Recap on stability of points and sets

Team τ vertical presentation

Mohamed Elobaid

January 23, 2024

Prelude

we start with an “autonomous” system

$$\dot{x} = f(x) \qquad x \in \mathcal{X} \subseteq \mathbb{R}^n$$

we start with an "autonomous" system

$$\dot{x} = f(x) \qquad x \in \mathcal{X} \subseteq \mathbb{R}^n$$

For non-autonmous systems $\dot{x} = f(x, u),\ u \in \mathcal{U}$, typically $u = \gamma(x, x^d)$ (state or output feedback), i.e.

$$\dot{x} = f(x, \gamma(x, \cdot)) \quad \Longrightarrow \quad \dot{x} = \tilde{f}(x)$$

we start with an "autonomous" system

$$\dot{x} = f(x) \qquad x \in \mathcal{X} \subseteq \mathbb{R}^n$$

- Called linear system when $f(x) : \mathcal{X} \mapsto T_x\mathcal{X}$ is a linear mapping, i.e.

$$\dot{x} = Ax$$

- Called nonlinear if it is *not* linear!

we start with an "autonomous" system

$$\dot{x} = f(x) \qquad x \in \mathcal{X} \subseteq \mathbb{R}^n$$

- Called linear system when $f(x) : \mathcal{X} \mapsto T_x\mathcal{X}$ is a linear mapping, i.e.

$$\dot{x} = Ax$$

- Called nonlinear if it is *not* linear!

Example:

$$\dot{x}_1 = -x_2, \ \dot{x}_2 = x_1 \qquad\qquad \dot{x}_1 = x_2, \ \dot{x}_2 = -\alpha \sin x_1 - \beta x_2$$

linear dynamical system — nonlinear dynamical system

Stability of equilibrium points

Definition: equilibrium point [F. Bullo and A. D. Lewis]

a point $x_s \in \mathcal{X}$ is an equilibrium point if the trivial curve $x(t) = x_s,\ \forall t$ is an *integral curve* for $f(\cdot)$

Definition: equilibrium point [F. Bullo and A. D. Lewis]

a point $x_s \in \mathcal{X}$ is an equilibrium point if the trivial curve $x(t) = x_s,\ \forall t$ is an *integral curve* for $f(\cdot)$

- above is a fancy way of saying x_s is an equilibrium point (reads stationary/fixed/critical) **iff** $f(x_s) = 0$
- a consequence of the above for linear systems (i.e. $f(x) = Ax$) is that $x_s \in \mathcal{N}(A)$
- a consequence of the above is that linear systems has either single equilibrium (when $\rho(A) = n$) or
- all points lying in hyperplanes passing through the origin and defined by the basis of the null space are equilibria (infinite and contiguous).

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix} \implies x_s \in \mathsf{Span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$$

Definition: equilibrium point [F. Bullo and A. D. Lewis]

a point $x_s \in \mathcal{X}$ is an equilibrium point if the trivial curve $x(t) = x_s, \ \forall t$ is an *integral curve* for $f(\cdot)$

- when $f(\cdot)$ is nonlinear, non of the statements made concerning uniqueness, number or (in case there are multiple equilibria) being contiguous or belonging to a specific geometric structure hold true

It makes sense then to define stability in terms of small perturbations in the initial conditions with respect to equilibria (reads trivial integral curves).

Definition: stability [Lyaponuv]

An equilibrium point x_s is

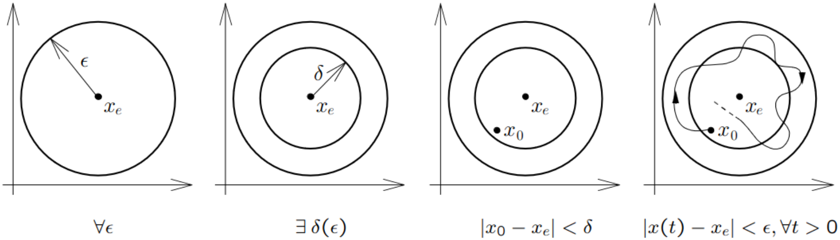
- **stable** if for any neighbourhood V of x_s exists a neighbourhood W of x_s s.t. $\forall x_0 \in W :\ t \mapsto \phi_t^f(x_0) \in V$.
- **Unstable** if it is *not* stable
- **locally asymptotically stable** if stable and $t \mapsto \phi_t^f(x_0)$ *converges* to x_s

