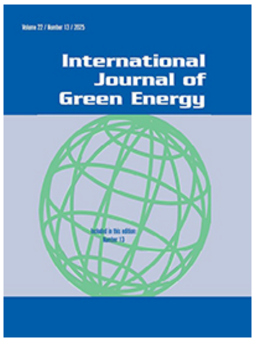

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Comparing the transition from fossil fuels to green energy in the United States and Africa

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ABSTRACT
Previous studies have rarely evaluated the effectiveness of the Power Africa policy. This study applies a three-dimensional Lotka-Volterra model to assess the short-term substitution of green energy for fossil fuels. Equilibrium analysis is used to compare the future trajectory predictions of green energy between Africa and the United States to identify what Africa needs to improve. We find that fossil fuel utilization leads to an increase in nuclear energy consumption in both the United States and Africa. Both regions replace fossil fuels with alternative clean energy sources during periods of inadequate electricity supply. Renewable and nuclear energy exhibit distinct relationship trends in the two regions. For the circular economy-based United States, renewable energy usage increases alongside nuclear energy. The United States tends to supplement power generation with renewable energy when nuclear power is insufficient. Conversely, in the linear economy-based Africa, renewable energy usage decreases as nuclear energy usage increases. This contrast highlights that the United States protects the environment by mitigating potential harm from nuclear waste and radiation. The equilibrium analysis results show a lower equilibrium ratio of green energy to fossil fuels in Africa than in the United States, since Africa lacks a stable circular economy system. The consumption of fossil fuels and nuclear energy in the United States ultimately decreases at specific equilibrium levels, indicative of restructuring the economy in an environmentally sustainable manner. However, the energy consumption of low-cost fossil fuels is predicted to increase in Africa. Implicitly, Africa should improve to develop more green power capacity.

KEYWORDS
Circular economy; equilibrium; green energy; kuznets curve; renewable energy

Introduction

This study analyzes whether the United States and Africa differ in the short-term correlation and long-term equilibrium among fossil fuel, nuclear energy, and renewable energy because of the diverse levels of industrialization in the United States and African nations. Moss and Bazilian (2018) indicated that the "Power Africa" signed by the United States in June 2013 is motivated to enhance the use of green energy for power generation across Africa. Allela (2021) expressed that the United States President Obama revised the aim of "Power Africa" to further improve new and clean electric capacity in Africa. According to the literature above, the United States is actively promoting green energy power generation across Africa. Moss and Bazilian (2018) and Allela (2021) lack quantitative analysis. Therefore, the novelty of this study lies in comparing the differences in green energy power generation between Africa and the United States, in order to understand the effectiveness of the U.S.'s "Power Africa" initiative and explore the path for future improvements in green energy in Africa.

The Global Risks Report published by the World Economic Forum in 2013 warned that economic crises combined with extreme weather events, including rising sea levels, threaten the world's major coastal cities. Zhong and Haigh (2013) have contended that with the increased atmospheric CO_2 concentrations, the atmosphere will absorb more outgoing longwave radiation and trap more heat, leading to further climate warming. According to data from the Global Carbon Project, the primary source of CO_2 emissions is the combustion of fossil fuels. Al-Harbi, Shams, and Alhajri (2020) propose greenhouse gas emissions in urban environments, and Soytas, Sari, and Ewing (2007) have contended that CO_2 emissions are affected by energy consumption. Multiple studies have revealed that energy consumption from power generation produces large CO_2 emissions (Adu et al. 2024). Stern (2007) estimated that global warming would cause economic losses in global GDP per capita, ranging from 5.3% to 13.8% by 2200. Thus, Khan, Ponce, and Yu (2021) proposed that the Paris Climate Conference (COP: 21) was held in 2015. The signatory countries of post-Paris agreement promised to reduce carbon emissions and to achieve sustainability targets. Cumulative consumption statistics from the US Energy Information Administration database revealed that the United States accounts for more than 20% of global fossil fuel consumption. Therefore, the United States has a responsibility to help reduce CO_2 emissions for global carbon reduction. However, Chaturvedi et al. (2014) argued that profit-oriented power plants incur considerable expenses related to reducing CO_2 emissions, and Rinne (2019) further contended that the

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electricity demand is inelastic. Numerous studies have described the difficulty of reducing carbon emissions through electricity savings.

Czepło and Borowski (2024) have proposed that fossil fuels could be replaced with photovoltaic installations for electricity generation. Borowski and Karlikowska (2023) propose the innovative renewable energy "hydrogen" to achieve energy transformation and climate neutrality. This implementation can help reduce the greenhouse effect and achieve the climate agreement goals (Solarin, Bello, and Olabisi 2021). Low-carbon energy sources comprise renewable and nuclear energy sources. Renewable energy, which is also known as green energy, includes hydro energy, wind energy, solar thermal and photovoltaic energy, geothermal energy, and biomass energy. Biomass is composed of wood and wood-derived fuel, agricultural products, solid waste, and biofuels. Bretschger (2024) discussed the energy transition decarbonized with regenerative energies. Cohen and Caron (2018) argued that the widespread adoption of wind energy in the United States could replace fossil fuels for electricity generation. Price, Keppo, and Dodds (2023) found that the cost-effectiveness of nuclear power depends on the operating costs, construction costs, and competing technologies, as well as on the availability and feasibility of interconnector expansion. Deschenes, Malloy, and McDonald (2023) highlighted the impact of renewable portfolio standards on solar and wind capacity and generation. Chen (2024) examined the impacts of nuclear, fossil fuel, and renewable energy consumption amidst Germany's Energiewende and found that renewable energy increases Germany's Load Capacity Factor by 0.03%. However, Tsai, Chang, and Chang (2016) debated the feasibility of replacing fossil fuels with low-carbon energy sources. Similarly, Chapagai et al. (2024) analyzed the bottlenecks of clean energy development. Moreover, Lenzen (2008) and Sovacool (2008) have argued that the construction of nuclear power plants requires a large amount of steel and cement, and the manufacturing processes of steel and cement mainly account for the consumption of fossil fuels. The carbon reduction measures implemented by the United States and African countries that signed the Paris Agreement merit discussion. Thus, the first motivation of this study is to examine the short-term impact of renewable or nuclear energy to reduce fossil fuels.

