CMPT 409/981: Quantum Circuits and Compilation
Assignment 1 Solutions

November 28, 2022

Question 1 [2 points]: GHZ states

We need to show that there are no states $|\psi\rangle = a|0\rangle + b|1\rangle$, $|\varphi\rangle = c|0\rangle + d|1\rangle$ and $|\theta\rangle = e|0\rangle + f|1\rangle$ such that $|\psi\rangle \otimes |\varphi\rangle \otimes |\theta\rangle = |GHZ\rangle$. Expanding the left hand side out,

$$|\psi\rangle \otimes |\varphi\rangle \otimes |\theta\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ace \\ acf \\ ade \\ adf \\ bce \\ bcf \\ bde \\ bdf \end{bmatrix}$$

Equality with the right asserts that $ace = bdf = \frac{1}{\sqrt{2}}$ and $acf = 0$. The former implies that a, b, c, d, e, f are all non-zero, so $acf \neq 0$ completing the proof.

Question 2 [3 points]: Entangling operations

Let $|\psi\rangle = |\phi\rangle = |+\rangle$. Then

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

As in the previous question we obtain a system of constraints $ac = ad = bc = -bd \neq 0$. Since all 4 are necessarily non-zero we can derive a contradiction by observing that $0 = ac - ad = a(c - d)$ and $0 = bc + bd = b(c + d)$, so $c - d = 0 = c + d$ which implies $c = 0$.

Question 3 [7 points]: Strange occurrences

1. (1 point) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$(A \otimes I)|\Phi^+\rangle = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

and

$$(I \otimes A^t)|\Phi^+\rangle = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

2. (3 points) The key to this problem is constructing the proper measurement operator, which in this case will be $M_{\Phi^+} = |\Phi^+\rangle\langle\Phi^+| \otimes I$ since we're measuring the first two qubits and leaving the third. The rest is (tedious) calculation:

$$\frac{M_{\Phi^+}|\psi\rangle|\Phi^+\rangle}{\sqrt{p(\Phi^+)}} = \frac{1}{\sqrt{p(\Phi^+)}} \cdot \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ b \end{bmatrix} = \frac{1}{\sqrt{p(\Phi^+)}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix}$$

Now observe that $p(\Phi^+) = \langle\psi|\langle\Phi^+|M_{\Phi^+}^\dagger M_{\Phi^+}|\psi\rangle|\Phi^+\rangle = \frac{|a|^2+|b|^2+|a|^2+|b|^2}{32} = \frac{1}{8}$. Putting this all together, the final state is

$$\frac{1}{\sqrt{p(\Phi^+)}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix} = |\Phi^+\rangle \otimes |\psi\rangle$$

3. (2 points) Note that $|\psi\rangle \otimes (A \otimes I|\Phi^+\rangle) = |\psi\rangle \otimes (I \otimes A^t|\Phi^+\rangle) = (I \otimes I \otimes A^t)|\psi\rangle \otimes |\Phi^+\rangle$. Now

$$\begin{aligned} (|\Phi^+\rangle\langle\Phi^+| \otimes I)(I \otimes I \otimes A^t)|\psi\rangle \otimes |\Phi^+\rangle &= (I \otimes I \otimes A^t)(|\Phi^+\rangle\langle\Phi^+| \otimes I)|\psi\rangle \otimes |\Phi^+\rangle \\ &= (I \otimes I \otimes A^t)|\Phi^+\rangle \otimes |\psi\rangle \\ &= |\Phi^+\rangle \otimes (A^t|\psi\rangle) \end{aligned}$$

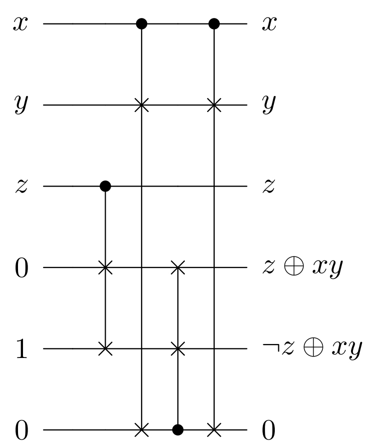
Note: this fact is usually stated in the language of categorical quantum mechanics. It tells us that, very loosely speaking, initialization and termination in the $|\Phi^+\rangle$ state *reverses the direction of the flow of information*. The categorical quantum mechanics theorists may get mad at this explanation so don't tell them I said it.

