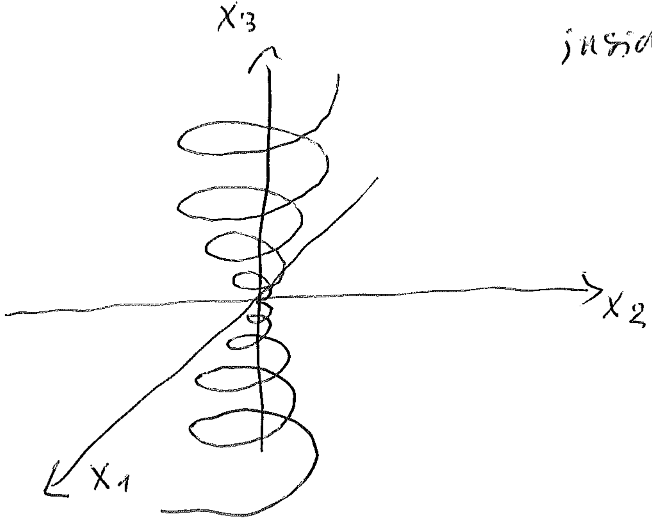
a spiral

inside $\{(x,y,z) \mid x^2+y^2=|z|\}$

a cone



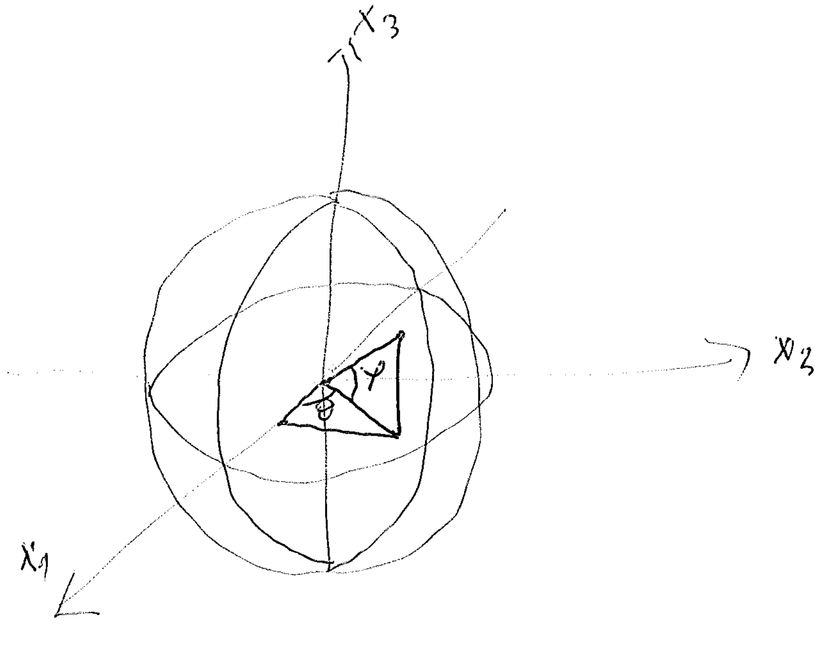
Def: A surface in $\mathbb{R}^q$ ($q \geq 2$) is a continuous function
$F: A \to \mathbb{R}^q$, D open, nonempty, in $\mathbb{R}^2$.

