Number Theory (Practice Sheet I) Solution

1. Does 17 divide each of these numbers?

   a) 68 b) 84 c) 357 d) 1001

   Ans:

   a) 68 % 17 = 0

      So, 17 | 68

   b) 84 % 17 = 16

      So, 17 does not divide 84

   c) 357 % 17 = 0

      So, 17 | 357

   d) 1001 % 17 = 15

      So, 17 does not divide 1001

2. Show that if a, b, and c are integers, where a = 0 and c = 0, such that ac | bc, then a | b.

   Ans:

   From Theorem 1(ii), we know that if a | b, then a | bc for all integers c;

   Now, let x as an integer.

   ac | bcx

   Now if we divide both sides by c,

   We get, a | bx

   Again using Theorem 1(ii) -

   We get a | b

3. What time does a 12-hour clock read

a) 80 hours after it reads 11:00?

b) 40 hours before it reads 12:00?

c) 100 hours after it reads 6:00?

Ans:

a) 11+80 = 91

91%12= 7

So, a 12- hour clock reads 7:00

b) 12-40=-28

-28%12= 8

So, a 12-hour clock reads 8:00

c) 6+100=106

106%12=10

So, a 12-hour clock reads 10:00

4. Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \le c \le 12$ such that

a) $c \equiv 9a \pmod{13}$.

b) $c \equiv 11b \pmod{13}$.

c) $c \equiv a + b \pmod{13}$.

d) c ≡ 2a + 3b (mod 13).

e) c ≡ a2 + b2 (mod 13).

f) c ≡ a3 − b3 (mod 13).

Ans:

a) 9 · 4 mod 13 = 36 mod 13 = 10

b) 11·9 mod 13 = 99 mod 13 = 8

c) 4 + 9 mod 13 = 13 mod 13 = 0

d) 2 · 4 + 3 · 9 mod 13 = 35 mod 13 = 9

e) 42 + 92 mod 13 = 97 mod 13 = 6

f) 43 - 93 mod 13 = -665 mod 13 = 11 (because -665 = -52 · 13 + 11)

5. Show that if n and k are positive integers, then n/k = (n − 1)/k + 1.

Ans:

The quotient n/k lies between two consecutive integers, say b-1 and b,

So, b - 1 < n/k<=b. [n/k]=b

So, n/k > b-1 ; n-1 >= k(b-1) ; (n-1)/k >= b-1

So, [b-1] = [(n-1)/k] and [n/k] = [b]

[(n-1)/k] <= (n-1)/k <n/k <= b

So, we can write [(n-1)/k] < b ;

[(n-1)/k] <= b-1

[(n-1)/k] = [n/k]-1

[n/k] = [(n-1)/k]+1

6. Find a div m and a mod m when

a) a = 228, m = 119.

b) a = 9009, m = 223.

c) a = −10101, m = 333.

d) a = −765432, m = 38271.

Ans:

a) Quotient: 1, Remainder: 109

b) Quotient: 40, Remainder: 89

c) Quotient: -31 , Remainder: 222

d) Quotient: -21 , Remainder: 38259

7. List all integers between−100 and 100 that are congruent to −1 modulo 25.

Ans:

[-76, -51, -26, -1, 24, 49, 74, 99]

An integer a is congruent to b modulo n if a-b is divisible by n.
So, we need to find integers such that a-(-1) is divisible by 25. This is

equivalent to:
a + 1 ≡ 25k [k is an integer]
a = 25k - 1
When k=-5, we get a = -126

When k=-4, we get a = -101

When k=-3, we get a = -76

When k=-2, we get a = -51

When k=-1, we get a = -26

When k=0, we get a = -1

When k=1, we get a = 24

When k=2, we get a = 49

When k=3, we get a = 74

When k=4, we get a = 99

So, the integers between −100 and 100 that are congruent to −1 modulo 25 are:

[-76, -51, -26, -1, 24, 49, 74, 99]

8. Show that if n | m, where n and m are integers greater than 1, and if a ≡ b (mod m), where a and b are integers, then a ≡ b (mod n).

Ans:

Here, given, n | m and

From a ≡ b (mod m), we get m |(a-b)

Now, using Theorem 1 (iii) -[if a | b and b | c, then a | c.]

we can write n | (a-b)

So, a ≡ b (mod n).

9. Determine whether the integers in each of these sets are pairwise relatively prime.

a) 11, 15, 19

b) 14, 15, 21

c) 12, 17, 31, 37

d) 7, 8, 9, 11

Ans:

a)Let us determine the prime factorization of each integer:

11=1*11

15=3*5

19=1*19

Let us use the prime factorizations to determine the greatest common divisor of each pair

of the given integers.

gcd(11,15)=1

gcd(15,19)=1

gcd(11,19)=1

The integers are then pairwise relatively prime, because all greatest common divisors are equal to 1.

b) Let us determine the prime factorization of each integer:

14=2*7

15=3*5

21=3*7

Let us use the prime factorizations to determine the greatest common divisor of each pair of the given integers.

gcd(14,15)=1

gcd(15,21)=3

gcd(14,21)=7

The integers are then not pairwise relatively prime, because there exists a pair of integers that has a greatest common divisor different from 1.

