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Exercise 1

a. $var(\bar{Y} + \hat{\beta}_1) = var(\bar{Y}) + var(\hat{\beta}_1) + 2\,cov(\bar{Y}, \hat{\beta}_1)$

$\bar{Y} = \frac{1}{n} \sum Y_i$

$var(\bar{Y}) = var\left(\frac{1}{n} \sum Y_i\right) =$

$\frac{1}{n^2} \sum var(Y_i) = \frac{1}{n^2} \sum \sigma^2$

$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

we know: $var(\hat{\beta}_1) = \frac{\sigma^2}{\sum (x_1 - \bar{x})^2}$

we know: $2\,cov(\bar{Y}, \hat{\beta}_1) = 0$

$var(\bar{Y} + \hat{\beta}_1) = \frac{\sigma^2}{n} + \frac{\sigma^2}{\sum (x_1 - \bar{x})^2}$

b. $E(\bar{Y} + \hat{\beta}_1)^2 =$

Rule: $var(x) = E(x^2) - E(x)^2$
$E(x^2) = var(x) + E(x)^2$

$var(\bar{Y} + \hat{\beta}_1) + \left[E(\bar{Y} + \hat{\beta}_1)\right]^2$

$var(\bar{Y} + \hat{\beta}_1) + \left[E(\bar{Y}) + E(\hat{\beta}_1)\right]^2 = \frac{\sigma^2}{n} + \frac{\sigma^2}{\sum(x_1 - \bar{x})^2} + \left[\hat{b}_0 + \hat{b}_1 x_i + b_1\right]^2$

Seen above

$\frac{\sigma^2}{n} + \frac{\sigma^2}{\sum(x_1 - \bar{x})^2}$

$\hat{b}_0 + \hat{b}_1 x_i$

b_1

c. $var(\hat{Y}_i - e_i) = var(\hat{Y}_i - (Y_i - \hat{Y}_i)) = var(\hat{Y}_i - Y_i + \hat{Y}_i) =$

$var(2\hat{Y}_i - Y_i) = var(2\hat{Y}_i) + var(Y_i) - 2\,cov(\hat{Y}_i, Y_i)$

$= 4\,var(\hat{Y}_i) + \underbrace{var(Y_i)}_{=\sigma^2} - 2\,cov(\hat{Y}_i, Y_i)$

Note: $\operatorname{cov}(y_i, \hat{y}_i) = \operatorname{cov}(y_i, x_i \hat{\beta}_1) = x_i \operatorname{cov}(y_i, \hat{\beta}_1) = x_i \operatorname{cov}\left(y_i, \frac{\sum x_j y_j}{\sum x_j^2}\right)$

$$= \frac{x_i}{\sum x_j^2} \operatorname{cov}(y_i, \sum x_j y_j) = \frac{x_i}{\sum x_j^2} \sum_j x_j \operatorname{cov}(y_i, y_j)$$

$$= \frac{x_i}{\sum x_j^2} \Big[x_i \operatorname{cov}(y_i, y_i) + \sum_{j \neq i} x_i \underbrace{\operatorname{cov}(y_i, y_j)}^{=0} \Big] = \frac{x_i^2}{\sum x_j^2} \operatorname{cov}(y_i, y_i)$$

$$= \frac{x_i^2}{\sum x_j^2} \sigma^2$$

Note: $\operatorname{var}(\hat{Y}_i) = \sigma^2 \sum \left(\frac{1}{n} + (x_i - \bar{x}) k_j\right)^2 = \sigma^2 \sum \left[\frac{1}{n^2} + [(x_i - \bar{x})]^2 k_i^2 + \frac{2}{n}(x_i - \bar{x}) k_j\right]$

$$= \sigma^2 \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right)$$

$4 \operatorname{var}(\hat{Y}_i) + \operatorname{var}(Y_i) - 2 \operatorname{cov}(\hat{Y}_i, Y_i) =$

$$4 \sigma^2 \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right) + \sigma^2 - \frac{x_i^2}{\sum x_j^2} \sigma^2$$

d. $\hat{y}_i = \sum h_{ij} y_j$ $\qquad$ $\hat{Y}_i = \bar{Y} + \hat{b}_1 (x_i - \bar{x})$

$$\bar{Y} + \hat{b}_1 (x_i - \bar{x}) = \sum h_{ij} y_j$$

$$n \sum y_i + \frac{\sum (x_i - \bar{x}) Y_i}{\sum (x_i - \bar{x})^2} (x_i - \bar{x}) = \sum h_{ij} y_j$$