$$D \subseteq A \subseteq \overline{D}.$$

Examples: (1) 2-dimensional sphere in $\mathbb{R}^3$

$F: [0, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3$

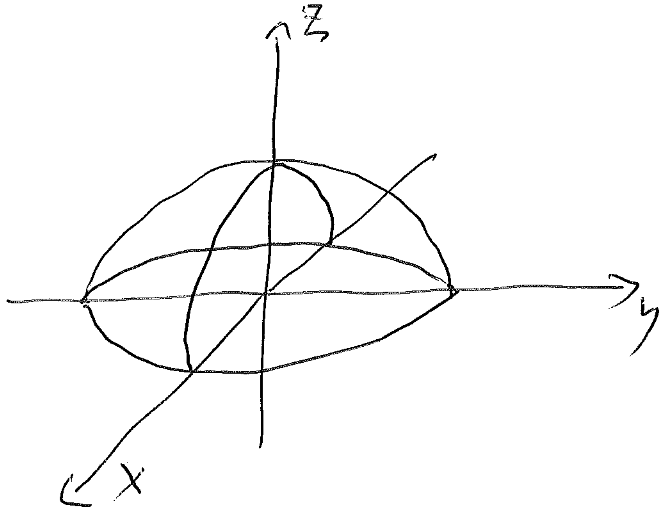
$F(\theta, \varphi) = (\cos\theta \cos\varphi, \sin\theta \cos\varphi, \sin\varphi)$



(2) $G : B \longrightarrow \mathbb{R}^3$

$G(x,y,z) = (x, y, \sqrt{1-x^2-y^2})$

$B := \{ (x,y) \mid x^2+y^2 \leq 1 \}$



Continuity and Topology

Theorem (Topological Characterisation of Continuity)

Let $(X;\rho)$, $(Y;\eta)$ be metric spaces.
$f: X \to Y$ a function. TFAE:

(i) f is continuous.

(ii) $\forall$ U open in Y, $f^{-1}(U)$ is open in X.

(iii) $\forall$ F closed in Y, $f^{-1}(F)$ is closed in X.

Proof: (i) $\Rightarrow$ (ii). Let U open in Y and $x \in f^{-1}(U)$,
i.e. $f(x) \in U$. Then $\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subseteq U$.
Since f is cont. at x $\exists$ $\delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq U$

hence $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))) \subseteq f^{-1}(U)$,

i.e. $f^{-1}(U)$ is open.

(ii) $\Rightarrow$ (i). Let $x \in X$ and $\varepsilon > 0$. Then $B_\varepsilon(f(x))$ is open in Y

hence $f^{-1}(B_\varepsilon(f(x)))$ is open in X.
Since $x \in f^{-1}(B_\varepsilon(f(x)))$ it follows that $\exists$ $\delta > 0$
s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$, i.e. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Hence f is cont. at each $x \in X$.

(ii) $\Leftrightarrow$ (iii). Since $f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A)$

#

<u>Corollary</u>: Let $\emptyset \neq D \subseteq \mathbb{R}^p$, $f: D \to \mathbb{R}^q$ a function.

TFAE:

(i) f is continuous.

(ii) $\forall$ U open in $\mathbb{R}^q$, $f^{-1}(U)$ is relatively open in D.

(iii) $\forall$ F closed in $\mathbb{R}^q$, $f^{-1}(F)$ is relatively closed in D.

<u>Corollary</u>: Let $\emptyset \neq D \subseteq \mathbb{R}^p$ open, $f: D \to \mathbb{R}^q$ function.

TFAE:

(i) f is continuous.

(ii) $\forall$ U open in $\mathbb{R}^q$, $f^{-1}(U)$ is open in $\mathbb{R}^p$.

(iii) $\forall$ F closed in $\mathbb{R}^q$, $f^{-1}(F)$ is closed in $\mathbb{R}^p$.

Continuity and Compactness

Theorem: Let $(X;\rho)$, $(Y;\gamma)$ be metric spaces. $f: X \to Y$ a continuous function and K compact in X. Then $f(K)$ is compact.

Proof: Let $\{U_i \mid i \in J\}$ be an open (in Y) covering of $f(K)$:
- $\forall i \in J$, U_i is open in Y
- $f(K) \subseteq \bigcup_{i \in J} U_i$

Then $\forall i \in J$, $f^{-1}(U_i)$ is open in X and

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\Big(\bigcup_{i \in J} U_i\Big) = \bigcup_{i \in J} f^{-1}(U_i),$$

hence $\{f^{-1}(U_i) \mid i \in J\}$ is an open covering of K.

