Interpolation Error :

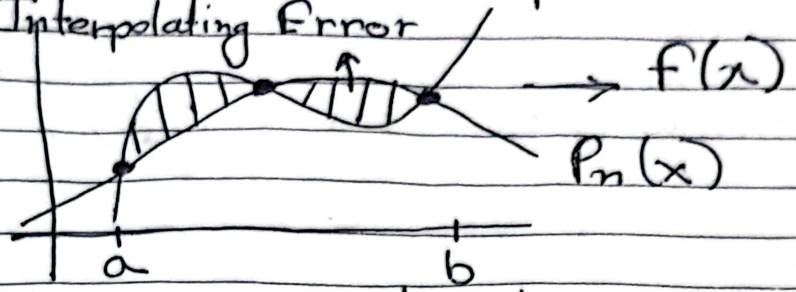
If we recall from Taylor Series,

$$|f(x) - P_n(x)| \rightarrow \text{Error Bound/ Lagrange form of the reminder}$$

$$= \frac{f^{(n+1)}(\xi_0)}{(n+1)!}(x - x_0)^{n+1}$$

↳ one node

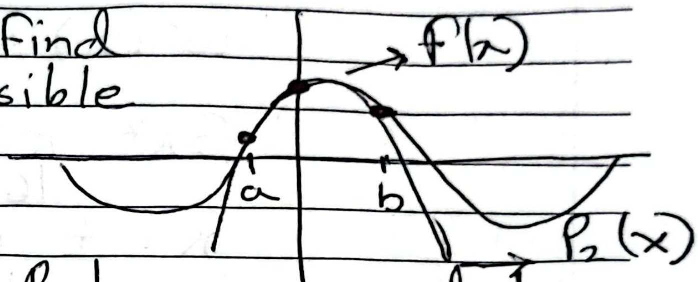
We will see it for multiple nodes,



$f(x) = \cos x$ nodes $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

$P_2(x) = \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 + 1$,

Our target is to find the maximum possible error.



Cauchy's Theorem :

$$|f(x) - P_n(x)| = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_n)$$

Between a and b, the error is very less.

Error increasing Rate = Exponential outside the range.

$f(x) = \cos x$ nodes $= \left\{ -\frac{\pi}{4}, 0, \frac{\pi}{4} \right\}$

Interval $[a, b]$

$$|f(x) - P_2(x)| = \left| \frac{f'''(\xi)}{3!}\left(x+\frac{\pi}{4}\right)(x-0)\left(x-\frac{\pi}{4}\right) \right|$$

Maximum possible error / Upper bound error

$$|f(x) - P_2(x)| \leq \left| \frac{f'''(\xi)}{6}\left(x^2 - \frac{\pi^2}{16}\right)(x) \right|$$

$$\leq \left| \frac{\sin(\xi)}{6} \right| \times \left| \left(x^3 - \frac{\pi^2}{16}x\right) \right|$$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$\xi \in [-1, 1]$

Max: for $\frac{\sin(\xi)}{6}$,

$\sin(1) = 0.8415$

$\sin(-1) = -0.8415$

$\frac{0.8415}{6}$

Max: for $\left(x^3 - \frac{\pi^2}{16}x\right)$,

$w(x) = x^3 - \frac{\pi^2}{16}x$

$w'(x) = 3x^2 - \frac{\pi^2}{16}$

$3x^2 - \frac{\pi^2}{16} = 0$] for min, max

$x = \pm \frac{\pi}{4\sqrt{3}}$

$w\left(\frac{\pi}{4\sqrt{3}}\right) = -0.186$

$w\left(-\frac{\pi}{4\sqrt{3}}\right) = 0.186$

$w(-1) = 0.3831$

$w(1) = -0.3831$.

CamScanner

$$|f(x) - P_2(x)| \leq \frac{0.8415}{6} \times 0.3831$$
$$\leq 0.0537$$

↳ Maximum Error

(b) If interval is not given then what to consider?

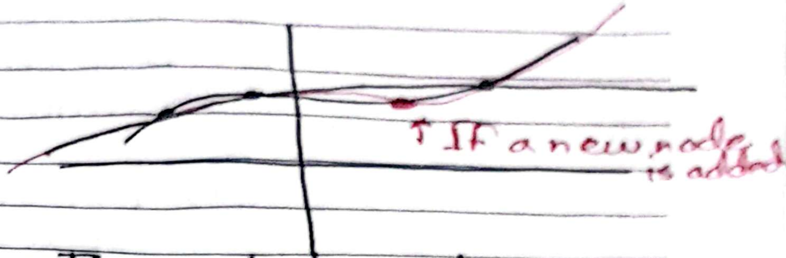
Then in that case the interval is to be considered for the given nodes.

