Analytic Solution for Spinning Book

Matthew Earl

11/21/2021

A book freely spinning is rotating with some initial angular velocity $\boldsymbol{\omega_0}$. If a frame is attached to the book at the mass center with its base vectors aligned with the book's axes of principal inertia. This means that the inertia tensor with respect to the rotating frame is given by:

$$\boldsymbol{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Because this frame is attached to the body and aligned with the principal directions of inertia, we safely apply Euler's equations:

$$\sum M_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3$$

$$\sum M_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1$$

$$\sum M_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2$$

The only force acting on this book if we throw it in the air is gravity (Assuming air resistance negligible). This force does not produce a moment as it acts through the mass center of the book. Thus, Euler's equations reduce to:

$$\dot{\omega}_1 = \lambda_1\omega_2\omega_3$$

$$\dot{\omega}_2 = \lambda_2\omega_3\omega_1$$

$$\dot{\omega}_3 = \lambda_3\omega_1\omega_2$$

Where $\lambda_1 = \frac{(I_2 - I_3)}{I_1}, \lambda_2 = \frac{(I_3 - I_1)}{I_2}, \lambda_3 = \frac{(I_1 - I_2)}{I_3}$

Note that in this scenario the angular velocity $\boldsymbol{\omega}$, the kinetic energy T, and the angular momentum $\boldsymbol{H_G}$ will all have a constant magnitude even though the direction of angular velocity and angular momentum may change.

This conservation of energy tells us that the kinetic energy is equal to the initial kinetic energy.

$$T_0 = \frac{1}{2}\boldsymbol{\omega}(0) \cdot \boldsymbol{H_G}(0) = \frac{1}{2}\boldsymbol{\omega}(t) \cdot \boldsymbol{H_G}(t)$$

The conservation of angular momentum likewise:

$$H_0 = |\boldsymbol{I} \cdot \boldsymbol{\omega}(\boldsymbol{0})| = |\boldsymbol{I} \cdot \boldsymbol{\omega}(t)|$$

If we expand the kinetic energy equation, we get:

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T_0 \tag{1}$$

Similarly expanding the angular momentum equation, we get:

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = H_0^2 \tag{2}$$

If we consider $(2) - I_3(1)$ we get that:

$$I_1(I_1 - I_3)\omega_1^2 + I_2(I_2 - I_3)\omega_2^2 = H_0^2 - 2T_0I_3$$

$$\omega_2^2 = \frac{H_0^2 - 2T_0I_3}{I_2(I_2 - I_3)} - \frac{I_1(I_1 - I_3)}{I_2(I_2 - I_3)}\omega_1^2$$

$$\omega_2 = \sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)\left(\frac{H_0^2 - 2T_0I_3}{-I_1I_2\lambda_2} - \omega_1^2\right)} \tag{ω_2}$$

Similarly if we consider $(2) - I_2(1)$ we get that:

$$I_1(I_1 - I_2)\omega_1^2 + I_3(I_3 - I_2)\omega_3^2 = H_0^2 - 2T_0I_2$$

$$\omega_3^2 = \frac{H_0^2 - 2T_0I_2}{I_3(I_3 - I_2)} - \frac{I_1(I_1 - I_2)}{I_3(I_3 - I_2)}\omega_1^2$$

$$\omega_3 = \sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)\left(\frac{H_0^2 - 2T_0I_2}{I_1I_3\lambda_3} - \omega_1^2\right)} \tag{ω_3}$$

With these expressions for ω_2 and ω_3 we can uncouple the euler equation in the 1 direction:

$$\dot{\omega_1} = \lambda_1\omega_2\omega_3$$

$$= \lambda_1\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)\left(\frac{H_0^2 - 2T_0I_3}{-I_1I_2\lambda_2} - \omega_1^2\right)\left(-\frac{\lambda_3}{\lambda_1}\right)\left(\frac{H_0^2 - 2T_0I_2}{I_1I_3\lambda_3} - \omega_1^2\right)}$$

$$= \sqrt{\lambda_2\lambda_3(a^2 - \omega_1^2)(b^2 - \omega_1^2)}$$

Where I have introduced the convenience notation:

$$a^2 = \frac{H_0^2 - 2T_0I_3}{-I_1I_2\lambda_2} \qquad b^2 = \frac{H_0^2 - 2T_0I_2}{I_1I_3\lambda_3}$$

The differential equation has been reduced to a separable differential equation!

$$\frac{d\omega_1}{\sqrt{(a^2-\omega_1^2)(b^2-\omega_1^2)}} = \sqrt{\lambda_2\lambda_3}dt$$

$$\int_0^{\omega_1} \frac{d\omega_1}{\sqrt{(a^2-\omega_1^2)(b^2-\omega_1^2)}} = \sqrt{\lambda_2\lambda_3}t$$

We can now integrate the left-hand side. We will get three very different answers depending on the ordering of a and b. We need to consider three cases: i) $a = b$, ii) $a > b$, and $a < b$.