Since K is compact $\exists\, i_1, \dots, i_n \in J$ s.t.

$$K \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}) = f^{-1}\Big(\bigcup_{k=1}^{n} U_{i_k}\Big)$$

hence $f(K) \subseteq f\Big(f^{-1}\Big(\bigcup_{k=1}^{n} U_{i_k}\Big)\Big) \subseteq \bigcup_{k=1}^{n} U_{i_k}$.

\#

Corollary: Let $f: X \to \mathbb{R}$ a continuous function and K compact, nonempty in X. Then:
- f is bounded on K, i.e. $f(K)$ is bounded in $\mathbb{R}$.
- the extreme values of f on K are attained, i.e. $\exists\, x_m, x_M \in K$ s.t.

$$f(x_m) = \inf_K f, \quad f(x_M) = \sup_K f$$

<u>Proof</u>: $f(K)$ is compact in $\mathbb{R}$, hence closed and bounded.

- $f(K)$ bounded $\Rightarrow$ $\inf_K f,\ \sup_K f \in \mathbb{R}$.
- $\inf_K f \underset{(!)}{\in} \overline{f(K)} = f(K)$, hence $\exists\, x_m \in K$ s.t. $\inf_K f = f(x_m)$.
- $\sup_K f \in \overline{f(K)} = f(K)$, hence $\exists\, x_M \in K$ s.t. $\sup_K f = f(x_M)$.

#

<u>Corollary</u>: Let $f : X \to \mathbb{R}^q$ be a continuous function.
Then $\forall$ K nonempty and compact in X
$\exists$ $x_m, x_M \in K$ s.t.

$$\|f(x_m)\| = \inf_K \|f\|, \quad \|f(x_M)\| = \sup_K \|f\|.$$

<u>Proof</u>:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "\|f\|" '] & \mathbb{R}^q \arrow[d, "\|\cdot\|"] \\ & \mathbb{R} \end{tikzcd}$$

continuous

#

Continuity and Connectedness

Theorem: $(X;\rho)$, $(Y;\eta)$ metric spaces
$f: X \to Y$ continuous function
C connected in X.
Then $f(C)$ is connected in Y.

Proof: By contrapositive, assume that $f(C)$ is separated,
hence: $\exists\ U, V$ open in Y s.t.

- $f(C) \subseteq U \cup V$
- $f(C) \cap U \neq \emptyset$
- $f(C) \cap V \neq \emptyset$
- $f(C) \cap U \cap V = \emptyset$

Then $f^{-1}(U)$, $f^{-1}(V)$ are open in X

- $C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
- $\emptyset \neq f^{-1}(f(C) \cap U) = f^{-1}(f(C)) \cap f^{-1}(U)$
- we prove $C \cap f^{-1}(U) \neq \emptyset$.
  Since $f(C) \cap U \neq \emptyset$, $\exists\ y \in f(C)$ and $y \in U$
  hence $\exists\ x \in C$ s.t. $f(x) \in U$, hence $x \in f^{-1}(U)$,
  i.e. $x \in C \cap f^{-1}(U)$.
- similarly $C \cap f^{-1}(V) \neq \emptyset$
- $C \cap f^{-1}(U) \cap f^{-1}(V) = \emptyset$

Assume $C \cap f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$, then $\exists\, x \in C$ s.t.
$x \in f^{-1}(U) \cap f^{-1}(V)$

hence $f(x) \in f(C)$ and $f(x) \in f(f^{-1}(U)) \subseteq U$
$f(x) \in f(f^{-1}(V)) \subseteq V$

i.e. $f(x) \in f(C) \cap U \cap V$, contradiction! $\not$

Thus, $f^{-1}(U)$ and $f^{-1}(V)$ separates C, contradiction!