Previous studies have further investigated the differences in environmental protection between developed and developing countries according to the environmental Kuznets curve. Kim, Suen, and Lin (2019) applied the environmental Kuznets curve to international trade and asserted that high-income countries tend to import products with high carbon emissions during production. Chishti, Dogan, and Zaman (2023) mentioned that a circular economy has a vital impact on the production and consumption of sustainable energy, which implies that economic policy is inseparable from energy policy. Yamaka, Chimprang, and Klinlumpu (2022) revealed the existence of sustainable growth in high sustainable development index countries and suggested that low sustainable development index countries should initiate and adopt effective policies for energy usage optimization. Zeng et al. (2022) stated that the United States has implemented the circular economy framework and initiated the Comprehensive Environmental Response and Compensation and Liability Act.

Ponce et al. (2021) illustrate that green energy measures ensure economic growth. The 1% increase in renewable energy consumption led to a 1% growth in GDP. Sariatli (2017) compares the differences between the circular economy and the linear economy. The circular economy implements the waste-elimination concept and thus is sustainable. In contrast, the linear economy model follows a take-make-dispose pattern, which is non-sustainable. Ogunmakinde, Egbelakin, and Sher (2022) mentioned that Africa has not yet transformed into a circular economy environment. The United States belongs to a circular-based economy, while Africa tends to be a linear-based economy. Following Ruiz et al. (2025) to explore differential impact of resources policy between developed and developing countries, the second motivation of this study is to examine whether the United States can efficiently utilize energy resources in an environmentally friendly way to sustain at the ultimate energy equilibrium levels, but African energy consumption fluctuates due to the lack of a circular economy. The long-term energy consumption between the United States and Africa, countries undergoing various circular economy levels, is compared using the equilibrium analysis of the three-dimensional Lotka-Volterra models.

This study contributes to the literature by investigating both short-term and long-term correlations between these energy sources through the three-dimensional Lotka-Volterra model. Sodero and Rabinovich (2017); Lukas et al. (2017) have applied the Bass (1969) model to predict new product diffusion patterns, but the Bass model does not consider competition and dependence between energy sources. Because of the competition in the diffusion of fossil fuel, renewable energy, and nuclear energy sources, alternative models must be developed to solve this problem. Kamimura, Guerra, and Sauer (2006); Paç, Savin, and Velu (2018) have used the binary Lotka-Volterra model to describe the competitive behaviors between two or more groups. Tsai and Chen (2020) employed the binary Lotka-Volterra model to demonstrate that CO_2 emissions from Taiwan and China, which are highly dependent on industry and trade, exhibit significant interaction effects. However, the binary Lotka-Volterra model in previous studies cannot simultaneously reflect the practical substitution of fossil fuels, renewable energy, and nuclear energy. To resolve the drawback of the binary model, the present study employs a modified three-dimensional bioecological Lotka-Volterra model to analyze the substitution correlations between fossil fuels, renewable energy sources, and nuclear energy sources in the United States and Africa. This study contributes to the extant literature by investigating the diffusion of energy sources concerning their interactive influence, which reflects the reality of generating power from various energy resources in the short run. Furthermore, this study conducts the equilibrium analysis to compare the long-term energy trajectory between the United States and Africa.

This study has four purposes. First, this study examines the short-term effectiveness of measures to replace fossil fuels with renewable or nuclear energy to reduce CO_2 emissions. Second, this study compares the potential for the growth of renewable or nuclear power between the United States and Africa. This study simulates the long-term trajectory of nuclear energy or renewable energy in Africa to illustrate how African renewable or nuclear power generation fails to achieve economies of scale stably. Third, the Jacobian Matrix and Lyapunov Function are employed to investigate the equilibrium relationship between different energy sources and to determine whether the levels of future energy consumption can be maintained at stable equilibrium points. Fourth, this study implements an analysis of forecast accuracy to identify whether the three-dimensional Lotka-Volterra model, which incorporates interactions between energy sources, is more accurate than the traditional Bass diffusion model, which does not incorporate interactions between energy sources. Unlike previous studies, this study incorporates substitution relationships between multiple energy sources to predict individual energy consumption. As stated by Ratnatunga and Balachandran (2009), measuring energy consumption is crucial. This study uses the concept of co-competition relationships to analyze the trade-off decision-making patterns of African and United States policymakers with respect to energy choices. The proposed three-dimensional Lotka-Volterra Model is to fully capture the co-competition relationships between energy sources and their practical implications. This study finds that the short-term replacement of nuclear energy with renewable power plants in the United States will lead to nuclear decline in the long term by employing equilibrium analysis.

Methodology

Data and sample

This study employs annual data from British Petroleum's Statistical Review of World Energy from 1984 to 2022. The renewable energy sources in our study include green energy such as solar, wind, geothermal, and biomass, but exclude hydropower. This study estimates the model parameters based on the data during the training period from 1965 to 2019, and predicts the energy consumption during the testing period from 2020 to 2022. This study implements the three-dimensional Lotka-Volterra model to evaluate energy sources' short-term and long-term relationships. The research process flowchart is presented in Figure 1.

