Def: • A <u>curve</u> in $\mathbb{R}^q$ is a continuous function

$\gamma : I \to \mathbb{R}^q$, $q \geq 1$. where I is an interval

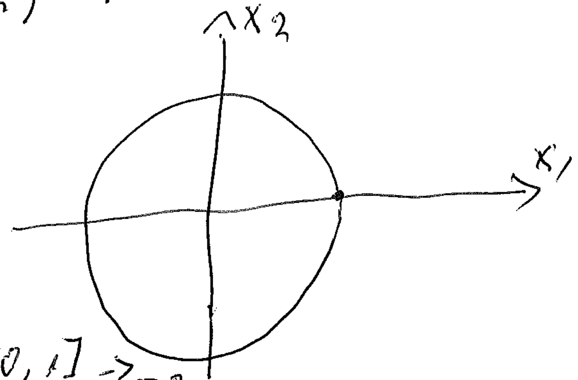
• If the interval $I = [a, b]$ is compact, then the curve has <u>endpoints</u> $x = \gamma(a)$ and $y = \gamma(b)$.

In this case we say that γ is a <u>path</u> joining x and y

• A curve with endpoints x and y is called <u>closed</u> if $x = y$.

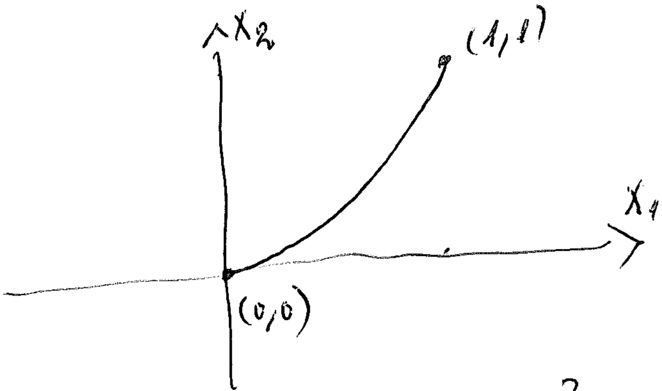
<u>Examples</u>: (1) $\gamma(t) = (\cos t, \sin t)$, $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$\gamma(0) = \gamma(2\pi)$ so γ is a closed curve.



(2) $\gamma(t) = (t^2, t^3)$, $\gamma : [0, 1] \to \mathbb{R}^2$

$\gamma(0) = (0, 0)$ $\gamma(1) = (1, 1)$ a curve with endpoints but not closed



(3) $\gamma(t) = (t\cos t, t\sin t, t)$ $\gamma : \mathbb{R} \to \mathbb{R}^3$

is a curve with no endpoints

<u>Examples</u>: (1) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

- f continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$
- $(0,0)$ is an accum. point for $\mathbb{R}^2$ and f does not have a limit $(x,y) \to (0,0)$

$$\begin{array}{ll} x=0 \quad y \to 0 & f(0,y) = 0 \to 0 \\ y=0 \quad x \to 0 & f(x,0) = 0 \to 0 \\ x=y \to 0 & f(x,y) = \frac{1}{2} \to \cancel{0}\,\frac{1}{2} \end{array}$$

(2) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

- f is continuous on $\mathbb{R}^2$
- at $(x_0,y_0) \neq (0,0)$, clear (!)
- at $(0,0)$

$$|f(x,y)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{2} = \frac{\|(x,y)\|_2}{2}$$

(ii) Let $x_0 \in D$. Then

f is cont. at $x_0 \iff \forall j = 1, \dots, q$, f_j is cont. at x_0.

proof: (i) "$\Longrightarrow$". Assume that $\lim_{x \to x_0} f(x) = y_0$ and use the $\|\cdot\|_\infty$.

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\| < \delta$ then $\|f(x) - y^{(0)}\|_\infty < \varepsilon$.

