HW7

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Exercise a

$$\frac{(Y - \gamma_0 1 - Z\beta_{(0)} \overset{\pm[1\hat{\gamma}_0 + Z\hat{\beta}_{(0)}]}{)}'(Y - \gamma_0 1 - Z\beta_{(0)} \overset{\pm[1\hat{\gamma}_0 + Z\hat{\beta}_{(0)}]}{)}}{\sigma^2} =$$

$$\frac{(Y - \gamma_0 1 - Z\beta_{(0)} + 1\hat{\gamma}_0 + Z\hat{\beta}_{(0)} - 1\hat{\gamma}_0 - Z\hat{\beta}_{(0)})'(Y - \gamma_0 1 - Z\beta_{(0)} + 1\hat{\gamma}_0 + Z\hat{\beta}_{(0)} - 1\hat{\gamma}_0 - Z\hat{\beta}_{(0)})}{\sigma^2} =$$

$$\frac{(e - \gamma_0 1 - Z\beta_{(0)} + 1\hat{\gamma}_0 + Z\hat{\beta}_{(0)})'(e - \gamma_0 1 - Z\beta_{(0)} + 1\hat{\gamma}_0 + Z\hat{\beta}_{(0)})}{\sigma^2} =$$

$e'1 = 0$
$1'e = 0$

$$\frac{(e - [\gamma_0 - \hat{\gamma}_0]1 - Z[\beta_{(0)} - \hat{\beta}_{(0)}])'(e - [\gamma_0 - \hat{\gamma}_0]1 - Z[\beta_{(0)} - \hat{\beta}_{(0)}])}{\sigma^2} =$$

$$\frac{(e' + [\hat{\gamma}_0 - \gamma_0]1' + [\hat{\beta}_{(0)} - \beta_{(0)}]'Z')(e + [\hat{\gamma}_0 - \gamma_0]1 + Z[\hat{\beta}_{(0)} - \beta_{(0)}])}{\sigma^2} =$$

$$\frac{e'e}{\sigma^2} + \overbrace{\frac{[\hat{\gamma}_0 - \gamma_0]e'1}{\sigma^2}}^{=0} + \overbrace{\frac{e'Z[\hat{\beta}_{(0)} - \hat{\beta}_{(0)}]}{\sigma^2}}^{=0} + \overbrace{\frac{[\hat{\gamma}_0 - \gamma_0]1'e}{\sigma^2}}^{=0} + \frac{n[\hat{\gamma}_0 - \gamma_0]^2}{\sigma^2} +$$

$$\overbrace{\frac{[\hat{\gamma}_0 - \gamma_0]1'Z[\hat{\beta}_{(0)} - \beta_{(0)}]}{\sigma^2}}^{=0} + \overbrace{\frac{[\hat{\beta}_{(0)} - \beta_{(0)}]'Z'e}{\sigma^2}}^{=0} + \overbrace{\frac{[\hat{\gamma}_0 - \gamma_0][\hat{\beta}_{(0)} - \beta_{(0)}]'Z'1}{\sigma^2}}^{=0} +$$

$$\frac{[\hat{\beta}_{(0)} - \beta_{(0)}]'Z'Z[\hat{\beta}_{(0)} - \beta_{(0)}])}{\sigma^2} =$$

$$\frac{e'e}{\sigma^2} + \frac{n[\hat{\gamma}_0 - \gamma_0]^2}{\sigma^2} + \frac{[\hat{\beta}_{(0)} - \beta_{(0)}]'Z'Z[\hat{\beta}_{(0)} - \beta_{(0)}])}{\sigma^2}$$

$$\hat{\gamma}_0 = \bar{Y} \rightarrow \hat{\gamma}_0 \sim N(\gamma_0, \tfrac{\sigma^2}{n}) \Rightarrow \frac{\sqrt{n}(\hat{\gamma}_0 - \gamma_0)}{\sigma} \sim N(0,1) \Rightarrow \frac{n[\hat{\gamma}_0 - \gamma_0]}{\sigma^2} \sim \chi^2_1$$

$$Y \sim N_n(\gamma_0 1 + Z\beta_{(0)}, \sigma^2 I) \longrightarrow \frac{(Y - \gamma_{(0)}1 - Z\beta_{(0)})'(Y - \gamma_0 1 - Z\beta_{(0)})}{\sigma^2} \sim \chi^2_n$$

$\hat{\beta}_{(0)} \sim N(\beta(0), \sigma^2(Z'Z)^{-1})$

let $V = (Z'Z)^{1/2}(\hat{\beta}_{(0)} - \beta_{(0)})$

$E(V) = (Z'Z)^{1/2}(\beta_{(0)} - \beta_{(0)}) = 0$

$\mathrm{var}(V) = (Z'Z)^{1/2}\,\mathrm{var}(\hat{\beta}_{(0)})[(Z'Z)^{1/2}]'$

$= \sigma^2 (Z'Z)^{1/2}(Z'Z)^{-1}(Z'Z)^{1/2}$

$= \sigma^2 (Z'Z)^{1/2}(Z'Z)^{-1/2}(Z'Z)^{-1/2}(Z'Z)^{1/2}$

$= \sigma^2 I$

$V \sim N_k(\underline{0}, \sigma^2 I)$ $\qquad \dfrac{V'V}{\sigma^2} \sim \chi^2_k$

$$\frac{V'V}{\sigma^2} = \frac{(\hat{\beta}_{(0)} - \beta_{(0)})'(Z'Z)^{1/2}(Z'Z)^{1/2}(\hat{\beta}_{(0)} - \beta_{(0)})}{\sigma^2} = \frac{(\hat{\beta}_{(0)} - \beta_{(0)})' Z'Z (\hat{\beta}_{(0)} - \beta_{(0)})}{\sigma^2}$$

