Madeline Blasingame 205412082

a. $X_1 = [\underline{1} \ \underline{x_1} \ \underline{x_2} \ x_3]$ $\qquad H_1 = X_1(X_1'X_1)^{-1}X_1'$ $\qquad \hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$

$X_2 = [\underline{x_4} \ \underline{x_5} \ \underline{x_6}]$

Regress X_2 on X_1

$(I-H_1)X_2 = [(I-H_1)x_4 \quad (I-H_1)x_5 \quad (I-H_1)x_6] = X_2^*$

Regress Y on X_1

$(I-H_1)Y = Y^*$

Regress Y^* on X_2^*

$\hat{\beta}_{2\cdot1} = (X_2^{*\prime}X_2^*)^{-1}X_2^{*\prime}Y^* = [((I-H_1)X_2)'((I-H_1)X_2)]^{-1}((I-H_1)X_2)'(I-H_1)Y$

$= X_2'(I-H_1)(I-H_1)X_2$

Regress Y on X_1

$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'Y$

Regress Y on X_2

$\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'Y$

b. $W = \begin{bmatrix} X & Z \end{bmatrix}$ $\qquad W'W = \begin{bmatrix} X' \\ Z' \end{bmatrix} \begin{bmatrix} X & Z \end{bmatrix} = \begin{array}{cc} A_{11} & A_{12} \\ \left[\begin{matrix} X'X \\ Z'X \end{matrix}\right. & \left.\begin{matrix} X'Z \\ Z'Z \end{matrix}\right] \\ A_{21} & A_{22} \end{array}$

If all inverses exist:

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{W'W}^{-1} = \underbrace{\begin{bmatrix} A_{11}^{-1} + B_{12} B_{22}^{-1} B_{21} & -B_{12} B_{22}^{-1} \\ -B_{22}^{-1} B_{21} & B_{22}^{-1} \end{bmatrix}}_{[W'W]^{-1}}$$

$B_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$

$B_{12} = A_{11}^{-1} A_{12}$

$B_{21} = A_{21} A_{11}^{-1}$

the last diagonal element of $[W'W]^{-1}$

$$B_{22}^{-1} = \left[A_{22} - A_{21} A_{11}^{-1} A_{12}\right]^{-1}$$

Note: $H_1 = X(X'X)^{-1}X'$

$$= \left[Z'Z - Z'X[X'X]^{-1} X'Z\right]^{-1} = \left[Z'Z - Z'H_1 Z\right]^{-1}$$

$$Z^* = (I - H_1) Z$$

$$\frac{1}{Z^{*\prime} Z^*} = \frac{1}{Z'(I-H_1)(I-H_1)Z} = \frac{1}{Z'(I-H_1)Z} = \frac{1}{Z'Z - Z'H_1 Z} =$$

this is a scalar

$\left[Z'Z - Z'H_1 Z\right]^{-1} \rightarrow$ therefore, the last diagonal element of $(W'W)^{-1}$ is equal to $\dfrac{1}{Z^{*\prime} Z^*}$

c. $H_{12} = X_1 X_2 (X_2' X_1' X_1 X_2)^{-1} X_2' X_1$

$\hat{\beta}_{2\cdot1} = (X_2^{*'} X_2^{*})^{-1} X_2^{*'} Y^{*}$

$(Y - X_2 \beta_{2\cdot1})' (I - H_1)(Y - X_2 \hat{\beta}_{2\cdot1}) =$

$(Y' - \beta_{2\cdot1}' X_2')(I - H_1)(Y - X_2 \hat{\beta}_{2\cdot1}) =$

$(Y' - Y^{*'} X_2^{*} (X_2^{*'} X_2^{*})^{-1} X_2')(I - H_1) (Y - X_2 (X_2^{*'} X_2^{*})^{-1} X_2^{*'} Y^{*})$

d. we know:

$$\hat{\beta}_{(0)} = (X_{(0)}^{*'} X_{(0)}^{*-1}) X_{(0)}^{*'} Y^*$$

where $X_{(0)}^* = (I - \frac{1}{n} 11') X_{(0)}$

$Y^* = (I - \frac{1}{n} 11') Y$

$$Var(\hat{\beta}_{(0)}) = var((X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'} Y^*))$$