Let $j \in \{1, \dots, q\}$. then

$$|f_j(x) - y_j^{(0)}| \leq \|f(x) - y^{(0)}\| < \varepsilon.$$

"$\Longleftarrow$" Assume that ~~$\lim_{x \to x_0} f(x) = y$~~ $\forall j = 1, \dots, q$

$$\lim_{x \to x_0} f_j(x) = y_j^{(0)}.$$

Then $\forall \varepsilon > 0\ \exists \delta_j > 0$ s.t. $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\|_\infty < \delta_j$ then $|f_j(x) - y_j^{(0)}| < \varepsilon$.

Take $\delta := \min\{\delta_1, \dots, \delta_q\} > 0$.

Then $\forall j = 1, \dots, q$, if $\|x - x_0\|_\infty < \delta$

then $|f_j(x) - y_j^{(0)}| < \varepsilon$.

hence $\|f(x) - y^{(0)}\|_\infty = \max_{j=1}^{q} \{|f_j(x) - y_j^{(0)}|\} < \varepsilon$

(ii) . if x_0 isolated, nothing to prove.

. if x_0 accum. point for D, we use (i).

#

(5) (Composition of Functions)

Let $\underset{\rho}{X} \xrightarrow{f} \underset{\eta}{Y} \xrightarrow{g} \underset{\theta}{Z}$ functions between metric spaces.

~~(i) Assume that~~ ~~x_0 is an accumulation point for X~~

~~f is continuous~~

~~$\lim_{x \to x_0} f(x) = y_0 \in Y$~~

~~y_0 is an accumulation point for Y~~

~~$\lim_{y \to y_0} g(y) = z_0 \in Z$~~

~~Then $\lim_{x \to x_0} (g \circ f)(x) = z_0$~~

(ii) Assume that $x_0 \in X$

f is cont. at x_0

g is cont. at $f(x_0)$

Then $g \circ f$ is continuous at x_0.

(6) (Functions Between Euclidean Spaces)

Let $D \subseteq \mathbb{R}^p$ and $f : D \to \mathbb{R}^q$ be a function,

hence $f(x) = (f_1(x), \dots, f_q(x))$, $\forall x \in D$

where $f_j : D \to \mathbb{R}$ function $\forall j = 1, \dots, q$.

(i) Let $x_0 \in D'$, i.e. x_0 is an accumulation point for D

and $y^{(0)} = (y_1^{(0)}, \dots, y_q^{(0)}) \in \mathbb{R}^q$.

Then $\lim_{x \to x_0} f(x) = y^{(0)} \iff \forall j = 1, \dots, q$ $\lim_{x \to x_0} f_j(x) = y_j^{(0)}$.

hence: $\eta(f(x_n), y_0) < \varepsilon$.

"$\Longleftarrow$" Assume that $\forall (x_n)_{n\geq 1}$ seq. with all elements in X
s.t. $x_n \neq x_0 \ \forall n \in \mathbb{N}$
and $\lim_{n\to\infty} x_n = x_0$
we have $\lim_{n\to\infty} f(x_n) = y_0$.

By contradiction assume that $f(x)$ does not conv. to y_0 as x approaches x_0.

Then $\exists \ \varepsilon_0 > 0$ s.t. $\forall \delta > 0 \ \exists \ x \in X \setminus \{x_0\}$ with
$\rho(x, x_0) < \delta$ and $\eta(f(x), y_0) \geq \varepsilon_0$
$\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$ hence $\exists \ x_n \in X \setminus \{x_0\}$ with
$\rho(x_n, x_0) < \delta = \frac{1}{n}$ and $\eta(f(x_n), y_0) \geq \varepsilon_0$

hence $x_n \xrightarrow[n\to\infty]{\rho} x_0$ but $f(x_n) \overset{\eta}{\underset{n\to\infty}{\not\longrightarrow}} y_0$.