$$\Rightarrow \frac{(\hat{\beta}_{(0)} - \beta_{(0)})' Z'Z (\hat{\beta}_{(0)} - \beta_{(0)})}{\sigma^2} \sim \chi^2_k$$

$$\underbrace{\frac{(Y - \gamma_0 1 - Z\beta_{(0)})'(Y - \gamma_0 1 - Z\beta_{(0)})}{\sigma^2}}_{\substack{Q \\ \downarrow \\ \chi^2_n}} = \underbrace{\frac{e'e}{\sigma^2}}_{Q_1} + \underbrace{\frac{n[\hat{\gamma}_0 - \gamma_0]^2}{\sigma^2}}_{\substack{Q_2 \\ \downarrow \\ \chi^2_1}} + \underbrace{\frac{[\hat{\beta}_{(0)} - \beta_{(0)}]' Z'Z [\hat{\beta}_{(0)} - \beta_{(0)}]}{\sigma^2}}_{\substack{Q_3 \\ \downarrow \\ \chi^2_k}}$$

e, $\hat{\gamma}_0$, and $\hat{\beta}_{(0)}$ are independent

$Q = Q_1 + Q_2 + Q_3$

$M_Q(t) = M_{Q_1}(t) \cdot M_{Q_2}(t) \cdot M_{Q_3}(t)$

$$M_{Q_1}(t) = \frac{M_Q(t)}{M_{Q_2}(t) \cdot M_{Q_3}(t)} = \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}(1-2t)^{-k/2}} = (1-2t)^{-\frac{(n-k-1)}{2}}$$

$$= Q_1 \sim \chi^2_{n-k-1} \Rightarrow \frac{e'e}{\sigma^2} = \frac{(n-k-1)s_e^2}{\sigma^2} \sim \chi^2_{n-k-1}$$

Exercise 6

$$L = f(Y) = (2\pi)^{-n/2} |\sigma^2 V|^{-1/2} e^{-1/2 (Y-\mu 1)'(\sigma^2 V)^{-1}(Y-\mu 1)}$$

$$\ln(L) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\ln|V| - \frac{1}{2\sigma^2}(Y-\mu 1)' V^{-1}(Y-\mu 1)$$

$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{n}\ln|V| - \frac{1}{2\sigma^2}(Y'V^{-1}Y - 2\mu 1'V^{-1}Y + \mu^2 1'V^{-1}1)$$

$$\frac{\partial \ln(L)}{\partial \mu} = \frac{-1}{2\sigma^2}(0 - 21'V^{-1}Y + 2\mu 1'V^{-1}1) = 0$$

$$2\mu 1'V^{-1}1 = 21'V^{-1}Y$$

$$\hat{\mu} = \frac{1'V^{-1}Y}{1'V^{-1}1}$$

$$\frac{\partial \ln(L)}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1) = 0$$

$$\frac{1}{2\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1) = \frac{n}{2\sigma^2}$$

$$(Y-\mu 1)'V^{-1}(Y-\mu 1) = \frac{n\, 2\sigma^4}{2\sigma^2}$$

$$\hat{\sigma}^2 = \frac{(Y-\hat{\mu}1)'V^{-1}(Y-\hat{\mu}1)}{n}$$

$$I(\theta) = -E\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 \ln L}{\partial \sigma^{2(2)}} \end{bmatrix} =$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1)$$

$$\frac{\partial^2 \ln L}{\partial \sigma^{2(2)}} = \frac{n}{2\sigma^4} - \frac{(Y-\mu 1)'V^{-1}(Y-\mu 1)}{\sigma^6}$$

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{2\sigma^2}(2\mu 1'V^{-1}1 - 21'V^{-1}Y)$$

$$\frac{\partial^2 \ln L}{\partial\mu\,\partial\sigma^2} = \frac{(2\mu 1'V^{-1}1 - 2 1'V^{-1}Y)}{2\sigma^4}$$

$$\frac{\partial^2 \ln L}{\partial\mu^2} = \frac{-1'V^{-1}1}{\sigma^2}$$

$$\frac{\partial \ln L}{\partial\sigma^2\partial\mu} = \frac{\partial}{\partial\mu}\left(\frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1)\right)$$

$$= \frac{\partial}{\partial\mu}\left(\frac{1}{2\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1)\right)$$

$$= \frac{1}{2\sigma^4}\cdot\frac{\partial}{\partial\mu}\left(Y'V^{-1}Y - Y'V^{-1}\mu 1 - \mu 1'V^{-1}Y + \mu^2 1'V^{-1}1\right)$$

$$= \frac{1}{2\sigma^4}\left(0 - \underbrace{Y'V^{-1}1}_{\text{scalar}} - \underbrace{1'V^{-1}Y}_{\text{scalar}} + 2\mu 1'V^{-1}1\right)$$

$$= \frac{1}{2\sigma^4}\left(-2Y'V^{-1}1 + 2 1'V^{-1}1\right) = \frac{\mu 1'V^{-1}1 - Y'V^{-1}1}{\sigma^4}$$

$$I(\theta) = -E\begin{bmatrix} \frac{-1'V^{-1}1}{\sigma^2} & \frac{(2\mu 1'V^{-1}1 - 2 1'V^{-1}Y)}{2\sigma^4} \\ \frac{\mu 1'V^{-1}1 - Y'V^{-1}1}{\sigma^4} & \frac{n}{2\sigma^4} - \frac{(Y-\mu 1)'V^{-1}(Y-\mu 1)}{\sigma^6} \end{bmatrix}$$

