A Design Study Approach to Classical Control

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Homework B.6

Defining the states as $\tilde{x} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$, the input as $\tilde{u} = \tilde{F}$, and the measured output as $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$, find the linear state space equations in the form

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y} &= C\tilde{x} + D\tilde{u}.\end{aligned}$$

Solution

The linear state space equations can be derived in two different ways: (1) directly from the linearized equations of motion, and (2) by linearizing the nonlinear equations of motion.

Starting with the linear state space equation in Equation (4.12) and solving for $(\ddot{\tilde{z}}, \ddot{\tilde{\theta}})^\top$ gives

$$\begin{aligned}\begin{pmatrix}\ddot{\tilde{z}} \\ \ddot{\tilde{\theta}}\end{pmatrix} &= \frac{\begin{pmatrix} m_1\frac{\ell^2}{4} & -m_1\frac{\ell}{2} \\ -m_1\frac{\ell}{2} & (m_1+m_2)\end{pmatrix}}{m_1^2\frac{\ell^2}{4} + m_1 m_2\frac{\ell^2}{4} - m_1^2\frac{\ell^2}{4}} \begin{pmatrix} -b\dot{\tilde{z}} + \tilde{F} \\ m_1 g\frac{\ell}{2}\tilde{\theta}\end{pmatrix} \\ &= \begin{pmatrix} -\frac{b}{m_2}\dot{\tilde{z}} + \frac{1}{m_2}\tilde{F} - \frac{m_1 g}{m_2}\tilde{\theta} \\ 2\frac{b}{m_2\ell}\dot{\tilde{z}} - \frac{2}{m_2\ell}\tilde{F} + \frac{(m_1+m_2)g}{m_2\ell}\tilde{\theta}\end{pmatrix}.\end{aligned}$$

Therefore, defining $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)^\top \stackrel{\triangle}{=} (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$, and $\tilde{u} = \tilde{F}$ gives

$$\dot{\tilde{x}} \stackrel{\triangle}{=} \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{pmatrix} = \begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \\ \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} \tilde{x}_3 \\ \tilde{x}_4 \\ -\frac{b}{m_2}\tilde{x}_3 + \frac{1}{m_2}\tilde{u} - \frac{m_1 g}{m_2}\tilde{x}_2 \\ \frac{2b}{m_2\ell}\tilde{x}_3 - \frac{2}{m_2\ell}\tilde{u} + \frac{2(m_1+m_2)g}{m_2\ell}\tilde{x}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m_1 g}{m_2} & -\frac{b}{m_2} & 0 \\ 0 & \frac{2(m_1+m_2)g}{m_2\ell} & \frac{2b}{m_2\ell} & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m_2} \\ -\frac{2}{m_2\ell} \end{pmatrix} \tilde{u}.$$

Assuming that the measured output of the system is $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$, the linearized output is given by

$$\tilde{y} = \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tilde{u}.$$

Therefore, the linearized state space equations are given by

$$\begin{aligned} \dot{\tilde{x}} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m_1 g}{m_2} & -\frac{b}{m_2} & 0 \\ 0 & \frac{2(m_1+m_2)g}{m_2\ell} & \frac{2b}{m_2\ell} & 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m_2} \\ -\frac{2}{m_2\ell} \end{pmatrix} \tilde{u} \\ \tilde{y} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tilde{x}. \end{aligned} \tag{1}$$

Alternatively, the linearized state space equations can be derived from the nonlinear equations of motion. Starting with Equation (3.2) and solving for $(\ddot{z}, \ddot{\theta})^\top$ gives

$$\begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} = \frac{\begin{pmatrix} m_1\frac{\ell^2}{4} & -m_1\frac{\ell}{2}\cos\theta \\ -m_1\frac{\ell}{2}\cos\theta & (m_1+m_2) \end{pmatrix}}{m_1 m_2 \frac{\ell^2}{4} + m_1^2\frac{\ell^2}{4}\sin^2\theta} \begin{pmatrix} m_1\frac{\ell}{2}\dot{\theta}^2 - b\dot{z} + F \\ m_1 g\ell\sin\theta \end{pmatrix}$$

