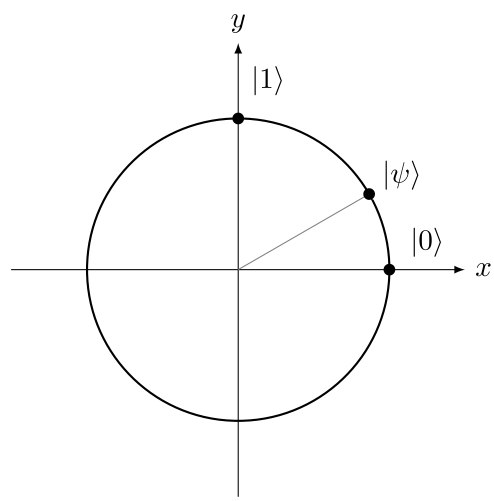
CMPT 476/981: Introduction to Quantum Algorithms
Assignment 3

Due **February 15th, 2024 at 11:59pm on coursys**
Complete individually and submit in PDF format.

Question 1 [4 points]: Projectors

Let $|\psi\rangle$ be a unit vector in $\mathbb{C}^d$ and $|\psi^\perp\rangle$ be a unit vector which is orthogonal to $|\psi\rangle$.

1. Let $P = |\psi\rangle\langle\psi|$. Compute $(I - 2P)|\psi\rangle$ and $(I - 2P)|\psi^\perp\rangle$.
2. Show that $(I - 2|\psi\rangle\langle\psi|)$ is unitary whenever $|\psi\rangle$ is a unit vector.
3. Suppose a single qubit has state $|\psi\rangle \in \mathbb{R}^2$ — that is, $|\psi\rangle$ is a unit vector in $\mathbb{R}^2$ where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can be viewed as the unit vector along the positive x-axis, and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the unit vector along the positive y axis. This is the two-dimensional picture of a quantum state which we've used in class:



   What is the geometric interpretation of the transformation $I - 2|0\rangle\langle 0|$ in $\mathbb{R}^2$?

4. Does the transformation $I - 2|0\rangle\langle 0|$ have a similar geometric interpretation in the Bloch sphere? Why or why not?

Solution. 1.

$$(I - 2|\psi\rangle\langle\psi|)|\psi\rangle = |\psi\rangle - 2|\psi\rangle = -|\psi\rangle$$
$$(I - 2|\psi\rangle\langle\psi|)|\psi^\perp\rangle = |\psi^\perp\rangle - 2|\psi\rangle\langle\psi|\psi^\perp\rangle = |\psi^\perp\rangle$$

2. $(I - 2|\psi\rangle\langle\psi|)^\dagger = I - 2|\psi\rangle\langle\psi|$ so, expanding the product:

$$\begin{aligned}(I - 2|\psi\rangle\langle\psi|)(I - 2|\psi\rangle\langle\psi|) &= I - 4|\psi\rangle\langle\psi| + 4\langle\psi|\psi\rangle|\psi\rangle\langle\psi| \\ &= I - 4|\psi\rangle\langle\psi| + 4|\psi\rangle\langle\psi| \\ &= I\end{aligned}$$

3. $I - 2|0\rangle\langle 0|$ is a reflection (specifically, a Householder reflection) along the hyperplane perpendicular to $|0\rangle$ — in $\mathbb{R}^2$ this is the y-axis. In particular, $(I - 2|0\rangle\langle 0|)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$ which is trivially a reflection along the y-axis.

4. No. In the Bloch sphere picture,

$$\begin{aligned}(I - 2|0\rangle\langle 0|)(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle) &= -\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \\ &= -(\cos(\theta/2)|0\rangle + e^{i(\phi+\pi)}\sin(\theta/2)|1\rangle\end{aligned}$$

which corresponds to a rotation of the state by 180^o around the z-axis. Note that while this state **can** be viewed as a reflection of $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, the reflection is along the $x - z$ plane rotated by ϕ degrees. Since in general different vectors would be reflected along different planes, it can not be a simple reflection.

