A Design Study Approach to Classical Control

Randal W. Beard Timothy W. McLain
Brigham Young University

Updated: December 28, 2020

Homework F.4

Since f_r and f_ℓ appear in the equations as either $f_r + f_\ell$ or $d(f_r - f_\ell)$, it is easier to think of the inputs as the total force $F \stackrel{\triangle}{=} f_r + f_\ell$, and the torque $\tau = d(f_r - f_\ell)$. Note that since

$$\begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ d & -d \end{pmatrix} \begin{pmatrix} f_r \\ f_\ell \end{pmatrix},$$

that

$$\begin{pmatrix} f_r \\ f_\ell \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ d & -d \end{pmatrix}^{-1} \begin{pmatrix} F \\ \tau \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2d} \\ \frac{1}{2} & -\frac{1}{2d} \end{pmatrix} \begin{pmatrix} F \\ \tau \end{pmatrix}.$$

Therefore, in subsequent exercises, if we determine F and τ, then the right and left forces are given by

$$\begin{aligned} f_r &= \frac{1}{2}F + \frac{1}{2d}\tau \\ f_\ell &= \frac{1}{2}F - \frac{1}{2d}\tau. \end{aligned}$$

(a) Find the equilibria of the system.

(b) Linearize the equations about the equilibria using Jacobian linearization.

(c) If possible, linearize the system using feedback linearization.

Solution

Using the expression $F = f_r + f_l$ and $\tau = d(f_r - f_l)$, the equations of motion are given by

$$\begin{pmatrix} m_c + 2m_r & 0 & 0 \\ 0 & m_c + 2m_r & 0 \\ 0 & 0 & J_c + 2m_r d^2 \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{h} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -F\sin\theta - \mu\dot{z} \\ -(m_c + 2m_r)g + F\cos\theta \\ \tau \end{pmatrix}.$$

At equilibria when $\dot{z} = \ddot{z} = \dot{h} = \ddot{h} = \dot{\theta} = \ddot{\theta} = 0$ we have

$$-F_e \sin\theta_e = 0 \tag{1}$$
$$-(m_c + 2m_r)\, g + F_e \cos\theta_e = 0 \tag{2}$$
$$\tau_e = 0. \tag{3}$$

From Equation (1), either $\theta_e = 0$ or $F_e = 0$, since $F_e = 0$ would make Equation (2) impossible to satisfy, we conclude that $\theta_e = 0$. From (1), $F_e = (m_c + 2m_r)\, g$.

To linearize, define

$$\begin{aligned} \tilde{\theta} &\stackrel{\triangle}{=} \theta - \theta_e &\quad\Rightarrow\quad& \theta = \theta_e + \tilde{\theta} \\ \tilde{F} &= F - F_e &\quad\Rightarrow\quad& F = F_e + \tilde{F}. \end{aligned}$$

In addition

$$\begin{aligned} F\sin\theta &\approx F_e \sin\theta_e + \frac{\partial}{\partial F}\left(F\sin(\theta)\right)\Big|_e \tilde{F} + \frac{\partial}{\partial \theta}\left(F\sin(\theta)\right)\Big|_e \tilde{\theta} \\ &= F_e \sin\theta_e + \sin\theta_e \tilde{F} + F_e \cos\theta_e \tilde{\theta} \\ &= F_e \tilde{\theta} \end{aligned}$$

and

$$\begin{aligned} F\cos\theta &\approx F_e \cos\theta_e + \frac{\partial}{\partial F}\left(F\cos\theta\right)\Big|_e \tilde{F} + \frac{\partial}{\partial \theta}\left(F\cos\theta\right)\Big|_e \tilde{\theta} \\ &= F_e \cos\theta_e + \cos\theta_e \tilde{F} - F_e \sin\theta_e \tilde{\theta} \\ &= F_e + \tilde{F}. \end{aligned}$$

Therefore, the linear equations of motion are

$$
\begin{aligned}
(m_c + 2m_r)\,\ddot{\tilde{z}} + \mu\dot{\tilde{z}} &= -F_e\tilde{\theta} \\
(m_c + 2m_r)\,\ddot{\tilde{h}} &= \tilde{F} \\
\left(J_c + 2m_r d^2\right)\ddot{\tilde{\theta}} &= \tilde{\tau}.
\end{aligned}
$$

Since the two nonlinearities are in the equations of motion cannot simultaneously be eliminated by correct choice of the single input F, feedback linearization is not a viable option for this system.