

## “Experimental Data Processing”

### Topic 5

#### **"Model construction at state space under uncertainty"**

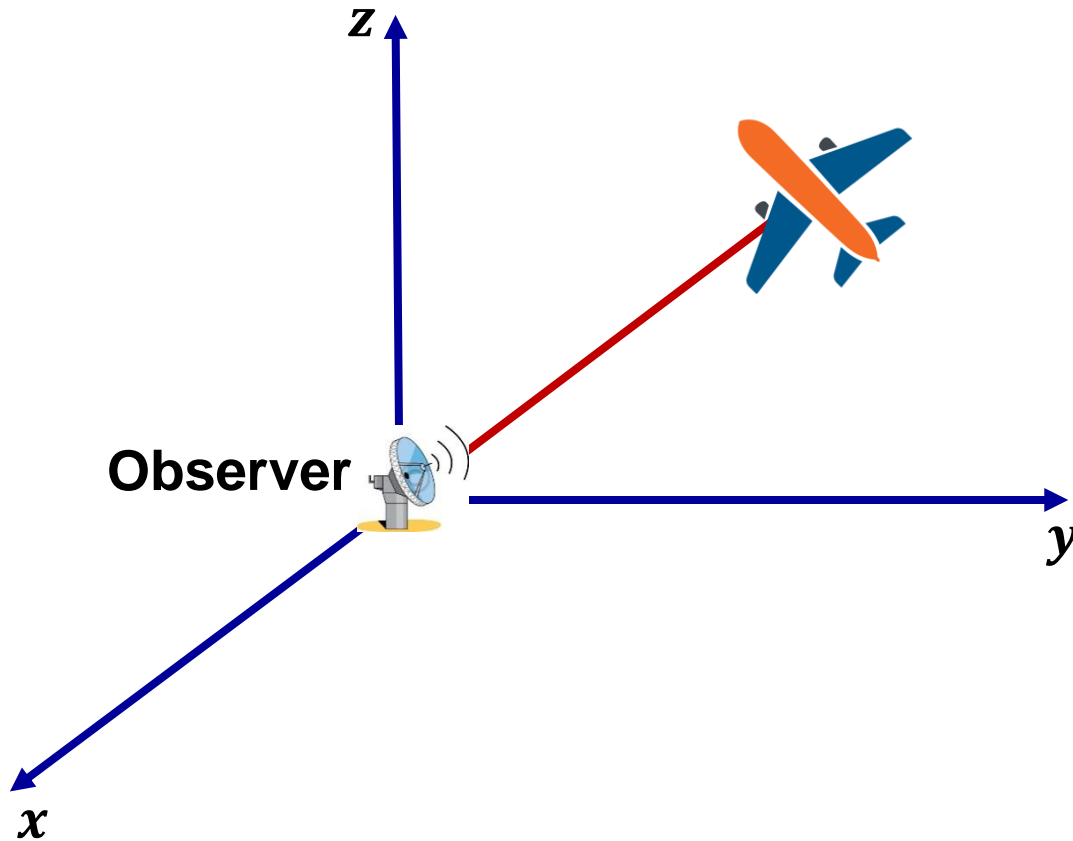
#### **I. Extended Kalman filter for navigation and tracking**

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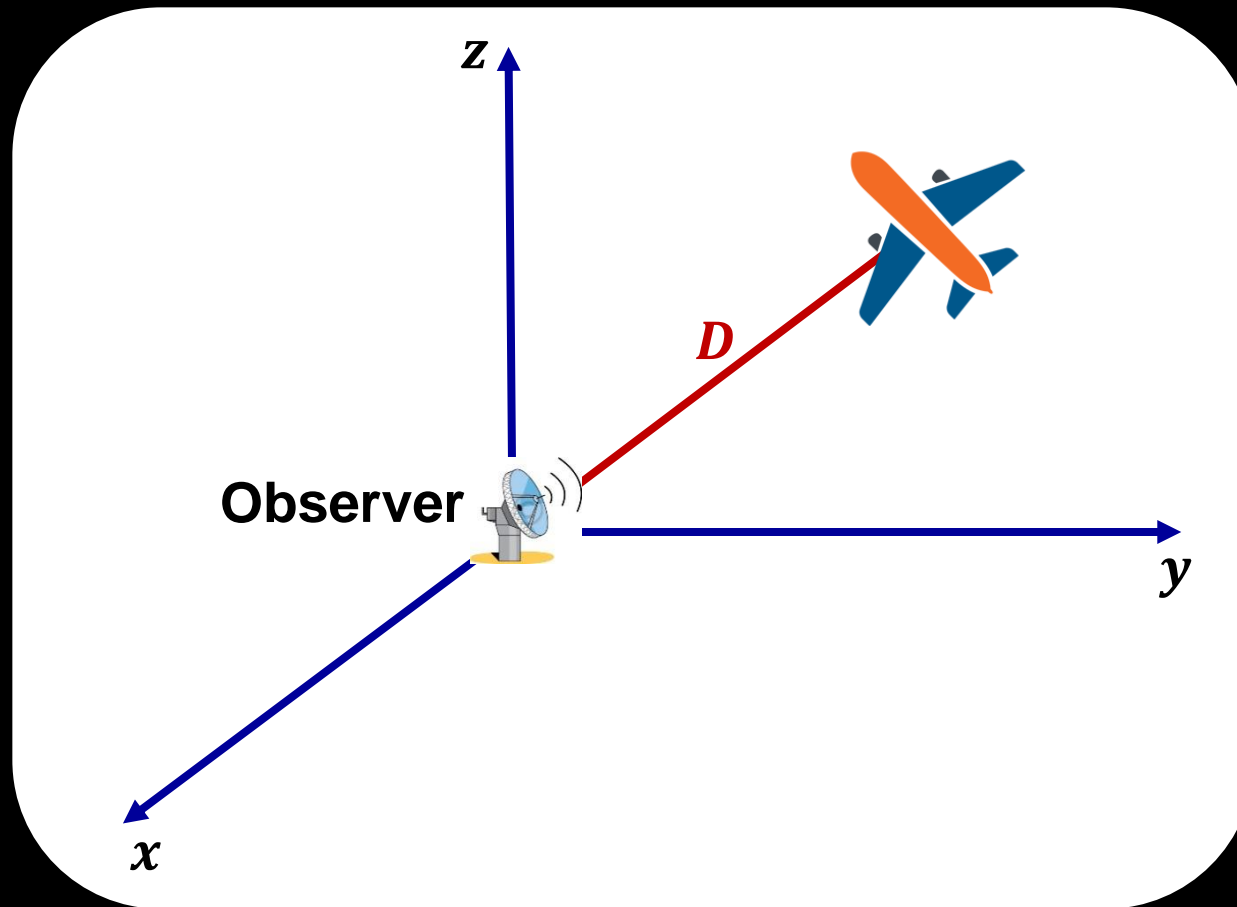
# State of a moving object is characterized by state vector in Cartesian coordinate system