First, the model parameters are estimated. This study constructs a three-dimensional Lotka-Volterra model to reflect the real environment by using the consumption of fossil fuels, nuclear energy, and renewable energy to estimate the model parameters. We employ the t-statistic to test the significance of the parameters related to the interactions between energy sources to analyze the potential for replacing fossil fuels with nuclear energy or renewable energy in the short term. Second, we implement an analysis of forecast accuracy to compare forecast accuracy between the three-dimensional Lotka-Volterra model, which incorporates the interaction relationships between energy sources, and the traditional Bass diffusion model, which does not incorporate the aforementioned interaction relationships. Finally, this study implements equilibrium analysis to test whether a stable replacement relationship is maintained among energy sources in the long term.

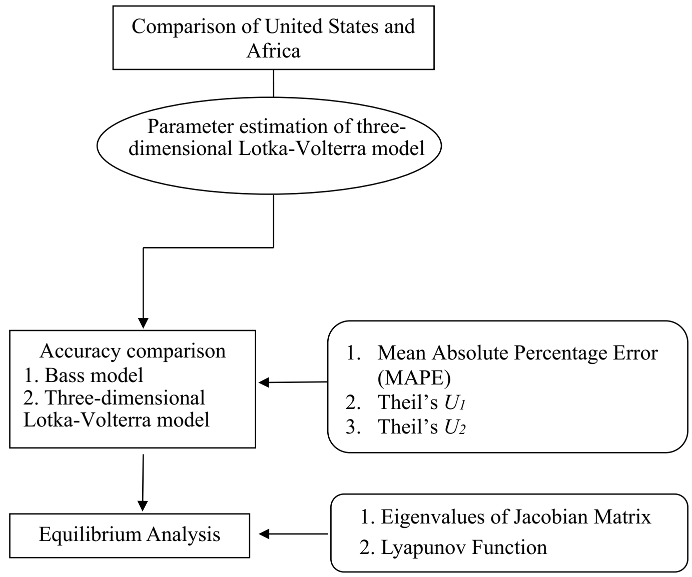

Figure 1. Research process flowchart.

Parameter estimation for the three-dimensional Lotka-Volterra model

To overcome the drawbacks of Tsai, Chang, and Chang (2016) two-dimensional Lotka-Volterra model, this study employs the three-dimensional Lotka-Volterra model to describe the interactions between energy sources. The model is based on the logistic equation, and the interactions between energy sources can be expressed using the following differential equations:

$$\frac{dX_1}{dt} = a_1X_1 - c_{11}X_1^2 - c_{12}X_1X_2 - c_{13}X_1X_3 \tag{1}$$

$$\frac{dX_2}{dt} = a_2X_2 - c_{22}X_2^2 - c_{21}X_2X_1 - c_{23}X_2X_3 \tag{2}$$

$$\frac{dX_3}{dt} = a_3X_3 - c_{33}X_3^2 - c_{31}X_3X_1 - c_{32}X_3X_2 \tag{3}$$

where X_1, X_2, and X_3 represent the consumption of fossil fuels, nuclear energy, and renewable energy, respectively, at time t. In addition, X_iX_j in Eqs. (1) to (3) is an interaction term between energy sources. The multi-competitive model includes pure competition, predation, mutualism, commensalism, and amensalism. The Lotka-Volterra Eqs. (1) to (3) are continuous-time models. However, because this study employs discrete-time data, the continuous-time Lotka-Volterra models must be transformed into discrete-time versions. This study adopts the three-dimensional Lotka-Volterra model developed by Liu and Gopalsamy (1997), transforming Eqs. (1) to (3) into three discrete equations, which can be expressed as follows:

$$X_1(t+1) = \frac{\alpha_1 X_1(t)}{1 + \beta_1 X_1(t) + \sum_{j\neq 1} \gamma_{1j} X_j(t)} \tag{4}$$

$$X_2(t+1) = \frac{\alpha_2 X_2(t)}{1 + \beta_2 X_2(t) + \sum_{j\neq 2} \gamma_{2j} X_j(t)} \tag{5}$$

$$X_3(t+1) = \frac{\alpha_3 X_3(t)}{1 + \beta_3 X_3(t) + \sum_{j\neq 3} \gamma_{3j} X_j(t)} \tag{6}$$

In Eqs. (4) to (6), α and β are parameters for a single energy source i, and γ represents the interaction effect between energy sources. The coefficient conversion relationship between Eqs. (4) to (6) of the three-dimensional Lotka-Volterra model and the original Eqs. (1) to (3) can be expressed as follows:

$$a_i = \ln \alpha_i \tag{7}$$

$$c_{ii} = \frac{\beta_i a_i}{\alpha_i - 1} = \frac{\beta_i \ln \alpha_i}{\alpha_i - 1} \tag{8}$$

$$c_{ij} = \gamma_{ij} \frac{c_{ii}}{\beta_i} = \frac{\gamma_{ij} \ln \alpha_i}{\alpha_i - 1} \tag{9}$$

After parameter estimation, t-tests are employed to evaluate the significance of coefficients and to analyze the interactions between fossil fuels, renewable energy, and nuclear energy consumption in the United States and Africa. We examine whether renewable energy or nuclear energy can effectively replace fossil fuels in the short run.

