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Exercise 1 205412882

$Y = X\beta + Z\gamma + \epsilon$ where X is $n \times k \times 1$ and Z is $n \times r$

Let $\hat{\delta}_G = \begin{bmatrix} \hat{\beta}_G \\ \hat{\gamma}_G \end{bmatrix}$ be the estimator of $\delta = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}$

$$\text{var}(\hat{\delta}_G) = \begin{bmatrix} \text{var}(\hat{\beta}_G) & \text{cov}(\hat{\beta}_G, \hat{\gamma}_G) \\ \text{cov}(\hat{\gamma}_G, \hat{\beta}_G) & \text{var}(\hat{\gamma}_G) \end{bmatrix} \qquad \text{cov}(\hat{\beta}_G, \hat{\gamma}_G)' = \text{cov}(\hat{\beta}_G, \hat{\gamma}_G)$$

First we use partial regression to estimate $\hat{\gamma}_G$

residuals: Regress Y on $X \rightarrow e$ $\quad (e = (I-H)Y)$

Regress Z on $X \rightarrow Z^*$ $\quad (Z^* = (I-H)Z)$

Regress e on $Z \longrightarrow \hat{\gamma}_G = (Z^{*\prime}Z^*)^{-1}Z^{*\prime}e$

$$\begin{aligned}
\text{var}(\hat{\gamma}_G) &= (Z^{*\prime}Z^*)^{-1}Z^{*\prime}\,\text{var}(e)\,Z^*\,(Z^{*\prime}Z^*)^{-1} \\
&= (Z^{*\prime}Z^*)^{-1}Z^{*\prime}\,\sigma^2(I-H)\,Z^*\,(Z^{*\prime}Z^*)^{-1} \\
&= \sigma^2\,[Z'(I-H)(I-H)Z]^{-1}\,Z'(I-H)(I-H)(I-H)Z\,[Z'(I-H)(I-H)Z]^{-1} \\
&= \sigma^2\,\underbrace{[Z'(I-H)Z]^{-1}\,Z'(I-H)Z}_{I}\,[Z'(I-H)Z]^{-1} \\
&= \sigma^2\,[Z'(I-H)Z]^{-1}
\end{aligned}$$

Note: $R = I - X(X'X)^{-1}X' = I - H$

$$\text{var}(\hat{\gamma}_G) = \sigma^2[Z'RZ]^{-1} = \sigma^2 M$$

Note: $M = [Z'RZ]^{-1}$

$$\hat{\beta}_G = \hat{\beta}_1 - (X'X)^{-1}X'Z\hat{\gamma}_G$$

Note: $\hat{\gamma}_G = (Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)Y = (X'X)^{-1}X'Y - (X'X)^{-1}X'Z\,(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)Y$

$$\text{var}(\hat{\beta}_G) = \text{var}(\hat{\beta}_1) + \text{var}((X'X)^{-1}X'Z\hat{\gamma}_G) - 2\,\text{cov}\left((X'X)^{-1}X'Y,\ (X'X)^{-1}X'Z\,(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)Y\right)$$

$$\begin{aligned}
\text{cov}\left((X'X)^{-1}X'Y,\ (X'X)^{-1}X'Z\,(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)Y\right) &= (X'X)^{-1}X\,\text{var}(Y)\,[(X'X)^{-1}X'Z\,(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)]' \\
&= \sigma^2(X'X)^{-1}X'(I-H)Z^*\,(Z^{*\prime}Z^*)^{-1}Z'X(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}\underbrace{[(I-H)X]'}_{0}Z^*\,(Z^{*\prime}Z^*)^{-1}Z'X(X'X)^{-1} = 0
\end{aligned}$$

$$\Rightarrow var(\hat{\beta}_u) = var(\hat{\beta}_1) + var((X'X)^{-1}X'Z\,\hat{\gamma}_u) - 2(0)$$
$$= \sigma^2 (X'X)^{-1} + (X'X)^{-1}X'Z\, Var(\hat{\gamma}_u)\,[(X'X)^{-1}X'Z]'$$