#

(4) (Sequential Characterisation of Continuity)
Let $f: X \to Y$ function and $x_0 \in X$.
Then f is continuous at $x_0 \iff \forall (x_n)_{n\geq 1}$ seq. in X
s.t. $x_n \xrightarrow[n]{\rho} x_0$
we have
$f(x_n) \xrightarrow[n]{\eta} f(x_0)$

(2) $f : X \to Y$, $g : Y \to Z$ functions s.t

(i) $x_0 \in X$, accum. point for X

(ii) f cont. at x_0

(iii) $f(x_0)$ accum. point for Y

(iv) $\lim_{y \to f(x_0)} g(y) = z_0$

Then $g \circ f : X \to Z$ has the property

$$\lim_{x \to x_0} (g \circ f)(x) = z_0$$

If instead of (iv) we assume

(iv)' g continuous at $f(x_0)$

Then $g \circ f$ is continuous at x_0

(3) $f : X \to Y$ function

$x_0 \in X$ accumulation point and $y_0 \in Y$

Then $\lim_{x \to x_0} f(x) = y_0 \iff \forall (x_n)_{n \geq 1}$ with all $x_n \in X$

and $x_n \neq x_0 \ \forall n$

and $\lim_{n \to \infty} x_n = x_0$

we have $\lim_{n \to \infty} f(x_n) = y_0$

<u>proof</u>: "$\Rightarrow$". $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \in X \setminus \{x_0\}$

if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \varepsilon$.

Taken $(x_n)_{n \geq 1}$ a sequence in X

$x_n \neq x_0 \ \forall n$

and $\lim_{n \to \infty} x_n = x_0$

Then $\exists N_\delta \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N_\delta$ then

$\rho(x_n, x_0) < \delta$

The last characterisation shows that continuity is a topological concept.

<u>Definition</u>: Let $(X; \mathcal{T})$ and $(Y; \mathcal{Y})$ be topological spaces. Let $f: X \to Y$ be a function and $x_0 \in X$. f is <u>continuous</u> at x_0 if $\forall\, U \in \mathcal{Y}$ s.t. $f(x_0) \in U$ $\exists\, V \in \mathcal{T}$ s.t. $x_0 \in V$ and $f(V) \subseteq U$.

Continuity is related to the more general concept of limit.

<u>Definition</u>: Let $(X; \rho)$ and $(Y; \eta)$ be two metric spaces. Let $f: X \to Y$ be a function and $x_0 \in X$, $y_0 \in Y$.

f <u>has limit</u> y_0 <u>at</u> x_0 if:

- x_0 is an accumulation point for X
- $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t. $\forall\, x \in X \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \varepsilon$.

<u>Remark</u>: If f has limit y_0 at x_0 then y_0 is unique hence we can denote: $\lim\limits_{x \to x_0} f(x) = y_0$.

<u>Facts</u>: (1) $f: X \to Y$, metric spaces, $x_0 \in X$.

(i) if x_0 is isolated in X then f is continuous at x_0

(ii) if x_0 is an accum. point for X then f is cont. at x_0 iff $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Continuous Functions

Def: Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces, $f: X \to Y$ and $x_0 \in X$.

- f is continuous at x_0 if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \in X$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0)) < \varepsilon$
- f is continuous if f is continuous at all $x_0 \in X$.

Remark: We may consider the "more general" setting $f: D (\subseteq X) \to Y$, $x_0 \in D$.

f is continuous at x_0 if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \in D$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0)) < \varepsilon$.

But this is not more general than the definition since it coincides with the case when D is considered as a metric space with the metric $\rho|D \times D$.

Facts: Let $(X;\rho)$, $(Y;\eta)$ be metric spaces $f: X \to Y$ and $x_0 \in X$. TFAE:

(i) f is continuous at x_0

(ii) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.

(iii) $\forall U \in \mathcal{U}(f(x_0)) \ \exists V \in \mathcal{U}(x_0)$ s.t. $f(V) \subseteq U$.

(iv) $\forall U$ open in Y s.t. $f(x_0) \in U \ \exists V$ open in X s.t. $x_0 \in V$ and $f(V) \subseteq U$.