State  
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

**D** 1. Estimation of coordinates using measurements of distance  $D$ , azimuth  $\beta$ , and angle of elevation  $\varepsilon$



State  
vector

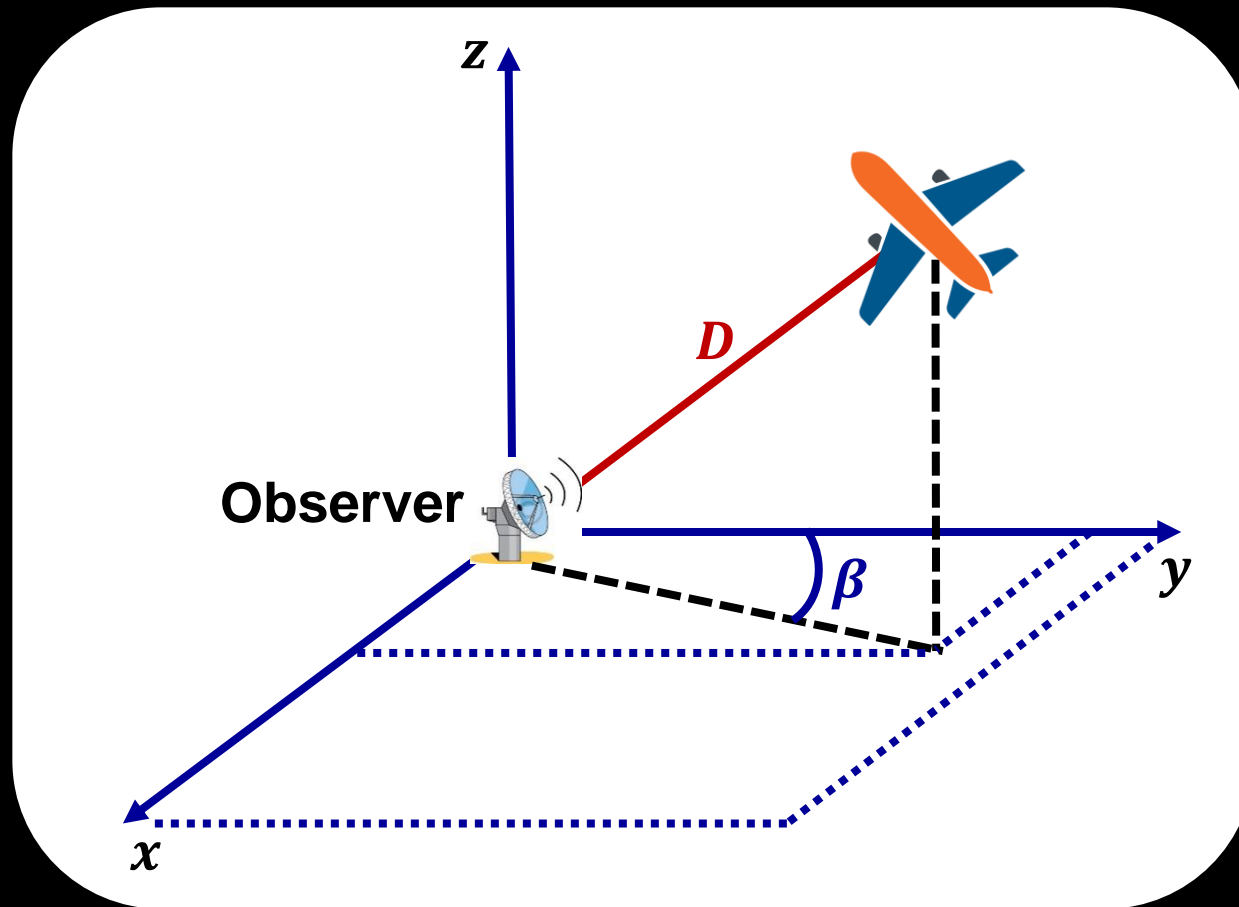
$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

Range  
 $D$

Distance from  
an observer to  
a moving object

$$D_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$$

**$\beta$**  1. Estimation of coordinates using measurements of distance  $D$ , azimuth  $\beta$ , and angle of elevation  $\varepsilon$



