SIMON FRASER UNIVERSITY
School of Computing Science

CMPT 476/981– MIDTERM EXAM
Introduction to Quantum Algorithms

Instructor: Matt Amy

2024/02/29

Name: ____

Student Number: _____________________________________

Instructions:

- **1 double-sided sheet of 8.5x11" paper is permitted as a cheat-sheet**
- A **non-programmable** calculator is permitted
- **No other aids are permitted**
- Print your **full name** and **student ID number** in the space above
- There are 10 pages including this cover page and 8 questions
- The total number of points is 46.
- You will have **110** minutes
- **Good luck!**

Distribution of Marks

Question	Points	Score
1	10	
2	7	
3	7	
4	6	
5	5	
6	2	
7	4	
8	5	
Total:	46	

1. (10 points) Short answers, 1 point each

   (a) What is the dimension of the state space of n qubits?

   2^n

   (b) What is the definition of a unitary operator (you do not need to define the dagger $(\cdot)^\dagger$)

   $U^\dagger = U^{-1}$

   (c) What is the maximum number of dimensions a single particle's quantum state can have?

   There is no maximum number

   (d) What is the probability of measuring $|0\rangle$ in the state $a|0\rangle + b|1\rangle + c|2\rangle$?

   $|a|^2$

   (e) Write the state $a|0\rangle + b|1\rangle + c|2\rangle$ as a vector.

   $\begin{bmatrix} a & b & c \end{bmatrix}^T$

   (f) Give one way in which quantum computation is different from probabilistic computation.

   States in quantum computation can have "negative probabilities"

   (g) Normalize the vector $\sqrt{5}|0\rangle + \sqrt{-11}|1\rangle$

   $\frac{\sqrt{5}}{4}|0\rangle + \frac{\sqrt{-11}}{4}|1\rangle$

   (h) Complete the expression:

   $$e^{i\theta} = \cos\theta + i\sin\theta$$

   (i) Complete the expression:

   $$\mathrm{Tr}\left(\begin{bmatrix} 3 & 5 & 1 \\ 0 & 4 & 9 \\ 5 & 5 & 5 \end{bmatrix}\right) = 12$$

   (j) Give one way a (quantum) controlled gate $c - U$ is different from a classically controlled gate U^x — i.e. applying a gate depending on the value of a classical bit $x \in \{0, 1\}$.

   A quantum controlled gate can be applied with a superposition of different control values

2. (a) (3 points) Calculate the probabilities of obtaining each result when measuring the state

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{-4i}{5}|1\rangle$$

in the basis

$$\{|A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |B\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\}$$

Probability of measuring A:

$$\begin{aligned}
|\langle A|\psi\rangle|^2 &= |(\frac{1}{\sqrt{2}}\langle 0| + \frac{-i}{\sqrt{2}}\langle 1|)(\frac{3}{5}|0\rangle + \frac{-4i}{5}|1\rangle)|^2 \\
&= |\frac{3}{5\sqrt{2}}\langle 0|0\rangle + \frac{-4i}{5\sqrt{2}}\langle 0|1\rangle + \frac{-3i}{5\sqrt{2}}\langle 1|0\rangle + \frac{-4}{5\sqrt{2}}\langle 1|1\rangle|^2 \\
&= |\frac{-1}{5\sqrt{2}}|^2 \\
&= \frac{1}{50}
\end{aligned}$$

Probability of measuring B:

$$|\langle B|\psi\rangle|^2 = ||^2 = |\frac{-3i}{5\sqrt{2}}\langle 0|0\rangle + \frac{-4i}{5\sqrt{2}}\langle 1|1\rangle|^2 = |\frac{-7i}{5\sqrt{2}}|^2 = \frac{49}{50}$$

(b) (4 points) Calculate the probabilities and give the corresponding final state if the first qubit of the state

$$|\psi\rangle = \frac{1}{5}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle + \frac{1}{\sqrt{5}}|10\rangle + \frac{4}{5}|11\rangle$$

is measured in the computational basis.

