

CHAPTER 1

Continuous Functions in Euclidean Spaces

1. Continuous Functions

DEFINITION 1.1. Let $(X; \rho)$ and $(Y; \eta)$ be two metric spaces, $f : X \rightarrow Y$ and $x_0 \in X$.

- f is *continuous* at x_0 if $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in X$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0)) < \epsilon$
- f is *continuous* if f is continuous at all $x_0 \in X$.

REMARK 1.2. We may consider the "more general" setting

$$f : D (\subseteq X) \rightarrow Y, x_0 \in D.$$

f is *continuous* at x_0 if $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in D$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), f(x_0)) < \epsilon$.

But this is not more general than the definition since it coincides with the case when D is considered as a metric space with the metric $\rho|_{D \times D}$.

FACT 1.3. Let $(X; \rho)$ and $(Y; \eta)$ be two metric spaces, $f : X \rightarrow Y$ and $x_0 \in X$. TFAE:

- (i) f is continuous at x_0
- (ii) $\forall \epsilon > 0 \exists \delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$
- (iii) $\forall U \in \mathcal{V}(f(x_0)) \exists V \in \mathcal{V}(x_0)$ such that $f(V) \subseteq U$.
- (iv) $\forall U$ open in Y such that $f(x_0) \in U \exists V$ open in X such that $x_0 \in V$ and $f(V) \subseteq U$.

The last characterization shows that continuity is a topological concept.

DEFINITION 1.4. Let $(X; \mathcal{T})$ and $(Y; \mathcal{Y})$ be topological spaces. Let $f : X \rightarrow Y$ be a function and $x_0 \in X$.

f is *continuous* at x_0 if $\forall U \in \mathcal{Y}$ such that $f(x_0) \in U \exists V \in \mathcal{T}$ such that $x_0 \in V$ and $f(V) \subseteq U$.

Continuity is related to the more general concept of limit.

DEFINITION 1.5. Let $(X; \rho)$ and $(Y; \eta)$ be topological spaces, $D \subseteq X$. Let $f : D \rightarrow Y$ be a function and $x_0 \in X, y_0 \in Y$.

f has limit y_0 at x_0 if:

- x_0 is an accumulation point for D
- $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

REMARK 1.6. If f has limit y_0 at x_0 then y_0 is unique, hence we can denote:

$$\lim_{x \rightarrow x_0} f(x) = y_0.$$

FACT 1.7.

(1) $f : D (\subseteq X) \rightarrow Y$, metric spaces, $x_0 \in X$.

(i) if x_0 is isolated in D then f is continuous at x_0

(ii) if x_0 is an accum. point for D then f is cont. at x_0 iff

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

(2) $f : D \rightarrow Y$ function, $x_0 \in D$ accumulation point and $y_0 \in Y$. Then

$$\lim_{x \rightarrow x_0} f(x) = y_0 \Leftrightarrow \forall (x_n)_{n \geq 1} \text{ with all } x_n \in D$$

$$\text{and } x_n \neq x_0 \forall n$$

$$\text{and } \lim_{n \rightarrow \infty} x_n = x_0$$

$$\text{we have } \lim_{n \rightarrow \infty} f(x_n) = y_0.$$

PROOF. " \Rightarrow " $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\rho(x, x_0) < \delta$ then $\eta(f(x), y_0) < \epsilon$.

Take $(x_n)_{n \geq 1}$ a sequence in X , $x_n \neq x_0 \forall n$ and $\lim_{n \rightarrow \infty} x_n = x_0$.

Then $\exists N_\delta \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, if $n \geq N_\delta$ then $\rho(x_n, x_0) < \delta$ hence:

$$\eta(f(x_n), y_0) < \epsilon.$$

" \Leftarrow " Assume that $\forall (x_n)_{n \geq 1}$ seq. with all elements in D such that $x_n \neq x_0 \forall n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} x_n = x_0$ we have $\lim_{n \rightarrow \infty} f(x_n) = y_0$.

By contradiction assume that $f(x)$ does not converge to y_0 as x approaches x_0 .

Then $\exists \epsilon_0 > 0$ such that $\forall \delta > 0 \exists x \in D \setminus \{x_0\}$ with $\rho(x, x_0) < \delta$ and $\eta(f(x), y_0) \geq \epsilon_0$.