$a = -\frac{\pi}{4}$, $b = \frac{\pi}{4}$.

Convergence : $|f(x) - P_n(x)|$

nodes ↑ error ↓

nodes ∞ error = 0

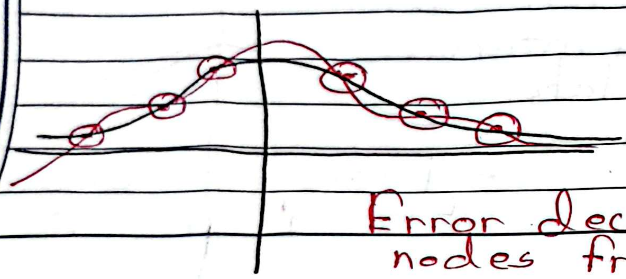


However this is not always true. There are some exceptions.

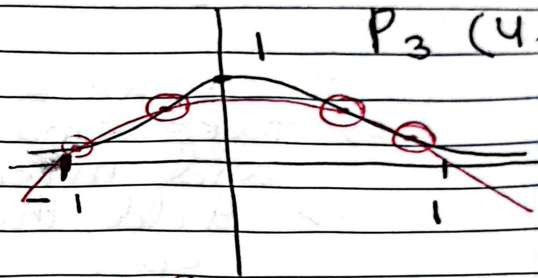
The polynomial converges towards the newer node.

$f(x) = \frac{1}{1+25x^2}$ interval $[-1, 1]$

For 6 nodes.



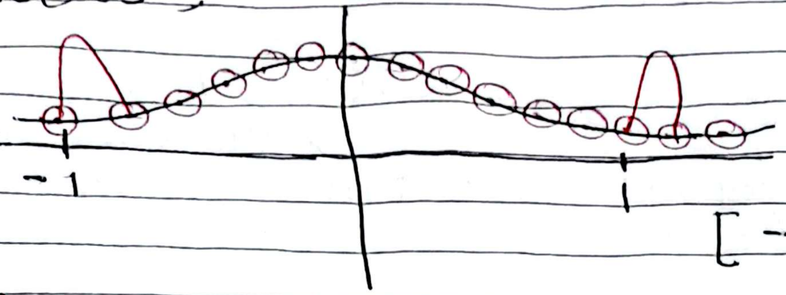
P_3 (4 nodes)



Error decreases if we increase nodes from nodes 4 to nodes 6

15 nodes,

Equally spaced nodes



Equally spaced nodes. [-0.5, 0, 0.5, 1] - nodes

There is a spike in the polynomial at the end points of -1 and 1. The error is most at these two points.

↳ The phenoma is known as Runge Phenomena.

(I) Depends on the function. (Symmetric function)

(II) Depends on the nodes (Equally spaced)

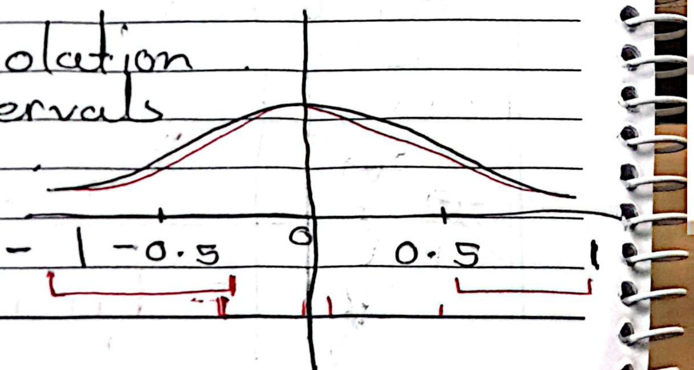
** Nodes need to be increased at the end points to avoid Runge Phenoma. (interval)

Solution:

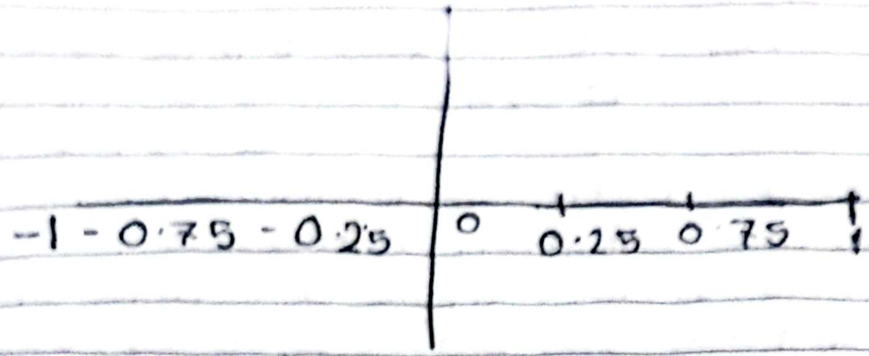
(I) Piece wise Interpolation.

