Chapter - 07
Numerical Integration

$$I(f) = \int_a^b f(x)\,dx$$

↳ We will see how to approximate $f(x)$

Previously we have seen that in order to approximate $f(x)$ we have used polynomial. We will be using the same concept here.

$f(x)$ will be approximated to $P_n(x)$

$\int_a^b P_n(x) \rightarrow$ We will integrate this polynomial.

If we want to represent the polynomial in terms of Lagrange basis then,

$$P_1(x) = l_0(x)\,f(x_0) + l_1(x)\,f(x_1)$$

$$\int_a^b P_n(x) \Rightarrow \int_a^b \sum_{k=0}^{n} l_k(x)\,f(x_k)\,dx.$$

$$\boxed{I_n = \int_a^b \sum_{k=0}^{n} l_k(x)\,f(x_k)\,dx}$$

Weight function : $\delta_k = \int_a^b l_k(x)\, dx$

** If the nodes are of equal dictant this formula is called $\boxed{\text{Newton's Cotes formula}}$

$$I_n = \int_a^b \sum_{k=0}^{n} l_k(x)\, f(x_k)\, dx$$

$$\boxed{I_n = \sum_{k=0}^{n} \delta_k f(x_k)}$$

Closed Newton's -Cotes Formula :

$[a, b]$

$x_0 \leftarrow \quad \rightarrow x_n$

$$a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$$

$$\boxed{h = \frac{b-a}{n}}$$

open Newton's - cotes Formula :

$[a, b]$

$$a < x_0 < x_1 < x_2 \ldots < x_n < b$$

$$\boxed{h = \frac{b-a}{n+2}}$$

Trapezium Rule [Closed Newton's-Cotes formula] :

$\boxed{\text{with } n = 1}$

$n = 1$ means that two node points are given. So the interpolating polynomial will be of degree 1 and there will be 2 nodes.

$$h = \frac{b-a}{n} = \frac{b-a}{1} = b-a.$$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-b}{a-b}.$$

$$\hookrightarrow \begin{bmatrix} x \in [a,b] \\ x_0 = a \\ x_1 = b \end{bmatrix}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$= \frac{x-a}{b-a}.$$

$$\delta_0 = \int_a^b l_0(x)\, dx$$

$$\delta_0 = \frac{1}{a-b} \int_a^b (x-b)\, dx$$

$$\delta_0 = \frac{1}{a-b} \left[\frac{1}{2} x^2 - bx \right]_a^b$$

$$\delta_0 = \frac{b-a}{2}.$$

$$\delta_1 = \frac{1}{b-a} \int_a^b (x-a)\, dx .$$

$$\delta_1 = \frac{1}{b-a} \left[\frac{1}{2} x^2 - ax \right]_a^b$$

$$\delta_1 = \frac{b-a}{2} .$$

$$I_n = \int_a^b P_1(x)\, dx$$

$$I_n = \int_a^b l_0(x) f(x_0) + l_1(x) f(x_1)\, dx .$$

$$I_n = \int_a^b l_0(x) f(x_0)\, dx + \int_a^b l_1(x) f(x_1)\, dx .$$

$$I_n = \delta_0 f(x_0) + \delta_1 f(x_1)$$

$$\boxed{I_n = \frac{b-a}{2} \left(f(a) + f(b) \right)} \quad \boxed{\text{for } n = 1}$$

↳ Closed Newton's - Cotes formula with $n = 1$.

Example : $[0, 2]$ and $f(x) = e^x$.

1) Numerical Integration using Trapezium Rule .

2) Actual value .

3) % Error .