If $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ then

$$|\psi\rangle = \sqrt{|a|^2 + |b|^2}|0\rangle \otimes \left(\frac{a|0\rangle + b|1\rangle}{\sqrt{|a|^2 + |b|^2}}\right) + \sqrt{|c|^2 + |d|^2}|1\rangle \otimes \left(\frac{c|0\rangle + d|1\rangle}{\sqrt{|c|^2 + |d|^2}}\right)$$

Then the partial measurement rule applies with $p(0) = |a|^2 + |b|^2$ and $p(1) = |c|^2 + |d|^2$.

Measurement result 0:

- Probability is $p(0) = |\frac{1}{5}|^2 + |\frac{\sqrt{3}}{5}|^2 = \frac{4}{25}$
- Final state is $\frac{\frac{1}{5}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle}{\sqrt{p(0)}} = \frac{5}{2}(\frac{1}{5}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle) = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|01\rangle$

Measurement result 1:

- Probability is $p(1) = |\frac{1}{\sqrt{5}}|^2 + |\frac{4}{5}|^2 = \frac{21}{25}$
- Final state is $\frac{\frac{1}{\sqrt{5}}|10\rangle + \frac{4}{5}|11\rangle}{\sqrt{p(1)}} = \frac{5}{\sqrt{21}}(\frac{1}{\sqrt{5}}|10\rangle + \frac{4}{5}|11\rangle) = \frac{\sqrt{5}}{\sqrt{21}}|00\rangle + \frac{4}{\sqrt{21}}|01\rangle$

3. (a) (2 points) Write the following 3-qubit state as a linear combination over the **3-qubit computational (binary) basis**:

$$\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ i \\ -1 \\ 0 \\ -i \\ 0 \end{bmatrix}$$

$$\frac{1}{2}\left(|001\rangle + i|011\rangle - |100\rangle - i|110\rangle\right)$$

(b) (3 points) Write the following 2-qubit operator as a linear combination over the computational basis of 2-qubit operators $\{|ij\rangle\langle lk| \mid i, j, l, k \in \{0, 1\}\}$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i\sqrt{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\left(-i\sqrt{2}|00\rangle\langle 11| + |01\rangle\langle 01| + i|01\rangle\langle 10| + i|10\rangle\langle 01| + |10\rangle\langle 10| + i\sqrt{2}|11\rangle\langle 00|\right)$$

(c) (2 points) To combat noise and decoherence, we often encode a **logical** qubit inside a **subspace** of a larger Hilbert space. Suppose we use the subspace $\mathrm{span}(\{|00\rangle, |11\rangle\})$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$ to encode one logical qubit, where the "0" state is taken to be $|00\rangle$ and the "1" state is $|11\rangle$. What 1 qubit gate (i.e. a 2×2 unitary transformation) does the operator in the previous part perform **on this subspace?**

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i\sqrt{2} \\ i\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

4. (a) (2 points) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} e & f \\ g & h \\ i & j \end{bmatrix}.$$

Write down the matrix $A \otimes B$.

$$\begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ai & aj & bi & bj \\ ce & cf & de & df \\ cg & ch & dg & dh \\ ci & cj & di & dj \end{bmatrix}$$

(b) (4 points) Let $\mathbb{C}^{m\times n}$ denote the space of complex-valued matrices with m rows and n columns, and define the following constants:

$$\begin{array}{lll} |\psi\rangle \in \mathbb{C}^2 = \mathbb{C}^{2\times 1} & A \in \mathbb{C}^{2\times 2} & E \in \mathbb{C}^{4\times 8} \\ |\phi\rangle \in \mathbb{C}^8 = \mathbb{C}^{8\times 1} & B \in \mathbb{C}^{8\times 8} & I \in \mathbb{C}^{2\times 2} \\ |\Delta\rangle \in \mathbb{C}^4 = \mathbb{C}^{4\times 1} & C \in \mathbb{C}^{4\times 4} & \\ |\zeta\rangle \in \mathbb{C}^4 = \mathbb{C}^{4\times 1} & D \in \mathbb{C}^{4\times 4} & \end{array}$$

