HW10

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Exercise 1

a. $r_{yx_1}^2 = \dfrac{SSE(x_2, x_3, x_4) - SSE(x_1, x_2, x_3, x_4)}{SSE(x_2, x_3, x_4)}$

$$= \frac{t^2}{t^2 + (n-k-1)} = \frac{(-5.529)^2}{(-5.529)^2 + 354} = 0.79491$$

b. $VIF_1 = (n-1) S_{x_1}^2 V_{11} = (359-1)(0.7300493)(7.007232 \times 10^{-3})$
$= 1.8515$

c. $\hat{\delta}_1 = \hat{\beta}_1 \sqrt{\Sigma (x_{i1} - \bar{x}_1)^2}$

$$\begin{aligned} var(\hat{\delta}_1) &= \Sigma (x_{i1} - \bar{x})^2 \, var(\hat{\ell}_1) \\ &= (n-1) S_{x_1}^2 \, var(\hat{\ell}_1) \\ &= (359-1)(0.7300493)(1.91208)^2 \\ &= 966.8161 \end{aligned}$$

d. $\hat{\gamma}_0 = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}_1 + \hat{\beta}_2 \bar{X}_2 + \ldots + \hat{\beta}_k \bar{X}_k$

we cannot compute $\hat{\gamma}_0$ because while we are given $\hat{\beta}$ values we are not given the sample means $\bar{X}_1 \ldots \bar{X}_4$

Exercise 2

Let $Y_1 = X_1 \overset{k+1 \times 1}{\overline{\beta_1}} + \varepsilon_1$ $\qquad \beta_1 = \begin{bmatrix} \beta_1^1 \\ \beta_1^2 \end{bmatrix} \begin{matrix} r \times 1 \\ k+1-r \times 1 \end{matrix}$ $\qquad \beta_2 = \begin{bmatrix} \beta_2^1 \\ \beta_2^2 \end{bmatrix} \begin{matrix} r \times 1 \\ k+1-r \times 1 \end{matrix}$

$Y_2 = X_2 \beta_2 + \varepsilon_2$

Write the 2 models in vector form $Y = X\beta + \varepsilon$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \overbrace{X_1^1 \; X_1^2}^{X_1} & 0 \quad 0 \\ 0 \quad 0 & \underbrace{X_2^1 \; X_2^2}_{X_2} \end{bmatrix} \begin{bmatrix} \beta_1^1 \\ \beta_1^2 \\ \beta_2^1 \\ \beta_2^1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

$$Y \quad = \quad X \quad \beta + \varepsilon$$

Suppose you want to test $H_0: \beta_1^1 = \beta_2^1$

Write this in the form $H_0: C\beta = \gamma$

$\beta_1^1 = \beta_2^1 \rightarrow \beta_1^1 - \beta_2^1 = 0$

$\rightarrow C\beta = \beta_1^1 - \beta_2^1 = 0 \qquad C \in \mathbb{R}^{r \times 2(k+1)}$

$$C = \begin{bmatrix} I_r & \underset{r \times (k+1-r)}{\underset{\downarrow}{0}} & -I_r & \underset{r \times (k+1-r)}{\underset{\downarrow}{0}} \end{bmatrix}$$

In this case $r = 3 \quad k = 6$

$$\Longrightarrow \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 3

1. consider a linear model with $n = 2m$ in which

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, 2, \dots, m$$
$$y_i = \beta_0 + \beta_2 x_i + \varepsilon_i, \quad i = m+1, m+2, \dots, n$$

$$\sum_{i=1}^{m} x_i = \sum_{i=m+1}^{n} x_i = 0$$

$$\hookrightarrow \; 1' \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = 1' \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ x_n \end{bmatrix} = 0$$

write the vector in form $Y = X\beta + \varepsilon$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \\ \vdots \\ y_n \end{bmatrix} = \left[\begin{array}{ccc} 1 & x_1 & \\ 1 & \vdots & 0_{m\times 1} \\ \vdots & x_m & \\ \hline 1 & & x_{m+1} \\ 1 & 0_{m\times 1} & \vdots \\ & & x_n \end{array}\right] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \\ \varepsilon_{m+1} \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$Y \quad = \quad X \quad \beta \quad + \quad \varepsilon$$

2. $\hat{\beta} = (X'X)^{-1}X'Y$

$$= \left(\underset{3\times 6}{\begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ x_1 & \dots & x_m & 0 & \dots & 0 \\ 0 & \dots & 0 & x_{m+1} & \dots & x_n \end{bmatrix}} \underset{6\times 3}{\begin{bmatrix} 1 & x_1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & x_m & 0 \\ 1 & 0 & x_{m+1} \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_n \end{bmatrix}} \right)^{-1} \begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ x_1 & \dots & x_m & 0 & \dots & 0 \\ 0 & \dots & 0 & x_{m+1} & \dots & x \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum_1^m x_i & \sum_{m+1}^n x_i \\ \sum_1^m x_i & \sum_1^m x_i^2 & 0 \\ \sum_{m+1}^n x_i & 0 & \sum_{m+1}^n x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{m+1} & \cdots & x \end{bmatrix} \bar{Y}$$

Note: $x_1 + \dots x_m = x_{m+1} + \dots + x_n = 0$

$$\begin{array}{cc} 3\times 6 & 6\times 1 \end{array}$$

$$= \begin{bmatrix} n & 0 & 0 \\ 0 & \sum_1^m x_i^2 & 0 \\ 0 & 0 & \sum_{m+1}^n x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{m+1} & \cdots & x \end{bmatrix} \bar{Y} = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1/\sum_1^m x_i^2 & 0 \\ 0 & 0 & 1/\sum_{m+1}^n x_i^2 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{m+1} & \cdots & x \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1/\sum_1^m x_i^2 & 0 \\ 0 & 0 & 1/\sum_{m+1}^n x_i^2 \end{bmatrix} \begin{bmatrix} \sum_1^n y_i \\ \sum_1^m x_i y_i \\ \sum_{m+1}^n x_i y_i \end{bmatrix} = \begin{bmatrix} \bar{Y} \\ \dfrac{\sum_1^m x_i y_i}{\sum_1^m x_i^2} \\ \dfrac{\sum_{m+1}^n x_i y_i}{\sum_{m+1}^n x_i^2} \end{bmatrix} = \hat{\beta}$$

Exercise 4

Suppose a new case (Y_0, X_0) is added to the data (Y, X). Show that the new error sum of squares will increase by

$$\frac{e^2}{1+X_0'(X'X)^{-1}X} \quad \text{where} \quad e = Y_0 - X_0'\hat{\beta}_{old} \quad \text{and} \quad \hat{\beta}_{old} = (X'X)^{-1}X'Y$$

From handout #48:

$$\hat{\beta}_{new} = \hat{\beta}_{old} + \frac{(X'X)^{-1}X_0 e}{1+h} \qquad \text{where } h = X_0'(X'X)^{-1}X_0$$

$$SSE_{new} = (Y - X\hat{\beta}_{new})'(Y - X\hat{\beta}_{new}) + (\overbrace{Y_0 - X_0'\hat{\beta}_{old}}^{=e} - \overbrace{X_0'(X'X)^{-1}X_0}^{=h}\frac{e}{1+h})^2$$

new residual corresponding to y_0

$$= (\overbrace{y - X\hat{\beta}_{old}}^{\underline{e}} - \frac{X(X'X)^{-1}X_0 e}{1+h})'(\overbrace{y - X\hat{\beta}_{old}}^{\underline{e}} - \frac{X(X'X)^{-1}X_0 e}{1+h}) + (e - \frac{he}{1+h})^2$$

$$= SSE_{new} = (\underline{e} - \frac{X(X'X)^{-1}X_0 e}{1+h})'(\underline{e} - \frac{X(X'X)^{-1}X_0 e}{1+h}) + (e - \frac{he}{1+h})^2$$

Note: $\underline{e}'X = 0$, $X'\underline{e} = 0$

$$= \underline{e}'\underline{e} - \frac{\overbrace{\underline{e}'X}^{=0}(X'X)^{-1}X_0 e}{1+h} - \frac{e'X_0'(X'X)^{-1}\overbrace{X'\underline{e}}^{=0}}{1+h} + \frac{e'X_0'\overbrace{(X'X)^{-1}X'X}^{I}(X'X)^{-1}X_0 e}{(1+h)^2} + (e - \frac{he}{1+h})^2$$

$$= \underline{e}'\underline{e} + \frac{e'\overbrace{X_0'(X'X)^{-1}X_0}^{h} e}{(1+h)^2} + (e - \frac{he}{1+h})(e - \frac{he}{1+h})$$

