A Design Study Approach to Classical Control

Randal W. Beard   Timothy W. McLain
Brigham Young University

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Homework E.6

Defining the states as $\tilde{x} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$, the input as $\tilde{u} = \tilde{F}$, and the measured output as $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$, find the linear state space equations in the form

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y} &= C\tilde{x} + D\tilde{u}.\end{aligned}$$

Solution

Starting with the linear state space equations derived in Homework E.5, and solving for $(\ddot{\tilde{z}}, \ddot{\tilde{\theta}})^\top$ gives

$$\begin{aligned}\ddot{\tilde{z}} &= -g\tilde{\theta} \\ \ddot{\tilde{\theta}} &= \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{F} - \frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{z}.\end{aligned}$$

Defining $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)^\top \triangleq (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$, and $\tilde{u} = \tilde{F}$ gives

$$\dot{\tilde{x}} \triangleq \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{pmatrix} = \begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \\ \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} \tilde{x}_3 \\ \tilde{x}_4 \\ -g\tilde{x}_2 \\ \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{u} - \frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{x}_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -g & 0 & 0 \\ -\frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2} \end{pmatrix} u.$$

Assuming that the measured output of the system is $\tilde{y} = (\tilde{z}, \tilde{\theta})^\top$, the linearized output is given by

$$\tilde{y} = \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tilde{u}.$$

Therefore, the linear state space equations are given by

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$
$$\tilde{y} = C\tilde{x} + D\tilde{u},$$

where

$$A \triangleq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -g & 0 & 0 \\ -\frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2} & 0 & 0 & 0 \end{pmatrix}$$

$$B \triangleq \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2} \end{pmatrix}$$

$$C \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$D \triangleq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$