4. (1 point) Let $U = (H \otimes I)CNOT$. It can readily be observed that

$$U|\Phi^+\rangle = |00\rangle \qquad U|\Phi^-\rangle = |10\rangle$$
$$U|\Psi^+\rangle = |01\rangle \qquad U|\Psi^-\rangle = |11\rangle$$

Question 4 [2 points]: Another universal gate?

Observe that the Toffoli $|x\rangle|y\rangle|z\rangle \mapsto |x\rangle|y\rangle|z \oplus xy\rangle$ can be constructed from FREDKIN gates and ancillas as follows

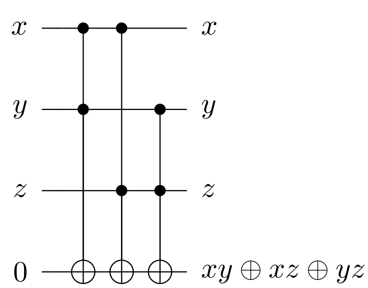


Question 5 [4 points]: Majority rules

The *majority* function $maj : \{0,1\}^3 \to \{0,1\}$ computes the majority value of 3 bits — that is, if 2 or more of x, y, z are 1, then $maj(x, y, z) = 1$, otherwise $maj(x, y, z) = 0$.

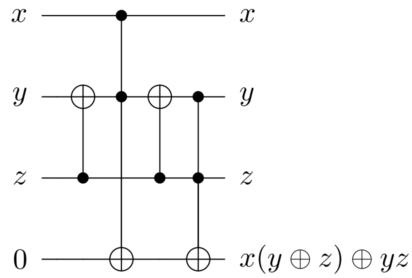
1. (1 point) Compute $xy \oplus xz \oplus yz$ for the only 4 unique cases $x = y = z = 0$, $x = y = z = 1$, $x = 1, y = z = 0$, $x = y = 1, z = 0$.

2. (1 point)

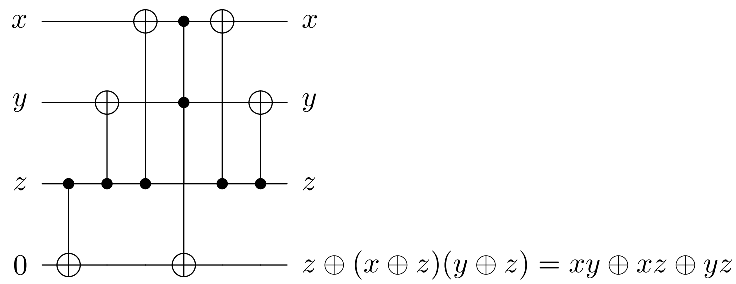


3. (1 point) $maj(x, y, z) = x(y \oplus z) \oplus yz$

4. (1 point)



5. (2 bonus points)



Question 6 [10 points]: Diagonalization

1. (1 point) $|+_Y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$, $|-_Y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}$ suffice.

2. (1 point) $(X \otimes Z)(Z \otimes X) = XZ \otimes ZX = (-ZX) \otimes (-XZ) = (Z \otimes X)(X \otimes Z)$

3. (3 points) The following suffice:

$$v_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}, v_2 = \frac{1}{2}\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}, v_3 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, v_4 = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

4. (2 points) $U = \begin{bmatrix}v_1 & v_2 & v_3 & v_4\end{bmatrix}$ suffices since v_1 and v_2 are $+1$ eigenvectors of $X \otimes Z$, and v_1, v_3 are $+1$-eigenvectors of $Z \otimes X$.

5. (3 points) Since $X \otimes Z$ and $Z \otimes X$ commute,

$$\begin{aligned}
e^{i(\theta_1 X\otimes Z + \theta_2 Z \otimes X)} &= e^{i(\theta_1 X \otimes Z)} e^{i(\theta_2 Z \otimes X)} \\
&= U e^{i(\theta_1 Z \otimes I)} U^\dagger U e^{i(\theta_2 I \otimes Z)} U^\dagger \\
&= U e^{i(\theta_1 Z \otimes I)} e^{i(\theta_2 I \otimes Z)} U^\dagger
\end{aligned}$$

For the final step, it suffices to observe that $e^{i(A\otimes B)} = e^{iA} \otimes e^{iB}$ when A and B are diagonal, since

$$e^{i(A\otimes B)} = \sum_{i,j} e^{ia_i} e^{ib_j} |i\rangle |j\rangle = e^{iA} \otimes e^{iB}$$

by definition of the tensor product on $e^{iA} = \sum_i e^{ia_i}|i\rangle$ and $e^{iB} = \sum_j e^{ib_j}|j\rangle$.