State  
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

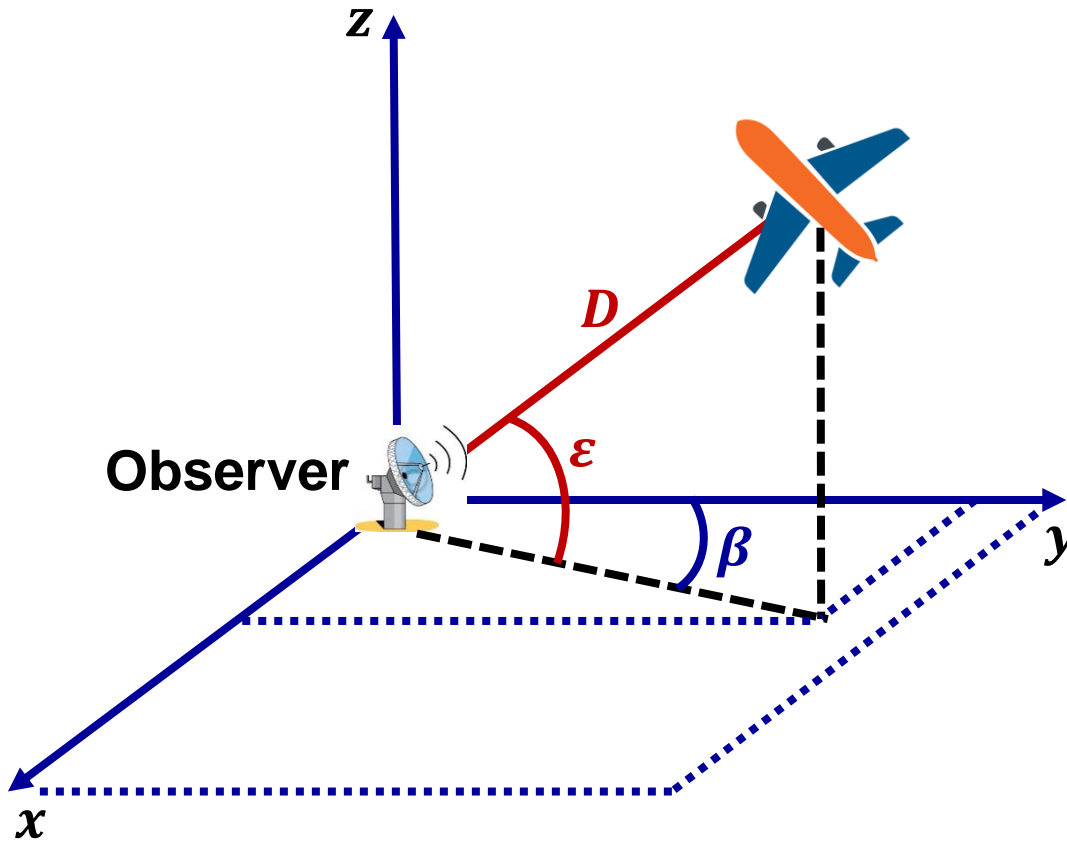
Azimuth  
 $\beta$

Angle between direction  
of North and projection  
line in horizontal plane

$$\beta_i = \arctg \left( \frac{x_i}{y_i} \right)$$

$\varepsilon$ 

# 1. Estimation of coordinates using measurements of distance $D$ , azimuth $\beta$ , and angle of elevation $\varepsilon$



State  
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

Angle of  
elevation  $\varepsilon$

Angle between the  
horizontal plane and  
direction of an object

$$\varepsilon_i = \arcsin \left( \frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}} \right)$$

# 1. Estimation of coordinates using measurements of distance $D$ , azimuth $\beta$ , and angle of elevation $\varepsilon$

Measurement equation

$$z_i = h(X_i) + \eta_i$$

Measurement  
vector  $z_i$

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \\ \varepsilon_i^m \end{bmatrix}$$

# 1. Estimation of coordinates using measurements of distance $D$ , azimuth $\beta$ , and angle of elevation $\varepsilon$

Measurement equation

$$z_i = h(X_i) + \eta_i$$

Measurement  
vector  $z_i$

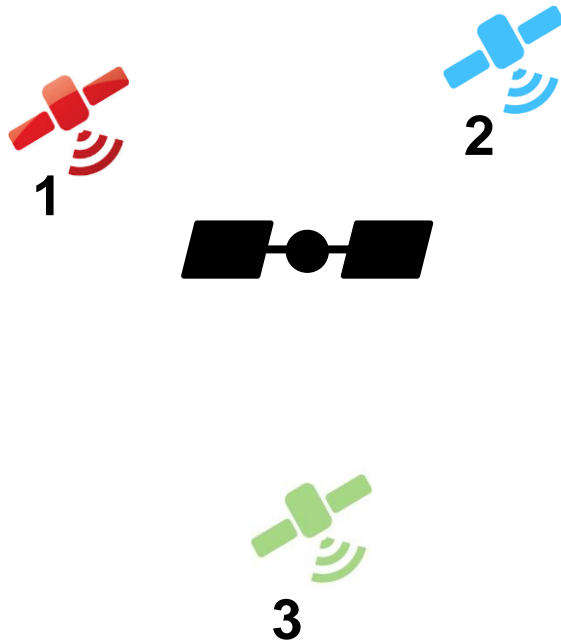
Nonlinear  
function  $h(X_i)$

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \\ \varepsilon_i^m \end{bmatrix}$$

$$h(X_i) = \begin{bmatrix} \sqrt{x_i^2 + y_i^2 + z_i^2} \\ \arctg\left(\frac{x_i}{y_i}\right) \\ \arcsin\left(\frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}}\right) \end{bmatrix}$$

