A Design Study Approach to Classical Control

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Homework E.2

(a) Using the configuration variables z and θ, write an expression for the kinetic energy of the system. Assume that the ball is a point mass without rolling inertia. In general, this term will be small and will not have a large effect on the dynamics of the ball-beam system.

(b) Create an animation of the ball on beam system. The inputs should be z and θ. Turn in a screen capture of the animation.

Solution

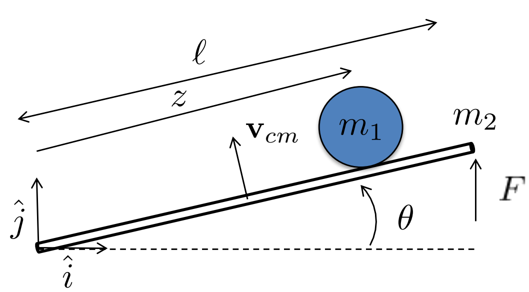

Figure 1: Computing the kinetic energy for the ball on beam system.

Define the inertial coordinate frame as in Figure 1, with $\hat{k}$ out of the page.

Since there are two masses, the total kinetic energy of the system is given by

$$K = \frac{1}{2} m_{ball} \mathbf{v}_{ball}^\top \mathbf{v}_{ball} + \frac{1}{2} \boldsymbol{\omega}_{ball}^\top J_{ball} \boldsymbol{\omega}_{ball} + \frac{1}{2} m_{beam} \mathbf{v}_{beam}^\top \mathbf{v}_{beam} + \frac{1}{2} \boldsymbol{\omega}_{beam}^\top J_{beam} \boldsymbol{\omega}_{beam}.$$

We will assume that the ball moves without rolling and therefore that $\boldsymbol{\omega}_{ball} = 0$.

The horizontal position of the ball m is given by

$$\mathbf{p}_{ball} = \begin{pmatrix} z(t)\cos\theta(t) \\ z(t)\sin\theta(t) \\ 0 \end{pmatrix}.$$

Similarly, the position of the center of mass of the beam is given by

$$\mathbf{p}_{beam} = \begin{pmatrix} \frac{\ell}{2}\cos\theta(t) \\ \frac{\ell}{2}\sin\theta(t) \\ 0 \end{pmatrix}.$$

Differentiating we obtain

$$\mathbf{v}_{ball} = \begin{pmatrix} \dot{z}\cos\theta - z\dot{\theta}\sin\theta \\ \dot{z}\sin\theta + z\dot{\theta}\cos\theta \\ 0 \end{pmatrix}$$

$$\mathbf{v}_{beam} = \begin{pmatrix} -\frac{\ell}{2}\dot{\theta}\sin\theta \\ \frac{\ell}{2}\dot{\theta}\cos\theta \\ 0 \end{pmatrix}.$$

The angular velocity of the beam is given by

$$\boldsymbol{\omega}_{beam} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}.$$

Therefore, the kinetic energy is given by

$$\begin{aligned} K = \frac{1}{2} m_1 &\left[(\dot{z}\cos\theta - z\dot{\theta}\sin\theta)^2 + (\dot{z}\sin\theta + z\dot{\theta}\cos\theta)^2 \right] \\ &+ \frac{1}{2} m_2 \left[(-\frac{\ell}{2}\dot{\theta}\sin\theta)^2 + (\frac{\ell}{2}\dot{\theta}\cos\theta)^2 \right] + \frac{1}{2} J_z \dot{\theta}^2, \end{aligned}$$

where J_z is the $(3,3)$ element of the inertia matrix J_{beam}. Modeling the beam as a thin rod we have

$$J_z = \frac{m_2 \ell^2}{12}.$$

Therefore

$$\begin{aligned}
K &= \frac{1}{2}m_1\left[\dot{z}^2\cos^2\theta - 2z\dot{z}\dot{\theta}\sin\theta\cos\theta + z^2\dot{\theta}^2\sin^2\theta + \dot{z}^2\sin^2\theta + 2z\dot{z}\dot{\theta}\sin\theta\cos\theta + z^2\dot{\theta}^2\cos^2\theta\right] \\
&\quad + \frac{1}{2}m_2\left[\frac{\ell^2}{4}\dot{\theta}^2\sin^2\theta + \frac{\ell^2}{4}\dot{\theta}^2\cos^2\theta\right] + \frac{1}{2}J_z\dot{\theta}^2, \\
&= \frac{1}{2}m_1\left(\dot{z}^2 + z^2\dot{\theta}^2\right) + \frac{1}{2}\frac{m_2\ell^2}{3}\dot{\theta}^2 \\
&= \frac{1}{2}m_1\dot{z}^2 + \frac{1}{2}\left(\frac{m_2\ell^2}{3} + m_1 z^2\right)\dot{\theta}^2.
\end{aligned}$$