Problem-solving

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In the class we have covered such topics as vector and matrix norms, quantum fidelity and established the relation between fidelity and norm of the purified states (Uhlmann's theorem).

Remark 0.1. Quantum fidelity is defines as

$$F(\rho, \sigma) = \operatorname{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}. \tag{1}$$

Problem 1. For two one-cubit (one spin) states ρ_a, ρ_b written as vectors on Block sphere,

$$\rho_a = \frac{1 + \vec{a} \cdot \vec{\sigma}}{2}, \quad \rho_b = \frac{1 + \vec{b} \cdot \vec{\sigma}}{2}, \tag{2}$$

calculate quantum fidelity in terms of $\vec{a}, \vec{b}$.

Problem 2. For the two states (2) prove

$$F(\rho_a, \rho_b) = \operatorname{Tr}(\rho_a \rho_b) + 2\sqrt{\det(\rho_a) \det(\rho_b)}. \tag{3}$$

Remark 0.2. Besides quantum fidelity, another very useful way to quantify how close two quantum states to each other is the so-called trace distance, defined as

$$T(\rho, \sigma) = \frac{1}{2} ||\rho - \sigma||_1 = \frac{1}{2} \operatorname{Tr} \sqrt{(\rho - \sigma)^2}. \tag{4}$$

Problem 3. Calculate trace distance $T(\rho_a, \rho_b)$ for two states (2).

Problem 4. Using the results of problems 1 and 2 find the inequalities satisfied by the trace distance and quantum fidelity

$$LHS(F(\rho_a, \rho_b)) \leq T(\rho_a, \rho_b) \leq RHS(F(\rho_a, \rho_b)), \tag{5}$$

where $LHS(x)$ and $RHS(x)$ are some algebraic functions.

Problem 5. Define the "angle" between two quantum states (density matrices) ρ, σ via

$$\Theta(\rho, \sigma) = \arccos F(\rho, \sigma). \tag{6}$$

Prove the triangle inequality

$$\Theta(\rho, \sigma) \leq \Theta(\rho, \tau) + \Theta(\sigma, \tau). \tag{7}$$

Problem 6. Consider two states ρ_I, ρ_{II} of a two-spin system. You can parametrize these states for example in this way,

$$\rho = \sum_{i,j=0}^{4} \rho_{ij}\, \sigma^i \otimes \sigma^j, \tag{8}$$

where $\sigma^0 = 1$ and $\rho_{00} = 1/4$. Introduce the reduced states

$$\tilde{\rho} = \mathrm{Tr}_2(\rho) = \frac{1 + 2\sum_{i=1}^{3} \rho_{i0}\, \sigma^i}{2}, \tag{9}$$

where the trace is over the "second" spin. Prove that quantum fidelity of the reduced states is higher than the fidelity of the original states

$$F(\tilde{\rho}_I, \tilde{\rho}_{II}) \geq F(\rho_I, \rho_{II}). \tag{10}$$