#

Consequences: (1) $f : (X, g) \to \mathbb{R}$ continuous
C connected in X
Then $f(C)$ is an interval.

(2) Let I be an interval in $\mathbb{R}$ and
$\gamma : I \to \mathbb{R}^d$ continuous (a curve).
Then $\gamma(I)$ is connected.

Definition: A subset $S \subseteq (X, g)$ is called pathwise connected if $\forall\, a, b \in S$ $\exists$ a path (i.e. continuous)
$\gamma : [0,1] \to S$ s.t.
$a = \gamma(0)$ and $b = \gamma(1)$.

(3) If S is pathwise connected then it is connected.

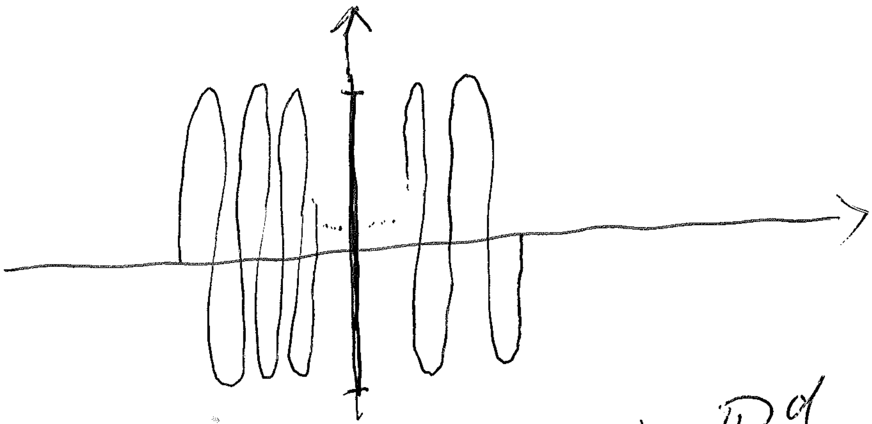
proof: Assume S is not connected, let U, V open in X and separating S

Then $\exists$ $a \in S \cap U$ and $b \in S \cap V$.
Since S pathwise connected $\exists$ $\gamma : [0,1] \to S$ continuous
s.t. $\gamma(0) = a$ and $\gamma(1) = b$.
But then U and V separates $\gamma([0,1])$,
contradiction!

#

<u>Example</u>: A set in $\mathbb{R}^2$ that is connected but not pathwise connected.

$$S = \{(0,y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin \tfrac{1}{x}) \mid -\tfrac{1}{\pi} \leq x \leq \tfrac{1}{\pi},\ x \neq 0\}$$



(4) Assume that D is open in $\mathbb{R}^d$.
Then D is connected iff D is pathwise connected.

<u>Proof</u>: "$\Leftarrow$". Holds in general.
"$\Rightarrow$" On D we define a relation: $x \sim y$ if $\exists$ $\gamma : [0,1] \to D$ continuous s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

- $\sim$ is an equivalence relation on D

- $\forall x \in D$ its equivalence class $[x]_p$ is open set.
- If D is not pathwise connected then there exists at least two different cosets w.r.t. $\overset{P}{\sim}$, hence D is disconnected. #

Uniform Continuity

Def: Let $(X;\rho)$ and $(Y;\eta)$ be two metric spaces.
A function $f: X \to Y$ is uniformly continuous
if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $\forall\, x_1, x_2 \in X$,
if $\rho(x_1, x_2) < \delta$ then $\eta(f(x_1), f(x_2)) < \varepsilon$.

Theorem: If $\underset{\rho}{X} \xrightarrow{f} \underset{\eta}{Y}$ is continuous and
X is compact then f is uniformly continuous.