we know: $E[Y] = \mu 1$

$Y \sim N(\mu 1, \sigma^2 V)$

let $A = V^{-1/2}(Y-\mu 1) \rightarrow A \sim N(0, \sigma^2 I) \rightarrow \frac{A'A}{\sigma^2} \sim \chi^2_n$

$$\therefore \frac{(Y-\mu 1)'V^{-1}(Y-\mu 1)}{\sigma^2} \sim \chi^2_n \Rightarrow E\left[\frac{(Y-\mu 1)'V^{-1}(Y-\mu 1)}{\sigma^2}\right] = n$$

$$E\left[\frac{1}{\sigma^4}\left[\frac{n}{2} - \frac{Y-\mu 1)'V^{-1}(Y-\mu 1)}{\sigma^2}\right]\right] = \frac{1}{\sigma^4}\cdot\frac{-n}{2} = \frac{-n}{2\sigma^4}$$

$$I(\theta) = \begin{bmatrix} \frac{\overbrace{1'V^{-1}1}^{\text{scalar}}}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix} \rightarrow I^{-1}(\theta) = \begin{bmatrix} \frac{\sigma^2}{1'V^{-1}1} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

$$\operatorname{var}(\hat{\mu}) = \operatorname{var}\left[\frac{1'V^{-1}Y}{1'V^{-1}1}\right]$$

Let $A = \dfrac{1'V^{-1}}{1'V^{-1}1}$ $\rightarrow$ $\operatorname{var}(AY) = A\operatorname{var}(Y)A' = \sigma^2 AVA'$

$$= \sigma^2 \frac{1'V^{-1}VV^{-1}1}{(1'V^{-1}1)^2} = \sigma^2 \frac{\overset{\text{Scalar}}{\overline{1'V^{-1}1}}}{(1'V^{-1}1)^2} =$$

$$= \frac{\sigma^2}{(1'V^{-1}1)}$$

Yes, $\hat{\mu}$ is an efficient estimator of μ.

Exercise c

$Y = (Y_1, Y_2, \ldots, Y_n)' \sim N(\mu, \sigma^2 I)$

Theorem: Let $Y \sim N_n(0,1)$ and A be a symmetric, idempotent matrix. Then $Y'AY \sim \chi^2_r$ where r is the number of eigenvalues of A equal to 1. The other $n-r$ eigenvalues are equal to 0.

S^2 is the sample variance of $Y_1, Y_2, \ldots, Y_n$

$$S^2 = \frac{(Y - \frac{1}{n}11'Y)'(Y - \frac{1}{n}11'Y)}{n-1} \rightarrow \frac{(n-1)S^2}{\sigma^2} = \frac{(Y-\bar{Y})'(Y-\bar{Y})}{\sigma^2}$$

$$= \frac{[(I - \frac{1}{n}11')Y]'(I - \frac{1}{n}11')Y}{n-1} = \frac{Y'(I - \frac{1}{n}11')(I - \frac{1}{n}11')Y}{n-1}$$

$$= \frac{Y'(I - \frac{1}{n}11')Y}{n-1} \Rightarrow \frac{(n-1)S^2}{\sigma^2} = \frac{Y'(I - \frac{1}{n}11')Y}{\sigma^2}$$

show: $Y'(I - \frac{1}{n}11')Y = (Y - \mu 1)'(I - \frac{1}{n}11')(Y - \mu 1)$

$$= (Y' - \mu 1')(I - \frac{1}{n}11')(Y - \mu 1)$$

$$= (Y' - Y'\tfrac{1}{n}11' - \mu 1' + \tfrac{1}{n}\mu 1'11')(Y - \mu 1) \qquad \text{Note: } 1'1 = n$$

$$= (Y' - \tfrac{1}{n}Y'11' - \mu 1' + \mu 1')(Y - \mu 1)$$

$$= (Y' - \tfrac{1}{n}Y'11')(Y - \mu 1)$$

$$= (Y'Y - \mu Y'1 - \tfrac{1}{n}Y'11'Y + \mu \tfrac{1}{n}Y'11'1)$$

$$= (Y'Y - \mu Y'1 + \mu Y'1 - \tfrac{1}{n}Y'11'Y)$$

$$= (Y'Y - \tfrac{1}{n}Y'11'Y) = Y(I - \tfrac{1}{n}11')Y$$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} = \frac{Y'(I - \frac{1}{n}11')Y}{\sigma^2} = \frac{(Y - \mu 1)'(I - \frac{1}{n}11')(Y - \mu 1)}{\sigma^2}$$

Let $Y^* = \dfrac{(Y - \mu 1)}{\sigma} \rightarrow \dfrac{(n-1)S^2}{\sigma^2} = Y^{*\prime}(I - \frac{1}{n}11')Y^*$

$Y \sim N(\mu, \sigma^2 I)$

$\Rightarrow Y^* \sim N(0, I)$

Let $A = (I - \frac{1}{n}11')$ this is an idempotent and symmetric matrix

Using the theorem: $Y^{*\prime}(I - \frac{1}{n}11')Y^* \sim \chi^2_r$

Solve for r:

$$r = tr(I - \tfrac{1}{n}11') = tr(I) - \underline{tr(\tfrac{1}{n}11')} =$$

$$\tfrac{1}{n} tr(11') = \tfrac{1}{n} tr(1'1) = \tfrac{1}{n} \cdot n = 1$$

$$= n-1$$

$$\Rightarrow \quad Y^* (I - \tfrac{1}{n}11') Y^* \sim \chi^2_{n-1}$$

$$\Rightarrow \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Exercise d

$AX = \lambda X$

$X'AX = \lambda X'X$

$X'AAX = \lambda X'X$ because A is idempotent

$(AX)'(AX) = \lambda X'X$ because A is symmetric

$(\lambda X')(\lambda X) = \lambda X'X$ Note: $AX = \lambda X$

$\lambda^2 X'X = \lambda X'X$

$\lambda^2 = \lambda \Rightarrow \lambda = 0, 1$

$tr(A) = tr(P\Lambda P') = tr(\Lambda PP') = tr(\Lambda)$

Note: $PP' = I$ because P is an orthogonal matrix

Note: $A = P\Lambda P'$ (spectral decomposition because A is symmetric)

