CHAPTER 1

Continuous Functions in Euclidean Spaces

1. Continuous Functions

DEFINITION 1.1. Let $(X; \rho)$ and $(Y; \eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$.

- f is *continuous* at x_0 if $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall x \in X$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0)) < \epsilon$
- f is *continuous* if f is continuous at all $x_0 \in X$.

REMARK 1.2. We may consider the "more general" setting

$$f : D \; (\subseteq X) \to Y, \; x_0 \in D.$$

f is *continuous* at x_0 if $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall x \in D$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0)) < \epsilon$.

But this is not more general than the definition since it coincides with the case when D is considered as a metric space with the metric $\rho|_{D \times D}$.

FACT 1.3. *Let $(X; \rho)$ and $(Y; \eta)$ be two metric spaces, $f : X \to Y$ and $x_0 \in X$. TFAE:*

(i) *f is continuous at x_0*
(ii) *$\forall \epsilon > 0 \; \exists \delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$*
(iii) *$\forall U \in \mathcal{V}(f(x_0)) \; \exists V \in \mathcal{V}(x_0)$ such that $f(V) \subseteq U$.*
(iv) *$\forall U$ open in Y such that $f(x_0) \in U \; \exists V$ open in X such that $x_0 \in V$ and $f(V) \subseteq U$.*

The last characterization shows that continuity is a topological concept.

DEFINITION 1.4. Let $(X; \mathcal{T})$ and $(Y; \mathcal{Y})$ be topological spaces. Let $f : X \to Y$ be a function and $x_0 \in X$.

f is *continuous* at x_0 if $\forall U \in \mathcal{Y}$ such that $f(x_0) \in U \; \exists V \in \mathcal{T}$ such that $x_0 \in V$ and $f(V) \subseteq U$.

Continuity is related to the more general concept of limit.

DEFINITION 1.5. Let $(X; \rho)$ and $(Y; \eta)$ be topological spaces, $D \subseteq X$. Let $f : D \to Y$ be a function and $x_0 \in X, y_0 \in Y$.

f has *limit y_0 at x_0* if:

- x_0 is an accumulation point for D
- $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

REMARK 1.6. If f has limit y_0 at x_0 then y_0 is unique, hence we can denote:

$$\lim_{x \to x_0} f(x) = y_0.$$

FACT 1.7.

(1) $f : D \, (\subseteq X) \to Y$, metric spaces, $x_0 \in X$.
   (i) if x_0 is isolated in D then f is continuous at x_0
   (ii) if x_0 is an accum. point for D then f is cont. at x_0 iff

$$\lim_{x \to x_0} f(x) = f(x_0).$$

(2)

(3) $f : D \to Y$ function, $x_0 \in D$ accumulation point and $y_0 \in Y$. Then

$$\lim_{x \to x_0} f(x) = y_0 \ \Leftrightarrow \ \forall (x_n)_{n \geq 1} \text{ with all } x_n \in D$$

$$\text{and } x_n \neq x_0 \ \forall n$$

$$\text{and } \lim_{n \to \infty} x_n = x_0$$

$$\text{we have } \lim_{n \to \infty} f(x_n) = y_0.$$

PROOF.   "$\Rightarrow$" $\forall \epsilon > 0 \ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.
        Take $(x_n)_{n \geq 1}$ a sequence in X, $x_n \neq x_0 \ \forall n$ and $\lim_{n \to \infty} x_n = x_0$.
        Then $\exists N_\delta \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N_\delta$ then $\rho(x_n, x_0) < \delta$ hence: $\eta(f(x_n), y_0) < \epsilon$.
"$\Leftarrow$" Assume that $\forall (x_n)_{n \geq 1}$ seq. with all elements in D such that $x_n \neq x_0 \ \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x_0$ we have $\lim_{n \to \infty} f(x_n) = y_0$.
        By contradiction assume that $f(x)$ does not converge to y_0 as x approaches x_0.
        Then $\exists \epsilon_0 > 0$ such that $\forall \delta > 0 \ \exists x \in D \setminus \{x_0\}$ with $\rho(x, x_0) < \delta$ and $\eta(f(x), y_0) \geq \epsilon_0$.
        $\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$ hence $\exists x_n \in D \setminus \{x_0\}$ with $\rho(x_n, x_0) < \delta = \frac{1}{n}$ and $\eta(f(x_n), y_0) \geq \epsilon_0$
        hence $x_n \xrightarrow[n \to \infty]{\rho} x_0$ but $f(x_n) \xcancel{\xrightarrow[n \to \infty]{\eta}} y_0$.

$\square$

(4) (Sequential Characterization of Continuity) Let $f : X \to Y$ function and $x_0 \in X$.

$$\text{Then } f \text{ is continuous at } x_0 \Leftrightarrow \forall (x_n)_{n \geq 1} \text{ seq. in } X$$

$$\text{such that } x_n \xrightarrow[n]{\rho} x_0$$

$$\text{we have } f(x_n) \xrightarrow[n]{\eta} y_0.$$

(5) (Composition of Functions) Let $X \xrightarrow[\rho]{f} Y \xrightarrow[\eta]{g} Z \underset{\theta}{} $ functions between metric spaces.