$\forall n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$ hence $\exists x_n \in D \setminus \{x_0\}$ with $\rho(x_n, x_0) < \delta = \frac{1}{n}$ and $\eta(f(x_n), y_0) \geq \epsilon_0$

hence $x_n \xrightarrow[n \rightarrow \infty]{\rho} x_0$ but $f(x_n) \not\xrightarrow[n \rightarrow \infty]{\eta} y_0$.

□

(3) (Sequential Characterization of Continuity) Let $f : X \rightarrow Y$ function and $x_0 \in X$.

Then f is continuous at $x_0 \Leftrightarrow \forall (x_n)_{n \geq 1}$ seq. in X

such that $x_n \xrightarrow[n]{\rho} x_0$

we have $f(x_n) \xrightarrow[n]{\eta} y_0$.

(4) (Composition of Functions) Let $X \xrightarrow[\rho]{f} Y \xrightarrow[\theta]{g} Z$ functions between metric spaces.

(i)

(ii)

Assume that $x_0 \in X$ *f is cont. at x_0* *g is cont. at $f(x_0)$* *Then $g \circ f$ is continuous at x_0 .*

(5) (Functions Between Euclidean Spaces) Let $D \subseteq \mathbb{R}^p$ and $f : D \rightarrow \mathbb{R}^q$ be a function, hence $f(x) = (f_1(x), \dots, f_q(x))$, $\forall x \in D$ where $f_j : D \rightarrow \mathbb{R}$ function $\forall j = 1, \dots, q$.

(i) Let $x_0 \in D'$, i.e. x_0 is an accumulation point for D and

$y^{(0)} = (y_1^{(0)}, \dots, y_q^{(0)}) \in \mathbb{R}^q$. Then

$$\lim_{x \rightarrow x_0} f(x) = y^{(0)} \Leftrightarrow \forall j = 1, \dots, q \quad \lim_{x \rightarrow x_0} f_j(x) = y_j^{(0)}.$$

(ii) Let $x_0 \in D$. Then

$$f \text{ is cont. at } x_0 \Leftrightarrow \forall j = 1, \dots, q, f_j \text{ is cont. at } x_0.$$

PROOF.

(i)

" \Rightarrow ". Assume that $\lim_{x \rightarrow x_0} f(x) = y^{(0)}$ and use the $\|\cdot\|_\infty$.

$\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\| < \delta$ then

$$\|f(x) - y^{(0)}\|_\infty < \epsilon.$$

Let $j \in \{1, \dots, q\}$, then

$$|f_j(x) - y_j^{(0)}| \leq \|f(x) - y^{(0)}\|_\infty < \epsilon.$$

" \Leftarrow ". Assume that $\forall j = 1, \dots, q$

$$\lim_{x \rightarrow x_0} f_j(x) = y_j^{(0)}.$$

Then $\forall \epsilon > 0 \exists \delta_j > 0$ such that $\forall x \in D \setminus \{x_0\}$, if $\|x - x_0\|_\infty < \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$.

Take $\delta := \min\{\delta_1, \dots, \delta_q\} > 0$.

Then $\forall j = 1, \dots, q$, if $\|x - x_0\|_\infty < \delta \leq \delta_j$ then $|f_j(x) - y_j^{(0)}| < \epsilon$ hence

$$\|f(x) - y^{(0)}\|_\infty = \max_{j=1}^q \{|f_j(x) - y_j^{(0)}|\} < \epsilon.$$

- (ii) • if x_0 isolated, nothing to prove.
 • if x_0 accum. point for D , we use (i).

□

EXAMPLE 1.8. (1) $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

- f continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$

- $(0, 0)$ is an accum. point for \mathbb{R}^2
and f does not have a limit $(x, y) \rightarrow (0, 0)$

$$x = 0, y \rightarrow 0 \Rightarrow f(0, y) = 0 \rightarrow 0$$

$$y = 0, x \rightarrow 0 \Rightarrow f(x, 0) = 0 \rightarrow 0$$

$$x = y \rightarrow 0 \Rightarrow f(x, y) = \frac{1}{2} \rightarrow \frac{1}{2}$$

(2) $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

- f continuous on \mathbb{R}^2
- at $(x_0, y_0) \neq (0, 0)$, clear
- at $(0, 0)$

$$|f(x, y)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{2} = \frac{\|(x, y)\|_2}{2}$$

DEFINITION 1.9. • A *curve* in \mathbb{R}^2 is a continuous function $\gamma : I \rightarrow \mathbb{R}^q, q \geq 1$ where I is an interval.

