CNOT-dihedral circuits

Another class of circuits which can be **efficiently represented** and **optimally** (in some cases) synthesized are **CNOT-dihedral circuits**

Def'n (CNOT-dihedral)

CNOT-dihedral circuits of order n are those over $\{X_n, CNOT_n, R_n = R_z(\pi/n)\}$

Fact

$\{X_n, R_n\}$ generate the **dihedral group** of order n, D_n

Fact

CNOT and X together generate the **affine** permutations

$$(A, \vec{b}) \in GA(n, \mathbb{Z}_2) \simeq GL(n, \mathbb{Z}_2) \ltimes \mathbb{Z}_2^n$$
$$(A, \vec{b})\vec{x} = A\vec{x} \oplus \vec{b}$$

E.g. $X|x\rangle = |x \oplus 1\rangle = |(I, 1)x\rangle$

Intuitively then, CNOT-dihedral circuits compute an affine on basis states, together with a **phase** which is a sum of phases on affine combinations of basis states

e.g.  sends $|x\rangle|y\rangle \xrightarrow{CNOT} |x\rangle|x \oplus y\rangle$
$\xrightarrow{T} \omega^{x \oplus y}|x\rangle|x \oplus y\rangle$

 sends $|x\rangle \xrightarrow{X} |x \oplus 1\rangle$
$\xrightarrow{T} \omega^{x \oplus 1}|x \oplus 1\rangle$

Def^n

Given $\vec{x}, \vec{y} \in \mathbb{Z}_2^n$, $\chi_{\vec{y}}(\vec{x}) = \vec{x} \cdot \vec{y} = x_1 y_1 \oplus \cdots \oplus x_n y_n$

Representation of CNOT-dihedral groups

Prop (phase polynomial representation)

Let C be a circuit over $\{X, CNOT, R_m\}$. Then

$$[\![C]\!] : |\vec{x}\rangle \longmapsto e^{2\pi i/m\, P(\vec{x})} |A\vec{x} + \vec{b}\rangle$$

where $A \in GL(n, \mathbb{Z}_2)$, and $P : \mathbb{Z}_2^n \to \mathbb{Z}_m$ can be written as

$$P(\vec{x}) = \sum_{\vec{y}} a_{\vec{y}}\, x_{\vec{y}}(\vec{x}), \quad a_{\vec{y}} \in \mathbb{Z}_m$$

We call P a **phase polynomial** and $(P, A, \vec{b})$ a **phase polynomial representation** of C

Pf.

By induction on the gates of g. Let

$$[\![C]\!] : |\vec{x}\rangle \longmapsto e^{2\pi i/m\, P(\vec{x})} |A\vec{x} + \vec{b}\rangle$$

Then $[\![X_i \cdot C]\!]|\vec{x}\rangle \longmapsto e^{2\pi i/m\, P(\vec{x})}(X_i |A\vec{x} + \vec{b}\rangle)$

$$\longmapsto e^{2\pi i/m\, P(\vec{x})} |A\vec{x} + (\vec{b} + \vec{e}_i)\rangle$$

$$[\![CX_{ij} \cdot C]\!]|\vec{x}\rangle \longmapsto e^{2\pi i/m\, P(\vec{x})}(CX_{ij} |A\vec{x} + \vec{b}\rangle)$$

$$\longmapsto e^{2\pi i/m\, P(\vec{x})} |E_{ij} A\vec{x} + E_{ij}\vec{b}\rangle$$

$$[\![(R_k)_i \cdot C]\!]|\vec{x}\rangle \longmapsto e^{2\pi i/m\, P(\vec{x})}((R_k)_i |A\vec{x} + \vec{b}\rangle)$$

$$\longmapsto e^{2\pi i/m\, P(\vec{x})}\, e^{2\pi i/m (A_i \vec{x} \oplus b_i)} |A\vec{x} + \vec{b}\rangle$$