Definition: stability [Lyaponuv]

An equilibrium point x_s is

- **stable** if for any neighbourhood V of x_s exists a neighbourhood W of x_s s.t. $\forall x_0 \in W :\ t \mapsto \phi_t^f(x_0) \in V$.
- **Unstable** if it is *not* stable
- **locally asymptotically stable** if stable and $t \mapsto \phi_t^f(x_0)$ *converges* to x_s

A more recent approach is the use of $\epsilon,\ \delta$ notation



Definition: stability [Lyaponuv]

An equilibrium point x_s is

- **stable** if for any neighbourhood V of x_s exists a neighbourhood W of x_s s.t. $\forall x_0 \in W : \; t \mapsto \phi_t^f(x_0) \in V$.
- **Unstable** if it is *not* stable
- **locally asymptotically stable** if stable and $t \mapsto \phi_t^f(x_0)$ *converges* to x_s

- x_s is stable if it is possible to arbitrarily bound the solution to the neighborhood of x_s by suitably selecting x_0
- stability is the property of a point (later set) and **not of the system**
- Unstability does **not** mean solutions diverge away from x_s (e.g. limit cycles)
- convergence does **not** mean stability (e.g. quasi-asymptotically stable points)

Standard tools :

- Direct Lyaponuv method (building a Lyaponuv candidate and reasoning about energy and disspation)
- Invariant set theorem and La Salle's corollary to reason about cases where $\dot{V}(x) \leq 0$
- Indirect Lyaponuv method (reasoning about the stability of a nonlinear system by looking at its linearization)

Others

- Central Manifold theorem (when Indirect method is in-conclusive)
- Barbalat Lemma extending the Invariant set theorems to time-varying systems
- ...

Stability of sets

Definition: set stability [N. P. Bhatia, G. P. Szegö[1]]

A *closed, positively invariant* $\Gamma \subset \mathcal{X}$ is

- **stable** if for any neighbourhood $V(\Gamma)$ then $\forall t: \; t \mapsto \phi_t^f(V) \in W_\epsilon(\Gamma)$.
- **Unstable** if it is *not* stable
- **attractor** if $\forall x_0 \in V(\Gamma) : \lim_{t\to\infty} |\phi_t^f(x_0)\|_\Gamma = 0$ (i.e. converges to Γ).
- **locally asymptotically stable** if stable and attractive

[1] This simpler version of the defintion can be found in Elhawary and Maggiore (TAC)

Definition: set stability [N. P. Bhatia, G. P. Szegö[1]]

A *closed, positively invariant* $\Gamma \subset \mathcal{X}$ is

- **stable** if for any neighbourhood $V(\Gamma)$ then $\forall t: \ t \mapsto \phi_t^f(V) \in W_\epsilon(\Gamma)$.
- **Unstable** if it is *not* stable
- **attractor** if $\forall x_0 \in V(\Gamma) : \lim_{t\to\infty} |\phi_t^f(x_0)\|_\Gamma = 0$ (i.e. converges to Γ).
- **locally asymptotically stable** if stable and attractive

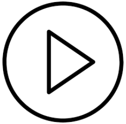
- when Γ is compact (bounded), the definition mirrors that for equilibria
- when Γ is unbounded, domain of attraction not necessary for attractivity!

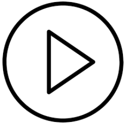
[1] This simpler version of the defintion can be found in Elhawary and Maggiore (TAC)

$$\dot{x} = \begin{pmatrix} -x_2 \\ x_1 \\ x_3 x_4 \\ -2\log(\frac{x_3}{x_1^2+x_2^2} - x_4) - 2x_4 \end{pmatrix}$$

$$\Gamma = \{x \in \mathcal{X} \subset \mathbb{R}^4 : x_1^2 + x_2^2 - x_3 = x_4 = 0\}$$

Check the details in Elobaid et. al (Control Systems Letters)





Definition: Stability and attractivity near a set [Elhawary and Maggiore]

For $\Gamma_1, \Gamma_2 : \Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ closed positively invariant sets