let $A = (X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'}$

$$= var(AY^*)$$

$$= A\, var(Y^*) A'$$

let $B = (I - \frac{1}{n} 11')$

$$= AB\, var(Y) B' A'$$

$$= AB \sigma^2 B' A'$$

$$= \sigma^2 (X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'} (I - \tfrac{1}{n} 11')(I - \tfrac{1}{n} 11') [(X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'}]'$$

$$= \sigma^2 (X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'} (I - \tfrac{1}{n} 11') [(X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'}]'$$

$$= \sigma^2 (X_{(0)}^{*'} X_{(0)}^{*})^{-1} X_{(0)}^{*'} (I - \tfrac{1}{n} 11') X_{(0)}^{*} (X_{(0)}^{*'} X_{(0)}^{*})^{-1}$$

$$= \sigma^2 (X_{(0)} (I - \tfrac{1}{n} 11')(I - \tfrac{1}{n} 11') X_{(0)})^{-1} X_{(0)} (I - \tfrac{1}{n} 11')(I - \tfrac{1}{n} 11')(I - \tfrac{1}{n} 11') X_{(0)} (X_{(0)}^{*'} X_{(0)}^{*})^{-1}$$

$$\sigma^2 (X_{(0)} (I - \tfrac{1}{n} 11') X_{(0)})^{-1} (X_{(0)} (I - \tfrac{1}{n} 11') X_{(0)}) (X_{(0)}^{*'} X_{(0)}^{*})^{-1}$$

$$= \sigma^2 I (X_{(0)}^{*'} X_{(0)}^{*})^{-1} = \sigma^2 (X_{(0)}^{*'} X_{(0)}^{*})^{-1}$$

we know:

$$\hat{\beta}_0 = (1'1)^{-1} 1'Y - (1'1)^{-1} 1' X_{(0)} \hat{\beta}_{(0)}$$

$$= \overbrace{(1'1)^{-1} 1Y}^{A} - \overbrace{(1'1)^{-1} 1' X_{(0)} (X_{(0)}^{*'} X_{(0)}^{*-1}) X_{(0)}^{*'} (I - \tfrac{1}{n} 11')}^{B} Y$$

$$var(\hat{\beta}_0) = var(AY) - var(BY) + 2 cov(AY, BY)$$

$$= A\, var(Y) A' - B\, var(Y) B' + 2A\, var(Y) B'$$

$$= A \sigma^2 I A' - B \sigma^2 I B' + 2A \sigma^2 I B'$$

$$= \sigma^2 AA' - \sigma^2 BB' + 2\sigma^2 AB'$$

$$= \sigma^2 (AA' - BB' + 2AB')$$

e. $(Y-\hat{Y}_c)'(Y-\hat{Y}_c) = (Y-\hat{Y})'(Y-\hat{Y}) + (\hat{Y}-\hat{Y}_c)(\hat{Y}-\hat{Y}_c)$

we know: $Y-\hat{Y}_c = e_c$

$Y-\hat{Y} = e$

$\longrightarrow e_c'e_c = e'e + (\hat{Y}-\hat{Y}_c)$

From lecture we know:

$$e_c'e_c = e'e + (c\hat{\beta}-\gamma)'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)$$

So we need to show:

$$(\hat{Y}-\hat{Y}_c)'(\hat{Y}-\hat{Y}_c) = (c\hat{\beta}-\gamma)'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)$$

From lecture we also know:

$$\hat{Y}_c = X\hat{\beta}_c = X\hat{\beta} - X(X'X)^{-1}c'(c(X'X)^{-1}c')(c\hat{\beta}-\gamma)$$

and $\hat{Y} = X\hat{\beta}$

$$\hat{Y}-\hat{Y}_c = X\hat{\beta} - [X\hat{\beta} - X(X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)]$$

$$= X(X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)$$

$$(\hat{Y}-\hat{Y}_c)'(\hat{Y}-\hat{Y}_c) = [X(X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)]'[X(X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)]$$

$$= [(c\hat{\beta}-\gamma)'[c(X'X)^{-1}c']^{-1}c\underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)]$$

$$= [(c\hat{\beta}-\gamma)'[c(X'X)^{-1}c']^{-1}\underbrace{c(X'X)^{-1}c'[c(X'X)^{-1}c']^{-1}}_{I}(c\hat{\beta}-\gamma)]$$

$$= [(c\hat{\beta}-\gamma)'[c(X'X)^{-1}c']^{-1}(c\hat{\beta}-\gamma)]$$

$\therefore (Y-\hat{Y}_c)'(Y-\hat{Y}_c) = (Y-\hat{Y})'(Y-\hat{Y}) + (\hat{Y}-\hat{Y}_c)(\hat{Y}-\hat{Y}_c)$

