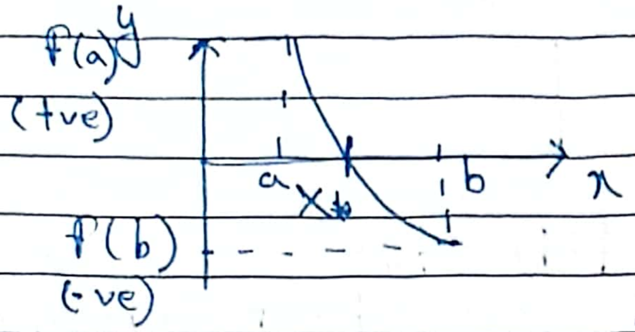
Bisection Method [Root finding Algorithm]

The solution of x is known as roots.



$x_* \rightarrow$ Root.

$f(x) = x^2 - 2x$
$f(x) = 0$
$x(x-2) = 0$
$x = 0, x = 2$.

We cannot solve this always easily because sometimes the functions are infinitely large

So, we use iterative method which tries to approximate a root. The solution is approximate not exact. This is Root Finding Algorithm.

In these type of problems, an interval will be given.

$x \in [a, b]$

The root somewhere lies between this interval.

The first thing to do is check the validity of the interval

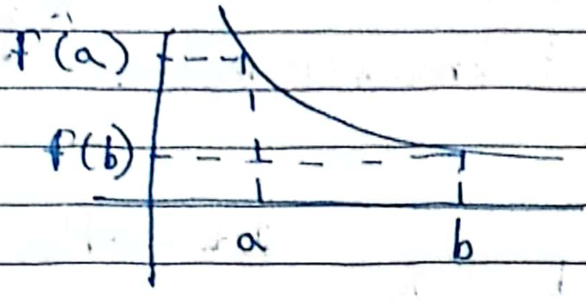
$x \in [a, b] \rightarrow$ Valid / Invalid

$f(a) * f(b) = -ve$

$+ve * -ve = -ve$

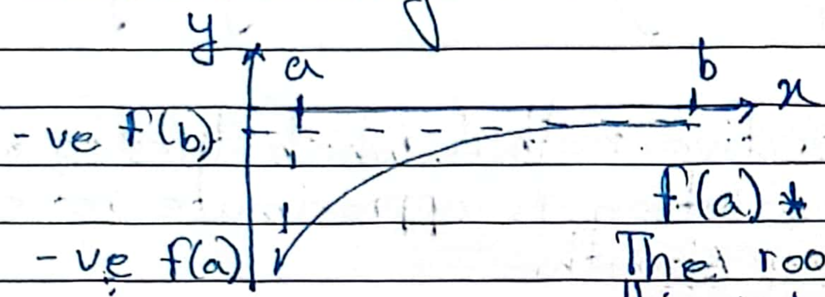
This is a valid range because $f(a) * f(b)$ is negative which indicates that the sign of y changed. So at some point the value of y / f(x) became zero where we found the root.

For example :



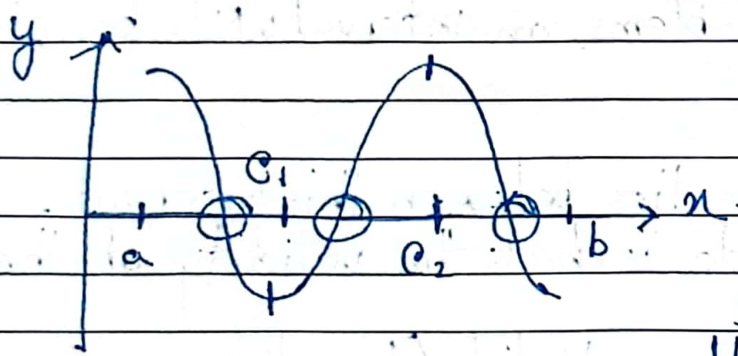
$f(a) * f(b) = +ve$

In this graph the value of y never became zero so the graph never touched the y axis from where we could find the x root ∴ This is not a valid range



$f(a) * f(b) : +ve$

The root doesn't lie within this interval.