Then: (i) $\forall x \in A$, C_x is connected.

(ii) $\forall x \in A$, C_x is the largest connected subset of A that contains x

(iii) $\forall x, y \in A$, if $C_x \cap C_y \neq \emptyset$ then $C_x = C_y$.

(iv) For $x, y \in A$ define $x \overset{A}{\simeq} y$ if $C_x = C_y$. Then $\overset{A}{\simeq}$ is an equivalence relation in A.

<u>Definition</u>: $A \subseteq X$ metric space. The cosets w.r.t $\overset{A}{\simeq}$ are called <u>connected components</u>.

(4) Let $\{A_i\}_{i\in J}$ be a collection of subsets of a metric space $(X;\rho)$ s.t.:

- $\forall i\in J$, A_i is connected.
- $\bigcap_{i\in J} A_i \neq \emptyset$.

Then $\bigcup_{i\in J} A_i$ is connected.

Proof. Assume that $A=\bigcup_{i\in J} A_i$ is separated, hence $\exists$ U, V open in X s.t. $U\cap A\neq\emptyset$, $V\cap A\neq\emptyset$, $U\cap V=\emptyset$ and $A\subseteq U\cup V$.

~~Let~~ $x\in\bigcap_{i\in J} A_i$ hence $x\in U$ or $x\in V$.

If $x\in U$ let $y\in V\cap A\neq\emptyset$.

Since $y\in A=\bigcup_{i\in J} A_i \Rightarrow \exists\, j\in J$ s.t. $y\in A_j$

Then: $U\cap A_j\neq\emptyset$, since $x\in U\cap A_j$

$V\cap A_j\neq\emptyset$, since $y\in V\cap A_j$

$U\cap V=\emptyset$ and $A_j\subseteq A\subseteq U\cup V$,

hence A_j is separated, contradiction.

Similar argument holds for $x\in V$.

#

(5) ~~Let~~ $A\subseteq X$, metric space.

$\forall x\in A$ let $C_x := \bigcup_{\substack{x\in C\subseteq A\\ \text{connected}}} C$. ~~Then~~

Let $f : A \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \in U \cap A \\ 1, & x \in V \cap A. \end{cases}$$

- f is continuous on A

Let $x_0 \in A$. either $x_0 \in U \cap A$.
since U open, $\exists\ \delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \subseteq U$

Then $\forall\ \varepsilon > 0 \quad \forall\ x \in (x_0 - \delta, x_0 + \delta) \cap A \subseteq U \cap A$

$$|\underbrace{f(x)}_{=0} - \underbrace{f(x_0)}_{=0}| = 0 < \varepsilon$$

- or $x_0 \in V \cap A$
since V open, $\exists\ \delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \subseteq V$

Then $\forall\ \varepsilon > 0 \quad \forall\ x \in (x_0 - \delta, x_0 + \delta) \cap A \subseteq V \cap A$

$$|\underbrace{f(x)}_{=1} - \underbrace{f(x_0)}_{=1}| = 0 < \varepsilon.$$

- Since A is an interval and f is continuous, by the Intermediate Value Property, $f(A) = \{0, 1\}$ is an interval, contradiction!

#

(3) Let $\emptyset \neq A \subseteq \mathbb{R}$. Then A is connected iff A is an interval.

proof: "$\Longrightarrow$". Suppose $\emptyset \neq A \subseteq \mathbb{R}$ is connected and let

$$a = \inf A \quad \text{and} \quad b = \sup A.$$

Then $a, b \in ext(\mathbb{R})$ and, since $A \neq \emptyset$, $a \leq b$.
To prove that A is an interval we should prove that $(a, b) \subseteq A$. (!)
To this end, assume that $(a,b) \not\subseteq A$, hence
$\exists\ x \in (a, b)$ s.t. $x \notin A$.
Consider $U = (-\infty, x)$ and $V = (x, +\infty)$ and observe that:

- U and V are open
- $U \cap A \neq \emptyset$ since $a = \inf A < x$
- $V \cap A \neq \emptyset$ since $b = \sup A > x$
- $A \subseteq \mathbb{R} \setminus \{x\} = U \cup V$
- $U \cap V = \emptyset$

hence A is not connected, contradiction!