F. From lecture we know:

$$R^2_{XZ} = R^2_X + \frac{Z^{*\prime} Z^* \hat{c}^2}{SST} \qquad SST = (Y-\bar{Y})'(Y-\bar{Y})$$

show: $R^2_{XZ} = R^2_X + (1 - R^2_X) r^2_{yX|Z}$ $\qquad R^2_X = 1 - \frac{e'e}{SST}$

$$\hat{c} = \frac{Z^{*\prime} e}{Z^{*\prime} Z^*}$$

$$r^2_{YX_k | X_1, \ldots X_{k-1}} = \frac{(Y^{*\prime} X_k^*)^2}{(X_k^{*\prime} X_k^*)(Y^{*\prime} Y^*)}$$

$$\Rightarrow \quad r^2_{YZ|X} = \frac{(Y^{*\prime} Z^*)^2}{(Z^{*\prime} Z^*)(Y^{*\prime} Y^*)}$$

we need to show: $\frac{Z^{*\prime} Z^* \hat{c}^2}{SST} = (1 - R^2_X) r^2_{yX|Z}$

$$\hat{c} = \frac{Z^{*\prime} e}{Z^{*\prime} Z^*} \rightarrow \frac{Z^{*\prime} Z^* \hat{c}^2}{SST} = \frac{Z^{*\prime} Z^*}{SST} \frac{(Z^{*\prime} e)^2}{(Z^{*\prime} Z^*)^2} = \frac{(Z^{*\prime} e)^2}{SST\ (Z^{*\prime} Z^*)^2}$$

$e = (I - H)Y = Y^*$

$$= \frac{(Z^{*\prime} Y^*)^2}{SST\ (Z^{*\prime} Z^*)^2} \quad \text{they are scalars:} \quad = \frac{(Y^* Z^*)^2}{SST\ (Z^{*\prime} Z^*)}$$

$$(1 - R^2_X) r^2_{yX|Z} = 1 - \left[1 - \frac{e'e}{SST}\right] \frac{(Y^{*\prime} Z^*)^2}{(Z^{*\prime} Z^*)(Y^{*\prime} Y^*)} = \frac{e'e}{SST} \frac{(Y^{*\prime} Z^*)^2}{(Z^{*\prime} Z^*)(Y^{*\prime} Y^*)}$$

$$= \frac{(Y^{*\prime} Y^*)(Y^{*\prime} Z^*)^2}{SST\ (Z^{*\prime} Z^*)(Y^{*\prime} Y^*)} = \frac{(Y^* Z^*)^2}{SST\ (Z^{*\prime} Z^*)}$$

g. From lecture we know:

$$(\hat{Y}-\bar{Y})'(\hat{Y}-\bar{Y})$$
$$(X\beta-\bar{Y})'(X\beta-\bar{Y})$$
$$(\beta'X'-\bar{Y})(X\beta-\bar{Y})$$
$$\beta'X'X\beta-\bar{Y}\beta'X'-\bar{Y}X\beta+\bar{Y}^2$$

Note: $\hat{Y}=X\beta$

$$(X'X)^{-1}X'Y$$
$$Y'(X'X)^{-1}X$$

$$SSR=\hat{\beta}'X'X\hat{\beta}-n\bar{y}^2$$

$$E[SSR]=E[\hat{\beta}'X'X\hat{\beta}-n\bar{y}^2]=E[\underbrace{\hat{\beta}'X'X\hat{\beta}}_{\text{scalar}}]-nE[\bar{y}^2]$$

