HW9

Madeline Blasingame
205412882

Exercise 1

$Y = X\beta + \epsilon \qquad E[\epsilon] = 0, \quad var(\epsilon) = \sigma^2 I$

$Y^* = X^*\beta + \epsilon^*$

a. $\epsilon^* = \Gamma\epsilon$

$$E[\epsilon^*] = \Gamma E[\epsilon] = \Gamma(0) = 0$$

$$var(\epsilon^*) = \Gamma\, var(\epsilon)\Gamma' = \sigma^2 \Gamma\Gamma' = \sigma^2 I$$

b. $\hat{\beta} = (X'X)^{-1}X'Y$

$$\hat{\beta}^* = (X^{*\prime}X^*)^{-1}X^{*\prime}Y^* \qquad X^* = \Gamma X \qquad Y^* = \Gamma Y$$

$$= (X'\Gamma'\Gamma X)^{-1} X'\Gamma'\Gamma Y$$

$$= (X'X)^{-1}X'Y = \hat{\beta}$$

$$S_e^2 = \frac{e'e}{n-k-1} \qquad S_e^{2*} = \frac{e^{*\prime}e^*}{n-k-1}$$

we need to show $e'e = e^{*\prime}e^*$

we know $e = Y - \hat{Y} = Y - X\hat{\beta}$

$e^* = Y^* - \hat{Y}^* = Y^* - X^*\hat{\beta}^*$ we already showed $\hat{\beta}^* = \hat{\beta}$

$$e^* = (Y^* - X^*\hat{\beta}) = (\Gamma Y - \Gamma X\hat{\beta})$$

$$= \Gamma(Y - X\hat{\beta}) = \Gamma e$$

$$e^{*\prime}e^* = (\Gamma e)'(\Gamma e)$$

$$= e'\Gamma'\Gamma e$$

$$= e'e$$

$$\Rightarrow \quad S_e^2 = S_e^{2*}$$

Exercise 2

When weighting by $1/c_1^2 \; .. \; 1/c_n^2$:

$$V = \begin{bmatrix} 1/c_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/c_n^2 \end{bmatrix}$$

$$\hat{\beta}_{WLS} = (X'V^{-1}X)^{-1}X'V^{-1}Y$$
$$var(\hat{\beta}_{WLS}) = Se^2_{WLS}\,(X'V^{-1}X)^{-1}$$

when weighting by $a/c_1^2 \ldots a/c_n^2$:

$$V^* = \begin{bmatrix} a/c_1^2 & & 0 \\ & \ddots & \\ 0 & & a/c_n^2 \end{bmatrix} = a \begin{bmatrix} 1/c_1^2 & & \\ & \ddots & \\ & & 1/c_n^2 \end{bmatrix} \Rightarrow V^* = aV$$

$$\begin{aligned}
\hat{\beta}^*_{WLS} &= (X'V^{*-1}X)^{-1}X'V^{*-1}Y \\
&= (X'(aV)^{-1}X)^{-1}X'(aV)^{-1}Y \\
&= (\tfrac{1}{a}X'V^{-1}X)^{-1}X'\tfrac{1}{a}V^{-1}Y \\
&= a(X'V^{-1}X)^{-1}X'\tfrac{1}{a}V^{-1}Y \\
&= \tfrac{a}{a}(X'V^{-1}X)^{-1}X'V^{-1}Y \\
&= (X'V^{-1}X)^{-1}X'V^{-1}Y = \hat{\beta}_{WLS}
\end{aligned}$$

$$Se^2_{WLS} = \frac{(Y - X\hat{\beta}_{WLS})'V^{-1}(Y - X\hat{\beta}_{WLS})}{n-k-1}$$

$$Se^{2*}_{WLS} = \frac{(Y - X\hat{\beta}^*_{WLS})'V^{*-1}(Y - X\hat{\beta}^*_{WLS})}{n-k-1} = \frac{(Y - X\hat{\beta}_{WLS})'(aV)^{-1}(Y - X\hat{\beta}_{WLS})}{n-k-1}$$

$$= \frac{1}{a}\frac{(Y - X\hat{\beta}_{WLS})'V^{-1}(Y - X\hat{\beta}_{WLS})}{n-k-1} = \frac{1}{a}Se^2_{WLS}$$

$$\begin{aligned}
var(\hat{\beta}^*_{WLS}) = Se^{2*}_{WLS}(X'V^{*-1}X)^{-1} &= \frac{1}{a}Se^2_{WLS}(X'(aV)^{-1}X)^{-1} \\
&= \frac{1}{a}Se^2_{WLS}(\tfrac{1}{a}X'V^{-1}X)^{-1} \\
&= \frac{a}{a}Se^2_{WLS}(X'V^{-1}X)^{-1} \\
&= Se^2_{WLS}(X'V^{-1}X)^{-1} = var(\hat{\beta}_{WLS})
\end{aligned}$$