"$\Longleftarrow$". Asume that A is an interval, that is, (a, b), $(a, b]$, $[a, b]$, or $[a, b)$.
Assume that A is not connected, hen $\exists\ U, V$ open in $\mathbb{R}$ s.t.: $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $A \subseteq U \cup V$ and $U \cap V = \emptyset$.

Connected Sets

Let $(X; \mathcal{T})$ be a topological space and $A \subseteq X$.

Def: A is called separated if $\exists\ U, V \in \mathcal{T}$ s.t.

- $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$
- $A \cap (U \cap V) = \emptyset$
- $A \subseteq U \cup V$.

Def: A is called connected if it is not separated.

Def: $B \subseteq A$ is called relatively open w.r.t. A and $\mathcal{T}$ if $\exists\ U \in \mathcal{T}$ s.t. $B = U \cap A$.

Facts: (1) Let $\mathcal{T}_A := \{ B \subseteq A \mid B \text{ relatively open w.r.t. } A \text{ and } \mathcal{T} \}$

Then $\mathcal{T}_A$ is a topology on A, called the topology induced by $\mathcal{T}$ on A

(2) A is separated iff $A = B \cup C$ s.t.

- $B, C \in \mathcal{T}_A$
- $B, C \neq \emptyset$
- $B \cap C = \emptyset$

"$\Leftarrow$" Assume that A is closed and bounded.

Let $(a_n)_{n\geq 1}$ be a sequence in A, bounded, hence the sequence $(a_n)_{n\geq 1}$ is bounded $\underset{B.-W.}{\Longrightarrow} \exists\ (a_{k_n})_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n]{} a \in \mathbb{R}^d$. But A is closed, hence $a \in A$. $\#$

<u>Remark</u>: The previous result is more general, namely in any metric space, compactness is the same with sequential compactness, but this goes beyond the topics of this course.

Def.: $A \subseteq (X; \rho)$ metric space

A is sequentially compact if $\forall (a_n)_{n\geq 1}$ with $a_n \in A \ \forall n \in \mathbb{N}$ $\exists (a_{k_n})_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n]{} a \in A$.

(2) A is sequentially compact in $\mathbb{R}^d$ iff A is compact.

Proof: By Heine-Borel, A is compact iff it is closed and bounded.

"$\Rightarrow$" Assume that A is sequentially compact

. A is bounded.

If not, then $\forall n \in \mathbb{N} \ \exists a_n \in A$ s.t. $\|a_n\| \geq n$. Then $(a_n)_{n\geq 1}$ is a sequence in A having no convergent subsequence, contradiction! $\dashv$

. A is closed.

Let $(a_n)_{n\geq 1}$ in A s.t. $a_n \xrightarrow[n]{} a \in \mathbb{R}^d$. Since A is sequentially bounded $\exists$ a subsequence $(a_{k_n})_{n\geq 1}$ converging to $b \in A$. But $a_{k_n} \xrightarrow[n]{} a$ hence $a = b \in A$.

Hence any limit point of A is in A, i.e. $\bar{A} = A$. $\dashv$

Then $x \in K \subseteq \bigcup_{V \in \mathcal{V}} V$ hence $\exists\, V \in \mathcal{V}$ s.t. $x \in \underset{\text{open}}{V}$,

hence $\exists\, r > 0$ s.t. $B_r(x) \subseteq V$.

Since $\sqrt{d}\, \frac{\ell}{2^{n+1}} \xrightarrow[n \to \infty]{} 0$ it follows that $\exists\, n \geq 1$

s.t. $\sqrt{d}\, \frac{\ell}{2^{n+1}} < r \Rightarrow K \cap C_n \subseteq C_n \subseteq B_r(x) \subseteq V$,

contradiction with the assumption that $K \cap C_n$ does not have any finite subcover.