□

Question 2 [3 points]: Parity measurement

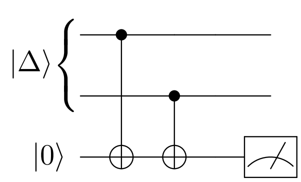
1. How is a parity measurement of two qubits different from measuring both bits in the computational basis **and then taking their parity**?

2. Devise a circuit using $CNOT$ gates and computational basis measurement which measures the parity of two qubits **without measuring either qubit itself**.

   Hint: you will need to use an *ancilla* — i.e. an additional qubit initialized to $|0\rangle$:

Solution. 1. The computational basis measurement is in general more destructive than a parity measurement. Consider the bell state $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ which has parity 0. A computational basis measurement yields state either $|00\rangle$ or $|11\rangle$ with probability 1/2 respectively. Conversely, a parity measurement leaves the state unchanged.

2. The following circuit computes the parity of two qubits without measuring either qubit



Let $|\Delta\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ be an arbitrary two qubit state. Then a parity measurement yields 0 with probability $|a|^2 + |d|^2$ and leaves the first two qubits in the (normalized) state $a|00\rangle + d|11\rangle$, or 1 with probability $|b|^2 + |c|^2$ and final state of the first two qubits $b|01\rangle + c|10\rangle$, appropriately normalized. For the same input state $|\Delta\rangle$, the circuit above has state $a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle$ just before measurement. Then a computational basis measurement on the third qubit yields 0 with probability $|a|^2 + |d|^2$ and 1 with probability $|b|^2 + |c|^2$ which agrees with the parity measurement.

$\square$

Question 3 [1 points]: Mixed states

Calculate the density matrix of the following ensembles.

1. $\{(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|+\rangle, 1)\}$
2. $\{(|0\rangle, \frac{1}{2}), (|+\rangle, \frac{1}{2})\}$
3. $\{(|00\rangle, \frac{1}{2}), (|01\rangle, \frac{1}{4}), (|10\rangle, \frac{1}{4})\}$

Solution. 1. With the vector as written:

$$\begin{aligned}
\frac{1}{2}(|0\rangle + |+\rangle)(\langle 0| + \langle +|) &= \frac{1}{2}(|0\rangle + \frac{|0\rangle + |1\rangle}{\sqrt{2}})(\langle 0| + \frac{\langle 0| + \langle 1|}{\sqrt{2}}) \\
&= \frac{1}{2}((1 + 1/\sqrt{2})|0\rangle + \frac{1}{\sqrt{2}}|1\rangle)(1 + 1/\sqrt{2})\langle 0| + \frac{1}{\sqrt{2}}\langle 1|) \\
&= \frac{1}{2}((1 + 1/\sqrt{2})^2|0\rangle\langle 0| + (1/2 + 1/\sqrt{2})|1\rangle\langle 0| + (1/2 + 1/\sqrt{2})|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1| \\
&= \frac{1}{2}((1 + \sqrt{2} + 1/2)|0\rangle\langle 0| + (1/2 + 1/\sqrt{2})|1\rangle\langle 0| + (1/2 + 1/\sqrt{2})|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1| \\
&= \frac{3 + 2\sqrt{(2)}}{4}|0\rangle\langle 0| + \frac{1 + \sqrt{2}}{4}|1\rangle\langle 0| + \frac{1 + \sqrt{2}}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 1|
\end{aligned}$$

which has matrix representation

$$\begin{bmatrix} \frac{3+2\sqrt{(2)}}{4} & \frac{1+\sqrt{2}}{4} \\ \frac{1+\sqrt{2}}{4} & \frac{1}{4} \end{bmatrix}$$