## 2. Estimation of coordinates using measurements only of distance $D$

Three navigation stations  
measure distance  $D$   
to a moving object

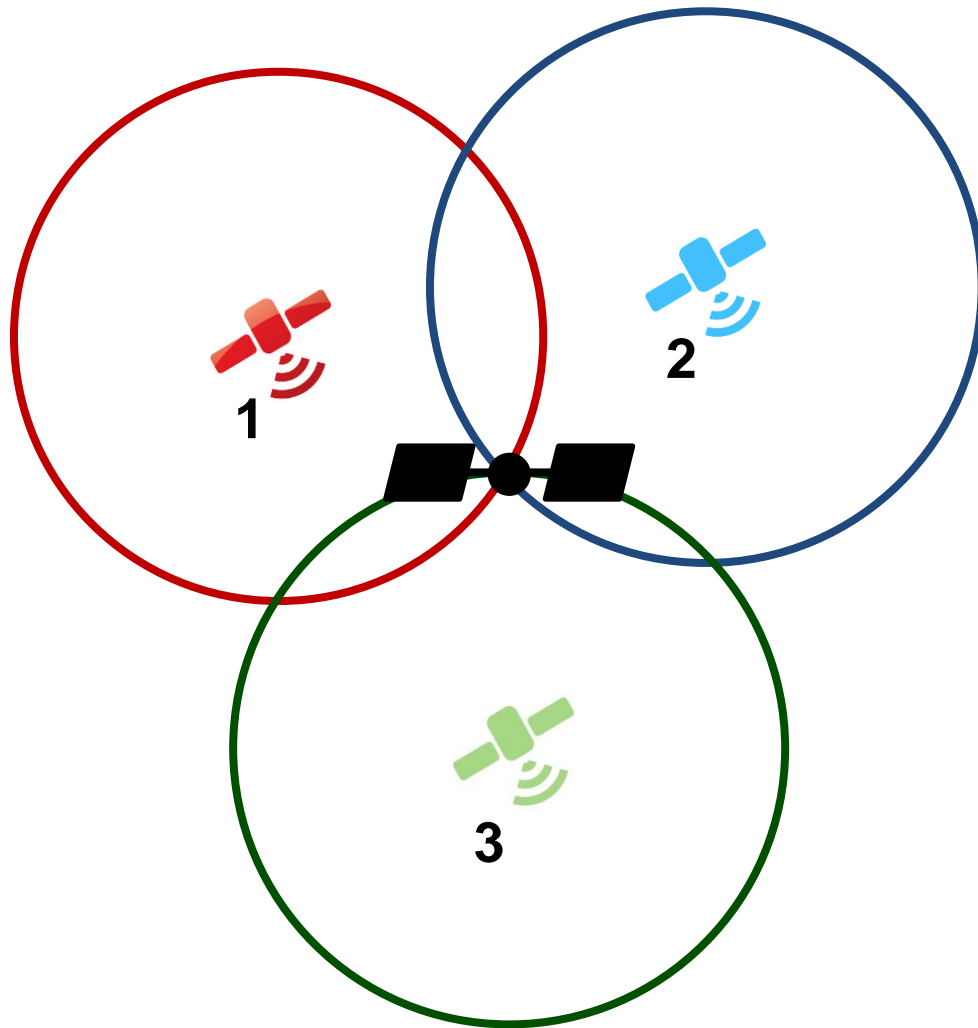


Moving object with  
unknown coordinates  $x, y, z$

Navigation stations with  
known coordinates  
 $(x_1, y_1, z_1); (x_2, y_2, z_2),$   
 $(x_3, y_3, z_3)$



## 2. Estimation of coordinates using measurements only of distance $D$



The position of a satellite is at the intersecting points of circles



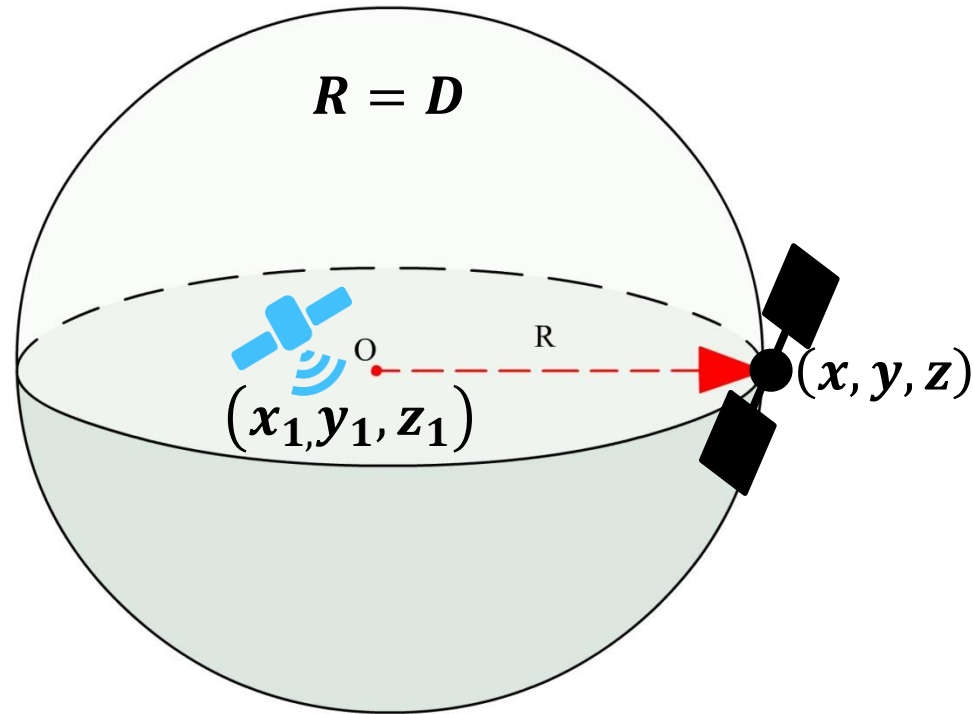
Moving object with unknown coordinates  $x, y, z$



