A Design Study Approach to Classical Control

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Homework D.9

(a) When the controller for the mass spring damper is PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the case where a constant disturbance acts at the input to the plant (for example an inaccurate knowledge of the spring constant in this case). What is the steady state error to a constant input disturbance when the integrator is not present, and when it is present?

Solution

The closed loop system is shown in Figure **??**.

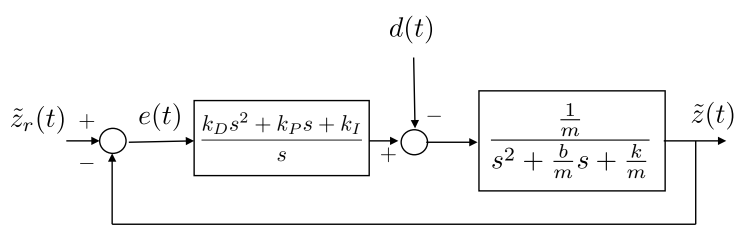

Figure 1: Closed-loop system for problem mass-spring-damper system with PID control.

Without the integrator, the open-loop transfer function is given by

$$P(s)C(s) = \left(\frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m})} \right) (k_D s + k_P).$$

The system has no free integrators and is therefore type 0, which, from Table **9-1** implies that the tracking error when the input is a step is

$$\lim_{t\to\infty} e(t) = \frac{1}{1+M_p} = \frac{1}{1+\lim_{s\to 0} P(s)C(s)} = \frac{k}{k+k_P}.$$

The tracking error when the input is a ramp, parabola, or higher order polynomial, is ∞, meaning that $z(t)$ and $z_r(t)$ diverge as $t \to \infty$.

With the integrator, the open loop transfer function is

$$P(s)C(s) = \left(\frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m})} \right) \left(\frac{k_D s^2 + k_P s + k_I}{s} \right).$$

which has one free integrator and is therefore type 1. Therefore, from Table **9-1** the tracking error when the input is either a step is zero, and the tracking error when the input is a ramp is

$$\lim_{t\to\infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s\to 0} sP(s)C(s)} = \frac{k}{k_I}.$$

The tracking error when the input is t^2 or a higher order polynomial, is ∞.

For the input disturbance, the transfer function from $D(s)$ to $E(s)$ is given by

$$E(s) = \frac{P(s)}{1+P(s)C(s)} D(s).$$

Without the integrator, and when $D(s) = \frac{A}{s^{q+1}}$, the steady state error is given by

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{P}{1+PC}\frac{A}{s^q} \\
&= \lim_{s\to 0} \frac{\left(\frac{\frac{1}{m}}{s^2+\frac{b}{m}s+\frac{k}{m}}\right)}{1+\left(\frac{\frac{1}{m}}{s^2+\frac{b}{m}s+\frac{k}{m}}\right)(k_D s + k_P)}\frac{A}{s^q} \\
&= \lim_{s\to 0} \frac{\frac{1}{m}}{\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)+\frac{1}{m}(k_D s + k_P)}\frac{A}{s^q} \\
&= \lim_{s\to 0} \frac{1}{k+k_P}\frac{A}{s^q} \\
&= \frac{A}{k+k_P},
\end{aligned}$$

if $q=0$. Therefore, to an input disturbance the system is type 0. The steady state error when a constant step of size A is placed on $d(t)$ is $\frac{A}{k+k_P}$.

With the integrator, and when $D(s)=\frac{A}{s^{q+1}}$, the steady state error is given by

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{P}{1+PC}\frac{A}{s^q} \\
&= \lim_{s\to 0} \frac{\left(\frac{\frac{1}{m}}{s^2+\frac{b}{m}s+\frac{k}{m}}\right)}{1+\left(\frac{\frac{1}{m}}{s^2+\frac{b}{m}s+\frac{k}{m}}\right)\left(\frac{k_D s^2 + k_P s + k_I}{s}\right)}\frac{A}{s^q} \\
&= \lim_{s\to 0} \frac{s\frac{1}{m}}{s\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)+\frac{1}{m}(k_D s^2 + k_P s + k_I)}\frac{A}{s^q} \\
&= \lim_{s\to 0} \frac{\frac{1}{m}}{s\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)+\frac{1}{m}(k_D s^2 + k_P s + k_I)}\frac{A}{s^{q-1}} \\
&= \lim_{s\to 0} \frac{1}{k_I}\frac{A}{s^{q-1}} \\
&= \frac{A}{k_I},
\end{aligned}$$

if $q=1$. Therefore, to an input disturbance the system is type 1. The steady-state error when $d(t)$ is a constant step is zero, and the steady-state error when $d(t)$ is a ramp of slope of size A is $\frac{A}{k_I}$.