A Design Study Approach to Classical Control

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Homework B.2

Using the configuration variables z and θ, write an expression for the kinetic energy of the system.

Solution

$0.5\ 6_{d}esign_{s}tudies/figures/hw_{p}endulum_{k}inetic_{e}nergy.pdf Pendulumonacart.Computethevelocity$
is given by

$$\mathbf{p}_1 = \begin{pmatrix} z(t) + \frac{\ell}{2}\sin\theta(t) \\ \frac{\ell}{2}\cos\theta(t) \\ 0 \end{pmatrix},$$

and the horizontal position of the cart with mass m_2 is given by

$$\mathbf{p}_2 = \begin{pmatrix} z(t) \\ 0 \\ 0 \end{pmatrix}.$$

Differentiating to obtain the velocities of m_1 and m_2 we obtain

$$\mathbf{v}_1 = \begin{pmatrix} \dot{z} + \frac{\ell}{2}\dot{\theta}\cos\theta \\ -\frac{\ell}{2}\dot{\theta}\sin\theta \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} \dot{z} \\ 0 \\ 0 \end{pmatrix}.$$

Since we are modeling the cart and the bob as point masses and the rod as massless, there is no associated moments of inertia. Therefore the kinetic energy of the system is given by

$$\begin{aligned}
K &= \frac{1}{2}m_1\mathbf{v}_1^\top\mathbf{v}_1 + \frac{1}{2}m_2\mathbf{v}_2^\top\mathbf{v}_2 \\
&= \frac{1}{2}m_1\left[(\dot{z} + \frac{\ell}{2}\dot{\theta}\cos\theta)^2 + (-\frac{\ell}{2}\dot{\theta}\sin\theta)^2\right] + \frac{1}{2}m_2\dot{z}^2 \\
&= \frac{1}{2}m_1\left[\dot{z}^2 + 2\frac{\ell}{2}\dot{z}\dot{\theta}\cos\theta + \frac{\ell^2}{4}\dot{\theta}^2\cos^2\theta + \frac{\ell^2}{4}\dot{\theta}^2\sin^2\theta\right] \\
&\quad + \frac{1}{2}m_2\dot{z}^2 \\
&= \frac{1}{2}(m_1+m_2)\dot{z}^2 + \frac{1}{2}m_1\frac{\ell^2}{4}\dot{\theta}^2 + m_1\frac{\ell}{2}\dot{z}\dot{\theta}\cos\theta.
\end{aligned} \tag{1}$$