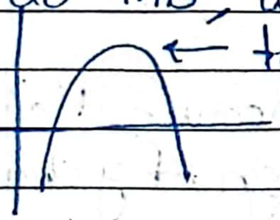


[a, b]

Multiple roots.

So we need to divide the given interval into sub-intervals

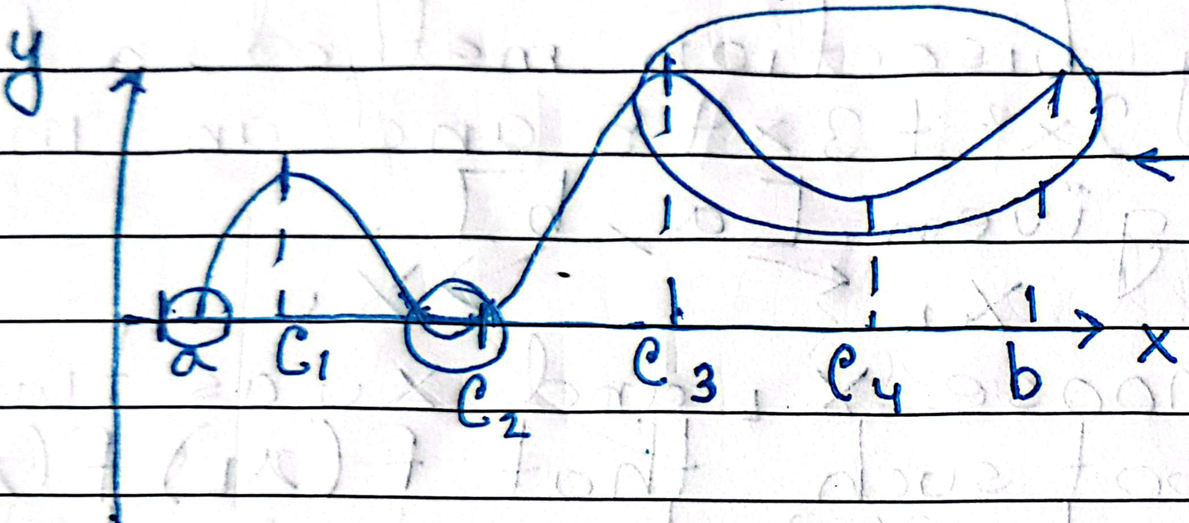
In order to do this, we need to label the turning points.



← turning point

$[a, b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, b]$

↳ sub-interval



Avoid this interval because we can see no root here.

$[a, b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, c_3]$

Bisection Method [Root finding Algorithm]

When bisection method is going to be applied an equatio will be given, $f(x) = 2x^2 + 3x + 5$ and an interval will be given, $[a, b]$.

Complete Algorithm for the Bisection Method

Step 1 : Choose x_l and x_u as two gusses for the root such that $f(x_l) f(x_u) < 0$ in other words, $f(x)$ changes sign between x_l and x_u.

For the given range, $[a, b]$

x_l (x lower) $\quad$ x_u (x upper)

$f(x_l) f(x_u) < 0$ cheaks the validity of the given interval $[a, b]$.

Step 2 : Estimate the root x_m of the equation as the mid-point between x_l and x_u, $x_m = \dfrac{x_l + x_u}{2}$

$$= \frac{a+b}{2}$$

This is similar to binary search. The interval is divided and checks whether the interval lies in the left side or right side.