$$=E[tr(\hat{\beta}'X'X\hat{\beta})]-nE[\bar{y}^2]$$
$$=E[tr(\hat{\beta}\hat{\beta}'X'X)]-nE[\bar{y}^2]$$
$$=tr(E[\hat{\beta}\hat{\beta}'X'X])-nE[\bar{y}^2]$$
$$=tr(X'X\,E[\hat{\beta}\hat{\beta}'])-nE[\bar{y}^2]$$
$$=tr(X'X(\sigma^2(X'X)^{-1}+\beta\beta')-nE[\bar{y}^2]$$
$$=tr(\sigma^2I+X'X\beta\beta')-nE[\bar{y}^2]$$
$$=tr(\sigma^2I)+tr(X'X\beta\beta')-nE[\bar{y}^2]$$
$$=n\sigma^2+tr(X'X\beta\beta')-nE[\bar{y}^2]$$
$$=n\sigma^2+tr(\underbrace{\beta'X'X\beta}_{\text{scalar}})-nE[\bar{y}^2]$$
$$=n\sigma^2+\beta'X'X\beta-nE[\bar{y}^2]$$
$$=n\sigma^2+\beta'X'X\beta-n\left[\frac{\sigma^2}{n}+\frac{1}{n^2}(1'X\beta)^2\right]$$
$$=n\sigma^2+\beta'X'\beta-\sigma^2+\frac{1}{n}(1'X\beta)^2$$

Note: $E[\hat{\beta}\beta']=var(\hat{\beta})+E(\beta)E(\beta)'$
$$=\sigma^2(X'X)^{-1}+\beta\beta'$$

Note: $tr(\sigma^2I)=n\sigma^2$

Note: $E[\bar{y}^2]=var(\bar{y})+E[\bar{y}]^2$
$$=\frac{\sigma^2}{n}+E[\tfrac{1}{n}1'Y]^2$$
$$=\frac{\sigma^2}{n}+\left[\frac{1}{n}1'E(Y)\right]^2$$
$$=\frac{\sigma^2}{n}+\frac{1}{n^2}(\underbrace{1'X\beta}_{\text{scalar}})^2$$

h. $Y = X\beta + \epsilon$ $\qquad \epsilon \sim N(0, \sigma^2 I)$ $\qquad \beta \sim N(\beta, \sigma^2 (X'X)^{-1})$

$$Q = \begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix}$$

$E[Q] = C\,E[\hat{\beta}] = C\beta$

$\text{var}(Q) = \text{var}(C\hat{\beta}) = C\,\text{var}(\hat{\beta})C'$

$= \sigma^2 C\,(X'X)^{-1} C'$

$Q \sim N(C\beta, \sigma^2 C\,(X'X)^{-1}C')$

when $C = \begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix}$

$K = 3$

$$\sigma^2 \underbrace{\begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} V_{00} & V_{01} & V_{02} & V_{03} \\ V_{10} & V_{11} & V_{12} & V_{13} \\ V_{20} & V_{21} & V_{22} & V_{23} \\ V_{30} & V_{31} & V_{32} & V_{33} \end{bmatrix}}_{(X'X)^{-1}} \underbrace{\begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}}_{C'} = \sigma^2 \begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} V_{00} - 2V_{01} + 3V_{02} + V_{03} \\ V_{10} - 2V_{11} + 3V_{12} + V_{13} \\ V_{20} - 2V_{21} + 3V_{22} + V_{23} \\ V_{30} - 2V_{31} + 3V_{32} + V_{33} \end{bmatrix}$$

$$= \sigma^2 \left[(V_{00} - 2V_{01} + 3V_{02} + V_{03}) - 2(V_{10} - 2V_{11} + 3V_{12} + V_{13}) + 3(V_{20} - 2V_{21} + 3V_{22} + V_{23}) + (V_{30} - 2V_{31} + 3V_{32} + V_{33}) \right]$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} =$$

$$Q \sim N\left(\hat{\beta}_0 - 2\hat{\beta}_1 + 3\hat{\beta}_2 + \hat{\beta}_3,\ \sigma^2 \left[(V_{00} - 2V_{01} + 3V_{02} + V_{03}) - 2(V_{10} - 2V_{11} + 3V_{12} + V_{13}) + 3(V_{20} - 2V_{21} + 3V_{22} + V_{23}) + (V_{30} - 2V_{31} + 3V_{32} + V_{33}) \right] \right)$$

Read in and create data

```
a <- read.table("http://www.stat.ucla.edu/~nchristo/statistics_c173_c273/jura.txt", head
er=TRUE)
y <- a$Pb

x1 <- a$Cd
x2 <- a$Co
x3 <- a$Cr
x4 <- a$Cu
x5 <- a$Ni
x6 <- a$Zn

ones <- rep(1, nrow(a))
X <- as.matrix(cbind(ones,x1, x2, x3, x4, x5, x6))
X1 <- as.matrix(cbind(ones,x1, x2, x3, x4))
X2 <- as.matrix(cbind(x5, x6))
```