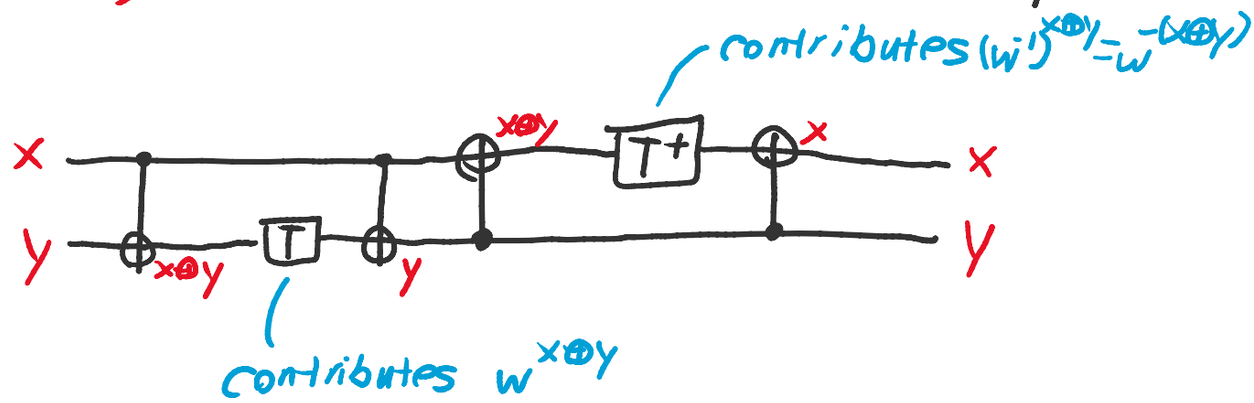
$$\longmapsto e^{2\pi i/m (P(\vec{x}) + (-1)^{b_i} x_{A_i}(\vec{x}) + b_i)} |A\vec{x} + \vec{b}\rangle$$

Note that $(-1)^{b_i} x_{A_i}(\vec{x}) + b_i$ is itself a prose polynomial

Annotated circuits

We can find a Phase polynomial representation by annotating the states of qubits (in the comp. basis)

E.g.



Hence,

$$|x\rangle|y\rangle \longmapsto w^{x\oplus y} w^{-(x\oplus y)} |x\rangle|y\rangle = |x\rangle|y\rangle$$

Prop.

Given a circuit C over $\{X, CNOT, R_m\}$, a phase polynomial representation of C can be computed in time and space linear in the volume of C,

$$|C| = (\text{num qubits of } C) \cdot (\text{num gates of } C)$$

We call the bitstrings $\vec{y} \in \mathbb{Z}_q^n$ where $q_{\vec{y}} \neq 0$ the Support of a phase polynomial P, denoted $\text{supp}(P)$. While phase polynomials are NOT unique, e.g.

$$\sum_{\vec{y}} x_{\vec{y}}(\vec{x}) \equiv 0 \bmod 8 \quad \forall \vec{x}$$

The canonical phase polynomial of C, obtained by annotating the circuit and collecting the phases satisfies

$$|\text{supp}(P)| \leq \#\, R_n \text{ gates in } C$$

Synthesis of CNOT-dihedral ops

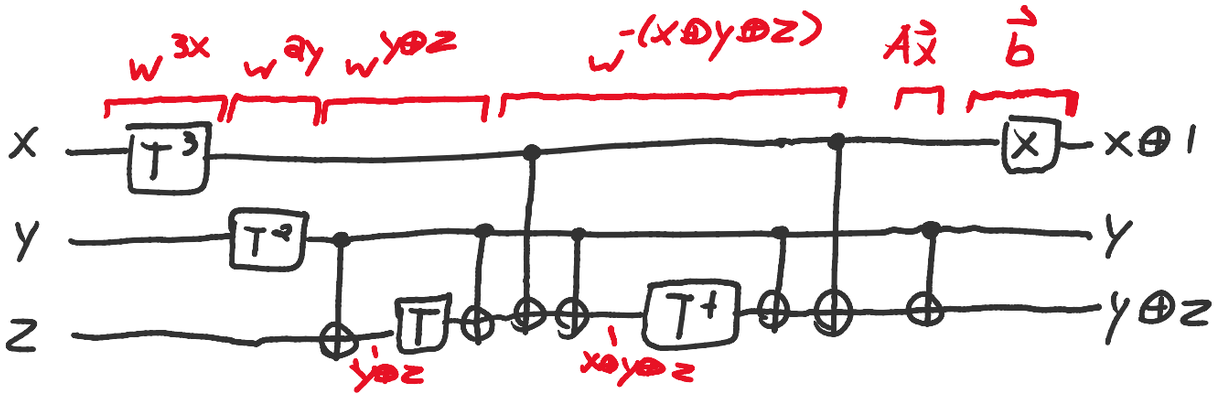
Intuitively, a phase polynomial representation $(P, A, \vec{b})$ corresponds to a sequence of phase rotations on parities of $\vec{x}$, followed by a final Affine permutation, so we can easily synthesize such a circuit.

E.g.