Note: Λ is a diagonal matrix with eigenvalues of A on the diagonal. We already showed the eigenvalues of A are 0 or 1 so the trace of Λ will be a sum of 1s and 0s meaning trace (Λ) = # of 1 eigenvalues $\Rightarrow$ $tr(A)$ = # of 1 eigenvalues

$H = X(X'X)^{-1}X'$ $\rightarrow$ H is symmetric and idempotent

$tr(H) = tr(X(X'X)^{-1}X') = tr((X'X)^{-1}X'X) = tr(I) = k+1$

$\hookrightarrow$ H has $k+1$ eigenvalues equal to 1

$I-H$ $\rightarrow$ is symmetric and idempotent

$tr(I-H) = tr(I) - tr(H) = n-k-1$

$\hookrightarrow$ $I-H$ has $n-k-1$ eigenvalues equal to 1

Exercise e

$y_i = \beta_0 + \beta_1 X_i + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma)$

$y_i \sim N(\beta_0 + \beta_1 X_i, \sigma)$

Find the information Matrix

$I(\beta_0, \beta_1, \sigma^2) =$

$$-E\left[\begin{array}{cc|c} \frac{\partial \ln L}{\partial \beta_0^2} & \frac{\partial \ln L}{\partial \beta_0 \partial \beta_1} & \frac{\partial \ln L}{\partial \beta_0 \partial \sigma^2} \\ \frac{\partial \ln L}{\partial \beta_1 \partial \beta_0} & \frac{\partial \ln L}{\partial \beta_1^2} & \frac{\partial \ln L}{\partial \beta_1 \partial \sigma^2} \\ \hline \frac{\partial \ln L}{\partial \sigma^2 \partial \beta_0} & \frac{\partial \ln L}{\partial \sigma^2 \partial \beta_1} & \frac{\partial \ln L}{\partial \sigma^2} \end{array}\right]$$

$$\ln L = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum (Y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial \ln L}{\partial \beta_0} = \frac{1}{\sigma^2}\sum (Y_i - \beta_0 - \beta_1 x_i) = \frac{\sum Y_i}{\sigma^2} - \frac{n\beta_0}{\sigma^2} - \frac{\beta_1 \sum x_i}{\sigma^2}$$

$$\frac{\partial \ln L}{\partial \beta_1} = \frac{1}{\sigma^2}\sum (Y_i - \beta_0 - \beta_1 X_i) X_i = \frac{\sum Y_i X_i}{\sigma^2} - \frac{\beta_0 \sum X_i}{\sigma^2} - \frac{\beta_1 \sum X_i^2}{\sigma^2}$$

$$\frac{\partial \ln L}{\sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum (Y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial^2 \ln L}{\partial \beta_0^2} = \frac{-n}{\sigma^2} \rightarrow -E\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_1} = \frac{-\sum x_i}{\sigma^2} \rightarrow -E\left[-\frac{\sum x_i}{\sigma^2}\right] = \frac{\sum x_i}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta_0 \partial \sigma^2} = \frac{-2}{\sigma^4}\sum (Y_i - \beta_0 - \beta_1 x_i) \rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \beta_0 \partial \sigma^2}\right] = \frac{-2}{\sigma^4}\sum E(e_i) = 0$$

$$\frac{\partial^2 \ln L}{\partial \beta_1 \partial \beta_0} = \frac{-\sum X_i}{\sigma^2} \rightarrow -E\left[\frac{-\sum X_i}{\sigma^2}\right] = \frac{\sum X_i}{\sigma^2}$$

$$\mathrm{var}(X) = E[X^2] - E[X]^2$$

$$\frac{\partial^2 \ln L}{\partial \beta_1^2} = \frac{-\sum X_i^2}{\sigma^2} \rightarrow -E\left[\frac{-\sum X_i^2}{\sigma^2}\right] = \frac{\sum X_i^2}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta_1 \partial \sigma^2} = -\frac{2}{\sigma^4}\sum (Y_i - \beta_0 - \beta_1 X_i) X_i \rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \beta_1 \partial \sigma^2}\right] = \frac{2}{\sigma^4}\sum E[e_i] x_i = 0$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_0} = \frac{-1}{\sigma^4}\sum (Y_i - \beta_0 - \beta_1 X_i) \rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_0}\right] = \frac{1}{\sigma^4}\sum E[e_i] = 0$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_1} - \frac{1}{\sigma^4}\sum (Y_i - \beta_0 - \beta_1 X_i) X_i \rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_1}\right] = \frac{1}{\sigma^4}\sum E[e_i] x_i = 0$$

$$\frac{\partial^2 \ln L}{\partial \sigma^{2(2)}} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum (Y_i - \beta_0 - \beta_1 X_i)^2$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \sigma^2(2)}\right] = -\left[\frac{n}{2\sigma^4} - \frac{\Sigma E[e_i^2]}{\sigma^6}\right] = -\left[\frac{n}{2\sigma^4} - \frac{\Sigma var(e_i) + E[e_i]^2}{\sigma^6}\right] = -\left[\frac{n}{2\sigma^4} - \frac{n\sigma^2}{\sigma^6}\right] =$$

$$-\left[\frac{n}{2\sigma^4} - \frac{n}{\sigma^4}\right] = -\left[\frac{-n}{2\sigma^4}\right] = \frac{n}{2\sigma^4}$$

$$\Rightarrow \quad I(\beta_0, \beta_1, \sigma^2) = \left[\begin{array}{cc|c} \frac{n}{\sigma^2} & \frac{\Sigma x_i}{\sigma^2} & 0 \\ \frac{\Sigma x_i}{\sigma^2} & \frac{\Sigma x_i^2}{\sigma^2} & 0 \\ \hline 0 & 0 & \frac{n}{2\sigma^4} \end{array}\right] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$I^{-1}(\beta_0, \beta_1, \sigma^2) = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{n}{\sigma^2} & \frac{\Sigma x_i}{\sigma^2} \\ \frac{\Sigma x_i}{\sigma^2} & \frac{\Sigma x_i^2}{\sigma^2} \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\frac{n\Sigma x_i^2}{\sigma^4} - \frac{(\Sigma x_i)^2}{\sigma^4}} \begin{bmatrix} \frac{\Sigma x_i^2}{\sigma^2} & -\frac{\Sigma x_i}{\sigma^2} \\ -\frac{\Sigma x_i}{\sigma^2} & \frac{n}{\sigma^2} \end{bmatrix}$$

Note: $n\Sigma x_i^2 - (x_i)^2 = n\Sigma(x_i - \bar{x})^2$

$$A^{-1} = \frac{\sigma^4}{n\Sigma(x_i-\bar{x})^2} \begin{bmatrix} \frac{\Sigma x_i^2}{\sigma^2} & -\frac{\Sigma x_i}{\sigma^2} \\ -\frac{\Sigma x_i}{\sigma^2} & \frac{n}{\sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2 \Sigma x_i^2}{n\Sigma(x_i-\bar{x})^2} & \frac{-\sigma^2\bar{x}}{\Sigma(x_i-\bar{x})^2} \\ \frac{-\sigma^2\bar{x}}{\Sigma(x_i-\bar{x})^2} & \frac{\sigma^2}{\Sigma(x_i-\bar{x})^2} \end{bmatrix}$$

Note: $\sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\Sigma(x_i-\bar{x})^2}\right) = \sigma^2\left(\frac{\Sigma(x_i-\bar{x})^2 + n\bar{x}^2}{n\Sigma(x_i-\bar{x})^2}\right) =$

$$\sigma^2\left(\frac{\Sigma x_i^2 - 2\bar{x}\Sigma x_i + n\bar{x}^2 + n\bar{x}^2}{n\Sigma(x_i-\bar{x})^2}\right) = \sigma^2\left(\frac{\Sigma x_i^2 - 2n\bar{x}^2 + 2n\bar{x}^2}{n\Sigma(x_i-\bar{x})^2}\right) = \frac{\sigma^2\Sigma x_i^2}{n\Sigma(x_i-\bar{x})^2}$$

$$A^{-1} = \begin{bmatrix} \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\Sigma(x_i-\bar{x})^2}\right) & \frac{-\sigma^2\bar{x}}{\Sigma(x_i-\bar{x})^2} \\ \frac{-\sigma^2\bar{x}}{\Sigma(x_i-\bar{x})^2} & \frac{\sigma^2}{\Sigma(x_i-\bar{x})^2} \end{bmatrix}$$

$$\Longrightarrow \quad I^{-1}(\theta) = \begin{bmatrix} \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i-\bar{x})^2}\right) & \frac{-\sigma^2\bar{x}}{\sum(x_i-\bar{x})^2} & 0 \\ \frac{-\sigma^2\bar{x}}{\sum(x_i-\bar{x})^2} & \frac{\sigma^2}{\sum(x_i-\bar{x})^2} & 0 \\ 0 & 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

$$var(\hat{\beta}_0) = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i-\bar{x})^2}\right)$$

$$var(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i-\bar{x})^2}$$

Therefore, $\hat{\beta}_0$ and $\hat{\beta}_1$ are efficient estimators for β_0 and β_1 respectively.

Exercise F

$c\hat{\beta} - \gamma$ is a function of $\hat{\beta}$ and $\hat{\beta}$ follows a multivariate normal distribution meaning that $c\hat{\beta} - \gamma$ also follows a multivariate normal distribution

$$E[c\hat{\beta} - \gamma] = cE[\hat{\beta}] - \gamma = c\beta - \gamma = \gamma - \gamma = 0$$

Note: $c\beta = \gamma$

$$\text{var}[c\hat{\beta} - \gamma] = c\,\text{var}(\hat{\beta})c' = \sigma^2 c(x'x)^{-1}c'$$

$$c\hat{\beta} - \gamma \sim N(0, \sigma^2 c(x'x)^{-1}c')$$

Note: $c(x'x)^{-1}c'$ is a symmetric matrix

Let $v = [c(x'x)^{-1}c']^{-1/2}(c\hat{\beta} - \gamma)$

$$\begin{aligned} E[v] &= [c(x'x)^{-1}c']^{-1/2} E[c\hat{\beta} - \gamma] \\ &= [c(x'x)^{-1}c']^{-1/2} cE[\hat{\beta}] - \gamma \\ &= [c(x'x)^{-1}c']^{-1/2} c\beta - \gamma \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{var}(v) &= \text{var}\left([c(x'x)^{-1}c']^{-1/2}(c\hat{\beta} - \gamma)\right) \\ &= \text{var}\left([c(x'x)^{-1}c']^{-1/2}(c\hat{\beta} - \gamma)\right) \\ &= [c(x'x)^{-1}c']^{-1/2}\text{var}(c\hat{\beta} - \gamma)[c(x'x^{-1})c']^{-1/2} \\ &= \sigma^2[c(x'x)^{-1}c']^{-1/2}[c(x'x)^{-1}c'][c(x'x)^{-1}c']^{-1/2} \\ &= \sigma^2[c(x'x)^{-1}c']^{-1/2}[c(x'x)^{-1}c']^{1/2}[c(x'x)^{-1}c']^{1/2}[c(x'x)^{-1}c']^{-1/2} \\ &= \sigma^2 I_m \end{aligned}$$

$$\Rightarrow v \sim N(0, \sigma^2 I_m) \Rightarrow \frac{v'v}{\sigma^2} \sim \chi^2_m$$

$$\Rightarrow \frac{(c\hat{\beta} - \gamma)'[c(x'x)^{-1}c']^{-1/2}[c(x'x)^{-1}c']^{-1/2}(c\hat{\beta} - \gamma)}{\sigma^2}$$

$$= \frac{(c\hat{\beta} - \gamma)'[c(x'x)^{-1}c']^{-1}(c\hat{\beta} - \gamma)}{\sigma^2} \sim \chi^2_m$$

Exercise 9

From lecture we know:

$$e_c'e_c = e'e + (c\hat{\beta}+\gamma)'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)$$

$$\Rightarrow \underset{Q}{\frac{e_c'e_c}{\sigma^2}} = \underset{Q_1}{\frac{e'e}{\sigma^2}} + \underset{Q_2}{\frac{(c\hat{\beta}+\gamma)'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)}{\sigma^2}}$$

From f we know:

$$\frac{(c\hat{\beta}+\gamma)'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)}{\sigma^2} \sim \chi^2_m$$

Note: $S_e^2 = \frac{e'e}{n-k-1} \rightarrow e'e = (n-k-1)S_e^2$

$$\Rightarrow \frac{e'e}{\sigma^2} = \frac{(n-k-1)S_e^2}{\sigma^2} \sim \chi^2_{n-k-1}$$ From lecture

$$\begin{aligned} cov(\hat{\beta}, e) &= cov((X'X)^{-1}X'Y, (I-H)Y) \\ &= (X'X)^{-1}X' \, var(Y)\,(I-H) \\ &= \sigma^2 (X'X)^{-1}X'(I-H) \\ &= \sigma^2 (X'X)^{-1}(X'-X'H) = \sigma^2 (X'X)^{-1}(X'-X') = 0 \end{aligned}$$

$\Rightarrow$ $\hat{\beta}, e$ are independent

$\Rightarrow$ Q_1, Q_2 are independent because Q_1 is a function of e and Q_2 is a function of $\hat{\beta}$

$$M_Q(t) = M_{Q_1}(t) \cdot M_{Q_2}(t)$$

$$M_Q(t) = (1-2t)^{-\frac{(n-k-1)}{2}} \cdot (1-2t)^{-m/2} = (1-2t)^{-\frac{(n+m-k-1)}{2}}$$

$$\Rightarrow Q = \frac{e_c'e_c}{\sigma^2} \sim \chi^2_{n+m-k-1}$$

From HW6:

$$S_{ec}^2 = \frac{e_c'e_c}{n+m-k-1} = \frac{\sigma^2}{n+m-k-1} Q$$

$$\Rightarrow \quad M_{S_{ec}^2}(t) = M_{\frac{\sigma^2}{n+m-k-1}Q}(t) = M_Q\left(\frac{\sigma^2}{n+m-k-1}t\right)$$

$$= \left(1 - \frac{2\sigma^2}{n+m-k-1}t\right)^{-\frac{(n+m-k-1)}{2}}$$

$$\Rightarrow \quad S_e^2 \sim \Gamma\left(\frac{n+m-k-1}{2}, \frac{2\sigma^2}{n+m-k-1}\right)$$

Exercise h

From lecture we know:

$$\hat{\beta}_c = \hat{\beta} - (X'X)^{-1} C' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)$$

$$E[\hat{\beta}_c] = E[\hat{\beta} - (X'X)^{-1} C' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - \gamma)]$$

$$= E[\hat{\beta}] - (X'X)^{-1} C [C(X'X)^{-1}C']^{-1} (C E[\hat{\beta}] - \gamma)$$

Note: $C\beta = \gamma$

$$= \beta - (X'X)^{-1} C [C(X'X)^{-1}C']^{-1} (C\beta - \gamma)$$

$$= \beta - (X'X)^{-1} C [C(X'X)^{-1}C']^{-1} (\gamma - \gamma)$$

$$= \beta$$

$$\mathrm{var}[\hat{\beta}_c] = \mathrm{var}(\hat{\beta} - (X'X)^{-1} C' [C(X'X^{-1}C']^{-1} C\hat{\beta} + (X'X)^{-1} C' [C(X'X)^{-1}C]^{-1} \gamma)$$

$$= \mathrm{var}(\underbrace{[I - (X'X)^{-1} C' [C(X'X)^{-1}C']^{-1} C]}_{A} \hat{\beta})$$

Note: A is a $K+1 \times K+1$ matrix

$$= A \, \mathrm{var}(\hat{\beta}) A'$$

$$= \sigma^2 A (X'X)^{-1} A'$$

$\hat{\beta}_c$ is a function of $\hat{\beta}$ and $\hat{\beta}$ follows a multivariate normal distribution meaning that $\hat{\beta}_c$ also follows a multivariate normal distribution.

$$\Rightarrow \hat{\beta}_c \sim N(\hat{\beta}, \sigma^2 A (X'X)^{-1} A')$$

Note: $A(X'X)^{-1}A'$ is a symmetric matrix

Let $V = [A(X'X)^{-1}A']^{-1/2} (\hat{\beta}_c - \beta)$

$$E[V] = [A(X'X)^{-1}A']^{-1/2} E[\hat{\beta}_c] - [A(X'X)^{-1}A]^{-1/2} \beta$$

$$= [A(X'X)^{-1}A']^{-1/2} \hat{\beta} - [A(X'X)^{-1}A]^{-1/2} \beta$$

$$= 0$$

$$\mathrm{var}[V] = \mathrm{var}([A(X'X)^{-1}A']^{-1/2} \hat{\beta}_c - [A(X'X)^{-1}A']^{-1/2} \beta$$

$$= \mathrm{var}([A(X'X)^{-1}A']^{-1/2} \hat{\beta}_c)$$

$$= [A(X'X)^{-1}A']^{-1/2} \mathrm{var}(\hat{\beta}_c) [A(X'X)^{-1}A']^{-1/2}$$

$$= \sigma^2 [A(X'X)^{-1}A']^{-1/2} [A(X'X)^{-1}A'] [A(X'X)^{-1}A']^{-1/2}$$

$$= \sigma^2 [A(X'X)^{-1}A']^{-1/2} [A(X'X)^{-1}A']^{1/2} [A(X'X)^{-1}A']^{1/2} [A(X'X)^{-1}A']^{-1/2}$$

$$= \sigma^2 I_{K+1}$$

$$\Rightarrow V \sim N(0, \sigma^2 I) \Rightarrow \frac{V'V}{\sigma^2} \sim \chi^2_{K+1}$$

$$\Rightarrow \frac{(\hat{\beta}_c - \beta)' [A(X'X)^{-1}A']^{-1/2} [A(X'X)^{-1}A']^{-1/2} (\hat{\beta}_c - \beta)}{\sigma^2} \sim \chi^2_{k+1}$$

$$\Rightarrow \frac{(\hat{\beta}_c - \beta)' [A(X'X)^{-1}A']^{-1} (\hat{\beta}_c - \beta)}{\sigma^2} \sim \chi^2_{k+1}$$

where $A = I - (X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C$

Exercise i

canonical Form:

$$e_c = Y - X\hat{\beta}_c \quad \text{with} \quad X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \quad \hat{\beta} = \begin{bmatrix} \hat{\beta}_{1c} \\ \hat{\beta}_{2c} \end{bmatrix}$$

Meaning e_c using the canonical model can be written as:

$$e_c = Y - X_1 \hat{\beta}_{1c} - X_2 \hat{\beta}_{2c}$$

From lecture we know:

$\hat{\beta}_{1c} = C_1^{-1}[\gamma - C_2 \hat{\beta}_{2c}]$ when C_1 is singular

$Y_r = Y - X_1 C_1^{-1} \gamma$

$X_{2r} = X_2 - X_1 C^{-1} C_2$

$$\Rightarrow e_c = Y - X_1 C_1^{-1}[\gamma - C_2 \hat{\beta}_{2c}] - X_2 \hat{\beta}_{2c}$$

$$= \underbrace{Y - X_1 C_1^{-1}\gamma}_{Y_r} + X_1 C_1^{-1} C_2 \hat{\beta}_{2c} - X_2 \hat{\beta}_{2c}$$

$$= Y_r - \underbrace{[X_2 - X_1 C_1^{-1} C_2]}_{X_{2r}} \hat{\beta}_{2c}$$

$$= Y_r - X_{2r} \hat{\beta}_{2c}$$

$$\Rightarrow e_c = Y_r - X_{2r} \hat{\beta}_{2c}$$

using the constrained form we get $Y_r = X_{2r} \hat{\beta}_{2c} + \epsilon$ so the fitted model is $Y_r - \hat{Y}_r = Y_r - X_{2r} \hat{\beta}_{2c}$

$$Y_r - X_{2r} \hat{\beta}_{2c} = Y_r - X_{2r} \hat{\beta}_{2c}$$

Therefore, the residuals of the constrained least squares model are the same as those obtained using the canonical form of the model.

Exercise j

$$Y_r \sim N(X_{2r}\beta_2, \sigma^2 I_n) \implies \frac{(Y_r - X_{2r}\beta_2)'(Y_r - X_{2r}\beta_2)}{\sigma^2} \sim \chi^2_n$$

$$\frac{(Y_r \underset{\pm X_{2r}\hat\beta_{2c}}{-} X_{2r}\beta_2)'(Y_r \underset{\pm X_{2r}\hat\beta_{2c}}{-} X_{2r}\beta_2)}{\sigma^2} = \frac{(Y_r - X_{2r}\hat\beta_{2c} - X_{2r}\beta_2 + X_{2r}\hat\beta_{2c})'(Y_r - X_{2r}\hat\beta_{2c} - X_{2r}\beta_2 + X_{2r}\hat\beta_{2c})}{\sigma^2}$$

Note: $e_c = Y_r - X_{2r}\hat\beta_{2c}$ (from Qi)

$$= \frac{(e_c + X_{2r}(\hat\beta_{2c} - \beta_2))'(e_c + X_{2r}(\hat\beta_{2c} - \hat\beta_2))}{\sigma^2}$$

$$= \frac{e_c'e_c}{\sigma^2} + \underbrace{\frac{e_c'X_{2r}(\hat\beta_{2c} - \hat\beta_2)}{\sigma^2}}_{=0} + \underbrace{\frac{(\hat\beta_{2c} - \beta_2)'X_r'e_c}{\sigma^2}}_{=0} + \frac{(\hat\beta_{2c} - \beta_2)X_{2r}'X_{2r}(\hat\beta_{2c} - \hat\beta_2)}{\sigma^2}$$

