A Design Study Approach to Classical Control

Randal W. Beard   Timothy W. McLain
Brigham Young University

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Homework F.6

(a) Defining the longitudinal states as $\tilde{x}_{lon} = (\tilde{h}, \dot{\tilde{h}})^\top$ and the longitudinal input as $\tilde{u}_{lon} = \tilde{F}$ and the longitudinal output as $\tilde{y}_{lon} = \tilde{h}$, find the linear state space equations in the form

$$\dot{\tilde{x}}_{lon} = A\tilde{x}_{lon} + B\tilde{u}_{lon}$$
$$\tilde{y}_{lon} = C\tilde{x}_{lon} + D\tilde{u}_{lon}.$$

(b) Defining the lateral states as $\tilde{x}_{lat} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$ and the lateral input as $\tilde{u}_{lat} = \tilde{\tau}$ and the lateral output as $\tilde{y}_{lat} = (\tilde{z}, \tilde{\theta})^\top$, find the linear state space equations in the form

$$\dot{\tilde{x}}_{lat} = A\tilde{x}_{lat} + B\tilde{u}_{lat}$$
$$\tilde{y}_{lat} = C\tilde{x}_{lat} + D\tilde{u}_{lat}.$$

Solution

The longitudinal equations of motion are

$$(m_c + 2m_r)\ddot{\tilde{h}} = \tilde{F},$$

which implies that

$$\ddot{\tilde{h}} = \left(\frac{1}{m_c + 2m_r}\right)\tilde{F}.$$

Let $\tilde{x}_{lon} = (\tilde{h}, \dot{\tilde{h}})^\top \stackrel{\triangle}{=} (\tilde{x}_1, \tilde{x}_2)^\top$ and $\tilde{u} = \tilde{F}$ and $\tilde{y} = \tilde{h}$, then

$$\dot{\tilde{x}}_{lon} = \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} \dot{\tilde{h}} \\ \ddot{\tilde{h}} \end{pmatrix} = \begin{pmatrix} \tilde{x}_2 \\ \left(\frac{1}{m_c+2m_r}\right)\tilde{F} \end{pmatrix} = \begin{pmatrix} \tilde{x}_2 \\ \left(\frac{1}{m_c+2m_r}\right)\tilde{u} \end{pmatrix}.$$

Therefore, the state space equations are

$$\begin{aligned} \dot{\tilde{x}}_{lon} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{x}_{lon} + \begin{pmatrix} 0 \\ \frac{1}{m_c+2m_r} \end{pmatrix} \tilde{u} \\ \tilde{y} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \tilde{x}_{lon} + 0\tilde{u}. \end{aligned}$$

The lateral equations of motion are

$$\begin{aligned} (m_c + 2m_r)\ddot{\tilde{z}} &= -F_e\tilde{\theta} - \mu\dot{\tilde{z}} \\ (J_c + 2m_r d^2)\ddot{\tilde{\theta}} &= \tilde{\tau}. \end{aligned}$$

Solving for the highest order derivatives on the left hand side gives which implies that

$$\begin{aligned} \ddot{\tilde{z}} &= -\left(\frac{F_e}{m_c + 2m_r}\right)\tilde{\theta} - \left(\frac{\mu}{m_c + 2m_r}\right)\dot{\tilde{z}} \\ \ddot{\tilde{\theta}} &= \left(\frac{1}{J_c + 2m_r d^2}\right)\tilde{\tau}. \end{aligned}$$

Let $\tilde{x}_{lat} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top \stackrel{\triangle}{=} (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)^\top$ and $\tilde{u} = \tilde{\tau}$ and $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$, then

$$\dot{\tilde{x}}_{lat} = \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{pmatrix} = \begin{pmatrix} \tilde{x}_3 \\ \tilde{x}_4 \\ -\left(\frac{F_e}{m_c+2m_r}\right)\tilde{x}_2 - \left(\frac{\mu}{m_c+2m_r}\right)\tilde{x}_3 \\ \left(\frac{1}{J_c+2m_r d^2}\right)\tilde{u} \end{pmatrix}.$$

Therefore, the state space equations are

$$\begin{aligned} \dot{\tilde{x}}_{lat} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \left(-\frac{F_e}{m_c+2m_r}\right) & \left(-\frac{\mu}{m_c+2m_r}\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tilde{x}_{lat} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{1}{J_c+2m_r d^2}\right) \end{pmatrix} \tilde{u} \\ \tilde{y} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tilde{x}_{lat} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tilde{u}. \end{aligned}$$