Case i): $a = b$

$$\int_0^{\omega_1} \frac{d\omega_1}{\sqrt{(a^2-\omega_1^2)(b^2-\omega_1^2)}} = \int_0^{\omega_1} \frac{d\omega_1}{a^2-\omega_1^2}$$

$$= \frac{1}{a}\operatorname{arctanh}\left(\frac{\omega_1}{a}\right)$$

Hence:

$$\omega_1(t) = a\tanh\left(a\sqrt{\lambda_2\lambda_3}t\right)$$

Plugging this expression into (ω_2) yields:

$$\omega_2 = \sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)\left(a^2 - a^2\tanh^2(a\sqrt{\lambda_2\lambda_3}t)\right)}$$

$$\omega_2(t) = a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)}\operatorname{sech}(a\sqrt{\lambda_2\lambda_3}t)$$

Similarly plugging into (ω_3) yields:

$$\omega_3 = a\sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)}\ \operatorname{sech}(a\sqrt{\lambda_2\lambda_3}t)$$

Case ii) $a < b$:

$$\int_0^{\omega_1} \frac{d\omega_1}{\sqrt{(a^2-\omega_1^2)(b^2-\omega_1^2)}} = \int_0^{\omega_1} \frac{d\omega_1}{ab\sqrt{\left(1-\frac{\omega_1^2}{a^2}\right)(1-\frac{\omega_1^2}{b^2})}}$$

$$= \int_0^{\frac{\omega_1}{a}} \frac{du}{b\sqrt{(1-u^2)(1-\frac{a^2}{b^2}u^2)}}$$

$$= \frac{1}{b} arcsn\left(\frac{\omega_1}{a}, \frac{a}{b}\right)$$

$$arcsn\left(\frac{\omega_1}{a}, \frac{a}{b}\right) = b\sqrt{\lambda_2\lambda_3}t$$

$$\omega_1 = a\ sn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)$$

From which we conclude:

$$\omega_2 = a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)\left(1 - sn^2\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)\right)}$$

$$= a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)}\, cn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)$$

$$\omega_3 = \sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)\left(b^2 - a^2 sn^2\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)\right)}$$

$$= b\sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)}\ dn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)$$

Case iii) $a > b$:

$$\int_0^{\omega_1} \frac{d\omega_1}{\sqrt{(a^2-\omega_1^2)(b^2-\omega_1^2)}} = \int_0^{\omega_1} \frac{d\omega_1}{ab\sqrt{\left(1-\frac{\omega_1^2}{a^2}\right)(1-\frac{\omega_1^2}{b^2})}}$$

$$= \int_0^{\frac{\omega_1}{b}} \frac{du}{a\sqrt{(1-u^2)(1-\frac{b^2}{a^2}u^2)}}$$

$$= \frac{1}{a} arcsn\left(\frac{\omega_1}{b}, \frac{b}{a}\right)$$

$$arcsn\left(\frac{\omega_1}{b}, \frac{b}{a}\right) = a\sqrt{\lambda_2\lambda_3}t$$

$$\omega_1 = b\ sn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)$$

From which we conclude:

$$\omega_2 = a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)\left(1-\frac{b^2}{a^2}sn^2\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)\right)}$$

$$= a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)}dn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)$$

$$\omega_3 = \sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)\left(b^2-b^2sn^2\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)\right)}$$

$$= b\sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)}\ cn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)$$

In conclusion we have the three solutions:

Case i)

$$\omega_1 = a\tanh\left(a\sqrt{\lambda_2\lambda_3}t\right)$$

$$\omega_2 = a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)}\operatorname{sech}(a\sqrt{\lambda_2\lambda_3}t)$$

$$\omega_3 = a\sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)}\operatorname{sech}(a\sqrt{\lambda_2\lambda_3}t)$$

Case ii)

$$\omega_1 = a\, sn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)$$

$$\omega_2 = a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)}\, cn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)$$

$$\omega_3 = b\sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)}\, dn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)$$

Case iii)

$$\omega_1 = b\, sn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)$$

$$\omega_2 = a\sqrt{\left(-\frac{\lambda_2}{\lambda_1}\right)}\, dn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)$$

$$\omega_3 = b\sqrt{\left(-\frac{\lambda_3}{\lambda_1}\right)}\, cn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)$$

In conclusion we have the three solutions:

Case i)

$$\theta_1 = \frac{1}{\sqrt{\lambda_2\lambda_3}}\log(\cosh(a\sqrt{\lambda_2\lambda_3}\,t))$$

$$\theta_2 = \frac{2}{\sqrt{-\lambda_1\lambda_3}}\arctan\left(\tanh\left(\frac{a}{2}\sqrt{\lambda_2\lambda_3}t\right)\right)$$

$$\theta_3 = \frac{2}{\sqrt{-\lambda_1\lambda_2}}\arctan\left(\tanh\left(\frac{a}{2}\sqrt{\lambda_2\lambda_3}t\right)\right)$$

Case ii)

$$\theta_1 = \frac{1}{\sqrt{\lambda_2\lambda_3}}\log\left(dn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right) - \frac{a}{b}cn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)\right)$$

$$\theta_2 = \frac{1}{\sqrt{-\lambda_1\lambda_3}}\arccos\left(dn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)\right)$$

$$\theta_3 = \frac{1}{\sqrt{-\lambda_1\lambda_2}}\arcsin\left(sn\left(b\sqrt{\lambda_2\lambda_3}t, \frac{a}{b}\right)\right)$$

Case iii)

$$\theta_1 = \frac{1}{\sqrt{\lambda_2\lambda_3}}\log\left(dn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right) - \frac{b}{a}cn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)\right)$$

$$\theta_2 = \frac{1}{\sqrt{-\lambda_1\lambda_3}}\arcsin\left(sn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)\right)$$

$$\theta_3 = \frac{1}{\sqrt{-\lambda_1\lambda_2}}\arccos\left(dn\left(a\sqrt{\lambda_2\lambda_3}t, \frac{b}{a}\right)\right)$$