$$= \frac{\begin{pmatrix} m_1^2\frac{\ell^2}{4}\dot{\theta}^2\sin\theta - bm_1\frac{\ell^2}{4}\dot{z} + m_1\frac{\ell^2}{4}F - m_1^2\frac{\ell^2}{4}g\sin\theta\cos\theta \\ -m_1^2\frac{\ell^2}{4}\dot{\theta}^2\sin\theta\cos\theta - bm_1\frac{\ell^2}{4}\dot{z}\cos\theta - m_1\frac{\ell}{2}\cos\theta F + (m_1+m_2)m_1 g\frac{\ell}{2}\sin\theta \end{pmatrix}}{m_1 m_2\frac{\ell^2}{4} + m_1^2\frac{\ell^2}{4}\sin^2\theta}.$$

Defining $x \triangleq (z, \theta, \dot{z}, \dot{\theta})^\top$, $u = F$, and $y = (z, \theta)^2$ results in the nonlinear state space equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ \frac{m_1^2 \frac{\ell^2}{4} x_4^2 \sin x_2 - b m_1 \frac{\ell^2}{4} x_3 + m_1 \frac{\ell^2}{4} u - m_1^2 \frac{\ell^2}{4} g \sin x_2 \cos x_2}{m_1 m_2 \frac{\ell^2}{4} + m_1^2 \frac{\ell^2}{4} \sin^2 x_2} \\ \frac{-m_1^2 \frac{\ell^2}{4} x_4^2 \sin x_2 \cos x_2 - b m_1 \frac{\ell^2}{4} x_3 \cos x_2 - m_1 \frac{\ell}{2} \cos\theta u + (m_1 + m_2) m_1 g \frac{\ell}{2} \sin x_2}{m_1 m_2 \frac{\ell^2}{4} + m_1^2 \frac{\ell^2}{4} \sin^2 x_2} \end{pmatrix} \triangleq f(x, u)$$

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleq h(x, u).$$

Taking the Jacobians about the equilibrium $x_e = (z_e, 0, 0, 0)^\top$ gives

$$A \triangleq \left.\frac{\partial f}{\partial x}\right|_e = \left.\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\partial f_3}{\partial x_2} & \frac{-b m_1 \frac{\ell^2}{4}}{m_1 m_2 \frac{\ell^2}{4} + m_1 \frac{\ell^2}{4} \sin^2 x_2} & \frac{2 m_1^2 \frac{\ell}{2} x_4 \sin x_2}{m_1 m_2 \frac{\ell^2}{4} + m_1 \frac{\ell^2}{4} \sin^2 x_2} \\ 0 & \frac{\partial f_4}{\partial x_2} & \frac{-b m_1 \frac{\ell}{2} \cos x_2}{m_1 m_2 \frac{\ell^2}{4} + m_1^2 \frac{\ell^2}{4} \sin^2 x_2} & \frac{-2 m_1^2 \frac{\ell^2}{4} x_4 \sin x_2 \cos x_2}{m_1 m_2 \frac{\ell^2}{4} + m_1^2 \frac{\ell^2}{4} \sin^2 x_2} \end{pmatrix}\right|_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m_1 g}{m_2} & \frac{-b}{m_2} & 0 \\ 0 & \frac{2(m_1 + m_2) g}{m_2 \ell} & \frac{2b}{m_2 \ell} & 0 \end{pmatrix}$$

$$B \triangleq \left.\frac{\partial f}{\partial u}\right|_e = \left.\begin{pmatrix} 0 \\ 0 \\ \frac{m_1 \frac{\ell^2}{4}}{m_1 m_2 \frac{\ell^2}{4} + m_1^2 \frac{\ell^2}{4} \sin^2 x_2} \\ -\frac{m_1 \frac{\ell}{2} \cos\theta}{m_1 m_2 \frac{\ell^2}{4} + m_1^2 \frac{\ell^2}{4} \sin^2 x_2} \end{pmatrix}\right|_e = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m_2} \\ -\frac{2}{m_2 \ell} \end{pmatrix}$$

$$C \triangleq \left.\frac{\partial h}{\partial x}\right|_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$D \triangleq \left.\frac{\partial h}{\partial u}\right|_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

resulting in the linearized state space equations given in Equation (1).