A Design Study Approach to Classical Control

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Updated: August 19, 2016

Homework B.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces and damping forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Solution

Let P_0 be the potential energy when $\theta = 0$ and $z = 0$. Then the potential energy of the pendulum system is given by

$$P = P_0 + m_1 g \frac{\ell}{2}(1 - \cos\theta),$$

where $\ell/2(1-\cos\theta)$ is the height of the center of mass of the rod. Note that the potential energy decreases as θ increases.

The generalized coordinates for the system are the horizontal position z and the angle θ. Therefore, let $\mathbf{q} = (z, \theta)^\top$.

The external forces acting in the direction of z is $\tau_1 = F$, and the external torque acting in the direction of θ is $\tau_2 = 0$. Therefore, the generalized forces

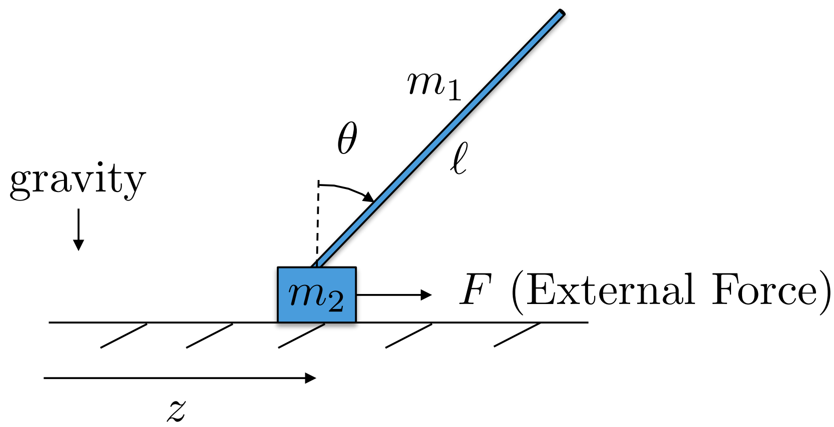

Figure 1: Pendulum on a cart.

are $\boldsymbol{\tau} = (F, 0)^\top$. A damping term acts in the direction of z, which implies that $-B\dot{\mathbf{q}} = (-b\dot{z}, 0)^\top$.

Using Equation (2.4), the kinetic energy can be written in terms of the generalized coordinates as

$$
\begin{aligned}
K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}(m_1 + m_2)\dot{z}^2 + \frac{1}{2}m_1\frac{\ell^2}{4}\dot{\theta}^2 + m_1\frac{\ell}{2}\dot{z}\dot{\theta}\cos\theta \\
&= \frac{1}{2}(m_1 + m_2)\dot{q}_1^2 + \frac{1}{2}m_1\frac{\ell^2}{4}\dot{q}_2^2 + m_1\frac{\ell}{2}\dot{q}_1\dot{q}_2\cos q_2,
\end{aligned}
$$

and the potential energy can be written as

$$
\begin{aligned}
P(\mathbf{q}) &= P_0 + m_1 g\frac{\ell}{2}(1 - \cos\theta) \\
&= P_0 + m_1 g\frac{\ell}{2}(1 - \cos q_2).
\end{aligned}
$$

The Lagrangian is therefore given by

$$
L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}(m_1 + m_2)\dot{q}_1^2 + \frac{1}{2}m_1\frac{\ell^2}{4}\dot{q}_2^2 + m_1\frac{\ell}{2}\dot{q}_1\dot{q}_2\cos q_2 - P_0 - m_1 g\frac{\ell}{2}(1 - \cos q_2).
$$

Therefore

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\mathbf{q}}} &= \begin{pmatrix} (m_1 + m_2)\dot{z} + m_1\frac{\ell}{2}\dot{\theta}\cos\theta \\ m_1\frac{\ell^2}{4}\dot{\theta} - m_1\frac{\ell}{2}\dot{z}\cos\theta \end{pmatrix} \\
\frac{\partial L}{\partial \mathbf{q}} &= \begin{pmatrix} 0 \\ -m_1\frac{\ell}{2}\dot{z}\dot{\theta}\sin\theta + m_1 g\frac{\ell}{2}\sin\theta \end{pmatrix}.
\end{aligned}
$$

Differentiating $\frac{\partial L}{\partial \dot{\mathbf{q}}}$ with respect to time gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} (m_1+m_2)\ddot{z} + m_1\frac{\ell}{2}\ddot{\theta}\cos\theta - m_1\frac{\ell}{2}\dot{\theta}^2\sin\theta \\ m_1\frac{\ell^2}{4}\ddot{\theta} + m_1\frac{\ell}{2}\ddot{z}\cos\theta - m_1\frac{\ell}{2}\dot{z}\dot{\theta}\sin\theta \end{pmatrix}.$$

Therefore the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} - B\dot{\mathbf{q}}$$

gives

$$\begin{pmatrix} (m_1+m_2)\ddot{z} + m_1\frac{\ell}{2}\ddot{\theta}\cos\theta - m_1\frac{\ell}{2}\dot{\theta}^2\sin\theta \\ m_1\frac{\ell^2}{4}\ddot{\theta} + m_1\frac{\ell}{2}\ddot{z}\cos\theta - m_1\frac{\ell}{2}\dot{z}\dot{\theta}\sin\theta + m_1\frac{\ell}{2}\dot{z}\dot{\theta}\sin\theta + m_1 g\frac{\ell}{2}\sin\theta \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} - \begin{pmatrix} b\dot{z} \\ 0 \end{pmatrix}.$$

Simplifying and moving all second order derivatives to the left and side, and all other terms to the right hand side gives

$$\begin{pmatrix} (m_1+m_2)\ddot{z} + m_1\frac{\ell}{2}\ddot{\theta}\cos\theta \\ m_1\frac{\ell^2}{4}\ddot{\theta} + m_1\frac{\ell}{2}\ddot{z}\cos\theta \end{pmatrix} = \begin{pmatrix} m_1\frac{\ell}{2}\dot{\theta}^2\sin\theta + F - b\dot{z} \\ -m_1 g\frac{\ell}{2}\sin\theta \end{pmatrix}.$$

Using matrix notation, this equation can be rearranged to isolate the second order derivatives on the left and side

$$\begin{pmatrix} (m_1+m_2) & m_1\frac{\ell}{2}\cos\theta \\ m_1\frac{\ell}{2}\cos\theta & m_1\frac{\ell^2}{4} \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} m_1\frac{\ell}{2}\dot{\theta}^2\sin\theta + F - b\dot{z} \\ -m_1 g\frac{\ell}{2}\sin\theta \end{pmatrix}. \tag{1}$$

Equation (1) represents the simulation model for the pendulum on a cart system.