Navigation stations with known coordinates  
 $(x_1, y_1, z_1); (x_2, y_2, z_2),$   
 $(x_3, y_3, z_3)$

## 2. Estimation of coordinates using measurements only of distance $D$

Equation  
of a 3-D sphere



Unknown coordinates  
 $x, y, z$  can be obtained  
by solving system  
of equations

$$\begin{aligned} D_1 &= \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \\ D_2 &= \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} \\ D_3 &= \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2} \end{aligned}$$

## 2. Estimation of coordinates using measurements only of distance $D$

Measurement equation

$$z_i = h(X_i) + \eta_i$$

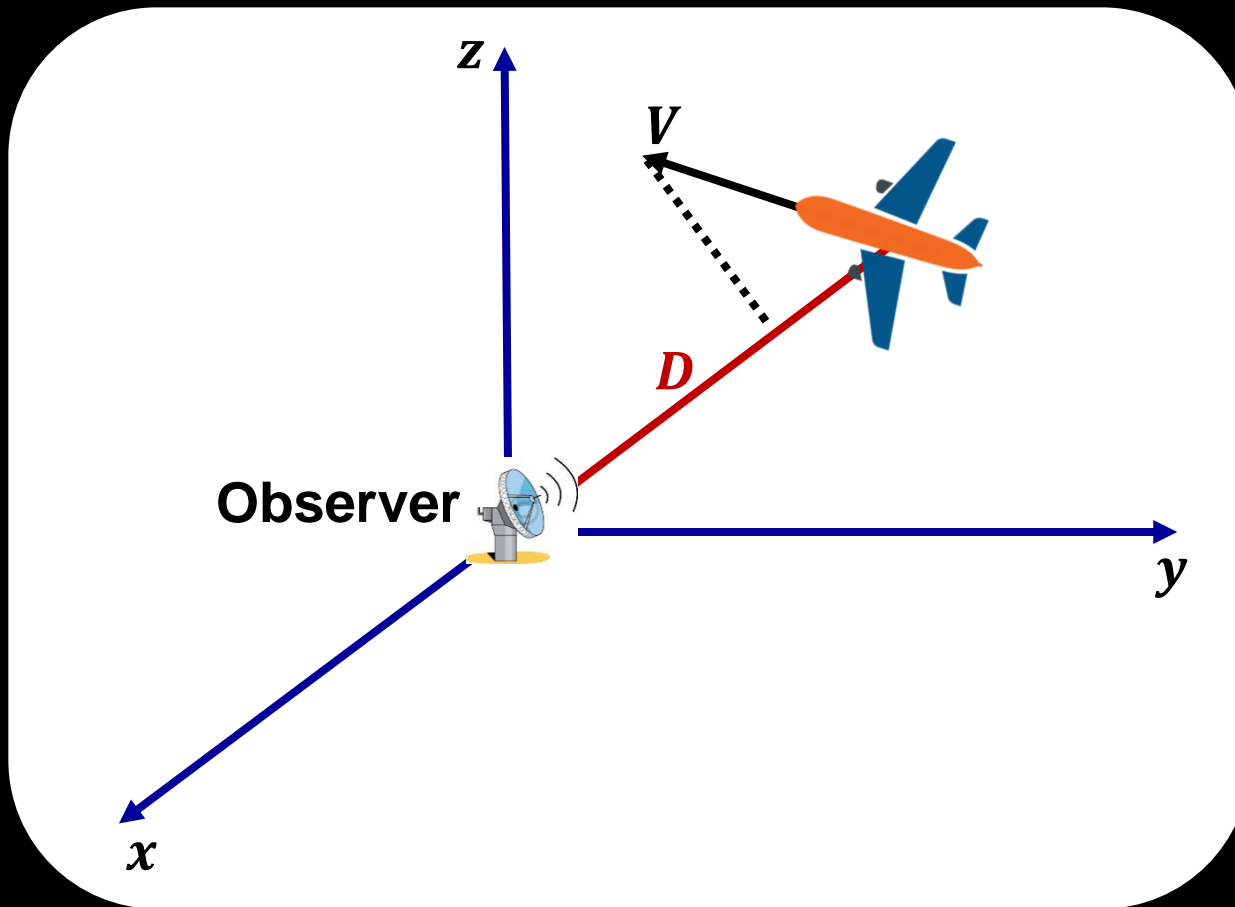
Measurement  
vector  $z_i$

$$z_i = \begin{bmatrix} D_1^m \\ D_2^m \\ D_3^m \end{bmatrix}$$

Nonlinear  
function  $h(X_i)$

$$h(X_i) = \begin{bmatrix} \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \\ \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} \\ \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2} \end{bmatrix}$$

### 3. Estimation of velocity using measurements of Doppler velocity



State  
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

Doppler  
velocity  $V_d$

Projection of  $V$  on  
vector  $D$  - radial  
component of  $V$

$$V_d = \frac{xV^x + yV^y + zV^z}{D}$$

### 3. Estimation of velocity using measurements of Doppler velocity

Measurement equation

$$z_i = h(X_i) + \eta_i$$

Measurement  
vector  $z_i$

$$z_i = |V_d^m|$$

Nonlinear  
function  $h(X_i)$

$$h(X_i) = \frac{xV^x + yV^y + zV^z}{\sqrt{x^2 + y^2 + z^2}}$$

## 4. Nonlinear model of a geostationary satellite orbit

System of differential equations in Celestial Reference System (CRS)