- Γ_2 stable near Γ_1 if $\forall x \in \Gamma_1,\ \forall x_0 \in B_\delta(\Gamma_1), \forall t > 0$ whenever $\phi_t^f(x_0) \subset B_c(x) \implies \phi_t^f(x_0) \subset B_\epsilon(\Gamma_2)$
- Γ_2 attractive near Γ_1 if $\forall x_0 \in V(\Gamma_1)$ we have $\lim_{t\to\infty} \|\phi_t^f(x_0)\|_{\Gamma_2} = 0$

Definition: Stability and attractivity near a set [Elhawary and Maggiore]

For $\Gamma_1, \Gamma_2 : \Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ closed positively invariant sets

- Γ_2 stable near Γ_1 if $\forall x \in \Gamma_1,\ \forall x_0 \in B_\delta(\Gamma_1), \forall t > 0$ whenever $\phi_t^f(x_0) \subset B_c(x) \implies \phi_t^f(x_0) \subset B_\epsilon(\Gamma_2)$
- Γ_2 attractive near Γ_1 if $\forall x_0 \in V(\Gamma_1)$ we have $\lim_{t\to\infty} \|\phi_t^f(x_0)\|_{\Gamma_2} = 0$

- above is a fancy way of saying; Γ_2 is stable near Γ_1 if trajectories starting in some neighbourhood of Γ_1 do not travel far from Γ_2 before leaving that neighbourhood of Γ_1
- above implies that stability of Γ_2 near Γ_1 is a *necessary condition* for stability of Γ_1

Definition: Stability and attractivity near a set [Elhawary and Maggiore]

For $\Gamma_1, \Gamma_2 : \Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ closed positively invariant sets

- Γ_2 stable near Γ_1 if $\forall x \in \Gamma_1,\ \forall x_0 \in B_\delta(\Gamma_1), \forall t > 0$ whenever $\phi_t^f(x_0) \subset B_c(x) \implies \phi_t^f(x_0) \subset B_\epsilon(\Gamma_2)$
- Γ_2 attractive near Γ_1 if $\forall x_0 \in V(\Gamma_1)$ we have $\lim_{t\to\infty} \|\phi_t^f(x_0)\|_{\Gamma_2} = 0$

Motivating examples

- if $\Gamma = \{x \in \mathcal{X}^m : x_1 = x_2 = \cdots = x_m\}$ for m *agents*, stabilizing Γ solves consensus and synchronization problems for multi-agent systems.
- for $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3$, if Γ_1 specifies a "kinematic" behavior of mobile agents, and Γ_2 specifies a *path* and Γ_3 a formation on the path $\rightarrow$ coorindated path-following
- ...

Definition: Stability and attractivity near a set [Elhawary and Maggiore]

For $\Gamma_1, \Gamma_2 : \Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ closed positively invariant sets

- Γ_2 stable near Γ_1 if $\forall x \in \Gamma_1,\ \forall x_0 \in B_\delta(\Gamma_1), \forall t > 0$ whenever $\phi_t^f(x_0) \subset B_c(x) \implies \phi_t^f(x_0) \subset B_\epsilon(\Gamma_2)$
- Γ_2 attractive near Γ_1 if $\forall x_0 \in V(\Gamma_1)$ we have $\lim_{t\to\infty} \|\phi_t^f(x_0)\|_{\Gamma_2} = 0$

- appeal to Lyaponuv methods, after restricting the dynamics to the given set, to test for stability
- similar machinery, **with care**, for criteria

In AMRs, if $\sigma(\cdot) : I \mapsto \mathbb{R}^2$ be a regular plananr curve parametrized s.t. $\sigma(I) = \{w \in \mathbb{R}^2 : s(w) = 0\}$. And let $\Gamma \subset \{q \in \mathcal{X} : (s \circ h)^{-1}(0)\}$ be positively invariant set, then stabilizing Γ via feedback solves path following problems for the planar curve.

Postlude

- the notion of stability is a property of points and sets
- stability concerns the behavior of the dynamics when initial conditions are not precisely the equilibria/positively-invariant-sets
- some peculiarities when studying equilibria and their stability between linear and nonlinear systems
- a more general treatment concerning sets. When the set is a single point, definitions and criteria reduces to that of equilibria
- the general study of stability of sets lends itself to more systematic treatment of complex control problems

Thanks for listening