Let $U:|xyz\rangle \longmapsto \omega^{3x+2y+(y\oplus z)-(x\oplus y\oplus z)}|(x\oplus 1)y(y\oplus z)\rangle$

($e^{\pi i/4}$)

We can implement U over $\{CNOT, X, T = R_8\}$ as so:



Prop

Let $P(\vec{x}) = \sum_{\vec{y}} a_{\vec{y}} x_{\vec{y}}(\vec{x})$, $a_{\vec{y}} \in \mathbb{Z}_m$ and $(A, \vec{b}) \in GA(n, \mathbb{Z}_2)$.

Then $U:|\vec{y}\rangle \longmapsto e^{2\pi i/m\, P(\vec{y})}|A\vec{x}+\vec{b}\rangle$ can be synthesized over $\{CNOT, X, R_m\}$ in time polynomial in n and with $|\mathrm{supp}(P)|$ R_n^k gates, $k \in \mathbb{Z}_m$

Note:

If $m = 2^8$, i.e. $R_m = T$, then any even power of T is a Clifford gate, and any odd power needs only 1 T gate to implement, since $T^{2k+1} = T \cdot T^{2k}$

Hence, T-count $= |\mathrm{supp}(P \bmod 2)|$

Synthesizing the Toffoli gate

Recall that  , where is the CCZ gate defined by

$$CCZ : |xyz\rangle \longmapsto (-1)^{xyz} |xyz\rangle$$

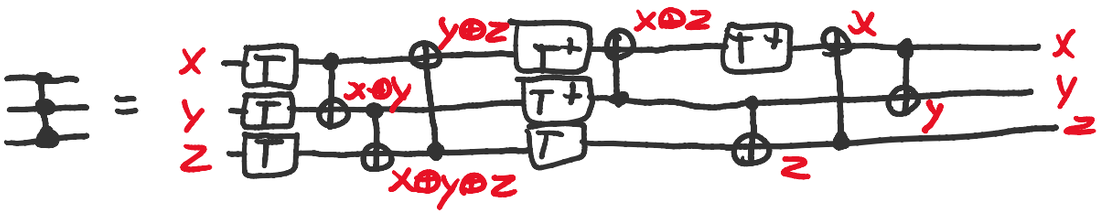
Taking the Fourier expansion* of $f(x,y,z) = xyz$ gives us

$$f(x,y,z) = \frac{1}{4}\left[x + y + z - (x \oplus y) - (x \oplus z) - (y \oplus z) + (x \oplus y \oplus z)\right]$$

↑ ↑ ↑ ↑ ↑ ↑ ↑

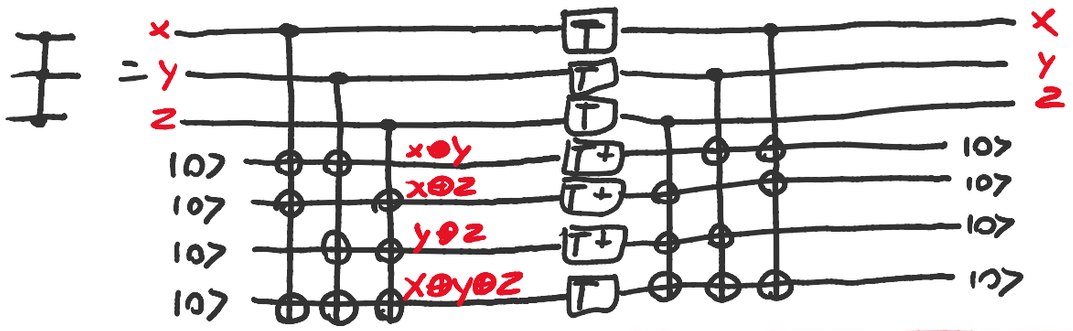
this is a phase polynomial over $\sqrt[4]{-1} = w = e^{\pi i/4}$

We can synthesize an explicit circuit for CCZ by preparing each parity and applying either a $w = e^{\pi i/4}$ or $w^{-1} = e^{\pi i/4}$ rotation (T or T^+).