With the corrected vector $\frac{i|0\rangle + |+\rangle}{\sqrt{2}}$

$$\begin{aligned}
\frac{1}{2}(i|0\rangle + |+\rangle)(-i\langle 0| + \langle +|) &= \frac{1}{2}(i|0\rangle + \frac{|0\rangle + |1\rangle}{\sqrt{2}})(-i\langle 0| + \frac{\langle 0| + \langle 1|}{\sqrt{2}}) \\
&= \frac{1}{2}((i + 1/\sqrt{2})|0\rangle + \frac{1}{\sqrt{2}}|1\rangle)(-i + 1/\sqrt{2})\langle 0| + \frac{1}{\sqrt{2}}\langle 1|) \\
&= \frac{1}{2}(|i + 1/\sqrt{2}|^2|0\rangle\langle 0| + (\frac{-i}{\sqrt{2}} + 1/2)|1\rangle\langle 0| + (\frac{i}{\sqrt{2}} + 1/2)|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1|) \\
&= \frac{1}{2}(\frac{3}{2}|0\rangle\langle 0| + (\frac{-i\sqrt{2} + 1}{2})|1\rangle\langle 0| + (\frac{i\sqrt{2} + 1}{2})|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1|) \\
&= \frac{3}{4}|0\rangle\langle 0| + (\frac{-i\sqrt{2} + 1}{4})|1\rangle\langle 0| + (\frac{i\sqrt{2} + 1}{4})|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 1|
\end{aligned}$$

which has matrix representation

$$\begin{bmatrix} \frac{3}{4} & \frac{i\sqrt{2}+1}{4} \\ \frac{-i\sqrt{2}+1}{4} & \frac{1}{4} \end{bmatrix}$$

2.

$$\begin{aligned}
\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}(|+\rangle\langle +|) &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\frac{\langle 0| + \langle 1|}{\sqrt{2}}) \\
&= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) \\
&= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|) \\
&= \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 1|
\end{aligned}$$

which has matrix representation

$$\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

3.

$$\frac{1}{2} + |00\rangle\langle 00| + \frac{1}{4}|10\rangle\langle 10| + \frac{1}{4}|01\rangle\langle 01|$$

which has matrix representation

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Question 4 [1 point]: Partial trace

Calculate the following reduced density matrix, taking A to be the first qubit (i.e. trace out the first qubit):

$$\mathrm{Tr}_A \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

Solution. Write in bra-ket notation.

$$\begin{aligned}
&\mathrm{Tr}_A \left(\frac{1}{2}|01\rangle\langle 01| - \frac{1}{2}|01\rangle\langle 10| - \frac{1}{2}|10\rangle\langle 01| + \frac{1}{2}|10\rangle\langle 10| \right) \\
&= \frac{1}{2}\langle 0|0\rangle|1\rangle\langle 1| - \frac{1}{2}\langle 0|1\rangle|1\rangle\langle 0| - \frac{1}{2}\langle 1|0\rangle|0\rangle\langle 1| + \frac{1}{2}\langle 1|1\rangle|0\rangle\langle 0| \\
&= \frac{1}{2}\langle 0|0\rangle|1\rangle\langle 1| + \frac{1}{2}\langle 1|1\rangle|0\rangle\langle 0| \\
&= \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|0\rangle\langle 0|
\end{aligned}$$

which has matrix representation

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$\square$

Question 5 [3 points]: Positivity of the density operator

An operator A is *positive-semidefinite* if $\langle v|A|v\rangle$ is real and non-negative for any vector $|v\rangle$ of appropriate dimension. That is, A is positive-semidefinite if and only if $\langle v|A|v\rangle \in \mathbb{R}^+$ where $\mathbb{R}^+$ are the non-negative real numbers for all vectors $|v\rangle$.

Show that the density matrix $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$ of an ensemble of pure states $\{(|\phi_i\rangle, p_i)\}$ is a positive-semidefinite operator.

Solution. Let $|v\rangle \in \mathcal{H}$.

$$\begin{aligned} \langle v|\rho|v\rangle &= \langle v|(\sum_i p_i|\phi_i\rangle\langle\phi_i|)|v\rangle \\ &= \sum_i p_i(\langle v|\phi_i\rangle\langle\phi_i|v\rangle \\ &= \sum_i p_i|\langle v|\phi_i\rangle|^2 \end{aligned}$$

Since each p_i is a non-negative real number (recall that $\{p_i\}$ forms a probability distribution) and

$$|a+bi|^2 = (a+bi)(a-bi) = a^2+b^2$$

is non-negative and real for any $a+bi \in \mathbb{C}$, the product $\langle v|\rho|v\rangle \in \mathbb{R}^+$. $\square$

Question 6 [4 points]: No-communication

Suppose Alice and Bob share some mixed state ρ on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Recall that the partial measurement of Alice's qubit in basis $\{|e_i\rangle\}$ corresponds to the projective measurement $\{P_i = |e_i\rangle\langle e_i| \otimes I\}$ which maps $\rho \mapsto \sum_i P_i\rho P_i$.