Analysis of forecast accuracy

Coya et al. (2024) examined the forecasting model's performance using statistical metrics based on the error of the actual and forecasted values. This study follows Coya et al. (2024) to compare the accuracy of our proposed three-dimensional Lotka-Volterra model, which incorporates substitution relationships between energy sources, with that of a traditional Bass model. In this study, a model is constructed to analyze the consumption of fossil fuels, nuclear energy, and renewable energy by integrating the framework of Bass (1969) diffusion theory. The model can be expressed as follows:

$$\frac{dN(t)}{dt} = (p + q\frac{N(t)}{M})(M - N(t)) \tag{10}$$

where $N(t)$ is the energy consumption at time t, and M is the maximum potential consumption of the Bass model. In addition, $(M - N(t))$ is the remaining consumption at time t. The parameters p and q indicate the different strengths affecting energy consumption, respectively. The parameter p represents the degree to which an energy source is used for the first time, and the parameter q represents the degree to which the energy source continues to be generated and consumed after being used for power generation. The regression (Eq. (11)) of the traditional Bass model is estimated using the ordinary least squares (OLS) method. When using discrete-time data to replace continuous data, the Bass regression can be represented as follows:

$$n(t) = pM + (q-p)N(t) - \frac{q}{M}[N(t)]^2 \tag{11}$$

To estimate the parameters p, q, and M from discrete-time data, we use the regression $n(t) = \delta + \theta N(t-1) + \kappa[N(t-1)]^2 + e_t$, where $n(t)$ represents the energy consumption in the interval $(t-1, t)$. Furthermore, $N(t-1)$ represents the cumulative consumption as of $t-1$, and e_t is the residual. Therefore, by using δ to estimate pM, θ to estimate $(q-p)$, and κ to estimate $-q/M$, we can estimate the parameters p and q and analyze the main factors affecting product diffusion through these parameters.

The accuracy of the three-dimensional Lotka-Volterra model and the traditional Bass diffusion model can be compared after applying the Lotka-Volterra mathematical framework for simulation analysis. This study employs annual data from British Petroleum's Statistical Review of World Energy from 1965 to 2022 to test the predictive ability of the model. This study estimates the model parameters based on the data during the training period from 1965 to 2019, predicts the energy consumption during the testing period from 2020 to 2022, and compares the predictive ability of the models using mean absolute percentage error (MAPE) and Theil's U. MAPE can be calculated as follows:

$$MAPE = \frac{1}{n}\sum_{t=1}^{n} \frac{\left| Z_t - \hat{Z}_t \right|}{Z_t} \times 100\% \tag{12}$$

where Z_t represents the actual value, $\widehat{Z}_t$ represents the predicted value, and n is the number of forecast periods. Martin and Witt (1989) and Lewis (1982) have used MAPE to evaluate the predictive ability of models, with an MAPE of <10% indicating excellent predictive ability, an MAPE between 10% and 20% indicating good predictive ability, an MAPE between 20% and 50% indicating reasonable predictive ability, and an MAPE >50% indicating incorrect prediction. Furthermore, this study evaluates the fit of the nonlinear regression model by using two Theil's U statistics. The two Theil's U statistics can be expressed as follows:

$$U_1 = \frac{\sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\widehat{Z}_t - Z_t\right)^2}}{\sqrt{\frac{1}{n}\sum_{t=1}^{n} Z_t^2} + \sqrt{\frac{1}{n}\sum_{t=1}^{n} \hat{Z}_t^2}} \tag{13}$$

$$U_2 = \sqrt{\frac{\frac{1}{n}\left[\sum_{t=1}^{n}\left(\hat{Z}_t - Z_t\right)^2\right]}{\sum_{t=1}^{n} Z_t^2}} \tag{14}$$

Equilibrium analysis

Previous literature lacks the equilibrium analysis of multi-dimensional models to predict the long-term energy trajectory. This study contributes to employing the estimated parameters in the three-dimensional Lotka-Volterra model to implement equilibrium analysis. The equilibrium point is the point at which long-term stability in the level of consumption can be maintained. The equilibrium point, that is, $\frac{dX_i}{dt} = 0$, can be expressed in the Appendix table. This study applies the Jacobian Matrix and the Lyapunov function to the model for determining whether the annual energy consumption in each region can reach equilibrium points and maintain stable levels of energy consumption over time.

Results

Results of parameter estimation in the United States

Table 1 presents the parameter estimations for the Lotka-Volterra model for the consumption of fossil fuels, nuclear energy, and renewable energy in the United States during the training period from 1984 to 2019. In Table 1, the self-influencing coefficient of the fossil fuel, β_1 is 5.6750×10^{-3} and significantly positive. Because the sign of c_{ii} is the same as that of β_i, c_{11} is significantly positive. This suggests that the United States constrains fossil fuel usage under the circular economy system, so a decrease in the additional amount of fossil fuels consumed as fossil fuel consumption increases, namely, the declining marginal fossil fuel consumption. In Table 1, the coefficient of the effect of fossil fuel on nuclear energy, γ_{21} is -4.8000×10^{-3}. Because the sign of c_i is the same as that of γ_i, c_{21} is significantly negative. The consumption of fossil fuels results in the increased consumption of nuclear energy. This is due to the United States' energy policy of replacing a portion of fossil fuels with nuclear energy. Additionally, the coefficient of the effect of nuclear energy on renewable energy, γ_{32}, is -4.5554×10^{-2} in Table 1 using the training period data from 1984 to 2019. This indicates that the consumption of nuclear energy increases the consumption of renewable energy. The United States' significant strides in electricity generation technology have paved the way for transitioning to renewable energy. This transition underscores the implementation of the circular economy concept, facilitating an environmentally sustainable economic framework. Because the sign of c_i is the same as that of γ_i, c_{32} is negative and significant, and c_{23} is nonsignificant. In other words, the consumption of nuclear energy increases renewable power generation, but the consumption of renewable energy does not influence nuclear power generation. The results suggest that the United States relies more on renewable power generation as nuclear consumption grows due to the fear of nuclear energy radiation and nuclear waste disposal. This finding aligns with the principles of the circular economy concept, which advocates for a more environmentally sustainable approach to energy management. From the perspective of the environmental Kuznets curve, the United States, being in a mature phase of economic development, reduces environmental degradation by mitigating potential harm from nuclear waste and radiation.