Note: $L = (X'X)^{-1}X'Z$

$$= \sigma^2 (X'X)^{-1} + L\, var(\hat{\gamma}_u) L'$$

$$var(\hat{\beta}_u) = \sigma^2 [(X'X)^{-1} + LML']$$

$$cov(\hat{\gamma}_u, \hat{\beta}_u) = cov((Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)Y,\ (X'X)^{-1}X - (X'X)^{-1}X'Z(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)Y)$$
$$= [(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)]\, var(Y)\, [-(X'X)^{-1}X'Z(Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)]$$
$$= -\sigma^2 (Z^{*\prime}Z^*)^{-1}Z^{*\prime}(I-H)(I-H)Z^*(Z^{*\prime}Z^*)^{-1}Z'X(X'X)^{-1}$$
$$= -\sigma^2 (Z'(I-H)Z)^{-1}(Z'(I-H)Z)(Z'(I-H)Z)^{-1}Z'X(X'X)^{-1}$$
$$= -\sigma^2 (Z'(I-H)Z)^{-1}Z'X(X'X)^{-1}$$
$$= -\sigma^2 ML' = \sigma^2(-ML')$$

Note: $L = (X'X)^{-1}X'Z$
$\Rightarrow L' = Z'X(X'X)^{-1}$

$$cov(\hat{\beta}_u, \hat{\gamma}_u) = cov(\hat{\beta} - (X'X)^{-1}X'Z\hat{\gamma}_u,\ \hat{\gamma}_u)$$
$$= cov(\hat{\beta}_1, \hat{\gamma}_u) - cov((X'X)^{-1}X'Z\,\hat{\gamma}_u,\ \hat{\gamma}_u)$$
$$= 0 - (X'X)^{-1}X'Z\, var(\hat{\gamma}_u)$$
$$= -L\sigma^2 M = \sigma^2(-LM)$$

Exercise 2

$Y = X\beta + Z\gamma + \epsilon$ $\qquad$ $\hat{\delta}_G = \begin{bmatrix} \hat{\beta}_G \\ \hat{\gamma}_G \end{bmatrix}$ is the estimator for $\hat{\delta} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}$

X is $n \times k+1$

Z is $n \times t$

Write the model as $Y = X\beta + HZ\gamma + (I-H)Z\gamma + \epsilon$

and $Y = X\alpha + (I-H)Z\gamma + \epsilon$

① $Y = X\beta + HZ\gamma + (I-H)Z\gamma + \epsilon$

$Y = X\beta + X(X'X)^{-1}X'Z\gamma + (I-H)Z\gamma + \epsilon$

$Y = X\underbrace{(\beta + (X'X)^{-1}X'Z\gamma)}_{\alpha} + (I-H)Z\gamma + \epsilon$

② $Y = X\alpha + (I-H)Z\gamma + \epsilon$ $\qquad$ where $\alpha = \beta + (X'X)^{-1}X'Z\gamma$

$\hat{\alpha}_G = \hat{\beta}_G + (X'X)^{-1}X'Z\hat{\gamma}_G$

use the normal equations to find $\hat{\alpha}_G$ and $\hat{\gamma}_G$ for $Y = X\alpha + (I-H)Z\gamma + \epsilon$

$$W'Wm = W'Y$$

$$\hat{m} = (W'W)^{-1}W'Y$$

Let $\hat{m} = \begin{bmatrix} \hat{\alpha}_G \\ \hat{\gamma}_G \end{bmatrix}$ and $W = \begin{bmatrix} \underline{X} & (\underline{I}-\underline{H})\underline{Z} \end{bmatrix}$

$$\begin{bmatrix} \hat{\alpha}_G \\ \hat{\gamma}_G \end{bmatrix} = \left(\begin{bmatrix} X' \\ Z'(I-H) \end{bmatrix} \begin{bmatrix} X & (I-H)Z \end{bmatrix} \right)^{-1} \begin{bmatrix} X' \\ Z'(I-H) \end{bmatrix} \underline{Y}$$

$$= \begin{bmatrix} X'X & X'(I-H)Z \\ Z'(I-H)X & Z'(I-H)Z \end{bmatrix}^{-1} \begin{bmatrix} X' \\ Z'(I-H) \end{bmatrix} \underline{Y}$$

$$= \begin{bmatrix} X'X & 0 \\ 0 & Z'(I-H)Z \end{bmatrix}^{-1} \begin{bmatrix} X' \\ Z'(I-H) \end{bmatrix} \underline{Y}$$

