A Design Study Approach to Classical Control

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Homework B.4

For the inverted pendulum,

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria using Jacobian linearization.

Note that for the inverted pendulum, since the nonlinearities are not all contained in the same channel as the control force F, the system cannot be feedback linearized.

Solution

The differential equations describing the inverted pendulum derived in HW **??**.3 are

$$\begin{aligned}(m_1+m_2)\ddot{z} + m_1\frac{\ell}{2}\ddot{\theta}\cos\theta &= m_1\frac{\ell}{2}\dot{\theta}^2\sin\theta - b\dot{z} + F \\ m_1\frac{\ell}{2}\ddot{z}\cos\theta + m_1\frac{\ell^2}{4}\ddot{\theta} &= m_1 g\frac{\ell}{2}\sin\theta.\end{aligned} \tag{1}$$

The equilibria values (z_e, θ_e, F_e) are found by setting $\dot{z} = \ddot{z} = \dot{\theta} = \ddot{\theta} = 0$ in Equation (1) to obtain

$$F_e = 0 \tag{2}$$

$$m_1 g\frac{\ell}{2}\sin\theta_e = 0. \tag{3}$$

Therefore, any triple (z_e, θ_e, F_e) satisfying Equations (2) and (3) is an equilibria, or in other words z_e can be any value, $F_e = 0$ and $\theta_e = k\pi$, where k is an integer.

To linearize around (z_e, θ_e, F_e) where k is an even integer, note that

$$
\begin{aligned}
\ddot{\theta}\cos\theta &\approx \ddot{\theta}_e \cos\theta_e + \frac{\partial}{\partial\theta}(\ddot{\theta}\cos\theta)\Big|_{(\ddot{\theta}_e,\theta_e)}(\theta-\theta_e) + \frac{\partial}{\partial\ddot{\theta}}(\ddot{\theta}\cos\theta)\Big|_{(\ddot{\theta}_e,\theta_e)}(\ddot{\theta}-\ddot{\theta}_e) \\
&= \ddot{\theta}_e\cos\theta_e - (\ddot{\theta}_e\sin\theta_e)\tilde{\theta} + (\cos\theta_e)\ddot{\tilde{\theta}} \\
&= \ddot{\tilde{\theta}} \\
\dot{\theta}^2\sin\theta &\approx \dot{\theta}_e^2\sin\theta_e + \frac{\partial}{\partial\theta}(\dot{\theta}^2\sin\theta)\Big|_{(\theta_e,\dot{\theta}_e)}(\theta-\theta_e) + \frac{\partial}{\partial\dot{\theta}}(\dot{\theta}^2\sin\theta)\Big|_{(\theta_e,\dot{\theta}_e)}(\dot{\theta}-\dot{\theta}_e) \\
&= \dot{\theta}_e^2\sin\theta_e + \dot{\theta}_e^2\cos\theta_e\tilde{\theta} + 2\dot{\theta}_e\sin\theta_e\dot{\tilde{\theta}} \\
&= 0, \\
\ddot{z}\cos\theta &\approx \ddot{z}_e\cos\theta_e + \frac{\partial}{\partial\theta}(\ddot{z}\cos\theta)\Big|_{(\ddot{z}_e,\theta_e)}(\theta-\theta_e) + \frac{\partial}{\partial\ddot{z}}(\ddot{z}\cos\theta)\Big|_{(\ddot{z}_e,\theta_e)}(\ddot{z}-\ddot{z}_e) \\
&= \ddot{z}_e\cos\theta_e - (\ddot{z}_e\sin\theta_e)\tilde{\theta} + (\cos\theta_e)\ddot{\tilde{z}} \\
&= \ddot{\tilde{z}} \\
\sin\theta &\approx \sin\theta_e + \frac{\partial}{\partial\theta}(\sin\theta)\Big|_{\theta_e}(\theta-\theta_e) \\
&= \sin\theta_e + (\cos\theta_e)\tilde{\theta} \\
&= \tilde{\theta},
\end{aligned}
$$

where we have defined $\tilde{\theta} \stackrel{\triangle}{=} \theta - \theta_e$ and $\tilde{z} = z - z_e$. Also defining $\tilde{F} = F - F_e$, and noting that

$$
\begin{aligned}
\theta &= \theta_e + \tilde{\theta} = \tilde{\theta} \\
\dot{\theta} &= \dot{\theta}_e + \dot{\tilde{\theta}} = \dot{\tilde{\theta}} \\
z &= z_e + \tilde{z} \\
\dot{z} &= \dot{z}_e + \dot{\tilde{z}} = \dot{\tilde{z}} \\
\ddot{z} &= \ddot{z}_e + \ddot{\tilde{z}} = \ddot{\tilde{z}} \\
F &= F_e + \tilde{F} = \tilde{F},
\end{aligned}
$$

we can write Equation (1) in its linearized form as

$$
\begin{aligned}
(m_1 + m_2)[\ddot{z}_e + \ddot{\tilde{z}}] + m_1\frac{\ell}{2}[\ddot{\tilde{\theta}}] &= m_1\frac{\ell}{2}[0] - b[\dot{z}_e + \dot{\tilde{z}}] + [F_e + \tilde{F}] \\
m_1\frac{\ell}{2}[\ddot{\tilde{z}}] + m_1\frac{\ell^2}{4}[\ddot{\theta}_e + \ddot{\tilde{\theta}}] &= m_1 g\frac{\ell}{2}[\tilde{\theta}],
\end{aligned}
$$

which simplifies to

$$
\begin{pmatrix} (m_1 + m_2) & m_1\frac{\ell}{2} \\ m_1\frac{\ell}{2} & m_1\frac{\ell^2}{4} \end{pmatrix} \begin{pmatrix} \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} -b\dot{\tilde{z}} + \tilde{F} \\ m_1 g\frac{\ell}{2}\tilde{\theta} \end{pmatrix}, \tag{4}
$$

which are the linearized equations of motion.