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{V} \\ \dot{\mathbf{V}} = -GM_o \frac{\mathbf{r}}{|\mathbf{r}|^3} + \mathbf{F}(\mathbf{r}, \mathbf{V}, t) \end{cases}$$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{V} = \begin{bmatrix} V^x \\ V^y \\ V^z \end{bmatrix}$$

Coordinates and components of velocity of a geostationary satellite in CRS

$$GM_o$$

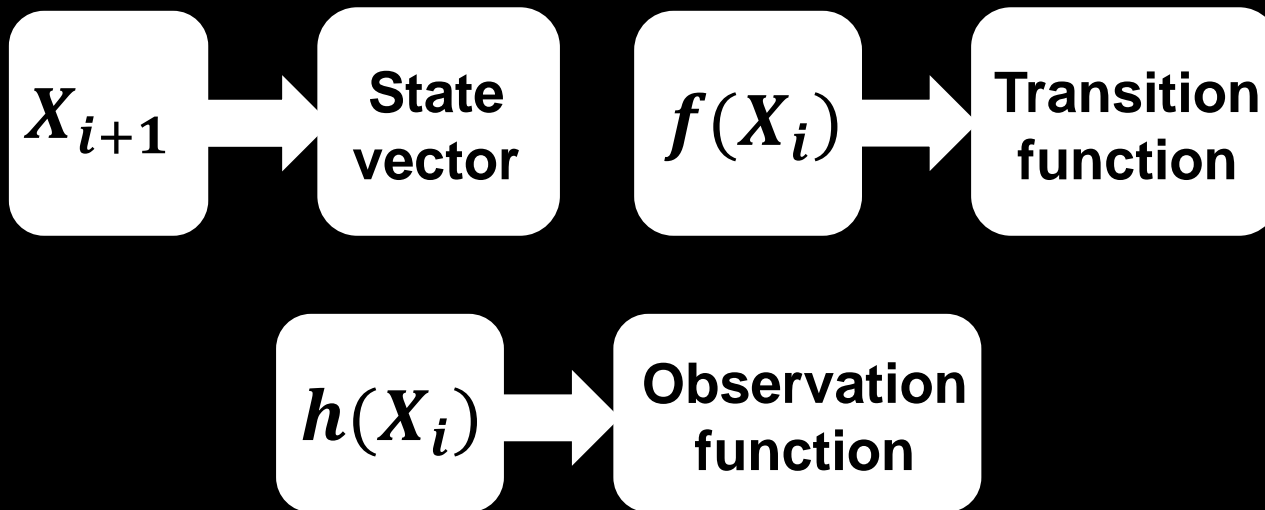
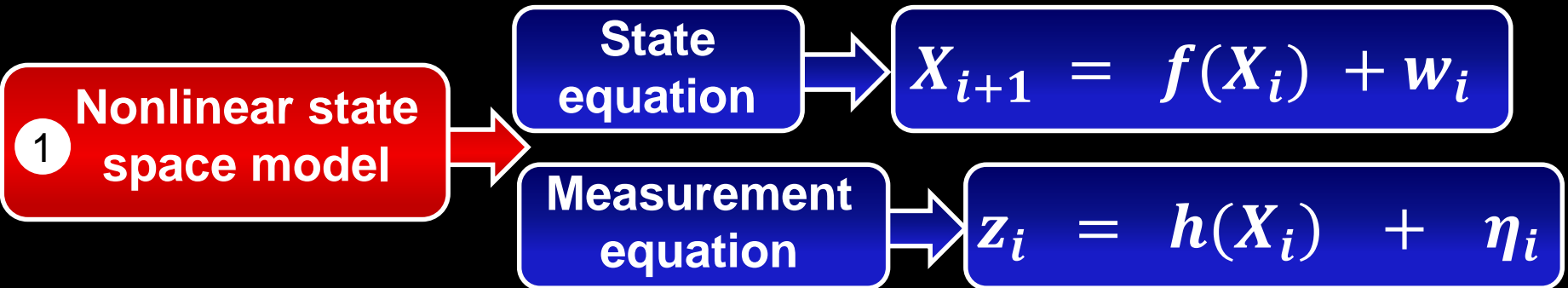
Gravitational constant multiplied by the Earth's mass

$$\mathbf{F}(\mathbf{r}, \mathbf{V}, t)$$

Disturbing acceleration related with:

- nonspherical gravity field of the Earth;
- gravitational disturbances of Moon and Sun;
- light pressure from Sun

# Extended Kalman filter



# Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$



Filtered and predicted estimates at time  $i$



# Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$



Filtered and predicted estimates at time  $i$

Let's produce Taylor series for  
 $f(X_i)$  and  $h(X_i)$  around estimates  $\hat{X}_{i,i}$  and  $\hat{X}_{i+1,i}$

# Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$

Filtered and predicted estimates at time  $i$

Let's produce Taylor series for  $f(X_i)$  and  $h(X_i)$  around estimates  $\hat{X}_{i,i}$  and  $\hat{X}_{i+1,i}$

State  
equation

$$f(X_i) \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i})$$

Measurement  
equation

$$h(X_{i+1}) \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_i} (X_{i+1} - \hat{X}_{i+1,i})$$

# Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$

Filtered and predicted estimates at time  $i$

Let's produce Taylor series for  $f(X_i)$  and  $h(X_i)$  around estimates  $\hat{X}_{i,i}$  and  $\hat{X}_{i+1,i}$

State  
equation

$$f(X_i) \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i})$$

Measurement  
equation

$$h(X_{i+1}) \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_i} (X_{i+1} - \hat{X}_{i+1,i})$$

Let's substitute these expressions for  $f(X_i)$  and  $h(X_{i+1})$  in state space model (1)

# Extended Kalman filter

**State  
equation**

$$X_{i+1} \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i}) + w_i$$

**Measurement  
equation**

$$z_{i+1} \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} (X_{i+1} - \hat{X}_{i+1,i}) + \eta_i$$

# Extended Kalman filter

State  
equation

$$X_{i+1} \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i}) + w_i$$

Measurement  
equation

$$z_{i+1} \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} (X_{i+1} - \hat{X}_{i+1,i}) + \eta_i$$

State  
equation

$$X_{i+1} \approx \frac{df(\hat{X}_{i,i})}{dX_i} X_i + w_i + f(\hat{X}_{i,i}) - \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i}$$

Measurement  
equation

$$z_{i+1} \approx \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} X_{i+1} + \eta_i + h(\hat{X}_{i+1,i}) - \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \hat{X}_{i+1,i}$$

Unknown terms

Known terms

# Extended Kalman filter

State  
equation

$$X_{i+1} \approx \frac{df(\hat{X}_{i,i})}{dX_i} X_i + w_i + u_i$$

Measurement  
equation

$$z_{i+1} \approx \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} X_{i+1} + \eta_{i+1} + y_{i+1}$$

Known values

$$u_i = f(\hat{X}_{i,i}) - \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i}$$

$$y_{i+1} = h(\hat{X}_{i+1,i}) - \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \hat{X}_{i+1,i}$$

# Recurrent algorithm of Kalman filter

## ① Prediction (extrapolation)

Prediction of state vector at time  $i$

$$\hat{X}_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i} + u_i$$

Prediction error covariance matrix

$$P_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} P_{i,i} \left( \frac{df(\hat{X}_{i,i})}{dX_i} \right)^T + Q_i$$

# Recurrent algorithm of Kalman filter

## ① Prediction (extrapolation)

Prediction of state vector at time  $i$

$$\hat{X}_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i} + u_i$$

Prediction error covariance matrix

$$P_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} P_{i,i} \left( \frac{df(\hat{X}_{i,i})}{dX_i} \right)^T + Q_i$$



More accurate prediction from state equation

$$\hat{X}_{i+1,i} = f(\hat{X}_{i,i})$$



# Recurrent algorithm of Kalman filter

## ② Filtration

### Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$\hat{X}_{i+1,i+1} = \hat{X}_{i+1,i} + K_{i+1}(z_{i+1} - h(\hat{X}_{i+1,i}))$$

Residual

Filter gain, weight of residual

$$K_{i+1} = P_{i+1,i} \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T \left[ \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) P_{i+1,i} \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T + R_i \right]^{-1}$$

Filtration error covariance matrix

$$P_{i+1,i+1} = \left[ I - K_{i+1} \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) \right] P_{i+1,i}$$

# Laboratory work 10

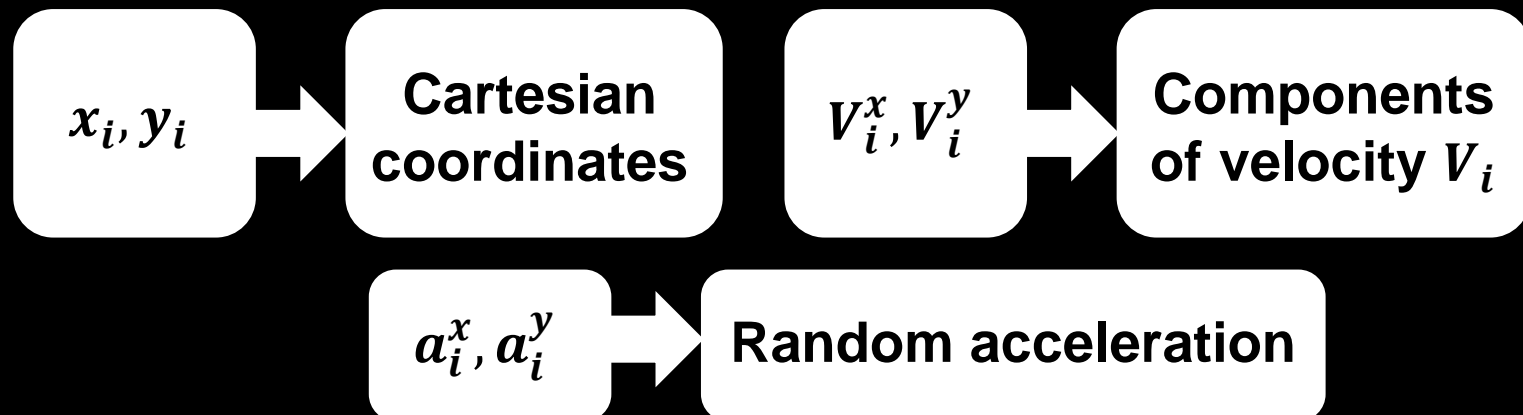
## Motion model is in Cartesian coordinate system