Exercise 2

a. $y_i - \bar{y} = \beta_0 + \beta_1 (x_i - \bar{x}) + \epsilon_i$

$y_i - \bar{y} \sim N(\beta_0 + \beta_1 (x_i - \bar{x}), \sigma^2)$

$$L = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{((y_i-\bar{y}) - \beta_0 - \beta_1(x_i-\bar{x}))^2}{2\sigma^2}}$$

$$\log(L) = \ell = \frac{-n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \Sigma (y_i - \bar{y}) - \beta_0 - \beta_1 (x_i - \bar{x}))^2$$

$$\frac{\partial \ell}{\partial \beta_0} = \frac{1}{\sigma^2} \Sigma (y_i - \bar{y}) - \beta_0 - \beta_1 (x_i - \bar{x}) = 0$$

$$\underbrace{\Sigma (y_i - \bar{y})}_{=0} - \Sigma \beta_0 - \beta_1 \underbrace{\Sigma (x_i - \bar{x})}_{=0} = 0$$

$$-\Sigma \beta_0 = 0$$

$$-n\hat{\beta}_0 = 0$$

$$\hat{\beta}_0 = 0$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{1}{\sigma^2} \Sigma (y_i - \bar{y}) - \beta_0 - \beta_1 (x_i - \bar{x}))(x_i - \bar{x}) =$$

$$\frac{1}{\sigma^2} \Sigma (y_i - \bar{y})(x_i - \bar{x}) - \beta_0 \underbrace{\Sigma (x_i - \bar{x})}_{=0} - \beta_1 \Sigma (x_i - \bar{x})^2 = 0$$

$$\Sigma (y_i - \bar{y})(x_i - \bar{x}) = \beta_1 \Sigma (x_i - \bar{x})^2$$

$$\beta_1 = \frac{\Sigma (y_i - \bar{y})(x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2}$$

b. non-centered:

$$\sum e_i^2 = \sum (Y_i - \hat{Y}_i)^2 = \sum (Y_i - \hat{b}_0 - \hat{b}_1 x_i)^2 = \sum (Y_i - \bar{y} + \hat{b}_1 (X_i - \bar{x}))^2$$

centered:

$$\sum e_i^2 = \sum (Y_i - (\hat{Y}_i - \bar{y}))^2 = \sum (Y_i - \hat{b}_0 - \hat{b}_1 X_i + \bar{y})^2$$

c. Let $Q_1 = \frac{(n-2) S_e^2}{\sigma^2}$ with $Q_1 \sim \chi^2_{n-2}$

Then: $S_e^2 = \frac{\sigma^2}{n-2} Q_1$

$$M_{S_e^2}(t) = M_{\frac{\sigma^2}{n-2} Q_1}(t) = M_{Q_1}\left(\frac{\sigma^2}{n-2} t \right.$$

$$M_{S_e^2}(t) = \left(1 - \frac{2\sigma^2}{n-2} t \right)^{-\frac{n-2}{2}}$$ therefore

$S_e^2 \sim \Gamma \left(\frac{n-2}{2}, \frac{2\sigma^2}{n-2} \right)$ and $var(S_e^2) = \frac{2\sigma^4}{n-2}$

Exercise 3

a. $E\left[\frac{\bar{Y}}{S_e}\right] = E(\bar{Y})\, E\left[\frac{1}{S_e}\right]$

$E(\bar{Y}) = \hat{\beta}_0 + \hat{\beta}_1 x_i$

$E\left(\frac{1}{S_e}\right) = E(S_e^2)^{-1/2}$ $k = -1/2$

$$= \frac{T\left(\frac{n-2}{2} - \frac{1}{2}\right)\left(\frac{2\sigma^2}{n-2}\right)^{-1/2}}{T\left(\frac{n-2}{2}\right)}$$

$$E\left[\frac{\bar{Y}}{S_e}\right] = (\hat{\beta}_0 + \hat{\beta}_1 x_i)\left[\frac{T\left(\frac{n-2}{2} - \frac{1}{2}\right)\left(\frac{2\sigma^2}{n-2}\right)^{-1/2}}{T\left(\frac{n-2}{2}\right)}\right]$$

b. $E\left[\frac{\beta_1^2}{SSE}\right] = E[\beta_1^2]\, E\left[\frac{1}{SSE}\right] = E[\beta_1^2]\, E\left[\frac{1}{\sum e_i^2}\right]$

c. $y_i = \frac{\theta}{2} x_i^2 + \epsilon_i \rightarrow y_i \sim N\left(\frac{\theta}{2} x_i^2, \sigma^2\right)$

Therefore the likelihood function equals:

$$L = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\sigma^2 (y_i - \frac{\theta}{2} x_i^2)^2}$$

$$\log(L) = l = \frac{-n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum (Y_i - \frac{\theta}{2} x_i^2)^2$$

$$\frac{\partial l}{\partial \theta} = -\frac{1}{\sigma^2} \sum (Y_i - \frac{\theta}{2} x_i^2)(-\frac{1}{2} x_i^2)$$

$-\frac{1}{2}(-\frac{\theta}{2} x_i^4)$

$$= -\frac{1}{\sigma^2} \sum -\frac{1}{2} x_i^2 y_i + \frac{\theta}{4} x_i^4$$

$$= \frac{1}{\sigma^2} \sum x_i^2 y_i - \frac{\theta}{2\sigma^2} \sum X_i^4 = 0$$

$$\frac{1}{\sigma^2}\sum x_i^2 y_i - \frac{\theta}{2\sigma^2}\sum x_i^4 = 0$$

$$\frac{1}{\sigma^2}\sum x_i^2 y_i = \frac{\theta}{2\sigma^2}\sum x_i^4$$

$$2\sum x_i^2 y_i = \theta \sum x_i^4$$

$$\hat{\theta} = \frac{2\sum x_i^2 y_i}{\sum x_i^4}$$

idk maybe it follows a normal distribution

Exercise 4

a. $\text{Min } Q = \sum \epsilon_i^2$

$\text{Min } Q = \sum (Y_i - \beta_0 - \beta_1 X_i)^2 =$

$\sum Y_i^2 - 2\beta_0 \sum Y_i - 2\beta_1 \sum X_i Y_i + n\beta_0^2 + 2\beta_0\beta_1 \sum X_i + \beta_1^2 \sum X_i^2$

$$\frac{\partial Q}{\partial \beta_1} = -2\sum x_i y_i + 2\beta_0 \sum X_i + 2\beta_1 \sum X_i^2 = 0$$

$$2\sum x_i y_i - 2\beta_0 \sum X_i = 2\beta_1 \sum X_i^2$$

$$\hat{\beta}_1 = \frac{\sum x_i y_i - \beta_0 \sum X_i}{\sum X_i^2}$$

$$\frac{\partial Q}{\partial \beta_0} = -2\sum Y_i - 2n\beta_0 + 2\beta_1 \sum X_i = 0$$

$$2n\beta_0 = 2\beta_1 \sum X_i - 2\sum Y_i$$

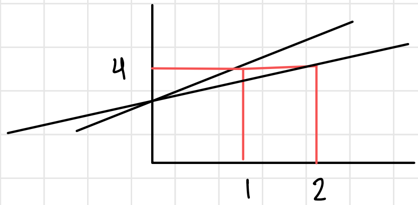
$$\hat{\beta}_0 = \frac{2\beta_1 \sum X_i - 2\sum Y_i}{2n}$$

b. $$R^2 = 1 - \frac{\sum e_i^2}{\sum (Y_i - \bar{Y})^2} = 1 - \frac{\sum (Y_i - \hat{Y}_i)^2}{\sum (Y_i - \bar{Y})^2} = \frac{\sum (Y_i - \beta_0 - \beta_1 X_i)^2}{\sum (Y_i - n\sum Y_i)^2}$$

c. $\hat{Y}_i = \sum h_{ij} y_j \qquad y_i = \beta_1 X_i + \epsilon_i$

Exercise 5

a. $\beta_0^* = \hat{\beta}_0$ and $\beta_1^* = \dfrac{\hat{\beta}_1}{z_i}$



$\dfrac{4}{1} \rightarrow \dfrac{2}{1}$

$\dfrac{1}{2}$

b. $R^2 = 1 - \dfrac{\Sigma e_i^2}{\Sigma (Y_i - \bar{Y})^2}$

$$\boxed{\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i = \bar{Y} + \hat{\beta}_1 (X_i - \bar{X})}$$

"Fitted value"

Model 1: $R^2 = 1 - \dfrac{\Sigma (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)}{\Sigma (Y_i - \bar{Y})^2} = 1 - \left[\dfrac{\Sigma Y_i - n\hat{\beta}_0 - \hat{\beta}_1 \Sigma X_i}{\Sigma (Y_i - \bar{Y})^2} \right]$

Model 2: $R^2 = 1 - \dfrac{\Sigma (Y_i - \beta_0^* - \beta_1^* X_i)}{\Sigma (Y_i - \bar{Y})^2} = 1 - \dfrac{\Sigma (Y_i - \hat{\beta}_0 - \frac{\hat{\beta}_1}{z_i} X_i)}{\Sigma (Y_i - \bar{Y})^2}$

$$= 1 - \left[\dfrac{\Sigma Y_i - n\hat{\beta}_0 - \frac{\hat{\beta}_1}{z_i} \Sigma X_i}{\Sigma (\hat{\beta}_0 + \frac{\hat{\beta}_1}{z_i} X_i - \bar{Y})^2} \right]$$

c. $\beta_1^* = \dfrac{\hat{\beta}_1}{z_i} \longrightarrow var(\beta_1^*) = var\left(\dfrac{\hat{\beta}_1}{z_i}\right) = z_i^2 \, var(\hat{\beta}_1)$

$$= z_i^2 \dfrac{\sigma^2}{\Sigma (X_i - \bar{X})^2}$$