(i)
(ii)

$$\textit{Assume that } x_0 \in X$$
$$f \textit{ is cont. at } x_0$$
$$g \textit{ is cont. at } f(x_0)$$

Then $g \circ f$ is continuous at x_0.

(6) *(Functions Between Euclidean Spaces) Let $D \subseteq \mathbb{R}^p$ and $f : D \to \mathbb{R}^q$ be a function, hence $f(x) = (f_1(x), \ldots, f_q(x))$, $\forall x \in D$ where $f_j : D \to \mathbb{R}$ function $\forall j = 1, \ldots, q$.*

(i) *Let $x_0 \in D'$, i.e. x_0 is an accumulation point for D and $y^{(0)} = \left(y_1^{(0)}, \ldots, y_q^{(0)} \right) \in \mathbb{R}^q$. Then*

$$\lim_{x \to x_0} f(x) = y^{(0)} \Leftrightarrow \forall j = 1, \ldots, q \ \lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

(ii) *Let $x_0 \in D$. Then*

$$f \textit{ is cont. at } x_0 \Leftrightarrow \forall j = 1, \ldots, q \ , f_j \textit{ is cont. at } x_0.$$

PROOF.

(i)
"$\Rightarrow$". Assume that $\lim_{x \to x_0} f(x) = y^{(0)}$ and use the $\| \cdot \|_\infty$.
$\forall \epsilon > 0 \ \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\| < \delta$ then $\|f(x) - y^{(0)}\|_\infty < \epsilon$.
Let $j \in \{1, \ldots, q\}$, then

$$|f_j(x) - y_j^{(0)}| \leq \|f(x) - y^{(0)}\|_\infty < \epsilon.$$

"$\Leftarrow$". Assume that $\forall j = 1, \ldots, q$

$$\lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

Then $\forall \epsilon > 0 \ \exists \delta_j > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\|_\infty < \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$.
Take $\delta := \min\{\delta_1, \ldots, \delta_q\} > 0$.
Then $\forall j = 1, \ldots, q$, if $\|x - x_0\|_\infty < \delta \leq \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$ hence

$$\|f(x) - y^{(0)}\|_\infty = \max_{j=1}^q \{|f_j(x) - y_j^{(0)}|\} < \epsilon.$$

(ii) • if x_0 isolated, nothing to prove.
  • if x_0 accum. point for D, we use (i).

$\square$

EXAMPLE 1.8.   (1) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

- f continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$
- $(0, 0)$ is an accum. point for $\mathbb{R}^2$

and f does not have a limit $(x, y) \to (0, 0)$

$$x = 0, \ y \to 0 \Rightarrow f(0, y) = 0 \to 0$$
$$y = 0, \ x \to 0 \Rightarrow f(x, 0) = 0 \to 0$$
$$x = y \to 0 \Rightarrow f(x, y) = \frac{1}{2} \to \frac{1}{2}$$

(2) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

- f continuous on $\mathbb{R}^2$
- at $(x_0, y_0) \neq (0, 0)$, clear
- at $(0, 0)$

$$|f(x, y)| = \frac{|xy|}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{x^2 + y^2}}{2} = \frac{\|(x, y)\|_2}{2}$$

DEFINITION 1.9. • A *curve* in $\mathbb{R}^2$ is a continuous function $\gamma : I \to \mathbb{R}^q, q \geq 1$ where I is an interval.

• If the interval $I = [a, b]$ is compact, then the curve has *endpoints* $x = \gamma(a)$ and $y = \gamma(b)$. In this case we say that γ is a *path* joining x and y.

• A curve with endpoints x and y is called *closed* if $x = y$.

EXAMPLE 1.10.   (1) $\gamma(t) = (\cos t, \sin t), \gamma : [0, 2\pi] \to \mathbb{R}^2$
        $\gamma(0) = \gamma(2\pi)$ so γ is a closed curve.
    (2) $\gamma(t) = (t^2, t^3), \gamma : [0, 1] \to \mathbb{R}^2$
        $\gamma(0) = (0, 0) \ \gamma(1) = (1, 1)$ a curve with endpoints but not closed.
    (3) $\gamma(t) = (t \cos t, t \sin t, t), \gamma : \mathbb{R} \to \mathbb{R}^3$
        is a curve with no endpoints
        a *spiral* inside $\{(x, y, z) | x^2 + y^2 = |z|\}$ a *cone*

DEFINITION 1.11. A *surface* in $\mathbb{R}^q$ $(q \geq 2)$ is a continuous function $F : A \to \mathbb{R}^q$, D open, nonempty in $\mathbb{R}^2$, $D \subseteq A \subseteq \bar{D}$.

EXAMPLE 1.12.   (1) *2-dimensional sphere in* $\mathbb{R}^3$

$$F : [0, 2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$$

$$F(\theta, \phi) = (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)$$

(2)
$$G : B \to \mathbb{R}^3$$
$$G(x, y, z) = \left(x, y, \sqrt{1 - x^2 - y^2} \right)$$
$$B = \left\{ (x, y) | x^2 + y^2 \le 1 \right\}$$

2. Continuity and Topology

3. Continuity and Compactness

4. Continuity and Connectedness

5. Uniform Continuity

6. Sequences of Functions

7. Series of Functions