1st part:

$$I_n = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

$$I_n = \frac{2-0}{2}\left(e^0+e^2\right)$$

$$I_n = 8.3891.$$

2nd part:

$$\int_0^2 e^x \, dx \quad \text{[Actual value]}$$

$$= \left[e^x\right]_0^2$$

$$= 6.3891.$$

3rd part:

$$\% \text{ Error} = \left|\frac{8.3891-6.3891}{6.3891}\right| \times 100\%$$

$$\% \text{ Error} = 31.65\%.$$

Upper Bound of Interpolation Error :

$$\left|\frac{f^{(n+1)}}{(n+1)!}(\xi)\right|_{\max\ \xi \in [a,b]} \underbrace{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}_{\omega(x)}$$

↳ Cauchy's Theorem

Example : $n = 1$ $f(x) = e^x$ $[0, 2]$

$$= \left|\frac{f^{(n+1)}}{(n+1)!}(\xi)\right|_{\max \in (a,b)}$$

$f(x) = e^x$
$f'(x) = e^x$
$f^2(x) = e^x$

$$= \left|\frac{f^2}{2!}(\xi)\right|$$

$$= \frac{1}{2!}e^{(2)}$$

$$\omega(x) = (x - x_0)(x - x_1)$$
$$\omega(x) = (x - 0)(x - 2)$$
$$\omega(x) = x^2 - 2x .$$

$$\int_0^2 \omega(x)\, dx$$

$$\int_0^2 (x^2 - 2x)\, dx$$

$$\left[\frac{1}{3}x^3 - \frac{2}{2}x^2\right]_0^2$$

$$= 4/3 .$$

upper bound $= \frac{1}{2!} e^{2} \left(\frac{4}{2}\right)$.

Composite Newton's Cotes Formula

This method improves the result without increasing the actual node numbers.

$[a, b]$ into m sub intervals equal widths.

$C_{1,m} \rightarrow$ composite newton's cotes formula notation.

$$C_{1,m}(f) = \sum_{i=0}^{m} l_{1i} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{m-1}) + f(x_m)\right]$$

$[0, 2]$ $f(x) = e^{x}$ $\boxed{m = 2}$.

$$h = \frac{b-a}{m} = \frac{2-0}{2} = 1.$$

x_0, x_1, x_2

$x_0 \rightarrow 0$, $x_1 \rightarrow$ unknown, $x_2 \rightarrow 2$

$$x_1 = x_0 + h$$
$$x_1 = 0 + 1$$
$$x_1 = 1$$

$$I_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$I_n = \frac{1}{2}\left[e^0 + 2(e^1) + e^2\right].$$

Ex: $[0, 2]$ $\quad f(x) = e^x \quad m = 4.$

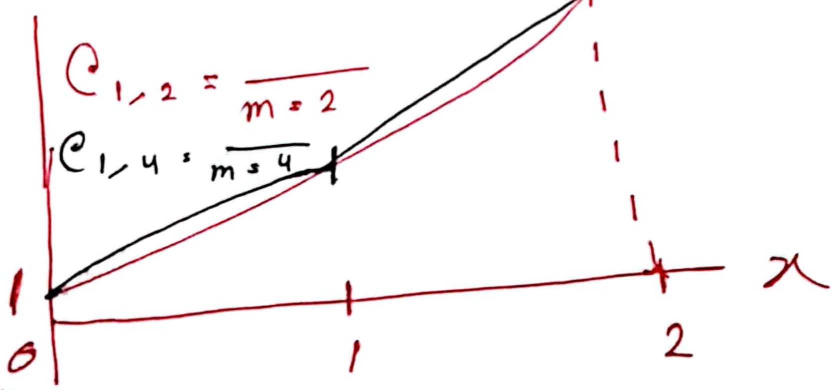
$$h = \frac{b-a}{m} = \frac{2}{4} = 0.5$$

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1.$$
$$x_3 = 1.5, \quad x_4 = 2.$$

$$I_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$I_n = \frac{0.5}{2}\left[e^0 + 2e^{0.5} + 2e^1 + 2e^{1.5} + e^2\right].$$

Note: * The errors decrease as m increases



* The error decreases by a factor of 4