A Design Study Approach to Classical Control

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Homework E.8

For the ball and beam problem, do the following:

(a) Using the principle of successive loop closure, draw a block diagram that uses PD control for both inner loop control and outer loop control. For design purposes, let the equilibrium position for the ball be $z_e = \frac{\ell}{2}$. The input to the outer loop controller is the desired ball position z_r and the output of the controller is the desired beam angle $\tilde{\theta}_r$. The input to the inner loop controller is the desired beam angle $\tilde{\theta}_r$ and the output is the force $\tilde{F}$ on the end of the beam.

(b) Focusing on the inner loop, find the PD gains k_{P_θ} and k_{D_θ} so that the rise time of the inner loop is $t_{r_\theta} = 1$ second, and the damping ratio is $\zeta_\theta = 0.707$.

(c) Find the DC gain k_{DC_θ} of the inner loop.

(d) Replacing the inner loop by its DC-gain, find the PD gains k_{P_z} and k_{D_z} so that the rise time of the outer loop is $t_{r_z} = 10 t_{r_\theta}$ and the damping ratio is $\zeta_z = 0.707$.

(e) Implement the successive loop closure design for the ball and beam in simulation where the commanded ball position is given by a square wave with magnitude 0.25 ± 0.15 meters and frequency 0.01 Hz. Use the actual position in the ball as a feedback linearizing term for the equilibrium force.

(f) Suppose that the size of the input force on the beam is limited to $F_{\max} =$ 15 N. Modify the simulation to include a saturation block on the force F. Using the rise time of the outer loop, tune the PD control law to get the fastest possible response without input saturation when a step of size 0.25 meter is placed on $\tilde{z}^r$.

Solution

The block diagram for the inner loop is shown in Figure 1, where

$$b_0 = \frac{\ell}{\frac{m_2 \ell^2}{3} + m_1 z_e^2}.$$

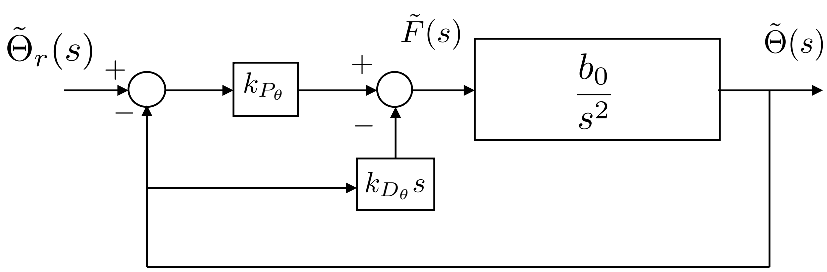

Figure 1: Block diagram for inner loop of the ball beam control.

The closed loop transfer function from $\tilde{\Theta}_r$ to $\tilde{\Theta}$ is given by

$$\tilde{\Theta}(s) = \frac{k_{P_\theta} b_0}{s^2 + k_{D_\theta} b_0 s + k_{P_\theta} b_0} \tilde{\Theta}_r(s).$$

Therefore the closed loop characteristic equation is

$$\Delta_{cl}(s) = s^2 + k_{D_\theta} b_0 s + k_{P_\theta} b_0.$$

The desired closed loop characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2,$$

where

$$\omega_{n_\theta} = \frac{2.2}{t_{r_\theta}} = 2.2$$
$$\zeta_\theta = 0.707.$$

Therefore

$$k_{P_\theta} = \frac{w_{n_\theta}^2}{b_0} = 1.8251$$
$$k_{D_\theta} = \frac{2\zeta_\theta \omega_{n_\theta}}{b_0} = 1.1730.$$

The DC gain of the inner loop is given by

$$k_{DC_\theta} = \frac{k_{P_\theta} b_0}{k_{P_\theta} b_0} = 1.$$

Replacing the inner loop by its DC gain, the block diagram for the outer loop is shown in Figure 2.

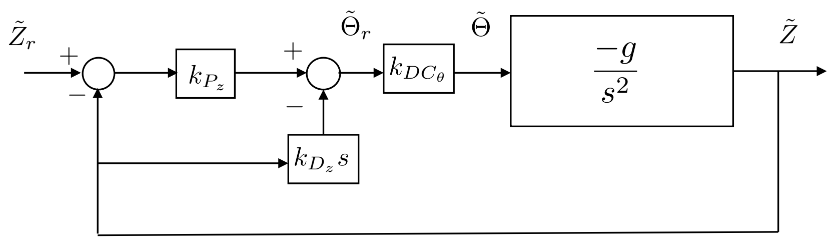

Figure 2: Block diagram for outer loop of ball on beam closed loop system.

The closed loop transfer function from z_r to z is given by

$$Z(s) = \frac{-g k_{P_z}}{s^2 - g k_{D_z} s - g k_{P_z}} Z_r(s).$$

Note that the DC gain for the outer loop is equal to one. The closed loop characteristic equation is therefore

$$\Delta_{cl}(s) = s^2 - g k_{D_z} s - g k_{P_z}.$$

The desired characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2,$$

where

$$t_{r_z} = 10 t_{r_\theta} = 10$$
$$\omega_{n_z} = \frac{2.2}{t_{r_z}} = 0.22$$
$$\zeta_z = 0.707.$$

The PD gains are therefore

$$
\begin{aligned}
k_{P_z} &= -\frac{\omega_{n_z}^2}{g} = -0.0049 \\
k_{D_z} &= -\frac{2\zeta_z \omega_{n_z}}{g} = -0.0317.
\end{aligned}
$$

The Simulink files are on the wiki page associated with the book.