A Design Study Approach to Classical Control

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Homework A.9

(a) When the controller for the single link robot arm is PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the case where a constant disturbance acts at the input to the plant (for example gravity in this case). What is the steady state error to a constant input disturbance when the integrator is not present, and when it is present?

Solution

The closed loop system is shown in Figure 1.

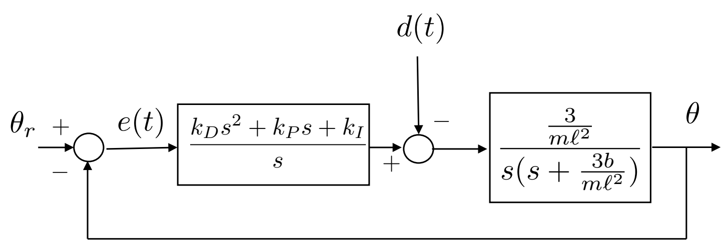

Figure 1: Closed loop system for problem HW **??**.13.

Without the integrator, the open-loop transfer function is given by

$$P(s)C(s) = \left(\frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right) (k_D s + k_P) \,.$$

The system has one free integrator and is therefore type 1, which, from Table **??** implies that the tracking error when the input is a step is zero, and the tracking error when the input is a ramp is

$$\lim_{t\to\infty} e(t) = \frac{1}{M_v} = \frac{1}{\lim_{s\to 0} sP(s)C(s)} = \frac{1}{\frac{k_P}{b}} = \frac{b}{k_P}.$$

The tracking error when the input is a parabola, or higher order polynomial, is ∞, meaning that $\theta(t)$ and $\theta^d(t)$ diverge as $t \to \infty$.

With the integrator, the open loop transfer function is

$$P(s)C(s) = \left(\frac{\frac{3}{m\ell^2}}{s(s + \frac{3b}{m\ell^2})} \right) \left(\frac{k_D s^2 + k_P s + k_I}{s} \right).$$

which has two free integrators and is therefore type 2. Therefore, from Table **??** the tracking error when the input is either a step or a ramp is zero, and the tracking error when the input is a parabola is

$$\lim_{t\to\infty} e(t) = \frac{1}{M_a} = \frac{1}{\lim_{s\to 0} s^2 P(s)C(s)} = \frac{1}{\frac{k_I}{b}} = \frac{b}{k_I}.$$

The tracking error when the input is t^3 or a higher order polynomial, is ∞.

For the input disturbance, the transfer function from $D(s)$ to $E(s)$ is given by

$$E(s) = \frac{P(s)}{1 + P(s)C(s)} D(s).$$

Without the integrator, and when $D(s) = \frac{A}{s^{q+1}}$, the steady state error is given by

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{P}{1+PC}\frac{A}{s^q} \\
&= \lim_{s\to 0}\left(\frac{\left(\frac{\frac{3}{m\ell^2}}{s(s+\frac{3b}{m\ell^2})}\right)}{1+\left(\frac{\frac{3}{m\ell^2}}{s(s+\frac{3b}{m\ell^2})(k_D s+k_P)}\right)}\right)\left(\frac{A}{s^q}\right) \\
&= \lim_{s\to 0}\left(\frac{\frac{3}{m\ell^2}}{s(s+\frac{3b}{m\ell^2})+\frac{3}{m\ell^2}(k_D s+k_P)}\right)\left(\frac{A}{s^q}\right) \\
&= \frac{A\frac{3}{m\ell^2}}{\frac{3k_P}{m\ell^2}} \\
&= \frac{A}{k_P},
\end{aligned}$$

if $q=0$. Therefore, to an input disturbance the system is type 0. The steady state error when a constant step of size A is placed on $d(t)$ is $\frac{A}{k_P}$.

With the integrator, and when $D(s)=\frac{A}{s^{q+1}}$, the steady state error is given by

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{P}{1+PC}\frac{A}{s^q} \\
&= \lim_{s\to 0}\left(\frac{\left(\frac{\frac{3}{m\ell^2}}{s(s+\frac{3b}{m\ell^2})}\right)}{1+\left(\frac{\frac{3}{m\ell^2}}{s(s+\frac{3b}{m\ell^2})}\right)\left(\frac{k_D s^2+k_P s+k_I}{s}\right)}\right)\left(\frac{A}{s^q}\right) \\
&= \lim_{s\to 0}\left(s\frac{\frac{3}{m\ell^2}}{s^2(s+\frac{3b}{m\ell^2})+\frac{3}{m\ell^2}(k_D s^2+k_P s+k_I)}\right)\left(\frac{A}{s^q}\right) \\
&= \frac{A\frac{3}{m\ell^2}}{\frac{3k_I}{m\ell^2}} \\
&= \frac{A}{k_I},
\end{aligned}$$

if $q=1$. Therefore, to an input disturbance the system is type 1. The steady state error when $d(t)$ is a constant step is zero, and the steady state error when $d(t)$ is a ramp of slope of size A is $\frac{A}{k_I}$.