A Design Study Approach to Classical Control

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Homework D.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces and damping forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

(e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable force input. The output should connect to the animation function developed in homework D.2.

Solution

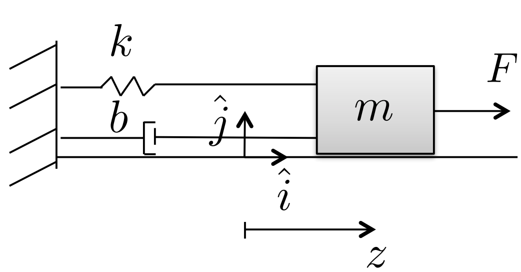

Figure 1: Computing the kinetic energy for the mass-spring-damper.

The generalized coordinates for the mass spring system is the position $\mathbf{q} = z$. The force F is applied in the direction of z and therefore, the generalized force is $\boldsymbol{\tau} = F$. There is a viscous friction force acting in the direction of $\dot{z}$ which implies that $-B\dot{\mathbf{q}} = -b\dot{z}$.

From homework D.1, the kinetic energy can be written in terms of the generalized coordinates as

$$K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\dot{z}^2 = \frac{1}{2}m\dot{q}^2.$$

The potential energy of the system is due to the spring force and is given by

$$P(\mathbf{q}) = \frac{1}{2}kz^2 = \frac{1}{2}kq^2,$$

where k is the spring constant. The Lagrangian is therefore given by

$$L(\mathbf{q}, \dot{\mathbf{q}}) = K - P = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kz^2.$$

Therefore

$$\begin{aligned} \frac{\partial L}{\partial \dot{\mathbf{q}}} &= m\dot{z} \\ \frac{\partial L}{\partial \mathbf{q}} &= -kz. \end{aligned}$$

Differentiating $\frac{\partial L}{\partial \dot{\mathbf{q}}}$ with respect to time gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = m\ddot{z}.$$

Therefore the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}$$

gives

$$m\ddot{z} + kz = F - b\dot{z}.$$