Step 3 : Now chekk the following

Case 1: If $f(x_l) \times f(x_m) < 0$ the new interval will be

$$x_l = x_l \qquad [x_l \ldots\ldots x_m \ldots\ldots x_u]$$

$$x_u = x_m$$

$$
\begin{array}{ccc}
 & [x_l, x_m] & \\
\swarrow & & \searrow \\
x_l & & x_u
\end{array}
$$

Case 2 : If $f(x_l) \times f(x_m) > 0$ the new interval will be

$$x_l = x_m$$

$$x_u = x_u$$

$$
\begin{array}{ccc}
 & [x_m, x_u] & \\
\swarrow & & \searrow \\
x_l & & x_u
\end{array}
$$

Case 3 : If $f(x_l) \times f(x_m) = 0$

so x_m is the root (This case does not generally occur but case 1 and case 2 is more ~~impo~~ frequently occuring)

Example : $f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$

$[0, 0.11]$ ← given interval

x_l x_u

** Firstly - we need to check the validity of the interval, but in question if it is not mentioned then no need to check it.

$$f(x_l) = 0^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

$$f(x_u) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4}$$
$$= -2.662\times10^{-4}$$

$f(x_l)\,f(x_u) < 0$ → The given interval is valid.

1st iteration:

$$x_m = \frac{0+0.11}{2} = 0.055$$

$$f(x_m) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4}$$
$$= 6.655\times10^{-5}$$

$$f(x_l)\,f(x_m) = (3.993\times10^{-4}) * (6.655\times10^{-5})$$

$f(x_l)\,f(x_m) = +ve$

$f(x_l)\,f(x_m) > 0$ [So Case 2 occurs here]

$x_l = x_m$ $\quad\quad$ $x_l = 0.055$

$x_u = x_u$ $\quad\quad$ $x_u = 0.11$

New interval is $[0.055, 0.11]$

The root never lies in the interval $[0, 0.055]$ so we have avoided this interval.

2nd iteration:

$$x_m = \frac{0.055 + 0.11}{2} = 0.0825$$

$$f(x_l)\, f(x_m) = f(0.055)\, f(0.0825)$$

$$f(x_l)\, f(x_m) = (6.655 \times 10^{-5})(-1.622 \times 10^{-4})$$

$$f(x_l)\, f(x_m) = -ve$$

Case 1 occurs here

$$x_l = x_l = 0.055$$

$$x_u = x_m = 0.0825$$

New interval $[0.055, 0.0825]$

The iteration will continue till root is found.

** Question: How many iterations it will take to find the root of the function / Find Iteration - n?

[Note: The root we find will be with some error bound]

Interval will be given $x \in [1.5, 3]$ and Machine Epsilon / scale invariant error will be given $= 1.1 \times 10^{-16}$

$$n \geq \frac{\log|b-a| - \log(\epsilon)}{\log 2} - 1$$

Note if accuracy is given then the formula will be:

$$n \geq \frac{\log|b-a| - \log(\epsilon)}{\log 2}$$

$$n \geq \frac{\log|3-1.5| - \log(1.1\times10^{-16})}{\log 2} - 1$$

$n \geq 52.5983$

$n \geq 53$ [Iteration can't be fractional]

If Accuracy was given 1.1×10^{-16},

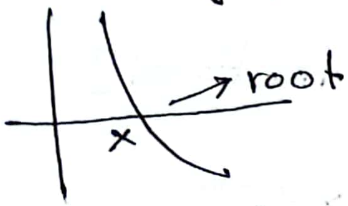
$$n \geq \frac{\log|3-1.5| - \log(1.1\times10^{-16})}{\log 2}$$

Drawback of Bisection Method :

It is comparatively slower .

Fixed Point Iteration [Root finding Algorithm]

So far we have seen the point at which $f(x) = 0$ which is the graph cuts the x-axis is the root.



$f(x) = 0$

In ~~root~~ fixed point iteration we will convert this $f(x)$ into $g(x)$ using algebric manipulation.

$g(x) = x$ [Fixed point]
↳ root of the actual $f(x)$

Example $f(x) = x^2 - 2x - 3$ $f(x) = 0$

1st way: $x^2 = 2x + 3$ [Make x the subject]

$x = \sqrt{2x+3}$

$g_1(x) = \sqrt{2x+3}$

2nd way: $x^2 - 2x - 3 = 0$

$x^2 - x - x - 3 = 0$

$-x = x + 3 - x^2$

$x = x^2 - x - 3$

$g_2(x) = x^2 - x - 3$

3rd way:

$$x^2 - 2x - 3 = 0$$

$$2x^2 - x^2 - 2x - 3 = 0$$

$$2x^2 - 2x = x^2 + 3$$

$$x(2x-2) = x^2 + 3$$

$$x = \frac{x^2+3}{2x-2}$$

$$g_3(x) = \frac{x^2+3}{2x-2}$$

** Question might ask you to find the actual value

$$x^2 - x - 3 = 0$$

$$x^2 - 3x + x - 3 = 0$$

$$x(x-3) + 1(x-3) = 0$$

$$(x-3)(x+1) = 0$$

$$x = -1,\ x = 3$$

Convergent

$g_1(x) = \sqrt{2x+3}$ $\quad x_0 = 0$ [keep in mind significant figures and decimal places]

$$g_1(0) = 1.73$$

$$g_1(1.73) = 2.54$$

$$g_1(2.54) = 2.84$$

$$g_1(2.84) = 2.95$$

$$g_1(2.95) = 2.98$$

$$g_1(2.98) = 3.00$$

$$\boxed{g_1(3.00) = 3.00}$$ ← root of $f(x)$

$g_2(x) = x^2 - x - 3 \rightarrow$ Divergent

$x_0 = 0$

$g_2(0) = -3$

$g_2(-3) = 9.0$

$g_2(9.0) = 69.0$

$g_2(69.0) = 4.69 \times 10^3$

$g_3(x) = \dfrac{x^2 + 3}{2x - 2}$ $x_0 = 0 \rightarrow$ Convergent

$g_3(0) = -1.50$

$g_3(-1.50) = -1.05$

$g_3(-1.05) = -1.00$ $\rightarrow$ fixed point

$g_3(-1.00) = -1.00 \rightarrow$ Root of the $f(x)$

If we take $x_0 = 42$,

$g(42) = 21.6$

$g(21.6) = 11.4$

$g(11.4) = 6.39$

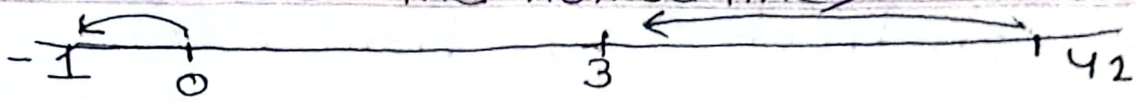
$g(6.39) = 4.07$

$g(4.07) = 3.19$

$g(3.19) = 3.01$

$g(3.01) = 3.00$

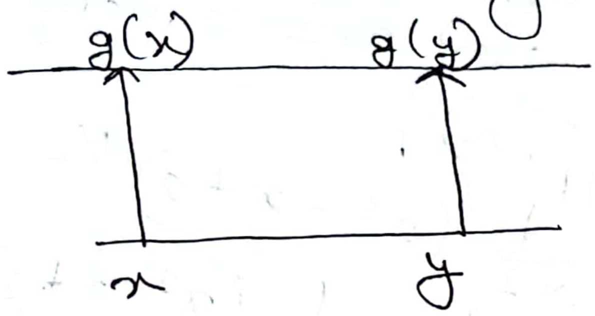
If we consider the number line



$x_0 = 0$ [It is closer to -1 so converges to the nearest root]

$x_0 = 42$ [It is closer to 3 so converges to the nearest root]

Contraction Mapping Theorem :



$$\lambda = \left| \frac{g(y) - g(x)}{y - x} \right|$$

$\lambda = |g'(x)|$ at the roots

[It indicates whether the function is convergent or divergent.

