HW 4

Exercise 1

a. $y_i = \beta_0 + \beta_1 x_i + \varepsilon$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_{11} \\ 1 & x_{21} \\ \vdots & \vdots \\ 1 & x_{n1} \end{bmatrix}}_{\underline{x}} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\underline{x}'\underline{x} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{21} & \cdots & x_{n1} \end{bmatrix} \begin{bmatrix} 1 & x_{11} \\ 1 & x_{21} \\ \vdots & \vdots \\ 1 & x_{n1} \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$(x'x)^{-1} = \frac{1}{n\sum x_i^2 - \sum x_i \sum x_i} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

$(x'x)^{-1}$ forms part of the solution to finding $\hat{\beta}_0, \hat{\beta}_1$ from the Normal Equations for simple regression

b. Find $\hat{\beta}_0$ and $\hat{\beta}_1$ using $\hat{\beta} = (x'x)^{-1} x'Y$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \in \mathbb{R}^{2\times 1} \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$$

$$x'Y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{21} & \cdots & x_{n1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_n \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

using $(x'x)^{-1}$ we found in part a:

$$\hat{\beta} = \frac{1}{n\sum x_i^2 - \sum x_i \sum x_i} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

$$= \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{bmatrix} \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

Note: $\sum (x_i - \bar{x})^2 =$
$\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) =$
$\sum x_i^2 - 2\bar{x}\sum x_i + \sum \bar{x}^2) =$
$\sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2) =$
$\sum x_i^2 - n\bar{x}^2 =$
$= \sum x_i^2 - \frac{1}{n}(\sum x_i)^2$

$\Rightarrow n\sum (x_i - \bar{x})^2 = n\sum x_i^2 - (\sum x_i)^2$

$\Rightarrow \sum x_i^2 = \sum (x_i - \bar{x})^2 + n\bar{x}^2$

$$= \frac{1}{n\sum (x_i - \bar{x})^2} \begin{bmatrix} (\sum Y_i)(\sum x_i^2) - n\bar{x}\sum x_i Y_i \\ -n\bar{x}\sum Y_i + n\sum x_i Y_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\bar{y}\sum x_i^2}{\sum (x_i - \bar{x})^2} - \frac{\bar{x}\sum x_i y_i}{\sum (x_i - \bar{x})^2} \\ \frac{-\bar{x}\sum Y_i + \sum x_i Y_i}{\sum (x_i - \bar{x})^2} \end{bmatrix} = \begin{bmatrix} \frac{\bar{y}\sum x_i^2}{\sum (x_i - \bar{x})^2} - \frac{\bar{x}\sum x_i y_i}{\sum (x_i - \bar{x})^2} \\ \frac{\sum x_i Y_i - n\bar{x}\bar{Y}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}} \end{bmatrix}$$

From the normal eq. we know:

$$\hat{\beta}_1 = \frac{\sum x_i Y_i - n\bar{x}\bar{Y}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}$$

using $\sum x_i^2 = \sum (x_i - \bar{x})^2 + n\bar{x}^2$:

$$= \begin{bmatrix} \frac{\bar{Y}\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} + \frac{\bar{Y}n\bar{x}^2}{\sum (x_i - \bar{x})^2} - \frac{\bar{x}\sum (x_i y_i)}{\sum (x_i y_i)} \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \bar{Y} + \frac{\bar{Y}n[\bar{x}]^2}{\sum (x_i - \bar{x})^2} - \frac{\bar{x}\sum (x_i Y_i)}{\sum (x_i - \bar{x})^2} \\ \hat{\beta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{Y} - \bar{x}\left[\frac{-n\bar{Y}\bar{x} + \sum x_i Y_i}{\sum (x_i - \bar{x})^2}\right] \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \bar{Y} - \bar{x}\left[\frac{-\sum \bar{x} Y_i + \sum x_i Y_i}{\sum (x_i - \bar{x})^2}\right] \\ \hat{\beta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{Y} - \bar{x}\left[\frac{\sum (x_i - \bar{x})Y_i}{\sum (x_i - \bar{x})^2}\right] \\ \hat{\beta}_1 \end{bmatrix}$$

using the normal equations:

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})Y_i}{\sum (x_i - \bar{x})^2}$$

$$= \begin{bmatrix} \bar{Y} - \hat{\beta}_1 \bar{x} \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \hat{\beta}$$

using the normal equations

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

Exercise 2

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i$$

$$\underset{\sim}{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \qquad \underset{\sim}{X} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{13} \\ 1 & X_{21} & X_{22} & X_{23} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & X_{n3} \end{bmatrix} \in \mathbb{R}^{K \times 4} \qquad \underset{\sim}{X}' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{11} & X_{21} & \cdots & X_{n1} \\ X_{12} & X_{22} & \cdots & X_{n2} \\ X_{13} & X_{23} & \cdots & X_{n3} \end{bmatrix} \in \mathbb{R}^{4 \times K}$$

$$X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{11} & X_{21} & \cdots & X_{n1} \\ X_{12} & X_{22} & \cdots & X_{n2} \\ X_{13} & X_{23} & \cdots & X_{n3} \end{bmatrix} \begin{bmatrix} 1 & X_{11} & X_{12} & X_{13} \\ 1 & X_{21} & X_{22} & X_{23} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & X_{n3} \end{bmatrix} = \begin{bmatrix} n & \sum X_{i1} & \sum X_{i2} & \sum X_{i3} \\ \sum X_{i1} & \sum X_{i1}^2 & \sum X_{i1} X_{i2} & \sum X_{i1} X_{i3} \\ \sum X_{i2} & \sum X_{i2} X_{i1} & \sum X_{i2}^2 & \sum X_{i2} X_{i3} \\ \sum X_{i3} & \sum X_{i3} X_{i1} & \sum X_{i3} X_{i2} & \sum X_{i3}^2 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$$

$$X'Y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{11} & X_{21} & \cdots & X_{n1} \\ X_{12} & X_{22} & \cdots & X_{n2} \\ X_{13} & X_{23} & \cdots & X_{n3} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum X_{i1} Y_i \\ \sum X_{i2} Y_i \\ \sum Y_{i3} Y_i \end{bmatrix} \in \mathbb{R}^{4 \times 1}$$

Exercise 3

a. $y = X\beta + \epsilon$ → Gauss-Markov conditions hold

Show $\hat{\beta} = \beta + A\epsilon$

Then show $\hat{Y} = X\beta + B\epsilon$

Find A, B

↓ plug in given value of y

$$\hat{\beta} = (X'X)^{-1}X'Y \rightarrow \hat{\beta} = (X'X)^{-1}X'(X\beta + \epsilon)$$
$$= (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\epsilon$$
$$= \beta + (X'X)^{-1}X'\epsilon$$
$$= \beta + A\epsilon \implies A = (X'X)^{-1}X'$$

$\hat{Y} = X\hat{\beta}$ using the definition of $\hat{Y}$

$$= X(\beta + A\epsilon)$$
$$= X\beta + \underbrace{XA}\epsilon$$
$$= X\beta + B\epsilon \quad = B = XA = X(X'X)^{-1}X'$$

b. Show $e'X = 0$ where $e = Y - X\hat{\beta}$ using the definition of e
$= Y - \hat{Y} \in \mathbb{R}^n$

$e = Y - \hat{Y} = Y - HY = (I_n - H)Y$ using the definition of $\hat{Y}$

both symmetric

↓

so $e'X = Y'(I - H)'X = Y'(I' - H')X \stackrel{\cdot}{=} Y'(I - H)X$
$$= Y'(X - HX) = Y'(X - \underbrace{X(X'X)^{-1}X'X}_{X})$$
$$= Y'(X - X) = 0$$

c. Show $e'\hat{Y} = 0$

$$e'\hat{Y} = e'X\hat{\beta} = (e'X)\hat{\beta} = 0\hat{\beta} = 0$$

Exercise 4

$$E[Y_i] = \beta_0 + \beta_1 x_i + \beta_2 (3x_i^2 - 2) \quad i = 1,2,3$$

$$E[\underline{Y}] = E[\underline{X}\underline{\beta} + \underline{\varepsilon}] = \underline{X}\,\underline{\beta}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}}_{\underline{X}} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\underline{\beta}} = \underbrace{\begin{bmatrix} \beta_0 - \beta_1 + \beta_2 \\ \beta_0 - 2\beta_2 \\ \beta_0 + \beta_1 + \beta_2 \end{bmatrix}}_{E[\underline{Y}]} \qquad X_1 = -1 \quad X_2 = 0 \quad X_3 = 1 \qquad \underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

solved for $\underline{X}$

$$\hat{\beta} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{Y}$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow (X'X)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_1 + Y_2 + Y_3 \\ Y_3 - Y_1 \\ Y_1 - 2Y_2 + Y_3 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \frac{1}{3}(Y_1 + Y_2 + Y_3) \\ \frac{1}{2}(Y_3 - Y_1) \\ \frac{1}{6}(Y_1 - 2Y_2 + Y_3) \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

$$\text{var}(\hat{\beta}) = \sigma^2 (X'X)^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{3} & 0 & 0 \\ 0 & \frac{\sigma^2}{2} & 0 \\ 0 & 0 & \frac{\sigma^2}{6} \end{bmatrix}$$