(T-depth 1 CCZ)

Alternatively, we can synthesize the CCZ (and hence Toffoli gate) in T-depth 1 by using an ancilla to "hold" each parity at once



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* Fourier expansion: given $f : \mathbb{Z}_2^n \to \mathbb{R}$, an expression of f as $f(\vec{x}) = \sum_{\vec{y}} \hat{f}(\vec{y}) \chi_{\vec{y}}(\vec{x})$

The phase polynomial method

We now have the ingredients for a phase gate reducing algorithm, by re-synthesizing CNOT-dihedral circuits. In particular, note

1. The phase polynomial representation $(P, A, \vec{b})$ of C can be computed efficiently
2. $|supp(P)| \leq$ # phase gates in C
3. The phase polynomial representation $(P, A, \vec{b})$ can be synthesized efficiently with $|supp(P)|$ phase gates

(phase gate, e.g. T gate, optimization for $\{CNOT, X, R_m\}$)

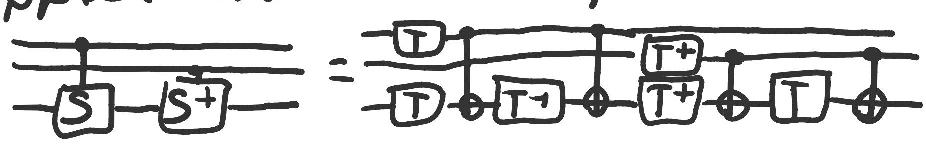
$$C \xrightarrow{\text{annotate}} (P, A, \vec{b}) \xrightarrow{\text{synthesize}} C'$$

$$R_m\text{-count}(C') \leq R_m\text{-count}(C)$$

profit ☺

(Example)

Suppose we want to optimize the circuit:

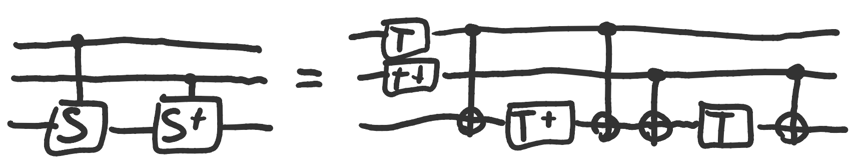


We start by computing the phase polynomial representation:

$$|xyz\rangle \longmapsto \omega^{x+z-(x\oplus z)-y-z+(y\oplus z)}|xyz\rangle$$

$$\longmapsto \omega^{x-(x\oplus z)-y+(y\oplus z)}|xyz\rangle$$

Re-synthesizing we get a circuit with 4 < 6 T gates.



Phase polynomial synthesis problems

More generally, we can ask what is the best circuit for $(P, A, \vec{b})$ in some cost metric? Synthesizing an optimal circuit for $(P, A, \vec{b})$ is known as a phase polynomial synthesis problem.

metric	result
R_m-depth	poly-time via Matroid partitioning (t-par)
R_m-count	equivalent to min-distance decoding for $RM^*(n, n-U_2(m)-1)$ (hard!)
CNOT-count	NP-hard in restricted cases asymptotically optimal heuristic (Gray-Synth)

Note: $U_2(m)$ is the 2-adic valuation of m (highest power of 2 that divides m)

(Upper & lower bounds on solutions)

metric	lower bound	upper bound
R_m-depth	$O(1)$ (with ancillas)	$O(1)$ (with ancillas)
R_m-count	Covering radius of $RM^*(n, n-U_2(m)-1)$	$O\left(\frac{n^{U_2(m)-1}}{U_2(m)!}\right)$ / $O(n^2)$ for T-count
CNOT-count	$O(n^2)$ (from gray code)	$O(n^2)$ (from gray code)

CNOT-minimal synthesis

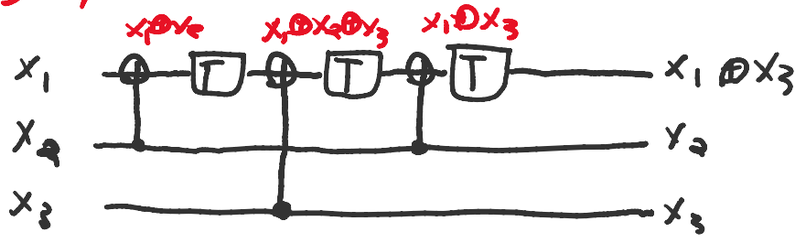
Problem:
Given a phase polynomial representation $(P, A, \vec{b})$, synthesize with minimal CNOT-count

Key idea:
we need to construct a tour of $\propto_{\vec{y}}(\vec{x})$ for each $\vec{y} \in \text{supp}\{P\}$ with the minimal CNOT-count

Ex.