#

Consequences of Heine-Borel Theorem

(1) (Separation): Let A compact, U open in $\mathbb{R}^d$ s.t. $A \subseteq U$.

Then $\exists\, V$ open with $\overline{V}$ compact s.t.

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Proof: $\forall\, a \in A \Rightarrow a \in \underset{\text{open}}{U} \Rightarrow \exists\, r_a > 0$ s.t. $B_{r_a}(a) \subseteq U$.

Then $\{B_{r_a/2}(a)\}_{a \in A}$ is an open cover of A,

hence $\exists\, a_1, \dots, a_n \in A$ s.t.

$$A \subseteq \bigcup_{i=1}^{n} B_{r_{a_i}/2}(a_i)$$

Take $V = \bigcup_{i=1}^{n} B_{\frac{r_{a_i}}{2}}(a_i)$ and then

$$\underset{\text{open}}{A \subseteq V} \subseteq \overline{V} = \underset{\text{compact}}{\bigcup_{i=1}^{n} \overline{B_{\frac{r_{a_i}}{2}}(a_i)}} \subseteq U. \quad (!)$$

#

"$\Leftarrow$" Let K be a closed and bounded subset in $\mathbb{R}^d$.
Let $\mathcal{V}$ be an open cover of K s.t. it does not contain any finite subcover.

A <u>d-cube</u> is a subset $C = [a_1, b_1] \times \dots \times [a_d, b_d]$ in $\mathbb{R}^d$
s.t. $b_j - a_j = \ell \quad \forall\, j = 1, \dots, d$
ℓ is the <u>~~side~~ length</u>.
$\sqrt{d} \cdot \ell$ is the largest diagonal length

Since K is bounded $\exists$ C a cube of side lenght ℓ
s.t. $K \subseteq C$.
Divide each side in half and get 2^d d-cubes of side length $\frac{\ell}{2}$

$C_{1,1}, C_{1,2}, \dots, C_{1,2^d}$

Then $C_{1,k} \cap K$ is closed and bounded, $k = 1, \dots, 2^d$
and at least one of them cannot be covered with finitely many subsets of $\mathcal{V}$ (!)
Let C_1 one of these s.t. $K \cap C_1 \neq \emptyset$ cannot be covered with finitely many subsets of $\mathcal{V}$.

By induction, we obtain a sequence of d-cubes
$$C_1 \supseteq C_2 \supseteq \dots \supseteq C_n \supseteq C_{n+1} \supseteq \dots$$
s.t. C_n has side length $\frac{\ell}{2^{n+1}}$, $\forall\, n \in \mathbb{N}$

and $\emptyset \neq C_n \cap K$ cannot be covered by a finitely many subsets in $\mathcal{V}$, $\forall\, n \in \mathbb{N}$

By Lemma $\exists\ x \in \bigcap_{n \geq 1} (K \cap C_n) = K \cap \bigcap_{n \geq 1} C_n$

The Heine-Borel Theorem

Lemma: Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of subsets in $\mathbb{R}^d$ s.t.

- $\forall n$, $A_n \neq \emptyset$, closed, and bounded.
- $\forall n$ $A_n \supseteq A_{n+1}$

Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Proof: By induction, $\exists$ $(x_n)_{n\geq 1}$ a sequence s.t.
$\forall n \in \mathbb{N}$, $x_n \in A_n$.
Then $\forall n \in \mathbb{N}$, $x_n \in A_1$, hence the sequence $(x_n)_{n\geq 1}$ is bounded $\underset{B\text{-}W}{\Longrightarrow}$ $\exists$ a subsequence $(x_{k_n})_{n\geq 1}$ s.t. $x_{k_n} \xrightarrow[n]{} x$.