- If the interval $I = [a, b]$ is compact, then the curve has *endpoints* $x = \gamma(a)$ and $y = \gamma(b)$. In this case we say that γ is a *path* joining x and y .
- A curve with endpoints x and y is called *closed* if $x = y$.

EXAMPLE 1.10. (1) $\gamma(t) = (\cos t, \sin t), \gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$

$\gamma(0) = \gamma(2\pi)$ so γ is a closed curve.

(2) $\gamma(t) = (t^2, t^3), \gamma : [0, 1] \rightarrow \mathbb{R}^2$

$\gamma(0) = (0, 0)$ $\gamma(1) = (1, 1)$ a curve with endpoints but not closed.

(3) $\gamma(t) = (t \cos t, t \sin t, t), \gamma : \mathbb{R} \rightarrow \mathbb{R}^3$

is a curve with no endpoints

a *spiral* inside $\{(x, y, z) | x^2 + y^2 = |z|\}$ a *cone*

DEFINITION 1.11. A *surface* in \mathbb{R}^q ($q \geq 2$) is a continuous function $F : A \rightarrow \mathbb{R}^q$, D open, nonempty in \mathbb{R}^2 , $D \subseteq A \subseteq \bar{D}$.

EXAMPLE 1.12. (1) *2-dimensional sphere in \mathbb{R}^3*

$$F : [0, 2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}^3$$

$$F(\theta, \phi) = (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)$$

(2)

$$G : B \rightarrow \mathbb{R}^3$$

$$G(x, y, z) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$$

$$B = \{(x, y) | x^2 + y^2 \leq 1\}$$

2. Continuity and Topology

THEOREM 2.1. (Topological Characterization of Continuity)

Let $(X; \rho)$, $(Y; \eta)$ be metric spaces. $f : X \rightarrow Y$ a function. TFAE:

- (i) f is continuous.
- (ii) For all U open in Y , $f^{-1}(U)$ is open in X .
- (iii) For all F closed in Y , $f^{-1}(F)$ is closed in X .

PROOF.

- (i) \Rightarrow (ii). Let U open in Y and $x \in f^{-1}(U)$, i.e. $f(x) \in U$.
 Then $\exists \epsilon > 0$ s.t. $B_\epsilon(f(x)) \subseteq U$.
 Since f is cont. at x $\exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(f(x)) \subseteq U$
 hence $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))) \subseteq f^{-1}(U)$, i.e. $f^{-1}(U)$ is open.
- (ii) \Rightarrow (i). Let $x \in X$ and $\epsilon > 0$. Then $B_\epsilon(f(x))$ is open in Y
 hence $f^{-1}(B_\epsilon(f(x)))$ is open in X .
 Since $x \in f^{-1}(B_\epsilon(f(x)))$ it follows that $\exists \delta > 0$
 s.t. $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$, i.e. $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.
 Hence f is cont. at each $x \in X$.
- (ii) \Leftrightarrow (iii). Since $f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A)$.

□

COROLLARY 2.2. Let $\emptyset \neq D \subseteq \mathbb{R}^p$, $f : D \rightarrow \mathbb{R}^q$ a function. TFAE:

- (i) f is continuous.
- (ii) For all U open in \mathbb{R}^q , $f^{-1}(U)$ is relatively open in D .
- (iii) For all F closed in \mathbb{R}^q , $f^{-1}(F)$ is relatively closed in D .

COROLLARY 2.3. Let $\emptyset \neq D \subseteq \mathbb{R}^p$ open, $f : D \rightarrow \mathbb{R}^q$ a function. TFAE:

- (i) f is continuous.
- (ii) For all U open in \mathbb{R}^q , $f^{-1}(U)$ is open in \mathbb{R}^p .
- (iii) For all F closed in \mathbb{R}^q , $f^{-1}(F)$ is closed in \mathbb{R}^p .