Question i

Part 1:

```
#H1 Matrix
H1 <- X1 %*% solve(t(X1) %*% X1) %*% t(X1)
#H matrix
H <- X %*% solve(t(X) %*% X) %*% t(X)

#H-H1 Matrix
mat <- round((H - H1) %*% (H - H1))

identical(round((H - H1)), mat)
```

```
## [1] TRUE
```

The matrix (H-H1)(H-H1) is the same as the matrix (H-H1) which means that (H-H1) must be idempotent.

Part 2:

```r
#Beta hat using full regression
beta_hat <- solve(t(X) %*% X) %*% t(X) %*% y

H1 <- X1 %*% solve(t(X1) %*% X1) %*% t(X1)
X22 <- (diag(nrow(a)) - H1) %*% X2
y11 <- (diag(nrow(a)) - H1) %*% y

#Beta hat 1 from partial regression
beta_hat2 <- solve(t(X22) %*% X22) %*% t(X22) %*% y11

#Beta hat 2 from partial regression
beta_hat1 <- solve(t(X1) %*% X1) %*% t(X1) %*% y - solve(t(X1) %*% X1) %*% t(X1) %*% X2
 %*% beta_hat2


beta_hat
```

```
##            [,1]
## ones 18.9100703
## x1   -1.5192122
## x2   -1.3608686
## x3   -0.1439250
## x4    0.9771788
## x5    0.4855964
## x6    0.3004051
```

```r
beta_hat1
```

```
##            [,1]
## ones 18.9100703
## x1   -1.5192122
## x2   -1.3608686
## x3   -0.1439250
## x4    0.9771788
```

```r
beta_hat2
```

```
##         [,1]
## x5 0.4855964
## x6 0.3004051
```

As you can see, the values of $\hat{\beta}_0 ... \hat{\beta}_5$ from full regression match the values of $\hat{\beta}_0 ... \hat{\beta}_3$ and $\hat{\beta}_4, \hat{\beta}_5$ found using partial regression.

Question j

Lagrange

```r
ones <- rep(1, nrow(a))
X <- as.matrix(cbind(ones,x1, x2, x3, x4))
C <- as.matrix(rbind(c(1,2,-3,5,-1), c(2,-1,1,3,0)))
beta_hat <- solve(t(X) %*% X) %*% t(X) %*% y
gamma <- as.matrix(rbind(5,10))

#Beta hat using the constrained model
beta_hat_c <- beta_hat - solve(t(X) %*% X) %*% t(C) %*% solve(C %*% solve(t(X) %*% X) %
*% t(C)) %*% (C %*% beta_hat - gamma)
beta_hat_c
```

```
##           [,1]
## ones 4.1814535
## x1   0.1587391
## x2   0.2727253
## x3   0.5077022
## x4   1.2192669
```

Canonical

```r
C1 <- as.matrix(rbind(c(1,2), c(2,-1)))
C2 <- as.matrix(rbind(c(-3,5,-1), c(1,3,0)))
X1 <- as.matrix(cbind(ones,x1))
X2 <- as.matrix(cbind(x2, x3, x4))


Yr <- y - X1 %*% solve(C1) %*% gamma
X2r <- X2 - X1 %*% solve(C1) %*% C2

#Beta hat 1 and beta hat 2 using the constrained model
beta_hat_2c <- solve(t(X2r) %*% X2r) %*% t(X2r) %*% Yr
beta_hat_1c <- solve(C1) %*% (gamma - C2 %*% beta_hat_2c)

beta_hat_1c
```

```
##           [,1]
## [1,] 4.1814535
## [2,] 0.1587391
```

```r
beta_hat_2c
```

```
##         [,1]
## x2 0.2727253
## x3 0.5077022
## x4 1.2192669
```

As you can see, the values of $\hat{\beta}_{c0} \ldots \hat{\beta}_{c4}$ obtained using Lagrange multipliers are the same as the values of $\hat{\beta}_{c0}, \hat{\beta}_{c1}$ and $\hat{\beta}_{c2} \ldots \hat{\beta}_{c4}$ obtained using the canonical form of the model.