Proof: Let $\varepsilon > 0$.
Since f is continuous on X, $\forall\, x \in X\ \exists\, \delta_x > 0$ s.t.
$\forall\, z \in X$ with $\rho(x, z) < \delta_x \Rightarrow \eta(f(x), f(z)) < \frac{\varepsilon}{2}$.
Since $\{ B_{\frac{\delta_x}{2}}(x) \mid x \in X \}$ is an open covering of X compact
it follows $\exists\, x_1, \dots, x_n \in X$ s.t.

$$X \subseteq \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$$

Let $\delta := \min\left\{ \frac{\delta_{x_i}}{2} \mid i = 1, \dots, n \right\} > 0$

and let $x, z \in X$ s.t. $\rho(x, z) < \delta$.
Then $\exists\, j \in \{1, \dots, n\}$ s.t. $x \in B_{\frac{\delta_{x_j}}{2}}(x_j)$

i.e. $\rho(x, x_j) < \frac{\delta_{x_j}}{2}$

Then $\rho(z, x_j) \leq \rho(z, x) + \rho(x, x_j) < \delta + \frac{\delta_{x_j}}{2} \leq \frac{\delta_{x_j}}{2} + \frac{\delta_{x_j}}{2} = \delta_{x_j}$

hence $\eta(f(x), f(x_j)) < \frac{\varepsilon}{2}$

and $\eta(f(z), f(x_j)) < \frac{\varepsilon}{2}$.

Then $\eta(f(x), f(z)) \leq \eta(f(x), f(x_j)) + \eta(f(z), f(x_j)) < \varepsilon$

#

<u>Theorem</u>: Let $(X; \rho)$ be a compact metric space.

$(Y; \eta)$ be a complete metric space

D ~~bounded~~ in X and $f: D \to Y$ a function.

TFAE: (i) f is uniformly continuous on D

(ii) $\exists \bar{f}: \bar{D} \to Y$ s.t.

. $\bar{f} | D = f$

. $\bar{f}$ continuous on $\bar{D}$.

<u>Proof</u>: (ii) $\Rightarrow$ (i). $\bar{D}$ closed in X compact hence $\bar{D}$ is compact $\Rightarrow \bar{f}$ unif. cont. on $\bar{D}$

$\Rightarrow f = \bar{f} | D$ unif. cont. on D.

(i) $\Rightarrow$ (ii). Assume that f is uniformly continuous.

. f maps Cauchy sequences from D to Cauchy sequences in Y

; ~~Let~~ $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $\forall x, z \in D$,

if $\rho(x, z) < \delta$ then $\eta(f(x), f(z)) < \varepsilon$.

Let $(x_n)_{n \geq 1}$ be ρ-Cauchy sequence in D

$\forall\, \varepsilon > 0$, with $\delta > 0$ as before, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, n \in \mathbb{N}$

if $m, n \geq N$ then $\rho(x_m, x_n) < \delta$

hence $\quad \eta(f(x_m), f(x_n)) < \varepsilon$.

—/

. Let $x \in \overline{D}$. Then $\exists\, (x_n)_{n \geq 1}$ in D s.t. $x_n \xrightarrow[n]{\rho} x$

hence $(x_n)_{n \geq 1}$ is ρ-Cauchy, hence $(f(x_n))_n$ is η-Cauchy

hence, since $(Y; \eta)$ is complete, $\exists\, y_x \in Y$ s.t.

$f(x_n) \xrightarrow[n]{\eta} y_x$. Define $\overline{f}(x) := y_x \in D$

The definition of $\overline{f}$ is correct.

; We use the interlacing method.

