Consider a fixed-point function $g(x) = (9x-1)^{1/3}$. The corresponding nonlinear function $f(x)$ has a solution $x_\star \in \mathbb{R}$.

(a) (2 marks) Show that $g(x)$ would lead to linear convergence if $x_\star > \frac{1}{9}\left(1+\sqrt{27}\right)$. Hint: Use the condition for linear convergence.

> **Solution:** Here, we have, $g'(x) = \frac{1}{3}\frac{9}{(9x-1)^{2/3}} = \frac{3}{(9x-1)^{2/3}}$. Therefore, for linear convergence, we must have,
>
> $$g'(x_\star) < 1 \implies \frac{3}{(9x_\star - 1)^{2/3}} < 1 \implies x_\star > \frac{1}{9}\left(1+\sqrt{27}\right).\checkmark$$

(b) (2 marks) Starting from $x_0 = 2.5$. find the value of $x_\star$ after 3 iterations upto 6 significant figures. Use the memory function CALC or equivalent to calculate easily.

> **Solution:** Here the iteration formula is, $x_{k+1} = g(x_k) = (9x_k - 1)^{1/3}$ for $k = 0, 1, 2, 3$. Starting with $x_0 = 2.5$, we find up to six significant figures,
>
> $$\begin{aligned} x_0 &= 2.5 \\ x_1 &= g(x_0) = 2.78065\checkmark \\ x_2 &= g(x_1) = 2.88553\checkmark \\ x_3 &= g(x_2) = 2.92284\checkmark \end{aligned}$$

Consider the fixed point function, $g(x) = \frac{2x+1}{\sqrt{x+1}}$. Show that to be super-linearly convergent, the root must satisfy $x\star = -3/2$. [4 marks]

$$g'(x) = \frac{2x+3}{2\left(\sqrt{x+1}\right)^3}$$

$$g'(x_*) = 0$$

$$\frac{2x_*+3}{2\left(\sqrt{x_*+1}\right)^3} = 0$$

$$\therefore x_* = \frac{-3}{2}$$