A Design Study Approach to Classical Control

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Homework A.6

For the feedback linearized equations of motion for the single link robot arm given in Equation (**??**), define the states as $x = (\theta, \dot{\theta})^\top$, and the input as $\tilde{u} = \tilde{\tau}$, and the measured output as $y = \theta$. Find the linear state space equations in the form

$$\begin{aligned}\dot{x} &= Ax + B\tilde{u} \\ y &= Cx + D\tilde{u}.\end{aligned}$$

Solution

The linear state space equations can be derived in two different ways: (1) directly from the linearized equations of motion, and (2) by linearizing the nonlinear equations of motion.

Starting with the linear state space equation in Equation (**??**) and solving for $\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{3}{m\ell^2}\tilde{\tau} - \frac{3b}{m\ell^2}\dot{\theta}.$$

Therefore,

$$\begin{aligned}\dot{x} &\triangleq \begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\dot{\theta} \\ \ddot{\theta}\end{pmatrix} = \begin{pmatrix}\dot{\theta} \\ \frac{3}{m\ell^2}\tilde{\tau} - \frac{3b}{m\ell^2}\dot{\theta}\end{pmatrix} = \begin{pmatrix}x_2 \\ \frac{3}{m\ell^2}\tilde{u} - \frac{3b}{m\ell^2}x_2\end{pmatrix} \\ &= \begin{pmatrix}0 & 1 \\ 0 & -\frac{3b}{2\ell}\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0 \\ \frac{3}{m\ell^2}\end{pmatrix}\tilde{u}.\end{aligned}$$

Assuming that the measured output of the system is $y = \theta$, the linearized output is given by

$$y = \theta = x_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (0)\tilde{u}.$$

Therefore, the linearized state space equations are given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ \frac{3g \sin\theta_e}{2\ell} & -\frac{3b}{2\ell} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \tilde{u} \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x. \end{aligned} \tag{1}$$

Alternatively, the state space equations can be found directly from the nonlinear equations of motion given in Equation (**??**), by forming the nonlinear state space model as

$$\dot{x} \triangleq \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \frac{3}{m\ell^2}\tau - \frac{3b}{m\ell^2}\dot{\theta} - \frac{3g}{2\ell}\cos\theta \end{pmatrix} \triangleq f(x, u).$$

Evaluating the Jacobians at the equilibrium $(\theta_e, \dot{\theta}_e, \tau_e) = (0, 0, \frac{mg\ell}{2})$ gives

$$\begin{aligned} A &= \frac{\partial f}{\partial x}\bigg|_e = \begin{pmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \dot{\theta}} \\ \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \dot{\theta}} \end{pmatrix}\bigg|_e \\ &= \begin{pmatrix} 0 & 0 \\ \frac{3g}{2\ell}\sin\theta & -\frac{3b}{m\ell^2} \end{pmatrix}\bigg|_e \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{3b}{m\ell^2} \end{pmatrix} \\ B &= \frac{\partial f}{\partial u}\bigg|_e = \begin{pmatrix} \frac{\partial f_1}{\partial \tau} \\ \frac{\partial f_2}{\partial \tau} \end{pmatrix}\bigg|_e \\ &= \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix}\bigg|_e \\ &= \begin{pmatrix} 0 \\ \frac{3b}{m\ell^2} \end{pmatrix}. \end{aligned}$$

Similarly, the output is given by $y = \theta = x_1 \triangleq h(x, u)$, which implies that

$$
\begin{aligned}
C &= \left.\frac{\partial h}{\partial x}\right|_e = \left.\begin{pmatrix} \frac{\partial h_1}{\partial \theta} & \frac{\partial h_1}{\partial \dot{\theta}} \end{pmatrix}\right|_e \\
&= \left.\begin{pmatrix} 1 & 0 \end{pmatrix}\right|_e \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \\
D &= \left.\frac{\partial h}{\partial u}\right|_e = 0.
\end{aligned}
$$

The linearized state space equations are therefore given by

$$
\begin{aligned}
\dot{\tilde{x}} &= \begin{pmatrix} 0 & 1 \\ \frac{3g \sin \theta_e}{2\ell} & -\frac{3b}{2\ell} \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \tilde{u} \\
\tilde{y} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \tilde{x},
\end{aligned}
$$

which is similar to Equation (1) but with linearized state and output.

The transfer function can be found from the linearized state space equations using Equation (**??**) as

$$
\begin{aligned}
H(s) &= C(sI - A)^{-1}B + D \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -\frac{3b}{m\ell^2} \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ 0 & s + \frac{3b}{m\ell^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix} \\
&= \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + \frac{3b}{m\ell^2} & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix}}{s^2 + \frac{3b}{m\ell^2} s} \\
&= \frac{\begin{pmatrix} s + \frac{3b}{m\ell^2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{3}{m\ell^2} \end{pmatrix}}{s^2 + \frac{3b}{m\ell^2} s} \\
&= \frac{\frac{3}{m\ell^2}}{s^2 + \frac{3b}{m\ell^2} s},
\end{aligned}
$$

which is identical to Equation (**??**).