**Dimensionless Heat Transfer Solution**

**Matt Earl**

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**Introduction**

Our goal is to solve the 1D conduction equation:

Subject to the boundary conditions: and

Subject to the initial condition:

The Dirichlet boundary condition maintains the temperature of the bottom of the rod at the base plate temperature. The Neumann boundary condition specifies that convection is occurring at the topmost point of the rod . In this equation the constant represents the thermal diffusivity of the material and is the convective coefficient at the end , is the thermal conductivity and the function represents rate of heat generation in units of .

I had developed an analytical solution using Fourier series to this problem and wrote a MATLAB code that tried to solve this problem. Unfortunately, there were some serious stability issues to this solution (i.e., it was real buggy). For instance, MATLAB would crash if the part length became too small. Similarly, if the time scale became too large the code would crash. Lastly the temperatures given by the model were unrealistic. It occurred to me just a couple of days ago that the problems in this solution may be resolved by nondimensionalizing the equation. This is a lot more straightforward of a task than I thought it would be. Without much effort, I was able to derive and solve the dimensionless form of the conduction equation and implement the solution in my code from over the summer. That’s what I am going to explain here.

**Derivation**

First let us normalize the conduction equation by introducing dimensionless variables:

The variable represents the hottest temperature achieved in the heating cycle. represents the length of the rod in , represents the length of the heating cycle in . With application of the chain rule, we can develop the dimensionless conduction equation.

For simplicity I will introduce two new variables and defined as

With these variables the differential equation can be expressed as:

The boundary and initial conditions need to be adjusted slightly to fit this dimensionless equation. It is easy to see that the Dirichlet boundary condition becomes . With a little work we can show that the Neumann boundary condition becomes where . For the sake of clarity, I will also differentiate the starting condition as . Thus, the original problem is recast in its dimensionless form (tildes have been dropped for simplicity’s sake) as:

Subject to the boundary conditions: and

Subject to the initial condition:

**Solution Structure**

This differential equation has a source term which means we will have to use a Green’s function approach to developing a solution. In this light, we will follow the approach of removing both types of inhomogeneity separately and superimposing the solutions.

First, we consider the system with no forcing function

Subject to the boundary conditions: and

Subject to the initial condition:

Secondly, we consider the system with a forcing function but no initial condition:

Subject to the boundary conditions: and

Subject to the initial condition:

If the solution of the first system is and the solution of the second system is it can be shown that the solution of the original problem is:

Now solving the problem has been broken down into solving these sub problems.

**Solving the first sub problem:**

Subject to the boundary conditions: and

Subject to the initial condition:

This is a linear P.D.E. so we assume a separable solution of the form

Thus, we get two differential equations which are very simple to solve:

We thus have that the solution is:

Application of the boundary conditions will help us to determine the eigenvalues and corresponding eigenfunctions.

First, we examine :

Hence is reduced to:

Secondly, we examine

The eigenvalues are thus the solutions of this nonlinear equation we will call these solutions where so that:

The corresponding eigenfunctions are thus:

We can thus rewrite the solution as:

Without going too deep into the theory of Fourier series we note that the eigenfunctions form an orthogonal set and we calculate the norm of this set below:

With this information we can write the particular solution of the first sub problem:

**Solving the Second Sub Problem**

Subject to the boundary conditions: and

Subject to the initial condition:

To solve this more complicated case we can use a standard Green’s function approach where we have to make a slight adjustment to the solution of the first sub problem.

If we write the solution of the first sub problem as:

Then the solution to the second problem is given by:

A simple proof of this statement is provided:

Thus is shown to solve the given differential equation. We can also check to see that the boundary conditions and initial conditions are indeed meant as well.

Thus we write explicitly as

**Whole Solution:**

According to what was stated in the solution structure section we finally have that the answer to the original system:

Subject to the boundary conditions: and

Subject to the initial condition:

Is given by the superposition of the two sub-solutions.

**Heat Generation Models:**

This extremely general solution presented above can be made much more practical if we try to match the heat generation input with a real laser model.

1. **Simple Laser**

One conventional model for heat generation from a constant laser input says that the heat generation rate drops off exponentially throughout the material:

In this case the laser is contacting the surface with max intensity at and drops off further into the material. Here we have the peak laser intensity , the reflection coefficient and the absorption coefficient of the material, and is the thermal conductivity of the material, and is the thermal diffusivity.

For our purposes this needs to be adjusted so that the max intensity occurs at .

The nondimensional version of this equation then is given by:

Let us call this monster constant:

And thus, the dimensionless laser model is:

In such a case the forced response of the system will be given as:

It is quite difficult to locate the optical properties relevant to this problem. The value of absorption varies astonishingly in scale with the wavelength of the light in question and with the material type, surface finish, phase etc. This number will have to be set to match experimental data. The DMLM machine used an Ytterbium fiber laser which tends to produce light with wavelength in . If the correct extinction coefficient was determined, we could use the relationship below to find the absorption.

It should also be noted that it is in fact disadvantageous to use the entirely nondimensionalized equation here. As the laser heat input model will actually give us the actual temperature distribution. We lose an important element if we divide each term through by . In this case we can simply multiply the dimensionless equation through by so the equation actually gives us the temperature distribution in Kelvin.

1. **Step Function**

A similar approach which greatly simplifies the laser physics uses a step function

Correspondingly we would have the dimensionless equation

In such a case the forced response of the system will be given as:

This method while simple seems to be surprisingly accurate.