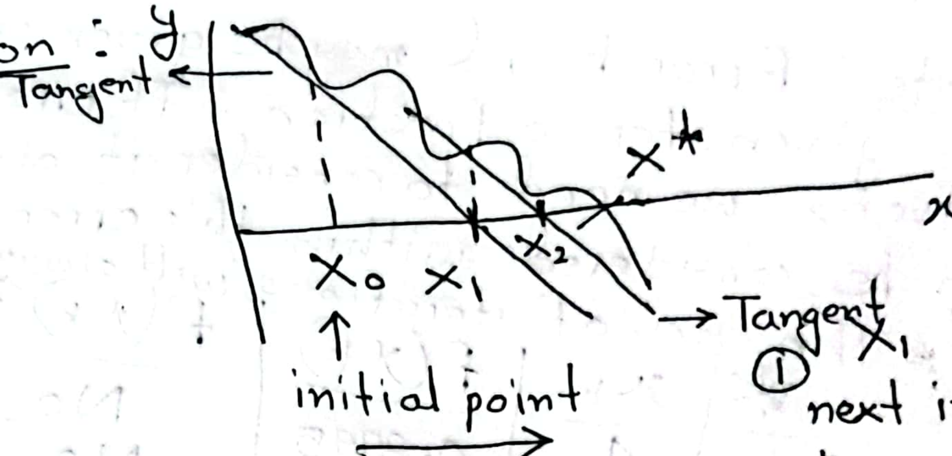
Newton's Method/ Newton Raphson Method

Advantage of this method : Super linear convergence
In this case, $\lambda = 0$.

Function :



(I) x_1 is my next iteration point. Draw a tangent
(II) x_2 is my next iteration point.

Newton's Method :

An initial point (x_0) will be given and we will draw a tangent for the corresponding point and it will cut the x-axis which will be our next iterative point (x_1, x_2) and continuing this process will help us reach our actual root.

If we keep continuing this process, we will reach our actual root.

Formula :

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Newton's Raphson method.

Example 1 : $f(x) = x^2 - 2xe^{-x} + e^{-2x}$

Initial point is given, $x_0 = 1$.

Error bound $= 1 \times 10^{-5}$.

Note: Error bound may be given or may not be given for a function. If error bound is given, we need to consider it and our root can be considered within this error bound.

** The no. of iterations will always be mentioned.

Table based approach:

k	x_k	$f(x_k)$	$f(x_k)$ < Error bound
0	1	0.3995	No
1	0.7689	0.093	No
2	0.6648	0.0266	No
⋮			
8	0.56	0.59×10^{-5}	Yes

Note: If $f(x_k) = 0$ it means x_k is the actual root. However, we will get the value with some error bound in numerical methods.

Calculation: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f'(x) = 2x - [2e^{-x} + 2xe^{-x}(-1)] + e^{-2x}(-2)$

$= 1 - \frac{0.3995}{1.7293} = \boxed{0.7689}$

$f'(x) = 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.7689 - \frac{f(0.7689)}{f'(0.7689)}$$

$= 0.6648$.

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$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 0.7689 - \dfrac{f(0.7689)}{f'(0.7689)}$

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Since for $k = 8$, we found that $f(x_k) <$ Error Bound, we can declare $x_k = 0.56$ as a solution for the function. [$x_k = 0.56$ is the root].

Example 2 : $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$

Initial point, $x_0 = 0.05$ [will always be given]

Show first 3 iterations with relative error at each iterations.

$$f'(x) = 3x^2 - 0.33x$$

Iteration 01 : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = 0.05 - \frac{f(0.05)}{f'(0.05)}$$

$$x_1 = 0.0624$$

for the relative error part, we need to compare between the current iteration value and the previous iteration value.

$$R.E. = |x_{new} - x_{old}|$$

$$= |0.0624 - 0.05|$$

$$= 0.0124.$$

Iteration - 02 : $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$x_2 = 0.0624 - \frac{f(0.0624)}{f'(0.0624)}$$

$$x_2 = 0.0623.$$

Relative Error $= |0.0624 - 0.0623|$
$= 0.0001.$

Iteration - 03 : $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$x_3 = 0.0623 - \frac{f(0.0623)}{f'(0.0623)}$$

$$x_3 = 0.0623.$$

Relative Error $= |0.0623 - 0.0623|$
$= 0.0$

Proof : super linear convergence

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

As we have learnt in fixed point
$$g(x) = x - \frac{f(x)}{f'(x)}$$
$$\lambda = |g'(x)| \rightarrow \text{check for every root values } [x*]$$

$$g(x) = x - \frac{f(x)}{f'(x)} \Big\} \text{ Apply u/v rule for this part.}$$

$$\lambda = |g'(x)| = 1 - \frac{f'(x)\,f'(x) - f(x)\,f''(x)}{[f'(x)]^2}$$

$$\lambda = |g'(x)| = \frac{f(x)\,f''(x)}{[f'(x)]^2} \Big] \text{ simplified version.}$$