If $\quad (x_n)_{n \geq 1}$ is a seq. in D, $\quad x_n \xrightarrow[n]{\rho} x$

$\quad\quad (z_n)_{n \geq 1}$ is a seq. in D, $\quad z_n \xrightarrow[n]{\rho} x$

Then $(t_n)_{n \geq 1}$ defined by

$$t_n = \begin{cases} x_{\frac{n}{2}}, & \text{if } n = 2k \\ z_{\frac{n-1}{2}}, & \text{if } n = 2k+1 \end{cases}$$

is a sequence in D and $\quad t_n \xrightarrow[n]{\rho} x \quad$ (1)

Then $\quad f(x_n) \xrightarrow[n]{\eta} y_x$, $\; f(z_n) \xrightarrow[n]{\eta} z_x$, $\; f(t_n) \xrightarrow[n]{\eta} t_x$

Since $(f(x_n))_{n \geq 1}$ and $(f(z_n))_{n \geq 1}$ are subsequences of $(f(t_n))_{n \geq 1}$ it follows $\quad y_x = t_x = z_x$

—/

- $\bar{f}$ is continuous on $\bar{D}$.

; We use sequential characterization of continuity.

Let $\bar{x} \in \bar{D}$ and $(\bar{x}_n)_{n\geq 1}$ a seq. in $\bar{D}$ s.t.

$$\bar{x}_n \xrightarrow[n]{\rho} \bar{x}.$$

Then $\forall\, n \geq 1 \quad \exists \quad x_n \in D$ s.t.

$$\left.\begin{aligned} &\rho(\bar{x}_n, x_n) < \frac{1}{n} \\ \text{and} \quad &\eta(\bar{f}(\bar{x}_n), f(x_n)) < \frac{1}{n} \end{aligned}\right\} \text{(!)}$$

Then

$$\rho(x_n, \bar{x}) \leq \rho(x_n, \bar{x}_n) + \rho(\bar{x}_n, \bar{x}) < \frac{1}{n} + \rho(\bar{x}_n, \bar{x}) \xrightarrow[n]{} 0$$

hence $x_n \xrightarrow[n]{\rho} \bar{x}$, and then, by def. of $\bar{f}$ we have $f(x_n) \xrightarrow[n]{\eta} \bar{f}(\bar{x})$.

and then

$$\eta(\bar{f}(\bar{x}_n), \bar{f}(\bar{x})) \leq \eta(\bar{f}(\bar{x}_n), f(x_n)) + \eta(f(x_n), \bar{f}(\bar{x}))$$

$$< \frac{1}{n} + \eta(f(x_n), \bar{f}(\bar{x})) \xrightarrow[n]{} 0$$

hence $\bar{f}(\bar{x}_n) \xrightarrow[n]{\eta} \bar{f}(\bar{x})$.

$\#$

Sequences of Functions

Let X be a nonempty set and $(Y;\eta)$ a metric space.
We consider sequences $(f_n)_{n\geq 1}$ of functions

$$f_n : X \longrightarrow Y, \quad n \in \mathbb{N}.$$

Definition: • $(f_n)_{n\geq 1}$ converges pointwise to $f: X \to Y$ if
$\forall x \in X \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\eta(f(x), f_n(x)) < \varepsilon$,
i.e. $\forall x \in X$, $(f_n(x))_{n\geq 1}$ converges to $f(x)$ in Y w.r.t. η.

• $(f_n)_{n\geq 1}$ converges uniformly to $f: X \longrightarrow Y$ if
$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\forall x \in X$, $\eta(f(x), f_n(x)) < \varepsilon$.

Facts: (1) If $\exists (\alpha_n)_{n\geq 1}$ s.t. $\alpha_n \geq 0 \ \forall n \in \mathbb{N}$
and $\alpha_n \xrightarrow[n]{} 0$
and $\eta(f(x), f_n(x)) \leq \alpha_n \ \forall n \in \mathbb{N}$ and $\forall x \in X$

Then $f_n \xrightarrow[n]{\text{unif}} f$.