We show that $x \in \bigcap_{n\geq 1} A_n$.
Let $n \in \mathbb{N}$ be arbitrary: Since $k_m \xrightarrow[m]{} +\infty$ $\exists$ $N \in \mathbb{N}$ s.t. $k_m \geq n$ $\forall m \geq N$

hence $(x_{k_m})_{m\geq N}$ is contained in A_n, closed,
hence $\lim_{\substack{m\to\infty \\ m\geq N}} x_{k_m} = x \in A_n$. $\#$

Theorem (Heine-Borel): A subset of $\mathbb{R}^d$ is compact iff it is closed and bounded.

Proof: "$\Longrightarrow$" Proven for any metric space.

(3) Let A be compact and C closed $\subseteq A$. Then C is compact.

proof: Let $\{U_i \mid i \in J\}$ be an arbitrary open cover for ~~A~~ C.

Then ~~A~~ $A \subseteq \underset{\text{open}}{(X \setminus C)} \cup \bigcup_{i \in J} C_i$

hence $\exists\ i_1, \dots, i_n \in J$ s.t.

$$C \subseteq A \subseteq (X \setminus C) \cup \bigcup_{j=1}^{n} C_{i_j}$$

Since $C \cap (X \setminus C) = \emptyset$ it follows $C \subseteq \bigcup_{j=1}^{n} C_{i_j}$.

#

Example: $\mathbb{N}$ with discrete metric δ is closed and bounded but not compact.

(4) Let K be a compact set and $g : K \to (0, +\infty)$ a function. Then $\exists\ x_1, \dots, x_m \in K$ s.t.

$$K \subseteq \bigcup_{j=1}^{m} B_{g(x_j)}(x_j)$$

proof: Consider $\{B_{g(x)}(x) \mid x \in K\}$ which is an open cover of K, compact, hence it has an subcover $\{B_{g(x_j)}(x_j) \mid j = 1, 2, \dots, m\}$.

#

~~(5) Let K be compact and U open s.t. $K \subseteq U$. Then $\exists\ V$ open~~

Let (X, ρ) be a metric space, T_ρ the induced topology

<u>Facts</u>: (1) If $A \subseteq X$ is compact then it is bounded,

i.e. $\exists\, x_0 \in X$ and $r > 0$ s.t. $A \subseteq B_r(x_0)$.

<u>proof</u>: Consider $x_0 \in X$ and $\{B_r(x_0) \mid r > 0\}$.

Then $A \subseteq X = \bigcup_{r>0} B_r(x_0)$, hence an open cover of A.

Then $\exists\, \{B_{r_j}(x_0)\}_{j=1}^{n}$ a finite subcover

$$A \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_0) = B_r(x_0) \quad \text{where } r := \max_{j=1}^{n}\{r_j\} < +\infty$$

#

(2) If $A \subseteq X$ is compact then A is closed.

<u>proof</u>: Assume A is compact.

By contradiction, assume A is not closed, hence

$\exists\, x \in \bar{A} \setminus A$.

Consider ~~$\{B_{\frac{1}{n}}(x)$~~ $U_n := \{y \in X \mid \rho(y, x) > \frac{1}{n}\}$, $n \in \mathbb{N}$

- $\forall n \in \mathbb{N}$, U_n is open

; let $z \in U_n$, $\rho(z, x) > \frac{1}{n}$.

Then $B_r(z) \subseteq U_n$, by Triangle Inequality

~~for~~ $0 < r < \rho(z, x) - \frac{1}{n}$

Then $A \subseteq X \setminus \{x\} = \bigcup_{n \in \mathbb{N}} U_n$, hence $\exists\, n_1, \dots, n_k \in \mathbb{N}$

s.t. $A \subseteq \bigcup_{j=1}^{k} U_{n_j} = U_n$, $n := \max\{n_1, \dots, n_k\} \in \mathbb{N}$

i.e. $\forall\, y \in A$, $\rho(y, x) > \frac{1}{n}$ hence

$B_{\frac{1}{2n}}(x) \cap A = \emptyset$,

contradiction with $x \in \bar{A}$

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