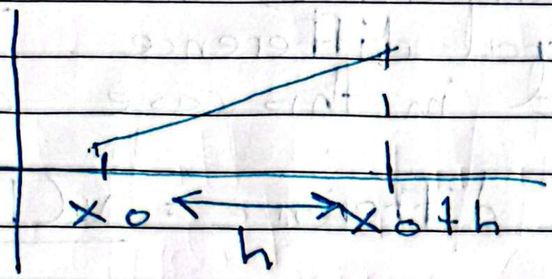
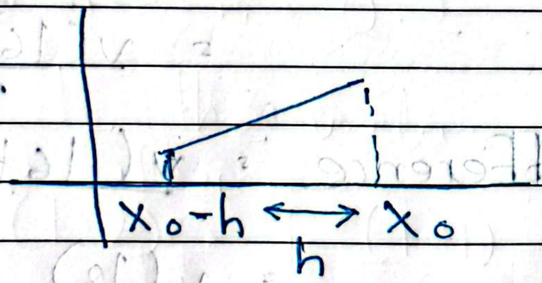
Numerical Differentiation :-

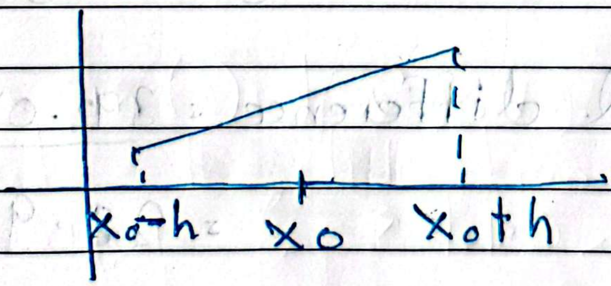
We will use forward differentiation when we know the current node and the future node.



We will use backward differentiation when we know the current node and the previous node.



We will use central differentiation when we know the current node, the previous node and the future node.



Why do we need numerical differentiation?

If we cannot completely differentiate the natural function then we do approximation by using forward difference, backward difference and central difference.

Formulas:

① Forward diff: $\underbrace{\dfrac{f(x_0+h) - f(x_0)}{h}}_{\text{Approximate value}} + \underbrace{\dfrac{f''(\xi)(-h)}{2}}_{\text{Truncation Error}}$

④ Difference between truncation error and rounding error?

Consider a certain part of the function

$f(x) = \underbrace{x + 2x^2 + 3x^3 + 4x^4} + \dots + 99x^{99}$

→ Truncation Error

$\underbrace{2.536824}_{2.54}$ → Rounding Error since computer can read upto 3 significant figures.

(ii) Central diff: $\underbrace{\dfrac{f(x_0+h) - f(x_0-h)}{2h}}_{\text{Approximated value}} + \underbrace{\dfrac{f'''(\xi)(h^2)}{3!}}_{\text{Truncation Error / Error Bound}}$

Not considering truncation error for backward difference.

Backward difference $= \frac{f(x_0) - f(x_0 - h)}{h}$ ↳ Approximate value

$+ \frac{f''(\xi)}{2}(-h)$

Forward diff : Error $\propto$ -h.

Backward diff : Truncation Error not needed

Central diff : Error $\propto h^2$

↳ Same as forward diff [Error $\propto$ -h] so truncation error is same.

x	1.0	1.2	1.4
$f(x)$	1.2717	1.3642	1.5496

step size = h = 0.2

(a) Calculate the value of $f'(1.2)$ using central difference.

$$f'(1.2) = \frac{f'(1.4) - f'(1.0)}{(2 \times 0.2)}$$

$$= \frac{1.5496 - 1.2717}{0.4}$$

$$= 0.69475$$

(b) $f(x) = x \sin(x) + x^2 \cos x$.

Compute the truncation error/error bound for using $\xi \rightarrow [1.0, 1.4]$.

central diff : $\left| f^3(\xi) \frac{h^2}{3!} \right|$

$f(x) = x\sin x + x^3\cos x$

$f'(x) = \sin x + x\cos x + 2x\cos x - x^2\sin x$

$f'(x) = \sin x + 3x\cos x - x^2\sin x$

$f''(x) = \cos x + 3\cos x - 3x\sin x - x^2\cos x - 2x\sin x$

$f''(x) = 4\cos x - 5x\sin x - x^2\cos x$

$f'''(x) = -4\sin x \cancel{-5\cos x} - 5x\cos x - 5\sin x$
$- 2x\cos x + x^2 \cancel{\cos}\sin x$

$f'''(x) = -9\sin x - 7x\cos x + x^2\sin x$

$$\left|\frac{f^3(\xi)}{3!} h^2\right|$$

$$\left|\frac{(0.2)^2}{6}\right| \left|9\sin(\xi) + 7(\xi)\cos(\xi) + \xi^2\sin\xi\right|$$

$$\left|\frac{(0.2)^2}{6}\right| \left|9\sin(1.4) + 7(1.4)\cos(1.0) + (1.4)^2 \sin(1.4)\right|$$

↳ upper bound

(Examples) Numerical Differentiation :

Example 1 : $f(x) = 2x^2 - e^x$.
Using this function approximate $f'(x)$ using central difference : $f'(2)$

h	$f'(x_0)$	Truncation Error
0.1	0.59865	$\lvert(0.61094-0.59865)\rvert = 0.01229$
0.01	0.6105	$\lvert(0.61094-0.6105)\rvert = 4.4\times10^{-4}$
0.001	0.61	$\lvert(0.61094-0.61)\rvert = 9.4\times10^{-4}$

$h = 0.1$

$$f'(x) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$f'(x) = \frac{f(2+0.1) - f(2-0.1)}{(2\times0.1)}$$

$$f'(x) = \frac{0.65383 - 0.5341}{0.2}$$

$$f'(x) = 0.59865.$$

$h = 0.01$

$f'(x) = f(x_0+h) - f(x_0-h)/2h$

$f'(x) = f(2+0.01) - f(2-0.01)/(2\times0.01)$

$f'(x) = 0.61688 - 0.60467/0.02$

$f'(x) = 0.6105.$

$h = 0.001$

$f'(x) = f(x_0+h) - f(x_0-h)/2h$

$f'(x) = f(0.001+2) - f(2-0.001)/(2\times0.001)$

$f'(x) = 0.61155 - 0.61033/2\times10^{-3}$

$f'(x) = 0.61$

Actual Derivative :

$f'(x) = 4x - e^x$

$f'(x) = 4(2) - e^2$

$f'(x) = 0.61094$ [Actual value]

Example 2 : $x_0 = 4.0, 4.1, 4.2, 4.3, 4.4$

$f(x_0) = 16, 18, 20, 21, 22$

(a) Using forward difference calculate the value of $f'(4.2)$.

(b) Using backward difference calculate the value of $f'(4.2)$

Notice that step size h is not mentioned. So h is the difference between the x nodes which is 0.1 in this case.

(a) $f'(4.2) = \dfrac{f(4.2+0.1) - f(4.2)}{\cancel{2\times}(0.1)}$

$f'(4.2) = \dfrac{f(4.3) - f(4.2)}{0.1}$

$f'(4.2) = \dfrac{21 - 20}{0.1}$

$f'(4.2) = \cancel{5}\ 10.$

(b) $f'(4.2) = \dfrac{f(4.2) - f(4.2-0.1)}{0.1}$

$f'(4.2) = \dfrac{20 - f(4.1)}{0.1}$

$f'(4.2) = \dfrac{20 - 18}{0.1}$

$f'(4.2) = 20$

Example 3 : $v(t) = 2000 \ln \left[\dfrac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t$

Calculate the value of $v'(16)$ using forward difference, backward difference and central difference. The step size is mentioned in this case $\Delta t = 2s$

ⓐ Forward difference $= \dfrac{v(16+2) - v(16)}{\Delta t}$

$= \dfrac{v(18) - v(16)}{2}$

ⓑ Backward difference $= \dfrac{v(16) - v(16-2)}{2}$

$= \dfrac{v(16) - v(14)}{2}$

ⓒ Central difference $= \dfrac{v(16+2) - v(16-2)}{(2 \times 2)}$

$= \dfrac{v(18) - v(14)}{4}$

ⓐ Forward difference $= \dfrac{453.0214897 - 392.0736914}{2}$

$= 30.47389915$.

ⓑ Backward difference $= \dfrac{392.0736914 - 334.244667}{2}$

$= 28.9145122$

ⓒ Central difference $= \dfrac{453.0214837 - 334.244667}{4}$

$= 29.6937$.

Richardson Extrapolation

$D_h \quad f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h}$

Representing $f(x_1+h)$ using Taylor series

$$f(x_1+h) = f(x_1) + f'(x_1)h + \frac{f^2(x_1)h^2}{2!} + \frac{f^3(x_1)h^3}{3!} + \frac{f^4(x_1)h^4}{4!} + \frac{f^5(x_1)h^5}{5!} + \frac{f^6(x_1)h^6}{6!} + \dots$$

Taylor Series: $f(x_1-h) = f(x_1) - f'(x_1)h + \frac{f^2(x_1)h^2}{2!} - \frac{f^3(x_1)h^3}{3!} + \frac{f^4(x_1)h^4}{4!} - \frac{f^5(x_1)h^4}{5!} + O(h^6) + \dots \quad O(h^6)$

$$D_h = \frac{1}{2h}\left[2f'(x_1)h + \frac{2f^3(x_1)h^3}{3!} + \frac{2f^5(x_1)h^5}{5!} + O(h^7)\right]$$

$$D_h = \underbrace{f'(x_1)}_{\text{Actual derivative}} + \underbrace{\frac{f^3(x_1)h^2}{3!} + \frac{f^5(x_1)h^4}{5!} + O(h^6)}_{\text{Error}}$$