(2) If $(X;\rho)$, $(Y;\eta)$ are metric spaces and functions $f_n : \cancel{X} \to Y$ continuous $\forall n \in \mathbb{N}$, $f: X \to Y$ s.t.
$f_n \xrightarrow[n]{\text{unif}} f$

Then f is continuous on X

Proof: Let $x_0 \in X$ and $\varepsilon > 0$.
$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ and all $x \in X$
$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

f_N is continuous at x_0:

$\exists\, \delta > 0$ s.t. $\forall\, x \in X$, $\rho(x, x_0) < \delta \Rightarrow \eta(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$

Then: $\forall\, x \in X$, if $\rho(x, x_0) < \delta$ then

$$\eta(f(x), f(x_0)) \leq \eta(f(x), f_N(x)) + \eta(f_N(x), f_N(x_0)) + \eta(f_N(x_0), f(x_0))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

#

Examples: (1) $f_n : \overline{B_1(0)} \to \mathbb{R}$, $\overline{B_1(0)} \subset \mathbb{R}^d$

$$f_n(x) = \|x\|_2^n \qquad f(x) = \begin{cases} 0, & x \in B_1(0) \\ 1, & x \in \partial B_1(0) \end{cases}$$

Then $f_n \xrightarrow[n]{\text{pointwise}} f$ but not uniformly.

(2) $B_1(0)$ in $\mathbb{R}^2$

$f_n : B_1(0) \to \mathbb{R}^2$ $\qquad f_n(x_1, x_2) = \left(\frac{x_1^2 - n x_2^2}{1 + n x_2^2}, \frac{n x_1}{1 + n x_1^2} \right)$

$f : B_1(0) \to \mathbb{R}^2$

$$f(x) = \begin{cases} \left(-1, \frac{1}{x_1}\right), & x_1 \neq 0,\ x_2 \neq 0 \\ (-1, 0) & x_1 = 0,\ x_2 \neq 0 \\ \left(x_1^2, \frac{1}{x_1}\right) & x_1 \neq 0\ \ x_2 = 0 \\ (0, 0) & x_1 = 0\ \ x_2 = 0 \end{cases}$$

$f_n \xrightarrow[n]{\text{pointwise}} f$ but not uniformly

Def: $f_n : \underset{\text{set}}{X} \longrightarrow \underset{\eta}{Y}$, $\forall n \in \mathbb{N}$.

$(f_n)_{n\geq 1}$ uniformly Cauchy if
$\forall \varepsilon > 0 \;\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$ we have

$$\eta(f_n(x), f_m(x)) < \varepsilon.$$

Facts: (1) If $f_n \xrightarrow[n]{\text{unif}} f$ then $(f_n)_{n\geq 1}$ is uniformly Cauchy.

(2) If $(Y; \eta)$ is complete and $(f_n)_{n\geq 1}$ is uniformly Cauchy then $\exists f : X \longrightarrow Y$ s.t. $f_n \xrightarrow[n]{\text{unif}} f$

Proof: Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \;\forall x \in X$

$$\eta(f_m(x), f_n(x)) < \frac{\varepsilon}{2} \qquad (*).$$

Then, $\forall x \in X$ $(f_n(x))_{n\geq 1}$ is Cauchy in Y, complete hence $\exists f(x) \in Y$ s.t. $f_n(x) \xrightarrow[n]{} f(x)$.
Due to the uniqueness of the limit in Y.

$$f : X \longrightarrow Y \text{ function}$$

s.t. $f_n \xrightarrow[n]{\text{pointwise}} f$.

Letting $m \longrightarrow +\infty$ in $(*)$: $\forall n \geq N \;\forall x \in X$

$$\eta(f(x), f_n(x)) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence $f_n \xrightarrow[n]{\text{unif.}} f$. #

Remark: $\forall z \in Y$ the map
$$Y \ni y \longmapsto \eta(y, z) \in \mathbb{R} \text{ is continuous.}$$

Def: Let $X \neq \emptyset$ and $(V, \|\cdot\|)$ a normed space.

$$\mathcal{B}(X;V) := \{ f : X \to V \mid f \text{ bounded function} \}$$

$$\forall f \in \mathcal{B}(X;V), \quad \|f\|_X := \sup_{x \in X} \|f(x)\|$$

Facts: (1) $\mathcal{B}(X,V)$ is a vector space
$\|\cdot\|_X$ is a norm, called the sup norm on X