3. Continuity and Compactness

THEOREM 3.1. Let $(X; \rho)$, $(Y; \eta)$ be metric spaces, $f : X \rightarrow Y$ a continuous function and K compact in X . Then $f(K)$ is compact.

PROOF. Let $\{U_i | i \in \mathcal{J}\}$ be an open (in Y) covering of $f(K)$:

- $\forall i \in \mathcal{J}, U_i$ is open in Y ;
- $f(K) \subseteq \bigcup_{i \in \mathcal{J}} U_i$.

Then $\forall i \in \mathcal{J}$, $f^{-1}(U_i)$ is open in X and

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{i \in \mathcal{J}} U_i\right) = \bigcup_{i \in \mathcal{J}} f^{-1}(U_i)$$

hence $\{f^{-1}(U_i) | i \in \mathcal{J}\}$ is an open covering of K .

Since K is compact $\exists i_1, \dots, i_n \in \mathcal{J}$ such that

$$K \subseteq \bigcup_{k=1}^n f^{-1}(U_{i_k}) = f^{-1}\left(\bigcup_{k=1}^n U_{i_k}\right)$$

hence $f(K) \subseteq f(f^{-1}(\bigcup_{k=1}^n U_{i_k})) \subseteq \bigcup_{k=1}^n U_{i_k}$. \square

COROLLARY 3.2. *Let $f : X \rightarrow \mathbb{R}$ a continuous function and K compact, nonempty in X . Then:*

- f is bounded on K , i.e. $f(K)$ is bounded in \mathbb{R} .
- The extreme values of f on K are attained, i.e. $\exists x_m, x_M \in K$ such that

$$f(x_m) = \inf_K f, \quad f(x_M) = \sup_K f.$$

PROOF. $f(K)$ is compact in \mathbb{R} , hence closed and bounded.

- $f(K)$ bounded $\Rightarrow \inf_K f, \sup_K f \in \mathbb{R}$.
- $\inf_K f \in f(\bar{K}) = f(K)$, hence $\exists x_m \in K$ such that $\inf_K f = f(x_m)$.
- $\sup_K f \in f(\bar{K}) = f(K)$, hence $\exists x_M \in K$ such that $\sup_K f = f(x_M)$. \square

COROLLARY 3.3. *Let $f : X \rightarrow \mathbb{R}^q$ a continuous function. Then for all K nonempty and compact in X $\exists x_m, x_M \in K$ such that*

$$\|f(x_m)\| = \inf_K \|f\|, \quad \|f(x_M)\| = \sup_K \|f\|.$$

PROOF.

$$\|f\| = \|\cdot\| \circ f : X \rightarrow \mathbb{R}^q \rightarrow \mathbb{R}.$$

\square

4. Continuity and Connectedness

THEOREM 4.1. *Let $(X; \rho)$, $(Y; \eta)$ be metric spaces, $f : X \rightarrow Y$ a continuous function, C connected in X . Then $f(C)$ is connected in Y .*

PROOF. By contrapositive, assume that $f(C)$ is separated, hence: there exist U, V open in Y such that

- $f(C) \subseteq U \cup V$
- $f(C) \cap U \neq \emptyset$
- $f(C) \cap V \neq \emptyset$
- $f(C) \cap U \cap V = \emptyset$

Then $f^{-1}(U)$, $f^{-1}(V)$ are open in X .

- $C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
- $\emptyset \neq f^{-1}(f(C) \cap U) = f^{-1}(f(C)) \cap f^{-1}(U)$

- We prove $C \cap f^{-1}(U) \neq \emptyset$.

Since $f(C) \cap U \neq \emptyset$, $\exists y \in f(C)$ and $y \in U$ hence $\exists x \in C$ such that $f(x) \in U$, hence $x \in f^{-1}(U)$, i.e. $x \in C \cap f^{-1}(U)$.

- Similarly $C \cap f^{-1}(V) \neq \emptyset$.

- Similarly $C \cap f^{-1}(U) \cap f^{-1}(V) = \emptyset$

Assume $C \cap f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$, then $\exists x \in C$ such that $x \in f^{-1}(U) \cap f^{-1}(V)$ hence $f(x) \in f(C)$ and $f(x) \in f(f^{-1}(U)) \subseteq U$, $f(x) \in f(f^{-1}(V)) \subseteq V$ i.e. $f(x) \in f(C) \cap U \cap V$, contradiction!