Exercise 3

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}Y$$

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'Y$$

$$\begin{aligned}
\operatorname{cov}(\hat{\beta}_{GLS}, \hat{\beta}_{GLS} - \hat{\beta}_{OLS}) &= \operatorname{cov}\left((X'V^{-1}X)^{-1}X'V^{-1}Y,\ (X'V^{-1}X)^{-1}X'V^{-1}Y - (X'X)^{-1}X'Y\right) \\
&= \operatorname{cov}\left((X'V^{-1}X)^{-1}X'V^{-1}Y,\ [(X'V^{-1}X)^{-1}X'V^{-1} - (X'X)^{-1}X']Y\right) \\
&= (X'V^{-1}X)^{-1}X'V^{-1}\operatorname{var}(Y)[(X'V^{-1}X)^{-1}X'V^{-1} - (X'X)^{-1}X']' \\
&= \sigma^2\left[(X'V^{-1}X)^{-1}X'V^{-1}V[V^{-1}X(X'V^{-1}X)^{-1} - X(X'X)^{-1}]\right] \\
&= \sigma^2[(X'V^{-1}X)^{-1}X'V^{-1}X(X'V^{-1}X)^{-1} - (X'V^{-1}X)^{-1}X'X(X'X)^{-1}] \\
&= \sigma^2[(X'V^{-1}X)^{-1} - (XV^{-1}X)^{-1}] \\
&= \sigma^2(0) \\
&= 0
\end{aligned}$$

Exercise 4

$Y = X\beta + \epsilon \qquad var(\epsilon) = \sigma^2 V$

$V = (1-p)I + p11' \quad$ with $\quad 0 \le p \le 1$

show $E[Se^2] = \sigma^2(1-p)$

we know: $se^2 = \dfrac{e'e}{n-k-1} = \dfrac{Y'(I-H)Y}{n-k-1}$

$$E[se^2] = E\left[\frac{Y'(I-H)Y}{n-k-1}\right] = \frac{1}{n-k-1} E[\underbrace{Y'(I-H)Y}_{\text{scalar}}]$$

$$\frac{1}{n-k-1} E[tr(Y'(I-H)Y)] = \frac{1}{n-k-1} E[tr((I-H)YY')] = \frac{1}{n-k-1} tr((I-H)E[YY'])$$

Note: $E[YY'] = var(Y) + E[Y]E[Y']$

$$= \frac{1}{n-k-1} tr((I-H)(\sigma^2 V + X\beta(X\beta)')) = \frac{1}{n-k-1} tr((I-H)(\sigma^2 V + X\beta\beta'X'))$$

$$= \frac{1}{n-k-1} tr[\sigma^2 V + X\beta\beta'X' - \sigma^2 HV - HX\beta\beta'X']$$

Note: $HX = X$

$$= \frac{1}{n-k-1} tr[\sigma^2 V + X\beta\beta'X' - \sigma^2 HV - X\beta\beta'X']$$

$$= \frac{1}{n-k-1} tr[\sigma^2 V - \sigma^2 HV] = \frac{1}{n-k-1} tr[\sigma^2(I-H)V]$$

$$= \frac{\sigma^2}{n-k-1} tr[(I-H)((1-p)I + p11')]$$

$$= \frac{\sigma^2}{n-k-1} tr[(I-H)(1-p)] + tr[(I-H)(p11')]$$

Note: $H1 = 1$

Note: $tr(I-H) = n-k-1$

$$= \frac{\sigma^2}{n-k-1} (1-p)\, tr[(I-H)] + tr[p11' - pH11']$$

$$= \frac{\sigma^2}{n-k-1} (1-p)(n-k-1) + p\, tr[11'] - p\, tr[11'] = \sigma^2(1-p) = E[se^2]$$

Exercise 5

$Y = \theta + \varepsilon \qquad var(\varepsilon) = \sigma^2 V$

Find $\hat{\theta}_{GLS}$ subject to $A\theta = 0$

$$V^{-1/2} Y = V^{-1/2}\theta + V^{-1/2}\varepsilon \qquad E[\varepsilon^*] = 0$$
$$Y^* = \theta^* + \varepsilon^* \qquad var[\varepsilon^*] = \sigma^2 I$$

$$\min Q = \varepsilon^{*\prime}\varepsilon^* - 2\lambda'[A\theta]$$
$$\min Q = (Y^* - \theta^*)'(Y^* - \theta^*) - 2\lambda'[A\theta] \qquad Y^* = V^{-1/2}Y$$
$$\min Q = (V^{-1/2}(Y-\theta))'(V^{-1/2}(Y-\theta)) - 2\lambda'[A\theta] \qquad \theta^* = V^{-1/2}\theta$$
$$= (Y-\theta)'V^{-1}(Y-\theta) - 2\lambda'[A\theta] =$$
$$= Y'V^{-1}Y - 2Y'V^{-1}\theta + \theta'V^{-1}\theta - 2\lambda'[A\theta]$$

$$\frac{\partial Q}{\partial \theta} = -2V^{-1}Y + 2V^{-1}\theta - 2A'\lambda = 0$$

$$V^{-1}\theta = A'\lambda + V^{-1}Y$$
$$\hat{\theta}_{GLS} = V(A'\lambda + V^{-1}Y)$$

$$A\hat{\theta}_{GLS} = AV(A'\lambda + V^{-1}Y) = 0$$
$$AVA'\lambda + AVV^{-1}Y = 0$$
$$AVA'\lambda + AY = 0$$
$$\lambda = -(AVA')^{-1}AY$$

$$\Rightarrow \quad \hat{\theta}_{GLS} = V(-A'(AVA')^{-1}AY + V^{-1}Y)$$
$$= -VA'(AVA')^{-1}AY + Y$$

$$Y = VA'\gamma + \varepsilon \qquad var(\varepsilon) = \sigma^2 V$$
$$V^{-1/2}Y = V^{-1/2}VA'\gamma + V^{-1/2}\varepsilon \qquad \text{where } V^{-1/2}VA' = X^* \qquad E[\varepsilon^*] = 0$$
$$Y^* = X^*\gamma + \varepsilon^* \qquad var[\varepsilon^*] = \sigma^2 I$$

From lecture: $\hat{\gamma}_{GLS} = (AVV^{-1}VA')^{-1}AVV^{-1}Y = (AVA')^{-1}AY$

Show: $Y - \hat{\theta}_{GLS} = VA'\hat{\gamma}_{GLS}$

$$
\begin{aligned}
Y - \hat{\theta}_{GLS} &= Y + VA'(AVA')^{-1}AY - Y \\
&= VA'(AVA')^{-1}AY \\
&= VA'\hat{\gamma}_{GLS}
\end{aligned}
$$

Exercise 6

$$y_i = \beta_1 x_i + \varepsilon_i \qquad E[\varepsilon_i] = 0 \qquad var(\varepsilon_i) = \sigma^2 x_i^2$$
$$E[\varepsilon_i \varepsilon_j] = 0$$

Transform the model:

$$Y_i / x_i = \beta_1 X_i / X_i + \varepsilon_i / X_i$$

$$Y_i^* = \beta_1 X_i^* + \varepsilon_i^* \qquad Y_i^* = Y_i / X_i \qquad X_i^* = 1$$
$$\varepsilon_i^* = \varepsilon_i / X_i$$

$$E[\varepsilon_i^*] = E[\varepsilon_i / X_i] = 1/X_i \; E[\varepsilon_i] = 0$$
$$var(\varepsilon_i^*) = var(\varepsilon_i / X_i) = \frac{1}{X_i^2} var(\varepsilon_i) = \frac{\sigma^2 X_i^2}{X_i^2} = \sigma^2$$

$$\min Q = \Sigma \varepsilon_i^{*2}$$
$$\min Q = \Sigma (Y_i^* - \beta_1)^2$$

$$\frac{\partial Q}{\partial \beta_1} = -2 \Sigma (Y_i^* - \beta_1) = 0$$

$$\Sigma (Y_i^* - \beta_1) = 0$$
$$\Sigma Y_i^* - \Sigma \beta_1 = 0$$
$$\Sigma Y_i^* = n \beta_1$$
$$\hat{\beta}_1 = \frac{\Sigma Y_i^*}{n} = \frac{\Sigma y_i / x_i}{n}$$

Assuming y_is are independent:

$$var(\hat{\beta}_1) = var\left(\frac{\Sigma y_i / x_i}{n}\right) = \frac{1}{n^2} var(\Sigma y_i / x_i) = \frac{1}{n^2} \Sigma var(y_i / x_i)$$

$$= \frac{1}{n^2} \Sigma \frac{1}{x_i^2} var(y_i) = \frac{1}{n^2} \Sigma \frac{1}{x_i^2} \sigma^2 x_i^2$$

$$= \frac{1}{n^2} \Sigma \sigma^2 = \frac{n \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Exercise 7

Show $var(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \bar{X}' D^{-1} R^{-1} D^{-1} \bar{X} \right)$ where D and R are defined as in the centered and scaled model