(2) If $(V; \|\cdot\|)$ is complete, then the normed space $(\mathcal{B}(X;V); \|\cdot\|_X)$ is complete.

proof: Let $(f_n)_{n \geq 1}$ a Cauchy sequence in $\mathcal{B}(X;V)$ w.r.t. the norm $\|\cdot\|_X$. Hence:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $\forall m, n \geq N$, $\|f_m - f_n\|_X < \varepsilon/2$

hence $\forall x \in X \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_X < \varepsilon/2$ (*)

Then: $\forall x \in X$ the sequence $(f_n(x))_{n \geq 1}$ is Cauchy in V, complete, hence $\exists f(x) \in V$ s.t.

$$\|f_n(x) - f(x)\| \xrightarrow[n \to \infty]{} 0$$

Then $X \ni x \longmapsto f(x) \in V$ is a function ①

$f \in \mathcal{B}(X;V)$ and $\|f_n - f\|_X \xrightarrow[n]{} 0$.

; $\forall x \in X$ we let $m \to \infty$ in $(*)$ hence

$$\|f(x) - f_n(x)\| \leq \frac{\varepsilon}{2} < \varepsilon \qquad (**)$$

hence $$\sup_{x \in X} \|f(x) - f_n(x)\| \leq \frac{\varepsilon}{2} < \varepsilon \qquad (***).$$

Also. $\forall x \in X$, letting $n = N$ in $(**)$ we have

$$\|f(x)\| \leq \|f(x) - f_N(x)\| + \|f_N(x)\| \leq \frac{\varepsilon}{2} + \|f_N\|_X < +\infty$$

hence $f \in \mathcal{B}(X; V)$.

On the other hand, from $(***)$, $\forall n \geq N$

$$\|f - f_n\|_X < \varepsilon,$$

hence $f_n \xrightarrow[n]{\|\cdot\|_X} f$.

#

(3) Let $(X; \rho)$ be a compact metric space, $(V; \|\cdot\|)$ Banach space.
and $\mathcal{C}(X; V) := \{f : X \to V \mid f \text{ continuous}\}$.

Then: • $\mathcal{C}(X; V) \subseteq \mathcal{B}(X; V)$ as a vector subspace

• $(\mathcal{C}(X; V); \|\cdot\|_X)$ is a complete normed space.

Series of Functions

Let $\emptyset \neq X$ be a set and $(V, \|\cdot\|_V)$ be a normed space and $(f_n)_{n\geq 1}$ a sequence of functions $f_n : X \to V$.

Definition: . A formal sum $\sum_{n=1}^{\infty} f_n$ is called a series of functions on X and valued in V.

$\forall\, n \in \mathbb{N}$ $\quad S_n = \sum_{k=1}^{n} f_n$,

$(S_n)_{n=1}^{\infty}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$

. The series $\sum_{n=1}^{\infty} f_n$ pointwise converges if the sequence $(S_n)_{n=1}^{\infty}$ converges pointwise to a function $f : X \to V$.

. The series $\sum_{n=1}^{\infty} f_n$ uniformly converges if $(S_n)_{i=1}^{\infty}$ converges uniformly to a function $f : X \to V$.

. The series $\sum_{n=1}^{\infty} f_n$ absolutely converges if $\sum_{n=1}^{\infty} \|f_n\|_V$ converges. This can be pointwise or uniformly.

Facts: Assume that the normed space $(V, \|\cdot\|)$ is complete.

(1) If $\sum_{n=1}^{\infty} f_n$ absolutely converges then it converges.

(2) If $\|f_n(x)\| \leq \alpha_n$ $\forall\, x \in X$ $\forall\, n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \alpha_n < +\infty$ then $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly.