A Design Study Approach to Classical Control

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Homework F.8

For the VTOL system, do the following:

(a) For the longitudinal (altitude) dynamics of the VTOL, suppose that the design requirements are that the rise time is $t_r \approx 8$ seconds, with a damping ratio of $\zeta = 0.707$. Find the desired closed-loop characteristic polynomial $\Delta_{cl}^d(s)$, and the associated pole locations. Find the proportional and derivative gains k_{P_h} and k_{D_h} to achieve these specifications, and modify the computer simulation from HW F.8 to verify that the step response satisfies the requirements.

(b) The lateral dynamics (side-to-side) will be designed using successive loop closure with the inner-loop using torque to regulate the roll angle, and the outer-loop using roll angle to regulate the position z. Focusing on the inner loop, find the PD gains k_{P_θ} and k_{D_θ} so that the rise time of the inner loop is $t_{r_\theta} = 0.8$ seconds, and the damping ratio is $\zeta_\theta = 0.707$.

(c) Find the DC gain k_{DC_θ} of the inner loop.

(d) Replacing the inner loop by its DC-gain, find the PD gains k_{P_z} and k_{D_z} so that the rise time of the outer loop is $t_{r_z} = 10 t_{r_\theta}$ and the damping ratio is $\zeta_z = 0.707$.

(e) Implement the successive loop closure design for the lateral control of the VTOL system where the commanded lateral position is given by a square wave with magnitude 3 ± 2.5 meters and frequency 0.08 Hz.

(f) Suppose that the size of the force on each rotor is constrained by $0 \leq f_\ell \leq f_{\max} = 10$ N, and $0 \leq f_r \leq f_{\max} = 10$ N. Modify the simulation to include saturations on the right and left forces f_ℓ and f_r. Using the rise time for the altitude controller, and the rise time for the outer loop of the lateral controller as tuning parameters, tune the controller to achieve the fastest possible response without input saturation.

Solution

A rise time for the altitude controller of $t_{r_h} \approx 8$ seconds, implies that the natural frequency is

$$\omega_n = 2.2/8 = 0.2750.$$

Therefore, the desired characteristic polynomial is

$$\Delta_{cl} = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 0.3889s + 0.0756. \tag{1}$$

From HW F.8 the actual closed loop characteristic polynomial is

$$\Delta_{cl}(s) = s^2 + 0.667k_{D_h}s + 0.667k_{P_h}. \tag{2}$$

Equating Equations (1) and (2) and solving for the gains gives

$$k_{P_h} = 0.1134$$
$$k_{D_h} = 0.5833.$$

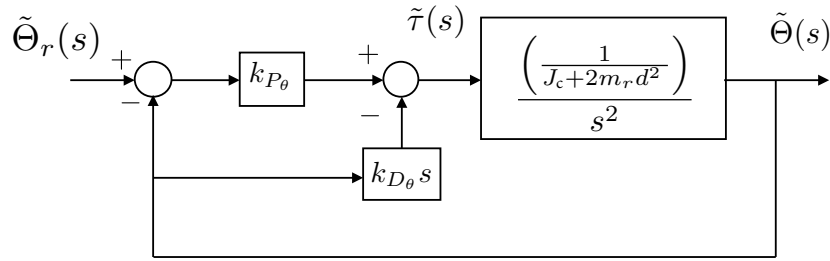The block diagram for the inner loop is shown in Figure 1.

Figure 1: Block diagram for inner loop of the VTOL lateral controller.

The closed loop transfer function from $\tilde{\Theta}_r$ to $\tilde{\Theta}$ is given by

$$\tilde{\Theta}(s) = \frac{\frac{k_{P_\theta}}{J_c + 2m_r d^2}}{s^2 + \frac{k_{D_\theta}}{J_c + 2m_r d^2}s + \frac{k_{P_\theta}}{J_c + 2m_r d^2}}\tilde{\Theta}_r(s).$$

Therefore the closed loop characteristic equation is

$$\Delta_{cl}(s) = s^2 + \frac{k_{D_\theta}}{J_c + 2m_r d^2}s + \frac{k_{P_\theta}}{J_c + 2m_r d^2}.$$

The desired closed loop characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2,$$

where

$$t_{r_\theta} = 0.8$$
$$\omega_{n_\theta} = \frac{2.2}{t_{r_\theta}} = 2.75$$
$$\zeta_\theta = 0.707.$$

Therefore

$$k_{P_\theta} = w_{n_\theta}^2 (J_c + 2m_r d^2) = 0.3721$$
$$k_{D_\theta} = \zeta_\theta \omega_{n_\theta} (J_c + 2m_r d^2) = 0.1913.$$

The DC gain of the inner loop is given by

$$k_{DC_\theta} = 1.$$

Replacing the inner loop by its DC gain, the block diagram for the outer loop is shown in Figure 2.
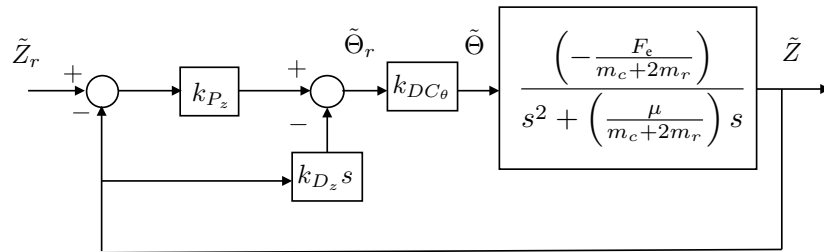

Figure 2: Block diagram for outer loop of VTOL lateral controller.

The closed loop transfer function from z_r to z is given by

$$Z(s) = \frac{b_1 k_{P_z}}{s^2 + (a_1 + b_1 k_{D_z})s + b_1 k_{P_z}} Z_r(s),$$

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where

$$a_1 = \frac{\mu}{m_c + 2m_r}$$

$$b_1 = \frac{-F_e}{m_c + 2m_r}.$$

The closed loop characteristic equation is therefore

$$\Delta_{cl}(s) = s^2 + (a_1 + b_1 k_{D_z})s + b_1 k_{P_z}.$$

The desired characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2,$$

where

$$t_{r_z} = 10 t_{r_\theta} = 8$$

$$\omega_{n_z} = \frac{2.2}{t_{r_z}} = 0.2750$$

$$\zeta_z = 0.707.$$

The PD gains are therefore

$$k_{P_z} = -\frac{\omega_{n_z}^2}{b_1} = -0.0077$$

$$k_{D_z} = -\frac{2\zeta_z \omega_{n_z} - a_1}{b_1} = -0.0328.$$

The Python, Matlab, and Simulink file associated with this problem is included on the wiki associated with the book.