Numerical Integration

$$I(f) = \int_a^b f(x)\,dx$$

f(x) will be approximated to a polynomial Pn(x)

$$I(f) = \int_a^b \underbrace{P_n(x)\,dx}$$

$l_k \leftarrow$ Lagrange Basis :

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$\underbrace{n=1}$

nodes = 2
lagrange basis = 2

$$P_n(x) = \sum_{k=0}^{n} l_k(x) f(x)$$

degree

$$I_n = \int_a^b \underbrace{\sum_{k=0}^{n} l_k(x) f(x)\,dx}$$

$\hookrightarrow$ weight function

$$\delta_k = \int_a^b l_k(x)\,dx \quad \delta_k$$

$\delta_k \longrightarrow$ weight function

If the nodes are of equal distance then this formula is called Newton's Cotes Formula

$$I_n = \sum_{k=0}^{n} \delta_k f(x_k)$$

*****Closed Newton's Cotes Formula:**
x = [a,b] a = x0, b = xn
a= x0 < x1 < x2 < x3 < ....<x(n-1)<xn = b
h = (b-a)/n

Open Newton's Cotes Formula:
x = [a,b]
a < x0 < x1 < x2 < x3< ....< xn < b
h = (b-a)/(n+2)

Trapezium Rule [Closed Newton's Cotes Formula]:
***with n = 1
n = 1 degree = 1 so polynomial will have 2 nodes.

Formula: In = (b-a)/2 (f(a) + f(b)) with n = 1

Example: f(x) = e^x with given [0,2]
i)Numerical Integration using Trapezium Rule/Closed Newton's Cotes Formula
ii) Actual value
iii) percentage error

i) [0,2] a = 0, b = 2
f(a) = f(0) = e^0 = 1
f(b) = f(2) = e^2
In = (b-a)/2 (f(a) + f(b))
In = (2 - 0) / 2 * (1 + e ^2)

b a f(a) f(b)

$= 8.3891$

ii) $\int_0^2 e^x\,dx = \left[e^x\right]_0^2 = 6.3891$

iii) $\%\text{error} = \left|\dfrac{8.3891-6.3891}{6.3891}\right| \times 100\% = 31.65\%$

Upper Bound of Interpolation Error: (max value)

Cauchy's Theorem:

| f^(n+1)/ (n+1)! (ξ) | $\underbrace{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}_{w(x)}$

Example: n = 1 , f(x) = e ^x [0,2] nodes = 2

|f^(n+1)/(n+1)! ($\xi| (x-x_0)(x-x_1)$

$\leftarrow$ apply integration

$$\left|\frac{f^{(2)}(\xi)}{2!}\right| = \frac{1}{2!}\left|f^{(2)}(\xi)\right|$$

$\hookrightarrow$ 2 bar differentiate

$f^{(2)} = e^x = e^0 = 1$

$= e^2 = 7.389 \leftarrow$ max value

$= \frac{1}{2!} \times e^2$

w(x) = (x- x0)(x -x1) $\leftarrow$ Integrate
= (x-0)(x-2)
= x^2 - 2x

$\int_0^2 w(x)\,dx$

$= 4/3$

(1/2!*e^2) * (4/3) = upper bound result

Composite Newton's Cotes Formula:

The method improves the result(reduces the percentage error) by keep the actual nodes number same.

[a,b] interval divided into m sub intervals

c1,m - notation $= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + f(x_n)\right]$

Example: f(x) = e^x   [0,2]   m = 2 $\leftarrow$ sub intervals

h = (b-a)/m = (2-0)/2 = 1 [nodes difference]

x0,x1,x2
x0 = 0 x1 = 1 x2 = 2

$x_1 = x_0 + h = 0+1 = 1$
$x_2 = x_1 + h = 1+1 = 2$

$x_0 = 0,\ x_1 = 2$
$[x_0, x_1] \cup [x_1, x_2]$

0,1 U 1,2

In = h/2 [f(x0) + 2f(x1) + f(x2)]

= 1/2 [e^0 + 2*e^1 + e^2]
= 6.9128

Ex: $f(x) = e^x\ [0,2]$
$m = 4$

$h = \frac{b-a}{m} = \frac{2-0}{4} = 0.5$

$x_0 = 0,\ x_1 = x_0 + h = 0+0.5 = 0.5,$
$x_2 = x_1 + h = 1,\ x_3 = x_2 + h = 1.5,$
$x_4 = x_3 + h = 1.5 + 0.5 = 2$

[0,2] m = 4
[0,0.5] U [0.5,1] U [1,1.5]U[1.5,2]

$$I_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

numerical integration

$$= \frac{0.5}{2}\left[e^0 + 2e^{0.5} + 2e^1 + 2e^{1.5} + e^2\right]$$

% error