A Design Study Approach to Classical Control

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Homework B.9

(a) When the inner loop controller for the inverted pendulum is PD control, what is the system type with respect to tracking of the inner loop? Characterize the steady state error when $\tilde{\theta}^r$ is a step, a ramp, and a parabola. What is the system type with respect to an input disturbance?

(b) When the outer loop controller for the inverted pendulum is PD control, what is the system type of the outer loop with respect to tracking? Characterize the steady state error when z^r is a step, a ramp, and a parabola. How does this change if you add an integrator? What is the system type with respect to an input disturbance for both PD and PID control?

Solution

The block diagram for the inner loop is shown in Figure 1.

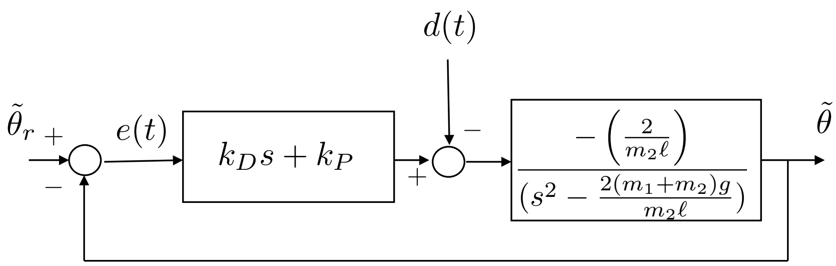

Figure 1: Inner loop system for problem HW B.9.

The open loop system is given by

$$P(s)C(s) = \left(\frac{-\frac{2}{m_2\ell}}{s^2 - \frac{2(m_1+m_2)g}{m_2\ell}} \right) (k_D s + k_P).$$

Since there are no free integrators, the system is type 0, and from Table 9.1 the tracking error when the input is a step is

$$\lim_{t\to\infty} e(t) = \frac{1}{1+M_p} = \frac{1}{1+\lim_{s\to 0} sP(s)C(s)} = \frac{1}{1+\frac{k_P}{(m_1+m_2)g}}.$$

The tracking error when the input is a ramp, or higher order polynomial, is ∞.

For the disturbance input, the steady state error to a step on $d(t)$ is

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} s \frac{P(s)}{1+P(s)C(s)} \frac{1}{s} \\
&= \lim_{s\to 0} \frac{\left(\frac{-\frac{2}{m_2\ell}}{s^2 - \frac{2(m_1+m_2)g}{m_2\ell}} \right)}{1 + \left(\frac{-\frac{2}{m_2\ell}}{s^2 - \frac{2(m_1+m_2)g}{m_2\ell}} \right) (k_D s + k_P)} \\
&= \lim_{s\to 0} \frac{\left(-\frac{2}{m_2\ell}\right)}{\left(s^2 - \frac{2(m_1+m_2)g}{m_2\ell}\right) + \left(-\frac{2}{m_2\ell}\right)(k_D s + k_P)} \\
&= \frac{\left(-\frac{2}{m_2\ell}\right)}{\left(-\frac{2(m_1+m_2)g}{m_2\ell}\right) + \left(-\frac{2k_P}{m_2\ell}\right)} \\
&= \frac{1}{(m_1+m_2)g + k_P}.
\end{aligned}$$

The system is type 0 with respect to the input disturbance.

The block diagram for the outer loop is shown in Figure 2.

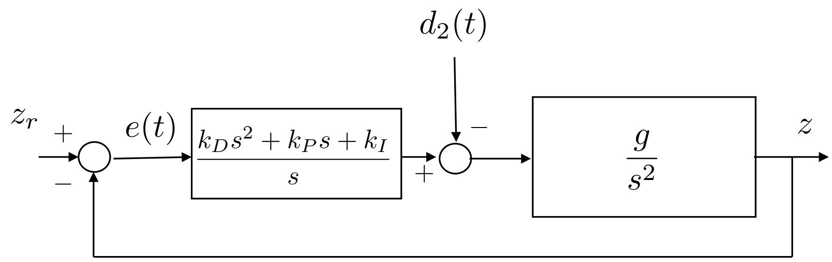

Figure 2: Outer loop system for problem HW B.9.

The open loop system is given by

$$P(s)C(s) = \left(\frac{g}{s^2}\right)\left(\frac{k_D s^2 + k_P s + k_I}{s}\right).$$

When $k_I = 0$ there are two free integrator in $P(s)C(s)$ and the system is type 2, and from Table 9.1 the tracking error when the input is a parabola is

$$\lim_{t\to\infty} e(t) = \frac{1}{M_a} = \frac{1}{\lim_{s\to 0} s^2 P(s)C(s)} = \frac{1}{k_P g}.$$

When $k_I \neq 0$, there are three free integrators in $P(s)C(s)$ and the system is type 3, with zero tracking error for a step, ramp, and parabola.

For the disturbance input, the steady state error when $D(s) = \frac{1}{s^{q+1}}$ is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{P(s)}{1 + P(s)C(s)} \frac{1}{s^{q+1}} = \lim_{s\to 0} \frac{\left(\frac{g}{s^2}\right)}{1 + \left(\frac{g}{s^2}\right)\left(\frac{k_D s^2 + k_P s + k_I}{s}\right)} \frac{1}{s^q}.$$

Without the integrator, i.e., when $k_I = 0$ we have

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{g}{s^2 + g\left(k_D s + k_P\right)} \frac{1}{s^q} \\ &= \lim_{s\to 0} \frac{g}{g\left(k_P\right)} \frac{1}{s^q}\end{aligned}$$

which is finite when $q = 0$. Therefore, the system is type 0 with respect to the input disturbance and the steady state error when $d_2(t)$ is a unit step is $1/k_P$.

When $k_I \neq 0$, we have

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{gs}{s^3 + g\left(k_D s^2 + k_P s + k_I\right)} \frac{1}{s^q} \\
&= \lim_{s\to 0} \frac{g}{g\left(k_I\right)} \frac{1}{s^{q-1}}
\end{aligned}$$

which is finite when $q = 1$. Therefore, the system is type 1 with respect to the input disturbance and the steady state error when $d_2(t)$ is a unit ramp is $1/k_I$.