Let $P(x_1, x_2, x_3) = x_1 \oplus x_2 + x_1 \oplus x_3 + x_1 \oplus x_2 \oplus x_3$

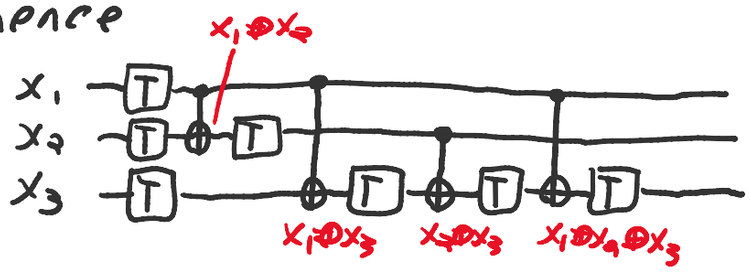
Since $\text{supp}\{P\} \simeq \mathbb{Z}_2^2$, we can construct a minimal tour using the gray code.



Ex.

Let $P(x_1, x_2, x_3) = x_1 + x_2 + x_3 + x_1 \oplus x_2 + x_1 \oplus x_3 + x_2 \oplus x_3 + x_1 \oplus x_2 \oplus x_3$

Now $\text{supp}\{P\} = \mathbb{Z}_2^3 \setminus \{0\}$. Any tour of $\text{supp}\{P\}$ hence requires at least $2^3 - 1$ CNOT gates. An optimal circuit is hence



Prop

Any set $X \simeq \mathbb{Z}_2^k$ (toured) can be enumerated optimally with $|X \setminus Y|$ CNOT gates where $Y \subseteq \mathbb{Z}_2^k$ is the set of inputs

Gray-synth

Basic idea

Attempt to decompose

$$S = \mathrm{supp}\{P\} = X_1 \cup X_2 \cup \cdots \cup X_m$$

where each $X_i \approx \mathbb{Z}_2^k$ for some k.

we do this by recursively selecting some qubit i which maximizes either the 0 or 1 co-factor,

$$i = \operatorname*{argmax}_i \max_{x \in \{0,1\}} |\{\vec{y} \in S \mid y_i = x\}|$$

Algorithm

queue

1. $C := []$, $Q := [(S, \{1, \ldots, n\}, e)]$
2. while $Q \neq []$
3. $\quad (S, I, j) = Q.\mathrm{pop}$
4. $\quad$ if $S = \emptyset$, then we're done
5. $\quad$ if $S = \{\vec{y}\}$, then
6. $\quad\quad$ for each j s.t. $\vec{y}_j = 1$
7. $\quad\quad\quad C := [CNOT_{j,i}] \mathbin{++} C$
8. $\quad\quad\quad$ for all remaining $\vec{y}'$, set $\vec{y}' := E_{i,j}\vec{y}'$
9. $\quad$ Else
10. $\quad\quad j := \operatorname{argmax}_{j \in I} \max_{x \in \{0,1\}} |\{\vec{y} \in S \mid y_j = x\}|$
11. $\quad\quad S_0 := \{\vec{y} \in S \mid y_j = 0\}$, $S_1 := \{\vec{y} \in S \mid y_j = 1\}$
12. $\quad\quad Q := [(S_0, I \setminus \{j\}, i \vee j), (S_1, I \setminus \{j\}, i \vee j)] \mathbin{++} Q$

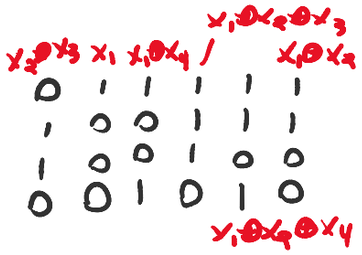
$(i \vee j = j$ if $i = e$, i otherwise$)$

Example

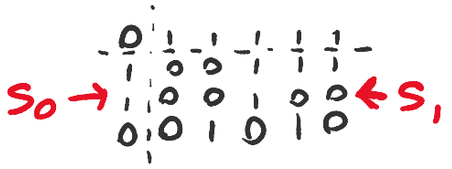
Suppose we wish to synthesize the phase polynomial

$$f(\vec{x}) = \frac{1}{8}\left[x_2 \oplus x_3 + x_1 + x_1 \oplus x_4 + x_1 \oplus x_2 \oplus x_3 + x_1 \oplus x_2 \oplus x_4 + x_1 \oplus x_2\right]$$

We can visualize Gray-synth as operating on a matrix with columns corresponding to each parity:



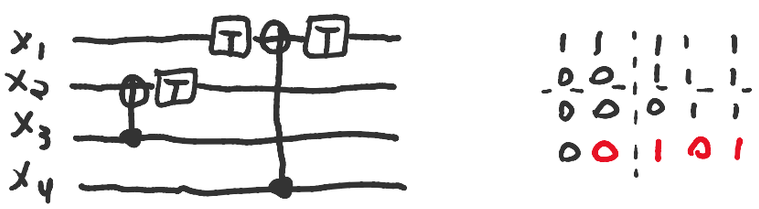
We first partition into S_0 & S_1 by the first row