Note:
$$e_c = y_r - X_{2r}\hat\beta_{2c}$$
$$= y_r - \underbrace{X_{2r}(X_{2r}'X_{2r})^{-1}X_{2r}}_{H_{2r}} y_r$$
$$= (I - H_{2r})y_r$$

$$\to \quad e_c'X_{2r} = y_r'(I - H_{2r})X_{2r}$$
$$= y_r'(X_{2r} - H_{2r}X_{2r})$$
$$= y_r'(X_{2r} - X_{2r}) = 0 = X_{2r}'e_c$$

$$\implies \frac{(Y_r - X_{2r}\beta_2)'(Y_r - X_{2r}\beta_2)}{\sigma^2} = \frac{e_c'e_c}{\sigma^2} + \frac{(\hat\beta_{2c} - \beta_2)X_{2r}'X_{2r}(\hat\beta_{2c} - \hat\beta_2)}{\sigma^2}$$

Note: $\hat\beta_{2c} \sim N(\beta_2, \sigma^2(X_{2r}'X_{2r})^{-1})$

Let $V = (X_{2r}'X_{2r})^{1/2}(\hat\beta_{2c} - \beta_2)$

$$E[V] = E[(X_{2r}'X_{2r})^{1/2}(\hat\beta_{2c} - \beta_2)]$$
$$= (X_{2r}'X_{2r})^{1/2}[E[\hat\beta_{2c}] - \beta_{2c}]$$
$$= (X_{2r}'X_{2r})^{1/2}[\beta_{2c} - \beta_{2c}] = 0$$

$$\mathrm{var}(V) = \mathrm{var}((X_{2r}'X_{2r})^{1/2}(\hat\beta_{2c} - \beta_2)) = \mathrm{var}((X_{2r}'X_{2r})^{1/2}\hat\beta_{2c} - (X_{2r}'X_{2r})^{1/2}\beta_2)$$
$$= \mathrm{var}((X_{2r}'X_{2r})^{1/2}\hat\beta_{2c})$$
$$= \sigma^2(X_{2r}'X_{2r})^{1/2}(X_{2r}'X_{2r})^{-1}(X_{2r}'X_{2r})^{1/2}$$
$$= \sigma^2(X_{2r}'X_{2r})^{1/2}(X_{2r}'X_{2r})^{-1/2}(X_{2r}'X_{2r})^{-1/2}(X_{2r}'X_{2r})^{1/2}$$
$$= \sigma^2 I$$

$$\implies V \sim N(0, \sigma^2 I) = \frac{V'V}{\sigma^2} \sim \chi^2_{k+1-m}$$

$$\Rightarrow \frac{(\hat\beta_{2c}-\beta_2)'(X_{2r}'X_{2r})^{1/2}\;(X_{2r}'X_{2r})^{1/2}\;(\hat\beta_{2c}-\beta_2)}{\sigma^2} = \frac{(\hat\beta_{2c}-\beta_2)'\;X_{2r}'X_{2r}\;(\hat\beta_{2c}-\beta_2)}{\sigma^2} \sim \chi^2_{k+1-m}$$

$$\underbrace{\frac{(Y_r - X_{2r}\beta_2)'(Y_r - X_{2r}\beta_2)}{\sigma^2}}_{\chi^2_n}^{Q} = \overbrace{\frac{e_c'e_c}{\sigma^2}}^{Q_1} + \underbrace{\frac{(\hat\beta_{2c}-\beta_2)X_{2r}'X_{2r}(\hat\beta_{2c}-\hat\beta_2)}{\sigma^2}}_{\chi^2_{k+1-m}}^{Q_2}$$

$$Q = Q_1 + Q_2$$

$$\begin{aligned}
\operatorname{cov}(e_c, \hat\beta_{2c}) &= \operatorname{cov}\left((I-H_{2r})y_r,\ (X_{2r}'X_{2r})^{-1}X_{2r}'y_r\right) \\
&= (I-H_{2r})\operatorname{var}(y_r)X_{2r}(X_{2r}'X_{2r})^{-1} \\
&= \sigma^2\underbrace{(I-H_{2r})X_{2r}}_{0}(X_{2r}'X_{2r})^{-1} = 0
\end{aligned}$$

$\Rightarrow$ $e_c, \hat\beta_{2c}$ are independent due to normality

$\Rightarrow$ $\dfrac{e_c'e_c}{\sigma^2}$, $\dfrac{(\hat\beta_{2c}-\beta_2)X_{2r}'X_{2r}(\hat\beta_{2c}-\hat\beta_2)}{\sigma^2}$ are independent because they are functions of $e_c, \hat\beta_{2c}$

Therefore, $M_Q(t) = M_{Q_1}(t)\cdot M_{Q_2}(t)$

or $M_{Q_1}(t) = \dfrac{M_Q(t)}{M_{Q_2}(t)}$

$$M_{Q_1}(t) = \frac{(1-2t)^{-n/2}}{(1-2t)^{-\frac{(k+1-m)}{2}}} = (1-2t)^{-\frac{(n-k-1+m)}{2}}$$

$$\Rightarrow Q_1 = \frac{e_c'e_c}{\sigma^2} \sim \chi^2_{n-k-1+m}$$

Note: $S_{ec}^2 = \dfrac{e_c'e_c}{n+m-k-1}$ so $\dfrac{(n-k-1+m)S_{ec}^2}{\sigma^2} = \dfrac{e_c'e_c}{\sigma^2} \sim \chi^2_{n-k-1+m}$