A Design Study Approach to Classical Control

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Homework C.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces and damping forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Solution

The generalized coordinates for the satellite attitude control problem are the angles θ and ϕ. Therefore, we define the generalized coordinates as $\mathbf{q} = (\theta, \phi)^\top$ We will assume the ability to apply torque on the main body (θ) of the satellite, but not on the solar panel. Therefore, the generalized force is $\boldsymbol{\tau} = (\tau, 0)^\top$. We will model the dissipation (friction) forces as proportional to the relative motion between the main body and the solar panel. Therefore, the friction term is given by

$$-B\dot{\mathbf{q}} = \begin{pmatrix} -b(\dot{\theta} - \dot{\phi}) \\ -b(\dot{\phi} - \dot{\theta}) \end{pmatrix} = -\begin{pmatrix} b & -b \\ -b & b \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}.$$

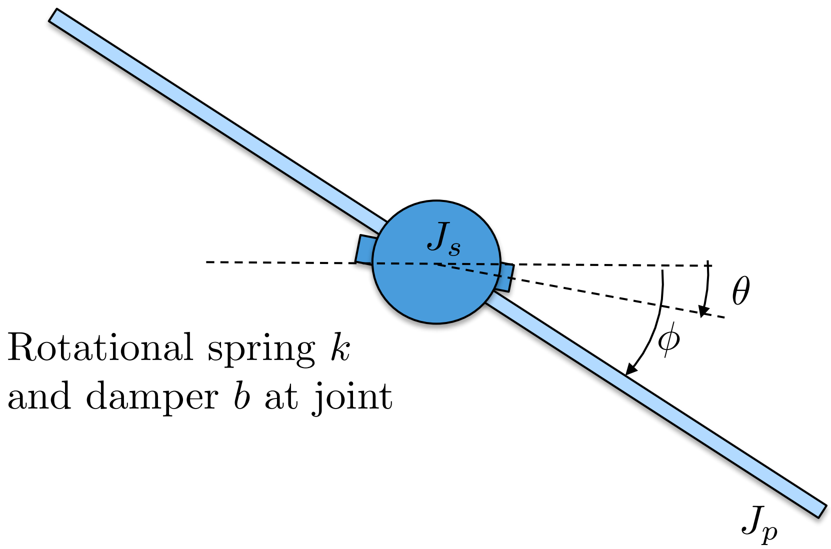

Figure 1: Satellite with flexible solar panels.

Using Equation (**??**), the kinetic energy can be written in terms of the generalized coordinates as

$$\begin{aligned} K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} J_s \dot{\theta}^2 + \frac{1}{2} J_p \dot{\phi}^2 \\ &= \frac{1}{2} J_s \dot{q}_1^2 + \frac{1}{2} J_p \dot{q}_2^2. \end{aligned}$$

The potential energy of the system is due to the spring force between the base and the solar panel, and is given by

$$\begin{aligned} P(\mathbf{q}) &= \frac{1}{2} k (\phi - \theta)^2 \\ &= \frac{1}{2} k (q_2 - q_1)^2, \end{aligned}$$

where k is the spring constant. The Lagrangian is therefore given by

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} J_s \dot{q}_1^2 + \frac{1}{2} J_p \dot{q}_2^2 - \frac{1}{2} k (q_2 - q_1)^2.$$

Therefore

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} J_s\dot{\theta} \\ J_p\dot{\phi} \end{pmatrix}$$
$$\frac{\partial L}{\partial \mathbf{q}} = \begin{pmatrix} k(\phi-\theta) \\ -k(\phi-\theta) \end{pmatrix}.$$

Differentiating $\frac{\partial L}{\partial \dot{\mathbf{q}}}$ with respect to time gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} J_s\ddot{\theta} \\ J_p\ddot{\phi} \end{pmatrix}.$$

Therefore the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}$$

gives

$$\begin{pmatrix} J_s\ddot{\theta} - k(\phi-\theta) \\ J_p\ddot{\phi} + k(\phi-\theta) \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix} + \begin{pmatrix} -b(\dot{\theta}-\dot{\phi}) \\ -b(\dot{\phi}-\dot{\theta}) \end{pmatrix}.$$

Simplifying and moving all second order derivatives to the left and side, and all other terms to the right hand side gives

$$\begin{pmatrix} J_s\ddot{\theta} \\ J_p\ddot{\phi} \end{pmatrix} = \begin{pmatrix} \tau - b(\dot{\theta}-\dot{\phi}) - k(\theta-\phi) \\ -b(\dot{\phi}-\dot{\theta}) - k(\phi-\theta) \end{pmatrix}.$$

Using matrix notation, this equation can be rearranged to isolate the second order derivatives on the left and side

$$\begin{pmatrix} J_s & 0 \\ 0 & J_p \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \tau - b(\dot{\theta}-\dot{\phi}) - k(\theta-\phi) \\ -b(\dot{\phi}-\dot{\theta}) - k(\phi-\theta) \end{pmatrix}. \tag{1}$$

Equation (1) represents the simulation model for the simplified satellite system.