$$= \begin{bmatrix} (X'X)^{-1} & 0 \\ 0 & (Z'(I-H)Z)^{-1} \end{bmatrix} \begin{bmatrix} X' \\ Z'(I-H) \end{bmatrix} \underline{Y}$$

$$= \begin{bmatrix} (X'X)^{-1}X'Y \\ (Z'(I-H)Z)^{-1}Z'(I-H)Y \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_G \\ \hat{\gamma}_G \end{bmatrix}$$

$$\hat{\gamma}_G = (Z'(I-H)Z)^{-1} Z'(I-H)Y$$
$$= [Z'(I-H)(I-H)Z]^{-1} Z'(I-H)(I-H)Y$$
$$= [Z^{*\prime} Z^*]^{-1} Z^{*\prime} e$$

Note:
$e = (I-H)Y$
$Z^* = (I-H)Z$

$$\Rightarrow \hat{\gamma}_G = [Z^{*\prime} Z^*]^{-1} Z^{*\prime} e$$

↳ This result is the same as the method of partial regression we discussed in class.

$$\alpha_G = (X'X)^{-1} X'Y = \hat{\beta}_G + (X'X)^{-1} X'Z \hat{\gamma}_G$$

$$\Rightarrow \hat{\beta}_G = (X'X)^{-1} X'Y - (X'X)^{-1} X'Z \hat{\gamma}_G$$
$$= (X'X)^{-1} X' [Y - Z\hat{\gamma}_G]$$
$$= (X'X)^{-1} X' [Y - Z[Z^{*\prime} Z^*]^{-1} Z^{*\prime} e]$$

$$\Rightarrow \hat{\beta}_G = (X'X)^{-1} X' [Y - Z[Z^{*\prime} Z^*]^{-1} Z^{*\prime} e]$$

Exercise 3

$Y = X\beta + \epsilon$ let $X = [X_1 \; X_2]$

Regress Y on X_1 to get $Y*$:

$(I - H_1)Y = Y*$

Regress each column of X_2 on X to get X_2^*

Let X_2 be $X_2 = [X_p \dots X_K]$

$X_2^* = (I - H_1)[X_p \dots X_K] = (I - H_1)X_p \dots (I - H_1)X_K$

Regress $Y*$ on X_2^* to get $\hat{\beta}_{2\cdot 1}$

$\hat{\beta}_{2\cdot 1} = (X_2^{*\prime} X_2^*)^{-1} X_2^{*\prime} Y*$

show that $Y* - X_2^* \hat{\beta}_{2\cdot 1}$ is orthogonal to X

when $A'B = 0 \Rightarrow A, B$ are orthogonal

Note: $(I - H_1)X =$
$(I - H_1)[X_1 \; X_2]$
$= (I - H_1)X_1 + (I - H_1)X_2$
$= [0 \; (I - H_1)X_2]$

$$\begin{aligned}
(Y* - X_2^* \hat{\beta}_{2\cdot 1})'X &= (Y*' - \hat{\beta}_{2\cdot 1}' X_2^{*\prime}) X \\
&= Y'(I - H_1)X - Y*' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2^{*\prime} X \\
&= Y'(I - H_1)X - Y*' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2'(I - H_1)X \\
&= Y'[0 \;\; (I - H_1)X_2] - Y*' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2'[0 \;\; (I - H_1)X_2] \\
&= [0 \;\; Y'(I - H_1)X_2] - [0 \;\; Y*' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2'(I - H_1)X_2] \\
&= [0 \;\; Y'(I - H_1)(I - H_1)X_2] - [0 \;\; Y*' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2'(I - H_1)(I - H_1)X_2] \\
&= [0 \;\; Y*' X_2^*] - [0 \;\; Y*' X_2^* \underbrace{(X_2^{*\prime} X_2^*) X_2^{*\prime} X_2^*}_{I_n}] \\
&= [0 \;\; Y*' X_2^*] - [0 \;\; Y*' X_2^*] \\
&= [0 \;\; 0]
\end{aligned}$$

