A Design Study Approach to Classical Control

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Homework C.12

(a) Modify the state feedback solution developed in Homework C.11 to add an integrator with anti-windup to the feedback loop for ϕ.

(b) Add a constant input disturbance of 1 Newtons-meters to the input of the plant and allow the plant parameters to vary up to 20%.

(c) Tune the integrator pole (and other gains if necessary) to get good tracking performance.

Solution

Step 1. The original state space equations are

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -0.0300 & 0.0300 & -0.0111 & 0.0111 \\ 0.1357 & -0.1357 & 0.0500 & -0.0500 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0.2105 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x.$$

The integrator will only be on $\phi = x_2$, therefore

$$C_r = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

The the augmented system is therefore

$$A_1 = \begin{pmatrix} A & \mathbf{0} \\ -C_r & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \\ -0.0300 & 0.0300 & -0.0111 & 0.0111 & 0 \\ 0.1357 & -0.1357 & 0.0500 & -0.0500 & 0 \\ 0 & -1.0000 & 0 & 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.2105 \\ 0 \\ 0 \end{pmatrix}$$

Step 2. After the design in HW C.11, the closed loop poles were located at $p_{1,2} = -1.0605 \pm j1.0608$, $p_{3,4} = -0.7016 \pm j0.7018$. We will add the integrator pole at $p_I = -1$. The new controllability matrix

$$\begin{aligned} \mathcal{C}_{A_1,B_1} &= [B_1, A_1B_1, A_1^2B_1, A_1^3B_1, A_1^4B_1] \\ &= \begin{pmatrix} 0 & 0.2105 & -0.0023 & -0.0062 & 0.0008 \\ 0 & 0 & 0.0105 & 0.0279 & -0.0034 \\ 0.2105 & -0.0023 & -0.0062 & 0.0008 & 0.0010 \\ 0 & 0.0105 & 0.0279 & -0.0034 & -0.0044 \\ 0 & 0 & 0 & 0.0105 & 0.0279 \end{pmatrix}. \end{aligned}$$

The determinant is nonzero, therefore the system is controllable.

The open loop characteristic polynomial

$$\begin{aligned} \Delta_{ol}(s) &= \det(sI - A_1) \\ &= s^5 + 0.0611s^4 + 0.1657s^3, \end{aligned}$$

which implies that

$$\mathbf{a}_{A_1} = \begin{pmatrix} 0.0611, & 0.1657, & 0, & 0, & 0 \end{pmatrix}$$

$$\mathcal{A}_{A_1} = \begin{pmatrix} 1 & 0.0611 & 0.1657 & 0 & 0 \\ 0 & 1 & 0.0611 & 0.1657 & 0 \\ 0 & 0 & 1 & 0.0611 & 0.1657 \\ 0 & 0 & 0 & 1 & 0.0611 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The desired closed loop polynomial

$$\begin{aligned}\Delta_{cl}^d(s) &= (s + 1.0605 - j1.0608)(s + 1.0605 + j1.0608)\ldots \\ &\quad (s + 0.7016 - j0.7018)(s + 0.7016 + j0.7018)(s + 1) \\ &= s^5 + 3.4038s^4 + 5.2934s^3 + 4.4761s^2 + 2.0221s + 0.4356,\end{aligned}$$

which implies that

$$\boldsymbol{\alpha} = \begin{pmatrix} 3.4038, & 5.2934 & 4.4761, & 2.0221, & 0.4356 \end{pmatrix}.$$

The augmented gains are therefore given as

$$\begin{aligned}K_1 &= (\boldsymbol{\alpha} - \mathbf{a}_{A_1})\mathcal{A}_{A_1}^{-1}\mathcal{C}_{A_1,B_1}^{-1} \\ &= \begin{pmatrix} 19.1496, & 43.4140, & 16.7190, & 111.6301, & -14.5200 \end{pmatrix}.\end{aligned}$$

Step 3. The feedback gains are therefore given by

$$\begin{aligned}K &= K_1(1:2) = \begin{pmatrix} 19.1496, & 43.4140, & 16.7190, & 111.6301 \end{pmatrix} \\ k_I &= K_1(3) = -14.5200.\end{aligned}$$

Alternatively, we could have used the following Matlab script

```
clear all

% initial conditions
P.theta0    = 0;
P.phi0      = 0;
P.thetadot0 = 0;
P.phidot0   = 0;

% system parameters known to controller
P.Js = 5; % kg m^2
P.Jp = 1;  % kg m^2
P.k = 0.15; % N m
P.b = 0.05; % N m s

% maximum torque
P.taumax = 5; %Nm

% sample rate for controller
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P.Ts = 0.01;

% gain for dirty derivative
P.sigma = 0.05;

% state space design
A = [...
    0, 0, 1, 0;...
    0, 0, 0, 1;...
    -P.k/P.Js, P.k/P.Js, -P.b/P.Js, P.b/P.Js;...
    P.k/P.Jp, -P.k/P.Jp, P.b/P.Jp, -P.b/P.Jp;...

];
B = [0; 0; 1/P.Js; 0];
C = [...
    1, 0, 0, 0;...
    0, 1, 0, 0;...
    ];

% form augmented system
Cout = [0,1,0,0];
A1 = [A, zeros(4,1); -Cout, 0];
B1 = [B; 0];

% tuning parameters
wn_th    = 0.6;
zeta_th = 0.707;
wn_phi    = 1.1;
zeta_phi  = 0.707;
integrator_pole = -1;

% gain selection
ol_char_poly = charpoly(A);
des_char_poly = conv(conv([1,2*zeta_th*wn_th,wn_th^2],...
                [1,2*zeta_phi*wn_phi,wn_phi^2]),...
                poly(integrator_pole));
des_poles = roots(des_char_poly);

% is the system controllable?
if rank(ctrb(A1,B1))≠5,
    disp('System Not Controllable');
else % if so, compute gains
    K1   = place(A1,B1,des_poles);
    P.K  = K1(1:4);
    P.ki = K1(5);
```

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end
```

The complete simulation files are on the wiki associated with this book.