A Design Study Approach to Classical Control

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Updated: December 28, 2020

Homework E.4

For the ball on beam system:

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria using Jacobian linearization.

(c) If possible, linearize the system using feedback linearization.

Solution

The differential equations describing the inverted derived in HW E.3 are

$$\begin{aligned} m_1\ddot{z} - m_1 z\dot{\theta}^2 + m_1 g \sin\theta &= 0 \qquad (1)\\ \left(\frac{m_2 l^2}{3} + m_1 z^2\right)\ddot{\theta} + 2m_1 z\dot{z}\dot{\theta} + m_1 g z\cos\theta + \frac{m_2 g l}{2}\cos\theta &= lF\cos\theta. \end{aligned}$$

The equilibria values (z_e, θ_e, F_e) are found by setting $\dot{z} = \ddot{z} = \dot{\theta} = \ddot{\theta} = 0$ in Equation (1) to obtain

$$m_1 g \sin\theta_e = 0 \qquad (2)$$

$$m_1 g z_e \cos\theta_e + \frac{m_2 g l}{2}\cos\theta_e = lF_e\cos\theta_e. \qquad (3)$$

Therefore, any triple (z_e, θ_e, F_e) satisfying Equations (2) and (3) is an equilibria, or in other words z_e can be any value, $\sin\theta_e = 0$ which implies that $\theta_e = 0$, and

$$F_e = \frac{m_1 g}{l} z_e + \frac{m_2 g}{2}.$$

To linearize around (z_e, θ_e, F_e) note that

$$
\begin{aligned}
z\dot{\theta}^2 &\approx z_e\dot{\theta}_e^2 + \frac{\partial}{\partial z}\left(z\dot{\theta}^2\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial \dot{\theta}}\left(z\dot{\theta}^2\right)\bigg|_e \dot{\tilde{\theta}} \\
&= z_e\dot{\theta}_e^2 + \dot{\theta}_e^2\tilde{z} + 2z_e\dot{\theta}_e\dot{\tilde{\theta}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta &\approx \sin\theta_e + \frac{\partial}{\partial \theta}\left(\sin\theta\right)\bigg|_e \tilde{\theta} \\
&= \sin\theta_e + \cos\theta_e\tilde{\theta} \\
&= \tilde{\theta}
\end{aligned}
$$

$$
\begin{aligned}
z\dot{z}\dot{\theta} &\approx z_e\dot{z}_e\dot{\theta}_e + \frac{\partial}{\partial z}\left(z\dot{z}\dot{\theta}\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial z}\left(z\dot{z}\dot{\theta}\right)\bigg|_e \dot{\tilde{z}} + \frac{\partial}{\partial z}\left(z\dot{z}\dot{\theta}\right)\bigg|_e \dot{\tilde{\theta}} \\
&= z_e\dot{z}_e\dot{\theta}_e + \dot{z}_e\dot{\theta}_e\tilde{z} + z_e\dot{\theta}_e\dot{\tilde{z}} + z_e\dot{z}_e\dot{\tilde{\theta}} \\
&= 0 \\
z\cos\theta &\approx z_e\cos\theta_e + \frac{\partial}{\partial z}\left(z\cos\theta\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial \theta}\left(z\cos\theta\right)\bigg|_e \tilde{\theta} \\
&= z_e\cos\theta_e + \cos\theta_e\tilde{z} - z_e\sin\theta_e\tilde{\theta} \\
&= z_e + \tilde{z}
\end{aligned}
$$

$$
\begin{aligned}
\cos\theta &\approx \cos\theta_e + \frac{\partial}{\partial \theta}\left(\cos\theta\right)\bigg|_e \tilde{\theta} \\
&= \cos\theta_e - \sin\theta_e\tilde{\theta} \\
&= 1 \\
z^2\ddot{\theta} &\approx z_e^2\ddot{\theta}_e + \frac{\partial}{\partial z}\left(z^2\ddot{\theta}\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial \ddot{\theta}}\left(z^2\ddot{\theta}\right)\bigg|_e \ddot{\tilde{\theta}} \\
&= z_e^2\ddot{\theta}_e + 2z_e\ddot{\theta}_e\tilde{z} + z_e^2\ddot{\tilde{\theta}} \\
&= z_e^2\ddot{\tilde{\theta}}
\end{aligned}
$$

where we have defined $\tilde{\theta} \stackrel{\triangle}{=} \theta - \theta_e$ and $\tilde{z} = z - z_e$. Also defining $\tilde{F} = F - F_e$, and noting that

$$\begin{aligned}
\ell F\cos\theta &= \ell F_e\cos\theta_e + \frac{\partial}{\partial F}(\ell F\cos\theta)\Big|_e(F-F_e) + \frac{\partial}{\partial\theta}(\ell F\cos\theta)\Big|_e(\theta-\theta_e)\\
&= \ell F_e\cos\theta_e + \ell\cos\theta_e\tilde{F} - \ell F_e\sin\theta_e\tilde{\theta}\\
&= \ell F_e + \ell\tilde{F},
\end{aligned}$$

we can write Equation (1) in its linearized form as

$$\begin{aligned}
m_1[\ddot{z}_e + \ddot{\tilde{z}}] - m_1[0] + m_1 g[\tilde{\theta}] = 0\\
\left(\frac{m_2 l^2}{3} + m_1[z_e^2]\right)[\ddot{\tilde{\theta}}] + 2m_1[0] + m_1 g[z_e + \tilde{z}] + \frac{m_2 g l}{2}[1] = l[F_e + \tilde{F}],
\end{aligned}$$

which simplifies to

$$\begin{aligned}
m_1\ddot{\tilde{z}} + m_1 g\tilde{\theta} = 0\\
\left(\frac{m_2 l^2}{3} + m_1 z_e^2\right)\ddot{\tilde{\theta}} + m_1 g\tilde{z} = l\tilde{F}.
\end{aligned} \tag{4}$$

which are the linearized equations of motion.

The nonlinearities are not all contained in the control channel and therefore cannot be feedback linearized using the techniques discussed in this book.