Note: $(I - H_1) = (I - H_1)(I - H_1)$

therefore, they are orthogonal

Exercise 4

Read in and create data

```
#Access the data:
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics_c173_c273/jura.txt", head
er=TRUE)

#Rename the variables:
y <- a$Pb

x1 <- a$Cd
x2 <- a$Co
x3 <- a$Cr
x4 <- a$Cu
x5 <- a$Ni
x6 <- a$Zn

#Full regression:
ones <- rep(1, nrow(a))
#create matrix X
X <- as.matrix(cbind(ones,x1, x2, x3, x4, x5, x6))
#solve for beta hat
beta_hat <- solve(t(X) %*% X) %*% t(X) %*% y

#define X1 and X2 for partial regression
X1 <- as.matrix(cbind(ones, x1, x2, x3, x4))
X2 <- as.matrix(cbind(x5, x6))

#define H1 and y star
H1 <- X1 %*% solve(t(X1) %*% X1) %*% t(X1)
y_star <- (diag(nrow(a)) - H1) %*% y

#define x2 star and solve for beta2 hat
X2_star <- (diag(nrow(a)) - H1) %*% X2
beta2_hat <- solve(t(X2_star) %*% X2_star) %*% t(X2_star) %*% y_star

beta2_hat
```

```
##         [,1]
## x5 0.4855964
## x6 0.3004051
```

```
beta_hat
```

```
##              [,1]
## ones 18.9100703
## x1   -1.5192122
## x2   -1.3608686
## x3   -0.1439250
## x4    0.9771788
## x5    0.4855964
## x6    0.3004051
```

```
#define x(o)
X0 <- as.matrix(cbind(x1, x2, x3, x4, x5, x6))
#find the mean sweeper matrix
mean_sweeper <- (diag(nrow(a)) - (1/nrow(a)) * as.matrix(cbind(rep(1, nrow(a)))) %*% as.
matrix(rbind(rep(1, nrow(a)))))

#find x(o) star and y star
X0_star <- (mean_sweeper %*% X0)
Y_star <- mean_sweeper %*% y

#solve for beta(o) hat
B0_hat <- solve(t(X0_star) %*% X0_star) %*% t(X0_star) %*% Y_star


B0_hat
```

```
##            [,1]
## x1 -1.5192122
## x2 -1.3608686
## x3 -0.1439250
## x4  0.9771788
## x5  0.4855964
## x6  0.3004051
```

```
#define X1 and X2
X1 <- as.matrix(cbind(ones, x1, x2))
X2 <- as.matrix(x3)

#Find H1
H1 <- X1 %*% solve(t(X1) %*% X1) %*% t(X1)
#find y star and x2_star
y_star <- (diag(nrow(a)) - H1) %*% y
X2_star <- (diag(nrow(a)) - H1) %*% X2

#solve for beta2 hat
beta2_hat <- solve(t(X2_star) %*% X2_star) %*% t(X2_star) %*% y_star
beta2_hat
```

```
##             [,1]
## [1,] 0.5134709
```

Exercise 5

a. $Y = X\beta + \epsilon \qquad C\beta = \gamma \qquad \gamma \neq 0$

C is an $m \times (K+1)$ matrix that can be partitioned into $[C_1 \; C_2]$ where C_2 is nonsingular

$$C = [C_1 \; C_2] \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \rightarrow \qquad C_1\beta_1 + C_2\beta_2 = \gamma$$

$$\beta_2 = C_2^{-1}[\gamma - C_1\beta_1]$$

using the same partition:

$$Y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

plug in β_2:

$$Y = X_1\beta_1 + X_2 C_2^{-1}[\gamma - C_1\beta_1] + \epsilon$$

$$Y = X_1\beta_1 + X_2 C_2^{-1}\gamma - X_2 C_2^{-1} C_1 \beta_1 + \epsilon$$

$$Y - X_2 C_2^{-1}\gamma = X_2\beta_1 - X_2 C_2^{-1} C_1\beta_1 + \epsilon$$

$$Y - X_2 C_2^{-1}\gamma = [X_2 - X_2 C_2^{-1} C_1]\beta_1 + \epsilon$$

$$Y_r = X_{1r}\beta_1 + \epsilon \rightarrow \text{same form as } Y = X\beta + \epsilon$$

$$\Rightarrow \quad \hat{\beta}_{1c} = (X_{1r}' X_{1r})^{-1} X_{1r}'^{-1} Y_r$$

$$\Rightarrow \quad \hat{\beta}_{2c} = C_2^{-1}[\gamma - C_1\hat{\beta}_{1c}]$$

In order to estimate β:

$$\hat{\beta}_c = \begin{bmatrix} \hat{\beta}_{1c} \\ \hat{\beta}_{2c} \end{bmatrix} = \begin{bmatrix} (X_{1r}' X_{1r})^{-1} X_{1r}'^{-1} Y_r \\ C_2^{-1}[\gamma - C_1\hat{\beta}_{1c}] \end{bmatrix}$$

b. Using the method of lagrange multipliers:

$$Q = (Y - X\beta)'(Y - X\beta) + 2\lambda'(C\beta - \gamma)$$

Minimize Q:

$$\frac{\partial Q}{\partial \beta} = -2X'Y + 2X'X\beta + 2C'\lambda = 0$$

Solve for β to get $\hat{\beta}_c$

$$\hat{\beta}_c = (X'X)^{-1}(X'Y - C'\lambda) \rightarrow C\hat{\beta}_c = C\hat{\beta} - C(X'X)^{-1}C'\lambda$$

$$\lambda = [C(X'X)^{-1}C']^{-1}(C\hat{\beta} - \gamma) = [C(X'X)^{-1}C']^{-1}C\hat{\beta} - [C(X'X)^{-1}C']^{-1}\gamma$$

show $SSE_c - SSE = \sigma^2 \lambda' [var(\lambda)]^{-1} \lambda$

we know:

$$e_c' e_c = e'e + (c\hat{\beta} - \gamma)' [c(X'X)^{-1}c']^{-1} (c\hat{\beta} - \gamma)$$

$$SSE_c = SSE + (c\hat{\beta} - \gamma)' [c(X'X)^{-1}c']^{-1} (c\hat{\beta} - \gamma)$$

$$SSE_c - SSE = (c\hat{\beta} - \gamma)' [c(X'X)^{-1}c']^{-1} (c\hat{\beta} - \gamma)$$

Note: $SSE = e'e$

$SSE_c = e_c' e_c$

$$var(\lambda) = [c(X'X)^{-1}c']^{-1} var(c\hat{\beta}) [c(X'X)^{-1}c']^{-1}$$

$$= [c(X'X)^{-1}c']^{-1} \sigma^2 c(X'X)^{-1}c' [c(X'X)^{-1}c']^{-1}$$

$$= \sigma^2 \underbrace{[c(X'X)^{-1}c']^{-1} c(X'X)^{-1}c'}_{I} [c(X'X)^{-1}c']^{-1}$$

$$= \sigma^2 [c(X'X)^{-1}c']^{-1}$$

Note: $[c(X'X)^{-1}c']^{-1\,\prime} = [c(X'X)^{-1}c']^{-1}$

$$\sigma^2 \lambda' [var(\lambda))]^{-1} \lambda$$

$$= \sigma^2 \lambda' [\sigma^2 [c(X'X)^{-1}c']^{-1}]^{-1} \lambda$$

$$= \sigma^2 \frac{1}{\sigma^2} \lambda' [c(X'X)^{-1}c'] \lambda$$

$$= (c\hat{\beta} - \gamma)' \underbrace{[c(X'X^{-1})c']^{-1} [c(X'X)^{-1}c']}_{I} [c(X'X)^{-1}c']^{-1} (c\hat{\beta} - \gamma)$$

$$= (c\hat{\beta} - \gamma)' [c(X'X)^{-1}c']^{-1} (c\hat{\beta} - \gamma)$$

$$= SSE_c - SSE$$

$$\Rightarrow SSE_c - SSE = \sigma^2 \lambda' [var(\lambda)]^{-1} \lambda$$

Exercise 6

a. $Y = X\beta + \epsilon \qquad \epsilon \sim N(0, \sigma^2 I)$

$$S^2 = \frac{SSE}{n-k+1} \qquad Se^2 = \frac{SSE}{n-k-1} \qquad Se^2 \text{ is unbiased} \Rightarrow E(Se^2) = \sigma^2$$

$$S^2 = \frac{se^2\,(n-k-1)}{n-k+1}$$

$$E(S^2) = E\left[\frac{Se^2\,(n-k-1)}{n-k+1}\right] = \frac{n-k-1}{n-k+1} E[Se^2] = \sigma^2 \left[\frac{n-k-1}{n-k+1}\right]$$

b. $Y = X\beta + \epsilon \qquad \epsilon \sim N(0, \sigma^2 I)$

$$X = \begin{bmatrix} 1 & X_0 \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_0 \\ \beta_{(0)} \end{bmatrix}$$

