forall \delta > 0\ \exists x \in D \setminus \{x_0\}$ with $\rho(x, x_0) < \delta$ and $\eta(f(x), y_0) \geq \epsilon_0$.

$\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$ hence $\exists x_n \in D \setminus \{x_0\}$ with $\rho(x_n, x_0) < \delta = \frac{1}{n}$ and $\eta(f(x_n), y_0) \geq \epsilon_0$

hence $x_n \xrightarrow[n \to \infty]{\rho} x_0$ but $f(x_n) \not\xrightarrow[n \to \infty]{\eta} y_0$.

$\square$

(4) *(Sequential Characterization of Continuity) Let $f : X \to Y$ function and $x_0 \in X$.*

$$\begin{aligned} \text{Then } f \text{ is continuous at } x_0 \Leftrightarrow\; & \forall (x_n)_{n \geq 1} \text{ seq. in } X \\ & \text{such that } x_n \xrightarrow[n]{\rho} x_0 \\ & \text{we have } f(x_n) \xrightarrow[n]{\eta} y_0. \end{aligned}$$

(5) *(Composition of Functions) Let $\underset{\rho}{X} \xrightarrow{f} \underset{\eta}{Y} \xrightarrow{g} \underset{\theta}{Z}$ functions between metric spaces.*

(i)
(ii)

Assume that $x_0 \in X$
f *is cont. at* x_0
g *is cont. at* $f(x_0)$

Then $g \circ f$ *is continuous at* x_0.

(6) *(Functions Between Euclidean Spaces) Let* $D \subseteq \mathbb{R}^p$ *and* $f : D \to \mathbb{R}^q$ *be a function, hence* $f(x) = (f_1(x), \ldots, f_q(x))$, $\forall x \in D$ *where* $f_j : D \to \mathbb{R}$ *function* $\forall j = 1, \ldots, q$.

(i) *Let* $x_0 \in D'$, *i.e.* x_0 *is an accumulation point for* D *and* $y^{(0)} = \left(y_1^{(0)}, \ldots, y_q^{(0)}\right) \in \mathbb{R}^q$. *Then*

$$\lim_{x \to x_0} f(x) = y^{(0)} \Leftrightarrow \ \forall j = 1, \ldots, q \ \lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

(ii) *Let* $x_0 \in D$. *Then*

$$f \text{ is cont. at } x_0 \Leftrightarrow \ \forall j = 1, \ldots, q \ , f_j \text{ is cont. at } x_0.$$

Proof.

(i)
"$\Rightarrow$". Assume that $\lim_{x \to x_0} f(x) = y^{(0)}$ and use the $\| \cdot \|_\infty$.
$\forall \epsilon > 0 \ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\| < \delta$ then $\|f(x) - y^{(0)}\|_\infty < \epsilon$.
Let $j \in \{1, \ldots, q\}$, then

$$|f_j(x) - y_j^{(0)}| \leq \|f(x) - y^{(0)}\|_\infty < \epsilon.$$

"$\Leftarrow$". Assume that $\forall j = 1, \ldots, q$

$$\lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

Then $\forall \epsilon > 0 \ \exists \delta_j > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\|_\infty < \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$.
Take $\delta := \min\{\delta_1, \ldots, \delta_q\} > 0$.
Then $\forall j = 1, \ldots, q$, if $\|x - x_0\|_\infty < \delta \leq \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$ hence

$$\|f(x) - y^{(0)}\|_\infty = \max_{j=1}^{q}\{|f_j(x) - y_j^{(0)}|\} < \epsilon.$$

(ii) • if x_0 isolated, nothing to prove.
• if x_0 accum. point for D, we use (i).

$\square$

EXAMPLE 1.8. (1) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

• f continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$
• $(0,0)$ is an accum. point for $\mathbb{R}^2$
and f does not have a limit $(x,y) \to (0,0)$

$$\begin{aligned} x = 0,\ y \to 0 &\Rightarrow f(0,y) = 0 \to 0 \\ y = 0,\ x \to 0 &\Rightarrow f(x,0) = 0 \to 0 \\ x = y \to 0 &\Rightarrow f(x,y) = \frac{1}{2} \to \frac{1}{2} \end{aligned}$$

(2) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

• f continuous on $\mathbb{R}^2$
• at $(x_0,y_0) \neq (0,0)$, clear
• at $(0,0)$

$$|f(x,y)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{2} = \frac{\|(x,y)\|_2}{2}$$

DEFINITION 1.9. • A *curve* in $\mathbb{R}^2$ is a continuous function $\gamma : I \to \mathbb{R}^q, q \geq 1$ where I is an interval.

• If the interval $I = [a,b]$ is compact, then the curve has *endpoints* $x = \gamma(a)$ and $y = \gamma(b)$. In this case we say that γ is a *path* joining x and y.

• A curve with endpoints x and y is called *closed* if $x = y$.

EXAMPLE 1.10. (1) $\gamma(t) = (\cos t, \sin t)\,, \gamma : [0,2\pi] \to \mathbb{R}^2$
$\gamma(0) = \gamma(2\pi)$ so γ is a closed curve.
(2) $\gamma(t) = (t^2,t^3)\,, \gamma : [0,1] \to \mathbb{R}^2$
$\gamma(0) = (0,0)$ $\gamma(1) = (1,1)$ a curve with endpoints but not closed.
(3) $\gamma(t) = (t\cos t, t\sin t, t)\,, \gamma : \mathbb{R} \to \mathbb{R}^3$
is a curve with no endpoints.

2. Continuity and Topology

3. Continuity and Compactness

4. Continuity and Connectedness

5. Uniform Continuity

6. Sequences of Functions

7. Series of Functions