We know: $Y = \gamma_0 1 + Z_s \delta_{(0)} + \varepsilon$

$$\hat{\beta}_0 = \hat{\gamma}_0 - \frac{1}{n} 1' X_{(0)} \hat{\beta}_{(0)}$$
$$= \bar{Y} - \frac{1}{n} 1' X_{(0)} D^{-1} \hat{\delta}_{(0)}$$

Note: $\hat{\beta}_{(0)} = D^{-1} \hat{\delta}_{(0)}$

$$\Rightarrow var(\hat{\beta}_0) = var(\bar{Y}) + var\left(\frac{1}{n} 1' X_{(0)} D^{-1} \hat{\delta}_{(0)}\right) - 2\,cov\left(\bar{Y}, \frac{1}{n} 1' X_{(0)} D^{-1} \hat{\delta}_{(0)}\right)$$

Note: $\bar{Y} = \frac{1}{n} 1' y$

$\hat{\delta}_{(0)} = R^{-1} Z_s' y$

$$cov\left(\bar{Y}, \frac{1}{n} 1' X_{(0)} D^{-1} \hat{\delta}_{(0)}\right) = cov\left(\frac{1}{n} 1' y, \frac{1}{n} 1' X_{(0)} D^{-1} R^{-1} Z_s' y\right)$$
$$= \frac{1}{n^2} 1' \, var(y) \, Z_s R^{-1} D^{-1} X_{(0)}' 1$$
$$= \frac{\sigma^2}{n^2} 1' Z_s R^{-1} D^{-1} X_{(0)}' 1$$

Note: $1' Z_s = 0$ $\qquad = 0$

$$\Rightarrow var(\hat{\beta}_0) = var(\bar{Y}) + var\left(\frac{1}{n} 1' X_{(0)} D^{-1} \hat{\delta}_{(0)}\right)$$
$$= \frac{\sigma^2}{n} + \frac{1}{n^2} 1' X_{(0)} D^{-1} R^{-1} Z_s' \, var(y) \, Z_s R^{-1} D^{-1} X_{(0)}' 1$$
$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n^2} 1' X_{(0)} D^{-1} R^{-1} Z_s' Z_s R^{-1} D^{-1} X_{(0)}' 1$$

Note: $Z_s' Z_s = R$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n^2} 1' X_{(0)} D^{-1} R^{-1} R R^{-1} D^{-1} X_{(0)}' 1$$
$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n^2} 1' X_{(0)} D^{-1} R^{-1} D^{-1} X_{(0)}' 1$$
$$= \sigma^2 \left(\frac{1}{n} + \frac{1}{n} 1' X_{(0)} D^{-1} R^{-1} D^{-1} \frac{1}{n} (1' X_{(0)}') \right)$$

Note: $\frac{1}{n} 1' X_{(0)} = \bar{X}'$

$$= \sigma^2 \left(\frac{1}{n} + \bar{X}' D^{-1} R^{-1} D^{-1} \bar{X} \right)$$

Exercise 8

we know: $H = X(X'X)^{-1}X'$ $\quad Y = X\beta + \epsilon$

If we partition X and β to get the centered model:

$Y = \gamma_0 1 + Z\beta_{(0)} + \epsilon$

such that $Z = (I - \frac{1}{n}11')X_{(0)} = (X_{(0)} - \frac{1}{n}11'X_{(0)})$

$W = [1 \;\; Z]$ $\quad W'W = \begin{bmatrix} 1' \\ Z' \end{bmatrix} [1 \;\; Z]$ $\quad H = W(W'W)^{-1}W'$

$$H = [1 \;\; Z] \begin{bmatrix} 1'1 & 1'Z \\ Z'1 & Z'Z \end{bmatrix}^{-1} \begin{bmatrix} 1' \\ Z' \end{bmatrix} = [1 \;\; Z] \begin{bmatrix} n & 0 \\ 0 & Z'Z \end{bmatrix}^{-1} \begin{bmatrix} 1' \\ Z' \end{bmatrix}$$

$$= [1 \;\; Z] \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & (Z'Z)^{-1} \end{bmatrix} \begin{bmatrix} 1' \\ Z' \end{bmatrix}$$

$$= \left[\tfrac{1}{n}1 \;\; Z(Z'Z)^{-1}\right] \begin{bmatrix} 1' \\ Z' \end{bmatrix}$$

$$= \frac{1}{n}11' + Z(Z'Z)^{-1}Z'$$

we know that $var(\hat{\beta}_{(0)}) = \sigma^2 (Z'Z)^{-1}$ because $var(\hat{\beta}_{(0)}) \geq 0$

$\Rightarrow (Z'Z)^{-1}$ is positive semi-definite.

we also know that because Z is a vector

$\Rightarrow Z(Z'Z)^{-1}Z' \geq 0$ (positive semi-definite)

$\therefore \; h_{ii} \geq \frac{1}{n}$