Normal Equations:

$$1'1\hat{\beta}_0 + 1'X_{(0)}\hat{\beta}_{(0)} = 1'Y$$

$$X_0'1\hat{\beta}_0 + X_{(0)}'X_{(0)}\hat{\beta}_{(0)} = X_0'Y$$

$$\hat{\beta}_0 = (1'1)^{-1}1'Y - (1'1)^{-1}1'X_{(0)}\hat{\beta}_{(0)}$$

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} Y - \frac{1}{n}1' \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{1k} \\ \vdots & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{nk} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$$

$$\hat{\beta}_0 = \bar{Y} - \frac{1}{n}1' \begin{bmatrix} X_{11}\hat{\beta}_1 + \dots + X_{1k}\hat{\beta}_k \\ X_{21}\hat{\beta}_1 + \dots + X_{2k}\hat{\beta}_k \\ \vdots \\ X_{n1}\hat{\beta}_1 + \dots + X_{nk}\hat{\beta}_k \end{bmatrix}$$

$$\hat{\beta}_0 = \bar{Y} - \frac{1}{n}\left[X_{11}\hat{\beta}_1 + \dots + X_{1k}\hat{\beta}_k + \dots + X_{n1}\hat{\beta}_1 + \dots + X_{nk}\hat{\beta}_k\right]$$

$$\hat{\beta}_0 = \bar{Y} - \frac{1}{n}\left[(X_{11} + \dots + X_{n1})\hat{\beta}_1 + \dots + (X_{1k} + \dots + X_{nk})\hat{\beta}_k\right]$$

$$\hat{\beta}_0 = \bar{Y} - \frac{1}{n}\left[\sum X_{i1}\hat{\beta}_1 + \dots + \sum X_{ik}\hat{\beta}_k\right]$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \frac{1}{n} \sum_i X_{i1} + \dots + \hat{\beta}_K \frac{1}{n} \sum_i X_{iK}$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1 + \dots + \hat{\beta}_K \bar{X}_K$$

Exercise 7

$Y = X\beta + \epsilon$ $\quad \epsilon \sim N(0, \sigma^2 I$ $\quad C = [C_1 \;\; C_2]$ $\quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

constraints: $C\beta = \gamma$

From 5a we know: (Assuming C_2 is invertible)

$\hat{\beta}_{1c} = (X_r'X_r)^{-1} X_r' Y_r$ $\quad$ where $\quad X_r = X_1 - X_2 C_2^{-1} C_1$

$\hat{\beta}_{2c} = C_2^{-1}(\gamma - C_1 \hat{\beta}_{1c})$ $\quad\quad Y_r = Y - X_2 C_2^{-1} \gamma$

$$\hat{\beta}_{1c} = (X_r'X_r)^{-1} X_r'(Y - X_2 C_2^{-1} \gamma)$$
$$= (X_r'X_r)^{-1} X_r' Y - (X_r'X_r)^{-1} X_r' X_2 C_2^{-1} \gamma$$

$$\text{var}(\hat{\beta}_{1c}) = \text{var}\left[(X_r'X_r)^{-1} X_r' Y - (X_r'X_r)^{-1} X_r' X_2 C_2^{-1} \gamma\right]$$
$$= [(X_r'X_r)^{-1} X_r'] \text{var}(Y) [(X_r'X_r)^{-1} X_r']'$$
$$= \sigma^2 (X_r'X_r)^{-1} \underbrace{X_r'X_r (X_r'X_r)^{-1}}_{I}$$
$$= \sigma^2 (X_r'X_r)^{-1}$$

$$\hat{\beta}_{2c} = C_2^{-1}(\gamma - C_1 \hat{\beta}_{1c}) = C_2^{-1}\gamma - C_2^{-1} C_1 \hat{\beta}_{1c}$$

$$\text{var}(\hat{\beta}_{2c}) = \text{var}\left[C_2^{-1}\gamma - C_2^{-1} C_1 \hat{\beta}_{1c}\right]$$
$$= C_2^{-1} C_1 \text{var}(\hat{\beta}_{1c}) (C_2^{-1} C_1)'$$
$$= \sigma^2 C_2^{-1} C_1 (X_r'X_r)^{-1} (C_2^{-1} C_1)'$$

