A Design Study Approach to Classical Control

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Homework F.5

For the planar VTOL system, note that the system naturally divides into so-called longitudinal dynamics (up-down motion) with total force $\tilde{F}$ as the input and altitude $\tilde{h}$ as the output, and lateral dynamics (side-to-side motion) with $\tilde{\tau}$ as the input and $\tilde{z}$ as the output, with $\tilde{\theta}$ as an intermediate variable.

(a) Start with the linearized equations of motion and use the Laplace transform to convert the equations of motion to the s-domain.

(b) For the longitudinal dynamics, find the transfer function from the input $\tilde{F}(s)$ to the output $\tilde{H}(s)$.

(c) For the lateral dynamics, find the transfer function from the input $\tilde{\tau}(s)$ to the intermediate state $\tilde{\Theta}(s)$ and the output $\tilde{Z}(s)$. Find the transfer function from $\tilde{\Theta}(s)$ to $\tilde{Z}(s)$.

(d) Draw a block diagram of the open loop longitudinal and lateral systems.

Solution

As shown in homework F.5, the feedback linearized model for the longitudinal dynamics is given by

$$(m_c + 2m_r)\ddot{\tilde{h}} = \tilde{F} \tag{1}$$

Taking the Laplace transform of Equation (1) and setting all initial conditions to zero we get

$$(m_c + 2m_r)s^2 H(s) = \tilde{F}(s).$$

Solving for $H(s)$ and putting the transfer function in monic form gives

$$H(s) = \frac{\left(\frac{1}{m_c+2m_r}\right)}{s^2}\tilde{F}(s). \tag{2}$$

where the expression in the parenthesis is the transfer function from $\tilde{F}$ to $\tilde{h}$, where $\tilde{F}$ indicate that we are working with feedback linearized control derived in homework F.5. The block diagram associated with Equation (2) is shown in Figure 1

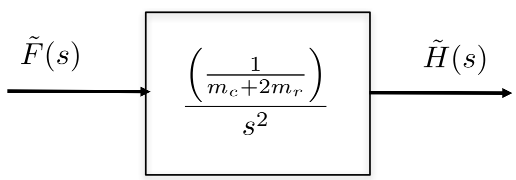

Figure 1: A block diagram of the longitudinal dynamics of the planar VTOL system.

From HW F.5, the linearized equations of motion for the lateral dynamics are given by

$$\begin{aligned}(m_c + 2m_r)\ddot{\tilde{z}} &= -F_e\tilde{\theta} - \mu\dot{\tilde{z}}\\ (J_c + 2m_r d^2)\ddot{\tilde{\theta}} &= \tilde{\tau}\end{aligned}$$

Taking the Laplace transform gives

$$(m_c + 2m_r)\, s^2 \tilde{Z}(s) = -F_0\tilde{\Theta}(s) - \mu s\tilde{Z}(s) \tag{3}$$

$$\left(J_c + 2m_r d^2\right) s^2\tilde{\Theta}(s) = \tilde{\tau}(s). \tag{4}$$

In matrix form we have

$$\begin{pmatrix}(m_c + 2m_r)\, s^2 + \mu s & F_e\\ 0 & (J_c + 2m_r d^2)\, s^2\end{pmatrix}\begin{pmatrix}\tilde{Z}(s)\\ \tilde{\Theta}(s)\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}\tilde{\tau}(s).$$

Defining

$$a_1 \triangleq m_c + 2m_r$$
$$a_2 \triangleq J_c + 2m_r d^2$$

and inverting the matrix on the left gives

$$\begin{aligned}
\begin{pmatrix} \tilde{Z} \\ \tilde{\Theta} \end{pmatrix} &= \begin{pmatrix} a_1 s^2 + \mu s & F_0 \\ 0 & a_2 s^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{\tau}(s) \\
&= \frac{\begin{pmatrix} a_2 s^2 & -F_0 \\ 0 & a_1 s^2 + \mu s \end{pmatrix}}{a_2 s^2 (a_1 s^2 + \mu s)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{\tau}(s) \\
&= \frac{\begin{pmatrix} -F_0 \\ a_1 s^2 + \mu s \end{pmatrix}}{a_2 s^2 (a_1 s^2 + \mu s)} \tilde{\tau}(s) \\
&= \begin{pmatrix} \frac{-\left(\frac{F_0}{a_1 a_2}\right)}{s^3 (s + \frac{\mu}{a_1})} \\ \frac{\left(\frac{1}{a_2}\right)}{s^2} \end{pmatrix} \tilde{\tau}(s),
\end{aligned}$$

which are the transfer function from τ to $\tilde{z}$ and $\tilde{\theta}$.

Returning to Equations (3) and (4) note that the system is naturally in cascade form. From Equation (4) the transfer function from $\tilde{\tau}$ to $\tilde{\theta}$ is

$$\tilde{\Theta}(s) = \frac{\left(\frac{1}{a_2}\right)}{s^2} \tilde{\tau}(s).$$

Similarly, from Equation (3), the transfer function from $\tilde{\theta}$ to $\tilde{z}$ is

$$Z(s) = \frac{-\left(\frac{F_0}{a_1}\right)}{s\left(s + \frac{\mu}{a_1}\right)} \tilde{\Theta}(s).$$

Therefore, the block diagram for linearized lateral subsystem is shown in Figure 2

The cascade approximation makes since physically because the torque on the planar VTOL system has an almost immediate effect on the angle of the aircraft, which leads to side to side motion.

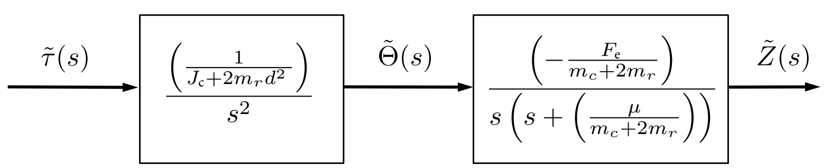

Figure 2: A block diagram of the lateral dynamics of the planar VTOL system.