Show that Bob's reduced density matrix is not affected by Alice measuring her qubit in any basis $\{|e_i\rangle\}$ of $\mathcal{H}_A$. Note: It may be helpful to assume that $\mathcal{H}_B$ has a basis $\{|f_j\rangle\}$.

Solution. Let $\{P_i = |e_i\rangle\langle e_i| \otimes I\}$ be the projectors for a measurement on Alice's qubit in basis $\{|e_i\rangle\}$. We need to show that

$$\mathrm{Tr}_A(\rho) = \mathrm{Tr}_A(\sum_i P_i\rho P_i)$$

Let $\rho = \sum_{ijlk} p_{ijlk}|e_i\rangle\langle e_j| \otimes |f_l\rangle\langle f_k|$. Then

$$\mathrm{Tr}_A(\rho) = \sum_{ijlk} p_{ijlk}\mathrm{Tr}(|e_i\rangle\langle e_j|)|f_l\rangle\langle f_k| = \sum_{ijlk} p_{ijlk}\langle e_j|e_i\rangle|f_l\rangle\langle f_k| = \sum_{ijlk|i=j} p_{ijlk}|f_l\rangle\langle f_k|$$

Now

$$\begin{aligned}
\sum_m P_m \rho P_m &= \sum_m P_m (\sum_{ijlk} p_{ijlk} |e_i\rangle\langle e_j| \otimes |f_l\rangle\langle f_k|) P_m \\
&= \sum_{mijlk} (|e_m\rangle\langle e_m| \otimes I)(|e_i\rangle\langle e_j| \otimes |f_l\rangle\langle f_k|)(|e_m\rangle\langle e_m| \otimes I) \\
&= \sum_{mijlk} p_{ijlk} |e_m\rangle\langle e_m|e_i\rangle\langle e_j|e_m\rangle\langle e_m| \otimes |f_l\rangle\langle f_k| \\
&= \sum_{mijlk|i=m=j} p_{ijlk} |e_m\rangle\langle e_m| \otimes |f_l\rangle\langle f_k|
\end{aligned}$$

Taking the partial trace over Alice's system of the final line above gives

$$\mathrm{Tr}_A(\sum_i P_i \rho P_i) = \sum_{mijlk|i=m=j} p_{ijlk} |f_l\rangle\langle f_k| = \sum_{ijlk|i=j} p_{ijlk} |f_l\rangle\langle f_k| = \mathrm{Tr}_A(\rho)$$

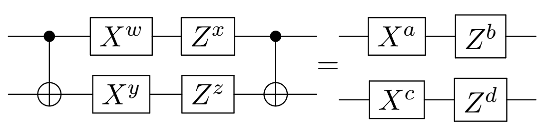
$\square$

Question 7 [6 points]: Teleportation-based protocols

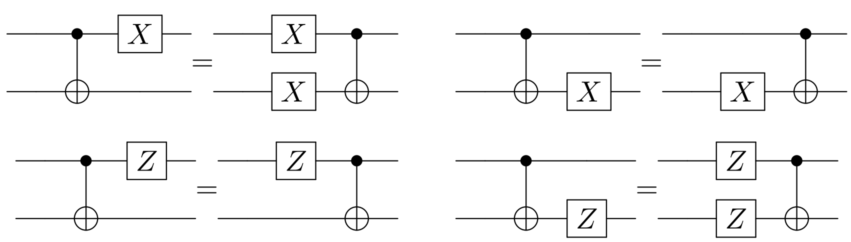
Suppose Alice has a qubit $|\psi\rangle$ and Bob has a qubit $|\phi\rangle$, and consider the following scenario:

- Alice and Bob have a classical communication channel
- Alice and Bob have shared access to an unlimited source of entangled qubits
- Alice and Bob do **not** have a quantum communication channel

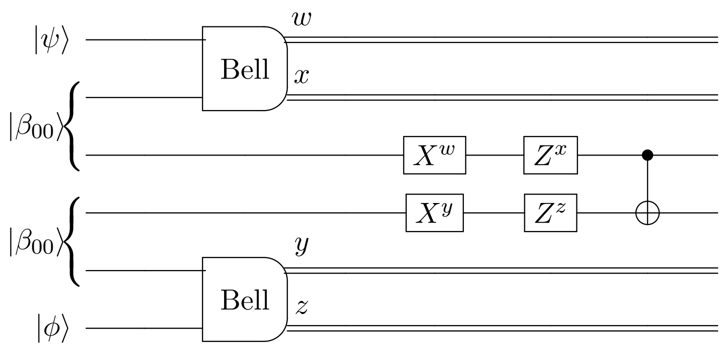
1. Describe a procedure by which Alice and Bob could apply a $CNOT$ gate to their pair of qubits — i.e. $CNOT(|\psi\rangle \otimes |\phi\rangle)$
2. Find values $a, b, c, d \in \{0, 1\}$ as functions of w, x, y, z such that



   You may find the following circuit equalities useful for this question:



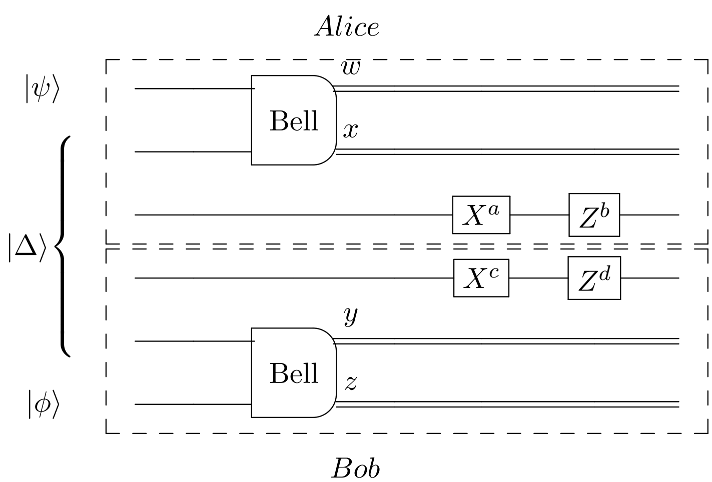
3. Explain why the following circuit would implement a $CNOT$ gate on the state $|\psi\rangle|\phi\rangle$



4. Let

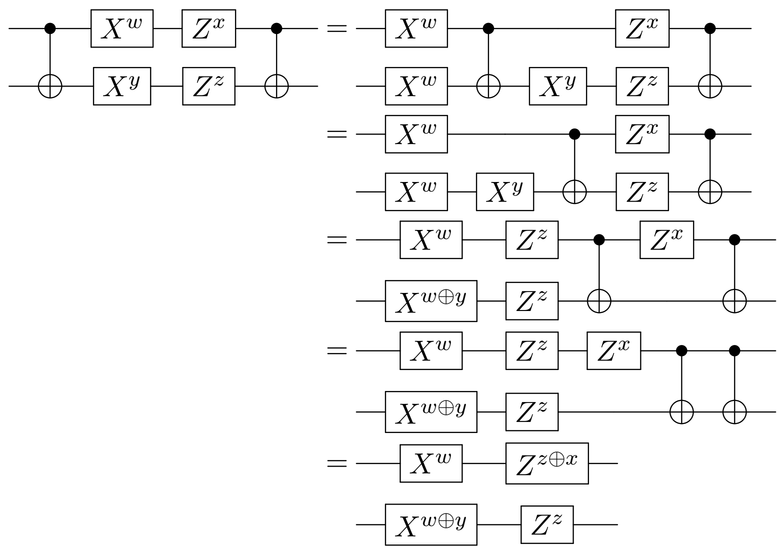
$$|\Delta\rangle = (I \otimes CNOT \otimes I)(|\beta_{00}\rangle \otimes |\beta_{00}\rangle) = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle\right)$$