$$\text{cov}(\hat{\beta}_{1c}, \hat{\beta}_{2c}) = \text{cov}(\hat{\beta}_{1c}, C_2^{-1}(\gamma - C_1 \hat{\beta}_{1c}))$$
$$= \text{cov}(\hat{\beta}_{1c}, C_2^{-1}\gamma - C_2^{-1} C_1 \hat{\beta}_{1c})$$
$$= \text{cov}(\hat{\beta}_{1c}, \hat{\beta}_{1c})(-C_2^{-1} C_1)'$$
$$= \text{var}(\hat{\beta}_{1c})(-C_2^{-1} C_1)'$$
$$= -\sigma^2 (X_r'X_r)^{-1} (C_2^{-1} C_1)'$$

Note: $\text{cov}(aX, bY)$
$= a\,\text{cov}(X, Y)\, b'$

$$\text{var}(\hat{\beta}_c) = \begin{bmatrix} \text{var}(\hat{\beta}_{1c}) & \text{cov}(\hat{\beta}_{1c}, \hat{\beta}_{2c}) \\ \text{cov}(\hat{\beta}_{1c}, \hat{\beta}_{2c})' & \text{var}(\hat{\beta}_{2c}) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 (X_r'X_r)^{-1} & -\sigma^2 (X_r'X_r)^{-1} (C_2^{-1} C_1)' \\ -\sigma^2 (C_2^{-1} C_1)(X_r'X_r)^{-1} & \sigma^2 C_2^{-1} C_1 (X_r'X_r)^{-1} (C_2^{-1} C_1)' \end{bmatrix}$$

Exercise 8

a. $Y = X\beta + \epsilon \qquad C\beta = \gamma$

From lecture, we know:

$$e_c'e_c = e'e + (C\hat{\beta} - \gamma)' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)$$

so, we need to show: $(C\hat{\beta} - \gamma)' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma) = (\hat{\beta}_c - \hat{\beta})' X'X (\hat{\beta}_c - \hat{\beta})$

Note: $\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}C' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)$

$$(\hat{\beta}_c - \hat{\beta})X'X(\hat{\beta}_c - \hat{\beta}) =$$

$$- (C\hat{\beta} - \gamma)' [C(X'X)^{-1}C]^{-1} C \underbrace{(X'X)^{-1}X'X}_{I} (\hat{\beta}_c - \beta)$$

$$= (C\hat{\beta} - \gamma)' \underbrace{[C(X'X)^{-1}C']^{-1} C(X'X)^{-1}C'}_{I} [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)$$

$$= (C\hat{\beta} - \gamma)' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)$$

$$\Rightarrow e_c'e_c = e'e + (\hat{\beta}_c - \hat{\beta})'X'X(\hat{\beta}_c - \hat{\beta})$$

b. Show that $e_c'e_c = SST$ when $C = [\underline{0} \;\; I_5]$ and $\gamma = 0$

From part a we know:

$$e_c'e_c = e'e + (\hat{\beta}_c - \hat{\beta})'X'X(\hat{\beta}_c - \hat{\beta})$$

Note: $SSR = (\hat{Y} - \bar{Y})'(\hat{Y} - \bar{Y}) = (\hat{Y} - \frac{1}{n}11'Y)'(\hat{Y} - \frac{1}{n}11'Y)$

Note: $SST = SSE + SSR$, so we need to show

$$(\hat{\beta}_c - \hat{\beta})'X'X(\hat{\beta}_c - \beta) = SSR$$

Note: $\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}C' [C'(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)$

Solve for $[C'(X'X)^{-1}C']$ using the inverse of a partition matrix:

$$C'(X'X)^{-1}C' = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} n & 1'X_{(0)} \\ X_{(0)}'1 & X_{(0)}'X_{(0)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = \left[X_{(0)}'X_{(0)} - \frac{1}{n} X_{(0)}'11'X_{(0)} \right]^{-1}$$

$$\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}C' \left[X_{(0)}'X_{(0)} - \frac{1}{n} X_{(0)}'11'X_{(0)} \right] (C\hat{\beta} - \gamma)$$