↳ take smaller intervals and interpolate.

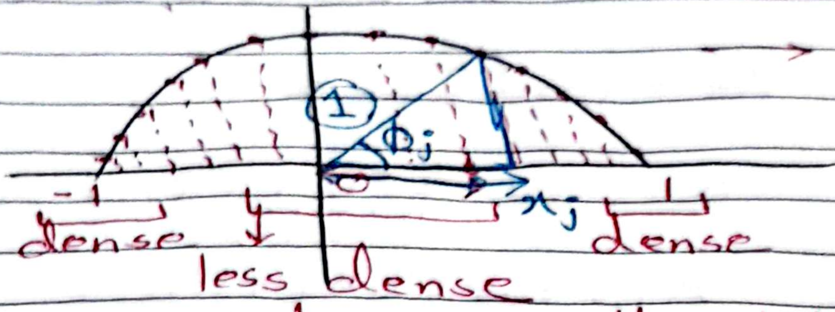
2> Add them up.



(ii) Non-equidistant nodes (Chebyshev's nodes)



(Normal approach)



Draw equally spaced points on the curve.

The more points near the interval the better.

radius is 1.

$$\Phi_j = \frac{(2j+1)}{2(n+1)}\pi \qquad x_j = ?$$

$$\cos \Phi_j = \frac{x_j}{1}$$

$$x_j = \cos \Phi_j$$

$$\boxed{x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)}.$$

Question :

$$F(x) = \frac{1}{1+25x^2} \leftarrow \text{Runge function}$$

$[-1, 1]$

$n = 3$. Define a polynomial?
Nodes are not given in this case.
In this case, nodes will be 4. Use Chebyshev's nodes to find the defined nodes, $[x_0, x_1, x_2, x_3]$

~~$x_0 = \cos (2 \times 0$~~

$$x_0 = \cos \frac{(2 \times 0 + 1)\pi}{2(3+1)} = \cos \frac{\pi}{8}.$$

$$x_1 = \cos \frac{(2 \times 1 + 1)\pi}{2 \times 4} = \cos \frac{3\pi}{8}.$$

$$x_2 = \cos \frac{5\pi}{8}.$$

$$x_3 = \cos \frac{7\pi}{8}.$$

$P_3(x) = ?$ You can use any interpolation method to define the polynomial.

Hermite Interpolation

$P_2(x)$
3 nodes / conditions

x	$f(x)$
3	15
5	27
9	35

The polynomials we have seen so far.

In case of Hermite interpolation, extra information of $f'(x)$ is provided.

$n = 2$

x	$f(x)$	$f'(x)$
3	15	40
5	27	20
9	35	37

3 conditions + 3 conditions
= 6 conditions

$P_{2n+1} \Rightarrow P_{2(2)+1} \Rightarrow P_5$

$n = 3$

x	$f(x)$	$f'(x)$
x_0	$f(x_0)$	$f'(x_0)$
x_1	$f(x_1)$	$f'(x_1)$
x_2	$f(x_2)$	$f'(x_2)$
x_3	$f(x_3)$	$f'(x_3)$
	4 conditions	4 conditions

$P_{2n+1} \Rightarrow P_{2(3)+1} \Rightarrow P_7$

Why do we need Hermite Interpolation?

According to Weisteress Approximation Theorem, $|f(x) - P_n(x)|$, a certain error is generated. If we increase the degree then the error decreases. Previously, the value of n was 3 for which there was an error. In case of Hermite Interpolation, the error decreases since n is equal 7 now and we could decrease the error using the same data points.