$*$ = new

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = (X'X)^{-1}X'Y \qquad \hat{\beta}^* = \begin{bmatrix} \hat{\beta}_0^* \\ \hat{\beta}_1^* \end{bmatrix} = (X^{*\prime}X^*)^{-1} X^* Y$$

we want to show: $\hat{\beta}_0 = \hat{\beta}_0^*$ and $\hat{\beta}_1 = \hat{\beta}_1^*$

$$X = \begin{bmatrix} 1 & x_1 & 3x_1^2 - 2 \\ 1 & x_2 & 3x_2^2 - 2 \\ 1 & x_3 & 3x_3^2 - 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad X^* = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

$$X^{*\prime} X^* = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow (X^{*\prime} X^*)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$X^{*\prime} Y = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_1 + Y_2 + Y_3 \\ Y_3 - Y_1 \end{bmatrix}$$

$$\hat{\beta}^* = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} Y_1 + Y_2 + Y_3 \\ Y_3 - Y_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(Y_1 + Y_2 + Y_3) \\ \frac{1}{2}(Y_3 - Y_1) \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0^* \\ \hat{\beta}_1^* \end{bmatrix}$$

$$\hat{\beta}_0 = \frac{1}{3}(Y_1 + Y_2 + Y_3) = \hat{\beta}_0^*$$
$$\hat{\beta}_1 = \frac{1}{2}(Y_3 - Y_1) = \hat{\beta}_1^*$$

Exercise 5

$Y = X\beta + \epsilon$

Gauss-Markov conditions hold

By definition: $H = X(X'X)^{-1}X'$

Let a be some column vector where the ith row is one and all other values are 0

$$a = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith row}$$

Therefore:

$$Ha = \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\text{hat matrix}} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = H_{\cdot i}$$

This means Ha will be equivalent to the ith column of H

In order to sum up the ith column we write:

$$1' H_{\cdot i} = \sum_{j=1}^{n} H_{ji}$$

We can generalize this to sum any column i (of matrix H) where $i = 1, 2, \dots, n$

$1' H_{\cdot i} = 1' Ha$ due to the fact $H_{\cdot i} = Ha$ (shown above)

$1' Ha = (H1)'a$ due to the fact $H = H'$: $(H1)'a = 1'H'a = 1'Ha$

We know that $H1 = 1$ due to the properties of H, therefore:

$(H1)'a = (1)'a = 1'a$

We also know $1'a = \sum_{i=1}^{n} a_i$ as shown above, a is a column vector where only the ith row is one and all other rows are 0

Therefore: $\sum_{i=1}^{n} a_i = 0 + \dots + 1 + \dots + 0 = 1$

Therefore, the sum of any column of H must equal 1.

When we remove the intercept from the model, the definition of H does not change, however the first column of X is no longer a column of 1s and the property $H1=1$ is no longer true. This puts a fault in the logic used above and means we are no longer able to know that the sum of the columns of H are equal to 1.

Question 5

With Intercept:

```
#Create the X matrix
X <- cbind(c(1,1,1), c(-1,0,1), c(1,-2,1))

#find the H (hat) matrix
H <- X %*% solve(t(X) %*% X) %*% t(X)

#find the column sums of H
colSums(H)
```

```
## [1] 1 1 1
```

Without Intercept:

```
#Create the X matrix
X <- cbind(c(-1,0,1), c(1,-2,1))

#find the H (hat) matrix
H <- X %*% solve(t(X) %*% X) %*% t(X)

#find the column sums of H
colSums(H)
```

```
## [1] -5.551115e-17  0.000000e+00 -5.551115e-17
```

Exercise 6

a. $H \in \mathbb{R}^{n\times n}$ s.t. $\underset{\sim}{\hat{Y}} = HY \in \mathbb{R}^{n\times 1}$

need to show: $\sum_{j}^{n} H_{ij}^2 = H_{ii}$

Recall: $\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y$

so $H = X(X'X)^{-1}X' \in \mathbb{R}^{n\times n}$

Note (Properties of H):

1. H is symmetric: $H' = H$
2. H is idempotent: $H^2 = HH = H$

Note:

Let $A \in \mathbb{R}^{m\times d}$, $B \in \mathbb{R}^{d\times k}$

1. $[AB]_{ij} = A_{i\cdot} B_{\cdot j}$ ← jth column

   ↗ ith row

2. $[AB]_{i\cdot} = A_{i\cdot} B$

3. $[AB]_{\cdot j} = AB_{\cdot j}$

Inner Product

$\langle v, v \rangle = \sum_{j=1}^{d} v_j^2$ for $v \in \mathbb{R}^d$

$$H_{ii} = [HH]_{ii} = \underbrace{H_{i\cdot}}_{\mathbb{R}^{1\times n}} \underbrace{H_{\cdot i}}_{\mathbb{R}^{n\times 1}} = H_{i\cdot}(H_{i\cdot})' = \langle H_{i\cdot}, H_{i\cdot} \rangle$$

$$= \sum_{j=1}^{n} H_{ij}^2$$

b. Show $SSE = Y'Y - \hat{\beta}'X'X\hat{\beta} = Y'Y - \hat{\beta}'X'Y = Y'Y - \hat{\beta}'X'\hat{Y}$

$$SSE = e'e$$
$$= Y'(I-H)'(I-H)Y$$
$$= Y'(I-H)(I-H)Y$$
$$= Y'(I-H)Y$$
$$= Y'Y - Y'HY$$
$$= Y'Y - Y'X(X'X)^{-1}X'Y$$
$$= \boxed{Y'Y - \hat{\beta}'X'Y}$$

Note: $e = (I-H)Y$

Note: $(I-H)' = (I-H)$

Note: $(I-H)(I-H) = (I-H)$

Note: $H = X(X'X)^{-1}X'$

Note: $\hat{\beta}' = Y'X(X'X)^{-1}$

$$= Y'Y - \hat{\beta}'\overbrace{X'X(X'X)^{-1}}^{I_n}X'Y$$

$$= \boxed{Y'Y - \hat{\beta}'X'X\hat{\beta}}$$

$$= \boxed{Y'Y - \hat{\beta}'X'\hat{Y}}$$

Note: $\hat{\beta} = (X'X)^{-1}X'Y$

Note: $\hat{Y} = X\hat{\beta}$

c. Show $(Y - X\beta)'(Y - X\beta) = (Y - X\hat{\beta})'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)$

$$Y - X\beta = Y - X\hat{\beta} + X(\hat{\beta} - \beta) = e + X(\hat{\beta} - \beta)$$

$$(Y - X\beta)'(Y - X\beta) = (e + X(\hat{\beta} - \beta))'(e + X(\hat{\beta} - \beta))$$

$$= (e' + (\hat{\beta}' - \beta')X')(e + X(\hat{\beta} - \beta))$$

$$= (e' + \hat{\beta}'X' - \beta'X')(e + X\hat{\beta} - X\beta)$$

$$= e'e + \underbrace{e'X\hat{\beta}}_{0} - \underbrace{e'X\beta}_{0} + \hat{\beta}'\underbrace{X'e}_{0} + \hat{\beta}'X'X\hat{\beta} - \hat{\beta}'X'X\beta - \beta'\underbrace{X'e}_{0} - \beta'X'X\hat{\beta} + \beta'X'X\beta$$

Note: $e'X = 0$ and $X'e = 0$

$$= e'e + \hat{\beta}'X'X\hat{\beta} - \hat{\beta}'X'X\beta - \beta'X'X\hat{\beta} + \beta'X'X\beta$$

Note: $e = Y - X\hat{\beta}$

Note: $(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)$

$$= (\hat{\beta}' - \beta')X'X(\hat{\beta} - \beta)$$

$$= (\hat{\beta}'X'X - \beta'X'X)(\hat{\beta} - \beta)$$

$$= \hat{\beta}'X'X\hat{\beta} - \hat{\beta}'X'X\beta - \beta'X'X\hat{\beta} + \beta'X'X\beta$$

$$= \boxed{(Y - X\hat{\beta})'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)}$$

Exercise 7

Load data and see the first 6 rows

```
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics100C/jura.txt", header=TRU
E)

#See the first six rows of the data:
head(a)
```

```
##       x     y Landuse Rock    Cd    Co    Cr    Cu    Ni    Pb    Zn
## 1 2.672 3.558       3    5 1.570  8.28 37.12 18.60 18.60 38.20  65.2
## 2 3.589 4.443       3    1 2.045 10.80 40.80 11.48 21.52 33.36 112.8
## 3 4.010 4.713       2    1 1.203 12.00 53.20 13.04 23.92 26.56  91.6
## 4 2.942 3.137       2    5 0.490 10.92 23.40  5.64 14.60 25.88  41.2
## 5 1.409 2.748       3    3 0.692  8.12 27.16 10.32 14.64 31.16  50.4
## 6 3.978 2.910       1    2 1.750  9.12 35.48  8.36 26.40 37.72  63.2
```

Create the X and Y matrices (assuming an intercept):

```
X <- as.matrix(cbind(rep(1, nrow(a)),a$Cd, a$Cu, a$Zn))
Y <- as.matrix(cbind(a$Pb))
```

1a. Compute (X'X)

```
xx <- t(X) %*% X
xx
```

```
##           [,1]       [,2]       [,3]       [,4]
## [1,]   359.000   462.4770   8467.184   27241.60
## [2,]   462.477   860.0015  11922.631   40964.38
## [3,]  8467.184 11922.6310 377220.935  803949.53
## [4,] 27241.600 40964.3842 803949.534 2407169.16
```

1b. Compute (X'X)^-1

```
xx_inverse <- solve(xx)
xx_inverse
```

```
##               [,1]          [,2]          [,3]          [,4]
## [1,]  0.0216942476  0.0018266062  1.554642e-04 -3.285175e-04
## [2,]  0.0018266062  0.0075924443  1.334245e-04 -1.944384e-04
## [3,]  0.0001554642  0.0001334245  1.225894e-05 -8.124196e-06
## [4,] -0.0003285175 -0.0001944384 -8.124196e-06  1.015543e-05
```

1c. Compute (X'Y)

```
xy <- t(X) %*% Y
xy
```

```
##            [,1]
## [1,]   19612.52
## [2,]   27548.77
## [3,]  680026.00
## [4,] 1731176.69
```

2. Compute B Hat (the beta_hat vector)

```
beta_hat <- solve(t(X) %*% X) %*% t(X) %*% Y
beta_hat
```

```
##            [,1]
## [1,] 12.7974915
## [2,] -0.8881627
## [3,]  0.9967046
## [4,]  0.2565816
```

3. Compute H (the hat matrix)

```
hat <- X %*% solve(t(X) %*% X) %*% t(X)
head(hat[,1])
```

```
## [1]  0.0047820244  0.0001749063 -0.0001138486  0.0030733254  0.0029572837
## [6]  0.0050189629
```

4. Compute Y hat (the fitted values)

```
Yhat <- X%*% beta_hat
head(Yhat)
```

```
##          [,1]
## [1,] 46.67090
## [2,] 51.36578
## [3,] 48.22894
## [4,] 28.55487
## [5,] 35.40059
## [6,] 35.79162
```

5. Compute e (the residuals)

```
e <- Y - Yhat
head(e)
```

```
##              [,1]
## [1,]  -8.470904
## [2,] -18.005776
## [3,] -21.668938
## [4,]  -2.674869
## [5,]  -4.240589
## [6,]   1.928383
```

Exercise 8

$SSE = e'e = (Y - X\hat{\beta})'(Y - X\hat{\beta}) = \Sigma (y_i - x_i\hat{\beta})^2$

$SSE_1 = (Y - Xc)'(Y - Xc)$

want to show: $SSE_1 - SSE = (\hat{\beta} - c)' X'X (\beta' - c)$ *

Recall: if $A \in \mathbb{R}^{m \times m}$ is positive, definite, then for all $V \in \mathbb{R}^{m \times 1}$ $V'AV \geq 0$
and $V \in \mathbb{R}^{m \times 1} \setminus 0$ $V'AV > 0$

excluding 0

claim: $X'X$ is positive, definite

Therefore * is showing that $SSE \leq SSE_1$ for any $c \in \mathbb{R}^p$

$$
\begin{aligned}
SSE_1 = (Y - Xc)'(Y - Xc) &= (Y - X\hat{\beta} + X\hat{\beta} - Xc)'(Y - X\hat{\beta} + X\hat{\beta} - Xc) \\
&= (e + X(\hat{\beta} - c))'(e + X(\hat{\beta} - c)) \\
&= \underbrace{e'e}_{SSE} + \underbrace{e'X}_{=0}(\hat{\beta} - c) + (\hat{\beta} - c)'\underbrace{X'e}_{=0} + (\hat{\beta} - c)'X'X(\hat{\beta} - c) \\
&= SSE + (\hat{\beta} - c)' X'X (\hat{\beta} - c) \\
\Rightarrow SSE_1 - SSE &= (\hat{\beta} - c)'X'X(\beta - c) \geq 0
\end{aligned}
$$

Recall HW1 Exercise 5

transpose of 0 is 0

$X'X \in \mathbb{R}^{p \times p}$ and is symmetric

columns are independent

when X is full-column rank and is symmetric $\Rightarrow X$ is positive definite