Results of parameter estimation in Africa

While previous studies by Moss and Bazilian (2018) and Allela (2021) only described the Power Africa framework in text and lacked data analysis, this study contributes by quantifying the results of Power Africa in terms of green energy generation in Africa. Table 2 presents the parameter estimation for the Lotka-Volterra model for energy consumption in Africa during the training period from 1984 to 2019. In Table 1, the self-influencing coefficient of the nuclear energy, β_1 is 1.2135×10^1 and significantly positive. Because the sign of c_{ii} is the same as that of β_i, c_{22} is significantly positive. This suggests that African capacity constraints of nuclear power cause a decrease in the additional amount of nuclear energy consumed as nuclear energy consumption increases, namely, the declining marginal nuclear consumption. Uranium reserves in Africa have led to the operation of nuclear power plants in Africa as early as 1984. Uranium mining and nuclear power generation in Africa have capacity constraints. As nuclear power generation approaches its saturation point, nuclear power generation will be curtailed. The coefficient of the effect of fossil fuels on nuclear energy, γ_{21}, is -4.4551×10^{-2}, which is significantly negative. This result indicates that the consumption of fossil fuels induces the increased consumption of nuclear energy. This result implies that Africa increasingly utilizes nuclear energy to generate power when fossil fuels are insufficient. The coefficient of the effect of fossil fuels on renewable energy, γ_{31}, is -5.1800×10^{-2}, which is significantly negative. This result indicates that the consumption of fossil fuels induces the increased consumption of renewable energy. This result implies that Africa increasingly utilizes renewable energy to generate power when fossil fuels are insufficient. Moreover, the coefficient of the effect of nuclear energy on renewable

Table 1. Parameter estimation for the three-dimensional Lotka – Volterra model in the United States from 1984 to 2019.

Energy	Parameter	Estimation	t-Value	Parameter	Estimation	Corresponding energy
Fossil fuel	a_1	1.3127	8.7849***	a_1	0.2721	
	β_1	5.6750×10^{-3}	1.8717*	c_{11}	0.0049	
	γ_{12}	-2.0438×10^{-3}	−1.4460	c_{12}	−0.0178	Nuclear
	γ_{13}	6.4500×10^{-3}	1.4656	c_{13}	0.0056	Renewable
Nuclear	a_2	9.9549×10^{-1}	8.7893***	a_2	−0.0045	
	β_2	4.9436×10^{-2}	5.3069***	c_{22}	0.0495	
	γ_{21}	-4.8000×10^{-3}	−2.4999**	c_{21}	−0.0048	Fossil fuel
	γ_{23}	1.2830×10^{-3}	−0.3301	c_{23}	−0.0013	Renewable
Renewable	a_3	9.2451×10^{-1}	5.1943***	a_3	−0.0785	
	β_3	1.0814×10^{-2}	2.1496**	c_{33}	0.0112	
	γ_{31}	1.9990×10^{-3}	0.5796	c_{31}	0.0021	Fossil fuel
	γ_{32}	-4.5554×10^{-2}	−1.8895*	c_{32}	−0.0474	Nuclear

*$p < .1$, **$p < .05$, ***$p < .01$.

Table 2. Parameter estimation for the three-dimensional Lotka – Volterra model in Africa from 1984 to 2019.

Energy	Parameter	Estimation	t-Value	Parameter	Estimation	Corresponding energy
Fossil fuel	α_1	1.0325	49.8469**	a_1	0.0320	
	β_1	-5.8400×10^{-5}	−0.0308	c_{11}	−0.0001	
	γ_{12}	4.3974×10^{-2}	0.2419	c_{12}	0.0431	Nuclear
	γ_{13}	2.1671×10^{-2}	0.3585	c_{13}	0.0213	Renewable
Nuclear	α_2	1.8903	4.0300**	a_2	0.6368	
	β_2	1.2135×10^{1}	2.7242*	c_{22}	8.6786	
	γ_{21}	-4.4551×10^{-2}	−1.7942	c_{21}	−0.0319	Fossil fuel
	γ_{23}	5.3824×10^{-1}	0.5487	c_{23}	0.3849	Renewable
Renewable	α_3	5.4470×10^{-1}	4.5910**	a_3	−0.6075	
	β_3	5.9736×10^{-1}	4.8116**	c_{33}	0.7971	
	γ_{31}	-5.1800×10^{-2}	−10.1976**	c_{31}	−0.0691	Fossil fuel
	γ_{32}	1.4816	3.1845*	c_{32}	1.9770	Nuclear

*$p < .1$, **$p < .05$, ***$p < .01$.

energy, γ_{32}, is 1.4816 in Table 2, which implies that the consumption of nuclear energy constrains the consumption of renewable energy. Nuclear energy consumption decreases renewable energy consumption in Africa since the operational cost of renewable power generation is expensive. This observation is consistent with the principles of the linear economy, which focuses on minimizing energy management costs at the expense of environmental sustainability. From the perspective of the Environmental Kuznets Curve, Africa, being in a developing phase of economic growth, experiences increased economic development at the cost of environmental degradation, including the potential harms associated with nuclear waste and radiation.