Thus, $f^{-1}(U)$ and $f^{-1}(V)$ separate C , contradiction! \square

COROLLARY 4.2. (1) Let $f : (X; \rho) \rightarrow \mathbb{R}$ continuous, C connected in X . Then $f(C)$ is an interval.
 (2) Let I be an interval in \mathbb{R} and $\gamma : I \rightarrow \mathbb{R}^d$ continuous (a curve). Then $\gamma(I)$ is connected.

DEFINITION 4.3. A subset $S \subseteq (X; \rho)$ is called *pathwise connected* if for all $a, b \in S$ there exists a (continuous) path $\gamma : [0, 1] \rightarrow S$ such that $a = \gamma(0)$ and $b = \gamma(1)$.

(3) If S is pathwise connected, then it is connected.

PROOF. Assume S is not connected, let U, V open in X and separating S . Then there exist $a \in S \cap U$ and $b \in S \cap V$. Since S is pathwise connected, there exists $\gamma : [0, 1] \rightarrow S$ continuous such that $\gamma(0) = a$ and $\gamma(1) = b$. But then U and V separate $\gamma([0, 1])$, contradiction! \square

EXAMPLE 4.4. A set in \mathbb{R}^2 that is connected but not pathwise connected.

$$S = \{(0, y) \mid -1 \leq y \leq 1\} \cap \{(x, \sin \frac{1}{x}) \mid -\frac{1}{\pi} < x < \frac{1}{\pi}, x \neq 0\}$$

(4) Assume that D is open in \mathbb{R}^d . Then D is connected iff D is pathwise connected.

PROOF. " \Leftarrow " Holds in general.

" \Rightarrow " On D we define a relation: $x \mathrel{\mathcal{L}} y$ if $\exists \gamma : [0, 1] \rightarrow D$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$.

- $\mathrel{\mathcal{L}}$ is an equivalence relation on D .
- $\forall x \in D$ its equivalence class $[x]_{\mathcal{L}}$ is an open set.
- If D is not pathwise connected then there exist at least two different cosets w.r.t. $\mathrel{\mathcal{L}}$, hence D is disconnected. \square

5. Uniform Continuity

DEFINITION 5.1. Let $(X; \rho)$ and $(Y; \eta)$ be two metric spaces. A function $f : X \rightarrow Y$ is *uniformly continuous* if $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x_1, x_2 \in X$, if $\rho(x_1, x_2) < \delta$ then $\eta(f(x_1), f(x_2)) < \epsilon$.

THEOREM 5.2. If $f : X \xrightarrow[\rho]{\eta} Y$ is continuous and X is compact, then f is uniformly continuous.

PROOF. Let $\epsilon > 0$. Since f is continuous on X , $\forall x \in X \exists \delta_x > 0$ such that $\forall z \in X$ with $\rho(x, z) < \delta_x \Rightarrow \eta(f(x), f(z)) < \epsilon/2$.

Since $\{B_{\delta_x/2}(x) | x \in X\}$ is an open covering of X compact, it follows that there exist $x_1, \dots, x_n \in X$ such that

$$X \subseteq \bigcup_{i=1}^n B_{\delta_{x_i}/2}(x_i)$$

Let $\delta := \min\{\frac{\delta_{x_i}}{2} | i = 1, \dots, n\} > 0$ and let $x, z \in X$ such that $\rho(x, z) < \delta$. Then $\exists j \in \{1, \dots, n\}$ such that $x \in B_{\delta_{x_j}/2}(x_j)$ i.e. $\rho(x, x_j) < \frac{\delta_{x_j}}{2}$. Then

$$\rho(z, x_j) \leq \rho(z, x) + \rho(x, x_j) < \delta + \frac{\delta_{x_j}}{2} \leq \frac{\delta_{x_j}}{2} + \frac{\delta_{x_j}}{2} = \delta_{x_j}$$

hence $\eta(f(x), f(x_j)) < \frac{\epsilon}{2}$ and $\eta(f(z), f(x_j)) < \frac{\epsilon}{2}$. Then

$$\eta(f(x), f(z)) \leq \eta(f(x), f(x_j)) + \eta(f(z), f(x_j)) < \epsilon$$

□

THEOREM 5.3. Let $(X; \rho)$ be a compact metric space, $(Y; \eta)$ be a complete metric space, $\emptyset \neq D \subseteq X$ and $f : D \rightarrow Y$ a function. TFAE:

- (i) f is uniformly continuous on D
- (ii) $\exists \bar{f} : \bar{D} \rightarrow Y$ such that $\bar{f}|_D = f$ and \bar{f} is continuous on \bar{D} .

