A Design Study Approach to Classical Control

Randal W. Beard Timothy W. McLain
Brigham Young University

Updated: December 28, 2020

Homework D.7

(a) Given the open loop transfer function from force to position in Homework D.5, find the open loop poles of the system, when the equilibrium position is $z_e = 0$.

(b) Using PD control architecture shown in Figure **7-2**, find the closed loop transfer function from z_r to z and find the closed loop poles as a function of k_P and k_D.

(c) Select k_P and k_D to place the closed loop poles at $p_1 = -1$ and $p_2 = -1.5$.

(d) Using the gains from part (c), implement the PD control for the mass spring damper in simulation and plot the response of the system to a 1 m step input.

Solution

The open loop transfer function from homework D.6 is

$$Z(s) = \left(\frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \right) F(s).$$

Using the parameters from Section D gives

$$Z(s) = \left(\frac{0.2}{s^2 + 0.1s + 0.6} \right) F(s).$$

The open loop poles are therefore the roots of the open loop polynomial

$$\Delta_{ol}(s) = s^2 + 0.1s + 0.6,$$

which are given by

$$p_{ol} = -0.05 \pm j0.773.$$

Using PD control, the closed loop system is therefore shown in Figure 1.

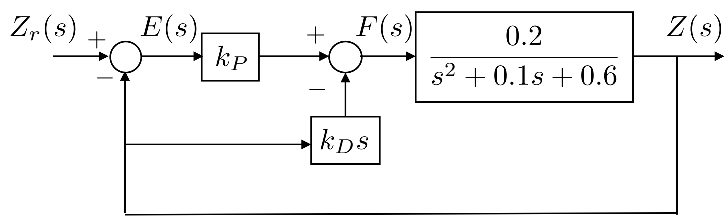

Figure 1: PD control of the mass spring damper.

The transfer function of the closed loop system is given by

$$\begin{aligned}
Z(s) &= \left(\frac{0.2}{s^2+0.1s+0.6}\right)(k_P(Z_r(s) - Z(s)) - k_D s Z(s)) \\
\implies (s^2+0.1s+0.6)Z(s) &= (0.2k_P(Z_r(s) - Z(s)) - 0.2k_D s Z(s)) \\
\implies \left[(s^2+0.1s+0.6) + (0.2k_D s + 0.2k_P)\right] Z(s) &= 0.2k_P Z_r(s) \\
\implies \left[s^2 + (0.1+0.2k_D)s + (0.6+0.2k_P)\right] Z(s) &= 0.2k_P Z_r(s) \\
\implies Z(s) &= \frac{0.2k_P}{s^2 + (0.1+0.2k_D)s + (0.6+0.2k_P)} Z_r(s).
\end{aligned}$$

Therefore, the closed loop poles are given by the roots of the closed loop characteristic polynomial

$$\Delta_{cl}(s) = s^2 + (0.1+0.2k_D)s + (0.6+0.2k_P)$$

which are given by

$$p_{cl} = -\frac{(0.1+0.2k_D)}{2} \pm \sqrt{\left(\frac{(0.1+0.2k_D)}{2}\right)^2 - (0.6+0.2k_P)}$$

If the desired closed loop poles are at -1 and -1.5, then the desired closed loop characteristic polynomial is

$$\begin{aligned}
\Delta_{cl}^d &= (s+1.5)(s+1) \\
&= s^2 + 2.5s + 1.5.
\end{aligned}$$

Equating the actual closed loop characteristic polynomial Δ_{cl} with the desired characteristic polynomial Δ_{cl}^d gives

$$s^2 + (0.1 + 0.2k_D)s + (0.6 + 0.2k_P) = s^2 + 2.5s + 1.5,$$

or by equating each term we get

$$\begin{aligned} 0.1 + 0.2k_D &= 2.5 \\ 0.6 + 0.2k_P &= 1.5. \end{aligned}$$

Solving for k_P and k_D gives

$$\begin{aligned} k_P &= 4.5 \\ k_D &= 12 \end{aligned}$$

The Simulink implementation is contained on the wiki associated with this book.