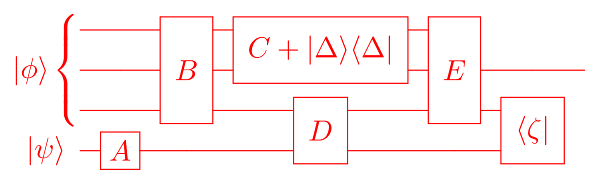
Correctly parenthesize the expression below to make it well-formed, keeping in mind that

- $A + B$ is well formed if and only if the dimensions of A and B are equal, and
- AB is well-formed if and only if the columns of A equal the rows of B.

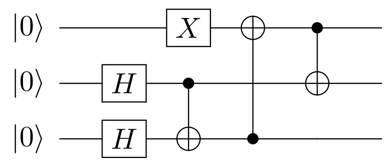
Hint: work from right to left

$$(\langle\zeta| \otimes I) \cdot (E \otimes I) \cdot ((C + |\Delta\rangle\langle\Delta|) \otimes D) \cdot (B \otimes I) \cdot (|\phi\rangle \otimes (A \cdot |\psi\rangle))$$

(c) (1 point (bonus)) Draw the expression in part (b) as a circuit diagram.



5. (5 points) Calculate the final state of the circuit below in the computational basis.



$$|000\rangle \xrightarrow{I\otimes H\otimes I} \frac{1}{\sqrt{2}}(|000\rangle + |010\rangle)$$

$$\xrightarrow{I\otimes I\otimes H} \frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle)$$

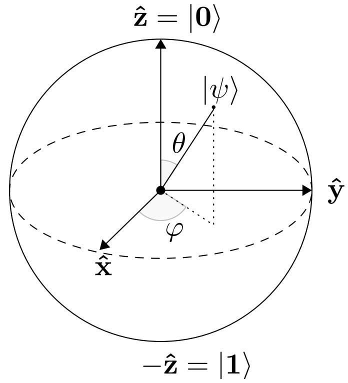
$$\xrightarrow{X\otimes I\otimes I} \frac{1}{2}(|100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

$$\xrightarrow{CNOT_{2,3}} \frac{1}{2}(|100\rangle + |101\rangle + |111\rangle + |110\rangle)$$

$$\xrightarrow{CNOT_{3,1}} \frac{1}{2}(|100\rangle + |001\rangle + |011\rangle + |110\rangle)$$

$$\xrightarrow{CNOT_{1,2}} \frac{1}{2}(|110\rangle + |001\rangle + |011\rangle + |100\rangle)$$

6. (2 points) Recall that in the Bloch sphere, a qubit has state $|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|1\rangle$ where θ is the angle the state makes with the positive z-axis and ϕ the angle it makes with the positive x-axis.



Implement a transformation that maps the $|0\rangle$ state to any point $\cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|1\rangle$ on the Bloch sphere using rotations around the x-, y-, and/or z-axes. Recall that the corresponding rotation matrices are defined as

$$R_x(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}, \qquad R_y(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}, \qquad R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

We can arrive at the point $|\psi\rangle$ by first rotating around the y axis θ degrees, then around the z axis ϕ degrees. In particular,

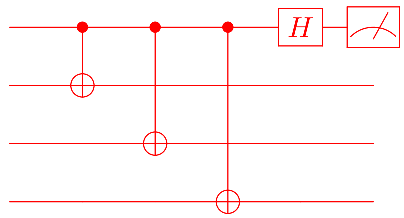
$$\begin{aligned} R_z(\varphi)R_y(\theta)|0\rangle &= R_z(\varphi)(\cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle) \\ &= e^{-i\varphi/2}\cos(\theta/2)|0\rangle + e^{i\varphi/2}\sin(\theta/2)|1\rangle \\ &= e^{-i\varphi/2}(\cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle) \\ &= |\psi\rangle \end{aligned}$$

where the last equality uses global phase invariance.