To compute the only parity in S_0, $x_2 \oplus x_3$, we perform a CNOT between 3 & 2, then update the matrix

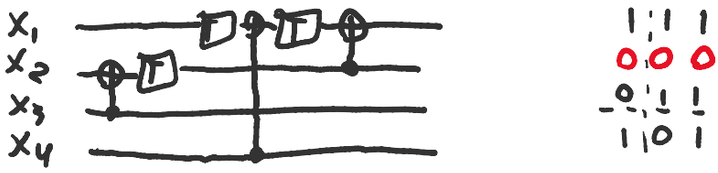
We're done with S_0 now, so we pop it off the queue and continue with the remaining, selecting row 2 to construct the next set of co-factors

The first parity of S_0 is already "computed", so we only need to perform a CNOT between row 4 & row 1:



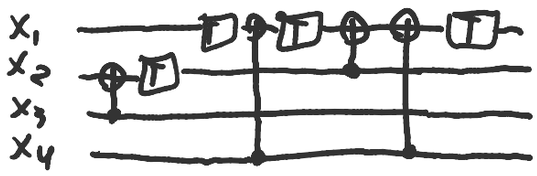
Example cont.

Continuing on with S_1, we zero out the second row with $CNOT_{2,1}$ then partition on row 3



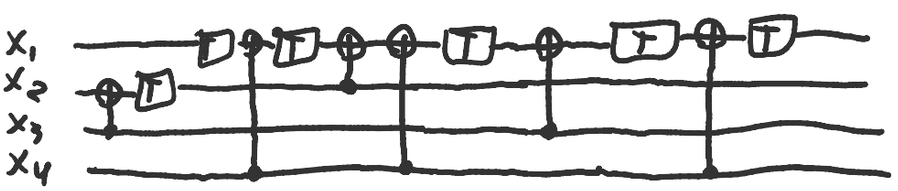
1 1 1
0 0 0
0 1 1
1 0 1

Next we $CNOT_{4,1}$ and move on to S_1



1 1
0 0
1 1
1 0

At this point we perform $CNOT_{3,1}$, and then divide into co-factors S_0 & S_1, which require 0 & 1 $CNOT_{n,1}$ resp

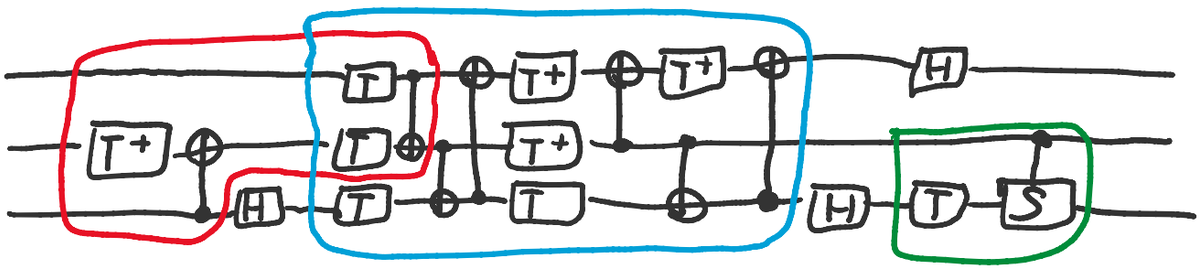


Choice of subcircuits

Even with optimal re-synthesis, different partitionings of a circuit may result in different costs.

E.g.

Consider the below circuit. There are 3 maximal CNOT-dihedral sub-circuits: 1, 2, & 3



Since 1 and 2 overlap, the order of re-synthesis matters — re-synthesizing 1 first may remove the gates in the intersection, but associating those gates with 2 instead could have led to better optimization.

One option is to mark gates in the intersection and try to find a global optimum over a set of synthesis problems. We're still just sweeping the floors on the Titanic though because the optimization fails to take into account the whole circuit. As in classical compilers, the best optimizations are whole program optimizations typically using static analysis to make it tractable.