Example : $f(x) = \sin(x)$ $f'(x) = \cos(x)$

Given,

		x	$f(x)$	$f'(x)$
$n = 1$	x_0	0	0	1
	x_1	$\pi/2$	1	0

$P_{2n+1} = P_{2(1)+1} = P_3$

$$P_3(x) = h_0(x) f(x_0) + h_1(x) f(x_1) + \hat{h}_0(x) f'(x_0) + \hat{h}_1(x) f'(x_1)$$

$$h_0(x) f(x_0) = 0$$

$$\hat{h}_1(x) f'(x_1) = 0$$

$$P_3(x) = h_1(x) f(x_1) + \hat{h}_0(x) f'(x_0)$$

$$h_k(x) = \{1 - 2(x - x_k) l'_k(x_k)\} (l_k(x))^2$$

$l_k(x) =$ Lagrange basis

$$\hat{h}_k(x) = (x - x_k)(l_k(x))^2$$

$$h_1(x) = \left\{1 - 2(x - \pi/2)\left(\frac{2}{\pi}\right)\right\}\left(\frac{2}{\pi}x\right)^2$$

$k = 1$

$$l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} = \frac{(x)}{(\pi/2)} = \frac{2}{\pi}x$$

$$l_1'(x) = \frac{2}{\pi}$$

$$\hat{h}_k(x) = (x - x_k)(l_k(x))^2$$

$$\hat{h}_0(x) = (x - x_0)(l_0(x))^2$$
$$= (x - 0)(l_0(x))^2$$
$$= (x)\left(1 - \frac{2}{\pi}x\right)^2$$

$$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} = \frac{(x - \pi/2)}{(-\pi/2)} = 1 - \frac{2}{\pi}x$$

$$P_3(x) = \left\{1 - 2\left(x - \frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right\}\left(\frac{2}{\pi}x\right)^2(1)$$
$$+ (x)\left(1 - \frac{2}{\pi}x\right)^2(1)$$

$$= \left\{1 - \frac{4}{\pi}\left(x - \frac{\pi}{2}\right)\right\}\left(\frac{2}{\pi}x\right)^2 + x\left(1 - \frac{2}{\pi}x\right)^2$$

Example :

	x	$f(x)$	$f'(x)$
x_0	-1	1	2
$n = 2$ x_1	0	0	2
x_2	1	1	0

Find Polynomial interpolation using Hermite basis ?

$$P_{2n+1} = P_{2(2)+1} = P_5$$

$$P_5(x) = h_0(x)f(x_0) + h_1(x)f(x_1)$$
$$+ h_2(x)f(x_2) + \hat{h}_0(x)f'(x_0)$$
$$+ \hat{h}_1(x)f'(x_1) + \hat{h}_2(x)f'(x_2)$$

$$P_5(x) = h_0(x)f(x_0) + h_2(x)f(x_2) +$$
$$\hat{h}_0(x)f(x_0) + \hat{h}_1(x)f'(x_1)$$

$$h_k(x) = \{1 - 2(x - x_k)l_k'(x_k)\}l_k^2(x)$$

$$\hat{h}_k(x) = (x - x_k)(l_k(x))^2$$

$$h_0(x) = \cancel{(x}\left[1 - 2(x+1)\left(-\frac{3}{2}\right)\right]\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)^2$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)}$$

$$l_0(x) = \frac{1}{2}x(x-1)$$

$$l_0(x) = \frac{1}{2}x^2 - \frac{1}{2}x.$$

$$l_0'(x) = \left(\frac{1}{2}\times 2\right)(x) - \frac{1}{2} = x - \frac{1}{2}.$$

$$l_0'(x_0) = -1 - \frac{1}{2} = -\frac{3}{2}$$

$$h_0(x) = [1 + 3(x+1)]\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)^2$$

(*) $$h_2(x) = \left[1 - 2(x-1)\left(\frac{3}{2}\right)\right]\left(\frac{1}{2}x^2 + \frac{1}{2}x\right)^2$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x)}{2(1)}$$

$$l_2(x) = \frac{1}{2}x(x+1) = \frac{1}{2}x^2 + \frac{1}{2}x$$

$$l_2'(x) = x + \frac{1}{2}.$$

$$l_2'(x_2) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$h_2(x) = (1 - 3(x-1))\left(\frac{1}{2}x^2 + \frac{1}{2}x\right)^2.$$

$$\hat{h}_k(x) = (x - x_k)(l_k(x))^2.$$

$$\hat{h}_0(x) = (x - x_0)(l_0(x))^2$$

$k = 0$ $$= (x+1)\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)^2$$

$\hat{h_1}(x) = (x - x_1)(l_1(x))^2$

$\hat{h_1}(x) = (x)(1-x^2)^2$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{(x+1)(x-1)}{(1)(-1)}$$

$l_1(x) = 1 - x^2$

$$P_5(x) = \left\{1 + 3(x+1)\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)^2\right\} + \left\{(1-3(x-1)\left(\frac{1}{2}x^2 + \frac{1}{2}x\right)^2\right\} + \left\{2(x+1)\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)^2\right\} + \left\{2x(1-x^2)^2\right\}$$