Results of forecast accuracy analysis

Table 3 presents the forecast accuracy results of the Lotka-Volterra model for the United States and Africa. The MAPEs of the predicted US consumption for fossil fuels, nuclear energy, and renewable energy during the training period are 1.5601%, 2.1210%, and 5.4243%, respectively. All the MAPE values for the Lotka-Volterra model predictions are lower than 10%, indicating excellent predictive accuracy. The MAPEs of the predicted African consumption for fossil fuels, nuclear energy, and renewable energy during the training period are 1.3101%, 12.2347%, and 19.4780%, respectively. According to Martin and Witt (1989), predictive ability can be expressed as excellent, good, or reasonable. The MAPEs of nuclear energy and renewable energy consumption in the United States are lower than those in Africa in the training period. The prediction for the United States is more accurate because the trends of nuclear and renewable energy use in the United States have been stable for many years, whereas the development trends of nuclear and renewable energy in Africa have a shorter history.

Overall, the United States nuclear and renewable energy forecast is more accurate than the African forecast. The values of U_1 and U_2 for nuclear and renewable energy are also smaller for the United States than for Africa in the training period, indicating more accurate results for the United States than for Africa, which is consistent with the MAPE values. According to Martin and Witt (1989) and Lewis (1982), the MAPE for renewable energy and fossil fuel consumption predictions falls at <10% for both models in the testing period, so there is not much difference in accuracy. However, the nuclear energy consumption prediction of the United States shows a MAPE of 1.2923%, which is much more accurate than Africa, which has a MAPE of 23.1400%. This result shows that the United States model has better prediction ability because the development trends of nuclear energy in Africa are more volatile and have a shorter history. Comparing Tables 3 and 4, in the United States, the renewable and nuclear energy consumption prediction of the Lotka-Volterra model is more accurate than the Bass model in the testing period. In Africa, the renewable consumption prediction of the Lotka-Volterra model is more accurate than the Bass model in the testing period.

Results of equilibrium analysis

Figure 2 illustrates the consumption of fossil fuels, nuclear energy, and renewable energy in the United States. The X-axis and Y-axis represent the time and energy consumption, respectively, of fossil fuels, nuclear energy, and renewable

Table 3. Predictive ability of the three-dimensional Lotka – Volterra model for energy source consumption.

	Training (1984–2019)			Testing (2020–2022)		
Energy	MAPE	U_1	U_2	MAPE	U_1	U_2
United States						
Fossil fuel	1.5601%	1.0008%	0.3383%	4.7716%	2.6489%	3.0687%
Nuclear	2.1210%	1.3208%	0.4464%	1.2923%	0.6592%	0.7638%
Renewable	5.4243%	1.6568%	0.5599%	9.7195%	5.1029%	5.8862%
Africa						
Energy	MAPE	U_1	U_2	MAPE	U_1	U_2
Fossil fuel	1.3101%	0.7914%	0.2675%	4.7453%	2.6673%	3.0877%
Nuclear	12.2347%	6.4612%	2.1752%	23.1400%	11.7085%	14.8804%
Renewable	19.4780%	3.5400%	1.1952%	7.9141%	4.5218%	5.4192%

Table 4. Predictive ability of the Bass model for energy source consumption.

	Training (1984–2019)			Testing (2020–2022)		
Energy	MAPE	U_1	U_2	MAPE	U_1	U_2
United States						
Fossil fuel	2.3294%	1.4080%	0.4750%	4.5227%	2.3655%	2.7085%
Nuclear	4.3402%	2.3416%	0.7890%	5.9757%	3.1013%	3.4745%
Renewable	22.5271%	7.1882%	2.3133%	12.5711%	5.8672%	7.1742%
Africa						
Energy	MAPE	U_1	U_2	MAPE	U_1	U_2
Fossil fuel	2.9501%	1.7529%	0.5845%	3.6990%	2.0869%	2.4525%
Nuclear	12.2752%	6.5932%	2.2162%	23.0999%	10.9011%	13.9239%
Renewable	17.5256%	13.5363%	4.0778%	25.4310%	15.4533%	20.5261%

energy. Energy consumption in the United States is predicted to reach a stable level surrounding 2095. Combining the results of Tables 1 and 2 that electricity generation in the United States transfers from fossil fuels to cleaner renewable energy alternatives in the short run, the consumption of fossil fuels is predicted to decrease and reach equilibrium points surrounding 2095. The equilibrium results support that the United States can efficiently utilize energy resources in an environmentally friendly way; its circular-based economy results in stable long-term energy consumption.

This work furthermore computed the negative real parts of the Jacobian matrix of the Lotka-Volterra equations, positive Lyapunov functions, and the negative first differentials of the Lyapunov functions in the United States. This proves the equilibrium stability. In the equilibrium state, the consumption levels of fossil fuels, nuclear energy, and renewable energy are 69.4612×10^{18} J, 6.8922×10^{18} J, and 9.2122×10^{18} J, respectively. Fossil fuels remain the main energy source, with a consumption level nearly 10 times more than that of nuclear energy and 7 times more than that of renewable energy. This manuscript further compares the equilibrium value relative to the energy consumption in the last year of the training period, namely, 2019. In 2019, in the United States, the consumption levels of fossil fuels, nuclear energy, and renewable energy are 78.8079×10^{18} J, 7.6000×10^{18} J, and 5.8286×10^{18} J, respectively, and each energy source has room for change. Compared with the energy consumption in 2019, the consumption of fossil fuels at the equilibrium point is predicted to decline by 11.8601%, the consumption of nuclear energy is predicted to decline by 9.3136%, and the consumption of renewable energy is predicted to rise by 58.0517%. The findings of Table 1 indicate that the United States replaces fossil fuels with nuclear energy and then replaces nuclear energy with renewable energy. In the long run, the usage of renewable energy surpasses that of nuclear energy, reflecting the short-term trend in the United States of increasing the usage of environmentally friendly renewable energy alongside nuclear power.