7. (4 points) Using $CNOT$ and H gates and computational-basis measurement, give a procedure to distinguish with **100% accuracy** between the following states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle), \qquad |\phi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle - |0011\rangle)$$

Apply the following circuit U:



and return $|\psi\rangle$ if the measurement result is $|0\rangle$, or $|\phi\rangle$ otherwise.

proof: Note that the first 3 $CNOT$ gates send $|1100\rangle$ to $|1011\rangle$ and $|0011\rangle$ to $|0011\rangle$. Hence the $CNOT$ gates map $|\psi\rangle$ to $\frac{1}{\sqrt{2}}(|1011\rangle + |0011\rangle) = |+\rangle \otimes |011\rangle$ and $|\phi\rangle$ to $\frac{1}{\sqrt{2}}(|1011\rangle - |0011\rangle) = -|-\rangle \otimes |011\rangle$. Therefore, after the final H gate the first qubit is in the state $|0\rangle$ if the original state was $|\psi\rangle$, and $|1\rangle$ if the original state was $|\phi\rangle$.

8. Suppose Alice and Bob share an EPR pair $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Suppose Bob applies a unitary U to his qubit and then keeps it in storage.

   (a) (4 points) Suppose a year has passed and Alice creates some qubit in the state $|\psi\rangle$. She then measures this qubit with her half of the EPR pair in the Bell basis to teleport it to Bob, and obtains measurement result β_{00}. What is the resulting state of Bob's qubit? If it helps, recall the definition of the Bell basis:

$$\begin{aligned} |\beta_{00}\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad & |\beta_{01}\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\beta_{10}\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad & |\beta_{11}\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}$$

   First we write the state of all 3 qubits with Alice's qubits in the Bell basis. Let $|\psi\rangle = a|0\rangle + b|1\rangle$.

$$\begin{aligned} |\psi\rangle \otimes (I \otimes U)|\beta_{00}\rangle &= (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes U|0\rangle + |1\rangle \otimes U|1\rangle) \\ &= \frac{1}{\sqrt{2}}(a|00\rangle \otimes U|0\rangle + a|01\rangle \otimes U|1\rangle + b|10\rangle \otimes U|0\rangle + b|11\rangle \otimes U|1\rangle) \\ &= \frac{1}{2}(a(|\beta_{00}\rangle + |\beta_{10}\rangle) \otimes U|0\rangle + a(|\beta_{01}\rangle + |\beta_{11}\rangle) \otimes U|1\rangle \\ &\quad + b(|\beta_{01}\rangle - |\beta_{11}\rangle) \otimes U|0\rangle + b(|\beta_{00}\rangle - |\beta_{10}\rangle) \otimes U|1\rangle) \\ &= \frac{1}{2}|\beta_{00}\rangle \otimes (aU|0\rangle + bU|1\rangle) + |\beta_{01}\rangle \otimes (aU|1\rangle + bU|0\rangle) \\ &\quad + |\beta_{10}\rangle \otimes (aU|0\rangle - bU|1\rangle) + |\beta_{11}\rangle \otimes (aU|1\rangle - bU|0\rangle) \end{aligned}$$

   So if Alice obtains measurement result $|\beta_{00}\rangle$ when measuring in the Bell basis, Bob has the state $aU|0\rangle + bU|1\rangle = U(a|0\rangle + b|1\rangle) = U|\psi\rangle$

   (b) (1 point) Argue whether or not the above could be considered to be a violation of **causality** — the notion that causes and effect occur in the order in which they happen.

   Yes, this could be viewed as a violation of causality, because it appears that the gate U which Bob applied to his qubit *before* $|\psi\rangle$ existed is actually applied *after* Alice creates the state $|\psi\rangle$. Of course, this must be preposterous and can't actually be what happens. What's more likely is that Bob's state $|\phi\rangle$ after teleportation is *related* to Alice's state $|\psi\rangle$ by the relation $|\phi\rangle = U|\psi\rangle$.

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