PROOF(ii) \Rightarrow (i) \bar{D} closed in X compact hence \bar{D} is compact $\Rightarrow \bar{f}$ unif. cont. on $\bar{D} \Rightarrow f = \bar{f}|_D$ unif. cont. on D .

(i) \Rightarrow (ii) Assume that f is uniformly continuous.

- f maps Cauchy sequences from D to Cauchy sequences in Y :
Let $\epsilon > 0$. Then $\exists \delta > 0$ such that $\forall x, z \in D$, if $\rho(x, z) < \delta$ then $\eta(f(x), f(z)) < \epsilon$.
Let $(x_n)_{n \geq 1}$ be a ρ -Cauchy sequence in D . $\forall \epsilon > 0$, with $\delta > 0$ as before, $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $m, n \geq N$ then $\rho(x_m, x_n) < \delta$ hence $\eta(f(x_m), f(x_n)) < \epsilon$.
- Let $x \in \bar{D}$. Then $\exists (x_n)_{n \geq 1}$ in D such that $x_n \xrightarrow[n]{\rho} x$ hence $(x_n)_{n \geq 1}$ is ρ -Cauchy, hence $(f(x_n))_{n \geq 1}$ is η -Cauchy hence, since $(Y; \eta)$ is complete, $\exists y_x \in Y$ such that $f(x_n) \xrightarrow[n]{\eta} y_x$. Define $\bar{f}(x) := y_x$. The definition of \bar{f} is correct:

We use the interlacing method.

If $(x_n)_{n \geq 1}$ is a seq. in D , $x_n \xrightarrow[n]{\rho} x$

$(z_n)_{n \geq 1}$ is a seq. in D , $z_n \xrightarrow[n]{\rho} x$

Then $(t_n)_{n \geq 1}$ defined by

$$t_n = \begin{cases} x_{\frac{n}{2}}, & \text{if } n = 2k \\ z_{\frac{n-1}{2}}, & \text{if } n = 2k + 1 \end{cases}$$

is a sequence in D and $t_n \xrightarrow[n]{\rho} x$.

Then $f(x_n) \xrightarrow[n]{\eta} y_x$, $f(z_n) \xrightarrow[n]{\eta} z_x$, $f(t_n) \xrightarrow[n]{\eta} t_x$. Since $(f(x_n))_{n \geq 1}$ and $(f(z_n))_{n \geq 1}$ are subsequences of $(f(t_n))_{n \geq 1}$, it follows that $y_x = t_x = z_x$.

- \bar{f} is continuous on \bar{D} :

We use the sequential characterization of continuity. Let $\bar{x} \in \bar{D}$ and $(\bar{x}_n)_{n \geq 1}$ a sequence in \bar{D} such that $\bar{x}_n \xrightarrow[n]{\rho} \bar{x}$.

Then $\forall n \geq 1 \exists x_n \in D$ such that $\rho(\bar{x}_n, x_n) < \frac{1}{n}$ and $\rho(\bar{f}(\bar{x}_n), f(x_n)) < \frac{1}{n}$. Then

$$\rho(x_n, \bar{x}) \leq \rho(x_n, \bar{x}_n) + \rho(\bar{x}_n, \bar{x}) < \frac{1}{n} + \rho(\bar{x}_n, \bar{x}) \rightarrow 0$$

hence $x_n \xrightarrow[n]{\rho} \bar{x}$, and then by def. of \bar{f} we have $f(x_n) \xrightarrow[n]{\rho} \bar{f}(\bar{x})$, and then

$$\eta(\bar{f}(\bar{x}_n), \bar{f}(\bar{x})) \leq \eta(\bar{f}(\bar{x}_n), f(x_n)) + \eta(f(x_n), \bar{f}(\bar{x})) < \frac{1}{n} + \eta(f(x_n), \bar{f}(\bar{x})) \rightarrow 0$$

hence $\bar{f}(\bar{x}_n) \xrightarrow[n]{\eta} \bar{f}(\bar{x})$.

