A Design Study Approach to Classical Control

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Homework F.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates and damping forces.

(c) Find the generalized forces. Note that the right and left forces are more easily modeled as a total force on the center of mass, and a torque about the center of mass.

(d) Derive the equations of motion for the planar VTOL system using the Euler-Lagrange equations.

(e) Referring to Appendices P.1, P.2, and P.3, write a class or s-function that implements the equations of motion. Simulate the system using a variable force inputs f_r and f_ℓ. The output should connect to the animation function developed in homework F.2.

Solution

The generalized coordinates for the system are the lateral position of the center pod z, the altitude of the center pod h, and the angle of the rotors θ. Therefore, let $\mathbf{q} = (z, h, \theta)^\top$.

Let P_0 be the potential energy when $z = 0$, $h = 0$, and $\theta = 0$. Then the potential energy of the planar VTOL system is the sum of the potential

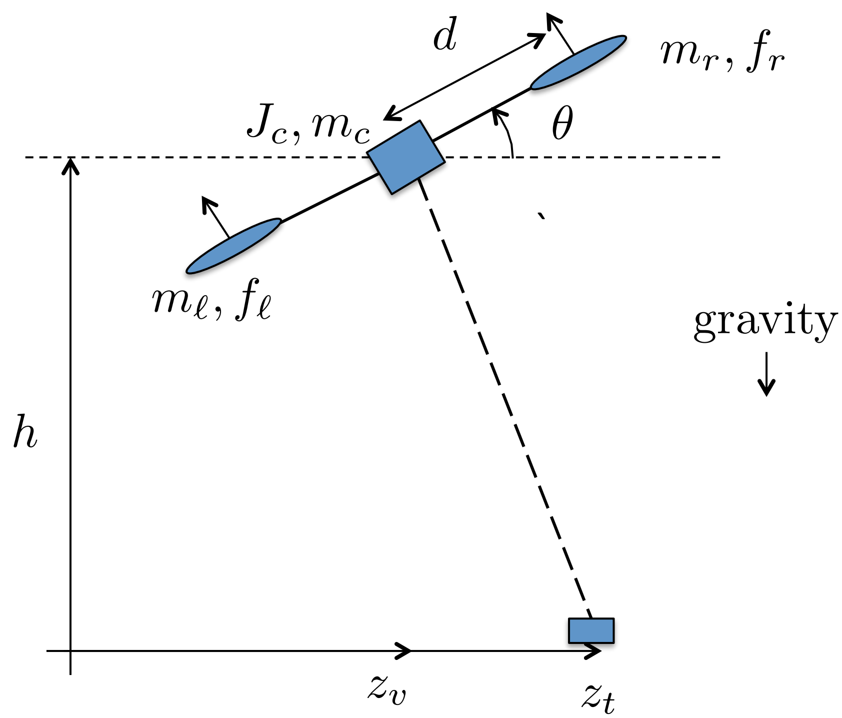

Figure 1: Finding the equations of motion for the planar VTOL system.

energy of the center of mass center pod, and the potential energy of each rotor, modeled as a point mass:

$$
\begin{aligned}
P &= P_0 + m_c g h + m_r g(h + d\sin\theta) + m_l g(h - d\sin\theta) \\
&= (m_c + 2m_r) g h + P_0.
\end{aligned}
$$

The external forces acting in the direction of z, h, and θ are

$$
\begin{aligned}
\tau_1 &= -(f_r + f_l)\sin\theta \\
\tau_2 &= (f_r + f_l)\cos\theta \\
\tau_3 &= d(f_r - f_l).
\end{aligned}
$$

Momentum drag induces a viscous friction term in the direction of z, therefore

$$
-B\dot{\mathbf{q}} = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{h} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -\mu\dot{z} \\ 0 \\ 0 \end{pmatrix}.
$$

From problem F.1, the kinetic energy is

$$K = \frac{1}{2}(m_c + 2m_r)\dot{z}^2 + \frac{1}{2}(m_c + 2m_r)\dot{h}^2 + \frac{1}{2}\left(J_c + 2m_r d^2\right)\dot{\theta}^2.$$

The Lagrangian is therefore given by

$$L = \frac{1}{2}(m_c+2m_r)\dot{z}^2 + \frac{1}{2}(m_c+2m_r)\dot{h}^2 + \frac{1}{2}(J_c+2m_r d^2)\dot{\theta}^2 - (m_c+2m_r)gh - P_0.$$

The Euler Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= \tau_1 - \mu\dot{z} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) - \frac{\partial L}{\partial h} &= \tau_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \tau_3,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial L}{\partial \dot{z}} &= (m_c + 2m_r)\dot{z} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) &= (m_c + 2m_r)\ddot{z} \\
\frac{\partial L}{\partial z} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{h}} &= (m_c + 2m_r)\dot{h} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) &= (m_c + 2m_r)\ddot{h} \\
\frac{\partial L}{\partial h} &= -(m_c + 2m_r)g
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= (J_c + 2m_r d^2)\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= (J_c + 2m_r d^2)\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= 0.
\end{aligned}$$

Therefore the equations of motion are

$$\begin{aligned}
(m_c + 2m_r)\ddot{z} &= -(f_r + f_l)\sin\theta - \mu\dot{z} \\
(m_c + 2m_r)\ddot{h} + (m_c + 2m_r)g &= (f_r + f_l)\cos\theta \\
\left(J_c + 2m_r d^2\right)\ddot{\theta} &= d\left(f_r - f_l\right)
\end{aligned}$$

Using matrix notation, this equation can be rearranged to isolate the second order derivatives on the left and side

$$\begin{pmatrix} m_c + 2m_r & 0 & 0 \\ 0 & m_c + 2m_r & 0 \\ 0 & 0 & J_c + 2m_r d^2 \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{h} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -(f_r + f_l)\sin\theta - \mu\dot{z} \\ -(m_c + 2m_r)g + (f_r + f_l)\cos\theta \\ d\left(f_r - f_l\right) \end{pmatrix}. \tag{1}$$

Equation (1) represents the simulation model for the ball on beam system.