$\lambda = 0$ [Super linear Convergent]

$0 < \lambda < 1$ [Linear Convergent]

$\lambda \geq 1$ [Divergent]

$$f(x) = x^3 - 2x^2 - x + 2$$

(a) Find the roots of the solution.

$$x^2(x-2) - 1\cdot(x-2) = 0$$

$$(x^2 - 1)(x-2) = 0$$

$$x = 1, -1, 2$$

(b) Construct 3 $g(x)$ from $f(x)$

$$g_1(x) = \sqrt{1/2\,(x^3 - x + 2)}$$
$$g_2(x) = x^3 - 2x^2 + 2$$
$$g_3(x) = \frac{-2}{x^2 - 2x - 1}$$

(c) Determine which $g(x)$ are convergent and which $g(x)$ are divergent.

$$\lambda = |g_1'(x)|$$
$$g_1(x) = \frac{1}{\sqrt{2}}(x^3 - x + 2)^{1/2}$$
$$g_1'(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{2}(x^3 - x + 2)^{-1/2}(3x^2 - 1)$$
$$g_1'(x) = \frac{(3x^2 - 1)}{2\sqrt{2}\,(x^3 - x + 2)^{1/2}}$$
$$\lambda = |g_1'(x)| = \left| \frac{3x^2 - 1}{2\sqrt{2}\,(x^3 - x + 2)^{1/2}} \right|$$

$x = 1 \longrightarrow 0.5$ (linear convergence)

$x = -1 \longrightarrow 0.6$ (linear convergence)

$x = 2 \longrightarrow 3.75$ (divergent)

$\lambda = |g_2'(x)| = |3x^2 - 4x| =$ [check λ values for x]

$x = -1 \rightarrow 7 \geq 1$ [Divergent]

$x = 1 \rightarrow 1 \geq 1$ [Divergent]

$x = 2 \rightarrow 4 \geq 1$ [Divergent]

$\lambda = |g_3'(x)| = -\dfrac{2}{x^2 - 2x - 1}$

$= -2 \times (x^2 - 2x - 1)^{-1}$

$= -2 \times -1 (x^2 - 2x - 1)^{-2} (2x - 2)$

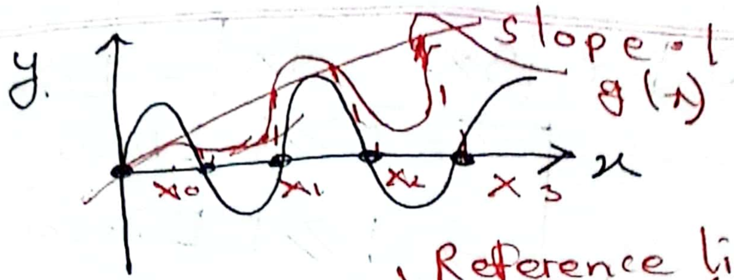
$= \dfrac{2(2x-2)}{(x^2 - 2x - 1)^2}$

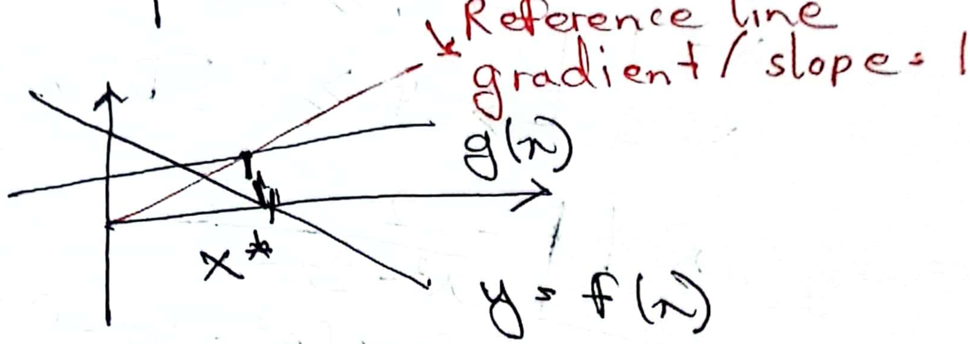
$\lambda = |g_3'(x)| = \left| \dfrac{2(2x-2)}{(x^2 - 2x - 1)^2} \right|$

$x = -1 \rightarrow 2 \geq 1$ [Divergent]

$x = 1 \rightarrow 0$ [Super linear convergent]

$x = 2 \rightarrow 4 \geq 1$ [Divergent]





The points at which the g(x) graph cuts the slope = 1. line / reference line we get the actual roots for the function at those points