Compared with the prior article by Moss and Bazilian (2018), which only exhibits the past data without prediction of the future development, Figure 3 reveals Africa's predicted fossil fuel, nuclear energy, and renewable energy consumption. Comparing Figures 2 and 3, the predicted consumption of fossil fuels, nuclear energy, and renewable energy in Africa is approximately forty percent, 1/70th, and 1/7th, respectively, of that in the United States. The United States produces and consumes significantly more energy due to its advanced energy infrastructure and innovative technological ability, and environmental awareness than Africa. The prediction of all three energy sources in Africa is predicted to reach their equilibrium states around 2260. In the equilibrium state, the consumption levels of fossil fuels, nuclear energy, and renewable energy are 27.5607×10^{18} J, 1.1501×10^{17} J, and 1.3425×10^{18} J, respectively. Fossil fuels remain the main energy source, with a consumption level nearly 239 times more than that of nuclear energy and a consumption level nearly 20 times more than that

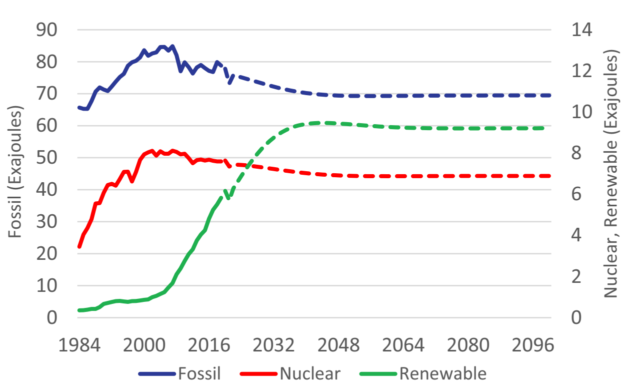

Figure 2. Forecasted consumption of fossil fuels, nuclear energy and renewable energy in United States.

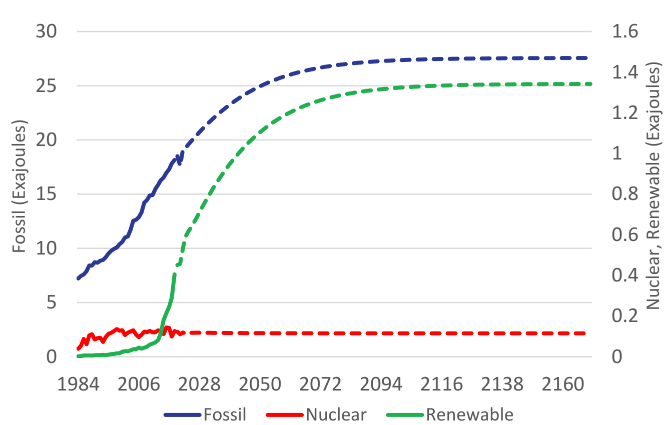

Figure 3. Forecasted consumption of fossil fuels, nuclear energy, and renewable energy in the Africa.

of renewable energy. The results suggest that fossil fuels dominate the power generation. The findings of a lower equilibrium ratio of green energy to fossil fuels in Africa than in the United States are because Africa lacks a stable circular economy system, following Chishti, Dogan, and Zaman (2023). This manuscript further compares the equilibrium value relative to the energy consumption in the last year of the training period, namely, 2019. In 2019, in Africa, the consumption levels of fossil fuels, nuclear energy, and renewable energy are 18.1556×10^{18} J, 1.2671×10^{17} J, and 4.0527×10^{17} J, respectively, and each energy source has room for change. Compared with the energy consumption in 2019, the consumption of fossil fuels at the equilibrium point is predicted to rise by 51.8026%, the consumption of nuclear energy is predicted to decline by 9.2347%, and the consumption of renewable energy is predicted to rise by 231.2521%. This forecast of growing renewable energy generation corresponds to the validity of "Power Africa."

The real parts of the eigenvalues of the Jacobian matrix at the equilibrium point are negative both in the United States and in Africa. Additionally, the long-term equilibrium relationships between energy sources in the United States and Africa are supported by both the positive Lyapunov function and the negative derivative of the Lyapunov function. Therefore, considering the current state of technology and energy supplies, the three energy sources in the United States and Africa eventually reach a steady equilibrium in the prediction model. Both renewable energy consumption increases in the United States and in Africa. This research further compares the results of Figures 2 and 3, and two obvious differences exist between the United States and Africa. First, fossil fuel and nuclear energy consumption in the circular-based economy of the United States stably declined, but fossil fuel consumption increased in Africa under the linear economy framework. The increase in African fossil fuel consumption implies that Africa prioritizes the economy over environmental protection and fails to utilize energy as efficiently as the United States. Second, fossil fuel consumption finally persists at nearly 10 times more than that of nuclear energy and 7 times more than that of renewable energy in the United States. By contrast, fossil fuels eventually remain at a consumption level nearly 239 times more than that of nuclear energy and a consumption level nearly 20 times more than that of renewable energy in Africa. The energy structure is different between the American circular economy and the African linear economy. The consumption of fossil fuels and nuclear energy in the United States ultimately decreases at specific equilibrium levels, indicative of restructuring the economy in an environmentally sustainable manner. However, the energy consumption of low-cost fossil fuels is predicted to increase in Africa. Previous studies by Moss and Bazilian (2018) and Allela (2021) only described the Power Africa framework in text and lacked data analysis. This study contributes by quantifying the partial effectiveness of Power Africa in terms of green energy generation in Africa. Our findings for a lower ratio of green energy in Africa than in the United States suggest that Africa should make greater use of green energy in the future.

