Chapter - 5

Linear system of equations

For every equation, there are three unknown variables

$$\begin{cases} x + 2y + 3z = 0 \\ x - 2y + 2z = 4 \\ 2x_1 + 12x_2 - 2x_3 = 4 \end{cases}$$

Conditions to solve these equations:

(I) determinent should not be zero.

(II) should be a square matrix

The solution of the linear system is by finding inverse matrix:

$$\boxed{x = A^{-1} b}$$

Basic properties of matrix A:

(I) A is a square matrix of order $n \times n$.

(II) A^T is the transpose of A, hence $(a^T)_{ij} = a_{ji}$

(III) A is symmetric if $A = A^T$

(IV) A is non-singular iff $\exists$ a solution $x \in R^n$ for every $b \in R^n$.

(V) A is non-singular iff $\det(A) \neq 0$

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(vi) A is non-singular if and only if there exists a unique inverse A^{-1} such that $AA^{-1} = A^{-1}A = I$

[5] Augmented Matrix :

The first step is to write the coefficients.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

Square matrix [3 × 3]

To solve an equation system, it must always be a square matrix [n × n]

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \rightarrow \text{Augmented Matrix}$$

Gaussian Elimination Method:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

(F) The first step is to convert this to Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right]$$

(4) We apply Gaussian Elimination method to convert this matrix into upper triangular matrix.

Note that there will be no change in the first row.

We need to apply row operation from ~~second~~ first row so that all elements below u_{11} become zero.

The upper triangular matrix will be as such

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

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Start applying row operation

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \xrightarrow[R_3 = R_3 - \left(\frac{2}{1}\right)R_1]{R_2 = R_2 - \left(\frac{1}{1}\right)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right]$$

Multiplier

Multiplier

We need to apply 1st row operation in each case to make all elements below U_{11} 0.

We need to continue the process in order to make elton elements below U_{22} 0. Now we will make comparison of row 3 with row 2.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right] \xrightarrow{R_3 = R_3 - \left(\frac{8}{-4}\right)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right]$$

upper triangular matrix

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \uparrow & \uparrow & \uparrow & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right] \rightarrow \text{Augmented Matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$$-2x_3 = 12$$
$$-4x_2 + x_3 = 4$$
$$x_1 + 2x_2 + x_3 = 0$$

$$x_3 = -6$$
$$x_2 = -2.5$$
$$x_1 = 11$$

Example : The upward velocity of a rocket is given at three different times in Table 1.

Time (s)	Velocity (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as :

$$v(t) = b_1 t^2 + b_2 t + b_3 \quad 5 \leq t \leq 12$$

8 · Find the values of b_1, b_2 and b_t using the Gaussian elimination method. Find the velocity at $t = 6, 7.5, 9, 11$ s.

$V(5) = 25b_1 + 5b_2 + b_3 = 106.8$ —(i)
$V(8) = 64b_1 + 8b_2 + b_3 = 177.2$ —(ii)
$V(12) = 144b_1 + 12b_2 + b_3 = 279.2$ —(iii)

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right] \rightarrow \text{Augmented matrix}$$

$R_2 = R_2 - \left(\frac{64}{25}\right)R_1 \qquad R_3 = R_3 - \left(\frac{144}{25}\right)R_1$

↳ 1st row operation

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.208 \\ 0 & -16.8 & -4.76 & -335.968 \end{array}\right]$$

$R_3 = R_3 - \left(\frac{-16.8}{-4.8} \times R_2\right)$

↳ 2nd row operation.

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.208 \\ 0 & 0 & 0.7 & 0.76 \end{array}\right]$$

$0.7b_3 = 0.76 \qquad b_3 = 1.0857$

$-4.8b_2 + (-1.56)b_3 = -96.208 \qquad b_2 = 19.69$

$25b_1 + 5b_2 + b_3 = 106.8 \qquad b_1 = 0.2906.$

$v(t) = 0.2906t^2 + 19.69t_2 + 1.0857$

At $t = 6$,

$v(6) = 0.2906(6)^2 + 19.69(6) + 1.0857$

$v(6) = 129.6873$ m/s.

