A Design Study Approach to Classical Control

Randal W. Beard   Timothy W. McLain
Brigham Young University

Updated: April 27, 2016

Homework A.2

Using the configuration variable θ, write an expression for the kinetic energy of the system.

Solution

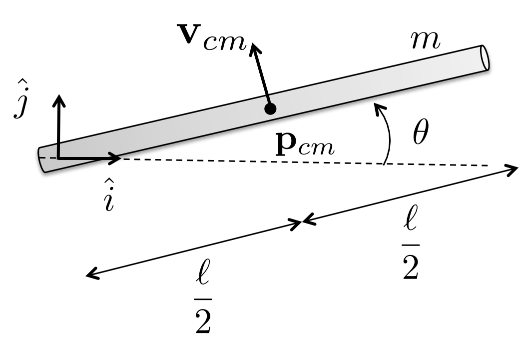

Figure 1: Single link robot arm. Compute the velocity of the center of mass and the angular velocity about the center of mass.

The first step is to define an inertial coordinate system as shown in Figure 1. If the pivot point is the origin of the coordinate system as shown in Figure 1, then the position of the center of mass is

$$\mathbf{p}_{cm}(t) = \begin{pmatrix} \frac{\ell}{2}\cos\theta(t) \\ \frac{\ell}{2}\sin\theta(t) \\ 0 \end{pmatrix}.$$

Differentiating to obtain the velocity we get

$$\mathbf{v}_{cm} = \frac{\ell}{2}\dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

The angular velocity about the center of mass is given by

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}.$$

Observe that

$$\boldsymbol{\omega}^\top J\boldsymbol{\omega} = \begin{pmatrix} 0 & 0 & \dot{\theta} \end{pmatrix}\begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} = J_z\dot{\theta}^2,$$

therefore we only need the moment of inertia about the $\hat{k}$ axis J_z. From the inertia matrix for a thin rod given in Equation (**??**), we have that $J_z = \frac{m\ell^2}{12}$.

Therefore, the kinetic energy of the single link robot arm is

$$\begin{aligned} K &= \frac{1}{2}m\mathbf{v}_{cm}^\top\mathbf{v}_{cm} + \frac{1}{2}\boldsymbol{\omega}^\top J\boldsymbol{\omega} \\ &= \frac{1}{2}m\frac{\ell^2}{4}\dot{\theta}^2(\sin^2\theta + \cos^2\theta) + \frac{1}{2}\dot{\theta}^2\frac{m\ell^2}{12} \\ &= \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2. \end{aligned}$$