be a 4 qubit entangled state. Suppose Alice has the first two qubits of $|\Delta\rangle$ and Bob has the second two. Explain why the circuit below where a, b, c, d are the functions of w, x, y, z you gave in part 3 implements a *remote* $CNOT$ between their qubits — that is, applies $CNOT(|\psi\rangle \otimes |\phi\rangle)$ *without* Alice or Bob physically teleporting their qubits to one another.



Solution. 1. Alice can teleport her qubit $|\psi\rangle$ to Bob through a bell state, sending the appropriate clasical bits to complete the teleportation. Then Bob can simply apply CNOT to the pair consisting of his half of the bell state (which is now in state $|\psi\rangle$) and his qubit $|\phi\rangle$.

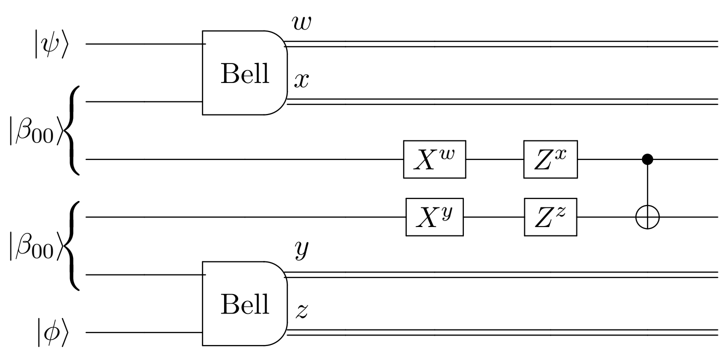
2. We can use the given circuit equalities to comute the CNOT through the circuit.



Thus $a = w, b = z \oplus x, c = w \oplus y, d = z$.

3. The first three qubit lines teleports the $|\psi\rangle$ state to the third qubit just before the CNOT gate, and likewise the bottom three teleport state $|\phi\rangle$ to the fourth qubit line just before the CNOT. Then applying CNOT on qubits three and four is exactly $CNOT(|\psi\rangle, |\phi\rangle)$.

4. By the circuit equality computed in part 2, we can show that the circuit in part 3 is equivalent to the circuit in part 4.



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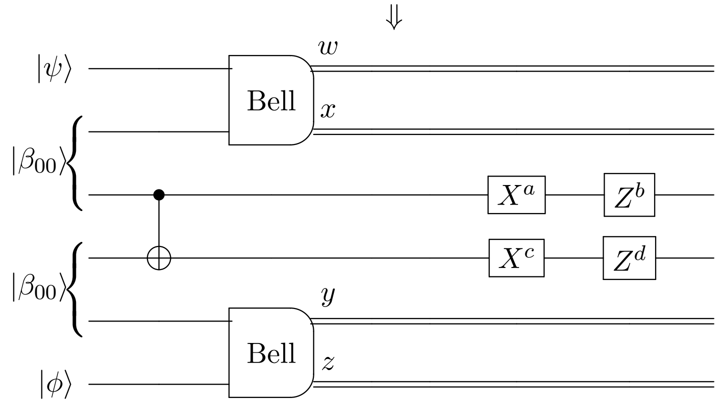

where the state after the CNOT on the inner four qubit lines is exactly $|\Delta\rangle$. The fact that the circuit computes $CNOT(|\psi\rangle, |\phi\rangle)$ then follows from part 3.

What makes this *remote* is that there is no obvious state teleportation through a bell pair from Alice to Bob. What we have instead is a pair of bell states which have apriori CNOT applied to one qubit respectively that can be "transfered" over to the qubits $|\psi\rangle, |\phi\rangle$ owned by Alice and Bob just by a total of 2 classical bits of communication.

$\square$