Clifford+T representations

To represent a **whole program** over $\{H, CNOT, T\}$ using phase polynomials, we can note that

$$H : |x\rangle \longmapsto \frac{1}{\sqrt{2}} \sum_{x' \in \mathbb{Z}_2} (-1)^{xx'} |x'\rangle$$

In particular, $\frac{1}{\sqrt{2}} \sum_{x' \in \mathbb{Z}_2} (-1)^{xx'} |x'\rangle = \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$

$= |+\rangle$ if $x = 0$, $|-\rangle$ if $x = 1$

Noting that $(-1)^{xx'} = i^{x + x' - (x \oplus x')}$, we almost have a **phase polynomial representation**, but with one extra variable (x') which is summed over (+ norm. factor)

Prop

Any circuit C over $\{H, CNOT, R_m\}$ can be written as

$$\text{①} \quad [\![C]\!] : |\vec{x}\rangle \longmapsto \frac{1}{\sqrt{2}^k} \sum_{\vec{x}' \in \mathbb{Z}_2^k} e^{2\pi i \frac{1}{m} P(\vec{x}, \vec{x}')} |A(\vec{x}, \vec{x}') + \vec{b}\rangle$$

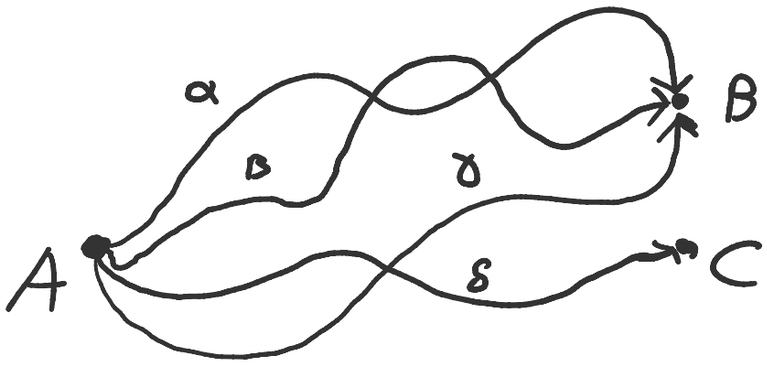
for some phase polynomial P and $A \in M_{n \times n+k}(\mathbb{Z}_2)$

(Sum-over-paths)

The representation ① is known as the (circuit) **sum-over-paths**. The summed variables x_i' are called **path variables** and can be thought of as identifying the branch taken (in superposition) at some **Hadamard** gate. The sum-over-paths corresponds to Feynman's **path integral formulation** of quantum mechanics.

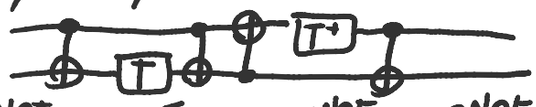
The Feynman path integral

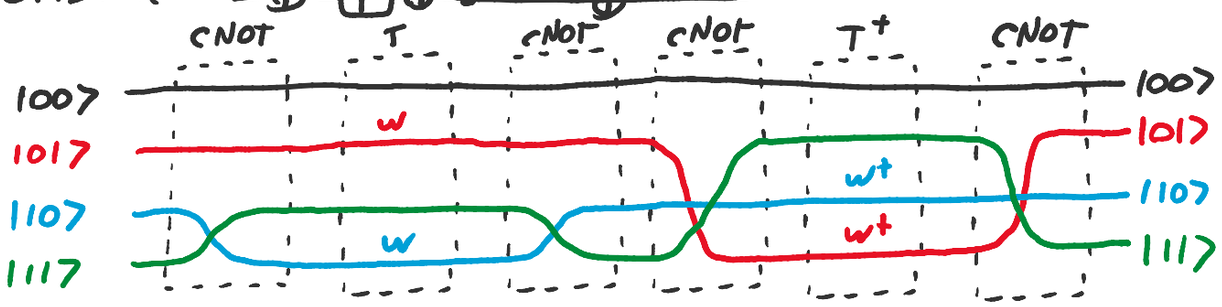
In Feynman's **path integral formulation** of QM, the evolution of a particle or system is viewed as the **sum/integral** over the different **(classical)** paths/evolutions it takes **in superposition.**



$$A \longmapsto (\alpha+\beta+\gamma)B + \delta C$$

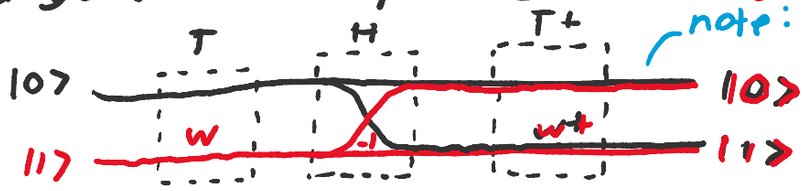
E.g.
We can analyze the evolution of circuits in the path integral style by tracking the evolution of an input basis state

Consider 



Each path ends in the state it started, and the red (01) and blue (10) paths both pick up a phase of $w \cdot w^\dagger = 1$, so the overall effect of the circuit is nothing.

