A Design Study Approach to Classical Control

Randal W. Beard   Timothy W. McLain
Brigham Young University

Updated: December 28, 2020

Homework D.6

For the equations of motion for the mass spring damper, define the states as $x = (z, \dot{z})^\top$, and the input as $u = F$, and the measured output as $y = z$. Find the linear state space equations in the form

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du.
\end{aligned}
$$

Solution

Starting with the linear state space equation given by

$$
\ddot{z} = -\frac{b}{m}\dot{z} - \frac{k}{m}z + \frac{1}{m}F,
$$

and defining $x = (z, \dot{z})^\top$ and $u = F$ we get

$$
\begin{aligned}
\dot{x}_1 &= \dot{z} = x_2 \\
\dot{x}_2 &= \ddot{z} = -\frac{b}{m}\dot{z} - \frac{k}{m}z + \frac{1}{m}F \\
&= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u \\
y &= z = x_1.
\end{aligned}
$$

In matrix form

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (0)u,
\end{aligned}
$$

or

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \\
B &= \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \\
C &= \begin{pmatrix} 1 & 0 \end{pmatrix}.
\end{aligned}
$$