$$x_i = x_{i-1} + V_{i-1}^x T + \frac{a_{i-1}^x T^2}{2}$$

$$V_i^x = V_{i-1}^x + a_{i-1}^x T$$

$$y_i = y_{i-1} + V_{i-1}^y T + \frac{a_{i-1}^y T^2}{2}$$

$$V_i^y = V_{i-1}^y + a_{i-1}^y T$$



# State-space model, state equation

State  
equation

$$X_i = \Phi_{i,i-1}X_{i-1} + Ga_{i-1}$$

State  
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \end{bmatrix}$$

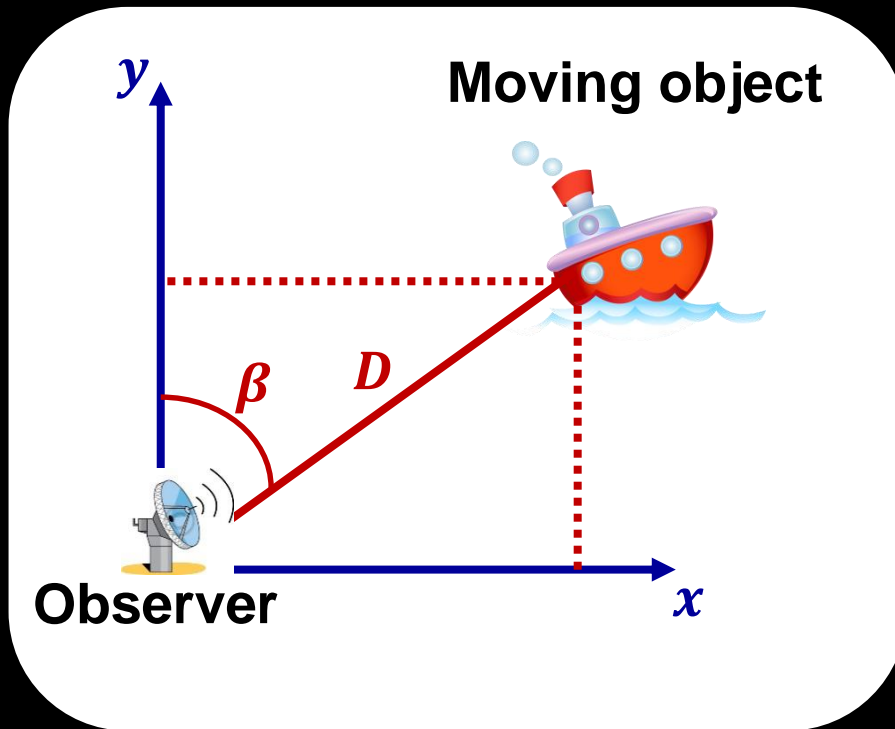
Transition  
matrix

$$\Phi_{i,i-1} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Input  
matrix

$$G = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix}$$

# State-space model, measurement equation



$$D = \sqrt{x^2 + y^2}$$

$$\beta = \arctg\left(\frac{x}{y}\right)$$

$$x = D \sin \beta$$
$$y = D \cos \beta$$

Measurement  
vector

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix}$$

$D_i^m$

Measurements of range  $D$

$\beta_i^m$

Measurements of azimuth  $\beta$

# State-space model, measurement equation

Measurement equation

$$z_i = h(X_i) + \eta_i$$

$$\eta_i = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Measurement  
vector  $z_i$

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix}$$

Nonlinear  
function  $h(X_i)$

$$h(X_i) = \begin{bmatrix} \sqrt{x_i^2 + y_i^2} \\ \arctg\left(\frac{x_i}{y_i}\right) \end{bmatrix}$$

# Recurrent algorithm of Kalman filter

## ① Prediction (extrapolation)

Prediction of state vector at time  $i + 1$  using  $i$  measurements

$$\hat{X}_{i+1,i} = \Phi_{i+1,i} \hat{X}_{i,i}$$

Prediction error covariance matrix

$$P_{i+1,i} = \Phi_{i+1,i} P_{i,i} \Phi_{i+1,i}^T + Q_i$$

$$P_{i+1,i} = E[(X_{i+1} - X_{i+1,i})(X_{i+1} - X_{i+1,i})^T]$$

$X_{i+1,i}$

First subscript  $i + 1$   
denotes time on which  
the prediction is made

Second subscript  $i$   
represents the number of  
measurements to get  $X_{i+1,i}$

# Recurrent algorithm of Kalman filter

## ② Filtration

### Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$\hat{X}_{i+1,i+1} = \hat{X}_{i+1,i} + K_{i+1}(z_{i+1} - h(\hat{X}_{i+1,i}))$$

Residual

Filter gain, weight of residual

$$K_{i+1} = P_{i+1,i} \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T \left[ \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) P_{i+1,i} \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T + R_i \right]^{-1}$$

Filtration error covariance matrix

$$P_{i+1,i+1} = \left[ I - K_{i+1} \left( \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) \right] P_{i+1,i}$$

# Recurrent algorithm of Kalman filter

Nonlinear  
function  $h(X_i)$

$$h(X_i) = \begin{bmatrix} \sqrt{x_i^2 + y_i^2} \\ \operatorname{arctg}\left(\frac{x}{y}\right) \end{bmatrix}$$

Derivative with respect  
to  $X_{i+1}$  at point  $\hat{X}_{i+1,i}$

$$\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} = \begin{bmatrix} \frac{x_{i+1,i}}{\sqrt{x_{i+1,i}^2 + y_{i+1,i}^2}} & 0 & \frac{y_{i+1,i}}{\sqrt{x_{i+1,i}^2 + y_{i+1,i}^2}} & 0 \\ \frac{y_{i+1,i}}{x_{i+1,i}^2 + y_{i+1,i}^2} & 0 & -\frac{x_{i+1,i}}{x_{i+1,i}^2 + y_{i+1,i}^2} & 0 \end{bmatrix}$$