Note: $h = X_0'(X'X)^{-1}X_0$

$$= \underline{e}'\underline{e} + \frac{he^2}{(1+h)^2} + e^2 - \frac{2he^2}{1+h} + \frac{hhe^2}{(1+h)^2} = \underline{e}'\underline{e} + e^2\left(1 - \frac{2h}{1+h} + \frac{h}{(1+h)^2} + \frac{hh}{(1+h)^2}\right)$$

$$= \underline{e}'\underline{e} + e^2\left(1 - \frac{2h}{(1+h)} + \frac{h+h^2}{(1+h)^2}\right) = \underline{e}'\underline{e} + e^2\left(1 - \frac{2h(1+h)}{(1+h)^2} + \frac{h+h^2}{(1+h)^2}\right)$$

$$= \underline{e}'\underline{e} + e^2\left(1 - \frac{2(h+h^2)}{(1+h^2)} + \frac{h+h^2}{(1+h^2)}\right) = \underline{e}'\underline{e} + e^2\left(1 - \frac{h+h^2}{(1+h^2)}\right)$$

$$= \underline{e}'\underline{e} + e^2 \left[\frac{(1+h)^2 - h - h^2}{(1+h)^2} \right] = \underline{e}'\underline{e} + e^2 \left[\frac{1+2h+h^2-h-h^2}{(1+h)^2} \right]$$

$$= \underline{e}'\underline{e} + e^2 \left[\frac{1-h}{(1-h)^2} \right] = \underline{e}'\underline{e} + \frac{e^2}{1-h} \qquad h = X_0'(X'X)^{-1}X_0$$

$$\Rightarrow SSE_{new} = SSE + \frac{e^2}{1 - X_0'(X'X)^{-1}X_0}$$

Exercise 5

$B \sim \text{beta}\left(\frac{1}{2}\alpha, \frac{1}{2}\beta\right) \qquad F = \dfrac{\beta B}{\alpha(1-B)} \qquad$ show that $F \sim F_{\alpha,\beta}$

$$F_F(f) = P(F \le f) = P\left(\frac{\beta B}{\alpha(1-B)} \le f\right) = P\left(\frac{B}{1-B} \le \frac{\alpha f}{\beta}\right)$$

$$= P\left(B \le \frac{\alpha f}{\beta}(1-B)\right) = P\left(B \le \frac{\alpha f}{\beta} - \frac{B\alpha f}{\beta}\right)$$

$$= P\left(B + \frac{B\alpha f}{\beta} \le \frac{\alpha f}{\beta}\right) = P\left(B\left(\frac{\beta+\alpha f}{\beta}\right) \le \frac{\alpha f}{\beta}\right)$$

$$= P\left(B \le \frac{\alpha f}{\beta+\alpha f}\right) \Rightarrow F_F(f) = F_B\left(\frac{\alpha f}{\beta+\alpha f}\right)$$

$$F_F(f) = F'_B\left(\frac{\alpha f}{\beta+\alpha f}\right) = \frac{\alpha(\beta+\alpha f) - \alpha(\alpha f)}{(\beta+\alpha f)^2} \underbrace{F'_B\left(\frac{\alpha f}{\beta+\alpha f}\right)}_{F_B\left(\frac{\alpha f}{\beta+\alpha f}\right) \to \text{pdf}}$$

$$f_F(f) = \frac{\alpha\beta}{(\beta+\alpha f)^2} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\alpha f}{\beta+\alpha f}\right)^{\alpha-1} \left(1 - \frac{\alpha f}{\beta+\alpha f}\right)^{\beta-1}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\alpha}{\beta}\right)^{\alpha/2} \left(\frac{\alpha f}{\beta+\alpha f}\right)^{\frac{\alpha}{2}-1} \left(1 + \frac{\alpha}{\beta}\left(\frac{\alpha f}{\beta+\alpha f}\right)\right]^{-\frac{1}{2}(\alpha+\beta)}$$

we know the pdf of $X \sim F_{n_1,n_2}$ is:

$$f(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{n_1/2} x^{\frac{n_1}{2}-1} \left(1 + \frac{n_1}{n_2}x\right)^{-1/2(n_1+n_2)}$$

$\Rightarrow$ if $X = \dfrac{\alpha f}{\beta+\alpha f}$, $n_1 = \alpha$, $n_2 = \beta$

then $\dfrac{\beta B}{\alpha(1-B)} \sim F_{\alpha,\beta}$