$\uparrow \; C = [0 \;\; I_5] \quad \gamma = 0$

$$\hat{\beta}_c = \hat{\beta} - (X'X)^{-1}C' \left[X_{(0)}'X_{(0)} - \frac{1}{n} X_{(0)}'11'X_{(0)} \right] \hat{\beta}_{(0)}$$

$$\Rightarrow (\hat\beta_c - \hat\beta)'(X'X)(\hat\beta_c - \hat\beta)$$
$$= \hat\beta_{(0)}'(X_{(0)}'(I - \tfrac{1}{n}11')X_{(0)})'c\ (X'X)^{-1}\ (X_{(0)}'(I - \tfrac{1}{n}11')X_{(0)})\hat\beta_{(0)}$$
$$= \hat\beta_{(0)}'(X_{(0)}(I - \tfrac{1}{n}11')X_{(0)})'\hat\beta_{(0)}$$

Note: $\hat Y = \beta_{(0)}1 + X_{(0)}\hat\beta_{(0)} \rightarrow \hat Y - \beta_{(0)}1 = X_{(0)}\hat\beta_{(0)}$

$$= (\hat Y - \beta_0 1)'(I - \tfrac{1}{n}11')(I - \tfrac{1}{n}11')(\hat Y - \beta_0 1)$$
$$= (\hat Y - \beta_0 1)'(I - \tfrac{1}{n}11')(\hat Y - \beta_0 1 - \tfrac{1}{n}11'\hat Y + \tfrac{1}{n}11'\beta_0 1)$$
$$= (\hat Y - \beta_0 1)'(I - \tfrac{1}{n}11')(\hat Y\ \beta_0 1 + \beta_0 1 - \tfrac{1}{n}11'Y)$$
$$= (\hat Y - \tfrac{1}{n}11'Y)'\ (\hat Y - \tfrac{1}{n}11'Y) = SSR$$

$$\Rightarrow e_c'e_c = SST$$

c. $R_c^2 = 1 - \dfrac{SSE_c}{SST}$ $\qquad R^2 = 1 - \dfrac{SSE}{SST}$

From a we know $SSE_c = SSE + \underbrace{(\hat\beta_c - \beta)X'X(\hat\beta_c - \beta)}_{K \geq 0}$ because $X'X$ is positive definite

$$\Rightarrow \frac{SSE_c}{SST} \geq \frac{SSE}{SST} \Rightarrow R_c^2 \leq R^2$$

d. $c\beta = \gamma$

From lecture we know:
$$e_c'e_c = e'e + (c\hat\beta - \gamma)'[c(X'X)^{-1}c']^{-1}(c\hat\beta - \gamma)$$

$$E[e_c'e_c] = E[e'e + (c\hat\beta - \gamma)'[c(X'X)^{-1}c']'(c\hat\beta - \gamma)]$$
$$= E[e'e] + E[(c\hat\beta - \gamma)'[c(X'X)^{-1}c']^{-1}(c\hat\beta - \gamma)]$$
$$= (n-k-1)\sigma^2 + E[(c\hat\beta - \gamma)'[c(X'X)^{-1}c']^{-1}(c\hat\beta - \gamma)]$$
$$= (n-k-1)\sigma^2 + E[tr((c\hat\beta - \gamma)'[c(X'X)^{-1}c']^{-1}(c\hat\beta - \gamma))]$$
$$= (n-k-1)\sigma^2 + tr([c(X'X)^{-1}c']^{-1}\ E[(c\hat\beta - \gamma)'(c\hat\beta - \gamma)])$$

Note: $E[(c\hat\beta - \gamma)'(c\hat\beta - \gamma)]) = var(c\hat\beta - \gamma) + E[c\hat\beta - \gamma]E[c\hat\beta - \gamma]'$

Note: $E[c\hat\beta - \gamma] = c\beta - \gamma = \gamma - \gamma = 0$

because $c\beta = \gamma$ by our constraint

$$= var(c\hat\beta - \gamma)$$
$$= c\,var(\hat\beta)c'$$
$$= \sigma^2 c(X'X)^{-1}c'$$

$$= (n-k-1)\sigma^2 + tr\left(\sigma^2[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}C'\right)$$
$$= (n-k-1)\sigma^2 + tr(\sigma^2 I_m)$$
$$= (n-k-1)\sigma^2 + \sigma^2 m$$
$$= (n+m-k-1)\sigma^2$$

$\Rightarrow \dfrac{e_c'e_c}{n+m-k-1}$ is the unbiased estimator of σ^2 when using the constraint $C\beta = \gamma$