CMPT 476/981: Introduction to Quantum Algorithms
Assignment 1

Due **January 18, 2024 at 11:59pm on coursys**
Complete individually and submit in PDF format.

Question 1 [3 points]: A universal classical gate

The $NAND$ gate is a classical gate with the following truth table:

x	y	$NAND(x,y)$
0	0	1
0	1	1
1	0	1
1	1	0

1. Show that the NOT gate can be implemented with NAND gates and FANOUT. You may draw a circuit or simply give the algebraic expression.
2. Show that the gate set $\{NAND, FANOUT\}$ is universal for classical computation by giving implementations of each gate in the universal gate set $\{AND, OR, NOT, FANOUT\}$.

Solution. 1. $NAND(x,x) = \neg(x \wedge x) = \neg x \vee \neg x = \neg x = NOT(x)$

2. <u>NOT</u>: Part 1.

   <u>FANOUT</u>: Trivial

   <u>AND</u>: $NOT(NAND(x,y)) = \neg(\neg(x \wedge y)) = x \wedge y = AND(x,y)$

   <u>OR</u>: $NAND(NOT(x), NOT(y)) = \neg(\neg x \wedge \neg y) = \neg\neg x \vee \neg\neg y = x \vee y = OR(x,y)$

□

Question 2 [6 points]: Dirac notation

Let $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle + \frac{-1}{\sqrt{3}}|2\rangle$, $|\phi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{-i}{\sqrt{2}}|2\rangle$ be two states of a **qutrit** (i.e. a three-level or three-dimensional system).

1. Give the explicit column vectors of $|\psi\rangle$ and $|\phi\rangle$
2. Calculate the following:
   - $\langle\psi|\psi\rangle$

- $\langle\phi|\phi\rangle$
- $\langle\psi|\phi\rangle$
- $|\psi\rangle\langle\phi|$
- $|\psi\rangle \otimes |\phi\rangle$

3. Is the vector $|\psi\rangle + |\phi\rangle$ a unit vector? If not, normalize it to get a unit vector.

Solution. 1.

$$|\psi\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} \qquad |\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\\ -i\end{bmatrix}$$

2. $\langle\psi|\psi\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & -i & -1\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} = \frac{1}{3}(1 + (-i)(i) + (-1)(-1)) = \frac{1}{3}(3) = 1$

   $\langle\phi|\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\\ -i\end{bmatrix} = \frac{1}{2}(0 + 1 + i(-i)) = \frac{1}{2}(2) = 1$

   $\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & i\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} = \frac{1}{\sqrt{6}}(0 + i - i) = 0$

   $|\psi\rangle\langle\phi| = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & i\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}0 & 1 & i\\ 0 & i & -1\\ 0 & -1 & -i\end{bmatrix}$

   $|\psi\rangle \otimes |\phi\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\\ -i\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}0\\ 1\\ -i\\ 0\\ i\\ 1\\ 0\\ -1\\ i\end{bmatrix}$

3. Let $|\theta\rangle = |\phi\rangle + |\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\\ -i\end{bmatrix} + \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{2}\\ \sqrt{3} + i\sqrt{2}\\ -\sqrt{2} - i\sqrt{3}\end{bmatrix}$

   Then $|\theta\rangle$ is not a unit vector since

   $\langle\theta|\theta\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{2} & \sqrt{3} - i\sqrt{2} & -\sqrt{2} + i\sqrt{3}\end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{2}\\ \sqrt{3} + i\sqrt{2}\\ -\sqrt{2} - i\sqrt{3}\end{bmatrix} = \frac{1}{6}(2 + (2 + 3) + (2 + 3)) = 2$

   So $\||\theta\rangle\| = \sqrt{2}$, so that by normalizing we have $|\theta'\rangle = \frac{1}{\sqrt{2}}|\theta\rangle$ is a unit vector.

$\square$

Question 3 [4 points]: Gates and measurement

Suppose we have a qubit initially in the state $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ for some $\theta \in \mathbb{R}$.

1. Calculate the probabilities of receiving result "0" or "1" if the qubit is measured.
2. Recall the definition of the Hadamard gate, which has the vectors $|+\rangle$ and $|-\rangle$ as its columns:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If we first apply the Hadamard gate to the initial state $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ and then measure, what are the probabilities of receiving the "0" and "1" results as an function of θ?
Note: this is the same thing as measuring the initial state in the $|+\rangle, |-\rangle$ basis.

