A Design Study Approach to Classical Control

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Updated: August 29, 2016

Homework B.8

For the inverted pendulum, do the following:

(a) Using the principle of successive loop closure, draw a block diagram that uses PD control for both inner loop control and outer loop control. The input to the outer loop controller is the desired cart position z^d and the output of the controller is the desired pendulum angle θ^d. The input to the inner loop controller is the desired pendulum angle θ^d and the output is the force F on the cart.

(b) Focusing on the inner loop, find the PD gains k_{P_θ} and k_{D_θ} so that the rise time of the inner loop is $t_{r_\theta} = 0.5$ seconds, and the damping ratio is $\zeta_\theta = 0.707$.

(c) Find the DC gain k_{DC_θ} of the inner loop.

(d) Replacing the inner loop by its DC-gain, find the PD gains k_{P_z} and k_{D_z} so that the rise time of the outer loop is $t_{r_z} = 10 t_{r_\theta}$ and the damping ratio is $\zeta_z = 0.707$.

(e) Implement the successive loop closure design for the inverted pendulum in Simulink where the commanded cart position is given by a square wave with magnitude 0.5 meters and frequency 0.01 Hz.

(f) Suppose that the size of the input force on the cart is limited to $F_{\max} =$ 5 N. Modify the Simulink diagram to include a saturation block on the force F. Using the rise time of the outer loop, tune the PD control law to get the fastest possible response without input saturation when a step of size 1 meter is placed on $\tilde{z}^r$.

Solution

The block diagram for the inner loop is shown in Figure 1.

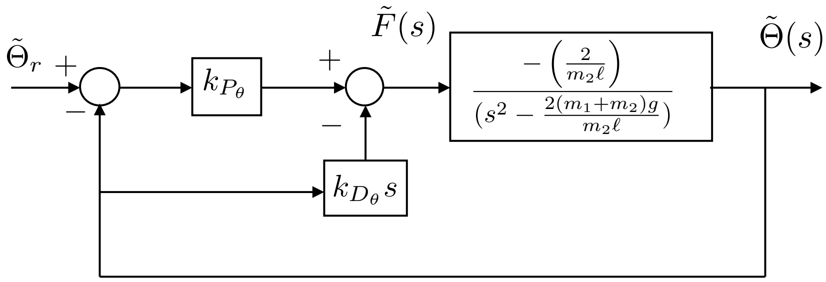

Figure 1: Block diagram for inner loop of inverted pendulum control

The closed loop transfer function from $\tilde{\Theta}^d$ to $\tilde{\Theta}$ is given by

$$\tilde{\Theta}(s) = \frac{-\frac{2k_{P_\theta}}{m_2\ell}}{s^2 - \frac{2k_{D_\theta}}{m_2\ell}s - \left(\frac{2(m_1+m_2)g}{m_2\ell} + \frac{2k_{P_\theta}}{m_2\ell}\right)}\tilde{\Theta}^d(s).$$

Therefore the closed loop characteristic equation is

$$\Delta_{cl}(s) = s^2 - \frac{2k_{D_\theta}}{m_2\ell}s - \left(\frac{2(m_1+m_2)g}{m_2\ell} + \frac{2k_{P_\theta}}{m_2\ell}\right).$$

The desired closed loop characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_\theta\omega_{n_\theta}s + \omega_{n_\theta}^2,$$

where

$$\omega_{n_\theta} = \frac{2.2}{t_{r_\theta}} = 4.4$$
$$\zeta_\theta = 0.707.$$

Therefore

$$k_{P_\theta} = -(m_1+m_2)g - \frac{m_2\ell\omega_{n_\theta}^2}{2} = -17.09$$
$$k_{D_\theta} = -\zeta_\theta\omega_{n_\theta}m_2\ell = -1.5554.$$

The DC gain of the inner loop is given by

$$k_{DC_\theta} = \frac{k_{P_\theta}}{(m_1+m_2)g + k_{P_\theta}} = 3.531.$$

Replacing the inner loop by its DC gain, the block diagram for the outer loop is shown in Figure 2.

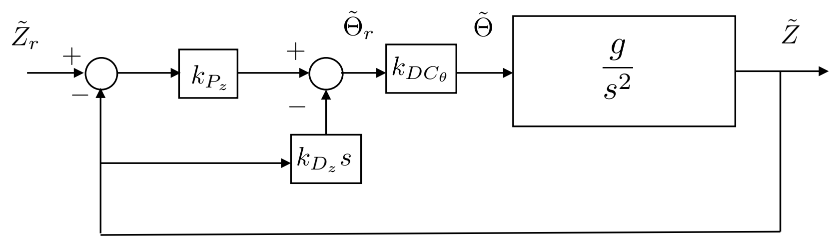

Figure 2: Block diagram for outer loop of inverted pendulum control

The closed loop transfer function from z^d to z is given by

$$Z(s) = \frac{gk_{DC_\theta}k_{P_z}}{s^2 + gk_{DC_\theta}k_{D_z}s + gk_{DC_\theta}k_{P_z}}Z^d(s).$$

Note that the DC gain for the outer loop is equal to one. The closed loop characteristic equation is therefore

$$\Delta_{cl}(s) = s^2 + gk_{DC_\theta}k_{D_z}s + gk_{DC_\theta}k_{P_z}.$$

The desired characteristic equation is

$$\Delta_{cl}^d(s) = s^2 + 2\zeta_z\omega_{n_z}s + \omega_{n_z}^2,$$

where

$$t_{r_z} = 10t_{r_\theta} = 5$$
$$\omega_{n_z} = \frac{2.2}{t_{r_z}} = 0.44$$
$$\zeta_z = 0.707.$$

The PD gains are therefore

$$
\begin{aligned}
k_{P_z} &= \frac{\omega_{n_z}^2}{g k_{DC_z}} = 0.0056 \\
k_{D_z} &= \frac{2\zeta_z \omega_{n_z}}{g k_{DC_\theta}} = 0.1134.
\end{aligned}
$$

See http://controlbook.byu.edu for the complete solution.