c) Let us determine the prime factorization of each integer:

12= 2*6

17= 1*17

31= 1*31

37= 1*37

Let us use the prime factorizations to determine the greatest common divisor of each pair

of the given integers.

gcd(12,17) =1

gcd(17,31) =1

gcd(31,37) =1

gcd(12,37) =1

gcd(17,37) =1

gcd(12,31) =1

The integers are then pairwise relatively prime, because all greatest common divisors are equal to 1.

d) Let us determine the prime factorization of each integer:

7= 1*7

8= 4*2

9= 3*3

11= 1*11

Let us use the prime factorizations to determine the greatest common divisor of each pair

of the given integers.

gcd(7,8) =1

gcd(7,9) =1

gcd(7,11) =1

gcd(8,9) =1

gcd(8,11) =1

gcd(9,11) =1

The integers are then pairwise relatively prime, because all greatest common divisors are equal to 1.

10.What is the greatest common divisor of 17 and 22?

Ans: The positive common divisors of 17 and 22 is 1 Hence, gcd(17,22) = 12.

11.Determine whether the integers 10, 17, and 21 are pairwise relatively prime and whether the integers 10, 19, and 24 are pairwise relatively prime.

Ans: Similar process as question-9

12.Use the Euclidean algorithm to find

a) gcd(12, 18).

b) gcd(111, 201).

c) gcd(1001, 1331).

d) gcd(12345, 54321).

3

e) gcd(1000, 5040).

f) gcd(9888, 6060)

Ans:

a) Using the Euclidean algorithm-

   18=12.1 + 6

   12=6.2 + 0

   Here, 6 is the last non-zero remainder. So, GCD(12,18) = 6

b) Using Euclidean algorithm-

   201 = 111.1 + 90

   111= 90.1 + 21

   90 = 21. 4 + 6

   21 = 6.3 + 3

   6 = 3.2 + 0

Here, 3 is the last non-zero remainder. So, GCD(201,111) = 3

c) Using Euclidean algorithm-

1331 = 1001.1 + 330

1001= 330.3 + 11

330 = 11.30 + 0

Here, 11 is the last non-zero remainder. So, GCD(1331,1001) = 11

d) Using Euclidean algorithm-

54321 = 12345.4 + 4941

12345 = 4941.2 + 2463

4941 = 2463.2 + 15

2463 = 15.164 + 3

15 = 3.5 + 0

Here, 3 is the last non-zero remainder. So, GCD(12345.54321) = 3

e) Using Euclidean algorithm-

5040 = 1000.5 +40

1000 = 40.25 + 0

Here, 40 is the last non-zero remainder. So, GCD(1000,5040) = 40

f) Using Euclidean algorithm-

9888 = 6060.1 + 3828

6060 = 3828.1 + 2232

3828 = 2232.1 + 1596

2232 = 1596.1 + 636

1596 = 636.2 + 324

636 = 324.1 + 312

324 = 312.1 + 12

312 = 12.26 + 0

Here, 12 is the last non-zero remainder. So, GCD(9888,6060) = 12

13.How many divisions are required to find gcd(34, 55) using the Euclidean algorithm?

Ans:

Using Euclidean algorithm-

55 = 34.1 + 21
34 = 21.1 + 13
21 = 13.1 + 8
13 = 8.1 + 5
8 = 5.1 + 3
5 = 3.1 + 2
3 = 2.1 + 1
2 = 1.2 + 0

Hence. GCD (34,55) is 1

So , 8 divisions are required to find gcd(34, 55) using the Euclidean algorithm

14.What are the greatest common divisors of these pairs of integers?

a) 37 · 53 · 73, 211 · 35 · 59

b) 11 · 13 · 17, 29 · 37 · 55 · 73

c) 2331, 2317

Ans:

a) These numbers have no common prime factors, so the gcd is 1 .

b) These numbers have no common prime factors, so the gcd is 1 .

c) 2331 = 3^2.7.37

2317 = 7.331

The second number has no prime factors of 3, so the gcd has no 3's. Since the first number has one factor of 7 and the second number has one as well. So the gcd is 7 .

15.Find gcd(92928, 123552) and lcm(92928, 123552), and verify that gcd(92928, 123552) · lcm(92928, 123552) = 92928 · 123552. [Hint: First find the prime factorizations of 92928 and 123552.]

Ans:

Prime factorization-

92928 = $2^8 . 3 . 11^2$

123552 = $2^5 . 3^3 . 11 . 13$

Now, GCD (92928,123552) = $2^{min(8,5)} . 3^{min(1,3)} . 11^{min(1,2)} . 13^{min(0,1)}$

= $2^5 . 3^1 . 11^1$

= 1056

LCM (92928,123552) = $2^{max(8,5)}. 3^{max(1,3)}. 11^{max(1,2)}. 13^{max(0,1)}$

= $2^8. 3^3. 11^2.13$

= 10872576

Now, gcd(92928, 123552) · lcm(92928, 123552) = $2^5. 3^1. 11^1.2^8. 3^3. 11^2.13$

= $(2^8. 3. 11^2) . (2^5. 3^3. 11. 13)$

= 92928 · 123552