□

6. Sequences of Functions

Let X be a nonempty set and $(Y; \eta)$ a vector space. We consider sequences $(f_n)_{n \geq 1}$ of functions

$$f_n : X \rightarrow Y, \quad n \in \mathbb{N}.$$

- DEFINITION 6.1.
- $(f_n)_{n \geq 1}$ converges pointwise to $f : X \rightarrow Y$ if $\forall x \in X \forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N, \eta(f(x), f_n(x)) < \epsilon$, i.e. $\forall x \in X, (f_n(x))_{n \geq 1}$ converges to $f(x)$ in Y with respect to η .
 - $(f_n)_{n \geq 1}$ converges uniformly to $f : X \rightarrow Y$ if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N, \forall x \in X, \eta(f(x), f_n(x)) < \epsilon$

FACT 6.2. (1) If there exists $(\alpha_n)_{n \geq 1}$ such that $\alpha_n \geq 0 \forall n \in \mathbb{N}$ and $\alpha_n \xrightarrow[n]{\rho} 0$ and $\eta(f(x), f_n(x)) \leq \alpha_n \forall n \in \mathbb{N}$ and $\forall x \in X$, then $f_n \xrightarrow[n]{\text{unif.}} f$.

- (2) If $(X; \rho)$ and $(Y; \eta)$ are metric spaces and functions $f_n : X \rightarrow Y$ continuous for all $n \in \mathbb{N}$, $f : X \rightarrow Y$ such that $f_n \xrightarrow[n]{\text{unif.}} f$, then f is continuous on X .

PROOF. Let $x_0 \in X$ and $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$

$$\eta(f_n(x), f(x)) < \frac{\epsilon}{3}.$$

f_N is continuous at x_0 , so there exists $\delta > 0$ such that for all $x \in X$, $\rho(x, x_0) < \delta$ implies $\eta(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$.

Then for all $x \in X$, if $\rho(x, x_0) < \delta$ then

$$\begin{aligned} \eta(f(x), f(x_0)) &\leq \eta(f(x), f_N(x)) + \eta(f_N(x), f_N(x_0)) + \eta(f_N(x_0), f(x_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

EXAMPLE 6.3. (1) $f_n : \overline{B_1(0)} \rightarrow \mathbb{R}$, $f_n(x) = \|x\|_2^n$,
 $\subseteq \mathbb{R}^d$

$$f(x) = \begin{cases} 0, & x \in B_1(0) \\ 1, & x \in \partial B_1(0) \end{cases}$$

Then $f_n \xrightarrow[n]{\text{pointwise}} f$ but not uniformly.

- (2) $B_1(0) \in \mathbb{R}^2$, $f_n : B_1(0) \rightarrow \mathbb{R}^2$, $f : B_1(0) \rightarrow \mathbb{R}^2$

$$f_n(x_1, x_2) = \left(\frac{x_1^2 - nx_2^2}{1 + nx_2^2}, \frac{nx_1}{1 + nx_1^2} \right)$$

$$f(x) = \begin{cases} (-1, \frac{1}{x_1}), & x_1 \neq 0, x_2 \neq 0 \\ (-1, 0), & x_1 = 0, x_2 \neq 0 \\ (x_1^2, \frac{1}{x_1}), & x_1 \neq 0, x_2 = 0 \\ (0, 0), & x_1 = 0, x_2 = 0 \end{cases}$$

$f_n \xrightarrow[n]{\text{pointwise}} f$ but not uniformly.

DEFINITION 6.4. $f_n : X \xrightarrow[\text{set}]{\eta} Y$, $\forall n \in \mathbb{N}$.

$(f_n)_{n \geq 1}$ is *uniformly Cauchy* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$\eta(f_n(x), f_m(x)) < \epsilon$$

.