Conclusion

This study compares the energy consumption substitution among various energy resources in terms of the short-term correlation and long-term equilibrium between the United States and Africa, because the two regions have diverse levels of circular economy. This study also finds out the effectiveness of the U.S.'s "Power Africa" initiative and explores the path for future improvements in green energy in Africa with data analysis, and contributes to the literature by constructing a three-dimensional Lotka-Volterra model to analyze the energy consumption of the United States and Africa. Predictive models must incorporate the interactions between renewable energy, nuclear energy, and fossil fuels. Thus, in this study, a three-dimensional model is constructed to investigate the substitution relationships in the short term between energy sources in the United States and Africa. The research results of parameter estimations reveal that the consumption of nuclear energy in the United States increases the consumption of renewable energy, but the consumption of fossil fuels in Africa increases the consumption of nuclear energy and renewable energy.

The first contribution of this study is the finding that both the United States and Africa have tended to make efforts to replace fossil fuels with other clean energy sources when the United States and Africa encounter insufficient electricity. The proposed three-dimensional Lotka-Volterra model is more accurate than the traditional Bass diffusion model in nuclear energy prediction in the United States. The second contribution of this study is identifying the short-term energy policy differences between the circular economy approach in the United States and the linear economy approach in Africa. The findings reveal that, in the United States, renewable energy usage increases alongside nuclear energy, while in Africa, renewable energy usage decreases as nuclear energy usage increases. The United States protects the environment by mitigating potential harm from nuclear waste and radiation, but Africa tends to choose lower-cost fossil fuels. This disparity highlights that the more economically developed United States adopts a more environmentally cautious approach compared to Africa, which is still in its stage of economic development.

The third contribution of this study is the finding of the difference in the long-term energy equilibrium between the circular-based economy of the United States and the linear-based economy of Africa. This paper, for the first time, conducts an equilibrium analysis of the three-dimensional Lotka-Volterra models to simulate the long-term energy trajectory. Our equilibrium analysis exhibits a sustained disparity of energy consumption patterns between the circular-based economy of the United States and the linear-based economy of Africa. Compared with Africa, the United States is in the mature stage of economic development and perfectly plans the power generation and utilization under the circular-based economy framework. Therefore, the equilibrium analysis indicates that the simulated consumption of renewable, nuclear, and fossil fuels in the United States reached a stable level in the long term. With technological development and increased environmental awareness, the consumption of nuclear energy and fossil fuels in the United States is predicted to decline, and the consumption of renewable energy is predicted to rise. However, Africa is still undergoing economic development under a linear economic framework, so the consumption of fossil fuels in Africa is predicted to increase. The findings of a lower equilibrium ratio of green energy to fossil fuels in Africa than in the United States are because Africa's lacks a stable circular economy system. Our empirical results suggest that Africa should improve to develop more green power capacity and establish a circular economy system following the United States' structure in the future.

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Author contribution

CRediT: **Bi-Huei Tsai:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix A.

	X_1	X_2	X_3
$X_1 = X_2 = X_3 = 0$	0	0	0
$X_1 \neq 0, X_2 = X_3 = 0$	$\frac{a_1}{b_1}$	0	0
$X_2 \neq 0, X_2 = X_3 = 0$	0	$\frac{a_2}{b_2}$	0
$X_3 \neq 0, X_1 = X_2 = 0$	0	0	$\frac{a_3}{b_3}$
$X_1 \& X_2 \neq 0, X_3 = 0$	$\frac{c_{12}a_2 - a_1 b_2}{c_{12}c_{21} - b_1 b_2}$	$\frac{c_{21}a_1 - a_2 b_1}{c_{12}c_{21} - b_1 b_2}$	0
$X_1 \& X_3 \neq 0, X_2 = 0$	$\frac{c_{13}a_3 - a_1 b_3}{c_{13}c_{31} - b_1 b_3}$	0	$\frac{c_{31}a_1 - a_3 b_1}{c_{13}c_{31} - b_1 b_3}$
$X_2 \& X_3 \neq 0, X_1 = 0$	0	$\frac{c_{23}a_3 - a_2 b_3}{c_{23}c_{32} - b_2 b_3}$	$\frac{c_{32}a_2 - a_3 b_2}{c_{23}c_{32} - b_2 b_3}$
$X_1 \& X_2 \& X_3 \neq 0$		X_1	
		$\frac{a_1(b_2 b_3 - c_{23}c_{32}) + a_2(c_{13}c_{32} - b_3 c_{12}) + a_3(c_{12}c_{23} - b_2 c_{13})}{b_1(b_2 b_3 - c_{23}c_{32}) + c_{12}(c_{23}c_{31} - b_3 c_{21}) + c_{13}(c_{21}c_{32} - b_2 c_{31})}$	
		X_2	
		$\frac{a_1(c_{23}c_{31} - b_3 c_{21}) + a_2(b_1 b_3 - c_{13}c_{31}) + a_3(c_{13}c_{21} - b_1 c_{23})}{b_1(b_2 b_3 - c_{23}c_{32}) + c_{12}(c_{23}c_{31} - b_3 c_{21}) + c_{13}(c_{21}c_{32} - b_2 c_{31})}$	
		X_3	
		$\frac{a_1(c_{21}c_{32} - b_2 c_{31}) + a_2(c_{12}c_{31} - b_1 c_{32}) + a_3(b_1 b_2 - c_{12}c_{21})}{b_1(b_2 b_3 - c_{23}c_{32}) + c_{12}(c_{23}c_{31} - b_3 c_{21}) + c_{13}(c_{21}c_{32} - b_2 c_{31})}$	