E.g.
Hadamard gates correspond to **branching paths**



note: each path has magnitude $\frac{1}{\sqrt{2}}$

phases don't easily slide/commute!

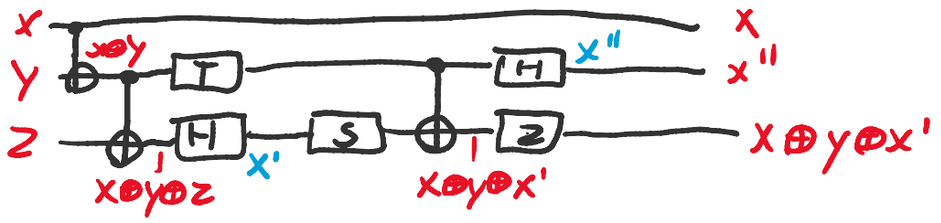
Annotated circuits

As with phase polynomial representations, the sum-over-paths can be determined by annotating

E.g.

Annotating ① basis state transformations, and
② branches,

the sum-over-paths representation of the following circuit can be read out by collecting the phase terms and summing over all branches:



SOP form: $|xyz\rangle \mapsto \frac{1}{2} \sum_{x',x''} \omega^{P(x,y,z,x',x'')} |x\, x''\, (x \oplus y \oplus x')\rangle$

$$P(x,y,z,x',x'') = x \oplus y + 4x'(x \oplus y \oplus z) + 2x' + 4(x \oplus y \oplus x') + 4|y''(x \oplus y)$$

$$= x \oplus y + 2x' + 2(x \oplus y \oplus z) + 6(x' \oplus x \oplus y \oplus z) + 2x'$$

$$4x + 4y + 4x' + 2x'' + 2(x \oplus y) + 6(x'' \oplus x \oplus y)$$

$$= x \oplus y + 2(x \oplus y \oplus z) + 6(x' \oplus x \oplus y \oplus z) + 4x + 4y + 2x'' + 2(x \oplus y) + 6(x'' \oplus x \oplus y)$$

This is tedious! Computers are much better

Prop.

The sum-over-paths representation of a Clifford+T circuit C can be computed in poly(|C|) time & space

Sum-over-paths re-synthesis

Synthesizing a circuit from a sum-over-paths is generically **hard** (at least **coNP-hard**)

Ex.

Consider a sum-over-paths with k distinct variables $\{x_i\}$ and phases **only of the form** $(-1)^{x_i x_j} = i^{x_i + x_j - (x_i \oplus x_j)}$. We can draw the phase polynomial as a **graph** where a phase of $(-1)^{x_i x_j}$ means an edge between x_i & x_j

$\longrightarrow (-1)^{x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4}$

we can implement phases by either

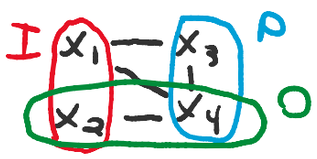
$CZ : |x_i\rangle|x_j\rangle \longmapsto (-1)^{x_i x_j}|x_i\rangle|x_j\rangle$, or

$H : |x_i\rangle \longmapsto \frac{1}{\sqrt{2}} \sum_{x_j} (-1)^{x_i x_j}|x_j\rangle$

Since H "**consumes**" an x_i, given subsets (I, P, O) of $\{x_i\}$ corr. to inputs, path variables, and outputs, synthesizing a circuit over $\{H, CZ\} \simeq \{H, CX, S\}$ amounts to finding an assignment $V(x_j) = x_i$ for all $x_j \in P$ s.t.

1. $V(x_j) = x_i \Rightarrow (x_i, x_j) \in E$
2. $V(x_j) = x_i = V(x_k) \Rightarrow x_j = x_k$
3. $V(\{x_i\}) \cap O = \emptyset$
4. $V(x_j) = x_i \Rightarrow x_i < x_j$ is a partial order

These constraints are unsatisfiable for the graph



But it can be satisfied by applying a **linear transform first**

$x_2 \to x_1 \oplus x_2$, i.e. $x_1 \to x_1$, $x_2 \to x_1 \oplus x_2$

Where do we go from here?

Classical program optimization also runs into inherent problems of intractability (or even uncomputability...). To solve this problem, they use static program analysis to prove properties which allow a compiler to safely perform an optimization.

Our final (lecture) topic will be the phase folding optimization which uses the sum-over-paths model to prove two phase gates can be merged into one.