Dominating factor in the error part so we need to remove this part.

$$D_{h/2} = f'(x_1) + \frac{f^3(x_1)}{3!}(h/2)^2 + \frac{f^5(x_1)}{5!}(h/2)^4 + O\left(\frac{h}{2}\right)^6$$

$$2^2 \times D_{h/2} = 2^2 f'(x_1) + \frac{f^3(x_1)(h^2)}{3!} + \frac{f^5(x_1)h^4}{5!}\cdot\frac{1}{2^2} + O\left(\frac{h^6}{2^4}\right)$$

$\cancel{D_n - 2^2 D_{n/2}}$

$2^2 D_{n/2} - D_n$

$= (2^2-1) f'(x_1) + \left(\frac{1}{2^2} - 1\right) \frac{f^5(x_1)}{5!} h^4 + O(h^6)$

$$\frac{2^2 D_{n/2} - D_n}{2^2-1} = \frac{(2^2-1)}{(2^2-1)} f'(x_1) + \frac{\left(\frac{1}{2^2} - 1\right)}{(2^2-1)} \frac{f^5(x_1)}{5!} h^4 + \frac{O(h^6)}{2^2-1}$$

$$D_h^1 = \frac{2^2 D_{h/2} - D_h}{2^2-1}.$$

Calculation :

$h = 0.1 \quad f(x)$

$D = f'(x) = \frac{f(x+h) - f(x-h)}{2h}$ } Approximate value

$$\boxed{D_h^{(1)} = \frac{2^n D_{h/2} - D_h}{2^n - 1}}$$

$f(x) = e^x \sin(x)$

$h = 0.5$

$h = 0.25$

Using central difference, find $f'(1)$.

For $h = 0.5$

$$f'(1) = \frac{f(1+0.5) - f(1-0.5)}{(2 \times 0.5)}$$

$$f'(1) = \frac{e^{1.5}\sin(1.5) - e^{0.5}\sin(0.5)}{(2\times 0.5)}$$

$f'(1) = 3.68.$

For $h = 0.25$,

$f'(1) = 3.7385$

We will use Richardson extrapolation to find the more accurate value.

$$D_h^{(1)} = \frac{2^2 D(h/2) - D(h)}{2^2 - 1}$$

$$D_h^{(1)} = \frac{4D(h/2) - D(h)}{3}$$

$$= \frac{4(3.738) - 3.68}{3}$$

$$= 3.757.$$

Example: $h = 0.1$ $f'(1) = 0.7$

$h = 0.2$ $f'(1) = 0.5$

Using Richardson Extrapolation find $f'(1)$.

$$D'_h = \frac{2^2 D(h/2) - D(h)}{2^2 - 1}$$

$$D'_h = \frac{4(0.7) - 0.5}{2^2 - 1}$$

$$D'_h = 0.77.$$

Example: $f(x) = e^{2x} + 3x$

$h = 1.2$

$h = 0.6$

Find $f'(2)$ using Richardson Extrapolation.

$$f'(2) = \frac{f(x+h) - f(x-h)}{2h} = 251.705$$

$$f'(2) = \frac{f(x+h) - f(x-h)}{2h} = 140.356$$

$$D_h' = \frac{2^2 D(h/2) - D(h)}{2^2 - 1}$$

$$= \cancel{(4 \times 0.6)}$$

$$= \frac{(4 \times 140.356) - 251.705}{3}$$

$$= 103.23$$

Example :

$f'(1)$ $h = 0.4$

$h = 0.2$

$h = 0.1$

x	$f(x)$
0.6	0.707178
0.8	0.8559892
0.9	0.925863
1.0	0.984007
1.1	1.033743
1.2	1.074575
1.4	1.127986

Using $h = 0.4$, find D_h

$$D_h = f'(1) = \frac{f(1+0.4) - f(1-0.4)}{(2 \times 0.4)} = 0.52601$$

$h = 0.2$,

$D_h = f'(1) = 0.5464$

$h = 0.1$

$D_h = f'(1) = 0.5394$

$$D'_h = \frac{4D(h/2) - D(h)}{3}$$

$$= \frac{4(0.5464) - 0.52601}{3}$$

$= 0.553$

$$D'_h = \frac{4(0.5394) - 0.5464}{3}$$

$= 0.537$

h	D_h		
0.4	0.52601	$D'_h = 0.553$	
0.2	0.5464		$D^2_h = 0.535933$
0.1	0.5394	$D'_h = 0.537$	

$$D^2_h = \frac{2^4 D'(h/2) - D(h)}{2^4 - 1}$$

$$D^2_h = \frac{16(0.537) - (0.553)}{2^4 - 1}$$

$= 0.535933$