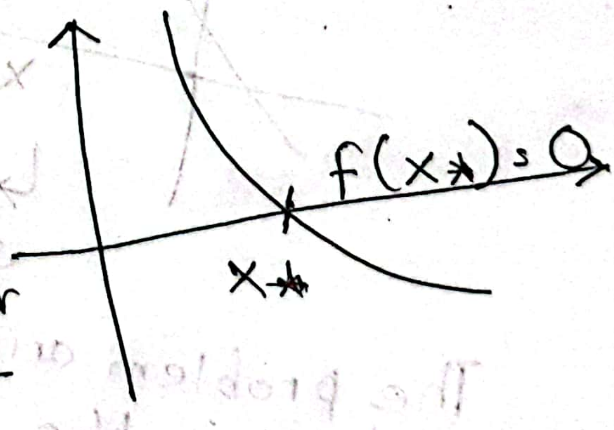
$x*$ is the root so we will give this as input

$$\lambda = |g'(x*)| = \frac{f(x*)\,f''(x*)}{[f'(x*)]^2}.$$

** for super linear convergence $\lambda = 0$ but how ?

$$\lambda = |g'(x*)| = \frac{0 * f''(x*)}{[f'(x*)]^2}$$

$\boxed{\lambda = 0}$ super linear convergence



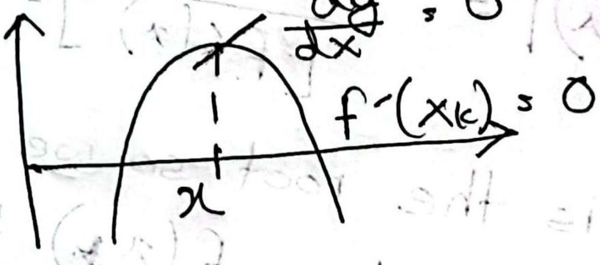
Advantage: Since Newton's method is super linear convergence, so we will find the root faster.

Disadvantage:

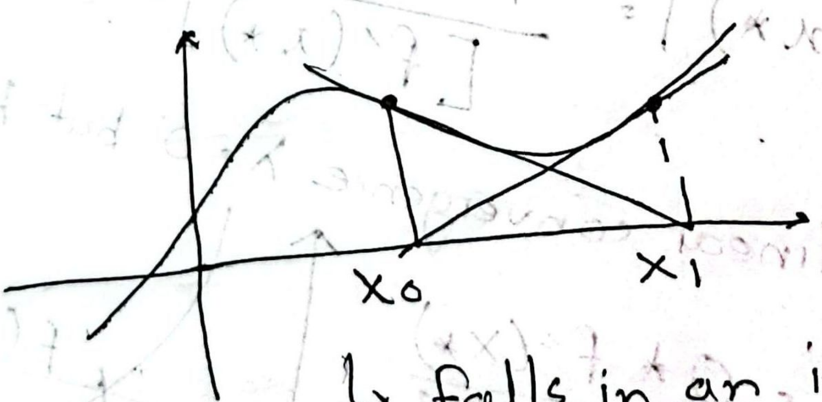
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

(I) If the $f'(x_k)$ becomes zero, then it is not possible to compute x_k since the function becomes undefined.

$f'(x_k) = 0$ [When turning point $\frac{dy}{dx} = 0$ is reached]



(II)



↳ falls in an infinite loop. So root can never be found.

The problem arises if we choose ~~an turning~~ initial point near the turning point.

Aitken Acceleration

It is the process of making a method faster. For example, if naturally 10 iterations are needed to find the solution of a function. Aitken acceleration would require 3/4 iterations to find the solution.

Aitken accelerated value of x $\rightarrow$ $$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

There are three different x values, x_k, x_{k+1}, x_{k+2}.

$$x_0 \xrightarrow{\text{Newton's Method}} x_1 \longrightarrow x_2 \Rightarrow \hat{x}_2 \xrightarrow{\text{Newton's method}} x_3 \longrightarrow x_4 \Rightarrow \hat{x}_4$$

Aitken acceleration

Once we find $\hat{x}_2$, we will disregard the previous x_2 and $\hat{x}_2$ will be our new x_2.

Example 1 : $f(x) = x^2 - 2xe^{-x} + e^{-2x}$

Initial point, $x_0 = 1$

Error Bound, $\epsilon = 1 \times 10^{-5}$

(i) Show 6 iterations by applying Newton Raphson.

(ii) Apply Aitken acceleration at appropriate position.

$$f'(x) = 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}$$

	k	x_k	$f(x_k)$	$f(x_k) < \epsilon$
Newton's Method	0	1	0.3995	No
	1	0.7685	0.09329	No
	2	0.66495	0.00326	No
Aitken acceleration	$\hat{2}$	0.5811	4.7600×10^{-4}	No
	3			
	4			
	$\hat{4}$			
	5			
	6			

$$\hat{X}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

$$\hat{x}_2 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

$$\hat{x}_2 = 1 - \frac{(0.7685 - 1)^2}{(0.66495 - 2 \times 0.7685 + 1)}$$

$$\hat{x}_2 = 0.5811$$

Advantage: If we used normal approach, it would require 818 iterations to get the error within 10^{-6} [reach root].

For Aitken acceleration, it requires just 8 iterations to get the same answer within 10^{-6} error bound. [reach root].

Secant Method :

Iterated value of x using secant method → $$X_{k+1} = X_k - \frac{f(X_k)(X_k - X_{k-1})}{f(X_k) - f(X_{k-1})}$$

Example : $x_0 = 0.03$ $x_{-1} = 0.02$ → given

Show 3 iterations for $f(x) = x^3 - 0.165x^2 + 3.993\times10^{-}$

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$x_1 = 0.03 - \frac{f(0.03)(0.03 - 0.02)}{f(0.03) - f(0.02)}$$

$$x_1 = 0.03 - \frac{3.8715\times10^{-5}}{-6.35\times10^{-5}}$$

$$x_1 = 0.639685.$$