FACT 6.5. (1) If $f_n \xrightarrow[n]{\text{unif.}} f$ then $(f_n)_{n \geq 1}$ is *uniformly Cauchy*.

- (2) If $(Y; \eta)$ is complete and $(f_n)_{n \geq 1}$ is *uniformly Cauchy* then $\exists f : X \rightarrow Y$ such that $f_n \xrightarrow[n]{\text{unif.}} f$.

PROOF. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, $x \in X$

$$(*) \quad \eta(f_m(x), f_n(x)) < \frac{\epsilon}{2}$$

Then, for each $x \in X$ $(f_n(x))_{n \geq 1}$ is Cauchy in Y complete, hence there exists $f(x) \in Y$ such that $f_n(x) \xrightarrow[n]{f(x)} f(x)$.

Due to the uniqueness of the limit in Y , there exists a function $f : X \rightarrow Y$ such that $f_n \xrightarrow[n]{\text{pointwise}} f$.

Letting $m \rightarrow +\infty$ in $(?)$, for all $n \geq N$, $x \in X$

$$\eta(f(x), f_n(x)) \leq \frac{\epsilon}{2} < \epsilon$$

Hence $f_n \xrightarrow[n]{\text{unif.}} f$. □

REMARK 6.6. For all $z \in Y$ the map $Y \ni y \mapsto \eta(y, z) \in \mathbb{R}$ is continuous.

DEFINITION 6.7. Let $X \neq \emptyset$ and $(V, \|\cdot\|)$ a normed space.

$$\mathcal{B}(X; V) := \{f : X \rightarrow V \mid f \text{ bounded function}\}$$

For all $f \in \mathcal{B}(X; V)$, $\|f\|_X := \sup_{x \in X} \|f(x)\|$.

FACT 6.8. (1) $\mathcal{B}(X; V)$ is a vector space. $\|\cdot\|_X$ is a norm, called the sup norm on X .

(2) If $(V; \|\cdot\|)$ is complete, then the normed space $(\mathcal{B}(X; V); \|\cdot\|_X)$ is complete.

PROOF. Let $(f_n)_{n \geq 1}$ be a Cauchy sequence in $\mathcal{B}(X; V)$ with respect to the norm $\|\cdot\|_X$. Hence:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N, \|f_m - f_n\|_X < \frac{\epsilon}{2}.$$

$$(*) \quad \text{hence } \forall x \in X \quad \|f_m(x) - f_n(x)\| \leq \|f_m - f_n\|_X < \frac{\epsilon}{2}$$

Then for all $x \in X$ the sequence $(f_n(x))_{n \geq 1}$ is Cauchy in V complete, hence there exists $f(x) \in V$ such that

$$\|f_n(x) - f(x)\| \xrightarrow[n \rightarrow \infty]{} 0$$

Then $X \ni x \mapsto f(x) \in V$ is a function.

- $f \in \mathcal{B}(X; V)$ and $\|f_n - f\|_X \xrightarrow[n]{} 0$.
- For all $x \in X$ let $n \rightarrow \infty$ in $(?)$ hence

$$(**) \quad \|f(x) - f_n(x)\| \leq \frac{\epsilon}{2} < \epsilon$$

hence

$$(***) \quad \sup_{x \in X} \|f(x) - f_n(x)\| \leq \frac{\epsilon}{2} < \epsilon$$

Also for all $x \in X$, letting $n = N$ in (??) we have

$$\|f(x)\| \leq \|f(x) - f_N(x)\| + \|f_N(x)\| \leq \frac{\epsilon}{2} + \|f_N\|_X < +\infty$$

hence $f \in \mathcal{B}(X; V)$.

On the other hand, from (??), for all $n \geq N$ $\|f - f_n\|_X < \epsilon$,

hence $f_n \xrightarrow{\|\cdot\|_X} f$. □

(3) Let $(X; \rho)$ be a compact metric space, $(V; \|\cdot\|)$ Banach space and

$$\mathcal{C}(X; V) := \{f : X \rightarrow V \mid f \text{ continuous}\}.$$

Then:

- $\mathcal{C}(X; V) \subseteq \mathcal{B}(X; V)$ as a vector subspace
- $(\mathcal{C}(X; V); \|\cdot\|_X)$ is a complete normed space.

7. Series of Functions