LU Decomposition

[lower triangular matrix and upper triangular matrix].

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

Problem with Gaussian Elimination:

If the value of b changes then we need to do the calculation (row operations) from the very beginning.

$$\text{coefficient matrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} = A^{(1)}$$

Concept of multipliers is very important.

$$R_2 - \left(\frac{1}{1}\right) R_1 = R_2 \quad \text{multiplier}$$

$$R_3 = R_3 - \left(\frac{2}{1}\right) R_1 \quad \text{multiplier}$$

We find $F^{(1)}$ using multipliers

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

negative values of multiplier

Diagonally all values will be 0

↳ In step 1, since we are not getting any multiplier so for the time being we are considering 0 instead of 12.

$$A^{(2)} = F^{(1)} \times A^{(1)}$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & \boxed{8} & -4 \end{bmatrix} \Longrightarrow R_3 = R_3 - \left(\frac{8}{-4}\right) R_2$$

$= -2$ → multiplier

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$[U] = A^{(3)} = F^{(2)} A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix}$$

↑ upper triangular matrix

$$[U] = A^{(3)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$ → upper triangular matrix

Lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Actual values of multiplier

↳ Diagonally all values will be 1.

$$[L] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

↳ given y values

↑ Temporary variables

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}.$$

<u>Calculation</u>: $(1 * a_1) + (0 * a_2) + (0 * a_3) = 0$

$$a_1 = 0$$

$$(1 * a_1) + (1 * a_2) + (0 * a_3) = 4$$

$$a_1 + a_2 = 4$$

$$a_2 = 4$$

$$(2 * a_1) + (-2 * a_2) + (1 * a_3) = 4$$

$$a_3 = 12.$$

$$[U]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

actual values of equation

Temporary variables

$$[U]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$$-2x_3 = 12$$

$$\boxed{x_3 = -6}$$

$$-4x_2 + x_3 = 4$$

$$\boxed{x_2 = -2.5}$$

$$x_1 + 2x_2 + x_3 = 12$$

$$\boxed{x_1 = 11}$$

Advantage: Only at the last step - we required b values.

We need to compute Lower triangular matrix and upper triangular matrix once only.

Summarization of the steps :

$$Ax = b$$

$\downarrow$ decompose

LU

$$L\boxed{Ux} = b$$

$\downarrow$

y

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

multipliers

temporary variables

$$Ux = y$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Find x_1, x_2, x_3

$2x + y + z = 10$
$3x + 2y + 3z = 18$
$x + 4y + 9z = 16$

Solve this linear system using LU decomposition.

$$A^{(1)} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

multiplier for $R_2 = 3/2$
multiplier for $R_3 = 1/2$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$$

negative values of multipliers

$$A^{(2)} = F^{(1)} A^{(1)}$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 3.5 & 8.5 \end{bmatrix}$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}$$

multipliers = 3.5/0.5
↓
consider negative value

$$[U] = A^{(3)} = F^2 A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 3.5 & 8.5 \end{bmatrix}$$

$$[U] = A^{(3)} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 0 & -2 \end{bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 0.5 & 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 0.5 & 7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

$y_1 = 10$

$y_2 = 3$

$y_3 = -10$

$$[U]\begin{bmatrix} \cancel{x_1} \\ \cancel{x_2} \\ \cancel{x_3} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ -10 \end{bmatrix}$$

$x_1 = 5$

$x_2 = 9$

$x_3 = 7$

Frobenius Matrix :

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} \quad F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}$$

$$L = (F^{(1)})^{-1}(F^{(2)})^{-1}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{matrix} \rightarrow R_2' = R_2 - \left(\frac{1}{1}\right)R_1 \\ \rightarrow R_3' = R_3 - \left(\frac{2}{1}\right)R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \rightarrow R_3' = R_3 - \left(\frac{8}{-4}\right)R_2$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = \left(F^{(1)}\right)^{-1} \left(F^{(2)}\right)^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$