Solution. 1. p(0) $= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

p(1) $= |\frac{e^{i\theta}}{\sqrt{2}}|^2 = \frac{e^{i\theta}e^{-i\theta}}{2} = \frac{1}{2}$

2. Let $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$

Then $H|\phi\rangle = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + e^{i\theta} \\ 1 - e^{i\theta} \end{bmatrix}$

$$\begin{aligned} p(0) &= |\frac{1 + e^{i\theta}}{2}|^2 \\ &= \frac{1}{4}(1 + \cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta) \\ &= \frac{1}{4}(1 + \cos\theta - i\sin\theta + \cos\theta + \cos^2\theta - i\cos\theta\sin\theta + i\sin\theta + i\cos\theta\sin\theta + \sin^2\theta) \\ &= \frac{1}{4}(2 + 2\cos\theta) \\ &= \frac{1}{2} + \frac{1}{2}\cos\theta \end{aligned}$$

and $p(1) = 1 - p(0) = \frac{1}{2} - \frac{1}{2}\cos\theta$
Indeed when $\theta = 0, |\phi\rangle = |+\rangle$ so that $p(0) = 1, p(1) = 0$ and likewise $\theta = \pi, |\phi\rangle = |-\rangle$ so that $p(0) = 0, p(1) = 1$ which is consistent with measurement in the $|+\rangle, |-\rangle$ basis.

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Question 4 [5 points]: Eigenvectors

Recall that an *eigenvector* of a matrix A is a vector $|v\rangle$ such that $A|v\rangle = \lambda|v\rangle$ for some scalar *eigenvalue* λ.

1. Let $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Find two **unit** vectors $|+_Y\rangle$, $|-_Y\rangle$ such that

$$\begin{aligned} Y|+_Y\rangle &= |+_Y\rangle \\ Y|-_Y\rangle &= -|-_Y\rangle \end{aligned}$$

2. Let U be the 2 by 2 matrix with columns $|+_Y\rangle$ and $|-_Y\rangle$. Is U unitary?

3. Calculate $U^\dagger Y U$.

Solution. 1. We compute eigenvectors for eigenvectors $1, -1$ in the usual way by computing the nullspace of $Id - Y$, $Id + Y$ respectively.

eigenvector $|+_Y\rangle$ Want $v = \begin{bmatrix} s \\ t \end{bmatrix}$ such that

$$\begin{aligned} (Id - Y)v &= 0 \\ \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} v &= 0 \\ \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} v &= 0 \end{aligned}$$

which gives $s + it = 0$, so choose solution $s = 1, t = i$ and normalize.

$|+_Y\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

eigenvector $|-_Y\rangle$ Similarly we can compute $|-_Y\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

2. Let $U = |+_Y\rangle\langle 0| + |-_Y\rangle\langle 1|$

$UU^\dagger = (|+_Y\rangle\langle 0| + |-_Y\rangle\langle 1|)(|0\rangle\langle +_Y| + |1\rangle\langle -_Y|) = |+_Y\rangle\langle +_Y| + |-_Y\rangle\langle -_Y| = Id$

since $\{|+_Y\rangle, |-_Y\rangle\}$ are an orthonormal basis for $\mathbb{C}^2$.

$U^\dagger U = (|0\rangle\langle +_Y| + |1\rangle\langle -_Y|)(|+_Y\rangle\langle 0| + |-_Y\rangle\langle 1|) = |0\rangle\langle 0| + |1\rangle\langle 1| = Id$

Since $U^\dagger U = UU^\dagger = Id$, U is unitary.

3. We compute the action of $U^\dagger Y U$ on $|0\rangle, |1\rangle$.

$U^\dagger Y U|0\rangle = U^\dagger Y|+_Y\rangle = U^\dagger|+_Y\rangle = |0\rangle$

$U^\dagger Y U|1\rangle = U^\dagger Y|-_Y\rangle = U^\dagger\left(-|-_Y\rangle\right) = -|1\rangle$

So $U^\dagger Y U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\square$